Alternative Approaches
to Comparative *n*th-Degree Risk Aversion

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(April, 2018)

Abstract: This paper extends the three main approaches to comparative risk aversion – the risk premium approach and the probability premium approach of Pratt (1964) and the comparative statics approach of Jindapon and Neilson (2007) – to study comparative *n*th-degree risk aversion. These extensions can accommodate trading off an *n*th-degree risk increase and an *m*th-degree risk increase for any m such that $1 \leq m < n$. It goes on to show that in the expected utility framework, all these general notions of comparative *n*th-degree risk aversion are equivalent, and can be characterized by the concept of (*n*/*m*)th-degree Ross more risk aversion of Liu and Meyer (2013).

Key Words: Risk aversion; Comparative risk aversion; Risk premium; Probability premium; Downside risk aversion

JEL Classification Codes: D81

*We thank Han Bleichrodt (the department editor), an associate editor and two reviewers for very constructive comments and suggestions. We also thank Christophe Courbage, Sebastian Ebert, Rachel Huang, Paan Jindapon, Jack Meyer, Mario Menegatti, Diego Nocetti, Richard Peter, Larry Tzeng and seminar/session participants at University of Tennessee, Virginia Tech, University of Colorado, National University of Singapore and WRIEC 2015 in Munich for insightful discussions and comments on earlier versions.

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1. Introduction

For more than half a century now economists have used the Arrow-Pratt measure of risk aversion to compare how risk averse two individuals are. There are good reasons why. For one thing, the mathematical characterization is very simple, based on a ratio of the first two derivatives of the utility function. Second, and perhaps more importantly, the comparison based on the Arrow-Pratt measure was accompanied by some mathematically-equivalent behavioral conditions. Specifically, being everywhere higher on the Arrow-Pratt measure is equivalent to having a larger risk premium or having a larger probability premium for an actuarially-neutral risk (Pratt 1964). These behavioral, or choice-based, measures of risk aversion – i.e., the risk premium and the probability premium – have the advantage of being readily computed and compared in experiments investigating the factors that affect the strength of risk aversion.[1]

Recent experimental studies have demonstrated, in various contexts, a salient aversion to risk increases of 3rd and even higher degrees.[2] Moreover, 3rd-degree risk aversion (i.e., downside risk aversion or prudence), or even higher-degree risk aversion, has been shown to play critical roles in some important models of decision making under risk. One example is the self-protection decision. While a risk averse individual does not necessarily invest more in self-protection than a risk neutral individual, a downside risk averse individual tends to invest less in self-protection than a downside risk neutral individual (Chiu 2005, Eeckhoudt and Gollier 2005, Menegatti 2009, Denuit et al. 2016, Crainich et al. 2016, and Peter 2017). Another example is the precautionary saving/effort decision. It has been shown that as future income undergoes an nth-degree risk increase, precautionary saving increases if and only if the utility function displays $(n+1)$th-degree risk aversion (Leland 1968, Sandmo 1970, Dreze and Modigliani 1972, Kimball 1990, Eeckhoudt and Schlesinger 2008, Eeckhoudt et al. 2012, Liu 2014, Wang et al. 2015, and Nocetti 2016).

Along with these interests in higher-degree risk aversion, a question arises as to how to compare two individuals' relative strength of higher-degree risk aversion. Pratt's risk premium approach to comparative risk aversion has been generalized to deal with random initial wealth

[1] For examples of choice-based risk aversion measures, see Holt and Laury (2002), Eckel and Grossman (2002), Andreoni and Sprenger (2011), Ebert and Wiesen (2014), Callen et al. (2014), and Grossman and Eckel (2015).
[2] For example, see Deck and Schlesinger (2010, 2014), Ebert and Wiesen (2011), Maier and Ruger (2011) and Noussair et al. (2014).

and comparative higher-degree risk aversion.[3] By comparison, his probability premium approach to comparative risk aversion, though extensively used in experiments investigating the strength of 2nd-degree risk aversion, has not played an important role in the study of comparative higher-degree risk aversion.

More recently, Jindapon and Neilson (2007) propose a new approach to comparative *n*th-degree risk aversion that is based on a comparative statics analysis. The intuition for this approach is the following. Given a continuous opportunity to reduce the *n*th-degree risk in one's wealth by incurring a monetary cost, a "more *n*th-degree risk averse" individual would choose to incur a larger monetary cost to reduce the *n*th-degree risk further. They show that, in the expected utility framework, an individual would always be willing to incur a larger monetary cost to reduce the *n*th-degree risk in wealth than another individual, if and only if the former is *n*th-degree Ross more risk averse than the latter.

As Liu and Meyer (2013) argue, however, all these existing approaches to comparative *n*th-degree risk aversion essentially quantify *n*th-degree risk aversion by the willingness to trade a 1st-degree risk increase with an *n*th-degree risk increase. They propose a notion of "risk tradeoff" – the ratio of the reduction in expected utility caused by an *n*th-degree risk increase to that caused by an *m*th-degree risk increase – in order to quantify *n*th-degree risk aversion through the willingness to trade an *m*th-degree risk increase with an *n*th-degree risk increase for any m such that $1 \le m < n$. They further show that an individual always has a larger tradeoff between an *n*th-degree risk increase and an *m*th-degree risk increase than another if and only if the former is (n/m)th-degree Ross more risk averse than the latter.

Nevertheless, Liu and Meyer's "risk tradeoff" approach to comparative risk aversion has an important limitation. Their "risk tradeoff" or "rate of substitution" is defined in the expected utility framework, and actually calculating it requires knowing the utility function of the decision maker, which greatly hinders the experimental implementation of their approach. In contrast, the risk premium and probability premium by Pratt (1964) and the optimal monetary investment in reducing the *n*th-degree risk of wealth by Jindapon and Neilson (2007) are all choice-based and

[3] For example, see Ross (1981), Machina and Neilson (1987), Modica and Scarsini (2005), Jindapon and Neilson (2007), Crainich and Eeckhoudt (2008), Li (2009), and Denuit and Eeckhoudt (2010).

can be assessed in appropriately designed experiments without requiring knowledge of a utility function.

This paper extends the three main behavioral approaches to comparative risk aversion – the risk premium approach and the probability premium approach of Pratt (1964) and the comparative statics approach of Jindapon and Neilson (2007) – to study comparative *n*th-degree risk aversion, accommodating trading an *m*th-degree risk increase with an *n*th-degree risk increase for any *m* such that $1 \leq m < n$. We show that, within the expected utility framework, behavior in all of these approaches is governed by the same mathematical condition based on the ratio of the *n*th and *m*th derivatives of the utility functions.[4] Accordingly, all of these behavioral conditions are equivalent, and decision analysts can treat all of them as appropriate avenues for measuring and interpreting higher-order comparative risk attitudes.

First, we consider a situation where the individual compares random initial wealth to a binary compound lottery where the "good" state has less *m*th-degree risk than initial wealth and the "bad" state has higher *n*th-degree risk than initial wealth, with $1 \leq m < n$.[5] Generalizing Pratt's probability premium, we look for the probability of the good state that makes the individual indifferent between initial wealth and the binary compound lottery. The relevant behavioral condition for one individual to be more *n*th-degree risk averse than another is that the former requires a higher probability on the good state than the latter for every initial wealth, every *n*th-degree risk increase, and every *m*th-degree risk decrease. We start with this generalized probability premium approach because it seems to be the most intuitive one, as well as the most convenient one to be implemented in experiments.

Second, we propose a notion of the path-dependent *m*th-degree risk premium for an *n*th-degree risk increase, and interpret the existing risk premium concepts as the 1st-degree risk premium along some special paths. The relevant behavioral condition for one individual to be

[4] Of course, these three approaches are not exhaustive. For example, Keenan and Snow (2009, 2016 and 2017), Crainich and Eeckhoudt (2011), Liu and Meyer (2012), Li and Liu (2014) and Huang and Stapleton (2015) investigate additional alternative approaches to comparative 3rd or higher-degree risk aversion. In particular, Li and Liu (2014) show that one individual has an everywhere larger ratio of the *n*th and *m*th derivatives of the utility function if and only if that individual has a larger monetary utility premium for an *n*th-degree risk than the other individual, a result very much in the spirit of the ones in this paper. As a behavioral condition, though, the monetary utility premium is not as standard or as useful for practitioners as the behavioral conditions we study here.

[5] The "good" or "bad" is from the perspective of an individual that is both *m*th-degree risk averse and *n*th-degree risk averse.

more nth-degree risk averse than another is that the former has a larger path-dependent mth-degree risk premium than the latter for every (random) initial wealth, every nth-degree risk increase, and every possible path of mth-degree increasing risk.

Third, we study a decision problem in which an individual faces an indexed path of random variables, and movements along the path involve precisely-defined reductions in nth-degree risk and increases in mth-degree risk. This formulation has as special cases both the comparative statics problem analyzed in Jindapon and Neilson (2007) and the portfolio choice problem analyzed in Pratt (1964), Ross (1981), and Machina and Neilson (1987). The relevant behavioral condition for one individual to be more nth-degree risk averse than another is that the former always chooses a random variable farther along the path than the latter. More simply, the nth-degree more risk averse individual chooses a random variable with less nth-degree risk but at the cost of more mth-degree risk.

Importantly, and unlike the derivative conditions, these general behavioral notions of comparative nth-degree risk aversion are all model-free; in particular, they do not depend on the expected utility framework to be well defined. As a result, they can be readily computed/tested based on decisions solicited in appropriately designed experiments. Nevertheless, the paper goes on to demonstrate that when the expected utility framework is assumed, all these general behavioral notions of comparative nth-degree risk aversion are equivalent, and can be characterized by the (n/m)th-degree Ross more risk aversion of Liu and Meyer (2013). Not only does this equivalence provide a unified choice-based justification for the concept of (n/m)th-degree Ross more risk aversion, it also provides practitioners with a variety of methods for assessing and explaining comparative higher-degree risk aversion.

The paper is organized as follows. Section 2 reviews notions of nth-degree increasing risk, nth-degree risk aversion, and (n/m)th-degree Ross more risk aversion. Section 3 presents three behavioral (i.e., choice-based) conditions comparing the nth-degree risk aversion of two individuals, as generalizations of the three main approaches to comparative risk aversion – the probability premium approach, the risk premium approach, and the comparative statics approach – respectively. The theorems in this section establish that, in the framework of expected utility, all these three behavioral conditions are equivalent and characterized by the (n/m)th-degree Ross more risk averse condition. Section 4 offers some conclusions.

2. *n*th-Degree Increases in Risk, *n*th-Degree Risk Aversion, and (*n*/*m*)th-Degree Ross More Risk Aversion.

Let $F(x)$ and $G(x)$ represent the cumulative distribution functions (CDFs) of two random variables whose supports are contained in a finite interval denoted $[a, b]$ with no probability mass at point a. This implies that $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Letting $F^{[1]}(x)$ denote $F(x)$, higher order cumulative functions are defined according to $F^{[k]}(x) = \int_a^x F^{[k-1]}(y)dy$, k = 2,3,…. Similar notation applies to $G(x)$ and other CDFs.

For any integer $n \geq 1$, Ekern (1980) gives the following definition.

Definition 1. $G(x)$ has more nth-degree risk (or is more nth-degree risky) than $F(x)$ if

$$G^{[k]}(b) = F^{[k]}(b) \qquad \text{for } k = 1, 2, \ldots, n, \text{ and} \tag{1}$$

$$G^{[n]}(x) \geq F^{[n]}(x) \qquad \text{for all } x \text{ in } [a, b] \text{ with “>” holding for some } x \text{ in } (a, b)\,. \tag{2}$$

Condition (1) guarantees that the first $n-1$ moments are held constant across the two distributions, and conditions (1) and (2) together imply that $F(x)$ dominates $G(x)$ in nth-degree stochastic dominance. Thus, the nth-degree risk increase is a special case of nth-degree stochastic dominance in which the first $n-1$ moments are kept the same. This general definition of nth-degree risk increases has many well-known notions of stochastic changes as special cases. An increase in 1st-degree risk is a first-order stochastically dominated shift, which visually entails a leftward shift in probability mass. It implies (but is not equivalent to) a reduction in the mean. An increase in 2nd-degree risk is simply the familiar risk increase of Rothschild and Stiglitz (1970) which corresponds to a sequence of mean-preserving spreads. It implies (but is not equivalent to) an increase in the variance. Similarly, an increase in 3rd-degree risk holds the first two moments constant and shifts risk from high-wealth levels to low-wealth levels, which is the downside risk increase of Menezes et al. (1980). It implies (but is not equivalent to) a reduction in rightward (positive) skewness.

The fact that an nth-degree risk increase requires that the first n – 1 moments remain constant places a restriction on the starting distribution $F(x)$. In particular, when $F(x)$ is degenerate, placing all of its probability mass on a single outcome x_0, only 1st-degree and 2nd-degree risk increases are possible. A 1st-degree risk increase would entail a first-order stochastically dominated shift, as usual, and a 2nd-degree one would involve a mean-preserving spread. A 3rd-degree or higher risk increase would require that the variance of the new distribution be the same as that of the original distribution, and a degenerate distribution has zero variance. Consequently, 3rd-degree or higher risk increases are only well-defined when the starting distribution is nondegenerate.

Ekern (1980) also provides a definition of nth-degree risk aversion when the preferences have an expected utility representation. For any utility function $u(x)$: $[a, b] \to \mathbb{R}$, assume that $u(x)$ has all possible derivatives and that they are continuous. Denote by $u^{(k)}(x)$ the kth derivative of $u(x)$, k = 1, 2, 3... .

<u>Definition 2.</u> Decision maker $u(x)$ is nth-degree risk averse if $(-1)^{n+1}u^{(n)}(x) > 0$ for all x in $[a, b]$.

We use u and v to refer to decision makers, and also use utility functions $u(x)$ and $v(x)$ to refer to decision makers with the corresponding utility functions when the expected utility framework is assumed. Note that $u(x)$ is said to be *weakly* nth-degree risk averse when the strict inequality in Definition 2 is replaced with a weak one. 1st-degree risk aversion corresponds to an everywhere increasing utility function, and the usual 2nd-degree risk aversion corresponds to a concave utility function. If an individual exhibits all possible degrees of risk aversion his utility function will have derivatives that alternate in sign, beginning with a positive first derivative.[6]

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[6] Almost all often-encountered utility functions satisfy this "mixed risk aversion" property (Brockett and Golden 1987, Caballe and Pomansky 1996). Eeckhoudt and Schlesinger (2006) and Menegatti (2015) provide significant new results on how these preferences may be characterized. For example, Menegatti (2015) shows that if the nth-order derivative of an increasing function $u(x)$ defined on $[0, \infty]$ is sign invariant then all the derivatives of orders from 2 to n alternate in sign. Note also that throughout this paper, and for Definition 2 in particular, the outcome variable x is assumed to belong to a bounded interval $[a, b]$. With the assumption of a bounded domain imposed in this paper, Menegatti's (2015) results do not apply.

The relationship between the two concepts in Definitions 1 and 2 is given in Lemma 1 below that is proved by Ekern (1980).[7]

Lemma 1. $G(x)$ has more nth-degree risk than $F(x)$ if and only if every nth-degree risk averse decision maker $u(x)$ prefers $F(x)$ to $G(x)$.

This result shows that nth-degree increases in risk are precisely the distribution changes that every nth-degree risk averse individual dislikes.

Another definition that is necessary for the analysis in this paper is (n/m)th-degree Ross more risk aversion, first described by Liu and Meyer (2013). Assume that m and n are two positive integers such that $1 \le m < n$, and let the two utility functions $u(x)$ and $v(x)$ each be both nth-degree and mth-degree risk averse on $[a, b]$. The following definition of (n/m)th-degree Ross more risk aversion is from Liu and Meyer (2013).

Definition 3. $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$ on $[a, b]$ if

$$\frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(y)} \ge \frac{(-1)^{n+1}v^{(n)}(x)}{(-1)^{m+1}v^{(m)}(y)} \qquad \text{for all } x, y \in [a, b], \tag{3}$$

or equivalently, if there exists $\lambda > 0$, such that $\frac{u^{(n)}(x)}{v^{(n)}(x)} \ge \lambda \ge \frac{u^{(m)}(y)}{v^{(m)}(y)}$ for all $x, y \in [a, b]$.

Definition 3 includes many existing notions of one utility function being more risk averse than another as special cases. For $n = 2$, $m = 1$ and $y = x$, condition (3) reduces to the familiar Arrow-Pratt more risk averse condition: $-\frac{u''(x)}{u'(x)} \ge -\frac{v''(x)}{v'(x)}$ for all $x \in [a,b]$. As Ross (1981) points out, the behavioral conditions related to this characterization must have nonstochastic initial wealth, and the stronger condition $-\frac{u''(x)}{u'(y)} \ge -\frac{v''(x)}{v'(y)}$ for all $x, y \in [a,b]$ – which is

[7] See also Denuit et al. (1999) and Jouini et al. (2013).

referred to in the literature as Ross more risk aversion – allows for random initial wealth. For m = 1, Definition (3) reduces to Ross more risk aversion when n = 2, to Ross more downside risk aversion when n = 3 (Modica and Scarsini 2005), and to Ross nth-degree more risk aversion for a general $n \geq 2$ (Jindapon and Neilson 2007, Li 2009, and Denuit and Eeckhoudt 2010).

The following lemmas regarding the (n/m)th-degree Ross more risk averse condition will be used in proving the main results in the paper. Specifically, Lemma 2 is useful when using (n/m)th-degree Ross more risk aversion as a sufficient condition, and Lemma 3 is useful when showing (n/m)th-degree Ross more risk aversion as a necessary condition.[8] A proof of Lemma 2 is given in Liu and Meyer (2013),[9] and a proof of Lemma 3 is provided in the appendix.

Lemma 2. $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$ on [a, b] if and only if there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n+1}\phi^{(n)}(x) \geq 0$ for all x in $[a, b]$ such that $u(x) \equiv \lambda v(x) + \phi(x)$.

Lemma 3. If $u(x)$ is NOT (n/m)th-degree Ross more risk averse than $v(x)$ on [a, b], then there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$ such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$(-1)^{n+1}\phi^{(n)}(x) < 0 \qquad \text{for all } x \in [a_1, b_1]$$
$$(-1)^{m+1}\phi^{(m)}(x) > 0 \qquad \text{for all } x \in [a_2, b_2]$$

Besides its usefulness for proofs, Lemma 2 provides a way to construct utility functions that are (n/m)th-degree Ross more risk averse than a given one, which might be useful for applied work. In addition, it is clear from Lemma 2 that assuming two utility functions to be ranked according to the (n/m)th-degree Ross more risk averse relation is by no means a vacuous assumption.

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[8] To appreciate the usefulness of these lemmas in proofs, it may be helpful to discuss their counterparts in the context of the more familiar Arrow-Pratt more risk aversion. By definition, $u(x)$ is Arrow-Pratt more risk averse than $v(x)$ if $-u''(x)/u'(x) \geq -v''(x)/v'(x)$ for all $x \in [a, b]$. Then the Arrow–Pratt counterpart of Lemma 2 is "$u(x)$ is Arrow-Pratt more risk averse than $v(x)$ if and only if the transformation function $T(y)$, defined through $u(x) \equiv T(v(x))$, satisfies $T''(y) \leq 0$ for all $y \in [v(a), v(b)]$," and the Arrow-Pratt counterpart of Lemma 3 is "If $u(x)$ is NOT Arrow-Pratt more risk averse than $v(x)$, then there exists $[a_1, b_1] \subset (a, b)$ such that for the transformation function $T(y)$ defined through $u(x) \equiv T(v(x))$, $T''(y) > 0$ for all $y \in [v(a_1), v(b_1)]$".

[9] See the proof of their Theorem 1.

3. Alternative Approaches to Comparative *n*th-Degree Risk Aversion

3.1. The Probability Premium Approach

Pratt (1964) proposes to use the probability premium as a measure of (global) risk aversion. Pratt defines the probability premium q according to the indifference condition

$$w : \begin{cases} w+\varepsilon & \text{with probability } \tfrac{1}{2}+q \\ w-\varepsilon & \text{with probability } \tfrac{1}{2}-q \end{cases} \tag{4}$$

where w is the nonrandom initial wealth and $\varepsilon > 0$ is a constant. Pratt (1964) further shows that, in the expected utility framework, an individual $u(x)$ always has a larger probability premium than another individual $v(x)$ – for all w and ε – if and only if the former is Arrow-Pratt more risk averse than the latter.

Unlike the risk premium approach that is discussed in the next subsection, the probability premium approach to comparative risk aversion has not played an important role in understanding comparative higher-degree risk aversion, even though probability-type measures of 2nd-degree risk aversion have been extensively used in experiments.[10] The reason for this is probably that the probability premium was not used by Ross (1981) in generalizing Pratt's analysis from nonstochastic initial wealth to random initial wealth, and one has to work with random initial wealth when studying 3rd- or even higher-degree risk increases.

We propose below a general formulation for using the probability premium to measure nth-degree risk aversion. Suppose that $\tilde{w}$ is initial wealth, $\tilde{y}$ is an nth-degree risk *increase* from $\tilde{w}$, and $\tilde{z}$ is an mth-degree risk *decrease* from $\tilde{w}$. For an individual who is both nth-degree and mth-degree risk averse, $\tilde{z} \succ \tilde{w} \succ \tilde{y}$, where "$\succ$" denotes the strict preference relationship. Consider a two-state compound lottery

$$\begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1-p \end{cases}.$$

[10]The only exceptions are Jindapon (2010) and Watt (2011), who propose alternative probability-type measures of downside risk aversion based on the risk apportionment framework of Eeckhoudt and Schlesinger (2006). In addition, Eeckhoudt and Laeven (2015) recently give a graphical representation of the probability premium.

As p increases continuously from 0 to 1, the above lottery goes from being dominated by $\tilde{w}$ to being preferred to $\tilde{w}$. Assuming continuity of preferences with respect to p, meaning that a very small change in p would not reverse a preference relation between the two lotteries compared, there exists a p such that

$$\tilde{w} \sim \begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1-p \end{cases}. \tag{5}$$

Formally, the *m*th-degree probability premium for an *n*th-degree risk increase is defined below.

Definition 4. Suppose that $\tilde{w}$ is the random initial wealth, $\tilde{y}$ is an *n*th-degree risk *increase* from $\tilde{w}$, and $\tilde{z}$ is an *m*th-degree risk *decrease* from $\tilde{w}$. The *m*th-degree probability premium for the *n*th-degree risk increase is the scalar p satisfying the indifference condition (5).

Note that for $n = 2$ and $m = 1$, the *m*th-degree probability premium for the *n*th-degree risk increase includes Pratt's probability premium as a special case. To see this, let $\tilde{w} = w$, $\tilde{z} = w+\varepsilon$ and $\tilde{y}$ have two outcomes, $w+\varepsilon$ and $w-\varepsilon$, with equal probability ½ . Then (5) becomes

$$w \sim \begin{cases} w+\varepsilon & \text{with probability } \frac{1}{2}+\frac{p}{2} \\ w-\varepsilon & \text{with probability } \frac{1}{2}-\frac{p}{2} \end{cases}$$

which is exactly the indifference condition (4) after relabeling $\frac{p}{2}$ as q.

It is straightforward to see that if the individual is both *m*th-degree and *n*th-degree risk averse, then any *m*th-degree probability premium for an *n*th-degree risk increase lies in (0, 1). Now consider two individuals, *u* and *v*, with different risk preferences. Given $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$, if p_u and p_v satisfy (5) for $\sim_u$ and $\sim_v$, respectively, and $p_u > p_v$, then this means that, compared to *v*, individual *u* requires a larger probability on the favorable state – in which an *m*th-degree risk decrease materializes – for the two-state compound lottery to be indifferent to the status quo.

If $p_u \geq p_v$ for all $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$, then *u* can be regarded as being more *n*th-degree risk averse than *v* when the necessary compensation to offset an *n*th-degree risk increase takes the form of an *m*th-degree risk *decrease*. The following theorem shows that in the framework of expected utility, the condition $p_u \geq p_v$ is characterized by (n/m)th-degree Ross more risk aversion. The proof of the theorem is in the appendix.

Theorem 1. Suppose that two expected utility maximizers $u(x)$ and $v(x)$ are each both *m*th-degree risk averse and *n*th-degree risk averse everywhere. The *m*th-degree probability premia satisfy $p_u \geq p_v$ for every $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is *n*th-degree more risky than $\tilde{w}$ and $\tilde{z}$ is *m*th-degree less risky than $\tilde{w}$, if and only if $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$.

Theorem 1 provides a straightforward way for understanding what (n/m)th-degree Ross more risk averse means. Individuals have initial random wealth given by $\tilde{w}$, and consider replacing it with a binary compound lottery that pays random variable $\tilde{z}$ in the good state and random variable $\tilde{y}$ in the bad state. What makes the bad state bad is that $\tilde{y}$ is *n*th-degree riskier than $\tilde{w}$, and what makes the good state good is that $\tilde{z}$ is *m*th-degree *less* risky than $\tilde{w}$. Choosing to move away from the status quo, then, involves trading off the *n*th-degree risk increase against the *m*th-degree risk reduction. The individual who is (n/m)th-degree Ross more risk averse requires a larger probability on the *m*th-degree risk reduction to keep him indifferent between the binary compound lottery and the status quo, which means for him the *n*th-degree risk increase weighs relatively more heavily in his decision than the *m*th-degree risk reduction does, compared to the other individual.

3.2 The Risk Premium Approach

The best-known approach to comparative risk aversion involves the risk premium. In the original Arrow-Pratt analysis, the decision-maker has nonstochastic initial wealth w and faces an additive mean-zero risk $\tilde{\varepsilon}$. The risk premium π is the payment that satisfies the indifference condition $w - \pi \sim w + \tilde{\varepsilon}$. Ross (1981) extends the Arrow-Pratt analysis to random starting wealth levels, and defines the risk premium π according to

$$\tilde{w} - \pi \sim \tilde{y}, \tag{6}$$

where $\tilde{w}$ is the random initial wealth and $\tilde{y}$ a Rothschild-Stiglitz risk increase from $\tilde{w}$. In the expected utility framework, Ross shows that an individual always has a larger risk premium than another – for all $\tilde{w}$ and $\tilde{y}$ – if and only if the former is Ross more risk averse than the latter.[11]

Machina and Neilson (1987) extend Ross (1981) by defining a random risk premium. More precisely, suppose that $\tilde{w}$ is the initial wealth, $\tilde{y}$ is a Rothschild-Stiglitz risk increase from $\tilde{w}$, and $\tilde{z}$ is a nonnegative random variable. The random risk premium π is a scalar satisfying the indifference condition

$$\tilde{w} - \pi\tilde{z} \sim \tilde{y}. \tag{7}$$

Machina and Neilson further show that, in the expected utility framework, an individual always has a larger random risk premium than another – for all $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ – if and only if the former is Ross more risk averse than the latter.

Note that the left-hand side of (6) or (7) ($\tilde{w} - \pi$ or $\tilde{w} - \pi\tilde{z}$) is a 1st-degree risk increase from $\tilde{w}$ when $\pi > 0$, and the right-hand side ($\tilde{y}$) is a 2nd-degree risk increase from $\tilde{w}$. So the risk premium conditions (6) and (7) involve trading off a 1st-degree risk increase against a 2nd-degree one, along their respective "path" of 1st-degree risk increases. Take (7), for example. The set $\{\tilde{w} - \pi\tilde{z}\}_{\pi \in \mathbb{R}^+}$, where $\mathbb{R}^+ = [0, \infty)$, constitutes a continuous, parameterized path indexed by the scalar π.[12] Along this path, higher values of π correspond to increases in 1st-degree risk, and identifying π in expression (7) is the same as finding the random variable on the path that is indifferent to $\tilde{y}$. A random variable further along the path involves more 1st-degree risk, and therefore a larger random risk premium, and consequently an individual who moves further along the path to reach indifference has a higher risk premium than one who does not move as far.

We can use this continuous path idea to formulate a general definition of the *path-dependent mth-degree risk premium* for an *n*th-degree risk increase, where $1 \leq m < n$, and use it to measure an individual's *n*th-degree risk aversion in terms of an *m*th-degree risk increase. Let

[11] This original notion of risk premium of Arrow-Pratt and Ross has been used to measure an individual's aversion to higher-degree risk increases by Modica and Scarsini (2005), Crainich and Eeckhoudt (2008), Li (2009) and Denuit and Eeckhoudt (2010).

[12] Continuity is for the space of probability distributions and with respect to the topology of weak convergence.

$\tilde{w}$ be the random initial wealth and $\tilde{y}$ be an *n*th-degree risk increase from $\tilde{w}$, and let $\{\tilde{x}(\pi)\}_{\pi\in[0,\,B]}$ denote a continuous path of random variables, parameterized by $\pi\in[0,B]\subset\mathbb{R}^{+}$, such that $\tilde{x}(0)=\tilde{w}$ and for every $\pi'>\pi\geq 0$ the random variable $\tilde{x}(\pi')$ has more *m*th-degree risk than $\tilde{x}(\pi)$ does, where $1\leq m<n$. We refer to $\{\tilde{x}(\pi)\}_{\pi\in[0,\,B]}$ as a *path of mth-degree increasing risk from* $\tilde{w}$.

<u>Definition 5.</u> Suppose that $\tilde{w}$ is the random initial wealth, $\tilde{y}$ is an *n*th-degree risk increase from $\tilde{w}$, and $\{\tilde{x}(\pi)\}_{\pi\in[0,\,B]}$ is a path of *m*th-degree increasing risk from $\tilde{w}$ with $\tilde{x}(B)\precsim\tilde{y}$. The *path-dependent mth-degree risk premium* is the scalar π satisfying the indifference condition

$$\tilde{x}(\pi)\sim\tilde{y}. \tag{8}$$

Obviously, $\{\tilde{w}-\pi\}_{\pi\in\mathbb{R}^{+}}$ and $\{\tilde{w}-\pi\tilde{\eta}\}_{\pi\in\mathbb{R}^{+}}$ are examples of paths of 1st-degree increasing risk from $\tilde{w}$. The following examples are some paths of *m*th-degree increasing risk from $\tilde{w}$ for $m\geq 2$. First, $\{\tilde{x}(\pi)=\tilde{w}+\pi\tilde{\varepsilon}\}_{\pi\in\mathbb{R}^{+}}$, where $\tilde{\varepsilon}$ is a mean-zero nondegenerate risk that is independent of $\tilde{w}$, is a path of 2nd-degree increasing risk from $\tilde{w}$. Second, suppose that $\tilde{z}$ (with CDF $H(x)$) has more *m*th-degree risk than $\tilde{w}$ (with CDF $F(x)$). Then $\{\tilde{x}(\pi)\}_{\pi\in[0,1]}$ is a path of *m*th-degree increasing risk from $\tilde{w}$ if $\tilde{x}(\pi)$ has a CDF of $\pi H(x)+(1-\pi)F(x)$. In fact, assuming the expected utility framework and representing the preferences by utility function $u(x)$, the path-dependent *m*th-degree risk premium for an *n*th-degree risk increase from $\tilde{w}$ to $\tilde{y}$ along this path is given by $\pi Eu(\tilde{z})+(1-\pi)Eu(\tilde{w})=Eu(\tilde{y})$ or

$$\pi=\frac{Eu(\tilde{w})-Eu(\tilde{y})}{Eu(\tilde{w})-Eu(\tilde{z})}\,. \tag{9}$$

Note that the ratio in (9) is the “rate of substitution” or “risk tradeoff” between an *n*th-degree risk increase and an *m*th-degree risk increase defined in Liu and Meyer (2013). So, their rate of substitution is the path-dependent *m*th-degree risk premium for an *n*th-degree risk increase along a special path of *m*th-degree increasing risk from $\tilde{w}$, $\{\tilde{x}(\pi)\}_{\pi\in[0,1]}$, as discussed above.

It is straightforward to see that if the individual is both *m*th-degree and *n*th-degree risk averse, any path-dependent *m*th-degree risk premium for an *n*th-degree risk increase must be positive. Now consider two individuals, *u* and *v*, with different risk preferences. Take as given $\tilde{x}$, $\tilde{y}$, and a path of *m*th-degree increasing risk from $\tilde{x}$, $\{\tilde{x}(\pi)\}_{\pi\in[0,B]}$. If π_u and π_v satisfy $\tilde{x}(\pi_u) \sim_u \tilde{y}$ and $\tilde{x}(\pi_v) \sim_v \tilde{y}$, respectively, and $\pi_u > \pi_v$, then this means that, compared to *v*, individual *u* must move further along the path of *m*th-degree increasing risk from $\tilde{x}$ before offsetting the disutility caused by the *n*th-degree increase in risk entailed in $\tilde{y}$. More to the point, and much like the original Arrow-Pratt case, individual *u* is willing to accept a larger *m*th-degree risk increase to avoid an *n*th-degree risk increase than individual *v*.

If $\pi_u \geq \pi_v$ for all $\tilde{x}$, $\tilde{y}$, and paths of *m*th-degree increasing risk from $\tilde{x}$, $\{\tilde{x}(\pi)\}_{\pi\in[0,B]}$, then *u* can be regarded as being more *n*th-degree risk averse than *v* when the willingness to pay for avoiding the *n*th-degree risk increase takes the form of an *m*th-degree risk increase. The following theorem provides a utility function-based characterization of the condition $\pi_u \geq \pi_v$, in the tradition of Pratt (1964), when the preferences of both *u* and *v* satisfy the axioms of expected utility, and are represented by utility functions $u(x)$ and $v(x)$, respectively. The proof of the theorem is in the appendix.

<u>Theorem 2.</u> Suppose that two expected utility maximizers $u(x)$ and $v(x)$ are each both *m*th-degree risk averse and *n*th-degree risk averse everywhere. The path-dependent *m*th-degree risk premia satisfy $\pi_u \geq \pi_v$ for every $\tilde{x}$, every $\tilde{y}$ that is *n*th-degree riskier than $\tilde{x}$ and every path of *m*th-degree increasing risk from $\tilde{x}$, $\{\tilde{x}(\pi)\}_{\pi\in[0,B]}$, if and only if $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$.

The condition that both individuals are everywhere *m*th-degree risk averse plays the same role that increasing utility functions play in the standard Arrow-Pratt characterization of 2nd-degree comparative risk aversion. There the increasing utility functions imply that the individual dislikes increases in the risk premium, and here the *m*th-degree risk aversion implies that the individual dislikes movements farther along the path of *m*th-degree increasing risk from $\tilde{x}$. We want to emphasize that the path-dependent *m*th-degree risk premium is a choice-based notion and that the condition for one individual's risk premium to be uniformly larger than another one's is independent of whether preferences can be represented within the expected utility framework.

3.3. The Comparative Statics Approach

Jindapon and Neilson (2007) construct a decision problem where an individual can reduce the *n*th-degree risk in the random wealth by incurring a monetary cost. They show that, in the expected utility framework, individual $u(x)$ would always want to incur a larger monetary cost, and hence to further reduce the *n*th-degree risk in the random wealth, if and only if the former is (n/1)th-degree Ross more risk averse than the latter. They refer to their analysis as the comparative statics approach to comparative *n*th-degree risk aversion.[13]

In Jindapon and Neilson's problem, the move to further reduce the *n*th-degree risk in the random wealth by incurring a larger monetary cost can be decomposed into an *n*th-degree risk decrease (an improvement) and a 1st-degree risk increase (a deterioration). The definition below provides a general notion of changes in a random distribution that can be decomposed into an improvement and a deterioration.

<u>Definition 6.</u> $F(x)$ is *jointly* less *n*th-degree risky and more *m*th-degree risky than $G(x)$, if there exists $H(x)$ such that $F(x)$ has less *n*th-degree risk than $H(x)$, and $H(x)$ has more *m*th-degree risk than $G(x)$.

Now consider a parameterized wealth path represented by $\tilde{w}(\alpha)$, where as α increases $\tilde{w}(\alpha)$ becomes jointly less *n*th-degree risky and more *m*th-degree risky. Moving down such a path, one reduces the *n*th-degree risk in wealth by increasing the *m*th-degree risk in wealth. Intuitively, if one individual always chooses a larger α than another individual for all such

[13] Watt and Vazquez (2013) provide an alternative comparative statics approach to comparative downside risk aversion.

parameterized wealth paths, it must be that the former is more *n*th-degree risk averse than the latter.

Adopting the expected utility framework, the problem of an individual $u(x)$ is

$$\max_{\alpha} \quad Eu\left[\tilde{w}(\alpha)\right] \tag{10}$$

The solution to (10) is assumed to be unique and is denoted α_u.[14] Similarly, denote the optimal choice of another individual $v(x)$ as α_v. In the appendix, we prove the following characterization theorem for $\alpha_u \geq \alpha_v$.

<u>Theorem 3.</u> $\alpha_u \geq \alpha_v$ for every wealth path $\tilde{w}(\alpha)$ where, as α increases, $\tilde{w}(\alpha)$ becomes jointly less *n*th-degree risky and more *m*th-degree risky, if and only if $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$.

There are a number of situations that can give rise to wealth paths $\tilde{w}(\alpha)$ where increases in α make wealth jointly less *n*th-degree risky and more *m*th-degree risky, and one of such situations is a direct generalization of the problem considered in Jindapon and Neilson (2007).[15] Suppose that total wealth consists of two independent components, i.e. $\tilde{w}(\alpha) = \tilde{w}_1(\alpha) + \tilde{w}_2(\alpha)$, and as α increases $\tilde{w}_1(\alpha)$ becomes less *n*th-degree risky and $\tilde{w}_2(\alpha)$ becomes more *m*th-degree risky. Note that the setup in Jindapon and Neilson (2007) is a special case of this situation where $m = 1$. It can be immediately checked that as α increases $\tilde{w}(\alpha)$ becomes jointly less *n*th-degree risky and more *m*th-degree risky. Then according to Theorem 3, an (n/m)th-degree Ross more risk averse individual would choose to have a less *n*th-degree risky first component and a more *m*th-degree risky second component.

Finally, it is important to point out that the assumption of a unique optimal solution to problem (10) is made for easy exposition rather than technical necessity. Indeed, Theorem 3, as well as its proof in the appendix, remains valid if the unique solution to (10) is replaced with a

---

[14] Without the expected utility approach, the value of α_u would be defined by the criterion $\tilde{w}(\alpha_u)$ is weakly preferred to $\tilde{w}(\alpha)$ for all α.

[15] Discussions of other situations where Theorem 3 is applicable are available from the authors upon request.

general set-valued solution. Such generalizability is important because global concavity of the objective function, which is often entailed by the assumption of a unique optimal solution, may impose strong constraints on the set of admissible utility functions for a given risk change or the set of risk changes for a given utility function.

4. Conclusion

More than half a century ago, Pratt (1964) uses two behavioral (or choice-based) conditions – which are based on the risk premium and the probability premium, respectively – to characterize the Arrow-Pratt more risk averse condition that is based on the famous Arrow-Pratt risk aversion measure, $-u''(x)/u'(x)$. These behavioral conditions regarding comparative risk aversion are important both because they have economic contents and because they can be readily implemented in experimental investigations into individual characteristics (e.g., gender, age, income, education, and religion) that affect the degree of risk aversion. These behavioral conditions do not depend on the expected utility framework to be meaningful and can be checked via experiments without explicit specifications of the utility function.

More recently, Liu and Meyer (2013) propose to use $\frac{(-1)^{n+1}u^{(n)}(x)}{(-1)^{m+1}u^{(m)}(x)}$ as the (*n/m*)th-degree risk aversion measure for *n*th-degree risk aversion, and generalize the Arrow-Pratt more risk averse condition and the Ross more risk averse condition to the (*n*/*m*)th-degree Ross more risk aversion.

This paper generalizes the three main existing (behavioral or choice-based) approaches to comparative risk aversion – the probability premium approach and the risk premium approach due to Pratt (1964) and the comparative statics approach due to Jindapon and Neilson (2007) – for comparative *n*th-degree risk aversion that can accommodate trading off an *n*th-degree risk increase and an *m*th-degree risk increase for any *m* such that $1 \le m < n$. It shows that when the expected utility framework is assumed, all these general notions of comparative *n*th-degree risk aversion are equivalent, and can be characterized by the (*n*/*m*)th-degree Ross more risk aversion.

In the future, economists and other social scientists, as well as financial analysts and decision analysts, may want to investigate the determining factors of the strength of 3^{rd}- and higher-degree risk aversion, just as what they have extensively done for the 2nd-degree risk aversion. It is our hope that the results in this paper will deepen the understanding of, and help in creating alternative measures for, the intensity of nth-degree risk aversion.

APPENDIX

Proof of Lemma 3.

If $u(x)$ is NOT (n/m)th-degree Ross more risk averse than $v(x)$ on $[a, b]$, then there exist some y and $z \in [a, b]$ such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \frac{u^{(m)}(z)}{v^{(m)}(z)}.$$

Obviously, for such y and z, there exists $\mu > 0$, such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)} \quad ,$$

which implies, due to continuity, that there exist $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$ such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)}$$

for all $y \in [a_1, b_1]$ and all $z \in [a_2, b_2]$.

Define $\phi(x) \equiv u(x) - \mu v(x)$. Differentiating yields

$$(-1)^{n+1}\phi^{(n)}(x) = (-1)^{n+1}u^{(n)}(x) - \mu(-1)^{n+1}v^{(n)}(x) < 0 \qquad \text{for all x} \in [a_1, b_1]$$
$$(-1)^{m+1}\phi^{(m)}(x) = (-1)^{m+1}u^{(m)}(x) - \mu(-1)^{m+1}v^{(m)}(x) > 0 \qquad \text{for all x} \in [a_2, b_2]$$

Q.E.D.

Proof of Theorem 1.

For any given $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is nth-degree more risky than $\tilde{w}$ and $\tilde{z}$ is mth-degree less risky than $\tilde{w}$, define

$$U(p) \equiv Eu(\tilde{w}) - [pEu(\tilde{z}) + (1-p)Eu(\tilde{y})].$$

Clearly, $U'(p) = Eu(\tilde{y}) - Eu(\tilde{z}) < 0$ because u(x) is both mth-degree risk averse and nth-degree risk averse. For the same reason, $U(0) > 0$ and $U(1) < 0$.

So $\exists\, p_u \in (0,1)$ such that $U(p_u) = 0$. $V(p)$ for v(x) can be similarly defined, and $\exists\, p_v \in (0,1)$ such that $V(p_v) = 0$.

<u>The "if" part:</u> Suppose that $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$. Then, from Lemma 2, there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x) \le 0$ and $(-1)^{n+1}\phi^{(n)}(x) \ge 0$ for all x in $[a, b]$ such that $u(x) \equiv \lambda v(x) + \phi(x)$. Note that $\phi(x)$ is both weakly mth-degree risk tolerant – meaning that $-\phi(x)$ is weakly mth-degree risk averse – and weakly nth-degree risk averse.

Evaluating $U(p)$ at p_v, we have

$$\begin{aligned} U(p_v) &= Eu(\tilde{w}) - [p_v Eu(\tilde{z}) + (1-p_v)Eu(\tilde{y})] \\ &= \lambda V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= p_v[E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1-p_v)[E\phi(\tilde{w}) - E\phi(\tilde{y})] \\ &\ge 0. \end{aligned}$$

The inequality above holds because (i) $\tilde{z}$ has less mth-degree risk than $\tilde{w}$ and $\tilde{y}$ has more nth-degree risk than $\tilde{w}$, and (ii) $\phi(x)$ is both weakly mth-degree risk tolerant and weakly nth-degree risk averse. Because $U(p)$ is strictly decreasing in p, we have $p_u \ge p_v$.

<u>The "only if" part:</u> Suppose that $p_u \ge p_v$ for all $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is nth-degree more risky than $\tilde{w}$ and $\tilde{z}$ is mth-degree less risky than $\tilde{w}$. To prove that $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$, assume otherwise. Then, according to Lemma 3, there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$, such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} &(-1)^{n+1}\phi^{(n)}(x) < 0 \qquad \text{for all x} \in [a_1, b_1] \\ &(-1)^{m+1}\phi^{(m)}(x) > 0 \qquad \text{for all x} \in [a_2, b_2] \end{aligned} \tag{A1}$$

Now denote the CDFs for $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ as $F(x)$, $G(x)$ and $H(x)$, respectively, and choose $F(x)$, $G(x)$ and $H(x)$ such that

$$\begin{cases} G^{[n]} - F^{[n]} > 0 & x \in (a_1, b_1) \\ G^{[n]} - F^{[n]} = 0 & x \notin (a_1, b_1) \end{cases} \qquad \begin{cases} F^{[m]} - H^{[m]} > 0 & x \in (a_2, b_2) \\ F^{[m]} - H^{[m]} = 0 & x \notin (a_2, b_2) \end{cases}. \tag{A2}$$

Evaluating $U(p)$ at p_v, we have

$$
\begin{aligned}
U(p_v) &= \mu V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\
&= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\
&= p_v[E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1-p_v)[E\phi(\tilde{w}) - E\phi(\tilde{y})] \\
&= p_v\int_a^b \phi(x)d[F(x)-H(x)] + (1-p_v)\int_a^b \phi(x)d[F(x)-G(x)] \\
&= p_v\int_a^b (-1)^{m+1}\phi^{(m)}(x)[H^{[m]}(x)-F^{[m]}(x)]dx + (1-p_v)\int_a^b (-1)^{n+1}\phi^{(n)}(x)[G^{[n]}(x)-F^{[n]}(x)]dx \\
&= p_v\int_{a_2}^{b_2} (-1)^{m+1}\phi^{(m)}(x)[H^{[m]}(x)-F^{[m]}(x)]dx + (1-p_v)\int_{a_1}^{b_1} (-1)^{n+1}\phi^{(n)}(x)[G^{[n]}(x)-F^{[n]}(x)]dx \\
&< 0.
\end{aligned}
$$

The inequality in above is from (A1) and (A2). Because $U(p)$ is strictly decreasing in *p*, we have $p_u < p_v$, a contradiction. Therefore, *u*(*x*) must be (*n*/*m*)th-degree Ross more risk averse than *v*(*x*). Q.E.D.

Proof of Theorem 2.

<u>The "if" part:</u> Suppose that *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*). Then, according to Lemma 2, there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x) \le 0$ and $(-1)^{n+1}\phi^{(n)}(x) \ge 0$ for all *x* in [*a*, *b*] such that $u(x) \equiv \lambda v(x) + \phi(x)$. Note that $\phi(x)$ is both weakly *m*th-degree risk tolerant – meaning that $-\phi(x)$ is weakly *m*th-degree risk averse – and weakly *n*th-degree risk averse.

For every $\tilde{w}$, every $\tilde{y}$ that is *n*th-degree riskier than $\tilde{w}$ and every path of *m*th-degree increasing risk from $\tilde{w}$, $\{\tilde{x}(\pi)\}_{\pi\in[0,B]}$, π_u and π_v satisfy $Eu(\tilde{x}(\pi_u)) = Eu(\tilde{y})$ and $Ev(\tilde{x}(\pi_v)) = Ev(\tilde{y})$, respectively, by definition. Further, we have

$$
\begin{aligned}
Eu(\tilde{x}(\pi_v)) &= \lambda Ev(\tilde{x}(\pi_v)) + E\phi(\tilde{x}(\pi_v)) \\
&= \lambda Ev(\tilde{y}) + E\phi(\tilde{x}(\pi_v)) \\
&\ge \lambda Ev(\tilde{y}) + E\phi(\tilde{y}) \\
&= Eu(\tilde{y}),
\end{aligned}
$$

where the inequality is from (i) $\tilde{x}(\pi_v)$ has more *m*th-degree risk than $\tilde{w}$ and $\tilde{y}$ has more *n*th-degree risk than $\tilde{w}$, and (ii) $\phi(x)$ is both weakly *m*th-degree risk tolerant and weakly *n*th-degree risk averse. Because $Eu\left(\tilde{x}(\pi)\right)$ is strictly decreasing in π, we have $\pi_u \geq \pi_v$.

<u>The "only if" part:</u> Suppose that the path-dependent *m*th-degree risk premia satisfy $\pi_u \geq \pi_v$ for every $\tilde{w}$, every $\tilde{y}$ that is *n*th-degree riskier than $\tilde{w}$ and every path of *m*th-degree increasing risk from $\tilde{w}$, $\left\{\tilde{x}(\pi)\right\}_{\pi\in[0,\,B]}$. Then, it must be the case that $\pi_u \geq \pi_v$ for every $\tilde{w}$, every $\tilde{y}$ that is *n*th-degree riskier than $\tilde{w}$ and a special path of *m*th-degree increasing risk from $\tilde{w}$ that is defined as follows.

Suppose that $\tilde{z}$ (with CDF H(x)) has more *m*th-degree risk than $\tilde{w}$ (with CDF F(x)). Then $\left\{\tilde{x}(\pi)\right\}_{\pi\in[0,1]}$ is a path of *m*th-degree increasing risk from $\tilde{w}$ if $\tilde{x}(\pi)$ has a CDF of $\pi H(x) + (1-\pi)F(x)$. And for u(x), the path-dependent *m*th-degree risk premium for an *n*th-degree risk increase from $\tilde{w}$ to $\tilde{y}$ along this path is given by $\pi Eu(\tilde{z}) + (1-\pi)Eu(\tilde{w}) = Eu(\tilde{y})$ or

$$\pi_u = \frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})}.$$

Similarly,

$$\pi_v = \frac{Ev(\tilde{w}) - Ev(\tilde{y})}{Ev(\tilde{w}) - Ev(\tilde{z})}.$$

So the given condition implies that $\frac{Eu(\tilde{w}) - Eu(\tilde{y})}{Eu(\tilde{w}) - Eu(\tilde{z})} \geq \frac{Ev(\tilde{w}) - Ev(\tilde{y})}{Ev(\tilde{w}) - Ev(\tilde{z})}$ for every $\tilde{w}$, every $\tilde{y}$ that is *n*th-degree riskier than $\tilde{w}$ and every $\tilde{z}$ that is *m*th-degree riskier than $\tilde{w}$. Note that the ratio on each side of the inequality is the "rate of substitution" between an *n*th-degree risk increase and an *m*th-degree risk increase defined in Liu and Meyer (2013). According to their Theorem 1, it must be the case that *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*).

Q.E.D.

Proof of Theorem 3.

<u>The "if" part:</u> Suppose that *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*). By Lemma 2, there exists $\lambda > 0$ and $\phi(x)$ such that $u(x) \equiv \lambda v(x) + \phi(x)$, where $(-1)^{m+1}\phi^{(m)}(x) \leq 0$ and

$(-1)^{n+1}\phi^{(n)}(x) \geq 0$ for all *x*. Note that $\phi(x)$ is both weakly *m*th-degree risk tolerant – meaning that $-\phi(x)$ is weakly *m*th-degree risk averse – and weakly *n*th-degree risk averse.

We use proof by contradiction. To prove $\alpha_u \geq \alpha_v$, assume $\alpha_u < \alpha_v$ instead. Note

$$Eu(\tilde{W}(\alpha_u)) - Eu(\tilde{W}(\alpha_v)) = \lambda\left[Ev(\tilde{W}(\alpha_u)) - Ev(\tilde{W}(\alpha_v))\right] + \left[E\phi(\tilde{W}(\alpha_u)) - E\phi(\tilde{W}(\alpha_v))\right].$$

The first bracket in the above expression is negative because the expected utility of v(x) is maximized at α_v. Under the assumption $\alpha_u < \alpha_v$, the second bracket is nonpositive because that $\phi(x)$ is both weakly *m*th-degree risk tolerant and weakly *n*th-degree risk averse, and that $\tilde{W}(\alpha_v)$ is sequentially less *n*th-degree risky and more *m*th-degree risky than $\tilde{W}(\alpha_u)$. So $Eu(\tilde{W}(\alpha_u)) - Eu(\tilde{W}(\alpha_v)) < 0$, which contradicts that α_u is the optimal choice for u(x). Therefore, it must be the case that $\alpha_u \geq \alpha_v$.

<u>The "only if" part:</u> Suppose that $\alpha_u \geq \alpha_v$ for every wealth path $\tilde{W}(\alpha)$ where, as α increases, $\tilde{W}(\alpha)$ becomes sequentially less *n*th-degree risky and more *m*th-degree risky. To prove that *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*), assume otherwise. Then, according to Lemma 3, there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$ such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} &(-1)^{n+1}\phi^{(n)}(x) < 0 \qquad \text{for all } x \in [a_1, b_1] \\ &(-1)^{m+1}\phi^{(m)}(x) > 0 \qquad \text{for all } x \in [a_2, b_2] \end{aligned} \tag{A3}$$

Because $\alpha_u \geq \alpha_v$, $\tilde{W}(\alpha_u)$ is sequentially less *n*th-degree risky and more *m*th-degree risky than $\tilde{W}(\alpha_v)$. So there exists $\tilde{Z}$ such that $\tilde{W}(\alpha_u)$ has less *n*th-degree risk than $\tilde{Z}$, and $\tilde{Z}$ has more *m*th-degree risk than $\tilde{W}(\alpha_v)$.

Denote the CDFs for $\tilde{W}(\alpha_u)$, $\tilde{W}(\alpha_v)$ and $\tilde{Z}$ as *F*(*x*), *G*(*x*) and *H*(*x*), respectively. Due to the arbitrariness of the wealth path $\tilde{W}(\alpha)$, we can choose *F*(*x*), *G*(*x*) and *H*(*x*) such that

$$\begin{cases} H^{[n]} - F^{[n]} > 0 & x \in (a_1, b_1) \\ H^{[n]} - F^{[n]} = 0 & x \notin (a_1, b_1) \end{cases} \qquad \begin{cases} H^{[m]} - G^{[m]} > 0 & x \in (a_2, b_2) \\ H^{[m]} - G^{[m]} = 0 & x \notin (a_2, b_2) \end{cases}. \tag{A4}$$

Then we have

$$
\begin{aligned}
& Eu\left(\tilde{w}(\alpha_u)\right) - Eu\left(\tilde{w}(\alpha_v)\right) \\
&= \mu\left[Ev\left(\tilde{w}(\alpha_u)\right) - Ev\left(\tilde{w}(\alpha_v)\right)\right] + \left[E\phi\left(\tilde{w}(\alpha_u)\right) - E\phi\left(\tilde{w}(\alpha_v)\right)\right] \\
&< E\phi\left(\tilde{w}(\alpha_u)\right) - E\phi\left(\tilde{w}(\alpha_v)\right) \\
&= \left[E\phi\left(\tilde{w}(\alpha_u)\right) - E\phi\left(\tilde{z}\right)\right] + \left[E\phi\left(\tilde{z}\right) - E\phi\left(\tilde{w}(\alpha_v)\right)\right] \\
&= \int_a^b \phi(x)d[F(x) - H(x)] + \int_a^b \phi(x)d[H(x) - G(x)] \\
&= \int_a^b (-1)^{n+1}\phi^{(n)}(x)[H^{[n]}(x) - F^{[n]}(x)]dx + \int_a^b (-1)^{m+1}\phi^{(m)}(x)[G^{[m]}(x) - H^{[m]}(x)]dx \\
&= \int_{a_1}^{b_1} (-1)^{n+1}\phi^{(n)}(x)[H^{[n]}(x) - F^{[n]}(x)]dx + \int_{a_2}^{b_2} (-1)^{m+1}\phi^{(m)}(x)[G^{[m]}(x) - H^{[m]}(x)]dx \\
&< 0.
\end{aligned}
\tag{A5}
$$

The first inequality above is from the fact that α_v is the optimal choice for v(x). The second inequality above is from (A3) and (A4). Note that (A5) implies that α_u is *not* the optimal choice for u(x), a contradiction. Therefore, it must be the case that $u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$. Q.E.D.

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