Portfolio Choice in the Model of Expected Utility with a Safety-First Component

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Abstract

The standard problem of portfolio choice between one risky and one riskless asset is analyzed in the model of expected utility with a safety-first component that is represented by the probability of final wealth exceeding a "safety" wealth level. It finds that a positive expected excess return remains sufficient for investing a positive amount in the risky asset except in the special situation where the safety wealth level coincides with the wealth obtained when the entire initial wealth is invested in the riskless asset. In this situation, the optimal amount invested in the risky asset is zero if the weight on the safety-first component is sufficiently large. Comparative statics analysis reveals that whether the optimal amount invested in the risky asset becomes smaller as the weight on the safety-first component increases depends on whether the safety wealth level is below the wealth obtained when the entire initial wealth is invested in the riskless asset. Further comparative statics analyses with respect to the safety wealth level and the degree of risk aversion in the expected utility component are also conducted.

Keywords: portfolio choice; expected utility; safety first; risk aversion
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1. Introduction

Whereas the majority of economists interpret risk as dispersion or variation in an outcome variable, many everyday decision makers tend to associate risk with the outcome failing to meet a certain "safety" level. Roy (1952) first proposes the safety-first principle of decision making under risk, according to which a decision maker minimizes (or maximizes) the probability of wealth falling below (or exceeding) an exogenously given safety level.[1] The safety-first principle has been given some more flexible interpretations since then. For example, Bawa (1978) interprets the goal of the safety-first model as being to minimize the probability of shortfall subject to the constraint that the mean wealth does not fall below an exogenously given amount, whereas Arzac and Bawa (1977), de Haan et al. (1994), and Jansen et al. (2000) interpret the goal of the safety-first model as being to maximize the mean wealth subject to the constraint that the probability of shortfall does not exceed an exogenously specified number.

These various interpretations to the safety-first principle either do not allow any tradeoffs between safety and the final wealth distribution, or allow only limited tradeoffs between the two. To remedy this deficiency, and building on the axiomatic foundation of Diecidue and van de Ven (2008), Levy and Levy (2009) propose a more general model of decision making under risk that includes both an expected utility component and a safety-first component, referred to as the EU-SF model. In this model, a decision maker's concern about the final wealth distribution per se (absent of any benchmark, reference point or safety level) is captured by the expected utility of the final wealth, and his concern about meeting a safety wealth level is captured by the probability of final wealth exceeding the safety level. Arguably, the EU-SF model retains the

[1] Roy's safety-first principle is equivalent to maximizing expected utility if the utility function has only two values: a lower value if disaster occurs and a higher value if it does not (Roy 1952). Moreover, in the special case where the critical return level equals the gross return on riskless investment, Roy's safety-first principle is equivalent to the mean-variance criterion (Levy and Sarnat 1972).

best of both the EU model and the SF model, and can accommodate the full range of tradeoffs between safety and the final wealth distribution.

This paper studies the standard problem of portfolio choice between one risky and one riskless asset in the EU-SF model. We focus on the portfolio choice decision to explore the implications of the EU-SF model for risk-taking behavior for two reasons: The portfolio choice between a risky and a riskless asset provides a classic tradeoff between risk and return, and it is mathematically equivalent to a few other decision problems in which the outcome variable is linear in the choice variable, such as the production decision of a competitive firm under price uncertainty or the insurance coverage decision under coinsurance (Dionne et al. 1993, Menegatti and Peter 2020). The work here complements analyses of portfolio choices in alternative "behavioral" decision models, which include Gomes (2005), Bernard and Ghossoub (2010), He and Zhou (2011), Fortin and Hlouskova (2011), Li (2011), Eeckhoudt et al. (2016), and Iwaki and Osaki (2014, 2017), among others. Our work may also shed light on the literature on the effect of relative wealth placement in a reference group on risk taking (Linde and Sonnemans 2012, Kuziemko et al. 2014, Chao et al. 2017, Hillebrandt and Steinorth 2020). In this literature, the investment decisions that are implemented through experiments are a simplified version of the portfolio decision considered in this paper, and the concern about relative wealth position is comparable to the safety-first concern emphasized here.

We find that, as in the traditional EU model, a positive expected excess return is sufficient for investing a positive amount in the risky asset in the EU-SF model, except in the special situation where the safety wealth level coincides with the wealth obtained when the entire initial wealth is invested in the riskless asset. In this situation, the optimal amount invested in the risky asset is zero if the weight on the safety-first component is sufficiently large.

To describe the comparative statics results obtained in the paper, denote the safety wealth level, the initial wealth and the rate of return of the riskless asset as S, w_0 and i, respectively. We find that whether the optimal amount invested in the risky asset becomes smaller as the weight on the safety-first component increases depends on whether $S < w_0(1+i)$ (i.e. whether the safety wealth level is below the wealth obtained when the entire initial wealth is invested in the riskless asset). Further comparative statics analyses with respect to the safety wealth level and the degree of risk aversion in the expected utility component are also conducted. We find that a marginal increase in the safety wealth level always induces a larger amount to be invested in the risky asset under the condition that the risky return follows a uniform distribution. We also find that the optimal amount invested in the risky asset becomes smaller as the investor becomes Ross more risk averse (in the EU component), under the condition $S > w_0(1+i)$.

The paper is organized as follows. The next section introduces the EU-SF model of decision making under risk. It also presents three properties of the EU-SF framework, each being about the implications of changing one of the three parameters in the model. Section 3 describes the portfolio choice problem in the EU-SF model, and studies the relationship between a positive expected excess return and a positive optimal investment in the risky asset. Section 4 presents results of comparative statics analyses with respect to the effects on the optimal investment in the risky asset from the parameters of the EU-SF model. Finally, Section 5 concludes.

2. The EU-SF Model of Decision Making under Risk

For some random wealth $\tilde{x}$, one measure of its "riskiness" is the probability of the wealth falling below an exogenously given "safety" level *S*, or $prob.\{\tilde{x} < S\}$.[2] The safety wealth level *S* may differ from one decision maker to another, and it may also be context-specific. It could literally mean the minimum wealth level for survival, and it could also mean the wealth representing the status quo or any other aspiration level falling short of which is interpreted as a failure. It could arise from a capital adequacy requirement by regulation, from an incentive mechanism for investment funds managers that has a specific performance target, or from a threshold asset balance specified in a commercial contract (such as the minimum balance in an investment account with margin calls).[3]

The so-called "safety-first principle" of Roy postulates that a decision maker would want the probability $prob.\{\tilde{x} < S\}$ to be as small as possible, or equivalently, he would want the probability $prob.\{\tilde{x} \geq S\}$ to be as large as possible. While the safety-first principle is intuitively appealing, casual observation suggests that a decision maker should also care about the overall distribution of the random wealth $\tilde{x}$. An example given in Levy and Levy (2009) well explains the need for looking beyond $prob.\{\tilde{x} \geq S\}$ when formulating the overall objective of a decision maker. Suppose that the safety wealth level is 0, and that random wealth $\tilde{x}$ yields \$0.01 with certainty, and random wealth $\tilde{y}$ yields -\$0.01 with a probability of 1% and \$1 million with a

[2] Note that the probability $\delta = prob.\{\tilde{x} < S\} = \int_{-\infty}^{S} (S - x)^0 dF(x)$ is also referred to as the "zeroth lower partial moment" with a target wealth level *S*, where F(x) is the CDF of $\tilde{x}$. It is one of a class of downside risk measures called "nth lower partial moment" (Menezes et al. 1980, and Danielsson et al. 2006). Also note that the probability δ is closely related to Value-at-Risk or VaR in that $VaR(\delta) = -S$ (Jansen et al. 2000, and Danielsson et al. 2006).

[3] For further discussions of the safety or aspiration level of wealth, see Fishburn (1977), Diecidue and van de Ven (2008), and Levy and Levy (2009).

probability of 99%. If the safety-first principle is strictly followed, $\tilde{x}$ would be preferred to $\tilde{y}$.[4] It seems, however, most people would choose $\tilde{y}$ over $\tilde{x}$ instead.

Through a series of experiments, Levy and Levy (2009) find strong evidence that decision makers care about safety. That is, the probability of achieving at least the safety wealth level plays an important role in the decision-making process. In the meantime, they also find evidence that decision makers do care about the entire distribution of random wealth beyond the information that is captured by the probability of being safe.

Consequently, Levy and Levy (2009) propose the model of expected utility with a safety-first component – the EU-SF model – that has both an EU component that captures the utility from the entire wealth distribution excluding the safety concern, and an SF component that is represented by the probability of being safe. Specifically, the goal for choosing among a set of random wealth variables is postulated to be maximizing

$$(1-\eta)\cdot E[U(\tilde{x})]+\eta\cdot prob.\{\tilde{x}\geq S\} \tag{1}$$

where $\tilde{x}$ is the random wealth, $U(w)$ is a normalized von Neumann-Morgenstern utility function of the decision maker, S is the safety wealth level, and $0<\eta<1$ is the weight on the SF component.[5] As explained in Levy and Levy (2009), the value of η for a specific individual can be estimated based on observed choices of the individual. The need for using a normalized utility function instead of an arbitrarily chosen von Neumann-Morgenstern utility function is

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[4] As pointed out in Levy and Levy (2009), this would imply highly unrealistic preferences that are indifferent between any two wealth levels above or equal to the safety level, and also between any two wealth levels below the safety level.

[5] Note that $\eta=0$ and $\eta=1$ correspond to the traditional EU model and the most narrowly interpreted SF model, respectively. We exclude these extreme cases to focus on the non-trivial situations where both the EU component and the SF component play a role.

caused by the addition of the SF component.[6] Suppose that all the random wealth variables considered here are contained in the interval $[a, b]$. Then any utility function of the individual, $u(w)$, can be turned into a normalized utility function $U(w) = \frac{u(w)-u(a)}{u(b)-u(a)}$. Importantly, $U(w)$, and hence the objective (1), is invariant to positive linear transformations of $u(w)$. It is assumed that $U(w)$, and hence the underlying $u(w)$, is strictly increasing and strictly concave throughout the paper.

Presenting the overall objective in the EU-SF model represented by (1) as a weighted sum of an EU component and an SF component naturally combines the two concerns of a decision maker: the final wealth distribution and the probability of the final wealth exceeding a safety level. As is explained in Levy and Levy (2009), however, the objective function (1) can also be obtained from decomposing the expected utility for a discontinuous utility function that has a positive jump at the safety wealth level S.[7]

The EU-SF model has the following features. First, by adding an EU component to the SF goal, the EU-SF model improves on the SF model of Roy (1952), provides an alternative to the constrained optimization models of Arzac and Bawa (1977), Bawa (1978), de Haan et al. (1994) and Jansen et al. (2000), and allows for the full range of tradeoffs between safety and the final wealth distribution. The increased flexibility facilitates accommodating a wider range of behavioral patterns.

Second, although both the EU-SF model and the prospect theory of Kahneman and Tversky (1979) can account for the empirical and experimental evidence that individuals tend to

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[6] The need to work with a normalized von Neumann-Morgenstern utility function in some situations, rather than just any von Neumann-Morgenstern utility function, is also emphasized by Liu and Wang (2017) and Argyris et al. (2018).

[7] See Levy and Levy (2009), as well as the discussion in Diecidue and van de Ven (2008) of the axiomatic foundation for the general EU model with a safety or aspirational level of wealth.

think in terms of a reference point or a benchmark wealth level, there exists empirical and experimental evidence that people display care about the probability of final wealth exceeding an aspiration level, a phenomenon that cannot be explained by the (cumulative) prospect theory.[8] The critical difference between these two types of decision models is that outcomes are interpreted as the gains/losses in the prospect theory, but as success/failure in addition to as absolute wealth levels in the EU-SF model.

Third, like both the EU model and the SF principle, the EU-SF model respects the first degree stochastic dominance, in the sense that a decision maker in the EU-SF model chooses $\tilde{y}$ over $\tilde{x}$ whenever $\tilde{y}$ dominates $\tilde{x}$ in the first degree.[9] Unlike the conventional EU model with concave utility functions, the EU-SF model does not necessarily respect the second degree stochastic dominance. That is, an EU-SF decision maker who maximizes (1) does not necessarily prefer $\tilde{y}$ to $\tilde{x}$ if $\tilde{y}$ dominates $\tilde{x}$ in the second degree (but not in the first degree). This is because the utility from the SF component may decrease with a mean-preserving contraction in the wealth distribution, even though the utility from the EU component will always increase with such a change under the stated assumptions of $U(w)$ being increasing and concave. In comparison, note that the mean-variance model and the (original) prospect theory do not even necessarily respect the first degree stochastic dominance, although the cumulative prospect theory does.

The objective function of an EU-SF decision maker (1) is fully specified by exogenous parameters η, S and $U(w)$. We conclude this section by presenting three properties of the EU-

[8] For example, see Lopes and Oden (1999) and Payne (2005).

[9] Bernard et al. (2015) show that optimal investment choices based on preferences respecting the first degree stochastic dominance can be rationalized with the expected utility paradigm.

SF framework, each being about the implications of changing one of the three parameters.[10] The proofs of these properties are given in the appendix.

Property 1. Suppose that two EU-SF decision makers DM_1 and DM_2 differ only in the weight on the safety-first component η, with objective functions $(1-\eta_1)\cdot E[U(\tilde{x})]+\eta_1 \cdot prob.\{\tilde{x}\geq S\}$ and $(1-\eta_2)\cdot E[U(\tilde{x})]+\eta_2 \cdot prob.\{\tilde{x}\geq S\}$, respectively, where $\eta_2 > \eta_1$. Then for any $\tilde{x}$ that second-degree stochastically dominates safety wealth *S*, $\tilde{x} \succ_2 S$ implies $\tilde{x} \succ_1 S$.

The intuition for Property 1 is the following. As explained earlier, an EU-SF decision maker does not necessarily like a second degree stochastic dominant move from *S* to $\tilde{x}$ because even though the EU component will always increase from such a move, the SF component may decrease. It is straightforward to see that if such a move is overall beneficial for DM_2 who places a larger weight on the SF component than DM_1, the move must also be overall beneficial for DM_1.

Property 2. Suppose that two EU-SF decision makers DM_1 and DM_2 differ only in safety wealth level *S*, with objective functions $(1-\eta)\cdot E[U(\tilde{x})]+\eta \cdot prob.\{\tilde{x}\geq S_1\}$ and $(1-\eta)\cdot E[U(\tilde{x})]+\eta \cdot prob.\{\tilde{x}\geq S_2\}$, respectively, where $S_2 > S_1$. Then

(i) for any $\tilde{x}$ and x_0 such that $x_0 < S_1 < S_2$, $\tilde{x} \succ_2 x_0$ implies $\tilde{x} \succ_1 x_0$;

(ii) for any $\tilde{x}$ and x_0 such that $S_1 \leq x_0 < S_2$, $\tilde{x} \succ_1 x_0$ implies $\tilde{x} \succ_2 x_0$;

(iii) for any $\tilde{x}$ and x_0 such that $S_1 < S_2 \leq x_0$, $\tilde{x} \succ_2 x_0$ implies $\tilde{x} \succ_1 x_0$.

[10] Note that these properties are about the implications of changing one of the modeling parameters. Importantly, they are not about characterizing such changes. For examples of characterizing differences in preferences, see Chateauneuf et al. (2005), Bommier et al. (2012), Cohen and Meilijson (2014) and Eeckhoudt et al. (2017), in addition to the classic example of Pratt (1964).

The intuition for Property 2 is the following. First note that a change in safety wealth S only affects the SF component, and does not affect the EU component, for the evaluation of any (random) payoff. For both cases (i) and (iii), whether safety wealth is S_1 or S_2 does not affect the SF component for the evaluation of x_0, but the SF component for the evaluation of $\tilde{x}$ is smaller with S_2. Therefore, if $\tilde{x}$ is preferred to x_0 by DM_2, the former must also be preferred to the latter by DM_1. For case (ii), on the other hand, while the SF component for random payoff $\tilde{x}$ decreases from S_1 to S_2, the SF component for payoff x_0 decreases by a larger amount from S_1 to S_2. Therefore, if $\tilde{x}$ is preferred to x_0 by DM_1, the former must also be preferred to the latter by DM_2.

When we consider a change in $U(w)$, a natural candidate is a $V(w)$ that is Arrow-Pratt more risk averse than $U(w)$. However, comparative statics analysis using the notion of Arrow-Pratt more risk averse fails to generate clear-cut statements in the context of the present paper. That is, being Arrow-Pratt more risk averse in the EU component is not sufficient to cause the optimal amount invested in the risky asset to decrease (or increase) in the EU-SF model.

As Eeckhoudt et al. (2017) demonstrate, using stronger notion of greater risk aversion often facilitates obtaining definitive comparative statics statements that would be unobtainable with notions of greater risk aversion that are not as strong, for a host of decision problems in the EU framework. Therefore, we consider here the implications of replacing $U(w)$ with a Ross more risk averse $V(w)$.[11] We begin with the following definition and lemma, from Ross (1981), for two strictly increasing and strictly concave utility functions u(w) and v(w).

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[11] Ross more risk aversion has an advantage over Arrow-Pratt more risk aversion in that the former can be generalized to compare the third and even higher degree risk aversion (Modica and Scarsini 2005, Jindapon and Neilson 2007, Li 2009, Denuit and Eeckhoudt 2010, and Liu and Meyer 2013).

Definition 1. (Ross) $v(w)$ is Ross more risk averse than $u(w)$ on [a, b] if

$$-\frac{v''(x)}{v'(y)} \geq -\frac{u''(x)}{u'(y)} \qquad \text{for all } x, y \in [a, b],$$

or equivalently, if there exists a constant $\lambda > 0$ such that $\frac{v''(x)}{u''(x)} \geq \lambda \geq \frac{v'(y)}{u'(y)}$ for all $x, y \in [a, b]$.

Lemma 1. (Ross) v(w) is Ross more risk averse than u(w) on [a, b] if and only if there exists a constant $\lambda > 0$ and $\phi(w)$ such that $v(w) \equiv \lambda u(w) + \phi(w)$, where $\phi'(w) \leq 0$ and $\phi''(w) \leq 0$ on [a, b].

Definition 1 and Lemma 1 apply to any two strictly increasing and strictly concave utility functions u(w) and v(w), regardless of whether they are normalized utility functions. For two normalized utility functions $U(w)$ and $V(w)$ – with $U(a) = V(a) = 0$ and $U(b) = V(b) = 1$ – we can prove a stronger lemma than Lemma 1. The proof of the lemma below is in the appendix.[12]

Lemma 2. Normalized utility function V(w) is Ross more risk averse than normalized utility function U(w) on [a, b] if and only if there exists a constant $\lambda \geq 1$ and $\phi(w)$ such that $V(w) \equiv \lambda U(w) + \phi(w)$, where $\phi'(w) \leq 0$ and $\phi''(w) \leq 0$ on [a, b].

Comparing Lemma 2 with Lemma 1, we have more information about λ in the former case: $\lambda \geq 1$ rather than λ being just a positive constant.

Property 3. Suppose that two EU-SF decision makers DM_U and DM_V differ only in the normalized utility function in the EU component, with objective functions $(1-\eta) \cdot E[U(\tilde{x})] + \eta \cdot prob.\{\tilde{x} \geq S\}$ and $(1-\eta) \cdot E[V(\tilde{x})] + \eta \cdot prob.\{\tilde{x} \geq S\}$, respectively, where V(w) is Ross more risk averse than U(w). Then for any $\tilde{x}$, $E(\tilde{x}) \succ_U \tilde{x}$ implies $E(\tilde{x}) \succ_V \tilde{x}$.

[12] The proof of Lemma 2 is based on an intermediate result that was independently established in Liu and Wang (2017) and Argyris et al. (2018) that if normalized utility function V(w) is Ross more risk averse than normalized utility function U(w) on [a, b], then V(w) - U(w) is concave on [a, b].

The intuition for Property 3 is the following. First note that a change from U(w) to V(w) only affects the EU component, and does not affect the SF component, for the evaluation of any (random) payoff. Intuitively, the relative attractiveness of the mean of $\tilde{x}$ over $\tilde{x}$ itself by the EU component is larger under the more risk averse V(w) than under the less risk averse U(w). As a result, if $E(\tilde{x})$ is preferred to $\tilde{x}$ by DM_U, the former must also be preferred to the latter by the more risk averse DM_V.

In the conventional EU model, V(w) is Ross more risk averse than U(w) if and only if decision maker V is always willing to pay more to avoid a Rothschild-Stiglitz risk increase than decision maker U (Ross 1981). With the added SF component in the EU-SF model, however, V(w) being Ross more risk averse than U(w) no longer has such a characterization. The present paper abstracts from the difficult problem of characterizing risk aversion and greater risk aversion in the EU-SF model, and instead has a more modest focus on the implications of replacing U(w) with a Ross more risk averse V(w) in the EU-SF model.[13]

3. Portfolio Choice in the EU-SF Model: The Non-Participation Puzzle

We consider the standard portfolio choice problem with one risky and one riskless asset in the EU-SF model, in order to obtain results that can be compared with those well-known results obtained in the EU model.[14] An investor with initial wealth $w_0 > 0$ decides how much of the initial wealth, denoted α, is invested in the risky asset, with the remaining wealth, $w_0 - \alpha$,

[13] For examples of characterizing risk aversion and greater risk aversion in multi-period/multi-component environment, see Bommier et al. (2012), Bommier and Le Grand (2014), Bommier et al. (2017) and Bommier and Le Grand (2019).

[14] Recently, Chiu et al. (2018) apply the EU-SF model to a dynamic portfolio problem which includes cash and various options in the portfolio. Their focus is on justifying investors' use of cash and these options.

being invested in the riskless asset. Suppose that the risky asset has a random return $\tilde{r}$, and the riskless asset has a sure return *i*. The end-of-period wealth can be expressed as

$$\tilde{w}(\alpha) = w_0(1+i) + \alpha\tilde{z}, \tag{2}$$

where $\tilde{z} = \tilde{r} - i$ is the excess return of the risky asset over the riskless asset.

In the EU-SF framework, the standard portfolio choice problem with one risky and one riskless asset is

$$\max_{\alpha\in[0,w_0]} \Gamma(\alpha) \equiv (1-\eta)\cdot E\left[U\left(\tilde{w}(\alpha)\right)\right] + \eta \cdot prob.\left\{\tilde{w}(\alpha) \geq S\right\}. \tag{3}$$ [15]

Throughout the paper we assume that the risky return $\tilde{r}$ is a continuous random variable. Therefore, $\Gamma(\alpha)$ is continuous on $[0, w_0]$, and a solution to (3) is guaranteed. Following the convention in the literature on portfolio choice decision, we assume that the expected excess return, $E(\tilde{z}) = E(\tilde{r}) - i$, is positive. In addition, we assume that $\tilde{r}$ does not dominate *i* in the first degree because otherwise the solution to (3) would trivially be $\alpha^* = w_0$.[16] Note that this latter assumption implies that $prob.\{\tilde{r} - i \geq 0\} < 1$.

In the conventional EU framework, the optimal amount invested in the risky asset is positive even for a risk-averse individual, as long as the expected excess return is positive.[17] According to Segal and Spivak (1990), while a risk averse EU maximizer displays second-order (global) risk aversion, he is nonetheless first-order (locally) risk neutral. That is, a DM with a differentiable risk-averse utility function would behave like a risk-neutral person when the risk

[15] Requiring $\alpha \in [0, w_0]$ is equivalent to disallowing short-selling or borrowing. This assumption is standard in a narrowly-focused portfolio choice problem.
[16] Moreover, the riskless asset would not exist in equilibrium in this case.
[17] For a comprehensive discussion of this result and other implications of the EU model for portfolio choice, see Gollier (2001) and Eeckhoudt et al. (2005). The counterpart of this result for the insurance problem in the EU framework is that the optimal insurance coverage is less than full even for a risk averse individual if the pricing of insurance is not actuarially fair.

involved is small. As a result, he would invest at least a small amount in the risky asset as long as its expected return is larger than the sure return on the riskless asset. However, this result is inconsistent with the empirical observation that a large proportion of the population does not hold any stock even though the expected return of stocks is substantially higher than that of bonds, generating the so-called "non-participation puzzle" (Mankiw and Zeldes 1991).[18]

It seems that adding a safety-first component to the conventional EU model would give the decision maker another reason to avoid risk. Therefore, it is interesting to explore a question raised in Levy and Levy (2009): to what degree can the EU-SF model explain the non-participation puzzle? We obtain the following proposition.

Proposition 1. (i) When the safety wealth is different from the end-of-period wealth that is obtained by investing the entire initial wealth in the riskless asset ($S \neq w_0(1+i)$), the optimal amount invested in the risky asset is positive; (ii) When the safety wealth equals the end-of-period wealth that is obtained by investing the entire initial wealth in the riskless asset ($S = w_0(1+i)$), the optimal amount invested in the risky asset is positive if the weight on the SF component, η, is sufficiently small, and is zero if η is sufficiently large.

Proof: First, note that regardless of the value of *S*,

$$\text{(4)} \qquad \left.\frac{dE\left[U\left(\tilde{w}(\alpha)\right)\right]}{d\alpha}\right|_{\alpha=0} = U'\left(w_0(1+i)\right)\left(E(\tilde{r})-i\right) > 0 .$$

[18] Segal and Spivak (1990) demonstrate that the rank-dependent expected utility model, which is more general than the EU model, can admit first-order risk aversion and hence can be used to explain the non-participation puzzle. In addition, background risks and transactions costs have also been used to explain the non-participation phenomenon in the stock market. On the other hand, Iwaki and Osaki (2017) show that in the phantom aversion decision model, which is also more general than the EU model, the optimal investment in the risky asset is positive as long as its expected excess return is positive.

So the maximal value of $E\left[U\left(\tilde{w}(\alpha)\right)\right]$ is reached at $\alpha > 0$, as is well known from the portfolio choice under the expected utility model. Denote

$$D = \max_{\alpha\in(0,w_0]} E\left[U\left(\tilde{w}(\alpha)\right)\right] - E\left[U\left(\tilde{w}(\alpha)\right)\right]\Big|_{\alpha=0} = \max_{\alpha\in(0,w_0]} E\left[U\left(\tilde{w}(\alpha)\right)\right] - U\left(w_0(1+i)\right) > 0 .$$

(i) $S \neq w_0(1+i)$. In this case, for sufficiently small α (including $\alpha = 0$) $prob.\left\{\tilde{w}(\alpha) \geq S\right\} = prob.\left\{w_0(1+i) + \alpha(\tilde{r} - i) \geq S\right\}$ is either constant at 1 (when $w_0(1+i) > S$) or constant at 0 (when $w_0(1+i) < S$). This means

(5) $$\frac{d\left[prob.\left\{\tilde{w}(\alpha) \geq S\right\}\right]}{d\alpha}\Bigg|_{\alpha=0} = 0 .$$

Combining (4) and (5), we have $\left.\dfrac{d\Gamma(\alpha)}{d\alpha}\right|_{\alpha=0} > 0$, and as a result, that the optimal amount invested in the risky asset is positive when $S \neq w_0(1+i)$.

(ii) $S = w_0(1+i)$. In this case, whether the optimal amount invested in the risky asset is positive depends on a comparison between $\max\limits_{\alpha\in(0,w_0]} \Gamma(\alpha)$ and $\Gamma(0)$.

First, $\Gamma(0) = (1-\eta)\cdot U\left(w_0(1+i)\right) + \eta$. On the other hand,

(6) $$\max_{\alpha\in(0,w_0]} \Gamma(\alpha) = (1-\eta)\cdot \max_{\alpha\in(0,w_0]} E\left[U\left(\tilde{w}(\alpha)\right)\right] + \eta \cdot prob.\left\{(\tilde{r} - i) \geq 0\right\}$$

due to the fact that, when $S = w_0(1+i)$, $prob.\left\{\tilde{w}(\alpha) \geq S\right\} = prob.\left\{(\tilde{r} - i) \geq 0\right\}$ for all $\alpha \in (0, w_0]$.

Therefore,

$$\max_{\alpha\in(0,w_0]} \Gamma(\alpha) - \Gamma(0) = (1-\eta)D + \eta\left[prob.\left\{(\tilde{r} - i) \geq 0\right\} - 1\right]$$

where D > 0 and $\left[prob.\{(\tilde{r}-i)\geq 0\}-1 \right]<0$. So $\max_{\alpha\in(0,w_0]} \Gamma(\alpha)-\Gamma(0)>0$ (i.e., the optimal amount invested in the risky asset is positive) if η is sufficiently small, and $\max_{\alpha\in(0,w_0]} \Gamma(\alpha)-\Gamma(0)<0$ (i.e., the optimal amount invested in the risky asset is zero) if η is sufficiently large. *Q.E.D.*

According to Proposition 1, non-participation can be explained within the EU-SF framework in the special situation where $S = w_0(1+i)$, i.e. where the safety wealth coincides with the final wealth obtained when the entire initial wealth is invested in the riskless asset. In this situation, the optimal amount invested in the risky asset is zero if the weight on the safety-first component η is sufficiently large. The intuition for this result is easy to understand. When $S = w_0(1+i)$, the probability of final wealth being no less than the safety wealth is one when $\alpha = 0$, but is smaller than one when $\alpha > 0$ under the assumption that $\tilde{r}$ does not first degree stochastically dominates *i* or $prob.\{(\tilde{r}-i)\geq 0\}<1$. As a result, $\alpha = 0$ is the optimal choice if the weight on the safety-first component is sufficiently large.

Also according to Proposition 1, the optimal amount invested in the risky asset should always be positive for a EU-SF decision maker whenever $S \neq w_0(1+i)$. The intuition for this result is also straightforward. Suppose, for example, $S > w_0(1+i)$. Then for either $\alpha = 0$ or a sufficiently small $\alpha > 0$, the probability of final wealth being no less than the safety wealth is zero. That is, the SF component is the same for $\alpha = 0$ or a sufficiently small $\alpha > 0$. As is well known, however, the EU component yields a larger expected utility in the latter case when the expected excess return on the risky asset is positive.

This result seems to suggest that the contribution of adding a safety-first component on the top of the expected utility model to solving the non-participation puzzle is limited to the very special case of $S = w_0(1+i)$. As a comparative statics result in the next section indicates, however, a larger weight on the safety-first component implies a smaller optimal amount invested in the risky asset when $S < w_0(1+i)$. In other words, for the case of $S < w_0(1+i)$, adding a safety-first component on the top of the expected utility model helps (partially) explain the non-participation puzzle by making the optimal amount invested in the risky asset smaller.

4. Comparative Statics Analyses

For comparative statics analysis, we are interested in how the optimal investment in the risky asset $\alpha*$ is affected by value changes in each of the three parameters in (1), namely η, *S*, and $U(w)$. First, adding a safety-first component to the conventional EU model naturally raises the question as to whether this would imply more or less risk-taking. In other words, it is of interest to study whether an EU-SF decision maker would invest more or less in the risky asset in response to an increase in the weight of the safety-first component η. In addition, the safety wealth level *S* is a critical part of the safety-first component, and it is of interest to see how the value of *S* influences the decision maker's propensity to take risk. Finally, a well-known comparative statics result for portfolio choice in the EU model is that a more risk averse individual in the Arrow-Pratt sense would invest a smaller amount in the risky asset. Here, one would want to investigate the effect on the optimal investment in the risky asset when the utility function $U(w)$ is replaced with a more risk averse $V(w)$.

In (3), $prob.\{\tilde{w}(\alpha) \geq S\} = prob.\left\{\tilde{r} \geq \dfrac{S - w_0(1+i)}{\alpha} + i\right\} = 1 - G\left(\dfrac{S - w_0(1+i)}{\alpha} + i\right)$ when $\alpha \neq 0$, where the function $G(r)$ is the CDF of $\tilde{r}$. As is established in Proposition 1, only in the special situation $S = w_0(1+i)$ can $\alpha = 0$ possibly be an optimal solution. Excluding this special situation and focusing on the situations where $S \neq w_0(1+i)$,[19] the individual's portfolio choice problem can be stated as choosing $\alpha \in (0, w_0]$ to maximize

$$\Gamma(\alpha) \equiv (1-\eta)\cdot E\left[U\left(w_0(1+i)+\alpha(\tilde{r}-i)\right)\right]+\eta\cdot\left[1-G\left(\frac{S-w_0(1+i)}{\alpha}+i\right)\right]. \tag{7}$$

The first- and second-order derivative of $\Gamma(\alpha)$ are, respectively,

$$\Gamma'(\alpha)=(1-\eta)\cdot E\left[U'\left(w_0(1+i)+\alpha(\tilde{r}-i)\right)(\tilde{r}-i)\right]+\eta\cdot g\left(\frac{S-w_0(1+i)}{\alpha}+i\right)\frac{S-w_0(1+i)}{\alpha^2}, \tag{8}$$

$$\begin{aligned}\Gamma''(\alpha)&=(1-\eta)\cdot E\left[U''\left(w_0(1+i)+\alpha(\tilde{r}-i)\right)(\tilde{r}-i)^2\right]\\ &\quad-\eta\cdot g'\left(\frac{S-w_0(1+i)}{\alpha}+i\right)\left(\frac{S-w_0(1+i)}{\alpha^2}\right)^2-2\eta\cdot g\left(\frac{S-w_0(1+i)}{\alpha}+i\right)\frac{S-w_0(1+i)}{\alpha^3}\end{aligned} \tag{9}$$

where $g(r)$ is the PDF of $\tilde{r}$.

Note that when $\eta = 0$, the DM's optimization problem (7) reduces to pure expected utility maximization. In this case, $\Gamma(\alpha)$ is globally concave (the first term in (9) is always negative), and comparative statics analysis can be conducted in the standard manner by comparing the first-order conditions that are respectively associated with two different values of the parameter in question.[20] When $0 < \eta \leq 1$, however, both the second and third term in (9) could be positive, rendering the standard approach to comparative statics analysis inapplicable.

We therefore resort to an alternative approach to comparative statics analysis developed by Topkis (1978) and Milgrom and Shannon (1994) that is based on a "single-crossing

[19]In the safety-first decision model, as well as the more general decision model with both an expected utility component and a safety-first component studied here, safety wealth S is exogenously determined. As a result, it would be extremely rare (with zero probability) to be in a situation where $S = w_0(1+i)$. We therefore exclude the case $S = w_0(1+i)$ in the comparative statics analyses in this section for mathematical convenience. On the other hand, for portfolio choice in some models with an endogenous reference wealth level, e.g. Bernard and Ghossoub (2010), $w_0(1+i)$ is a natural (benchmark) reference wealth level. Indeed, according to recent experimental evidence documented in Baillon et al. (2020), the most common reference points are the status quo.

[20] See Chuang et al. (2013) for an example of this standard approach to comparative statics analysis.

condition". This approach to comparative statics analysis does not require that the objective function be differentiable in the choice variable, and in fact, allows the choice variable to be discrete. The first-order condition and concavity of the objection function are not part of this comparative statics analysis.[21] For our purpose here, and following Wong (2017), the lemma below is adapted from Topkis (1978, 1998). A proof of the lemma is given in the appendix for completeness.

Lemma 3. (Topkis' Monotonicity Theorem) If the function $\Gamma(\alpha,\theta)$ satisfies the single-crossing condition in (α,θ), i.e.,

(10) $$\Gamma(\alpha_2,\theta_1)-\Gamma(\alpha_1,\theta_1)\geq 0 \;\Rightarrow\; \Gamma(\alpha_2,\theta_2)-\Gamma(\alpha_1,\theta_2)>0$$

for all $\alpha_2>\alpha_1$ (alternatively $\alpha_2<\alpha_1$) and $\theta_2>\theta_1$, then $\alpha^*(\theta_2)\geq\alpha^*(\theta_1)$ (alternatively $\alpha^*(\theta_2)\leq\alpha^*(\theta_1)$) for all $\theta_2>\theta_1$, where $\alpha^*(\theta)\in\arg\max_{\alpha\in(0,w_0]}\Gamma(\alpha,\theta)$.

4.1 Comparative Statics Analysis with Respect to the Effect of η

We start with the comparative statics analysis with respect to the effect of the weight given to the safety-first component η. The following proposition is established. A proof is provided in the appendix.

Proposition 2. (i) When the safety wealth is larger than the end-of-period wealth that is obtained by investing the entire initial wealth in the riskless asset ($S-w_0(1+i)>0$), the optimal amount invested in the risky asset is nondecreasing as the weight given to the safety-first component (η) gets larger; (ii) When the safety wealth is smaller than the end-of-period wealth that is obtained

[21] Nocetti (2016) applies this approach to comparative statics analysis when he investigates the effects of various exogenous changes in the risky environment for a large category of models of decision making under risk. Liu and Meyer (2017) implicitly adopt this approach when they study the effect of the intensity of risk aversion on the optimal choice in various decision problems.

by investing the entire initial wealth in the riskless asset ($S - w_0(1+i) < 0$), the optimal amount invested in the risky asset is nonincreasing as the weight given to the safety-first component (η) gets larger.

According to Proposition 2, the effect of an increase in the weight on the safety-first component η on the optimal amount invested in the risky asset α^* has the same sign as $S - w_0(1+i)$. The intuition for this result is straightforward. When $S - w_0(1+i) > 0$ ($S - w_0(1+i) < 0$), the larger (smaller) amount invested in the risky asset, the higher the probability of meeting the financial safety. Therefore, the optimal amount invested in the risky asset increases (decreases) in response to the weight on financial safety becoming heavier when $S - w_0(1+i) > 0$ ($S - w_0(1+i) < 0$).

Note that the increase in η studied in Proposition 2 includes an increase from $\eta = 0$ to $\eta > 0$ as a special case. Therefore, adding a safety-first concern in an otherwise EU maximizing DM's preferences has the effect of inducing him to take more (less) risk – as measured by the amount invested in the risky asset – if $S - w_0(1+i) > 0$ ($S - w_0(1+i) < 0$).

4.2 Comparative Statics Analysis with Respect to the Effect of *S*

The effect of the safety wealth level *S* on the optimal amount invested in the risky asset depends on the CDF of the risky return *G*(*r*) in complicated ways that are difficult to interpret. Nevertheless, while the comparative statics result with respect to *S* is not clear-cut in general, the following proposition can be established under the assumption that *G*(*r*) is the CDF of a uniform distribution. A proof is given in the appendix.

<u>Proposition 3.</u> Suppose that the risky return $\tilde{r}$ follows a uniform distribution. The optimal amount invested in the risky asset is nondecreasing as the safety wealth becomes larger (from S_1

to $S_2 > S_1$), as long as the relative position of the safety wealth to the end-of-period wealth that is obtained by investing the entire initial wealth in the riskless asset remains unchanged (i.e., either $S_2 > S_1 > w_0(1+i)$ or $S_1 < S_2 < w_0(1+i)$).

4.3 Comparative Statics Analysis with Respect to the Effect of $U(w)$

We now proceed to answer the question: how would the optimal amount invested in the risky asset change when the normalized utility function $U(w)$ is replaced with another normalized utility function $V(w)$ that is more risk averse? An important result from the conventional EU model is that an Arrow-Pratt more risk averse individual will optimally invest less in the risky asset.[22] So we would naturally explore the possibility of establishing the same result in the more general EU-SF model. Unfortunately, our effort of using the Arrow-Pratt notion of greater risk aversion turned out to be negative. That is, being Arrow-Pratt more risk averse is not sufficient to cause the optimal amount invested in the risky asset to decrease (or increase) in the EU-SF model.

Therefore, we consider the effect on the optimal amount invested in the risky asset when $U(w)$ is replaced with a Ross more risk averse $V(w)$. Denote the solution to the optimization problem (7) – with the normalized utility function $U(w)$ – as α_U, and the solution to (7) when $U(w)$ is replaced with another normalized utility function $V(w)$ as α_V. We prove the following proposition in the appendix.

Proposition 4. Suppose that the safety wealth is larger than the end-of-period wealth that is obtained by investing the entire initial wealth in the riskless asset ($S - w_0(1+i) > 0$). Then the

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[22] A direct application of this result is that under decreasing absolute risk aversion or DARA, the amount invested in the risky asset increases as an individual's initial wealth increases (Pratt 1964).

amount invested in the risky asset is nonincreasing if the normalized utility function $U(w)$ is replaced with another normalized utility function $V(w)$ that is Ross more risk averse.

Proposition 4 partially extends the corresponding result in the EU model – that greater Arrow-Pratt risk aversion implies a smaller amount invested in the risky asset – to the more general EU-SF model. The extension is partial because the Ross notion of greater risk aversion in Proposition 4 is stronger than the Arrow-Pratt notion of greater risk aversion, and because we can only obtain an unambiguous comparative statics statement for the case of $S - w_0(1+i) > 0$.

5. Conclusion

The portfolio choice decision has been extensively studied in the expected utility model (the EU model), with some interesting findings being obtained in the literature. For example, in the standard portfolio choice problem where an investor allocates a given amount of initial wealth between a risky asset and a riskless asset with the former having an expected return that is higher than the sure return of the latter, it is found that the amount invested in the risky asset is positive and that a more risk averse investor in the sense of Arrow (1974) and Pratt (1964) will invest less in the risky asset and more in the riskless asset.

In this paper we study the portfolio decision in the more general model of the expected utility with a safety-first component (the EU-SF model), to accommodate the empirical and experimental evidence that real-world decision makers care about the probability of final wealth exceeding a safety wealth level in addition to the distribution of the final wealth per se. We find that, as in the traditional EU model, a positive expected excess return is sufficient for investing a positive amount in the risky asset in the EU-SF model, except in the special situation where the safety wealth level coincides with the wealth obtained when the entire initial wealth is invested in the riskless asset (i.e., $S = w_0(1+i)$). In this situation, the optimal amount invested in the risky asset is zero if the weight on the safety-first component is sufficiently large.

In addition, we find that whether the optimal amount invested in the risky asset becomes smaller as the weight on the safety-first component increases depends on whether $S < w_0(1+i)$. Moreover, we find that a marginal increase in the safety wealth level always induces a larger amount to be invested in the risky asset under the condition that the risky return follows a uniform distribution. We also find that a Ross more risk averse individual will choose to invest a smaller amount in the risky asset in the situation $S > w_0(1+i)$.

Future research can study other economic and financial decisions in the EU-SF model, including the self-protection decision, the precautionary saving decision, and the insurance demand decision under deductible insurance arrangement or with the possibility of insurer default.

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Appendix

Proof of Property 1

Suppose that $\tilde{x}$ second-degree stochastically dominates *S* and $\tilde{x} \succ_2 S$. We then have

$$(1-\eta_2)\cdot E[U(\tilde{x})]+\eta_2 \cdot prob.\{\tilde{x}\geq S\} > (1-\eta_2)\cdot U(S)+\eta_2 ,$$

or equivalently,

$$(1-\eta_2)\cdot\left[E[U(\tilde{x})]-U(S)\right] > \eta_2\left[1-prob.\{\tilde{x}\geq S\}\right].$$

This implies that, for $\eta_2 > \eta_1$

$$(1-\eta_1)\cdot\left[E[U(\tilde{x})]-U(S)\right] \geq (1-\eta_2)\cdot\left[E[U(\tilde{x})]-U(S)\right] > \eta_2\left[1-prob.\{\tilde{x}\geq S\}\right] \geq \eta_1\left[1-prob.\{\tilde{x}\geq S\}\right]$$

,

where the first (weak) inequality is based on $E[U(\tilde{x})]-U(S)\geq 0$ ($\tilde{x}$ second-degree stochastically dominates *S*). Therefore, $\tilde{x} \succ_1 S$. *Q.E.D.*

Proof of Property 2

(i) $x_0 < S_1 < S_2$

Suppose $\tilde{x} \succ_2 x_0$, that is,

$$(1-\eta)\cdot E[U(\tilde{x})]+\eta \cdot prob.\{\tilde{x}\geq S_2\} > (1-\eta)\cdot U(x_0).$$

Then

$$(1-\eta)\cdot E[U(\tilde{x})]+\eta \cdot prob.\{\tilde{x}\geq S_1\} > (1-\eta)\cdot U(x_0),$$

or $\tilde{x} \succ_1 x_0$.

(ii) $S_1 \le x_0 < S_2$

Suppose $\tilde{x} \succ_1 x_0$, that is,

$$(1-\eta)\cdot E[U(\tilde{x})] + \eta \cdot prob.\left\{\tilde{x} \ge S_1\right\} > (1-\eta)\cdot U(x_0) + \eta\,.$$

Then

$$(1-\eta)\cdot E[U(\tilde{x})] + \eta \cdot prob.\left\{\tilde{x} \ge S_2\right\} > (1-\eta)\cdot U(x_0)\,,$$

or $\tilde{x} \succ_2 x_0$.

(iii) $S_1 < S_2 \le x_0$

Suppose $\tilde{x} \succ_2 x_0$, that is,

$$(1-\eta)\cdot E[U(\tilde{x})] + \eta \cdot prob.\left\{\tilde{x} \ge S_2\right\} > (1-\eta)\cdot U(x_0) + \eta\,.$$

Then

$$(1-\eta)\cdot E[U(\tilde{x})] + \eta \cdot prob.\left\{\tilde{x} \ge S_1\right\} > (1-\eta)\cdot U(x_0) + \eta\,,$$

or $\tilde{x} \succ_1 x_0$. *Q.E.D.*

Proof of Lemma 2

The "if" part of Lemma 2 is obtained immediately from Lemma 1. In the following, we prove the "only if" part: if normalized utility function V(w) is Ross more risk averse than normalized utility function U(w) on [*a*, b], then there exists a constant $\lambda \ge 1$ and $\phi(w)$ such that $V(w) \equiv \lambda U(w) + \phi(w)$, where $\phi'(w) \le 0$ and $\phi''(w) \le 0$ on [*a*, b].

According to Lemma 1, if utility function V(w) is Ross more risk averse than utility function U(w) on [a, b], then there exists a constant $\lambda > 0$ and $\phi(w)$ such that $V(w) \equiv \lambda U(w) + \phi(w)$, where $\phi'(w) \le 0$ and $\phi''(w) \le 0$ on [a, b]. We now demonstrate that it must be the case that $\lambda \ge 1$ when both U(w) and V(w) are normalized utility functions with $U(a) = V(a) = 0$ and $U(b) = V(b) = 1$.

We make use of an intermediate result that was independently established in Liu and Wang (2017) and Argyris et al. (2018) that if normalized utility function V(w) is Ross more risk averse than normalized utility function U(w) on [a, b], then V(w) - U(w) is concave on [a, b]. This result is established in the proof of their Proposition 1 in Liu and Wang (2017) and stated as their Lemma 1 in Argyris et al. (2018).

That V(w) - U(w) is concave on [a, b], together with $U(a) = V(a) = 0$ and $U(b) = V(b) = 1$, implies that

(A1) $$V'(a) - U'(a) \ge 0.$$

If $\lambda < 1$, however, we would have

(A2) $$V'(w) - U'(w) < V'(w) - \lambda U'(w) = \phi'(w) \le 0$$ for all w in [a, b],

which obviously contradicts (A1).

Therefore, it must be the case that $\lambda \ge 1$. *Q.E.D.*

Proof of Property 3

Suppose that for $\tilde{x}$, $E(\tilde{x}) \succ_U \tilde{x}$. That is,

$$(1-\eta) \cdot U\left[E(\tilde{x})\right] + \eta \cdot prob.\left\{E(\tilde{x}) \ge S\right\} > (1-\eta) \cdot E[U(\tilde{x})] + \eta \cdot prob.\left\{\tilde{x} \ge S\right\},$$

or equivalently,

$$(1-\eta)\cdot\left\{U\left[E(\tilde{x})\right]-E[U(\tilde{x})]\right\}>\eta\cdot prob.\left\{\tilde{x}\geq S\right\}-\eta\cdot prob.\left\{E(\tilde{x})\geq S\right\}$$

To show $E(\tilde{x})\succ_V \tilde{x}$, it is sufficient to show

$$V\left[E(\tilde{x})\right]-E[V(\tilde{x})]\geq U\left[E(\tilde{x})\right]-E[U(\tilde{x})].$$

According to Lemma 2, there exists a constant $\lambda\geq 1$ and $\phi(w)$ such that $V(w)\equiv\lambda U(w)+\phi(w)$, where $\phi'(w)\leq 0$ and $\phi''(w)\leq 0$. Then

$$\begin{aligned} V\left[E(\tilde{x})\right]-E[V(\tilde{x})] &= \lambda\left\{U\left[E(\tilde{x})\right]-E[U(\tilde{x})]\right\}+\phi\left[E(\tilde{x})\right]-E[\phi(\tilde{x})] \\ &\geq U\left[E(\tilde{x})\right]-E[U(\tilde{x})]. \end{aligned}$$

Q.E.D.

Proof of Lemma 3

We only prove the main version – the version outside the parentheses – of the lemma, because the alternative version – the version inside the parentheses – can be similarly proved.

We use proof by contradiction. Suppose $\alpha^*(\theta_1)>\alpha^*(\theta_2)$ instead. By the definition of $\alpha^*(\theta_1)$, $\Gamma(\alpha^*(\theta_1),\theta_1)-\Gamma(\alpha^*(\theta_2),\theta_1)\geq 0$. According to the single-crossing condition (10), we then have $\Gamma(\alpha^*(\theta_1),\theta_2)-\Gamma(\alpha^*(\theta_2),\theta_2)>0$, contradicting the definition of $\alpha^*(\theta_2)$.

Therefore, it must be the case that $\alpha^*(\theta_2)\geq\alpha^*(\theta_1)$. *Q.E.D.*

Proof of Proposition 2

We need to show that for $1\geq\eta_2>\eta_1\geq 0$, (i) $\alpha^*(\eta_2)\geq\alpha^*(\eta_1)$ if $S-w_0(1+i)>0$, and (ii) $\alpha^*(\eta_2)\leq\alpha^*(\eta_1)$ if $S-w_0(1+i)<0$.

From (7),

$$\Gamma(\alpha_2,\eta)-\Gamma(\alpha_1,\eta)=(1-\eta)\cdot\left\{E\left[U\left(w_0(1+i)+\alpha_2(\tilde{r}-i)\right)\right]-E\left[U\left(w_0(1+i)+\alpha_1(\tilde{r}-i)\right)\right]\right\}$$
$$+\eta\cdot\left[G\left(\frac{S-w_0(1+i)}{\alpha_1}+i\right)-G\left(\frac{S-w_0(1+i)}{\alpha_2}+i\right)\right]$$

(i) $S-w_0(1+i)>0$

In this case, $\left[G\left(\frac{S-w_0(1+i)}{\alpha_1}+i\right)-G\left(\frac{S-w_0(1+i)}{\alpha_2}+i\right)\right]>0$ whenever $\alpha_2>\alpha_1$.

Therefore, $\Gamma(\alpha_2,\eta_1)-\Gamma(\alpha_1,\eta_1)\geq 0 \Rightarrow \Gamma(\alpha_2,\eta_2)-\Gamma(\alpha_1,\eta_2)>0$ for all $\alpha_2>\alpha_1$. According to Lemma 3, we have $\alpha^*(\eta_2)\geq\alpha^*(\eta_1)$.

(ii) $S-w_0(1+i)<0$

In this case, $\left[G\left(\frac{S-w_0(1+i)}{\alpha_1}+i\right)-G\left(\frac{S-w_0(1+i)}{\alpha_2}+i\right)\right]>0$ whenever $\alpha_2<\alpha_1$.

Therefore, $\Gamma(\alpha_2,\eta_1)-\Gamma(\alpha_1,\eta_1)\geq 0 \Rightarrow \Gamma(\alpha_2,\eta_2)-\Gamma(\alpha_1,\eta_2)>0$ for all $\alpha_2<\alpha_1$. According to the alternative version – the version inside the parentheses – of Lemma 3, we have $\alpha^*(\eta_2)\leq\alpha^*(\eta_1)$.

Q.E.D.

Proof of Proposition 3

We need to show that for either $S_2>S_1>w_0(1+i)$ or $S_1<S_2<w_0(1+i)$, $\alpha^*(S_2)\geq\alpha^*(S_1)$.

From (7),

$$\Gamma(\alpha_2,S)-\Gamma(\alpha_1,S)=(1-\eta)\cdot\left\{E\left[U\left(w_0(1+i)+\alpha_2(\tilde{r}-i)\right)\right]-E\left[U\left(w_0(1+i)+\alpha_1(\tilde{r}-i)\right)\right]\right\}$$
$$+\eta\cdot\left[G\left(\frac{S-w_0(1+i)}{\alpha_1}+i\right)-G\left(\frac{S-w_0(1+i)}{\alpha_2}+i\right)\right]$$
$$=(1-\eta)\cdot\left\{E\left[U\left(w_0(1+i)+\alpha_2(\tilde{r}-i)\right)\right]-E\left[U\left(w_0(1+i)+\alpha_1(\tilde{r}-i)\right)\right]\right\}$$
$$+\eta\cdot g\cdot\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right)\left[S-w_0(1+i)\right],$$

where $g > 0$ is the constant density of the uniform distribution.

Therefore, $\Gamma(\alpha_2,S_1)-\Gamma(\alpha_1,S_1)\geq 0\Rightarrow\Gamma(\alpha_2,S_2)-\Gamma(\alpha_1,S_2)>0$ for all $\alpha_2>\alpha_1$ and $S_2>S_1>w_0(1+i)$ or $S_1<S_2<w_0(1+i)$. According to Lemma 3, we have $\alpha^*(S_2)\geq\alpha^*(S_1)$.

Q.E.D.

Proof of Proposition 4

Denote the optimal amount invested in the risky asset α_U under the normalized utility function $U(w)$ and α_V under the normalized utility function $V(w)$, and assume that $V(w)$ is Ross more risk averse than $U(w)$.

We provide a proof by contradiction that essentially makes use of the logic behind Lemma 3. From (7), we have

$$\Gamma(\alpha_U,U(w))-\Gamma(\alpha_V,U(w))=(1-\eta)\cdot\left\{E\left[U\left(w_0(1+i)+\alpha_U(\tilde{r}-i)\right)\right]-E\left[U\left(w_0(1+i)+\alpha_V(\tilde{r}-i)\right)\right]\right\}$$
$$+\eta\cdot\left[G\left(\frac{S-w_0(1+i)}{\alpha_V}+i\right)-G\left(\frac{S-w_0(1+i)}{\alpha_U}+i\right)\right]\geq 0, \tag{11}$$

where the inequality is from the definition of α_U.

Since $V(w)$ is Ross more risk averse than $U(w)$, according to Lemma 2, there exists a constant $\lambda\geq 1$ and $\phi(w)$ such that $V(w)\equiv\lambda U(w)+\phi(w)$, where $\phi'(w)\leq 0$ and $\phi''(w)\leq 0$ on [a, b]. Therefore,

$$\Gamma(\alpha_U,V(w))-\Gamma(\alpha_V,V(w))=(1-\eta)\cdot\lambda\left\{E\left[U\left(w_0(1+i)+\alpha_U(\tilde{r}-i)\right)\right]-E\left[U\left(w_0(1+i)+\alpha_V(\tilde{r}-i)\right)\right]\right\}$$

(12)

$$+(1-\eta)\cdot\left\{E\left[\phi\left(w_0(1+i)+\alpha_U(\tilde{r}-i)\right)\right]-E\left[\phi\left(w_0(1+i)+\alpha_V(\tilde{r}-i)\right)\right]\right\}+\eta\cdot\left[G\left(\frac{S-w_0(1+i)}{\alpha_V}+i\right)-G\left(\frac{S-w_0(1+i)}{\alpha_U}+i\right)\right].$$

Now assume that the conclusion in Proposition 4, namely $\alpha_V \le \alpha_U$, is not true. That is, assume $\alpha_V > \alpha_U$ instead. This has two implications for some items in (12). First, from Proposition 2 (the case of $S-w_0(1+i)>0$), we have $\alpha_V > \alpha_U \ge \bar{\alpha}$, where $E\left[U\left(w_0(1+i)+\alpha(\tilde{r}-i)\right)\right]$ is maximized at $\bar{\alpha}$. This, together with the strict concavity of $E\left[U\left(w_0(1+i)+\alpha(\tilde{r}-i)\right)\right]$, implies

(13) $$E\left[U\left(w_0(1+i)+\alpha_U(\tilde{r}-i)\right)\right]-E\left[U\left(w_0(1+i)+\alpha_V(\tilde{r}-i)\right)\right]>0.$$

Second, when $\alpha_V > \alpha_U$,

$$E\left[\phi\left(w_0(1+i)+\alpha_U(\tilde{r}-i)\right)\right]-E\left[\phi\left(w_0(1+i)+\alpha_V(\tilde{r}-i)\right)\right]$$

(14) $$=\left\{E\left[\phi\left(w_0(1+i)+\alpha_U(\tilde{r}-i)\right)\right]-E\left[\phi\left(w_0(1+i)+\alpha_V(\tilde{r}-\bar{r})+\alpha_U(\bar{r}-i)\right)\right]\right\}$$

$$+\left\{E\left[\phi\left(w_0(1+i)+\alpha_V(\tilde{r}-\bar{r})+\alpha_U(\bar{r}-i)\right)\right]-E\left[\phi\left(w_0(1+i)+\alpha_V(\tilde{r}-i)\right)\right]\right\}\ge 0,$$

where the expression in the first pair of curly brackets is nonnegative due to $\phi''(w)\le 0$, and the expression in the second pair of curly brackets is nonnegative due to $\phi'(w)\le 0$.

From (12), (13) and (14),

$$\Gamma(\alpha_U,V(w))-\Gamma(\alpha_V,V(w))\ge(1-\eta)\cdot\left\{E\left[U\left(w_0(1+i)+\alpha_U(\tilde{r}-i)\right)\right]-E\left[U\left(w_0(1+i)+\alpha_V(\tilde{r}-i)\right)\right]\right\}$$
$$+\eta\cdot\left[G\left(\frac{S-w_0(1+i)}{\alpha_V}+i\right)-G\left(\frac{S-w_0(1+i)}{\alpha_U}+i\right)\right]\ge 0,$$

which contradicts α_V being the optimal choice for *V*(w). Therefore, it must be the case that $\alpha_V \le \alpha_U$. *Q.E.D.*