The Increasing Convex Order and the Tradeoff of Size for Risk

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Abstract: One random variable is larger than another in the increasing convex order if that random variable is preferred or indifferent to the other by all decision makers with increasing and convex utility functions. Decision makers in this set prefer larger random variables and are risk loving. When a decision maker whose utility function is increasing and concave is indifferent between such a pair of random variables, a tradeoff of size for risk is revealed, and this information can be used to make comparative static predictions concerning the choices of others. For random variables ranked by the increasing convex order, the choices of all those who are strongly more (or less) risk averse can be predicted. Thus, the increasing convex order, together with Ross's (1981) definition of strongly more risk averse, can provide additional comparative static findings in a variety of decision problems. The analysis here discusses the decision to self-protect, and several others.

Key Words: the increasing convex order, increase in risk, stochastic dominance, strongly more risk averse, self-protection

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1. Introduction

Building decision models that involve tradeoffs is a distinguishing feature of economic analysis.[1] In models with randomness, the most important of these tradeoffs involves choosing between a larger and riskier random variable and a smaller and less risky one. Mean-variance (M-V) decision models represent this tradeoff in a very simple and direct fashion. M-V models measure size by the mean μ and risk by the standard deviation σ, and use a utility function $V(\sigma, \mu)$ to represent a decision maker's willingness to trade off size for risk.

In expected utility decision models, representing the size for risk tradeoff is a much more complex problem. There is no agreed upon measure of the size of a random variable, nor is there an agreed upon measure of the magnitude of risk.[2] First degree stochastic dominance provides a partial order for size, and the Rothschild and Stiglitz (1970) definition of an increase in risk provides a partial order for risk, but neither provides a measure. The analysis here employs the increasing convex order from the mathematical statistics literature to aid in modeling this size for risk tradeoff.

One random variable is said to be larger than another in the increasing convex order if that random variable is preferred or indifferent to the other by all decision makers with increasing and convex utility functions. Larger in the increasing convex order can also be characterized by a condition on cumulative distribution functions. This condition is similar in form to those used to define various forms of stochastic dominance. It is this characterization

[1] Greg Mankiw, in his Principles of Economics textbook, lists as Principle 1: "People Face Tradeoffs".

[2] Aumann and Serrano (2008) do provide a measure for risk, but their measure appears to combine both the elements of size and risk.

using cumulative distribution functions that is the fundamental tool for doing comparative static analysis.

While the particular group of decision makers associated with the increasing convex order is not one that is frequently analyzed in economic analysis, the increasing convex order proves to be very useful when discussing a decision maker's willingness to trade off size for risk. Recently, Liu and Meyer (2013) show that whenever a decision maker is observed to be indifferent between two random variables where one is larger than the other in the increasing convex order, a willingness to accept a reduction in size in order to also reduce risk is revealed. Liu and Meyer (L-M) measure this willingness to exchange size for risk using the rate of substitution of a first degree stochastic dominant increase in the random variable for a Rothschild and Stiglitz (1970) risk increase. More importantly for this analysis, L-M show that the magnitude of this rate of substitution increases as the decision maker becomes strongly more risk averse as defined by Ross (1981). Thus, for pairs of random variables where one is larger than the other in the increasing convex order, if any decision maker is indifferent between them, the choices of all those who are strongly more (or less) risk averse can be predicted. Such comparative static predictions are an integral part of the analysis of many economic decisions involving risk.

The increasing convex order is a term defined in the mathematical statistics literature. That literature also provides several equivalent ways to characterize this order, including a characterization that is based on the CDFs of random variables. The focus in the mathematical statistics literature, however, is on the partial order defined over random variables rather than using that order to frame the size for risk tradeoff. Although some of the alternative characterizations of the increasing convex order have been used before by others doing economic analysis, that work appears to be unaware of the full range of characterizations of the increasing convex order that are available, and does not refer to findings in the mathematical statistics literature. In particular, the CDF-based characterization of the increasing convex order that is emphasized here has not been used before for economic analysis. The primary reason for discussing the increasing convex order, and especially its CDF based characterization, is the comparative static statements that it can be used to demonstrate.

This paper does two main things and is organized as follows. First in Section 2, the increasing convex order is described and several equivalent characterizations are given. The majority of this material comes from the mathematical statistics literature. The terminology employed there is different from that used by most economists, but is easily translated into discussion that fits the more familiar stochastic dominance framework. The connection between the increasing convex order, an order over random variables, and Ross's strongly more risk averse order, an order over expected utility maximizing decision makers, is also reviewed.

After this review and summary, Section 3 presents an extensive discussion of how the increasing convex order, and Ross's strongly more risk averse order, can be used together to derive new comparative static statements in a variety of decision models. The section begins with the decision to self-protect. With self-protection, the least preferred outcome is made smaller by the amount paid for self-protection. This is an example of the so-called "left tail problem" where an activity which appears to be risk-reducing, in this case self-protection, shifts the worst outcome leftward. When the left tail problem occurs, it is well known that risk aversion can play an unusual role in the decision process. Some sufficiently risk averse decision makers, for instance, would not choose the apparently less risky alternative, self-protection. In addition, for any reference decision maker who chooses self-protection, there are always some decision makers who are more Arrow-Pratt risk averse who would not choose self-protection.[3] The increasing convex order provides a relatively simple condition on random variables that indicates when all those who are strongly more risk averse choose to self-protect whenever a reference decision make does so. This is true even when the left tail problem is present. Comparative static analysis in several other decision models is discussed as well. These include the insurance model with contract nonperformance by Doherty and Schlesinger (1990), the partial insurance model implicit in Ross (1981), and the decision model on risk-reducing investment by Jindapon and Neilson (2007).

2. Characterizations of the Increasing Convex Order

In terms of notation, let $\tilde{x}$ and $\tilde{y}$ denote two random variables with cumulative distribution functions (CDF) F(x) and G(x) respectively. Assume that the supports of all random variables lie in a bounded interval denoted [a, b] with no probability mass at the left endpoint a. This implies that F(a) = G(a) = 0 and F(b) = G(b) = 1. Let μ_F and μ_G denote the mean values of these alternatives, and σ_F^2 and σ_G^2 their variances. The convention used here is that the conditions are stated so that $\tilde{x}$ or F(x) dominates or is preferred to $\tilde{y}$ or G(x) rather than the reverse. It is assumed that any utility function u(x) is differentiable at least three times. Before presenting the increasing convex order, the definitions for three well known types of changes in a random variable are very briefly reviewed to establish a context for later findings.

[3] See Section 3.5 for a more complete discussion of the left tail problem and its implications.

The definition of first degree stochastic dominance (FSD) is provided by Hadar and Russell (1969) and Hanoch and Levy (1969). FSD is a well accepted way to define when one random variable is larger than another.

Definition 1: $\tilde{x}$ is larger than $\tilde{y}$ or $\tilde{x}$ dominates $\tilde{y}$ in FSD if G(x) ≥ F(x) for all x in [a, b].

It is well known that in expected utility (EU) decision models, F(x) dominates G(x) in FSD if and only if $E_F u(x) \geq E_G u(x)$ for all u(x) with u'(x) ≥ 0. It is also the case that FSD implies that $\mu_F \geq \mu_G$, but places no restriction on the relative sizes of σ_F^2 and σ_G^2.

Rothschild and Stiglitz (R-S) (1970) define what it means for one random variable to be riskier than another. Like FSD, this definition is well accepted as the way to define an increase in risk.

Definition 2: $\tilde{y}$ is riskier than $\tilde{x}$ if $\int_a^y [G(x) - F(x)]dx \geq 0$ for all y in [a, b] with equality holding at y = b.

It is also well known that G(x) is riskier than F(x) if and only if $E_F u(x) \geq E_G u(x)$ for all u(x) with u"(x) ≤ 0. The assumption that equality holds at y = b implies that $\mu_F = \mu_G$. R-S riskier implies that $\sigma_F^2 \leq \sigma_G^2$.

Second degree stochastic dominance (SSD), also defined by Hadar and Russell and Hanoch and Levy, is closely related to R-S riskier.

Definition 3: $\tilde{x}$ dominates $\tilde{y}$ in SSD if $\int_a^y [G(x) - F(x)]dx \geq 0$ for all y in [a, b].

It is the case that F(x) dominates G(x) in SSD if and only if $E_F u(x) \geq E_G u(x)$ for all u(x) with $u'(x) \geq 0$ and $u''(x) \leq 0$. SSD implies that $\mu_F \geq \mu_G$, and like FSD, SSD places no restriction on the relative sizes of σ_F^2 and σ_G^2.

The definition of larger in the increasing convex order is given below and comes from the mathematical statistics literature.[4]

Definition 4: Random variable $\tilde{y}$ is larger than $\tilde{x}$ in the increasing convex order if $E_G[u(x)] \geq E_F[u(x)]$ for all u(x) with $u'(x) \geq 0$ and $u''(x) \geq 0$.

Larger in the increasing convex order requires that $\mu_G \geq \mu_F$ which is the opposite from the requirement imposed by FSD or SSD. Like FSD and SSD, larger in the increasing convex order places no restriction on the relative sizes of σ_F^2 and σ_G^2. Obviously, the relation defined in Definition 4 is transitive, therefore creating a partial order, the "increasing convex order" over random variables. As is clear later (in Theorem 2), the change from F(x) to G(x), when G(x) is larger than F(x) in the increasing convex order, can be decomposed into a change becoming larger (in the FSD sense) and a change becoming riskier (in the R-S sense). Because larger is a good property when $u'(x) \geq 0$, and riskier is a bad property when $u''(x) \leq 0$, when a random variable becomes larger in the increasing convex order, the change is a combination of a good change and a bad change for those decision makers with $u'(x) \geq 0$ and $u''(x) \leq 0$. In other words, a pair of random variables linked by the increasing convex order presents a non-satiated and risk averse decision maker with a choice involving trading off size for risk.

[4] For the definition of larger in the increasing convex order and related results in the mathematical statistics literature, see Shaked and Shanthikumar (2007). A closely related concept is the so-called "stop-loss order" studied in the actuarial science literature (Denuit et al. 2005).

Several theorems in the mathematical statistics literature provide alternate ways to characterize the increasing convex order. The one given next takes a form similar to the stochastic dominance definitions just reviewed. Because the CDF-based characterization in Theorem 1 takes a different form than that in the mathematical statistics literature, an alternative and simple proof of Theorem 1 is provided in the Appendix.[5]

Theorem 1: Random variable $\tilde{y}$ is larger than $\tilde{x}$ in the increasing convex order if and only if $\int_a^y [G(x) - F(x)]dx \geq -Q$ for all y in [a, b] where $Q = \mu_G - \mu_F \geq 0$.

The condition in Theorem 1 is easily tested; it allows one to determine whether G(x) is larger than F(x) in the increasing convex order by direct computation.

Larger in the increasing convex order can also be characterized using a two-step procedure. The first of these steps is to make $\tilde{x}$ larger in the FSD sense, and the second step is to make this result riskier in the R-S sense, reaching $\tilde{y}$ in the end. The following theorem indicates that this characterization is indeed equivalent to the stated definition of larger in the increasing convex order. In addition, whether the increase in risk or the increase in size occurs first is not important. This result is also from Shaked and Shanthikumar (2007).

Theorem 2: The following two statements give equivalent characterizations of G(x) being larger than F(x) in the increasing convex order.

[5] The condition in Theorem 1 could be equivalently written as $\int_y^b [G(x) - F(x)]dx \leq 0$ for all y in [a, b]. While this is a more compact expression, we choose the form given in Theorem 1 to emphasize the fact that larger in the increasing convex order allows negative values for $\int_a^y [G(x) - F(x)]dx$.

i) There exists CDF H(x) such that G(x) is riskier than H(x) and H(x) dominates F(x) in the FSD.

ii) There exists CDF $H^*(x)$ such that G(x) dominates $H^*(x)$ in FSD and $H^*(x)$ is riskier than F(x).

Theorem 2 indicates that the two separate changes, becoming larger and becoming riskier, can be applied sequentially and in either order. Decomposing a relatively more complex change into two simpler changes seems to be quite natural. Indeed, in their pursuit of a better understanding of Ross's strongly more risk averse order, Ross (1981), Jewitt (1986) and Chiu (2005) all use this two-step decomposition. These studies do not, however, use the characterization given in Theorem 1 and this limits the analysis that can be carried out.

It is interesting to note that SSD changes to a CDF can also be decomposed into two separate parts. One can show that G(x) dominates F(x) in SSD if and only if there exists a CDF H(x) such that H(x) dominates F(x) in FSD and G(x) is <u>less</u> risky than H(x). SSD and larger in the increasing convex order are similar yet in some sense opposite concepts. Each involve increasing the size of the random variable, but one decreases and the other increases the riskiness.

If utility functions satisfy the usual assumptions made in economic analysis, $u'(x) \geq 0$ and $u''(x) \leq 0$, and $\tilde{y}$ is larger than $\tilde{x}$ in the increasing convex order, then some decision makers would choose $\tilde{y}$ over $\tilde{x}$ and others would choose $\tilde{x}$ over $\tilde{y}$. More importantly, when a particular decision maker chooses one over the other, information concerning that decision maker's rate of substitution or tradeoff of size for risk is obtained. This information is useful for predicting the choice between $\tilde{x}$ and $\tilde{y}$ for those decision makers with rates of substitution which are larger or smaller than that for the reference decision maker. This very simple and nontechnical statement is the essence of the way the concept of larger in the increasing convex order is combined with the Ross strongly more risk averse order to derive comparative static results. First a short review of the Ross order and facts concerning the rate of substitution of size for risk is needed.

<u>Definition 5:</u> (Ross) Suppose u'(x) > 0 and u"(x) < 0. u(x) is strongly more risk averse than v(x) on [a, b] if there exists a $\lambda > 0$ such that $\frac{u''(x)}{v''(x)} \geq \lambda \geq \frac{u'(y)}{v'(y)}$ for all x and y in [a, b].

The Ross strongly more risk averse order implies but is not implied by u(x) more Arrow-Pratt risk averse than v(x). The Ross strongly more risk averse order has several other equivalent characterizations including one based on the tradeoff between size and risk. Liu and Meyer (2013) discuss these characterizations and also define a rate of substitution T_u for an arbitrary FSD change and an arbitrary R-S risk change to a random variable. This definition is given below and indicates the rate at which a particular decision maker is willing to substitute the one change for the other, trading off size for risk.

<u>Definition 6</u>: For any increase in risk from F(x) to G(x) and for any FSD deterioration from F(x) to H(x), the rate of substitution of size for risk is given by

$$T_u(F(x), G(x), H(x)) = \frac{\int_a^b u(x)d[F(x) - G(x)]}{\int_a^b u(x)d[F(x) - H(x)]} = \frac{\int_a^b u'(x)[G(x) - F(x)]dx}{\int_a^b u'(x)[H(x) - F(x)]dx}$$

L-M show that u(x) is strongly more risk averse than v(x) on [a, b] if and only if $T_u \geq T_v$ for all F(x), G(x) and H(x) such that G(x) is riskier than F(x), and F(x) first-degree stochastically dominates H(x). That is, this rate of substitution provides another way to characterize the Ross strongly more risk averse order.

The following theorem uses the concept of larger in the increasing convex order to further explore the connection between the strongly more risk averse order of Ross, and the size-for-risk tradeoff. This theorem provides the comparative statics tool used in Section 3. The proof of Theorem 3 is given in the Appendix.

Theorem 3: If G(x) is larger than F(x) in the increasing convex order, then

i) $E_F v(x) \geq E_G v(x)$ implies $E_F u(x) \geq E_G u(x)$ for all u(x) who are strongly more risk averse than v(x).

ii) $E_F v(x) \leq E_G v(x)$ implies $E_F u(x) \leq E_G u(x)$ for all u(x) who are strongly less risk averse than v(x).

Combining conditions i) and ii), Theorem 3 implies that for any decision maker who is indifferent to a change from CDF F(x) to CDF G(x) which is larger in the increasing convex order, those strongly more risk averse than this person prefer F(x) (in some sense the smaller and less risky alternative), while those who are strongly less risk averse prefer G(x) (in some sense the larger and riskier option). The result in Theorem 3 has been previously given by Ross (1981), Jewitt (1986) and Chiu (2005), using the two-step decomposition definition of larger in the increasing convex order. What is new here is that Theorem 1 with its integral condition on cumulative distribution functions provides a concrete way to check for or impose the larger in the increasing convex order condition on random variables so that comparative static results can be derived.

This concludes the summary of the main results that are available in various places and using a variety of terminology concerning the larger in the increasing convex order for random variables. The connection of this order to Ross's strong more risk averse order is also reviewed. The remainder of the paper focuses on and uses the stochastic dominance like condition provided in Theorem 1 to determine the implications of these findings for comparative static analysis. The decision to self-protect is examined, and other decision models as well.

3. Self-Protection and Other Economic Applications

This section includes several comparative static theorems in a variety decision models that are obtained by making use of the concepts of the strongly more risk averse and the increasing convex orders.

3.1 Self-Protection

Ehrlich and Becker (1972) define self-protection as a costly action that reduces the probability that bad outcomes or losses occur. Dionne and Eeckhoudt (1985) use many examples to show that the relationship between the demand for self-protection and the risk aversion level of the decision maker is not a straightforward one. Most notably, as the decision maker becomes more Arrow-Pratt risk averse, higher levels of self-protection need not be selected. The reason for this is that increased self-protection reduces the probability of loss, but not the size of the loss. As a consequence, the smallest and least preferred outcome is shifted left by the cost of self-protection. In an expected utility decision setting, this leftward shift implies that there is always a decision maker who is sufficiently risk averse to not prefer the increase in self-protection. This is an example of the so-called left tail problem.

The literature since Dionne and Eeckhoudt has imposed complex joint conditions on risk aversion, prudence, and the relative sizes of the risk increases and decreases that result from an increase in self-protection as a way to overcome this comparative static difficulty.[6] The analysis here shows that when the self-protection model satisfies modest restrictions, decreased self-protection leads to an outcome variable which is larger in the increasing convex order, and hence Ross's strongly more risk averse order can be used to make straightforward comparative static statements. The comparative static predictions obtained here complement and augment those currently available.

Erhlich and Becker, and also Dionne and Eeckhoudt, conduct their analysis of self-protection in a simple decision model where there are just two possible outcomes, a loss of fixed size L, or no loss at all. Much of the research on self-protection has maintained this Bernoulli distribution assumption and the discussion here does also. The notation of Eeckhoudt and Gollier (2005) is used. Assume that a decision maker begins with certain wealth w and that this wealth is subject to loss $L > 0$ with probability p(e). The variable e represents the level of expenditure on self-protection chosen by the decision maker. The probability of loss p(e) is assumed to be decreasing in e. Final wealth W can take on one of two values, either (w - e) or (w - L - e).

Assume that e_1 and e_2 represent two levels of self-protection with $e_2 > e_1$. In addition, to simplify notation, let $p_1 = p(e_1)$ and $p_2 = p(e_2)$. The CDF for outcome variable W associated with the higher level of self-protection e_2 is denoted F(W), and that associated with e_1 as G(W). To apply Theorem 3, conditions are needed so that G(W) is larger than F(W) in the increasing

[6] For example, see Briys and Schlesinger (1990), Lee (1998), Jullien, Salanie and Salanie (1999), Chiu (2000), Eeckhoudt and Gollier (2005), Liu, Rettenmaier and Saving (2009) and Meyer and Meyer (2011).

convex order. The difference between these two CDFs, [G(W) - F(W)], is given below, and Figure 1 displays this information graphically.[7]

$$[G(W) - F(W)] = \begin{cases} 0 & \text{for } W < w - L - e_2 \\ -p_2 & \text{for } w - L - e_2 \leq W < w - L - e_1 \\ p_1 - p_2 & \text{for } w - L - e_1 \leq W < w - e_2 \\ p_1 - 1 & \text{for } w - e_2 \leq W < w - e_1 \\ 0 & \text{for } w - e_1 \leq W \end{cases}$$

Clearly [G(W) - F(W)] becomes negative first. That is, compared to G(W), the CDF associated with more self-protection F(W) has more of a left tail. This is the left tail problem. Because of this, a sufficiently risk averse decision maker chooses less self-protection and more risk averse persons may also choose less self-protection than less risk averse persons. It is also the case that for appropriate parameter values, G(W) is larger than F(W) in the increasing convex order and thus Theorem 3 can be used to indicate who would choose more self-protection based on the strongly more risk averse order. The integral condition in Theorem 1 provides a direct and easily computed method to find these parameter values. G(W) is larger than F(W) in the increasing convex order if and only if

$$(e_2 - e_1)p_2 \leq (w - p_1 \cdot L - e_1) - (w - p_2 \cdot L - e_2).$$

To interpret this condition, the RHS of this inequality is Q, the increase in the mean from reducing expenditure on self-protection, while the LHS is the largest negative value that $\int_a^y [G(x) - F(x)]dx$ attains for this particular [G(W) - F(W)]. Graphically, the requirement is that

[7] It is reasonable to assume that $w - L - e_1 < w - e_2$ or equivalently that $e_2 - e_1 < L$, since under no circumstance would a rational individual expend effort on self-protection beyond the size of loss, L.

area A be smaller in size than (area A) - (area B) + (area C), which reduces to area C being larger than area B. All areas are treated as positive values.

Each side of this inequality also has an economic interpretation. The RHS is the net expected cost of increased self-protection as measured by the reduced expected outcome. For this to be positive, it must be assumed that spending more on self-protection reduces the expected outcome; that is, the expected gain from increased self-protection is less than the cost incurred. This condition, that risk reduction is costly, holds at the margin for this and other risk reducing activities when the level is chosen by a risk averse decision maker.

The LHS of the inequality is a measure of the ineffectiveness of the increase in self-protection. It is the probability of the loss occurring even with the higher level of self-protection, p_2, multiplied by the extra expenditure on self-protection, $(e_2 - e_1)$. The LHS is the increase in the expected value of the wasted or ineffective expenditure on self-protection. Thus, the condition requires that the expected increase in wasted expenditure be less than the increase in net cost; that is the expected increase in cost is not a total waste.

Using Theorem 3, the following two comparative static theorems in the self-protection decision model are immediately available.

<u>Theorem 4</u>: Assume that $(e_2 - e_1)p_2 \leq (w - p_1 \cdot L - e_1) - (w - p_2 \cdot L - e_2)$. For any decision maker who chooses e_2 over e_1, all those who are strongly more risk averse also choose e_2 over e_1.

Theorem 5: Assume that $(e_2 - e_1)p_2 \le (w - p_1 \cdot L - e_1) - (w - p_2 \cdot L - e_2)$ for all $e_2 > e_1$.[8] For any decision maker who optimally chooses a level of self-protection e^*, all those who are strongly more risk averse choose a level of self-protection greater than or equal to e^*.

Theorem 4 is just a restatement of part i) of Theorem 3 in the self-protection context. Theorem 5 is the extension of this result to an optimal choice setting. The logic of the proof of Theorem 5 is straightforward. The condition $(e_2 - e_1)p_2 \le (w - p_1 \cdot L - e_1) - (w - p_2 \cdot L - e_2)$ for $e_2 > e_1$ implies that lower levels of self-protection yield larger random variables in the increasing convex order. Whenever any decision maker optimally chooses a level of self-protection, e^*, that decision reveals that e^* is preferred or indifferent to all available alternatives, both lower and higher values of e. Theorem 3 indicates that those decision makers who are strongly more risk averse than the reference person agree that e^* is preferred to all lower levels of e. Thus when these persons optimize they can only select a level of self-protection that is at least as large as e^*.

It is interesting to note that the way in which comparative static analysis is carried out using larger in the increasing convex order is somewhat different from the standard approach and is related to the procedure described by Ross in his Application I. Assumptions are made to ensure that for any two levels of the choice variable, the outcome variable for the one is larger than the other in the increasing convex order. Once this is accomplished, Theorem 3 is used to formulate and demonstrate comparative static theorems. This approach to comparative static analysis does not require that the outcome variable be differentiable in the choice variable, and in

[8] Note that this condition is equivalent to $p(e) \le 1+p'(e)L$ for all e. To give a general example in which this condition is satisfied, consider the self-protection technology $p(e) = p_0 /(1 + e)$, where p_0 is the probability of loss without self-protection i.e. when $e = 0$. A sufficient condition for $p(e) \le 1+p'(e)L$ for all e is that $L \le (1- p_0) / p_0$.

fact, allows the choice variable to be discrete. Differentiation and first and second order conditions are not part of this comparative static analysis.

An example where the test condition, $(e_2 - e_1)p_2 \leq (w - p_1 \cdot L - e_1) - (w - p_2 \cdot L - e_2)$, is met is the following. Suppose you own a bicycle whose value is \$400 and the probability of it being stolen if unlocked is .1. Thus, the expected value of this unlocked bike is \$360 and the expected loss is \$40. It is well known that if a bicycle lock that would completely eliminate theft were available, those who are sufficiently risk averse would buy this lock even if the price of the lock is more than the expected loss. Suppose now that the only lock available is one that does not completely eliminate theft, but does reduce the probability to .001, and that the price of this lock is e. The expected loss with the lock is \$.40. The test condition is $e(.001) \leq e - \$39.60$. The LHS is the expected amount lost if the lock purchase is ineffective, and the RHS is the loss in expected value from purchasing the lock. Solving for the critical value for e yields $e \geq \$39.64$. The inequality indicates that as long as the price of the lock exceeds \$39.64 then purchasing the lock leads to an outcome which is smaller in the increasing convex order. Therefore whenever a risk averse person buys the lock then so do all persons strongly more risk averse than that person. This example illustrates the general finding that when the left tail problem is small relative to the change in the mean, then if any person chooses the apparently safer choice, those who are strongly more risk averse would also make this same choice.

The theorems concerning self-protection are presented for those who are strongly more risk averse than the reference person. Similar results hold for those strongly less risk averse as well. These comparative static findings concerning who would choose more self-protection augment the existing ones and do not contradict or overlap with them.

3.2 The Insurance Decision with Contract Nonperformance

The self-protection example shows that even though the A-P more risk averse condition is not enough for obtaining definitive comparative statics results in the presence of the left tail problem, the strongly more risk averse condition of Ross is sufficient for this purpose as long as restrictions on the model parameters ensure that changing the size of the choice variable leads to an outcome variable which is larger in the increasing convex order.

There are other decision models where the least preferred outcome is shifted left by a change in a choice variable, and where the outcome variables, for certain parameter values, are related to one another by the increasing convex order. Doherty and Schlesinger's (1990) model of contract nonperformance, and the model of excluded losses used by Meyer and Meyer (2010) have this property. Models of information acquisition also are such that the information obtained does not prevent and may worsen the least preferred outcome. Theorem 3 can be a useful tool for obtaining additional comparative static results in these models.

To further illustrate the fact that the restrictions necessary to apply Theorem 3 are not onerous, consider a simplified version of Doherty and Schlesinger's (1990) insurance model of contract non-performance. Similar to the self-protection example, it has a left tail problem in that buying insurance shifts the worst outcome leftward due to the potential contract nonperformance. Specifically, suppose that the initial wealth is W and that a loss of size L occurs with probability p. Without insurance, the final wealth x follows a Bernoulli distribution whose CDF is denoted G(x): W occurs with probability (1 - p) and (W - L) with probability p. Buying a full coverage insurance policy with premium P changes the final wealth outcome variable to one described by a different Bernoulli distribution whose CDF is denoted F(x):

(W - P) occurs with probability $(1 - p + p \cdot q)$ and (W - P - L) with probability $p(1 - q)$, where q is the probability of insurer solvency conditional on the occurrence of a loss.

To apply Theorem 3, conditions are needed so that G(x) is larger than F(x) in the increasing convex order. [G(x) - F(x)] is given below, and Figure 2 displays this information graphically.

$$[G(x) - F(x)] = \begin{cases} 0 & \text{for } x < W - P - L \\ -p(1 - q) & \text{for } W - P - L \le x < W - L \\ p \cdot q & \text{for } W - L \le x < W - P \\ p - 1 & \text{for } W - P \le x < W \\ 0 & \text{for } W \le x \end{cases}$$

Applying the conditions in Theorem 1, it can be readily obtained that $Q = \mu_G - \mu_F \ge 0$ is equivalent to (area A) - (area B) + (area C) = $P - p \cdot q \cdot L \ge 0$ and that $\int_a^y [G(x) - F(x)]dx \ge -Q$ for all y in [a, b] is equivalent to (area A) $\le Q$ or $p(1 - q)P \le P - p \cdot q \cdot L$. Therefore, G(x) is larger than F(x) in the increasing convex order if and only if $P/L \ge p \cdot q/(1 - p + p \cdot q)$. In words, this condition requires that the ratio of P to L be no less than the ratio of the probability of a loss occurring and yet the insurer being solvent, to the total probability of the insurer being solvent regardless of whether a loss occurs.

The following theorem is immediately obtained from Theorem 3.

<u>Theorem 6</u>: Assume that $P/L \ge p \cdot q/(1 - p + p \cdot q)$. For any decision maker who chooses to buy insurance in the presence of contract nonperformance, all those who are strongly more risk averse also choose to buy insurance.

3.3 Partial Insurance

The next model illustrates the comparative static application of the increasing convex order in a decision model where the one outcome variable is larger than another in the increasing convex order, but the left tail problem is not exhibited. Instead, the model yields alternatives such that $\int_a^y [G(x) - F(x)]dx < 0$ at some point y beyond the left tail of the distributions. Theorem 3 allows violations of $\int_a^y [G(x) - F(x)]dx \geq 0$ to occur at any value for y in [a, b] as long as these violations are small relative to the increase in the mean value in going from F(x) to G(x). Hence, larger in the increasing convex order can overcome comparative static difficulties in this decision model as well.

Suppose a decision maker has an asset whose value is W when no loss occurs, but is subject to two different independently distributed losses of size L_1 and L_2 which occur with probabilities q_1 and q_2, respectively. With no insurance the possible outcomes are $(W - L_1 - L_2)$, $(W - L_2)$, $(W - L_1)$ and W and these occur with probabilities, $[q_1 \cdot q_2]$, $[q_2(1 - q_1)]$, $[q_1(1 - q_2)]$ and $[1 - q_1 - q_2 + q_1 \cdot q_2]$, respectively. Denote the CDF for no insurance as G(x). Assume now that full insurance coverage is available for the smaller of the two losses, which without loss of generality is L_1. Assume that the price of this insurance is P, and that no insurance is available for loss L_2. When the decision maker chooses to purchase this insurance the possible outcomes are $(W - L_2 - P)$ and $(W - P)$ and these outcomes occur with probabilities q_2 and $(1 - q_2)$, respectively. Denote the CDF that results from choosing to insure as F(x).

Again to apply Theorem 3, conditions are needed so that G(x) is larger than F(x) in the increasing convex order; that is no insurance results in a distribution that is larger in the increasing convex order than does choosing to insure. The difference between these two CDFs, [G(x) - F(x)], is given below, and Figure 3 displays this information graphically.

$$[G(x) - F(x)] = \begin{cases} 0 & \text{for } x < W - L_1 - L_2 \\ q_1 \cdot q_2 & \text{for } W - L_1 - L_2 \le x < W - L_2 - P \\ q_1 \cdot q_2 - q_2 & \text{for } W - L_2 - P \le x < W - L_2 \\ 0 & \text{for } W - L_2 \le x < W - L_1 \\ q_1 - q_1 \cdot q_2 & \text{for } W - L_1 \le x < W - P \\ q_1 + q_2 - 1 - q_1 \cdot q_2 & \text{for } W - P < x \le W \\ 0 & \text{for } W \le x \end{cases}$$

In this decision model, [G(x) - F(x)] becomes positive first, and therefore there is no left tail problem. It is still possible that G(x) is larger than F(x) in the increasing convex order, however. Indeed, following the conditions in Theorem 1, G(x) is larger than F(x) in the increasing convex order if and only if the insurance that is available is priced to be actuarially unfair in the sense that $P \ge q_1 \cdot L_1$. Graphically, the condition $Q = \mu_G - \mu_F \ge 0$ is equivalent to the requirement that (area A) - (area B) + (area C) - (area D) ≤ 0, and the condition $\int_a^y [G(x) - F(x)]dx \ge -Q$ for all y in [a, b] is equivalent to the requirement that (area A) - (area B) be larger than (area A) - (area B) + (area C) - (area D). Simple calculation shows that these requirements are satisfied if and only if $P \ge q_1 \cdot L_1$. The next theorem follows as a result. Obviously, requiring that the price of insurance be actuarially unfair is not an onerous restriction to place on this model.

<u>Theorem 7</u>: Assume that $P \ge q_1 \cdot L_1$. For any decision maker who chooses to insure against one loss in the presence of another independent loss, all those who are strongly more risk averse also choose to insure against the first loss.

3.4 The Comparative Statics Approach of Jindapon and Neilson (2007)

As a final comparative statics application of Theorem 3, consider the following decision problem analyzed by Jindapon and Neilson (2007). In contrast to the previous three comparative statics examples, the example here makes use of the larger in the increasing convex order concept without specifying the specific CDFs.

Suppose that G(x) is R-S riskier than F(x) and that by incurring a cost c(t), G(x) can be made into a less risky distribution tF(x) + (1 - t)G(x), where t is in [0, 1], c(0) = 0, c'(t) > 0 and c''(t) > 0. Final wealth W can be denoted as W = $\tilde{x}(t) - c(t)$, where $\tilde{x}(t)$ has a CDF [tF(x) + (1 - t)G(x)]. The decision maker is assumed to choose t to maximize expected utility. Jindapon and Neilson show that a Ross more risk averse decision maker always chooses a (weakly) larger t, incurring a higher cost to get a less risky outcome variable.

The same result can be obtained using Theorem 3. To begin, it can be shown that final wealth W = $\tilde{x}(t) - c(t)$ becomes larger in the increasing convex order as t becomes smaller. To see this, note that for $t_2 < t_1$, $\tilde{x}(t_2) - c(t_2)$ is R-S riskier than $\tilde{x}(t_1) - c(t_2)$, and $\tilde{x}(t_1) - c(t_2)$ dominates $\tilde{x}(t_1) - c(t_1)$ in FSD. Thus, according to Theorem 2, $\tilde{x}(t_2) - c(t_2)$ is larger than $\tilde{x}(t_1) - c(t_1)$ in the increasing convex order when $t_2 < t_1$.

To show that a strongly more risk averse individual u(x) always chooses a larger t than a less risk averse individual v(x), or $t_u \geq t_v$, assume otherwise, i.e., assume $t_u < t_v$. Then, $\tilde{x}(t_u) - c(t_u)$ is larger than $\tilde{x}(t_v) - c(t_v)$ in the increasing convex order. Because v(x) prefers $\tilde{x}(t_v) - c(t_v)$ to $\tilde{x}(t_u) - c(t_u)$ by the definition that t_v is the optimal choice for v(x), Theorem 3 indicates that the strongly more risk averse u(x) would also prefer $\tilde{x}(t_v) - c(t_v)$ to $\tilde{x}(t_u) - c(t_u)$, contradicting that t_u is the optimal choice for u(x). Therefore, it must be that $t_u \geq t_v$. That is, a Ross more risk averse decision maker always chooses a (weakly) larger t, incurring a higher cost to get a less risky outcome variable.

3.5 Some Further Discussion

For each of these examples, one alternative G(x) has a larger mean value than another alternative F(x), but G(x) does not dominate F(x) in SSD. When this is the case, for decision makers with u'(x) ≥ 0 and u"(x) ≤ 0, some choose F(x) over G(x) and others choose G(x) over F(x) depending on their size for risk tradeoff. Theorem 3 indicates that, under the condition that G(x) is larger than F(x) in the increasing convex order, if any particular decision maker is observed to choose F(x), then those who are strongly more risk averse also choose F(x).

Diamond and Stiglitz (1974) also provide a methodology for dealing with this same question, but the findings are quite different. The Diamond and Stiglitz definition of a mean utility preserving increase in risk is given below.

Definition 7: For a person with utility function v(x), $\tilde{y}$ is a mean utility preserving increase in risk from $\tilde{x}$ if $\int_a^y v'(x)[G(x) - F(x)]dx \geq 0$ for all y in [a, b] with equality holding at y = b.

As with Definitions 1 - 4, this definition is most easily interpreted when it is connected to EU maximization. Diamond and Stiglitz show that $E_F u(x) \geq E_G u(x)$ for all u(x) who are more Arrow-Pratt risk averse than v(x) if and only if G(x) is a mean utility preserving increase in risk from F(x) for v(x). Subsequent work (Meyer 1975, 1977) shows that for any F(x) and G(x) that cross a finite number of times, the CDF which becomes positive first is a mean utility preserving increase in risk from the other for some v(x). Based on this, comparative static statements can be made.

The following theorems are true, in the self-protection model and the insurance model with contract nonperformance, respectively.

Theorem 8: For any $e_2 > e_1$, there exists a v(x) who is indifferent between e_1 and e_2 and all those who are A-P more risk averse than v(x) choose e_1 over e_2.

Theorem 9: For any premium P, there exists a v(x) who is indifferent between buying and not-buying the full coverage and all those who are A-P more risk averse than v(x) choose not to buy.

Theorems 8 and 9 illustrate the left tail problem; those who are more risk averse choose less self-protection and decline to purchase insurance at all because the worst outcome becomes even worse with more self-protection or insurance. This is a counterintuitive finding since self-protection and insurance are thought to be risk reducing activities.

For the partial insurance decision, the left tail problem is not present and it is the case that there exists a v(x) who chooses to insure, and all more risk averse than v(x) also choose to insure. It is false, however, to conclude that if any v(x) chooses to insure then so do all who are more risk averse. That is, choosing to insure is a necessary but not sufficient condition for no insurance to be a mean utility preserving increase in risk from insurance. Thus, observing a decision maker choosing to insure must be paired with knowledge of that person's utility function to draw the conclusion that those more Arrow-Pratt risk averse also would choose to insure. The definition of larger in the increasing convex order does not involve a utility function and does not impose this requirement. Since no insurance is larger in the increasing convex order than insurance, when any decision maker chooses to insure all those who are strongly more risk averse do so as well.

4. Conclusion

When a decision maker prefers more to less and also dislikes risk, choosing between two random outcomes when one is larger than the other in the increasing convex order, is a decision that trades off size for risk. This paper uses the concept of the increasing convex order to provide a framework to study decision makers' tradeoff of size for risk and derive additional comparative static findings in a variety of decision models.

The first two examples illustrate how the larger in the increasing convex order concept can be used to address the left tail problem. In both the self-protection model and the insurance model with contract nonperformance, the risk-reducing action, self-protection or the purchase of insurance, shifts the least preferred outcome further to the left. As a consequence, it is not necessarily the case that the more risk averse decision-makers, in either the Arrow-Pratt or the Ross sense, want more self-protection or are more likely to purchase insurance. The larger in the increasing convex order, however, can accommodate this left tail problem and still generate interesting and new results. Specifically, with additional restrictions which seem minor, the risk-reducing activity makes the final wealth distribution smaller in the increasing convex order. Thus, with these additional restrictions, strongly more risk averse individuals always invest more in self-protection and purchase insurance. The third and fourth examples, concerning partial insurance and risk-reducing investment, respectively, indicate that the uses of larger in the increasing convex order go beyond dealing with left tail issues.

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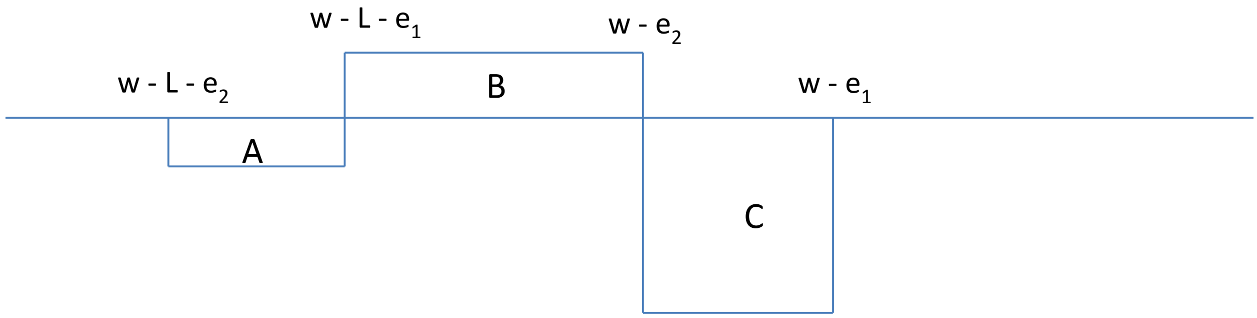

Figure 1

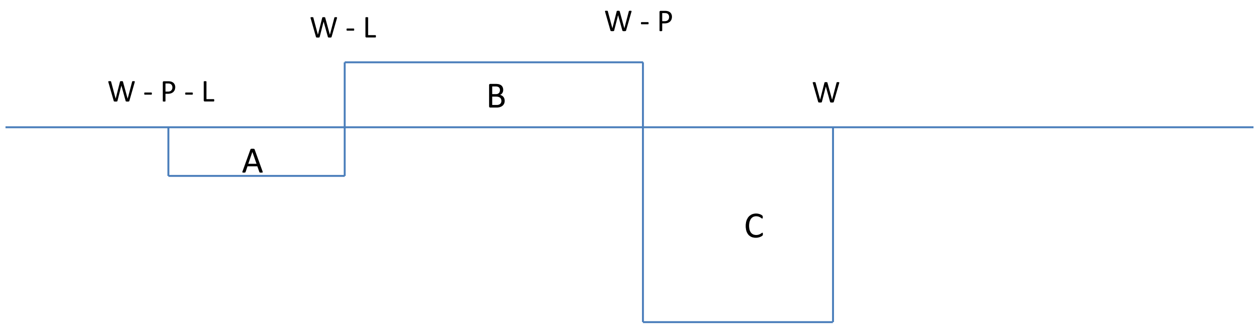

Figure 2

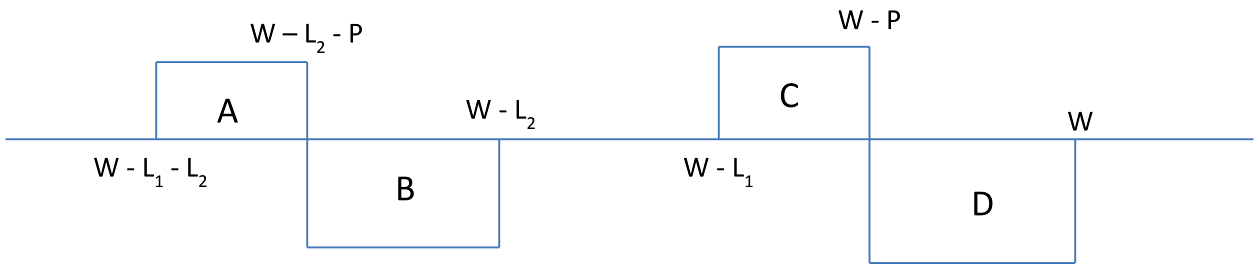

Figure 3

Appendix

Proof of Theorem 1

"if"

Suppose that $Q = \mu_G - \mu_F \geq 0$ and $\int_a^y [G(x) - F(x)]dx \geq -Q$ for all y in [a, b]. For every u(x) with $u'(x) \geq 0$ and $u''(x) \geq 0$, one obtains by using integration by parts twice

$$E_F u(x) - E_G u(x) = -u'(b)(\mu_G - \mu_F) - \int_a^b u''(y)[\int_a^y (G(x) - F(x))dx]dy$$

$$\leq -u'(b)(\mu_G - \mu_F) - \int_a^b u''(y)(-Q)dy = -u'(a)Q \leq 0.$$

"only if"

Suppose that $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \geq 0$ and $u''(x) \geq 0$. First, letting u(x) = x implies $Q = \mu_G - \mu_F \geq 0$. To show $\int_a^y [G(x) - F(x)]dx \geq -Q$ for all y in [a, b], use proof by contradiction. Assume that $\int_a^y [G(x) - F(x)]dx < -Q$ for some y in [a, b]. Then, due to continuity, there exists an interval $[\alpha, \beta] \subset (a, b)$ such that $\int_a^y [G(x) - F(x)]dx < -Q$ for all y in $[\alpha, \beta]$. Choose a special u(x) such that $u'(a) = 0$, $u''(x) > 0$ for $x \in (\alpha, \beta)$ and $u''(x) = 0$ otherwise. Then,

$$E_F u(x) - E_G u(x) = -u'(b)(\mu_G - \mu_F) - \int_a^b u''(y)[\int_a^y (G(x) - F(x))dx]dy$$

$$> -u'(b)Q - \int_a^b u''(y)(-Q)dy = -u'(a)Q = 0,$$

contradicting that $E_G u(x) \geq E_F u(x)$ for all u(x) with $u'(x) \geq 0$ and $u''(x) \geq 0$.

QED

Proof of Theorem 3

One only needs to prove i) because the proof of ii) is similar. Suppose that G(x) is larger than F(x) in the increasing convex order and $E_F v(x) \geq E_G v(x)$, the latter of which can be expressed as

$$-v'(b)(\mu_G - \mu_F) - \int_a^b v''(y)[\int_a^y (G(x) - F(x))dx]dy \geq 0.$$

Also note that G(x) being larger than F(x) in the increasing convex order implies $Q = \mu_G - \mu_F \geq 0$ and $\int_a^y [G(x) - F(x)]dx \geq -Q$ for all y in [a, b], according to Theorem 1.

Now let u(x) be a utility function that is strongly more risk averse than v(x). According to Ross (1981), there exists k > 0 and r(x), where $r'(x) \leq 0$ and $r''(x) \leq 0$, such that u(x) = kv(x) + r(x). Therefore,

$$E_F u(x) - E_G u(x) = -u'(b)(\mu_G - \mu_F) - \int_a^b u''(y)\left[\int_a^y (G(x) - F(x))dx\right]dy$$

$$= k\left\{-v'(b)(\mu_G - \mu_F) - \int_a^b v''(y)\left[\int_a^y (G(x) - F(x))dx\right]dy\right\}$$

$$- r'(b)(\mu_G - \mu_F) - \int_a^b r''(y)\left[\int_a^y (G(x) - F(x))dx\right]dy$$

$$\geq -r'(b)(\mu_G - \mu_F) - \int_a^b r''(y)\left[\int_a^y (G(x) - F(x))dx\right]dy$$

$$\geq -r'(b)(\mu_G - \mu_F) - \int_a^b r''(y)(-Q)dy = -r'(a)(\mu_G - \mu_F) \geq 0.$$

QED