ON THE SPECTRUM AND DENSITY OF STATES OF GRAPHS AND GROUPS.

A Thesis

by

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ABSTRACT

We study spectra and spectral measures of the discrete Laplacian and Markov operators on the Cayley graph of a finitely generated group G (and more generally the density of states of discrete periodic operators of finite order on G-periodic graphs). Several examples of computations of spectra and spectral measures of Cayley graphs are surveyed. Some well known theorems and applications of the theory to other areas of mathematics (Kesten's theorem, Kadison-Kaplansky conjecture, Property (T) and expander graphs) are described, following various papers, monographs and textbooks.

In the next chapter, we discuss an algebraic approach for the computation of the density of states: the use of finite support eigenfunctions, following the preprint paper of the author [1]. It is shown that eigenfunctions of λ with finite support are dense in the l^2-eigenspace of λ. Moreover, if G is a virtually polycyclic finitely generated group, then there are finitely many finite support eigenfunctions of λ up to translations and linear combinations. This property can be used to approximate the density of λ. When G has subexponential growth, the density of λ is obtained from a finite resolution by finitely generated free CG-modules (if it exists) of the CG-module of finite support eigenfunctions. Such a resolution always exists when G is abelian.

In the final chapter, we discuss some examples and suggest directions for further study of the use of finite support eigenfunctions.

DEDICATION

To my parents, for their love and support.

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1. PRELIMINARIES AND RELATED LITERATURE

This chapter serves as a brief overview of topics in spectral group theory. We begin with a introductory section on Cayley graphs. Then we move on to a slight generalization, G-periodic graphs and discuss periodic difference operators acting on them. We define the objects of interest: the spectrum and the density of states. We then survey examples of computations of spectra and density of states of infinite Cayley graphs. Moving on, we prove Kesten's characterization of amenability, a fundamental result which in many ways started the subject. We also discuss relations of spectral graph theory to the Kadison-Kaplansky conjecture, a conjecture with deep connections to non-commutative geometry. Finally, we slightly digress to discuss families of expander graphs, a powerful tool used in computer science and network theory. These graphs are often constructed using properties of infinite groups, such as Margulis' construction using Kazhdan's Property (T).

It should be noted that there are many more connections of the theory to other fields of mathematics that are not covered in this chapter. For instance, there are models in mathematical condensed matter physics which use the spectral theory of non-commutative groups, such as crystallographic groups or the discrete Heisenberg group (see [2] for instance). Furthermore, some problems in computer science (such as the Hanoi towers game) can be formulated as a problem about the Schreier graphs of self-similar groups (see [3] for instance). Finally, another connection is that with the K-theory and l^2 invariants of manifolds. For a survey see [4]; while for an example of an application of spectral group theory, see [5].

Some of the content of this chapter comes from the preprint [1].

1.1 Cayley Graphs of Finitely Generated Groups

This introductory section is meant for the reader who may be unfamiliar with geometric group theory. There are plenty of great texts in geometric group theory; for instance see [6] and [7].

First of all, we give some basic definitions. A **group** is a set along with an associative binary operation $\cdot : G \times G \to G$ which has an identity element and every element $g \in G$ has an inverse

g^{-1}. A **generating set** of G is a subset $S \subset G$ such that every element in G can be expressed as the product of elements in S and in $S^{-1} = \{s^{-1}|s \in S\}$. A group is **finitely generated** if it has a generating set which is finite, i.e. $|S| < \infty$. A group is **commutative** (also called **abelian**) whenever $gh = hg$ for all $g, h \in G$. Examples of commutative groups are the integers Z under addition, and the cyclic group of integers modulo $n \in N$, Z/n under the addition in modular arithmetic.

One of the main tools in the study of finitely generated groups is that of a Cayley graph. Given a finite generated set S of G, the **Cayley graph** $\Gamma(G, S)$ is a graph with set of vertices $V = G$ and set of edges

$$E = \{(g, sg)|s \in S \cup S^{-1},\ g \in G\}.$$

Note that the group G naturally acts on the Cayley graph by graph automorphisms via the formula: $g \cdot h := hg^{-1}$.

For example, the Cayley graphs of $\Gamma(Z, \{1\})$, $\Gamma(Z, \{2, 3\})$ and $\Gamma(Z/n, \{1\})$ are shown below. Observe how different $\Gamma(Z, \{1\})$ and $\Gamma(Z, \{2, 3\})$ look, even though the underlying group is the same (the generating sets are different). This is an important point. A property of a Cayley graph of a group may not be a property of the underlying group (since the Cayley graph of a group with another generating set may have different properties). Nonetheless, there are many graph theoretic invariants that turn out to be group thoeretic invariants as well. For example, the growth rate of the Cayley graph does not change with the generating set. More generally, coarse geometric invariants of Cayley graphs are group theoretic invariants (see [6] and [7] for more details).

For the shake of more examples, below are the Cayley graph of the free group of rank two $F_2 = Z * Z$, the free product of two cyclic groups $Z/2 * Z/3$, and the Heisenberg group $< x, y, z : z = [x, y], [x, z] = [y, z] = 1 >$ with respect to their usual generating sets (for brevity we omit further explanations).

Finally, below are the Cayley graphs of the wallpaper groups. A wallpaper group is a group acting freely by isometries on the Euclidean plane with a compact fundamental domain. It is well known that there are only 17 wallpaper groups up to group isomorphism. In [8], Coxeter and

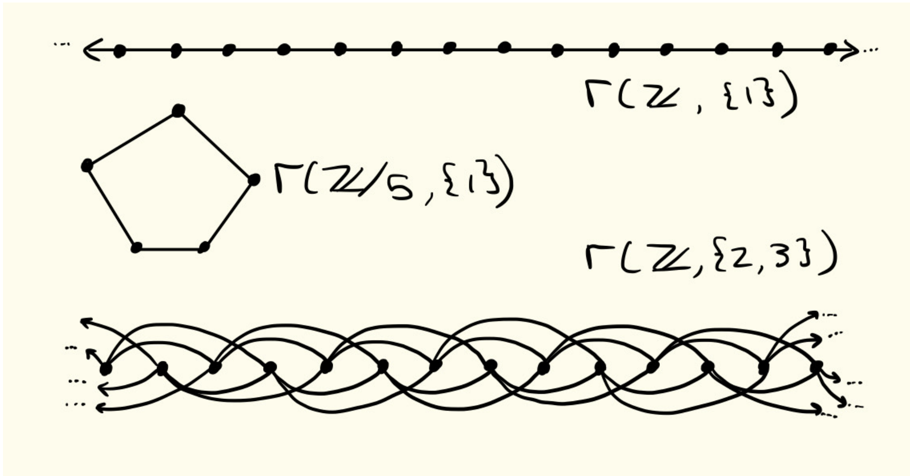

Figure 1.1: Examples of Cayley graphs

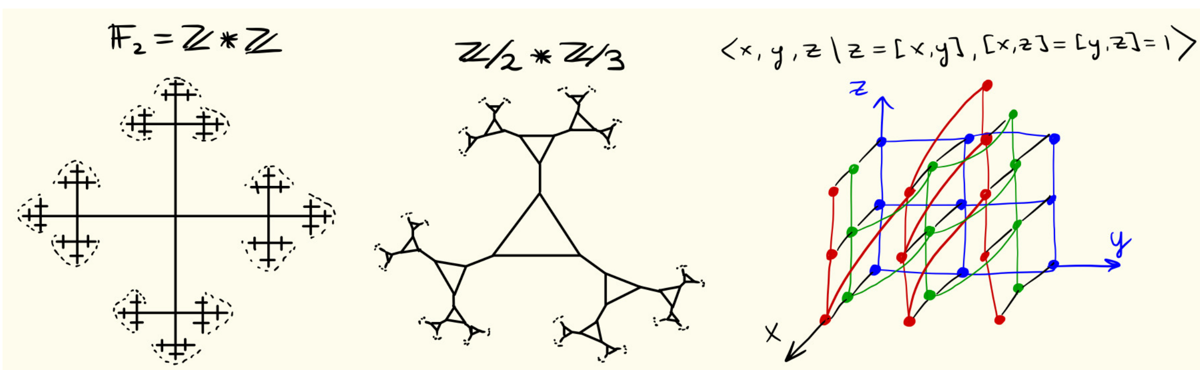

Figure 1.2: More examples of Cayley graphs

Moser find natural generating sets for all the 17 wallpaper groups and compute their Cayley graphs (shown below).

1.2 G-Periodic Graphs and Discrete Periodic Operators of Finite Order

In fact, it will be worthwhile to consider a more general version of graphs, that we will call G-periodic graphs. In fact, the Cayley graph of a finite extension of G will be a G-periodic graph. We begin with the most general setting.

Let $\Gamma = (V, E)$ be a **locally finite** graph (so the degree of each vertex is finite) with set of vertices V and set of edges E. Consider the Hilbert space of all complex valued square summable

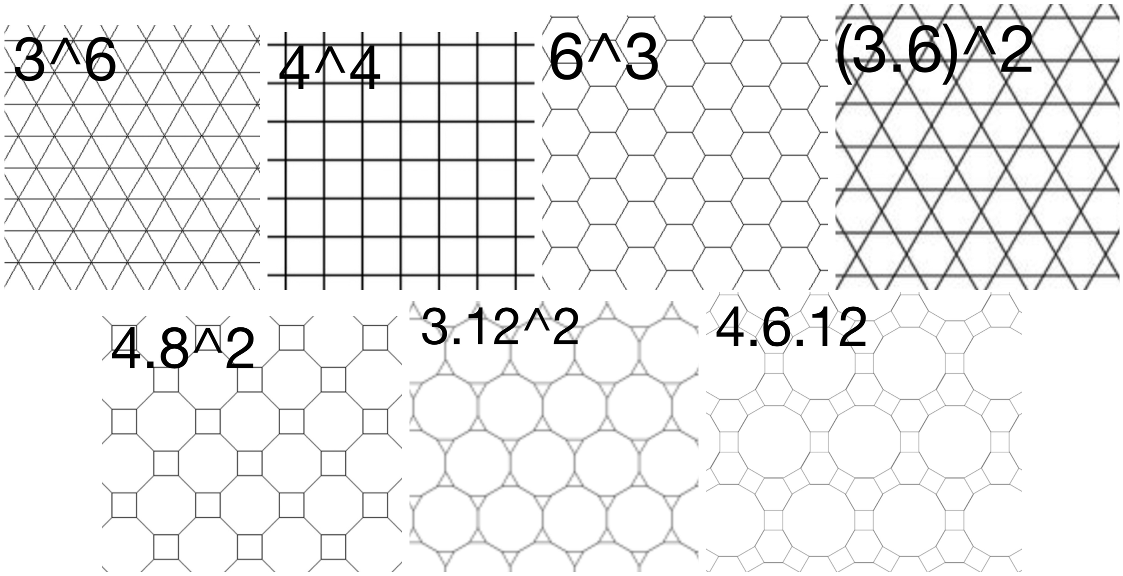

Figure 1.3: The seven Cayley graphs of the wallpaper groups.

functions on V

$$l^2(V) := \{f : V \to C : \sum_{v \in V} |f(v)|^2 < \infty\}$$

with the usual inner product

$$< f, g >:= \sum_{v \in V} f(v)\overline{g(v)}.$$

We define the **discrete Laplacian** $\Delta : l^2(V) \to l^2(V)$ as

$$\Delta f(v) := \frac{1}{\sqrt{deg_\Gamma v}} \sum_{w \sim v} (\frac{f(w)}{\sqrt{deg_\Gamma w}} - \frac{f(v)}{\sqrt{deg_\Gamma v}})$$

where $deg_\Gamma v$ is the degree of the vertex v.

It should be noted that there exist plenty of variations of this study [9, 10]. The most obvious one is to consider the **adjacency operator** A, **Markov operator** M and **Schrödinger operator** $\Delta + q$ (where $q : V \to R$ is bounded) defined as

$$Af(v) := \sum_{w \sim v} f(w) \qquad Mf(v) := \frac{1}{deg_\Gamma v} \sum_{w \sim v} f(w) \qquad (\Delta + q)f(v) := \Delta f(v) + q(v)$$

Another variation is to account for multiple edges, or to use transition probabilities.

Definition. Let G be a finitely generated group. A G**-periodic graph** is a graph $\Gamma = (V, E)$ which admits a free, cofinite and edge preserving action of G on the vertices V. More precisely:

i. the action of G is a free action on the set of vertices V.

ii. The orbit space V/G is finite.

iii. For all $g \in G, u, v, \in V$, $(u, v) \in E \implies (g \cdot u, g \cdot v) \in E$.

Choosing one vertex from each orbit of the group action, we obtain a **fundamental domain** $W \subset V$ which is a finite subset (by ii). If G is amenable (see definition below), we call Γ an amenable periodic graph, if G is abelian, we call Γ an abelian periodic graph, and so on.

Below is an abstract picture of a Z^2 periodic graph. The green enclosed region can be interpreted as a fundamental domain. This domain is then translated and then edges are attached (in red) between the vertices (in blue) in a periodic manner.

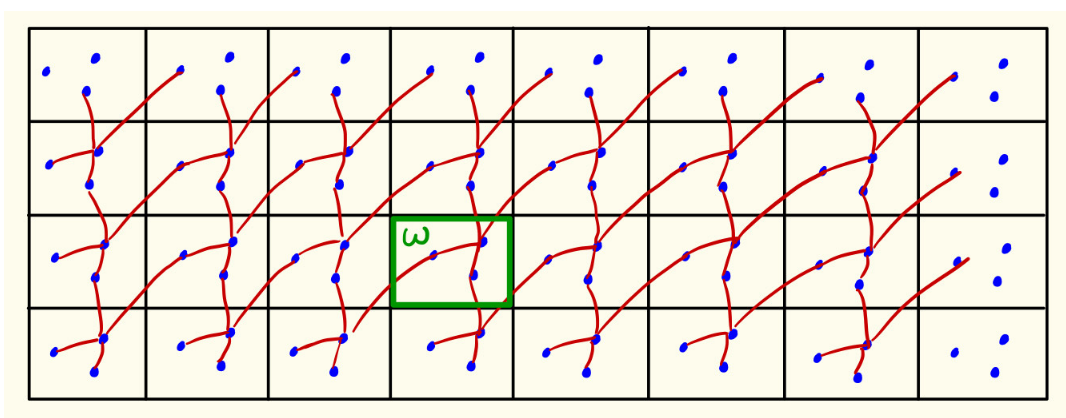

Figure 1.4: Abstract picture of a Z^2 periodic graph

For the rest of this section $\Gamma = (V, E)$ will always denote a G-periodic graph.

Definition. The **left-regular representation** of G associated to Γ is the map $\pi : G \to \mathcal{U}(l^2(V))$ defined by

$$\pi_g f(v) := f(g^{-1} \cdot v) \quad v \in V, f \in l^2(V), g \in G$$

It is a unitary representation of G into the space of bounded unitary operators on $l^2(V)$. An operator $T : l^2(V) \to l^2(V)$ is called **periodic** whenever it commutes with the left-regular representation, i.e. $\pi_g Tf = T\pi_g f$ for any $f \in l^2(V),\ g \in G$. Since G acting on V preserves edges, it follows that the discrete Laplacian is periodic.

For a group G with finite generating set S, we call G is **amenable** when there exists a sequence $\{F_j\}_{j=1}^{\infty}$ of finite subsets of G such that

$$\frac{|((S \cup S^{-1}) \cdot F_j) \setminus F_j|}{|F_j|} \to 0 \ \ as \ \ j \to \infty.$$

$\{F_j\}_j$ is called a **Følner sequence** for (G, S). Note that the amenability of G does not depend on the finite generating set S [7], because there are many more criteria for the amenability of a group which do not depend on the choice of a generating set (e.g. existence of an invariant mean).

Definition. For any vertices $u, v \in V$ denote by $d(v, w)$ the length of the shortest path in Γ from v to w, taking value ∞ when no such path exists. The **r-thick boundary** of a subset $F \subset V$ (where $r \in N$) is defined to be:

$$\partial_r F := \{v \in V \backslash F :\ there\ exists\ \ w \in F\ \ with\ \ d(v, w) \leq r\}$$

Note that $\{F_j\}_j$ is a Følner sequence for (G, S) when $|\partial_1 F_j|/|F_j| \to 0$ on the Cayley graph $\Gamma(G, S)$ (its set of vertices is G and set of edges is $\{(g, s \cdot g) \ :\ g \in G \ \ s \in S\}$). Also note that when $f \in l^2(V)$ is an eigenfunction and there exists $F \subset G$ with $f \equiv 0$ on $\partial_2 F$, then $I_F f$ is also an eigenfunction. Here, I_F is the projection operator onto functions supported on F, i.e. $I_F f(v) = f(v)$ when $v \in F$ and is 0 otherwise. In general, an operator $T : l^2(V) \to l^2(V)$ is said to be of **finite order r** if for any $f, g \in l^2(V), v \in V$,

$$f(w) = g(w)\ \ for\ all\ \ w\ \ with\ \ d(w, v) < r \implies Tf(v) = Tg(v).$$

The operator T then has the following property: when $f \in l^2(V)$ is an eigenfunction associated to T and there exists $F \subset V$ with $f \equiv 0$ on $\partial_r F$, then $I_F f$ is also an eigenfunction. The discrete Laplacian, the Adjacency, the Markov and the Schrödinger operators are all of order 2. Although we focus in this thesis on the discrete Laplacian and the Markov operator, all of the claims and techniques hold for periodic bounded self-adjoint operators of finite order on a periodic graph (with minor changes in the constants related to the order of the operator).

Lemma 1.2.1 (Thick Følner sequences)**.** *If G is amenable, then for any thickness $r \in N$, generating set S of G and fundamental domain W of Γ there exists $l \in N$ and a sequence of finite subsets $\mathcal{F}_j \subset G$ such that the sequence $F_j := \mathcal{F}_j \cdot W \subset V$ satisfies*

$$\partial_r F_j \subset (\partial_l \mathcal{F}_j) \cdot W \quad for\ all\ j, \quad and \quad \frac{|\partial_r F_j|}{|F_j|} \leq \frac{|(\partial_l \mathcal{F}_j) \cdot W|}{|F_j|} \to 0\ as\ j \to \infty.$$

*We call the sequence $\{F_j\}$ a **standard r-thick Følner sequence** w.r.t a fixed fundamental domain W and generating set S of G.*

Proof. The case when $r = 1$ and $\Gamma = \Gamma(G, S)$ follows by the definition of amenability. Next, lets look at the case where $r > 1$ and $\Gamma = \Gamma(G, S)$. Consider the generating set $S' := S \cup S^2 \cup ...S^r$ where $S^r := \{s_1^{\epsilon_1}...s_r^{\epsilon_r} : s_1, ...s_r \in S,\ \epsilon_1, ...\epsilon_r \in \{+1, -1\}\}$. Since G is amenable, there exists a 1-thick Følner sequence $\{F_j\}$ By construction, the 1-thick boundary of F_j with respect to $\Gamma(G, S^r)$ is precisely the r-thick boundary of F_j with respect to $\Gamma(G, S)$ therefore $\{F_j\}$ is our desired Følner sequence with $l = r$.

Next, we have the general case. Fix a fundamental domain W and a generating set S of G Each element $w \in W$ is connected to finitely many vertices $v \in V = \sqcup_{g \in G} g \cdot W$. So for each such v, there exists a unique $g_v \in G, w_v \in W$ such that $v = g_v \cdot w_v$. Consider $|g_v|$, which is the distance of g from 1 in the Cayley graph $\Gamma(G, S)$. Out of all $v \sim w$ and all $w \in W$ pick the largest value of $|g_v|$ and call it t. Inductively, it follows that if $u \in g_u \cdot W,\quad v \in g_v \cdot W$ and $d(u, v) \leq r$ then $|g_u g_v^{-1}| \leq rt$. It follows that for each $\mathcal{F} \subset G$, $\partial_r(\mathcal{F} \cdot W) \subset (\partial_l \mathcal{F}) \cdot W$. By the previous case, we may choose an $l := rt$ thick Følner sequence of $\Gamma(G,S)$, $\{\mathcal{F}_j\}$ and define $F_j := \mathcal{F}_j \cdot W$. We have:

$$\frac{|\partial_r F_j|}{|F_j|} \leq \frac{|(\partial_{rt}\mathcal{F}_j)\cdot W|}{|\mathcal{F}_j \cdot W|} = \frac{|\partial_{rt}\mathcal{F}_j||W|}{|\mathcal{F}_j||W|} = \frac{|\partial_{rt}\mathcal{F}_j|}{|\mathcal{F}_j|} \to 0$$

□

1.3 Spectra and Density of States

The discrete Laplacian Δ (and the Markov operator M provided the graph is regular) is a self-adjoint operator on $l^2(V)$. We consider the **spectrum** $sp(\Delta)$ of Δ, which is the set of all $\lambda \in C$ such that $(\Delta - \lambda I)$ does not have a bounded inverse. Since Δ is self-adjoint, $sp(\Delta) \subset R$. The **pure point spectrum** is the set of all $\lambda \in C$ for which $(\Delta - \lambda I)$ is not injective. An element λ of the pure point spectrum is called an **eigenvalue**. A function $f \in l^2(V)$ such that $(\Delta - \lambda I)f = 0$ is an **eigenfunction** of Δ corresponding to the eigenvalue λ and the space of all such functions is the **eigenspace** of Δ corresponding to λ.

We next discuss the concept of density. According to the spectral theorem of self-adjoint operators (see [11]), from Δ we obtain a spectral measure E whose input are Borel sets and outputs are projections on $l^2(V)$. In the case $B = \{\lambda\}$ where λ is an eigenvalue, $E(\{\lambda\})$ is the orthonormal projection onto the eigenspace of λ. We will denote this eigenspace by E_λ.

Definition. Fix a fundamental domain W of Γ. The **density** or **von Neumann trace** of a Borel subset $B \subset sp(\Delta)$ is

$$dk(B) := \frac{1}{|W|} tr(E(B)I_W)$$

where $tr(\cdot)$ is the usual trace of a Hilbert space operator and $I_W$is the standard projection $l^2(V)l^2(W)$ (which is a Hilbert-Schmidt operator hence $E(B)I_W$ is of trace class).

Note that we may commute the operators inside the trace: $tr(E(B)I_W) = tr(I_W E(B))$ (see [11]). From the spectral theorem, it follows that $dk(\cdot)$ is a measure on $sp(\Delta)$. It is well known (for instance see [12]) that this measure is purely continuous except a set of point masses which occur precisely at the point spectrum of Δ (i.e. the set of eigenvalues). When λ is an eigenvalue,

$dk(\{\lambda\})$ is called the **von Neumann dimension** of the eigenspace E_λ (see [12] for further context) and when $\Gamma = \Gamma(G,S)$ is a Cayley graph, $dk(B) =< E(B)\delta_1, \delta_1 >$ and is called the **spectral measure** of (G,S).

Note that if $H \leq G$ is a subgroup of finite index (so $|G/H| < \infty$), then H acts in any Cayley graph $\Gamma(G,S)$ of G. One can then check that the density of states of $\Gamma(G,S)$ with H as the acting group will be exactly the same as the density of states of $\Gamma(G,S)$ with G as the acting group.

1.4 Examples of Spectra of Cayley graphs

Surprisingly, there are not that many explicit computations of spectra and spectral measures (i.e. density of states) of Cayley graphs. To the best of the author's knowledge, the following is the list of the explicit computations throughout the literature (CORRECTION: the author was later notified of the article [13], which sadly, due to a lack of time, was unable to incorporate in the text).

1. The **free groups** F_k of rank $k \in N$. This was the first example of the spectrum of a group. It is due to Kesten [14]. With respect to the natural free generating set S, the spectrum of the Markov operator on the Cayley graph is

$$sp(M) = [-\frac{\sqrt{2k-1}}{k}, \frac{\sqrt{2k-1}}{k}].$$

and the spectral measure is absolutely continuous.

2. **Free products of cyclic groups**. In [15], Kuhn showed that for the free product of cyclic groups $Z/n * ... * Z/n =< a_1, ...a_r | a_1^n = ... = a_r^n = 1 >$, the spectrum of the Markov operator with respect to the generating set $\{a_1, a_1^2, ...a_1^{n-1}, a_2, a_2^2, ...a_2^{n-1}, ..., a_r, a_r^2, ...a_r^{n-1}\}$ is an closed interval inside $(-1,1)$. The numeric expressions for the endpoints of this interval are a bit involved. An interesting phenomenon is that, when we consider an anisotropic Markov operator (which has more weight toward certain generators), the spectrum can split into the union of two or more disjoint closed intervals.

3. **Virtually abelian groups** (i.e. finite extensions of commutative groups). Using the classical

Floquet-Bloch transform, the spectrum has a **band gap structure**, that is, it is the union of finitely many closed intervals. Moreover, the spectral measure is absolutely continuous expect possibly from a finite number of point masses. For more details, see the monograph of Berkolaiko and Kuchment [16].

4. The discrete **Heisenberg group**. In [17], Béguin, Valette and Zuk compute the spectrum of the Heisenberg group. With respect to the generating set from the presentation

$$< \ x, y : [x, [x, y]] = [y, [y, x]] = 1 \ >$$

the spectrum of the Markov operator is $[-1, 1]$. On the other hand, with respect to the generating set from the presentation

$$< \ x, y, z : z = [x, y], [x, z] = [y, z] = 1 \ >$$

the spectrum of the Markov operator is $[\frac{-1-\sqrt{2}}{3}, 1]$.

5. The **Lamplighter group**, which is the wreath product $Z/2 \wr Z$. In [18], Grigorchuk and Zuk realize the Lamplighter group as an automaton group, acting on the binary rooted tree by tree automorphisms. Using a matrix recursions technique, the spectral measure of the Lamplighter group is computed as the weak limit of the spectral measures of the Schreier graphs of the n^{th}-level stabilizers of the tree action. They find that $sp(M) = [-1, 1]$. Moreover, the spectral measure is discrete, and is given by the formula:

$$dk((-\infty, x]) = \bar{\sigma}(\frac{1}{\pi} arccos(x)) \ \ where \ \ \bar{\sigma}(z) := \sum_{q=2}^{\infty} \frac{|\{p | gcd(p, q) = 1 \ and \ p/q \leq z\}|}{2^q - 1} \ z \in [0, 1].$$

This is the first example of a group whose spectral measure is purely discrete. On the other hand, in [19], Grabowski and Virag find a generating set of the Lamplighter group whose spectral measure is singular continuous. Furthermore, in [12], Grigorchuk and Pittet show that the Lamplighter group (and more generally every amenable indicable group) has a generating set whose spectral measure is continuous. This implies that the continuity or the discreteness of the spectral measure of a group is not a group invariant.

6. The **Grigorchuk groups** $\{G_\omega\}_\omega$ with $\omega \in \Omega_2$. The Grigorchuk groups can be realized as groups acting by tree automorphisms on the infinite binary rooted tree [20]. For any infinite sequence $\omega \in \{0,1,2\}^\infty$ one can constructs G_ω. Let Ω_2 be the set of ω where two of the symbols $\{0,1,2\}$ appear infinitely often. The groups $\{G_\omega\}_\omega$ with $\omega \in \Omega_2$ are infinite torsion groups and all have intermediate growths which are in-equivalent for two different $\omega \in \Omega_2$ [20]. In particular, this means that the Cayley graphs of $\{G_\omega\}_\omega$ are not isomorphic. In [21], Dudko and Grigorchuk show that the spectrum of the Markov operator on $\Gamma(G_\omega, \{a_\omega, b_\omega, c_\omega, d_\omega\})$ is

$$[-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1].$$

An interesting consequence of this result is that there exists a continuum of Cayley graphs which are not isomorphic to each other yet their spectra are the same. Therefore, one cannot completely "Hear the Shape of a Group" [21]. Another example of this phenomenon is the countable family of groups Z^d where $d \in N$. In this case, basic Floquet-Bloch analysis reveals that $sp(M) = [-1, 1]$ for the standard generating set of Z^d. The same question about spectral measures (i.e. whether there exist two non-isomorphic Cayley graphs whose spectral measures are the same) remains open.

1.5 Kesten's Criterion of Amenability

In his paper [14], Kesten initiated the study of random walks on groups and their spectral properties. Later in his paper [22], he shows a spectral characterization of the amenability of a finitely generated group:

Theorem 1.5.1 (Kesten's characterization of amenability)**.** *Let G be a finitely generated group with generating set S. Then G is amenable if and only if the spectral radius of the Markov operator M is equal to 1.*

Kesten's original proof not only relies on the theory of random walks, but also to a weak Følner condition of amenability from Følner's original paper (weaker than the usual Følner condition). The proof of this weak Følner condition requires yet another condition of amenability from an earlier paper of Følner. In addition, the proof of Kesten requires plenty of Lemmas and long computations. Instead, we provide a shorter proof, following Pete's lecture notes [23]. The proof holds more generally for infinite reversible Markov chains.

Theorem 1.5.2 (Kesten-Cheeger-Dodziuk-Mohar)**.** *The following are equivalent for a connected regular graph* $\Gamma = (V, E)$*:*

1.(Edge Isoperimetric Inequality) There exists $\kappa > 0$ *such that for all finite subsets of the vertices* $F \subset\subset V$ *we have:*

$$\kappa|F| \leq |\partial_1 F|$$

where $\partial_1 F$ *denotes the 1-thick boundary of* F*.*

2.(Sobolev Inequality) There exists $\kappa > 0$ *such that for all finite supported functions* $f \in D(V)$*,*

$$\kappa \sum_v |f(v)| \leq \frac{1}{2} \sum_{v \sim w} |f(v) - f(w)|.$$

3.(Dirichlet Inequality) There exists $\tilde{\kappa} > 0$ *such that for all finite supported functions* $f \in D(V)$*,*

$$\tilde{\kappa} \sum_v |f(v)|^2 \leq \frac{1}{2} \sum_{v \sim w} |f(v) - f(w)|^2.$$

4.(Kesten's Criterion) There exists $\tilde{\kappa} > 0$ *such that the spectral radius of the Markov operator* M *satisfies* $r(M) \leq 1 - \tilde{\kappa}$*.*

Observe that by Følner's condition, a Cayley graph satisfies the edge isoperimetric inequality if and only if the underlying group is NOT amenable. Hence this theorem is a generalization of Kesten's theorem to infinite regular graphs.

Proof. $1 \implies 2$: First of all, since

$$\sum_{v\sim w} |f(v) - f(w)| \geq \sum_{v\sim w} |\ |f(v)| - |f(w)|\ |$$

without loss of generality we may assume that $f \geq 0$. We use a standard "wedding cake slices" argument from probability. We will look at the sets $\{v \in V : f(v) > t\}$ where $t > 0$.

$$\frac{1}{2}\sum_{v\sim w} |f(v) - f(w)| = \sum_{v} \sum_{w\sim v, f(w)\geq f(v)} |f(v) - f(w)|$$

$$= \sum_{v} \sum_{w\sim v, f(w)\geq f(v)} \int_0^\infty 1_{[f(v),f(w)]}(t)dt = \int_0^\infty \sum_{v} \sum_{w\sim v, f(w)\geq f(v)} 1_{[f(v),f(w)]}(t)dt$$

$$= \int_0^\infty \sum_{v} \sum_{w\sim v, f(w)\geq t\geq f(v)} 1dt = \int_0^\infty |\partial_1 S_t|dt \geq \kappa \int_0^\infty |S_t|dt = \kappa \sum_{v} |f(v)|.$$

$2 \implies 1$: Plug in $f = 1_F$.

$2 \implies 3$: This is a standard Cauchy-Schwartz space estimate: Again, by the triangle inequality, we may assume without loss of generality that $f \geq 0$.

$$\sum_{v} |f(v)|^2 = \sum_{v} f(v)^2 \leq \frac{1}{2\kappa}\sum_{v\sim w} |f(v)^2 - f(w)^2|$$

$$= \frac{1}{2\kappa}\sum_{v\sim w} |f(v) - f(w)| \cdot |f(v) + f(w)| \leq \frac{1}{2\kappa}\sqrt{\sum_{v\sim w} |f(v) - f(w)|^2}\sqrt{\sum_{v\sim w} |f(v) + f(w)|^2}$$

$$\leq \frac{1}{2\kappa}\sqrt{\sum_{v\sim w} |f(v) - f(w)|^2}\sqrt{2\sum_{v} |f(v)|^2}$$

After squaring and dividing by the l^2 norm of f we obtain:

$$\sum_{v} |f(v)|^2 \leq \frac{2}{\kappa^2}\frac{1}{2}\sum_{v\sim w} |f(v) - f(w)|^2.$$

$3 \implies 1$: Plug in $f = 1_F$.

$3 \iff 4$: Since the Markov operator is self adjoint, its spectral radius is equal to its operator norm and

$$r(M) = ||M|| = sup_{f\in l^2(V)} \frac{< Mf, f >}{< f, f >}$$

As a result, we have the following series of equivalences:

$$r(M) \leq 1 - \tilde{\kappa} \iff \forall f \in l^2(V) \quad < Mf, f > \leq (1 - \tilde{\kappa}) < f, f >$$

$$\iff \forall f \in D(V) \quad < Mf, f > \leq (1 - \tilde{\kappa}) < f, f >$$

$$\iff \forall f \in D(V) \quad \tilde{\kappa}||f||_2^2 \leq ||f||_2^2 - < Mf, f > .$$

Finally, observe that

$$||f||_2^2 - < Mf, f >= \sum_v (f(v)f(v) - \frac{1}{deg_\Gamma v} \sum_{w\sim v} f(v)f(w))$$

$$= \sum_v \frac{1}{deg_\Gamma v} \sum_{w\sim v} f(v)(f(v) - f(w))$$

$$= \frac{1}{2} \sum_v \frac{1}{deg_\Gamma v} \sum_{w\sim v} f(v)(f(v) - f(w)) + \frac{1}{2} \sum_w \frac{1}{deg_\Gamma w} \sum_{v\sim w} f(w)(f(w) - f(v))$$

$$= \frac{1}{2} \sum_{v\sim w} |f(v) - f(w)|^2.$$

□

Finally, we remark that Kesten showed in [14] that for any finitely generated group G and finite generating set S of size k, $r(M) \geq \frac{\sqrt{2k-1}}{k}$, with equality holding if and only if G is free and S is a free generating set. Therefore, the spectral radius not only quantifies the amenability of a group, but also whether a group is freely generated.

1.6 Relations to the Kadison-Kaplansky Conjecture

We now discuss an important connection between spectra of Cayley graphs and the Kadison-Kaplansky conjecture about group C^* algebras.

Definition. Given a finitely generated group G we can consider the left regular representation $\pi : G \to \mathcal{U}(l^2(G)) \subset \mathcal{B}(l^2(G))$ and take the closure of $\pi(G)$ under the strong operator topology (i.e. the topology where $T_n \to T$ whenever $||T_n - T|| \to 0$). This closure is called the **reduced group** C^* **algebra** of G and is denoted by $C_r^*(G)$.

Suppose now that G has **torsion**, that is, there exists $g \in G$ and $n \in N$ such that $g^n = 1$. Then $p = \dfrac{1}{n} \sum_{k=1}^{n} g^k$ is an **idempotent**, i.e. $p^2 = p$, in the group algebra (and hence also in the group C^* algebra). If $g \neq 1$, then $p \neq 0$ or 1, i.e. p is a **nontrivial idempotent**. It is natural to ask whether the converse holds. It turns out this is a difficult question, and it remains open for a general finitely generated group which is **torsion-free** (i.e. there is no torsion, i.e. no element has finite order).

Conjecture 1 (Kaplansky's Idempotent Conjecture)**.** ***If*** *G is a finitely generated torsion-free group and F is a field,* ***then*** *the group algebra $F[G]$ has no nontrivial idemptotents.*

Conjecture 2 (Kadison-Kaplansky Conjecture)**.** ***If*** *G is a finitely generated torsion-free group,* ***then*** *the reduced group C^* algebra $C_r^*(G)$ has no nontrivial idemptotents.*

We remark that one of the main conjectures in noncommutative geometry, the Baum-Connes Conjecture, implies the Kaplansky-Kadison Conjecture. The Baum-Connes Conjecture has been proven for wide classed of groups, including amenable groups, Gromov hyperbolic groups and their subgroups, groups acting properly on trees, lattices of $SO(n, 1)$ and $SU(n, 1)$ and many more groups. For more details see [24].

The main link between the Kadison-Kaplansky Conjecture and spectra of periodic operators on Cayley graphs is the following elementary lemma:

Lemma 1.6.1. *Let G be a finitely generated group and S a finite generating set and let M be the Markov operator on $\Gamma(G, S)$, or more generally any self adjoint operator in $C_r^*(G)$. If $sp(M)$ is disconnected, then $C_r^*(G)$ contains a nontrivial idempotent.*

Proof. The proof is a standard holomorphic functional calculus argument. If $sp(M) \subset [-1,1]$ is disconnected, say $\lambda \in [-1,1] - sp(M)$, since $sp(M)$ is closed we can find $\epsilon > 0$ such $(\lambda - \epsilon, \lambda + \epsilon) \subset [-1,1] - sp(M)$. We can therefore chose two contours on the complex plane Γ_1 and Γ_2 surrounding $[-1, \lambda - \epsilon]$ and $[\lambda + \epsilon, 1]$ respectively. Let ϕ be the analytic function on the bounding domains of Γ_1 and Γ_2 which is 0 on the bounding domain of Γ_1 and 1 on the bounding domain of Γ_2. By the holomorphic functional calculus, we can construct the operator $\phi(M) \in C_r^*(G)$ which will satisfy $\phi(M)\phi(M) = (\phi^2)(M) = \phi(M)$ and hence $\phi(M)$ is an idemptent. Finally, $\phi(M) \neq 0$ since $\phi(M)E([\lambda + \epsilon, 1]) = E([\lambda + \epsilon, 1]) \neq 0$ and $\phi(M) \neq 1$ since $\phi(M)E([-1, \lambda - \epsilon]) = 0$. Therefore $\phi(M)$ is a nontrivial idempotent. $\square$

Combining the above elementary lemma with the (not elementary) fact that amenable groups satisfy the Baum-Connes Conjecture (and hence also the Kadison-Kaplansky conjecture), the following corollaries follow:

Corollary 1.6.1.1. *If a finitely generated torsion free group G satisfies the Kadison-Kaplansky conjecture, then $sp(M)$ is a closed interval for all finite generating sets S.*

Corollary 1.6.1.2. *If G is a finitely generated amenable torsion free group and S is a finite generating set, then $sp(M) = [m, 1]$ for some $-1 \leq m \leq 1$.*

Corollary 1.6.1.3. *If G is a finitely generated amenable torsion free group, S is a finite generating set, and all the relations of S are generated by words of even length, then $\Gamma(G, S)$ is bipartite and hence $sp(M) = [-1, 1]$*

1.7 Digression: Property (T) and Expander Graphs

Another great example of the interplay between group theory and spectral graph theory is the construction of expander graphs. In this section, we follow [25].

Roughly speaking, expanders are special graphs which are sparse (have small degree) yet very well connected (each set of verices has a lot of neighbors relative to its size). They are objects of great importance in theoretical computer science and network theory (see [26] for more details).

Definition. The **Cheeger constant** of a connected finite graph $\Gamma = (V, E)$ is

$$h(\Gamma) := inf_{A \subset V} \frac{|E(A, V - A)|}{min(|A|, |V - A|)}$$

where $E(A, V - A)$ is the set of all edges connecting vertices in A with vertices in $V - A$.

A **family of expanders** with fixed degree $d \in N$ is a sequence of finite connected graphs $\Gamma_n = (V_n, E_n)$ such that:

1. $|V_n| \to \infty$ as $n \to \infty$.

2. Γ_n has degree d.

3. There exists a constant $\delta > 0$ such that $h(\Gamma_n) > \delta > 0$ for all n, i.e. $inf_n h(\Gamma_n) > 0$

Although is may not appear so at first sight, it is crucial to point out the expanders are a spectral property. To see this, we introduce more terminology. Given a connected regular finite graph $\Gamma = (V, E)$, we always have $0 \in sp(\Delta)$, $1 \in sp(M)$ and $\Delta = 1 - M$ (which means that $sp(\Delta) = 1 - sp(M)$). Since $< \Delta f, f > \geq 0$ for all $f \in l^2(V)$, the eigenvalues of Δ are non-negative. Denote the second smallest positive eigenvalue of Δ by $\lambda_1(\Gamma)$.

Since constant functions are the only eigenfunction of $\lambda = 0$ of Δ, removing this subspace from $l^2(V)$, we get:

$$\lambda_1(\Gamma) := inf\{\frac{< \Delta f, f >}{< f, f >} | \sum_{v \in V} f(v) = 0\}.$$

For each $A \subset V$, plugging in the function f which takes the value $|V - A|$ on A and the value $|A|$ on $V - A$, we obtain $\lambda_1(\Gamma) \leq 2h(\Gamma)$. We also have an estimate in the reverse direction:

Proposition 1.7.1 (Cheeger's Inequality)**.** *For every connected regular finite graph* $\Gamma = (V, E)$*,*

$$\lambda_1(\Gamma) \geq h(\Gamma)/2$$

Proof. See [25]. □

Corollary 1.7.1.1. *Let* $(\Gamma_n = (V_n, E_n))_n$ *be a sequence of connected finite graphs of fixed degree*

$d \in N$. *Then*

$$inf_n h(\Gamma_n) > 0 \iff inf_n \lambda_1(\Gamma) > 0 \iff sup_n sup\{|\lambda| : \lambda \in sp(M_n),\ \lambda \neq 1\} < 1$$

Although it is not hard to show that families of expanders exist (using the probabilistic method), finding explicit examples is a very nontrivial task. There are many different explicit constructions of expander graphs, many of which use Cayley and Schreier graphs of groups. In this section we will outline a construction due to Margulis which uses Kazhdan's Property (T) [25].

Definition. Let G be a finitely generated group. A unitary representation, $\rho : G \to \mathcal{U}(\mathcal{H})$ is said to have **almost invariant vectors** when for every $\epsilon > 0$ and finite subset $K \subset\subset G$, there exists a unit vector $v \in \mathcal{H}, ||v|| = 1$ with $||\rho(g)v - v|| < \epsilon$ for all $g \in K$.
We say that G has **Property (T)** if every unitary representation which has almost invariant vectors much have an invariant vector $v \in \mathcal{H}$, i.e. $\rho(g)v = v$ for all $g \in G$.

Proposition 1.7.2. *Infinite amenable groups do not have Property (T).*

Proof. By Følner's Criterion of amenability, for every $\epsilon > 0$ and finite $K \subset\subset G$ there exists finite $F \subset\subset G$ such that for all $g \in K$

$$|gF \,\Delta\, F| < \epsilon|F|$$

This means precisely that the unit vector $f(g) := \dfrac{1}{\sqrt{|F|}} 1_F(g)$ is an almost invariant vector of the left regular representation $\pi : G \to \mathcal{U}(l^2(G))$:

$$||\pi(g)f - f|| < \epsilon.$$

Therefore the left regular representation has almost invariant vectors. However, since the group is infinite and the action of G on itself is transitive, there are no invariant vectors in the left regular representation. Therefore G does not have Property (T). □

We remark that the converse is also true and is due to Hulanicki, though we will not need this result. In fact, one can obtain much deeper results via the notion of weak containment of representations. See [25] and [27] for more on this.

Theorem 1.7.3 (Hulanicki)**.** *G is amenable if and only if the left regular representation contains almost invariant vectors.*

Proposition 1.7.4. *Free groups do not have Property (T)*

Proof. Let F_k be free of rank $k \in N$. By a universal property argument, the abelianization $F_k/[F_k, F_k]$ is a free abelian group of rank k, and hence is infinite. Consider the left regular representation of $F_k/[F_k, F_k]$, and compose it with the natural quotient map:

$$\rho : F_k \to F_k/[F_k, F_k] \to \mathcal{U}(l^2(F_k/[F_k, F_k]))$$

Since $F_k/[F_k, F_k]$ is abelian, it is amenable, and hence ρ has almost invariant vectors. However, ρ cannot have any invariant vectors. Therefore F_k cannot have Property (T). $\square$

Unfortunately, we would have to go outside the scope of this thesis to prove the following fundamental result:

Theorem 1.7.5. *$SL_2(Z)$ does not have Property (T), while $SL_n(Z)$ does have Property (T) for $n \geq 3$.*

The following theorem is the key link between Property (T), a group theoretic property, and expanders, an object in spectral graph theory.

Theorem 1.7.6. *Let G be a finitely generated group with Property (T), and let $(N_k)_{k=1}^{\infty}$ be a (not necessarily nested) sequence of (not necessarily normal) subgroups of G of finite index going to infinity:*

$$for\ all\ k \quad N_k \leq G, \;\; [G : N_k] < \infty \;\; and \;\; [G : N_k] \to \infty \;\; as \;\; k \to \infty$$

Then for any finite generating set S of G, the family of Schreier graphs $(\Gamma(G, N_k, S))_k$ forms a family of expanders of fixed degree.

Proof. Recall that the Schreier graph $\Gamma_k := \Gamma(G, N_k, S)$ consists of vertices $V_k := G/N_k = \{gN_k | g \in G\}$ and edges $E_k = \{(sgN_k, gN_k) | g \in G, s \in S \cup S^{-1}\}$. Consider the subspace $l^2(V_k)$:

$$\mathcal{H}_k := \{f \in l^2(V_k) | \sum_{v \in V_k} f(v) = 0\}$$

Now, G acts on G/N_k and hence also on $l^2(V_k) = l^2(G/N_k)$. It is easy to check that $\mathcal{H}_k$ will be an invariant subspace of the action of G. We can therefore take the usual l^2 direct sum of all these representations and obtain an infinite dimensional unitary representation of G:

$$\mathcal{H} := \bigoplus_k \mathcal{H}_k, \quad \rho : G \to \mathcal{U}(\mathcal{H})$$

Since $l^2(V_k)$ splits as the direct sum of the constant functions and $\mathcal{H}_k$, $\mathcal{H}_k$ has no invariant vectors, and therefore $\mathcal{H}$ also has no invariant vectors.

By assumption, G has Property (T), and therefore $\rho : G \to \mathcal{U}(\mathcal{H})$ does not have almost invariant vectors. This means that there exists an $\epsilon > 0$ and finite $K \subset\subset G$ such that for all $f\mathcal{H}$ there exists $g \in K$ with

$$||\rho(g)f - f|| \geq \epsilon ||f|| \qquad (*).$$

Since K is finite, there exists $l \in N$ sich that K is contained in the ball of radius l in $\Gamma(G, S)$ centered at the identity.

We claim that for all $f \in \mathcal{H}$ there exists $s \in S \cup S^{-1}$ with

$$||\rho(s)f - f|| \geq \epsilon/l \ ||f||.$$

Lets prove this claim. Suppose the contrary, so there exists $f \in \mathcal{H}$ such that for all $s \in S \cup S^{-1}$ we have

$$||\rho(s)f - f|| < \epsilon/l \ ||f||.$$

Then for all $g \in K$, we may write $g = s_1 s_2 ... s_l$ where each s_i is in $s \in S \cup S^{-1} \cup \{1\}$.

$$||\rho(s_1 s_2 ... s_l)f - f|| \leq ||\rho(s_1 ... s_l)f - \rho(s_2 ... s_l)f|| + ||\rho(s_2 ... s_l)f - \rho(s_3 ... s_l)f|| + ... + ||\rho(s_l)f - f||$$

$$< \epsilon/l \ ||f|| + \epsilon/l \ ||f|| + ... + \epsilon/l \ ||f|| = \epsilon ||f||.$$

But this contradicts $(*)$ therefore the claim follows.

We can now show that $(\Gamma(G, N_k, S)_k$ is a family of expanders. The only thing to show is that $h(\Gamma(G, N_k, S) > \delta > 0$ for some fixed constant $\delta > 0$. Fix k and $A \subset V_k$ and let $n := |V_k|$ and $a := |A|$, Consider $f \in \mathcal{H}_k$ defined by:

$$f(v) = n - a \ \ if \ \ v \in A \qquad f(v) = a \ \ if \ \ v \in V - A$$

Then there exists $s \in S$ such that $||\rho(s)f - f|| \geq \epsilon/l \ ||f||$. This inequality is becomes:

$$n^2 |E_s(A, V - A)| \geq \frac{\epsilon^2}{l^2} \ na(n - a)$$

where $E_s(A, V - A)$ is the set of all edges connecting A to $V - A$ which have the form (sgN_k, gN_k) for some $g \in G$. Rearranging, we get:

$$|E_s(A, V - A)| \geq \frac{\epsilon^2}{l^2} \ a(1 - \frac{a}{n}) = \frac{\epsilon^2}{l^2} \ |A|(1 - \frac{|A|}{|V|})$$

If $|A| \leq |V|/2$, then

$$|E(A, V - A)| \geq |E_s(A, V - A)| \geq \frac{\epsilon^2}{l^2} \ |A|(1 - 1/2) = \frac{\epsilon^2}{2l^2} \ |A| = \frac{\epsilon^2}{2l^2} \ min(|V - A|, |A|).$$

If $|A| > |V|/2$, then $|V - A| \leq |V|/2$ and $|A| > |A - V|$ so

$$|E(A, V - A)| \geq \frac{\epsilon^2}{2l^2} \ |V - A| = \frac{\epsilon^2}{2l^2} \ min(|V - A|, |A|).$$

Since this holds for all subsets $A \subset V$, we conclude that $h(\Gamma_k) \geq \dfrac{\epsilon^2}{2l^2} > 0$, therefore we indeed obtain a family of expanders. $\square$

Example-Construction: Thanks to the above theorem, we have the following explicit construction of expanders. Consider $SL_3(Z)$, which has property (T) and is generated by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For each prime number p, consider the natural maps $\phi_p SL_3(Z) \to SL_3(Z/p)$, where Z/p is the finite field with p elements. Then $ker(\phi_p)$ is a subgroup of $SL_3(Z/p)$ of finite index and

$$[SL_3(Z/p) : ker(\phi_p)] = |SL_3(Z/p)| \to \infty \;\; as\ p\ prime\ \to \infty.$$

By the above theorem, the family of Cayley graphs $(\Gamma(SL_3(Z/p), \{A, B\}))_{p\ prime}$ is a family of expanders of fixed degree 4.

2. EIGENFUNCTIONS OF FINITE SUPPORT

All the content of this chapter comes from the preprint [1]. It should be noted that a lot of the ideas that follow are not original ideas of the author, and credit is attributed accordingly throughout the text.

2.1 Existence and Approximation in the Amenable Case

The following theorem is due to Kuchment [28] for the case when G is abelian. In [29], Veselić generalized the theorem to the case when G is amenable. Later in [30], Higuchi and Nomura gave a simpler proof using an argument of Delyon and Souillard in [31]. Here we provide the proof of Higuchi and Nomura.

Theorem 2.1.1 (Strong Localization of Eigenfunctions, Kuchment-Veselić)**.** *Let Γ be a G-periodic graph with amenable group G and let Δ be the Laplacian operator on it. If λ is an eigenvalue of Δ, then there exists an eigenfunction of λ which has finite support.*

Proof. We outline the proof of Higuchi and Nomura from [30]. Since λ is an eigenvalue, there exists $f \in l^2(V) \setminus \{0\}$ such that $\Delta f = \lambda f$. without loss of generality $||f|| = 1$. Fix a fundamental domain W. Since $f \neq 0$ and translations of eigenfunctions are still eigenfunctions, without loss of generality $f(w) \neq 0$ for some $w \in W$. Picking an orthonormal basis $\{\phi_i\}_i$ of $l^2(V)$ such that $\phi_1 = f$ we have:

$$dk(\{\lambda\}) = \frac{1}{|W|} tr(E(\{\lambda\})I_W) \geq \frac{< I_W E(\{\lambda\})f, f >}{|W|} = \frac{< I_W f, f >}{|W|} = \sum_{w \in W} |f(w)|^2 > 0$$

Next, since the action of G commutes with Δ, for all $g \in G$

$$|W| dk(\{\lambda\}) = tr(E(\{\lambda\})I_W) = tr(E(\{\lambda\})\pi_{g^{-1}}\pi_g I_W) = tr(E(\{\lambda\})I_{g \cdot W})$$

And hence for all finite subsets $\mathcal{F}$ of G, $dk(\{\lambda\}) = \dfrac{1}{|\mathcal{F} \cdot W|} tr(E(B)I_{\mathcal{F} \cdot W})$.

Now, pick a standard 2-thick Følner sequence $\{F_j\}_j$. where $F_j = \mathcal{F}_j \cdot W$ for all j and $l \in N$ is the corresponding constant from Lemma 3.1. Also, pick an orthonormal basis $\{\phi_1, ...\phi_{m_j}\}$ of $I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W} E(\{\lambda\}) l^2(V)$ and extend it to an orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ of $l^2(V)$. We have:

$$tr(I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W} E(\{\lambda\}) = \sum_{i=1}^{m_j} < I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W} E(\{\lambda\})\phi_i, \phi_i > + \sum_{i=m_j+1}^{\infty} < I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W} E(\{\lambda\})\phi_i, \phi_i >$$

$$= \sum_{i=1}^{m_j} < I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W} \phi_i, \phi_i > \leq m_j$$

Using the above estimate, we claim that there exists j such that $|\partial_2 F_j| < m_j$. If not, that $|\partial_2 F_j| \geq m_j$ for all j and hence

$$0 < dk(\{\lambda\}) = \frac{1}{|(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W|} tr(I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W} E(\{\lambda\}) \leq \frac{m_j}{|(\mathcal{F}_j \cup \partial_l \mathcal{F}_j)\cdot W|} \leq \frac{|\partial_2 F_j|}{|F_j|} \to 0$$

a contradiction. Picking j such that $|\partial_2 F_j| < m_j$, it follows that $\{I_{\partial_2 F_j}\phi_1, ...I_{\partial_2 F_j}\phi_{m_j}\}$ is a linearly dependent set so we can find $a_1, ...a_{m_j}$ not all zero such that

$$h := \sum_{i=1}^{m_j} a_j \phi_i \equiv 0 \ \ on \ \partial_2 F_j$$

It then follows that $I_{F_j} h$ is an eigenfunction with finite support inside F_j which is nonzero due to the independence of the ϕ_i on F_j. □

Denote by $D(\Gamma)$ all C-valued functions on the vertices V of Γ with finite support and by $D_\lambda(\Gamma)$ all the eigenfunctions of λ in $D(\Gamma)$. The following theorem is due to Kuchment for the abelian case [28] and Veselić for the amenable case [29]. Here, we provide a new proof using the argument of Delyon and Souillard [31].

Theorem 2.1.2 (Finite Support Approximation of Eigenfunctions, Kuchment-Veselić)**.** .
Let $G\Gamma$ be an G-periodic graph with amenable G and let Δ be the Laplacian operator on it with eigenvalue λ. If $f \in l^2(V)$ is an eigenfunction of λ, then for all $\epsilon > 0$ arbitrarily small, there exists

$g \in D_\lambda(\Gamma)$ *such that* $||f - g|| < \epsilon$, *i.e. the finite support eigenfunctions of* λ *are* l^2*-dense in the* l^2*-eigenspace of* λ.

Proof. Let $\mathcal{M}$ be the closure of $D(\Gamma)$ in $l^2(V)$ and E_λ be the eigenspace of λ. Suppose the contrary, i.e. $\mathcal{M} \neq E_\lambda$. Then the orthogonal complement of $\mathcal{M}$ (w.r.t. the subspace E_λ) is not empty, $\mathcal{N} := \mathcal{M}^\perp \neq \emptyset$ Note that since for all $g \in G \ \ \pi_g D(\Gamma) = D(\Gamma)$,

$$f \in \mathcal{N} \iff for\ all\ h \in D(\Gamma) < f, h >= 0$$

$$\iff for\ all\ h \in D(\Gamma) < \pi_g f, \pi_g h >= 0 \iff < \pi_g f, h >= 0$$

Hence $\mathcal{N}$ is also invariant under translations.

Next, consider the orthogonal projection onto $\mathcal{N}$, $P_\mathcal{N}$ and fix a fundamental domain W. Define the "density" of $\mathcal{N}$ as

$$d := \frac{1}{|W|} tr(I_W P_\mathcal{N})$$

where the trace is taken w.r.t. $l^2(V)$ Pick some $f \in \mathcal{N}$ with $||f|| = 1$ and translated so that $f(w) \neq 0$ for some $w \in W$. Then by extending $\{f\}$ to an orthonormal basis, we conclude that $d > 0$. Since $\mathcal{N}$ is invariant under translations, it also follows that $d = \dfrac{1}{|\mathcal{F} \cdot W|} tr(I_{\mathcal{F} \cdot W} P_\mathcal{N})$ for every finite subset F of G.

Choose a standard 2-thick Følner sequence $\{F_j\}_j$. where $F_j = \mathcal{F}_j \cdot W$ for all j, pick an orthonormal basis $\{\psi_1, ...\psi_{n_j}\}$ of $I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j) \cdot W} P_\mathcal{N} l^2(V)$ and extend it to an orthonormal basis $\{\psi_j\}_{j=1}^\infty$ of $l^2(V)$. We conclude that $tr(I_{(\mathcal{F}_j \cup \partial_l \mathcal{F}_j) \cdot W} P_\mathcal{N}) \leq n_j$. Similar to before, if for all j, $|\partial_2 F_j| \geq n_j$, then $0 < d < \dfrac{|\partial_2 F_j|}{|F_j|} \to 0$ This cannot happen, so there is some j such that $|\partial_2 F_j| \leq n_j$. We can then find $b_1, ...b_j$ not all zero such that

$$g := \sum_{i=1}^{n_j} b_j \psi_i \equiv 0 \ \ on\ \partial_2 F_j$$

Then $I_{F_j} g \in D(\Gamma)$. Here comes the contradiction. Since for all $i = 1...n_j$, $\psi_i \in I_{F_j} P_\mathcal{N} l^2(V) = I_{F_j} \mathcal{N}$, there exists $\bar{\psi}_i \in \mathcal{N}$ such that $I_{F_j} \bar{\psi}_i = \psi_i$ so $\sum_{i=1}^{n_j} b_j \bar{\psi}_i \in \mathcal{N}$. On the other hand, the ψ_i are linearly independent over F_j, $g \neq 0$. We have:

$$0 = < g, \sum_{i=1}^{n_j} b_j \bar{\psi}_i > = \sum_{v \in F_j} |g(v)|^2 \neq 0$$

This is a contradiction, therefore $\mathcal{N} = \emptyset \implies \mathcal{M} = E_\lambda$ and the claim follows. $\square$

2.2 Finite Generation in the Commutative Case

For this section, let $\Gamma = (V, E)$ be a Z^d-periodic graph with fundamental domain W.

The **Floquet-Bloch transform** of $f \in l^2(V)$ is a complex valued function with domain $V \times T^d$ (where T^d is the d dimensional torus)

$$\hat{f}(v, e^{ik}) := \sum_{g \in Z^d} f(g \cdot v) e^{-ik \cdot g}$$

where $v \in V$, $k \in Z^d$, $e^{ik} = (e^{ik_j})_{j=1}^d \in T^d$ and $k \cdot g$ is the standard dot product.

One can verify that for all $g \in Z^d, \hat{f}(g \cdot v, e^{ik}) = e^{ig} \hat{f}(v, e^{ik})$. This means that the entire function $\hat{f}$ may be recovered from its restriction to $W \times T^d$, hence from now on we will view $\hat{f}$ as a function on this restricted domain.

One can visualized the transform as shown below. For each vertex (purple, red, orange) in the fundamental domain (green box), we can consider the orbit under the Z^d action and then take "Fourier-like" sum along that orbit. We then obtain 3 functions on the unit torus, one for each element in the fundamental domain (or equivalently one for each orbit)

The following key theorem is a consequence of standard techniques from Fourier analysis. For a more detailed exposition see [16].

Proposition 2.2.1. *a) Inversion Formula: for all $v \in W, g \in Z^d$,*

$$f(g \cdot v) = \int_{[-\pi,\pi]^d} \hat{f}(v, e^{ik}) e^{ik \cdot g} dk$$

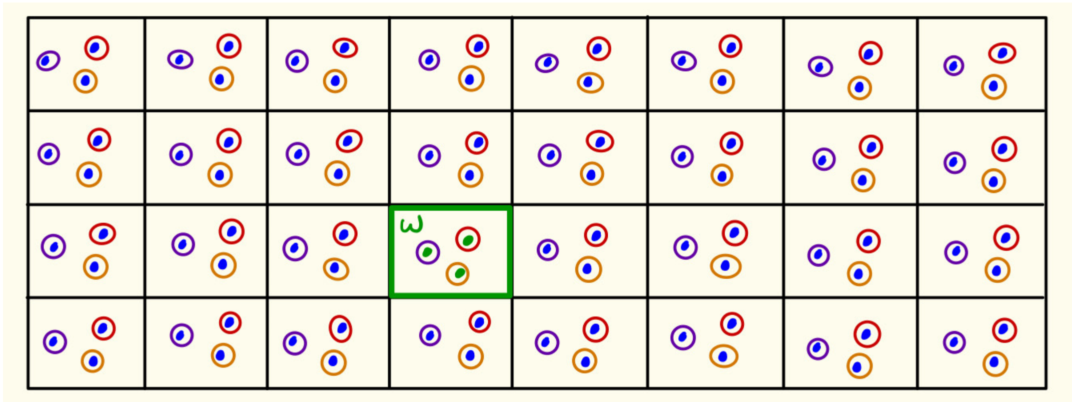

Figure 2.1: Abstract picture for the Floquet-Bloch transform

b) The map

$$f(u) \mapsto (2\pi)^{-n/2}\hat{f}(v, e^{ik})$$

is a unitary map from $l^2(V)$ to $L^2(T^d, C^{|W|})$, the space of all square summable functions from T^d to $C^{|W|}$.

As a result of the above theorem, by composing Δ with the Floquet-Bloch transform and its inverse, we may transform the discrete Laplacian $\Delta : l^2(\Gamma) \to l^2(\Gamma)$ to a corresponding self-adjoint map $\hat{\Delta} : L^2(T^d, C^{|W|}) \to L^2(T^d, C^{|W|})$.

Let $v_1 := (\delta_{1,i})_{i=1}^d, ...v_d := (\delta_{d,i})_{i=1}^d$ be the standard basis of Z^d and write $z_j := e^{iv_j}$. By definition, the image of $f \in D(\Gamma)$ under the Floquet-Bloch transform is a vector of size $n = |W|$ whose entries are Laurent polynomials in $(z_j)_j$ (that is, polynomials in $(z_j)_j \cup (z_j^{-1})_j$). We denote this ring of Laurent polynomials by $C[z_1^{\pm 1}, z_2^{\pm 1}, ..., z_d^{\pm 1}]$. The next two well known propositions from [28] allow us to pass questions about finite support eigenfunctions to questions in commutative algebra.

Proposition 2.2.2. *The map $\hat{\Delta} : L^2(T^d, C^{|W|}) \to L^2(T^d, C^{|W|})$ restricted to $\bigoplus_{k=1}^n C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$ is an $C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$-linear homomorphism from the $C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$-module $\bigoplus_{k=1}^n C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$ to itself.*

By the above proporition we may express $\hat{\Delta}$ as a $|W| \times |W|$ matrix whose entries are rational functions on $T^d \subset C^d$ expressed via Laurant polynomials. Classical Floquet-Bloch theory shows that λ is an eigenvalue iff $det(\hat{\Delta} - \lambda I)$ is the zero function on T^d. Moreover, λ lies in the spectrum $sp(\Delta)$ iff $det(\hat{\Delta} - \lambda I)$ has a zero. For more details see [16]. Moreover, we have:

Proposition 2.2.3 (Kuchment)**.** *A Z^d-periodic graph Γ with Laplacian Δ has an eigenvalue λ if and only if the map*

$$(\hat{\Delta} - \lambda I_n) : \bigoplus_{k=1}^{n} C[z_1^{\pm 1}, ..., z_d^{\pm 1}] \to \bigoplus_{k=1}^{n} C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$$

*is **not** injective. The kernel of this map corresponds to the finite support eigenfunctions of λ.*

In view of the formula

$$\hat{f}(g \cdot v, e^{ik}) = e^{ig}\hat{f}(v, e^{ik}) = z_1^{g_1}...z_d^{g_d}\hat{f}(v, e^{ik}) = z^g\hat{f}(v, e^{ik})$$

we see that when we multiply each component of $\hat{f}$ by the same monomial $z^g \in C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$ we are essentially translating f by $g \in Z^d$. It follows that when we multiply each component of $\hat{f}$ by an arbitrary element of $C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$, then we are taking linear combinations of translations of $\hat{f}$. Notice that $(\hat{\Delta} - \lambda I_n)\hat{f} = 0 \iff (\hat{\Delta} - \lambda I_n)z^g\hat{f} = 0$ hence translations of eigenfunctions are still eigenfunctions w.r.t. the same eigenvalue (alternatively one can just use the fact that Z^dV preserves edges). Therefore, if we wish to describe all finite support eigenfunctions of λ, it suffices to find them up to translations by Z^d.

The following proposition is due to Kuchement [28], but the proof presented here is new:

Proposition 2.2.4 (Kuchment)**.** *Let Γ be a Z^d-periodic graph with discrete Laplacian Δ on it and let λ be an eigenvalue of Δ. Then λ has finitely many finite support eigenfunctions up to translation and linear combinations. That is, there are eigenfunctions of λ $f^{(1)}, ...f^{(r)}$ such that every eigenfunction of λ f with finite support is the finite linear combination of translations of $f^{(1)}, ...f^{(r)}$.*

Proof. Suppose that we had, a priori, a finite support eigenfunction $f \in D_\lambda(\Gamma)$. Then we may translate it such that, without loss of generality, f has support in $\cup_{g\in N^d}$ -$g \cdot W$. This means that $\hat{f} \in C[z_1, ..., z_d]$. Next consider the entries of $(\hat{\Delta} - \lambda I_n)$ (which are elements of the ring $C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$), look at all the integer powers of $z_1, ...z_d$ in the terms of the entries and pick the smallest negative power $-P$ (set $P = 0$ if all the powers are non-negative). That way, the entries of $(z_1, ...z_d)^P(\hat{\Delta} - \lambda I_n)$ all lie in $C[z_1, ..., z_d]$. We conclude that the set of all eigenfunctions of λ whose support is finite and lies in $\cup_{g\in N^d}$ -$g \cdot W$ is the kernel of the $C[z_1, ..., z_d]$-linear map:

$$(z_1, ...z_d)^P(\hat{\Delta} - \lambda I_n) : \bigoplus_{k=1}^{n} C[z_1, ..., z_d] \to \bigoplus_{k=1}^{n} C[z_1, ..., z_d]$$

By the classical Hilbert Basis Theorem, every ideal of $C[z_1, ..., z_d]$ is finitely generated, i.e. $C[z_1, ..., z_d]$ is Noetherian. Every finitely generated module over a Noetherian ring is a Noetherian module. The kernel of $(z_1, ...z_d)^P(\hat{\Delta} - \lambda I_n)$ is certainly a submodule of the finitely generated $C[z_1, ..., z_d]$-module $\bigoplus_{k=1}^{n} C[z_1, ..., z_d]$, so it is finitely generated, say by generators $f^{(1)}, ...f^{(r)}$. But what does this mean? For every eigenfunction with Floquet-Bloch transform f there are $g_1, g_2, ...g_r \in C[z_1, ..., z_d]$ such that $f = g_1 f^{(1)} + g_2 f^{(2)} + ... + g_r f^{(r)}$. Breaking down $g_1, g_2, ...g_r$ into linear combinations of monomials, and noting that multiplication by z^g corresponds to translation in Γ by g, we see that f is the linear combination of translations of $f^{(1)}, ...f^{(r)}$, and the claim follows. □

2.3 Finite Generation in the Amenable Noetherian Case

In this section we generalize Proposition 3.4. Recall that the Hilbert basis theorem (polynomial rings are Noetherian) was the key ingredient in proving Proposition 3.4.

Throughout this section Γ is a G-periodic graph, where G is amenable.

Definition (Noncommutative Floquet-Bloch transform)**.** Associate to each $f \in D(\Gamma)$ a function $\hat{f} : V \to C[G]$ sending $v \in V$ to

$$\sum_{g\in G} f(g^{-1} \cdot v)g \in C[G].$$

Since f has finite support, the sum is finite and the map well defined. Notice that

$$for\ all\ \ h \in G\ \ (\hat{\pi_h f})(v) = \hat{f}(h^{-1} \cdot v) = h^{-1}\hat{f}(v).$$

Fix a fundamental domain W. In view of the above identity, we can recover $\hat{f} : V \to C[G]$ from $\hat{f}|_W$, i.e. the vector $(\hat{f}(w))_{w\in W} = (\sum_{g\in G} f(g \cdot w)g)_{w\in W} \in C[G]^{|W|}$.

It is easy to see that $\hat{\cdot} : D(\Gamma) \to C[G]^{|W|}$ is a bijective C-linear map. To the operator $\Delta : D(\Gamma) \to D(\Gamma)$ corresponds some other operator $\hat{\Delta} : C[G]^{|W|} \to C[G]^{|W|}$.

Proposition 2.3.1. *The map $\hat{\Delta}$ is a left $C[G]$-module homomorphism from the free $C[G]$-module $C[G]^{|W|}$ to itself.*

Proof. The simplify the notation in the calculation, we consider instead the adjacency operator $Af(x) := \sum_{y\sim x} f(y)$. Fix a fundamental domain $W = \{w_1, ...w_n\}$. For each $1 \le i, j \le n$, we can find unique $h^1_{i,j}, ...h^{m_{i,j}}_{i,j} \in G$ such that

$$\{v \in V | v \sim w_i\} = \{h^k_{i,j} \cdot w_j | 1 \le j \le n\ ,\ 1 \le k \le m_{i,j}\}.$$

Since the group action preserves edges, for all $g \in G$:

$$\{v \in V | v \sim g \cdot w_i\} = \{g \cdot h^k_{i,j} \cdot w_j | 1 \le j \le n\ ,\ 1 \le k \le m_{i,j}\}.$$

For each $f \in D(\Gamma)$ and $1 \le i \le n$:

$$\hat{A}\hat{f}(w_i) = (\hat{Af})(w_i) = \sum_{g\in G} Af(g^{-1} \cdot w_i)\ g = \sum_{g\in G}\sum_{j=1}^{n}\sum_{k=1}^{m_{i,j}} f(g^{-1}\ h^k_{i,j} \cdot w_j)\ g$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m_{i,j}}\sum_{g\in G} f(((h^k_{i,j})^{-1}g)^{-1}\ \cdot w_j)\ g = \sum_{j=1}^{n}\sum_{k=1}^{m_{i,j}}\sum_{h\in G} f(h^{-1} \cdot w_j)\ h^k_{i,j}h$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m_{i,j}} h^k_{i,j} \sum_{h\in G} f(h^{-1} \cdot w_j)\ h = \sum_{j=1}^{n} (\sum_{k=1}^{m_{i,j}} h^k_{i,j})\ \hat{f}(w_j)$$

Therefore we obtain a left $C[G]$-module homomorphism which is represented as a matrix of group algebra elements by $(\sum_{k=1}^{m_{i,j}} h_{i,j}^k)_{i=1,j=1}^{n,n}$. □

Recall that a ring R is **Noetherian** whenever every submodule of a finitely generated R-module is finitely generated.

Proposition 2.3.2. *Let Γ be an G-periodic graph with amenable group G, Δ be the discrete Laplacian on Γ and λ an eigenvalue of Δ.*
If *the group algebra $C[G]$ is Noetherian (in particular if* **G** *is virtually polycyclic),*
then *there are finitely many finite support eigenfunctions $f^{(1)},...f^{(r)} \in D_\lambda(\Gamma)$ such that every eigenfunction of λ with finite support on Γ is the finite linear combination of translations of $f^{(1)},...f^{(r)}$*

Proof. Note that $f \in D_\lambda(\Gamma)$ iff $(\Delta - \lambda I)f = 0$ iff $(\hat{\Delta} - \lambda I)\hat{f} = 0$ iff $f \in kernel(\hat{\Delta} - \lambda I)$. This kernel is a submodule of $C[G]^{|W|}$. Since $C[G]$ is Noetherian, $kernel(\hat{\Delta} - \lambda I)$ is finitely generated. Say, $kernel(\hat{\Delta} - \lambda I) =< \hat{f}^{(1)}, ..., \hat{f}^{(r)} >$. Then for all $f \in D_\lambda(\Gamma)$, there exist $h_1, ...h_r \in C[G]$ such that $\hat{f} = h_1\hat{f}^{(1)} + ... + h_r\hat{f}^{(r)}$. Write $h_1 = \sum_{g\in G} a_g g$ so that:

$$h_1\hat{f}^{(1)} = (\sum_{g\in G} a_g g)\hat{f}^{(1)} = \sum_{g\in G} a_g(g\hat{f}^{(1)}) = \sum_{g\in G} a_g(\widehat{\pi_g^{-1} f^{(1)}})$$

Hence to $h_1\hat{f}^{(1)}$ corresponds a function which is the finite linear combination of translations of $f^{(1)}$, and the theorem follows. □

The theorem of Paul Hall [32] states that the group algebra of a virtually polycyclic group is Noetherian (virtually means "finite extension of"). Recall that a finitely generated group G is **polycyclic** if it admits a subnormal series whose factors are all cyclic groups. Note that nilpotent groups are polycyclic, polycyclic groups are solvable and solvable groups are amenable. The lamplighter group is an example of a solvable group which is not polycyclic. Since finite extensions of amenable groups are amenable, it follows that virtually polycyclic groups are amenable, thus the above theorem applies to virtually polycyclic groups. Also note that by Gromov's Theorem, groups of polynomial growth are virtually nilpotent, hence also virtually polycylic. For more details on these claims see [7]. The point it that we have a large family of groups for which the group algebra CG is Noetherian.

2.4 Finite Support Approximation of Density

The following proposition allows us to study the density of an eigenvalue via its finite support eigenfunctions. Essentially, it connects the definition of density of states via a trace formula and the intuitive definition of density of states as the number of eigenfunctions per unit volume. The formula is often called **Shubin's formula**, named after Mikhail Shubin who used it in the study of almost periodic elliptic PDE [33]. For the case of G-periodic graphs and discrete periodic operators on them, there are many similar results which show that the CDF of the empirical density of states converges to the density of states at all points of continuity (see, for instance [29] and [9]). However, here we are exclusively interested at a single point of discontinuity, so those results do not apply.

Proposition 2.4.1. *Let Γ be an G-periodic graph with amenable G, discrete Laplacian Δ and let λ be an eigenvalue of Δ.*
If *every element in $D_\lambda(\Gamma)$ is the linear combination of translations (via the group action) of finitely many finite support eigenfunctions $f^{(1)},...f^{(r)}$ (in particular, if $C[G]$ is Noetherian),*
then *for any fundamental domain W and any generating set S of G there exists $j_0 \in N$ such that a standard j_0 thick Følner sequence $\{F_j\}$ of Γ satisfies the following formula:*

$$dk(\{\lambda\}) = lim_{j\to\infty} \frac{dim_C\{f \in D_\lambda(\Gamma) : sprt(f) \subset F_j\}}{|F_j|}$$

Proof. Consider the supports of $f^{(1)},...f^{(r)}$ which are all finite, hence we can find a finite $K \subset G$ such that $\cup_{i=1}^r sprt(f^{(i)}) \subset \cup_{g\in K} g \cdot W$ and let $j_0 := max_{g\in K}|g|$. Take a standard j_0 thick Følner sequence $\{F_j\}_j$. As we saw in section 2, $dk(\{\lambda\}) = \frac{1}{|F_j|} tr(E(\{\lambda\})I_{F_j})$ Define the C vector spaces

$$U_j := Span\{\pi_g f^i : g \in G,\ i = 1...r\ sprt(\pi_g f^i) \subset F_j\}$$

$$W_j := Span\{\pi_g f^i : g \in G,\ i = 1...r\ sprt(\pi_g f^i) \subset F_j \cup \partial_{j_0} F_j\}$$

By Theorem 2.3, the (closed) subspace of $l^2(V)$ generated by $\{\pi_g f^i : g \in G,\ i = 1...r\}$ is precisely E_λ. By our choice of j_0, for all $\pi_g f^{(i)} \in \{\pi_g f^{(i)} : sprt(\pi_g f^i) \not\subset F_j \cup \partial_{j_0} F_j, 1 \leq i \leq r\}$ wa have that $sprt(\pi_g f^{(i)}) \cap F_j = \emptyset$, hence $< I_{F_j} \pi_g f^{(i)}, \pi_g f^{(i)} >= 0$ and we conclude that $tr(E(\{\lambda\}) I_{F_j}) \leq dim_C W_j$. On the other hand, each element ϕ of an orthonormal basis for U_j satisfies $< I_{F_j} \phi, \phi >= ||\phi||^2 = 1$. We end up with the estimate:

$$dim_C U_j \leq tr(E(\{\lambda\}) I_{F_j}) \leq dim_C W_j$$

Next, we work on estimating $dim_C\{f \in E_\lambda : sprt(f) \subset F_j\}$. Obviously $U_j \subset \{f \in E_\lambda : sprt(f) \subset F_j\}$. On the other hand, if $f \in E_\lambda$ with $sprt(f) \subset F_j \cup \partial_{j_0} F_j$, f is the linear combination of translations of $f^{(1)}, ...f^{(r)}$. The values of f on $F_j \cup \partial_{j_0} F_j$ depend only through the terms $\pi_g f^i$ whose support is not disjoint from $F_j \cup \partial_{j_0} F_j$. Each such term should have support inside $F_j \cup \partial_{2j_0} F_j$, by the construction of j_0. Hence there is a function $f' \in W_j$ which is equal to f on $F_j \cup \partial_{j_0} F_j$. We get a map from $\{f \in E_\lambda : sprt(f) \subset F_j\}$ to W_j sending f to f' which is obviously injective. We conclude that: for all j

$$dim_C U_j \leq dim_C\{f \in E_\lambda : sprt(f) \subset F_j\} \leq dim_C W_j$$

Finally, we ask: what is $q_j := |\{\pi_g f^i : g \in G,\ i = 1...r\ sprt(\pi_g f^i) \not\subset F_j\ but\ sprt(\pi_g f^i) \subset F_j \cup \partial_{j_0} F_j\}|$? We get the obvious bound $q_j \leq |\partial_{j_0} \mathcal{F}_j| r$. Dividing by $|F_j|$, we take the limit as $j \to \infty$:

$$0 \leq lim_{j \to \infty} \frac{dim_C U_j - dim_C W_j}{|F_j|} \leq lim_{j \to \infty} \frac{q_j}{|F_j|} \leq lim_{j \to \infty} \frac{|\partial_{j_0} \mathcal{F}_|| r}{|F_j|} = 0$$

Therefore, by squeezing between U_j and W_j:

$$dk(\{\lambda\}) = lim_{j \to \infty} \frac{1}{|F_j|} tr(E(\{\lambda\}) I_{F_j}) = lim_{j \to \infty} \frac{dim_C\{f \in D_\lambda(\Gamma) : sprt(f) \subset F_j\}}{|F_j|}.$$

$\square$

2.5 A Free Resolution Formula

Whenever we have a G-periodic graph Γ with G finitely generated amenable, we know that the G-module of finite support eigenfunctions $K := kernel(\hat{\Delta} - \lambda I)$ is nonempty and it's dense in the l^2-eigenspace. The goal of this section is to use the algebraic structure of K to find the density of λ. We would like to apply Proposition 5.1 and use K to estimate $dim_C\{f \in D_\lambda(\Gamma) : sprt(f) \subset F_j\}$. The obvious way is to pick a generating set $f^{(1)}, f^{(2)}, ...f^{(r)}$ of K and count all $g \in G$ and $1 \leq i \leq r$ such that $sprt(\pi_g f^{(i)}) \subset F_j$. The issue is that the set of all those $\pi_g f^{(i)}$ may not be linearly independent, and hence our estimate can be far from optimal. This motivates us to consider syzygy modules.

Let R be a Noetherian ring. If $\{f^{(1)}, f^{(2)}, ...f^{(r)}\}$ is a finite generating set of a finitely generated R-module M, the **(first) syzygy module** of of M w.r.t. the generators $\{f^{(1)}, f^{(2)}, f^{(r)}\}$ is the set of all $g = (g_1, ...g_r) \in R^r$ such that

$$g_1 f^{(1)} + g_2 f^{(2)} + ... + g_r f^{(r)} = 0$$

and is denoted by $Syz(M)$. Via pointwise multiplication $Syz(M)$ is an R-module as well. Since R is Noetherian, $Syz(M) \subset R^r$ is finitely generated. Picking a finite set of generators for $Syz(M)$, we can consider its own syzygy module, $Syz(Syz(M))$, abbreviated by $Syz^2(M)$. By iteration we can define the **(higher) syzygy module** $Syz^k(M)$ (which is finitely generated) for any positive power k along with the conventions $Syz^1(M) = Syz(M)$ and $Syz^0(M) = M$.

A alternative way to describe syzygies is through free resolutions. Picking a finite generating set $\{f^{(1)}, f^{(2)}, ...f^{(r_0)}\}$ for M is equivalent to finding a surjection $R^{r_0} \to M \to 0$. The kernel of this map is precisely the first syzygy module, so we get the Short Exact Sequence $0 \to Syz(M) \to R^{r_0} \to M \to 0$. Iterating this proccess we get another Short Exact Sequence $0 \to Syz^2(M) \to R^{r_1} \to Syz(M) \to 0$ where we choose a generating set of $Syz(M)$ of length r_1. We end up with the following sequence of maps:

$$... \, Syz^r(M) \hookrightarrow R^{r_3}Syz^3(M) \hookrightarrow R^{r_2}Syz^2(M) \hookrightarrow R^{r_1}Syz(M) \hookrightarrow R^{r_0}M \to 0$$

Via composition we get a **free resolution** of M, that is, a Long Exact Sequence beginning with $\to R^{r_0} \to M \to 0$ and consequently consisting of free R-modules:

$$... \; \to R^{r_3} \to R^{r_2} \to R^{r_1} \to R^{r_0} \to M \to 0$$

Then $Syz^k(M)$ can be recovered as the kernels (or equivalently images) of each map. Note that when R is **not** Noetherian, we can still construct syzygy modules, however, we cannot guarrantee that the free modules in the resulting resolution will be finitely generated.

Hilbert's Syzygy Theorem (see [34]) asserts that the d^{th} syzygy module $Syz^d(M)$ of a finitely generated $C[z_1^{\pm1},...,z_d^{\pm1}]$ module M will always be free. This means that if we choose a free generating set for $Syz^d(M)$, then $Syz^{d+1}(M) = 0$. As a result the correspoding free resolution will terminate at the $(d+1)^{th}$ step.

Theorem 2.5.1. *Suppose that Γ is a Z^d-periodic graph with fundamental domain W and λ is an eigenvalue of the Laplacian Δ on Γ. Let K be the kernel of the map $\hat{\Delta} - \lambda I_n$ viewed as a map from the $R = C[z_1^{\pm1},...,z_d^{\pm1}]$-module $\bigoplus_{k=1}^{|W|} C[z_1^{\pm1},...,z_d^{\pm1}]$ to itself. Then the following formula about the density of $\{\lambda\}$ holds:*

$$dk(\{\lambda\}) = \frac{1}{|W|}\sum_{k=0}^{d}(-1)^k r_k$$

where $r_0,...r_d$ are the ranks of the free modules in a free resolution of K

$$0 \to R^{r_d} \to R^{r_{d-1}} \; ... \; \to R^{r_1} \to R^{r_0} \to K \to 0$$

In particular, $dk(\{\lambda\})$ is rational.

Until we state otherwise, $R = C[z_1^{\pm1},...,z_d^{\pm1}]$**.** We remark that there exist plenty of algorithms for the computation of K and its higher syzygy modules, as well as software for these algorithms (see [34]). We will use the following notation for an R-submodule M of a free R-module R^n ($n \in N$). For each monomial $z^l = z_1^{l_1}...z_d^{l_d}$ let $|z^l| := |l| = max_{1\le i\le d}|l_i|$. Next, for all $f_1 \in R$, let $|f_1|$ be the maximum length of the monomials it is comprised of and for all $f = (f_1, ...f_n) \in R^n$ let $|f| := max_{1\le k\le n}|f_k|$. Define

$$B(M,j) := \{f \in M : \ |f| \le j\} \qquad |B(M,j)| := dim_C(\{f \in M : \ |f| \le j\})$$

which are intuitively interpreted as balls in M centered at $0 \in M$. Note that the "length" $|f|$ of $f \in M$ is taken with respect to the free R-module R^n that M sits in. The proof of Theorem 6.1 will rely on the following estimate:

Lemma 2.5.2. *Let M be a submodule of the free R-module R^n ($R = C[z_1^{\pm 1}, ..., z_d^{\pm 1}]$) with syzygy module*

$$0 \to Syz(M) \to R^r \to M \to 0$$

Then there exists j_0 such that for all $j > j_0$ we have the estimate

$$(2(j-j_0)+1)^d r - |B(Syz(M), j-j_0)| \ \ \le \ \ |B(M,j)| \ \ \le \ \ (2j+1)^d r - |B(Syz(M), j)|$$

Proof of Lemma. Fix a generating set $\{f^{(1)}, ..., f^{(r)}\}$ of M. Let $j_0 := max_{1\le i\le r}|f^{(i)}|$. For all $j > j_0$, there are exactly $(2(j-j_0)+1)^d$ monomials z^k such that $|k| \le j - j_0$. For each such k and for every $1 \le i \le r$, we have $z^k f^{(i)} \in B(M,j)$. If $U_j := Span_C\{z^k f^{(i)} | 1 \le i \le r, |k| \le j - j_0\}$ is linearly independent over C, then certainly $r(2(j-j_0)+1)^d \le dim_C U_j \le |B(M,j)|$. However, this is true in general. Instead, the relations $\sum_\alpha z^{k_\alpha} f^{(i_\alpha)} = 0$ with $|k_\alpha| \le j - j_0$ are in 1-1 correspondence with the syzygies $(h_1, ...h_r) \in Syz(M)$ where $|h_i| \le j - j_0$ for all i. That is, the relations $\sum_\alpha z^{k_\alpha} f^{(i_\alpha)} = 0$ in 1-1 correspondence with $B(Syz(M), j - j_0)$. This correspondence is easily seen to be a linear map and hence we get a Short Exact Sequence:

$$0 \to B(Syz(M), j - j_0) \to B(R^r, j - j_0) \to U_j \to 0 \quad \Longrightarrow$$

$$(2(j-j_0)+1)^d r \; - \; |B(Syz(M), j-j_0)| \; = \; dim_C B(R^r, j-j_0) \; - \; dim_C B(Syz(M), j-j_0)$$

$$= \; dim_C U_j \; \leq \; |B(M,j)|.$$

For the second inequality, suppose that $f \in B(M,j)$. Since $M =< f^{(1)}, ..., f^{(r)} >$, there exist $h_1, ...h_r$ s.t. $f = h_1 f^{(1)} + ... + h_r f^{(r)}$. But f consists of entries with monomials z^k s.t. $|k| \leq j$ hence we may remove any monomials z^k from $h_1, ...h_r$ with $|k| > j$ and we still get $f = h_1 f^{(1)} + ... + h_r f^{(r)}$. This shows that $B(M,j) \subset W_j := Span_C\{z^k f^{(i)} | 1 \leq i \leq r, |k| \leq j\}$. In the exact same manner as with the first inequality, we get a Short Exact Sequence

$$0 \to B(Syz(M), j) \to B(R^r, j) \to W_j \to 0.$$

Therefore, the second inequality follows by taking dimensions:

$$|B(M,j)| \; \leq \; dim_C U_j \; = \; dim_C B(R^r, j) - dim_C B(Syz(M), j) \; = \; (2j+1)^d r - |B(Syz(M), j)|.$$

$\square$

Proof of Theorem. Using the lemma for each $Syz^i(K)$ for all $0 \leq i \leq d$, we obtain a j_0 value for each. Choose j_0 to be the maximum out of all these values and take the following dj_0-thick Følner sequence:

$$F_j := \mathcal{F}_j \cdot W := \{k \in Z^d : |k| \leq j\} \cdot W$$

Then for all j we have:

$$|F_j| = |W|(2j-1)^d \; , \quad \hat{F}_j = B(R^{|W|}, j) \; , \quad dim_C\{f \in D_\lambda(\Gamma) : sprt(f) \subset F_j\} = |B(K,j)|$$

By Theorem 5.2,

$$dk(\{\lambda\}) = lim_{j\to\infty}\frac{dim_C\{f \in D_\lambda(\Gamma) : sprt(f) \subset F_j\}}{|F_j|} = lim_{j\to\infty}\frac{|B(K,j)|}{|W|(2j+1)^d}$$

By the previous Lemma:

$$(2(j-j_0)+1)^d r_0 - |B(Syz(K), j-j_0)| \;\; \leq \;\; |B(K,j)| \;\; \leq \;\; (2j+1)^d r_0 - |B(Syz(K), j)|$$

Dividing by $(2j+1)^d$ and letting $j \to \infty$ we get

$$lim_{j\to\infty}\frac{|B(K,j)|}{(2j+1)^d} = r_0 - lim_{j\to\infty}\frac{|B(Syz(K),j)|}{(2j+1)^d}$$

By induction and since $Syz^{d+1}(K) = 0$,

$$lim_{j\to\infty}\frac{|B(K,j)|}{(2j+1)^d} = r_0 - r_1 + r_2 - ... + (-1)^d r_d$$

and the theorem follows. □

Now let $R = CG$ where G is a finitely generated group of subexponential growth. A group has subexponential growth whenever the growth function $\gamma_S(n) = \{g \in G : |g|_S = n\}$ with respect to any (equivalently all) finite generating sets S of G is a sequence of subexponential growth ($|g|_S$ is the distance of g from 1 along the Cayley graph $\Gamma(G, S)$). Groups of subexponenitial growth are always amenable [6]. We finish this section with a generalization of Theorem 6.1. The author believes that further study is needed. A key obstacle is the lack of Hilbert's Syzygy theorem to group algebras of non-abelian groups. Nonetheless, whenever the module of finite support eigenfunctions admits a finite resolution by finitely generated free CG-modules, we obtain the same conclusion as in Theorem 6.1.

Theorem 2.5.3. *Suppose that Γ is a G-periodic graph where G is a finitely generated group of subexponential growth and λ is an eigenvalue of Δ on Γ (or any periodic difference operator T of* *finite order). Let K be the CG-module of finite support eigenfunctions of λ.*

If *K admits a finite resolution by finitely generated free R-modules ($R = CG$)*

$$0 \to R^{r_d} \to R^{r_{d-1}} \ldots \to R^{r_1} \to R^{r_0} \to K \to 0$$

Then *the following formula about the density of $\{\lambda\}$ holds:*

$$dk(\{\lambda\}) = \frac{1}{|\Gamma/G|} \sum_{k=0}^{d} (-1)^k r_k.$$

In particular, $dk(\{\lambda\})$ is rational.

Proof. Note first that the hypothesis of Proposition 5.1 is satisfied since $r_0 < \infty$. Since G has subexponential growth, fixing any generating set S of G, the balls of radius n, $B(n) = \{g : |g|_S \leq n\}$ have a subsequence $F_j := B(n_j)$ which is a k-thick Folner sequence for all $k \in N$. This is a standard argument for groups of subexponential growth and can be found, for instance, in [6].

For each $g \in G$, let $|g| := min\{j : g \in B(n_j)\}$. For each $f_1 = \sum_g c_g g \in CG$, let $|f_1| := max\{|g| : c_g \neq 0\}$. Next, for each $f = (f_1, ...f_r)$ let $|f| := max\{|f_1|, ...|f_r|\}$. For each M is a submodule of R^n let

$$B(M, j) := \{f \in M : \ |f| \leq j\} \qquad |B(M, j)| := dim_C(\{f \in M : \ |f| \leq j\})$$

and finally, $\gamma(n) = |B(n)|$ is simply the growth function w.r.t. S. Similar to Lemma 6.2, one can show that then there exists j_0 such that for all $j > j_0$ we have the estimate

$$\gamma(n_j - n_{j_0})r - |B(Syz(M), j - j_0)| \ \leq \ |B(M, j)| \ \leq \ \gamma(n_j)r - |B(Syz(M), j)|$$

where $Syz(M)$ is the syzygy module w.r.t. a finite generating set of M of size $r < \infty$. Then the theorem follows from the same telescoping argument as before using the fact that for arbitrarily large $k \in N$:

$$lim_{j\to\infty}\frac{\gamma(n_j-k)}{\gamma(n_j)} = lim_{j\to\infty}\frac{\gamma(n_j)}{\gamma(n_j+k)} = 1 - lim_{j\to\infty}\frac{|\partial_k F_j|}{|\partial_k F_j \cup F_j|} = 1 - 0 = 1.$$

$\square$

3. EXAMPLES AND SUGGESTIONS FOR FURTHER STUDY

To illustrate the ideas of the previous section, we study the density of eigenvalues of the Laplacian on two Z^2-periodic graphs. Percolation, magnetic and spectral properties of these examples has been extensively studied. For deeper studies on similar graphs, see, for instance, [30, 35, 36]. Some of the content of this chapter comes from the preprint [1].

3.1 The graph of the Kagome lattice

We will find the eigenvalue (there is only one) of the Laplacian on the Kagome lattice, then classify all its finite support eigenfunctions up to translations and linear combinations and finally find the density of this eigenvalue.

Consider the periodic graph shown below in Figure 3.1. It is called the **Kagome lattice** or **trihexagonal** graph. It is a 4-regular periodic graph, has a group action by Z^2 which becomes multiplication by z_1 and z_2 under the Floquet-Bloch transform, and a fundamental domain $W = \{w_1, w_2, w_3\}$ shown below. We summarize our results for this example in following proposition:

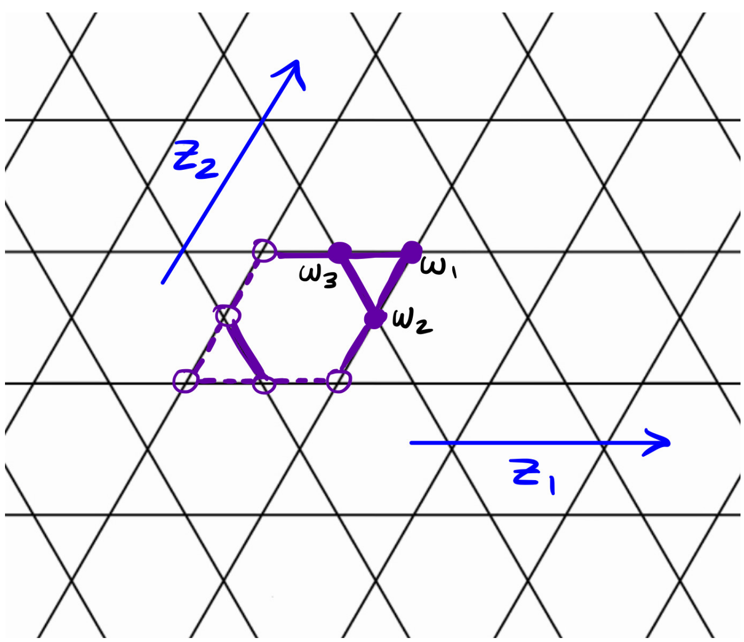

Figure 3.1: The Kagome lattice

Proposition 3.1.1. *The dicrete Laplacian on the trihexagonal graph or Kagome lattice has exactly one eigenvalue $\lambda = -2/3$ with density $dk(\{-2/3\}) = 1/3$. Moreover, every finite support eigenfunction is linear combination of translations of the eigenfunction with Floquet Bloch transform*

$$(\ z_1 - z_2, \ 1 - z_1, \ z_2 - 1 \ )$$

First, using the description of $\hat{\Delta}$ in the last section, we obtain:

$$\hat{\Delta} = \frac{1}{4} \begin{pmatrix} -4 & 1 + z_2 & 1 + z_1 \\ 1 + z_2^{-1} & -4 & 1 + z_1 z_2^{-1} \\ 1 + z_1^{-1} & 1 + z_1^{-1} z_2 & -4 \end{pmatrix}$$

where the matrix is ordered based on the ordering $\{w_1, w_2, w_3\}$ of the fundamental domain.

We next find the eigenvalues of Δ. We have:

$$det(\hat{\Delta} - \lambda I_3) = \frac{1}{4^3}((6+2\lambda)(z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_1 z_2^{-1} + z_2 z_1^{-1}) - 36 - 168\lambda - 192\lambda^2 - 64\lambda^3)$$

hence $det(\hat{\Delta} - \lambda I_3) \equiv 0$ iff $\lambda = -2/3$.

Moving on, we find all finite support eigenfunctions of λ. Suppose that $f' = (f_1', f_2', f_3')$ is the Floquet Bloch transform of an eigenfunction with finite support, so $f_1', f_2', f_3' \in C[z_1^{\pm 1}, z_2^{\pm 1}]$. By multiplying f' by $(z_1 z_2)^m$ for sufficiently large $m \in N$ we may assume that $(z_1 z_2)^m h \in \bigoplus_{k=1}^3 C[z_1, z_2]$. The function $f = (f_1, f_2, f_3) := (z_1, z_2)^m f'$ corresponds to the eigenfunction f' translated by m units in the vertical directions and m units in the horizontal direction and is still an eigenfunction. By Proposition 3.3, f_1, f_2, f_3 solve the following system of equations:

$$\frac{1}{4} \begin{pmatrix} 2 & 1 + z_2 & 1 + z_1 \\ 1 + z_2^{-1} & 2 & 1 + z_1 z_2^{-1} \\ 1 + z_1^{-1} & 1 + z_1^{-1} z_2 & 2 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff (z_2-1)f_1 = (z_1-z_2)f_3 \quad and \quad (z_2-1)f_2 = (1-z_1)f_3$$

Since (z_2-1), (z_1-z_2) and(z_2-1) are irreducible elements in $c[z_1, z_2]$

$$(z_1-z_2)|(z_2-1)f_1 \implies (z_1-z_2)|f_1$$

hence there is some $h \in C[z_1, z_2]$ such that $f_1 = (z_1-z_2)h$. Plugging in this equation to our system we get

$$f_3 = (z_2-1)h \quad and \quad f_2 = (1-z_1)h$$

Check that (f_1, f_2, f_3) is an eigenfunction forall $h \in C[z_1, z_2]$ (i.e. it solves the equation from Proposition 3.3.), so it follows that any finite support eigenfunction of Δ on the Kagome lattice has Floquet-Bloch transform of the form

$$(\ (z_1z_2)^{-m}(z_1-z_2)h,\ (z_1z_2)^{-m}(1-z_1)h,\ (z_1z_2)^{-m}(z_2-1)h\ ) \quad h \in C[z_1, z_2],\ m \in N$$

Therefore all finite support eigenfunctions of $\lambda = -2/3$ are translations and linear combinations of the single eigenfunction:

$$(\ z_1-z_2,\ 1-z_1,\ z_2-1\ )$$

With the use of the inversion formula of the Floquet-Bloch transform (Theorem 3.1), Figure 3.4 displays this eigenfunction, placing numbers next to the vertices in the support of this eigenfunction. Finally, we use Theorem 6.1 to find the density of $\{-2/3\}$. The module of finite support eigenfunctions is free and of rank 1 and the fundamental domain has 3 vertices, therefore $dk(\{-2/3\}) = 1/3$.

3.2 Periodic graph with 2 eigenfunctions

Similarly to the previous example, we can consider the following Z^2-periodic graph shown below in Figure 3.3. After expressing the Laplacian as a matrix of polynomials (via the Floquet Bloch transform) and then solving this systems of equations over the polynomial ring, we conclude

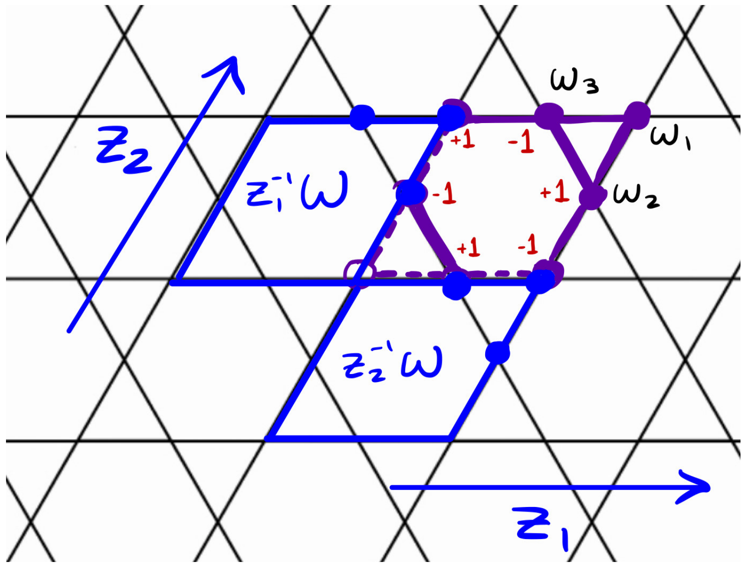

Figure 3.2: The unique eigenfunction up to translations and linear combinations

that there are two eigenfunctions of finite support up to translations and linear combinations, shown below in Figure 3.4.

3.3 Suggestions for further study

1.**Projectivity of the Module of Finite Support Eigenfunctions**

Similar in spirit to [2], whenever the Laplacian on a periodic graph has an eigenvalue, the corresponding eigenspace is a projective Hilbert-module over the reduced group C^* algebra. This follows from the fact that the spectral projection is a $C_r^*(G)$-module homomorphism. Given a finite support function, will the spectral projection to the eigenspace of $\lambda \in sp_{pp}(\Delta)$ be a function of finite support? If the answer to this question is yes, then the module of finite support eigenfunctions is a projective module over the group algebra. This can lead to relations of spectral group theory with the algebraic K-theory of group algebras.

2.**Homological Properties of Infinite Group Rings**

By the syzygy theorem of Hilbert, every finitely generated $C[Z^d]$ module admits a finite free resolution. Is there any other class of groups of subexponential growth, such that every finitely generated $C[G]$ module admits a finite resolution by free $C[G]$-modules of finite rank? Also, could

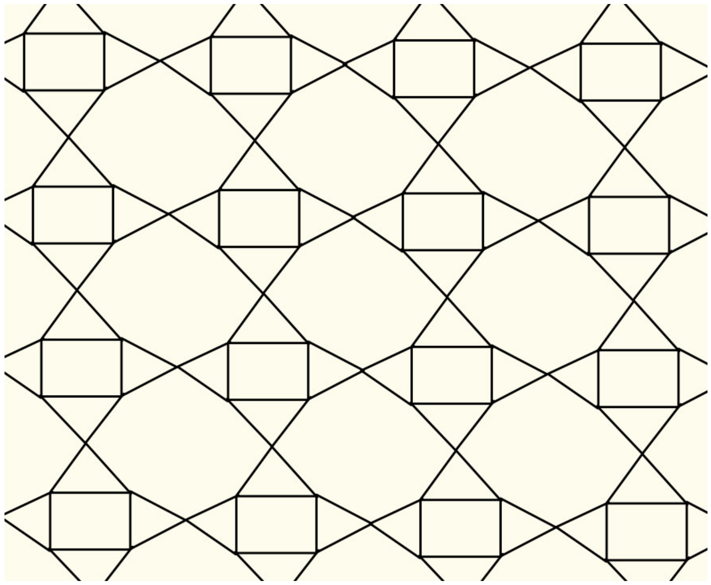

Figure 3.3: Another periodic graph

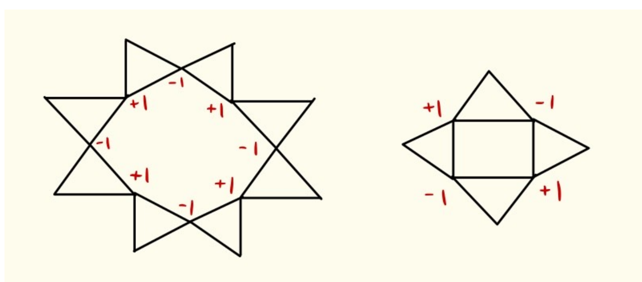

Figure 3.4: The two eigenfunctions up to translations and linear combinations

it happen that the module of finite support eigenfunctions is always free?

3.**Extending the Technique to Finite Support Approximate Eigenfunctions**

It is well known (see [11]) that λ lies the spectrum of a bounded linear operator A on a Hilbert space H if and only if it has approximate eigenvectors, that is, for every $\epsilon > 0$ there exists $x \in H$ such that $||Ax - \lambda x|| < \epsilon ||x||$.

In this thesis we focused on the eigenvalues (i.e. elements of the pure-point spectum) and how to use algebraic methods(Noetherian property, free resolutions, etc..) to study the density of states of these eigenvalues. It is natuaral to ask whether, through some clever way, one can compute the continuous part of the density of states via similar methods.

Moreover, one can study the Bloch variety in the case of Z^d periodic graphs. This variety (which is algebraic in the case of discrete graphs) is a central object in condensed matter physics and has recently been studied from the point of view of algebraic geometry (for instance see [37]). Each eigenvalue (i.e. element of the pure point spectrum) of Δ corresponds to a flat sheet on the Bloch variety. Can the method from the last chapter be extended (or at least related) to the study of algebraic properties of the Bloch variety?

4.**Search for more examples**

This final suggestion is a bit vague. Through experimentation, the author realized that "most" Z^2 periodic graphs tend to not have eigenvalues (perhaps this can be formalized in a probabilistic manner via combinatotics, or in a degeneracy manner via algebraic geometry). Can something be said about the finite support eigenfunctions on Z^2 periodic graphs? For instance, if a planar Z^2 periodic graph has no cycles of length 3, then does it have any finite support eigenfunctions? Also, are there interesting examples of periodic graphs with respect to the Heisenberg group whose discrete Laplacian has eigenvalues?

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