HYBRID AND NONADDITIVE QUANTUM CODES

A Dissertation

by

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ABSTRACT

Quantum error-correcting codes play a vital role in protecting quantum information from external noise and internal decoherence, and will be a critical tool in the construction of a universal fault-tolerant quantum computer. Most known quantum error-correcting codes belong to the well-studied class of quantum stabilizer codes, which have a well-behaved group structure that makes them easy to construct and practical to use. In this dissertation, we expand the theory of two generalizations of stabilizer codes: quantum-classical hybrid codes and nonadditive quantum codes.

Hybrid codes simultaneously encode both quantum and classical information together, which allows for some nontrivial advantage over coding schemes with separate transmission. As many quantum communications protocols involve both quantum and classical information, hybrid codes may be useful in designing more efficient schemes. We construct the first known families of genuine hybrid codes that are guaranteed to provide an advantage over quantum stabilizer codes, giving infinite families of both single-error detecting and correcting hybrid codes. We also generalize hybrid codes to allow for differing levels of protection of errors, and give a general construction of hybrid codes of this type from quantum subsystem codes. When used in conjunction with the class of Bacon-Casaccino subsystem codes, this provides for a method of constructing hybrid codes from pairs of classical linear codes. As an application of hybrid codes, we show how they can be used to protect against faulty syndrome measurement errors and inspire the construction of new quantum data-syndrome codes.

Finally, we investigate the Shor-Laflamme weight enumerators for both hybrid and nonadditive quantum codes. Weight enumerators are powerful tools that allow for the construction of linear-programming bounds on the parameters of quantum codes and let us rule out the existence of certain codes. In particular, we show that the weight enumerators of the nonadditive codeword stabilized quantum codes have a combinatorial interpretation analogous to that of quantum stabilizer codes, showing that they may be viewed as the distance enumerators of associated classical codes.

DEDICATION

To my family, who have always supported me.

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1. INTRODUCTION AND FUNDAMENTAL CONCEPTS

1.1 Motivation

The rapid advances made in the field of quantum computation in the past two and a half decades have rocketed us to the threshold of the Noisy Intermediate-Scale Quantum (NISQ) era of quantum computation [83], a period of time where quantum supremacy has been achieved (on approximately 50 to 100 noisy qubits) but large-scale fully fault-tolerant quantum computers have not been realized. In order to obtain the quadratic speedup for unstructured database search or the super-polynomial speedup for integer factorization promised by Grover [46] and Shor [95] respectively, both quantum error correction and fault-tolerance are necessary.

Both classical and quantum information are impacted by noise, although quantum information tends to be much less robust to noise, which may arise from the environment, nearby qubits, or even from within the qubit itself. In the case of classical information, the answer to the question of noise is to add redundancy to the system, and this is also the solution to protecting quantum information. However, the properties of quantum mechanics present several hurdles that need to be cleared first, especially the no-cloning theorem, which prevents the copying of quantum systems, and the collapse of a superposition of quantum states upon observation. In fact, until the construction of the first quantum error-correcting code by Shor [93], many skeptics pointed to these issues as proof of the infeasibility of quantum computation.

Since the introduction of this original 9-qubit Shor code, the field of quantum error correction has rapidly grown with the introduction of CSS codes [23, 98], which allow for the construction of quantum codes from classical linear codes, as well as their generalization the stabilizer codes [22, 39], which are so important as to be nearly synonymous with quantum error correction. Other more specialized constructions include subsystem codes [62], codeword stabilizd codes [29], entanglement-assisted codes [21], and topological codes [54].

Even though they operate primarily on quantum information, quantum computers still rely on classical information for their outputs and in some cases the controls for quantum gates, which may be affected by noise (e.g., the problem of faulty syndrome extraction [10, 36]). Similarly, some quantum communications protocols require the transmission of quantum and classical information across a quantum channel. To address these problems, the work in this dissertation primarily investigates hybrid quantum-classical codes that can protect both quantum and classical information together against the effects of noise. Additionally, the other focus in this work is investigating the weight enumerators of both hybrid quantum-classical codes and the nonadditive codeword stabilized quantum codes, which allow us to derive upper bounds on the existence of codes.

1.2 Quantum Information

The basic unit of quantum information is the qubit, which is a unit (column) vector in the Hilbert space $\mathbb{C}^2$. Using bra-ket notation from quantum mechanics, we define the computational basis of the Hilbert space to be

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

read as "ket 0" and "ket 1", which is the standard basis of the vector space. A qubit $|\psi\rangle$ can therefore be written as a linear combination of $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \text{ where } \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1.$$

A qubit written in this way is said to be a superposition of the basis states. This can be visualized as the qubit living on the surface of the Bloch sphere.

Multiple qubits can be combined together into a composite quantum system by taking a tensor product of the corresponding Hilbert spaces. A two qubit system is a unit vector in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$, which has the vectors $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$ as an orthonormal basis. For simplicity, and since this Hilbert space is isomorphic to $\mathbb{C}^4$, this basis is typically written as $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In general, an n-qubit system is the Hilbert space $\mathbb{C}^{2^n}$, with the computational basis indexed by the 2^n bit strings of length n.

An operator that maps a quantum state to a quantum state is modeled by a unitary matrix U, which means U satisfies $U^\dagger U = UU^\dagger = I$, where $U^\dagger$ is the conjugate transpose of U. Unitary operators on separate qubits can be combined by using the tensor product: for a three qubit system, the matrix $U \otimes I \otimes V$ applies the operator U to the first qubit, applies no operator (or the identity operator) to the second qubit, and the operator V to the third qubit. Three single-qubit operators that are especially important to quantum error correction are the Pauli matrices X, Y, and Z, where

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } Y = iXZ = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

The matrix X is known as the "bit-flip" operator since it maps $|0\rangle \mapsto |1\rangle$ and $|1\rangle \mapsto |0\rangle$, analogous to a classical bit-flip, while the matrix Z is known as the "phase-flip" operator, as it maps $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto -|1\rangle$. Another way of viewing it is the X, Y, and Z operators represent a rotation of π around their respective axes in the Bloch sphere. One fact about the Pauli matrices that will become useful later on when looking at quantum stabilizer codes in Section 1.3.2 is that they all anticommute with each other. Together with I, we denote the set of Pauli operators by $\mathcal{E} = \{I, X, Y, Z\}$.

Two other important unitary operators are the Hadamard and controlled-NOT (CNOT) gates, defined by

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Conjugating by Hadamard operators allow us to switch between X and Z operators, as

$$X = HZH \text{ and } Z = HXH.$$

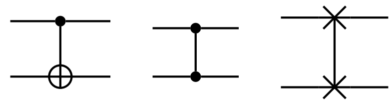

Figure 1.1: Quantum circuit diagrams for the CNOT, controlled-Z, and SWAP gates.

The CNOT gate is a two qubit operator that when applied to the computation basis states will perform the following mapping:

$$\begin{aligned}
|00\rangle &\mapsto |00\rangle\,, \\
|01\rangle &\mapsto |01\rangle\,, \\
|10\rangle &\mapsto |11\rangle\,, \\
|11\rangle &\mapsto |10\rangle\,.
\end{aligned}$$

The effect of this on the basis states is to apply an identity operator to the second qubit if the first qubit is "not set" (i.e., is the state $|0\rangle$), and to apply an X operator to the second qubit if the first qubit is "set" (i.e., is the state $|1\rangle$). The first and second qubits are referred to as the control and target respectively. Two qubit gates allow us to construct states that exhibit the uniquely quantum phenomenon of entanglement. Consider the state $|\Phi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ (obtained by applying H and CNOT in succession on the state $|00\rangle$), which we call entangled since its two qubits cannot be written as the tensor product of two single qubit states. One interesting and useful property of this state is that if one of the two qubits is measured, the superposition is immediately collapsed and the other quantum state is reduced to the identical state with probability 1. This makes entanglement useful for several quantum communication protocols including superdense coding [17] and quantum teleportation [16], as well as a resource to be used in entanglement-assisted quantum error correction [21].

The last two gates necessary for this dissertation are the controlled-Z gate and the SWAP gate. The SWAP gate simply swaps two qubits, while the controlled-Z gate applies a Z-gate on the target qubit if the control qubit is set, although there is no difference which qubit is the target and which is the control.

Another important type of operations are von Neumann measurements, which are represented by orthogonal projection matrices, that is a matrix P such that $P^2 = P = P^\dagger$. It is a basic rule that matrices of this form can be diagonalized so that $P = \sum_i |\phi_i\rangle \langle\phi_i|$, where $\{|\phi_i\rangle\}$ forms an orthonormal basis for the Hilbert space. Here $\langle\phi_i|$ (read as "bra ϕ_i") is the conjugate transpose of the state $|\phi_i\rangle$. When multiplied together, bra then ket, $\langle\phi|\psi\rangle$ is the standard complex inner product on the Hilbert space. Applying the operator P to a state $|\psi\rangle$ collapses the superposition and reduces it to the state $\frac{\langle\phi_i|\psi\rangle}{|\langle\phi_i|\psi\rangle|} |\phi_i\rangle$ with probability $|\langle\phi_i|\psi\rangle|^2$. As an illustrative example, if we take a measurement of the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ in the computational basis so that $P = |0\rangle \langle 0| + |1\rangle \langle 1|$, we will end up with the state $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$.

More information on quantum information may be found in the standard quantum computing textbook of Nielsen and Chuang [78], or the quantum information theory textbook of Wilde [99].

1.3 Quantum Error Correction

The noise that affects a quantum system can be modeled in several different ways. Noise can come internally through the process of amplitude damping, where the superposition of states tends to devolve away from the excited state $|1\rangle$ towards the lower-energy state $|0\rangle$. Noise can also come externally, through the interaction of the system with nearby systems or the environment. These new types of noise, combined with the rules of quantum mechanics, give us three main obstacles not present in classical error correction that must be overcome:

(1) the no-cloning theorem prevents the duplication of arbitrary quantum states,

(2) the errors that affect the system are continuous, and

(3) measuring quantum information destroys the system it is observing.

Thankfully, these obstacles can be overcome.

Firstly, the no-cloning theorem [32, 100] states there is no unitary operator that can take a general quantum state $|\psi\rangle$ tensored with an ancilla state $|0\rangle$ and "clone" the general quantum state onto the ancilla, that is there is no unitary U such that

$$|\psi\rangle \otimes |0\rangle \xrightarrow{U} |\psi\rangle \otimes |\psi\rangle$$

for an arbitrary $|\psi\rangle$. This prevents the use of basic repetition codes from classical coding theory. However, we can design unitaries that encode the basis states into encoded states on more qubits. For example, if $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we can use two two CNOT gates controlled by the first qubit and targeted on the second and third qubits respectively to perform the following mapping:

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle \to \alpha|000\rangle + \beta|111\rangle,$$

getting around obstacle (1).

The second obstacle is that as the errors that affect the system are continuous, there are uncountably infinite number of possible errors that must be corrected. We can avoid this situation by discretizing the errors and then correcting this discrete set of errors. The noise may be modeled in operator-sum representation as $\sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger$, and if we look at each operator E_i individually as it affects only the j-th physical qubit, we can write this as a linear combination of the Pauli operators I, X_j, Y_j, and Z_j, as they span the space of 2×2 complex matrices. This means that if we perform a measurement to extract the syndrome associated with E_i, we end up in one of the states $|\psi\rangle$, $X_j|\psi\rangle$, $Y_j|\psi\rangle$, or $Z_j|\psi\rangle$. This is true for each physical qubit, so we can construct an inverse operator, and since this is true for all E_j, we can recover the original state for any linear combination of the E_j.

Finally, the third obstacle can be overcome by only performing syndrome measurements that collapse the the superposition and entanglement between the ancilla qubits, which leaves the state in the form $|\psi\rangle \otimes |x\rangle$, where x is the bitstring associated with the measured syndromes. This limits the number of measurements we can perform to the number of ancilla we add to the system.

1.3.1 Quantum Error-Correcting Codes

An $(\!(n, K, d)\!)_2$ quantum code $\mathcal{C}$ is a K-dimensional subspace of a quantum system on n qubits $\mathcal{H} = \mathbb{C}^{2^n}$, with an orthonormal basis comprised of the quantum states $\{|c_1\rangle, \ldots, |c_k\rangle\}$, such that any error affecting less than $d - 1$ physical qubits can be detected (or equivalently, any error affecting less than $\lfloor d-1 \rfloor /2$ physical qubits can be corrected). Associated with the code is the orthogonal projector $P = \sum_{i=1}^{K} |c_i\rangle \langle c_i|$ that projects onto the subspace.

We define our error basis $\mathcal{E}_n$ to be the n-fold tensor product of elements from the Pauli group:

$$\mathcal{E}_n = \{\omega \cdot E_1 \otimes E_2 \otimes \cdots \otimes E_n \mid E_i \in \mathcal{E}, \omega \in \{\pm 1, \pm i\}\},$$

and we define the Hamming weight $\mathrm{wt}(E)$ of an element of $E \in \mathcal{E}_n$ to be the number of non-identity tensor components it contains. The Knill-Laflamme conditions give necessary and sufficient conditions for a set of errors $\mathcal{D}$ to be detectable by a quantum code $\mathcal{C}$:

Theorem 1 ([60]). *An error set $\mathcal{D}$ is detectable by a quantum code $\mathcal{C}$ with projector P if and only if*

$$PEP = \lambda_E P$$

holds for all $E \in \mathcal{D}$, where $\lambda_E \in \mathbb{C}$.

Roughly, this means that after the application of a recovery operator, a detectable error affects the code by no more than global multiplication by a scalar depending only on E (which is harmless as global scalars do not affect quantum information). We say the code has minimum distance d if and only if every $E \in \mathcal{E}_n$ of weight less than d is detectable.

Example 2. As an example, we present the original quantum code, Shor's $(\!(9, 2, 3)\!)_2$ code [93]. This code can be thought of as a 3-qubit bit-flip repetition code concatenated (in a coding theory sense) with a 3-qubit phase-flip repetition code. We denote the two encoded basis states by $|\overline{0}\rangle$ and

$|\overline{1}\rangle$, and define them as

$$\begin{aligned}|\overline{0}\rangle &= \frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)\otimes\left(|000\rangle + |111\rangle\right)\otimes\left(|000\rangle + |111\rangle\right),\\ |\overline{1}\rangle &= \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)\otimes\left(|000\rangle - |111\rangle\right)\otimes\left(|000\rangle - |111\rangle\right).\end{aligned}$$

We can correct any single bit-flip error (i.e., a single qubit Pauli-X error) on the first triplet of qubits by determining whether the first and second qubits are the same and then by determining whether the second and third qubits are the same, which can be done by measuring the operators Z_1Z_2 (i.e., Z-operators on the first and second qubits and identities everywhere else) and Z_2Z_3 respectively, revealing the erroneous qubit. Bit-flip errors on the second and third triplets may be corrected in the same way. Similarly, we can correct any single phase-flip error (Pauli Z-error) by noticing that a Z-error on any of the qubits of the first triplet will result in the same phase-flip, and similarly for the other two triplets of qubits. Therefore, it is only necessary to identify which triplet is affected and not the individual qubit. Therefore, it is sufficient to measure the operators $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$, in order to identify the block and correct the phase-error. Since Paul-Y errors are a combination of both X- and Z-type errors, they are also identified in this manner.

As we are able to correct any 1-qubit Pauli error, this minimum distance of Shor's 9-qubit code is at least 3.

1.3.2 Quantum Stabilizer Codes

Here we will focus on a particular class of quantum codes called stabilizer (or additive) codes, which are the quantum analog of linear and additive classical codes. Select $n-k$ independent commuting elements of $\mathcal{E}_n$ that are not scalar multiples of the identity and denote the group of order 2^{n-k} generated by them as $\mathcal{S}$, which we call the stabilizer group. The elements of $\mathcal{S}$ share 2^k eigenstates $|c_i\rangle$ with eigenvalue 1, so that for all $S \in \mathcal{S}$ we have $S\,|c_i\rangle = |c_i\rangle$. We define this set of eigenstates to be our stabilizer code $\mathcal{C}$, and since $K = 2^k$ we call it an $[\![n,k,d]\!]_2$ code.

In particular, the errors in the stabilizer group do not have any effect on the codewords, so it is unnecessary to detect them. Any error that anticommutes with at least one element of the stabilizer can be detected by making a measurement based off of the stabilizer generators. The only errors that cannot be detected are those that commute with every element of the stabilizer group, which are given by the centralizer $N(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{E}_n$:

$$N(\mathcal{S}) = \{F \in \mathcal{E}_n \mid FS = SF, \forall S \in \mathcal{S}\}\,.$$

The question now becomes how to identify elements of $N(\mathcal{S})$. It turns out that every stabilizer group is defined by a classical additive code C over $\mathbb{F}_4$, and that the normalizer is given by the dual code $C^{\perp}$. Since $\mathcal{S}$ is an abelian group, it follows that $\mathcal{S} \subseteq N(\mathcal{S})$, so $C \leq C^{\perp}$, that is C must be a self-orthogonal additive code. From this we can see that the minimum distance d of our quantum stabilizer code $\mathcal{C}$ can be described using purely classical codes:

$$d = \min\big\{\mathrm{wt}\big(C^{\perp} \setminus C\big)\big\}\,,$$

that is, it is the minimum weight of codewords in $C^{\perp} \setminus C$. The elements of $N(\mathcal{S})$ form the logical operators on the encoded qubits.

One purely quantum feature of quantum codes are impure errors: those undetectable errors in $\mathcal{S}$ that have weight less than d. This means that the minimum weight of the classical codes C and $C^{\perp}$ might both be larger than the minimum distance of the quantum code $\mathcal{C}$. In general, impure codes are much harder to both construct and analyze than pure codes, but as we show later they will prove to be necessary in the construction of hybrid codes.

Example 3. We now reinterpret Shor's 9-qubit code from Example 2 as a $[\![9,1,3]\!]_2$ quantum stabilizer code. To do this we need to identify the Pauli operators that stabilize the encoded basis states $|\overline{0}\rangle$ and $|\overline{1}\rangle$ (i.e., we wish to identify those Paul operators that have both $|\overline{0}\rangle$ and $|\overline{1}\rangle$ as eigenvectors with eigenvalue 1). Note that the eight operators we measured in Example 2 satisfy this requirement. We show this here for the operator Z_1Z_2:

$$\begin{aligned}
Z_1Z_2\left|\bar{0}\right\rangle &= \frac{1}{2\sqrt{2}}Z_1Z_2\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)\\
&= \frac{1}{2\sqrt{2}}Z_1\left(|000\rangle-|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)\\
&= \frac{1}{2\sqrt{2}}\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle+|111\rangle\right)\\
&= \left|\bar{0}\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
Z_1Z_2\left|\bar{1}\right\rangle &= \frac{1}{2\sqrt{2}}Z_1Z_2\left(|000\rangle-|111\rangle\right)\otimes\left(|000\rangle-|111\rangle\right)\otimes\left(|000\rangle-|111\rangle\right)\\
&= \frac{1}{2\sqrt{2}}Z_1\left(|000\rangle+|111\rangle\right)\otimes\left(|000\rangle-|111\rangle\right)\otimes\left(|000\rangle-|111\rangle\right)\\
&= \frac{1}{2\sqrt{2}}\left(|000\rangle-|111\rangle\right)\otimes\left(|000\rangle-|111\rangle\right)\otimes\left(|000\rangle-|111\rangle\right)\\
&= \left|\bar{1}\right\rangle.
\end{aligned}$$

Since these operators are all independent, they generate the entire stabilizer group. The logical X- and Z-opertors on the encoded logical qubit are given by $Z^{\otimes 9}$ and $X^{\otimes 9}$ respectively, and we will arrange the generators in a form similar to the generator matrices of classical linear codes:

$$\left(\begin{array}{ccccccccc}
X & X & X & X & X & X & I & I & I\\
I & I & I & X & X & X & X & X & X\\
Z & Z & I & I & I & I & I & I & I\\
I & Z & Z & I & I & I & I & I & I\\
I & I & I & Z & Z & I & I & I & I\\
I & I & I & I & Z & Z & I & I & I\\
I & I & I & I & I & I & Z & Z & I\\
I & I & I & I & I & I & I & Z & Z\\
\hline
X & X & X & X & X & X & X & X & X\\
Z & Z & Z & Z & Z & Z & Z & Z & Z
\end{array}\right).$$

Here, the eight generators of the stabilizers of the stabilizer group $\mathcal{S}$ are above the solid line, and all ten of the operators generate the centralizer $N(\mathcal{S})$. The minimum weight of this code is the minimum weight of the set $N(\mathcal{S})\setminus\mathcal{S}$, which is 3. Since there are multiple elements in the stabilizer with weight less than 3, this code is additionally an example of an impure quantum code.

Another class of quantum codes that will be important later on are a generalization of stabilizer codes known as sybsystem codes [62]. Briefly, subsystem codes can be thought of as stabilizer codes that have low weight elements in $N(\mathcal{S}) \setminus \mathcal{S}$. By moving an element from this set into $\mathcal{S}$ we (hopefully) can increase the minimum distance of the code, at the expense of giving up one of the logical qubits. Viewed in this way, we are increasing the minimum distance of the code by making the code “more impure".

1.3.3 Nonbinary Quantum Codes

We now turn to from binary quantum codes to the non-binary case. Here, the fundamental unit of quantum information is the q-level qudit, a unit vector in the Hilbert space $\mathbb{C}^q$, where q is a prime power. A nonbinary quantum code with parameters $((n, K, d))_q$ is a K-dimensional subspace $\mathcal{C}$ of a Hilbert space $\mathcal{H} = \mathbb{C}^{q^n}$ that can detect any errors on up to $d-1$ physical qudits. As in the binary case, we define the class of nonbinary stabilizer codes [7, 52] as the quantum analogues of classical additive codes, and we write their parameters as $[\![n, k, d]\!]_q$, where $k = \log_q(K)$. While in general K does not need to be an integral power of q, it will always be an integral power of p, the characteristic of the finite field $\mathbb{F}_q$.

Just as binary stabilizer codes are defined as the joint eigenspace of a subgroup of the n qudit error group generated by tensor products of the Pauli matrices, nonbinary stabilizer codes are definied in a similar way using nice error bases [56, 57, 59]. Let $\mathbb{F}_q$ be a finite field of characteristic p, where $q = p^\ell$. We define the trace function $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ by

$$\mathrm{tr}(x) = \sum_{i=0}^{\ell-1} x^{p^i}.$$

Let $a, b \in \mathbb{F}_q$ and denote by $|x\rangle$ the computational basis of $\mathbb{C}^q$ labeled by the elements $x \in \mathbb{F}_q$. The unitary operators $X(a)$ and $Z(b)$ are defined by

$$X(a)\,|x\rangle = |x+a\rangle \text{ and } Z(b)\,|x\rangle = \omega^{\mathrm{tr}(bx)}\,|x\rangle\,,$$

where ω is the primitive p-th root of unity $e^{2\pi i/p}$. The set

$$\mathcal{E} = \{X(a)\,Z(b) \mid a, b \in \mathbb{F}_q\}$$

forms a nice error basis for $\mathbb{C}^q$, modeling errors on a single qudit.

This can be extended to a system of n qudits by taking tensor products of the elements of $\mathcal{E}$: let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_q^n$ and define the unitary operators $X(\mathbf{a}) = X(a_1) \otimes \cdots \otimes X(a_n)$ and $Z(\mathbf{a}) = Z(a_1) \otimes \cdots \otimes Z(a_n)$. Then

$$\mathcal{E}_n = \left\{X(\mathbf{a})\,Z(\mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n\right\}$$

is a nice error basis for $\mathbb{C}^{q^n}$ and

$$G_n = \left\{\omega^c X(\mathbf{a})\,Z(\mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n, c \in \mathbb{F}_p\right\}$$

is the error group generated by the elements of $\mathcal{E}_n$ (if $\mathbb{F}_q$ has characteristic 2, replacing ω with i and letting $c \in \mathbb{Z}_4$ produces the standard version of complex Pauli matrices). When q is prime, the finite field $\mathbb{F}_q$ can be generated as an additive group by a single element, so $G_n = \langle \omega I, X_j(1)\,, Z_j(1) \mid j \in [n]\rangle$, where $X_j(1)$ operates only on the j-th qubit. When q is not a prime, but a prime power, any element in the field may be written as $a_0 + a_1\alpha + \cdots + a_{\ell-1}\alpha^{\ell-1}$ where $a_i \in \mathbb{F}_p$ and α is a root of an irreducible polynomial in $\mathbb{F}_p$ of degree ℓ. Using this we have $G_n = \langle \omega I, X_j(\alpha^i)\,, Z_j(\alpha^i) \mid 0 \leq i < \ell, j \in [n]\rangle$.

The weight $\mathrm{wt}(\cdot)$ of an element $E \in G_n$ is the number of tensor components of E that are not scalar multiples of the identity matrix. Any two elements E and E' of G_n, where $E =$

$\omega^c X(\mathbf{a})\, Z(\mathbf{b})$ and $E' = \omega^{c'} X(\mathbf{a}')\, Z(\mathbf{b}')$, satisfy the following commutation relation:

$$EE' = \omega^{\mathrm{tr}(\mathbf{b}\cdot\mathbf{a}' - \mathbf{b}'\cdot\mathbf{a})} E'E.$$

Let $\mathcal{S}$ be some abelian subgroup of G_n that does not contain a scalar multiple of the identity matrix. A stabilizer code $\mathcal{C}$ is the joint $+1$-eigenspace of $\mathcal{S}$, that is

$$\mathcal{C} = \bigcap_{E \in \mathcal{S}} \left\{ v \in \mathbb{C}^{q^n} \mid Ev = v \right\}.$$

The group $\mathcal{S}$ is called the stabilizer group of the code and has order q^{n-k}, generated by $\ell\,(n-k)$ elements of G_n.

The centralizer of the stabilizer group are those elements in G_n that commute with every element of $\mathcal{S}$, which is traditionally denoted by $N(\mathcal{S})$. The elements of $N(\mathcal{S})\,/\mathcal{S}Z(G_n)$, where $\mathcal{S}Z(G_n)$ is the group generated by $\mathcal{S}$ and the center $Z(G_n)$ of the group G_n, are cosets whose elements are Pauli operators on the logical qudits. We denote the logical operators on the i-th logical qudit by $\overline{X_i(a)}$ and $\overline{Z_i(b)}$, with $a, b \in \mathbb{F}_q$. These operators are not unique, as any element in the same coset will have the same effect on the quantum code $\mathcal{C}$. The labeling of these operators is somewhat arbitrary, their only requirement being that they satisfy the following commutation and non-commutation relations that generalize the commutation and anticommutation relations of the Pauli matrices: $\left[\overline{X_i(a)}, \overline{X_j(b)}\right] = 0$, $\left[\overline{Z_i(a)}, \overline{Z_j(b)}\right] = 0$, $\left[\overline{X_i(a)}, \overline{Z_j(b)}\right] = 0$ if $i \neq j$, and $\overline{X_i\,(a)}\ \overline{Z_i(b)} = \omega^{\mathrm{tr}(-b\cdot a)}\overline{Z_i(b)}\ \overline{X_i\,(a)}$. For example, the operators can be trivially relabeled by swapping the X_i and Z_i operators.

1.3.4 Weight Enumerators and Linear Programming Bounds

For classical codes, the distance between codewords is given by the Hamming distance:

$$d_H(x, y) = \left|\{i \mid x_i \neq y_i\}\right|.$$

The distance distribution A of an $(n, M, d)_q$ classical code C is a vector of length $(n+1)$, where

$$A_i = \frac{1}{M} \left| \{(x, y) \mid x, y \in C, d_H(x, y) = i\} \right|,$$

meaning that A_i is the number of codewords at distance i from each other, normalized by the size of the code. The polynomial

$$A(z) = \sum_{i=0}^{n} A_i z^i$$

is called the distance enumerator of the code. The minimum distance d of the code is the smallest index $i \neq 0$ such that A_i is non-zero.

The Hamming weight of a codeword is the distance from the all zero codeword, that is $\mathrm{wt}_H(x) = d_H(x, 0^n)$. If C is an additive code, that is a code which is closed under addition of the codewords, then A counts the number of codewords of each weight, so

$$A_i = \left| \{x \mid x \in C, \mathrm{wt}(x) = i\} \right|,$$

and we call A the weight distribution and $A(z)$ the weight enumerator of the code. The weight enumerator of an additive code C is connected to the weight enumerator $B(z)$ of its dual code $C^\perp$ by the MacWilliams identity [30, 69]:

$$B(z) = \frac{(1 + (q-1)\,z)^n}{M} A\left(\frac{1-z}{1+(q-1)\,z}\right).$$

For a nonadditive code, the MacWilliams identity may still be formally defined in the same way, although the resultant polynomial in general does not correspond to the distance enumerator of any code [68, 28].

Moving to quantum codes, we can define the weight of elements of nice error bases in two equivalent ways. Each element $E \in \mathcal{E}_n$ can be associated with a unique codeword $(a \mid b) = (a_1, \ldots, a_n \mid b_1, \ldots, b_n)$ of length $2n$. The distance between two codewords of this type is given by the symplectic distance:

$$d_s((a \mid b), (a' \mid b')) = |\{k \mid (a_k, b_k) \neq (a'_k, b'_k)\}| \,.$$

The symplectic weight $\mathrm{wt}_s(E)$ is then the distance of E from the operator $I^{\otimes n}$. We use this notation in Chapter 4. Alternatively, we can define the Hamming weight $\mathrm{wt}_H(E)$ as we did in Sections 1.3.1 and 1.3.3, as the number of non-Identity tensor compenents in an operator from a nice error basis. We use this notation in Chapter 2.

The most well studied class of quantum codes are the stabilizer codes [39]. Recall that the stabilizer code is the q^k-dimensional joint $+1$-eigenspace of $\mathcal{S}$ its stabilizer group. Associated with the stabilizer group is its centralizer $N(\mathcal{S})$, the group of all elements in the error group G_n that commute with every element in $\mathcal{S}$. These are the operators that act as the logical operators on the encoded states of the code.

Shor and Laflamme [94] defined a pair of weight enumerators $A(z)$ and $B(x)$ for quantum codes in the following fashion:

$$A_i = \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=i}} \mathrm{tr}(EP)\,\mathrm{tr}(E^*P)$$

and

$$B_i = \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=i}} \mathrm{tr}(EPE^*P)\,,$$

where P is the orthogonal projector onto the code $\mathcal{C}$. In general, the weight enumerators of quantum codes do not seem to admit as nice a combinatorial interpretation as they do for classical codes. However, for stabilizer codes there is such an interpretation, as $A(z)$ counts the number of elements of each weight in the stabilizer group $\mathcal{S}$ and $B(z)$ counts the number of elements in the centralizer $N(\mathcal{S})$ (both modulo the phases on the Pauli elements). Additionally, each element of the stabilizer and centralizer can be associated with a unique (up to phase) codeword of length $2n$.

Let C be the code containing the set of codewords associated with $\mathcal{S}$. Then its symplectic dual $C^{\perp}$ is the code associated with $N(\mathcal{S})$. Additionally, since $\mathcal{S} \leq N(\mathcal{S})$ (as $\mathcal{S}$ is Abelian), we have that $C \subseteq C^{\perp}$, that is C is self-orthogonal.

The two Shor-Laflamme weight enumerators are connected via the quantum MacWilliams identity:

$$B(z) = \frac{\left(1 + \left(q^2 - 1\right) z\right)^n}{M} A\left(\frac{1 - z}{1 + \left(q^2 - 1\right) z}\right),$$

which is the MacWilliams identity on an alphabet of size q^2. There are several other equalities and inequalities that are useful: $0 \leq A_i \leq B_i$ for all i follows from the Cauchy-Schwarz inequality, $A_0 = 1$, $\sum_{i=0}^{n} A_j = \frac{q^n}{K}$. Importantly, we use weight enumerators to completely determine the minimum distance of a quantum code, using the equalities $A_i = B_i$ for all $i < d$.

Since the parameters of any quantum code must satisfy each of these equalities and inequalities, we can set up a system of linear inequalities to get an upper bound on the parameters of quantum codes. Using linear programming, we can determine which sets of parameters do not satisfy these inequalities and therefore are not allowable parameters for quantum codes [11, 52]. These linear programming bounds often produce reasonably sharp bounds, and can often be improved with the addition of the quantum shadow inequalities of Rains [87].

1.4 Organization of Dissertation

The organization of the remaining of this dissertation is as follows: Chapter 2 covers the basics of quantum-classical hybrid codes and investigates their weight enumerators, before giving multiple families of genuine hybrid codes, the first known in the literature. Chapter 3.3 investigates hybrid codes where the quantum and classical information is protected to different degrees with two separate minimum distances for each, before giving a general construction for hybrid codes from the well-studied subsystem codes, particularly the Bacon-Shor subsystem codes, and then an application of protecting against faulty syndrome measurement errors with hybrid codes. Chapter 4 gives an interpretation for the Shor-Laflamme weight enumerators of the nonadditive codeword stabilized codes. Finally, in Chapter 5 we provided some concluding remarks and some future research directions.

2. CONSTRUCTIONS OF HYBRID CODES

The majority of this chapter came from the conference paper [73][1] and its expanded journal paper [75][2]. Section 2.3.2 is from [74][3] and Example 19 is adapted from [77].

2.1 Background

Hybrid codes simultaneously encode classical and quantum information into quantum digits such that the information is protected against errors when transmitted through a quantum channel. The simultaneous transmission of classical and quantum information was first investigated by Devetak and Shor [31], who characterized the set of admissible rate pairs. Notably, they showed that, at least for certain small error rates, time-sharing a quantum channel is inferior to simultaneous transmission. Constructions of hybrid codes were first studied by Kremsky, Hsieh, and Brun [61] in the context of entanglement-assisted stabilizer codes and by Bény, Kempf, and Kribs [18, 19] who outlined an operator-theoretic construction.

More recently, Grassl, Lu, and Zeng [41] gave linear programming bounds for a class of hybrid codes and constructed a number of hybrid stabilizer codes with parameters better than those of hybrid codes constructed from quantum stabilizer codes. In particular, these genuine hybrid codes outperform "trivial" hybrid codes regardless of the error rate of the channel. Additional work on hybrid codes has been done from both a coding theory approach [73] and from an operator-theoretic approach [70], as well as over a fully correlated quantum channel where the space of errors is spanned by $I^{\otimes n}$, $X^{\otimes n}$, $Y^{\otimes n}$, and $Z^{\otimes n}$ [66]. While they are still relatively unstudied, multiple uses for hybrid codes have already become apparent, including protecting hybrid quantum

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[3]Results reproduced with permission from "Nonbinary Error-Detecting Hybrid Codes" by Andrew Nemec and Andreas Klappenecker, 2020. *American Journal of Science and Engineering*, vol. 1(2) pp. 1-4. Copyright 2020 by AJSE.

memory [64] and constructing hybrid secret sharing schemes [106].

In this chapter we give some general results regarding hybrid codes, most notably that at least one of the quantum codes comprising a genuine hybrid code must be impure, as well as show that a hybrid code can always detect more errors than a comparable quantum code. We also generalize the weight enumerators given by Grassl et al. [41] for hybrid stabilizer codes to more general nonadditive hybrid codes and use them to derive linear programming bounds. Finally, we give multiple constructions for infinite families of hybrid codes with good parameters. The first of these families are single error-detecting hybrid stabilizer codes with parameters $[\![n, n-3\!:\!1, 2]\!]_2$ where the length n is odd, where an $[\![n, k\!:\!m, d]\!]_2$ hybrid code encodes k logical qubits and m logical bits into n physical qubits with minimum distance d, and derive a more general construction from this family that produces several new small-parametered hybrid codes. The second is a collection of families of single error-correcting hybrid codes constructed using stabilizer pasting, where we paste together stabilizers from Gottesman's $[\![2^j, 2^j - j - 2, 3]\!]_2$ stabilizers codes [37] and the small distance 3 hybrid codes from [41] and Examples 19, 20, and 21 in this chapter. Each of these families of hybrid codes were inspired by families of nonadditive quantum codes, especially those constructed by Rains [86] and Yu, Chen, and Oh [105].

2.1.1 Hybrid Codes

A quantum code is a subspace of a Hilbert space that allows for encoded quantum information to be recovered in the presence of errors on the physical qudits. Here our encoded message is a unit vector in the Hilbert space

$$H = \bigotimes_{\ell=1}^{n} \mathbb{C}^q \cong \mathbb{C}^{q^n}.$$

We say a quantum code has parameters $(\!(n, K)\!)_q$ if and only if it can encode a superposition of K orthogonal quantum states into n quantum digits with q levels.

Now suppose that we want to simultaneously transmit classical and quantum messages. Our goal will be to encode them into the state of n quantum digits that have q-levels each, so that the encoded message can be transmitted over a quantum channel. A hybrid code has the parameters

$(\!(n, K\!:\!M)\!)_q$ if and only if it can simultaneously encode one of M different classical messages and a superposition of K orthogonal quantum states into n quantum digits with q levels.

We can understand the hybrid code as a collection of M orthogonal K-dimensional quantum codes $\mathcal{C}_m$ that are indexed by the classical messages $m \in [M] := \{1, 2, \ldots, M\}$. If we want to transmit a classical message $m \in [M]$ and a quantum state $|\varphi\rangle$, then we need to encode $|\varphi\rangle$ into the quantum code $\mathcal{C}_m$. We will refer to the each of the quantum codes $\mathcal{C}_m$ as *inner codes* and the collection $\mathcal{C} = \{\mathcal{C}_m \mid m \in [M]\}$ as the *outer code*.

2.1.2 Error Detection

The encoded states will be subject to errors when transmitted through a quantum channel. Our first task will be to characterize the errors that can be detected by the hybrid code. We will set up a projective measurement that either upon receipt of a state $|\psi\rangle$ in H either (a) returns ϵ to indicate that an error happened or (b) claims that there is no error and returns a classical message m and a projection of $|\psi\rangle$ onto $\mathcal{C}_m$.

Let P_m denote the orthogonal projector onto the quantum code $\mathcal{C}_m$ for all integers m in the range $1 \leq m \leq M$. For distinct integers a and b in the range $1 \leq a, b \leq M$, the quantum codes $\mathcal{C}_a$ and $\mathcal{C}_b$ are orthogonal, so $P_b P_a = 0$. It follows that the orthogonal projector onto $\mathcal{C} = \bigoplus_{m=1}^{M} \mathcal{C}_m$ is given by

$$P = P_1 + P_2 + \cdots + P_M.$$

We define the orthogonal projection onto $\mathcal{C}^\perp$ by $P_\epsilon = 1 - P$. For the hybrid code $\{\mathcal{C}_m \mid m \in [M]\}$, we can define a projective measurement $\mathcal{P}$ that corresponds to the set

$$\{P_1, P_2, \ldots, P_M, P_\epsilon\}$$

of projection operators that partition unity.

We can now define the concept of a detectable error. An error E is called *detectable* by the hybrid code $\{\mathcal{C}_m \mid m \in [M]\}$ if and only if for each index a, b in the range $1 \leq a, b \leq M$, we have

$$P_b E P_a = \begin{cases} \lambda_{E,a} P_a & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases} \tag{2.1}$$

for some scalar $\lambda_{E,a}$.

The motivation for calling an error E detectable is the following simple protocol. Suppose that we encode a classical message m and a quantum state into a state $|v_m\rangle$ of $\mathcal{C}_m$, and transmit it through a quantum channel that imparts the error E. If the error is detectable, then measurement of the state $E|v_m\rangle = EP_m|v_m\rangle$ with the projective measurement $\mathcal{P}$ either

(E1) returns ϵ, which signals that an error happened, or

(E2) returns m and corrects the error by projecting the state back onto a scalar multiple $\lambda_{E,m}|v_m\rangle = P_m E P_m |v_m\rangle$ of the state $|v_m\rangle$.

The definition of a detectable error ensures that the measurement $\mathcal{P}$ will never return an incorrect classical message d, since $P_d E P_m |v_m\rangle = 0$ for all $d \neq m$, so the probability of detecting an incorrect message is zero. An error that is not detectable by the hybrid code can change the encoded classical, the encoded quantum information, or both.

The condition in Equation (2.1) is equivalent to the hybrid Knill-Laflamme condition [41, Theorem 4] for detectable errors: an error E is detectable by a hybrid code $\mathcal{C}$ with orthonormal basis states $\left\{ \left| c_i^{(a)} \right\rangle \mid i \in [K], a \in [M] \right\}$ if and only if

$$\left\langle c_i^{(b)} \middle| E \middle| c_j^{(a)} \right\rangle = \lambda_{E,a} \delta_{ij} \delta_{ab}. \tag{2.2}$$

Compared to the original Knill-Laflamme conditions for fully quantum codes [60] where the scalar only depended on the detectable error, these hybrid conditions allow for scalars $\lambda_{E,a}$ that may depend on both the detectable error E and the classical message a, allowing more flexibility in the design of codes. However, this flexibility comes at the price of no longer being able to send a superposition of all of the basis states.

The next proposition shows that hybrid codes can always detect more errors than a comparable quantum code that encodes both classical and quantum information. This is remarkable given that the advantages are much less apparent when one considers minimum distance, see [41].

Proposition 4. *The subset $\mathcal{D}$ of detectable errors of an $(\!(n, K\!:\!M)\!)_q$ hybrid code form a vector space of dimension*

$$\dim \mathcal{D} = q^{2n} - (MK)^2 + M.$$

In particular, an $(\!(n, K\!:\!M)\!)_q$ hybrid code with $M > 1$ can detect more errors than an $(\!(n, KM)\!)_q$ quantum code.

Proof. It is clear that any linear combination of detectable errors is detectable. If we choose a basis adapted to the orthogonal decomposition $H = \mathcal{C} \oplus \mathcal{C}^\perp$ with

$$\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots \oplus \mathcal{C}_M,$$

then an error E is represented by a matrix of the form

$$\begin{pmatrix} A & R \\ S & T \end{pmatrix},$$

where the blocks A and T correspond to the subspaces $\mathcal{C}$ and $\mathcal{C}^\perp$ respectively. Since E is detectable, the $MK \times MK$ matrix A must satisfy

$$A = \lambda_{E,1} 1_K \oplus \lambda_{E,2} 1_K \oplus \cdots \oplus \lambda_{E,M} 1_K,$$

where 1_K denote a $K \times K$ identity matrix, but R, S, and T can be arbitrary. Therefore, the dimension of the vector space of detectable errors is given by $q^{2n} - (MK)^2 + M$.

In the case of an $(\!(n, KM)\!)_q$ quantum code, A must satisfy $A = \lambda_E 1_{KM}$, so the vector space of detectable errors has dimension $q^{2n} - (KM)^2 + 1$, which is strictly less than $q^{2n} - (MK)^2 + M$ when $M > 1$. □

We briefly recall the concept of a nice error basis (see [56, 57, 59] for further details), so that we can define a suitable notion of weight for the errors. Let G be a group of order q^2 with identity element 1 and $\mathcal{U}(q)$ be the group of $q \times q$ unitary matrices. A *nice error basis* on $\mathbb{C}^q$ is a set $\mathcal{E} = \{\rho(g) \in \mathcal{U}(q) \mid g \in G\}$ of unitary matrices such that

(i) $\rho(1)$ is the identity matrix,

(ii) $\operatorname{Tr}\rho(g) = 0$ for all $g \in G \setminus \{1\}$,

(iii) $\rho(g)\rho(h) = \omega(g,h)\,\rho(gh)$ for all $g, h \in G$,

where $\omega(g,h)$ is a nonzero complex number depending on $(g,h) \in G \times G$; the function $\omega\colon G \times G \to \mathbb{C}^\times$ is called the factor system of ρ. We call G the *index group* of the error basis $\mathcal{E}$. The nice error basis that we have introduced so far generalizes the Pauli basis to systems with $q \geq 2$ levels.

We can obtain a nice error basis $\mathcal{E}_n$ on $H \cong \mathbb{C}^{q^n}$ by tensoring n elements of $\mathcal{E}$, so

$$\mathcal{E}_n = \mathcal{E}^{\otimes n} = \{E_1 \otimes E_2 \otimes \cdots \otimes E_n \mid E_k \in \mathcal{E}, 1 \leq k \leq n\}.$$

The weight of an element in $\mathcal{E}_n$ are the number of non-identity tensor components. We write $\operatorname{wt}(E) = d$ to denote that the element E in $\mathcal{E}_n$ has weight d. A hybrid code with parameters $(\!(n, K\!:\!M, d)\!)_q$ has *minimum distance* d if it can detect all errors of weight less than d.

Example 5. To construct our nonadditive hybrid code $\mathcal{C}$ we will combine two known degenerate stabilizer codes. The first code $\mathcal{C}_a$ is the $[\![6,1,3]\!]_2$ code constructed by extending the $[\![5,1,3]\!]_2$ Hamming code, see [22], where the stabilizer is given by

$$\langle XXZIZI, ZXXZII, IZXXZI, ZIZXXI, IIIIIX \rangle\,.$$

The second code $\mathcal{C}_b$ is a $[\![6,1,3]\!]_2$ code not equivalent to $\mathcal{C}_a$, see [91]. Its stabilizer is given by

$$\langle YIZXXY, ZXIIXZ, IZXXXX, IIIZIZ, ZZZIZI \rangle\,.$$

We can check that the resulting two codes are indeed orthogonal to each other. The resulting code $\mathcal{C}$ is a $(\!(6, 2\!:\!2, 1)\!)_2$ nonadditive hybrid code, since there are several errors of weight one such that $P_b E P_a \neq 0$, for example $E = IIIIXI$. This shows that even though $\mathcal{C}_a$ and $\mathcal{C}_b$ are optimal quantum codes on their own, together they make a hybrid code with an extremely poor minimum distance. Later we will see how to construct hybrid codes with better minimum distances.

2.1.3 Genuine Hybrid Codes

In general, it is not difficult to construct hybrid codes using quantum stabilizer codes. As Grassl et al. [41] pointed out, there are three simple constructions of hybrid codes that do not offer any real advantage over quantum error-correcting codes:

Proposition 6 ([41]). *Hybrid codes can be constructed using the following "trivial" constructions:*

1. *Given an $(\!(n, KM, d)\!)_q$ quantum code of composite dimension KM, there exisits a hybrid code with parameters $(\!(n, K\!:\!M, d)\!)_q$.*

2. *Given an $[\![n, k\!:\!m, d]\!]_q$ hybrid code with $k > 0$, there exists a hybrid code with parameters $[\![n, k-1\!:\!m+1, d]\!]_q$.*

3. *Given an $[\![n_1, k_1, d]\!]_q$ quantum code and an $[n_2, m_2, d]_q$ classical code, there exists a hybrid code with parameters $[\![n_1+n_2, k_1\!:\!m_2, d]\!]_q$.*

We say that a hybrid code is *genuine* if it cannot be constructed using one of the above constructions, following the work of Yu et al. on genuine nonadditive codes [105]. We also refer to a hybrid stabilizer code that provides an advantage over quantum stabilizer codes as a genuine hybrid stabilizer code. While all known genuine hybrid codes are in fact hybrid stabilizer codes, the linear programming bounds in Section 2.2.2 do not prohibit genuine nonadditive hybrid codes, and may give us some hints as to their parameters.

Multiple genuine hybrid stabilizer codes with small parameters were constructed by Grassl et al. in [41], all of which have degenerate inner codes. Having degenerate inner codes can allow for a more efficient packing of the inner codes inside the outer code than is possible when using nondegenerate codes, giving a hybrid code with parameters superior to those using the first construction of Proposition 6. However, they do not exclude the possibility that there is a genuine hybrid code where all of the inner codes are nondegenerate. Here, we show that for a genuine hybrid code, at least one of its inner codes must be impure. Recall that a quantum code is *pure* if trace-orthogonal errors map the code to orthogonal subspaces. A code that is not pure is called *impure*.

Proposition 7. *Suppose $\mathcal{C}$ is a genuine $(\!(n, K\!:\!M, d)\!)_q$ hybrid code. Then at least one inner code $\mathcal{C}_m$ of the hybrid code $\mathcal{C}$ is impure.*

Proof. Seeking a contradiction, suppose that every inner code of the hybrid code $\mathcal{C}$ is pure. For $m \in [M]$, let P_m denote the orthogonal projector onto the m-th inner code of the hybrid code $\mathcal{C}$. For every nonscalar error operator E of weight less than d, we have

$$P_a E P_b = 0,$$

where $a, b \in [M]$. Let $P = P_1 + P_2 + \cdots + P_M$ denote the projector onto the KM-dimensional vector space spanned by the inner codes. Then

$$PEP = 0,$$

so the image of P is an $(\!(n, KM, d)\!)_q$ quantum code, contradicting that the hybrid code $\mathcal{C}$ is genuine. □

Since for stabilizer codes the definitions of impure and degenerate codes coincide, genuine hybrid stabilizer codes necessarily require that one of the inner codes is degenerate. Therefore, one of the difficulties in constructing families of genuine codes is finding nontrivial degenerate codes. Unfortunately, there are few known families of impure or degenerate codes, see for example [4, 5], and they typically have minimum distances much lower than optimal quantum codes, suggesting they are not particularly suitable to use in constructing genuine hybrid codes.

2.1.4 Hybrid Stabilizer Codes

All of the hybrid codes constructed by Grassl et al. [41] were given using the codeword stabilizer (CWS)/union stabilizer framework, see [29, 42], which we will briefly describe here. Starting with a quantum code $\mathcal{C}_0$, we choose a set of M coset representatives t_i from the normalizer of $\mathcal{C}_0$ (we will always take t_1 to be I), and then construct the code

$$\mathcal{C} = \bigcup_{i \in [M]} t_i \mathcal{C}_0.$$

In the case of hybrid codes, $t_i\mathcal{C}_0$ are our inner codes and $\mathcal{C}$ is our outer code. If both $\mathcal{C}_0$ and $\mathcal{C}$ are stabilizer codes, we say that $\mathcal{C}$ is a hybrid stabilizer code.

The generators that define a hybrid code can be divided into those that generate the quantum stabilizer $\mathcal{S}_{\mathcal{Q}}$ which stabilizes the outer code $\mathcal{C}$ and those that generate the classical stabilizer $\mathcal{S}_{\mathcal{C}}$ which together with $\mathcal{S}_{\mathcal{Q}}$ stabilizes the inner code $\mathcal{C}_0$ [61]. The generators that define the $[\![7, 1\!:\!1, 3]\!]_2$ hybrid stabilizer code given in [41] are given in (2.3), where the generators of $\mathcal{S}_{\mathcal{Q}}$ are given above the dotted line, the generators of $\mathcal{S}_{\mathcal{C}}$ are between the dotted and solid line, the normalizer of the inner code $\mathcal{C}_0$ is generated by all elements above the double line, and the normalizer of the outer code is generated by all of the elements.

$$\left(\begin{array}{ccccccc}
X & I & I & Z & Y & Y & Z \\
Z & X & I & X & Z & I & X \\
Z & I & X & X & I & Z & X \\
Z & I & Z & Z & X & I & I \\
I & Z & I & Z & I & X & X \\
\hdashline
Z & I & I & I & I & I & X \\
\hline
I & I & I & X & Z & Z & X \\
I & I & I & Z & X & X & I \\
\hline\hline
I & I & I & I & X & Y & Y
\end{array}\right) \tag{2.3}$$

Following Kremsky et al. [61], we will often only include the stabilizer generators, as they are sufficient to fully define the hybrid code, as shown in the following proposition:

Proposition 8. *Let $\mathcal{C}$ be an $[\![n, k\!:\!m, d]\!]_p$ hybrid stabilizer code over a finite field of prime order p*

with quantum stabilizer $\mathcal{S}_\mathcal{Q}$ and classical stabilizer $\mathcal{S}_\mathcal{C} = \left\langle g_1^\mathcal{C}, \ldots, g_m^\mathcal{C} \right\rangle$. Then the stabilizer code $\mathcal{C}_c$ associated with classical message $c \in \mathbb{F}_p^m$ is given by the stabilizer

$$\left\langle \mathcal{S}_\mathcal{Q}, \omega^{c_1} g_1^\mathcal{C}, \ldots, \omega^{c_m} g_m^\mathcal{C} \right\rangle,$$

where c_i is the i-th entry of c and ω is a primitive complex p-th root of unity.

Proof. There are p^{k+m} basis states stabilized by $\mathcal{S}_\mathcal{Q}$. Each of these basis states is an eigenvector of $g_i^\mathcal{C}$, which naturally partitions the code into p cosets based on eigenvalues. Repeating this with all of the classical generators, we get p^m cosets of basis states, each of size p^k. Since v being an eigenvector of $g_i^\mathcal{C}$ with eigenvalue ω^{-1} means that it is a $+1$ eigenvector of $\omega g_i^\mathcal{C}$, therefore each coset is the $+1$ eigenspace of a stabilizer of the form $\left\langle \mathcal{S}_\mathcal{Q}, \omega^{c_1} g_1^\mathcal{C}, \ldots, \omega^{c_m} g_m^\mathcal{C} \right\rangle$, where the string $c \in \mathbb{F}_p^m$ can be used to index the stabilizer codes. □

2.2 Bounds for General Hybrid Codes

Weight enumerators for quantum codes were introduced by Shor and Laflamme [94], and as with their classical counterparts they can be used to give good bounds on code parameters using linear programming, see [11, 52]. Grassl et al. [41] gave weight enumerators and linear programming bounds for hybrid stabilizer codes, but these weight enumerators will not work for nonadditive hybrid codes such as the one given in Example 5. In this section, we define weight enumerators for general hybrid codes following the approach of Shor and Laflamme [94] and Rains [84] and use them to derive linear programming bounds for general hybrid codes.

2.2.1 Hybrid Weight Enumerators

For an $(\!(n, K{:}M, d)\!)_q$ hybrid code $\mathcal{C}$ defined by the projector $P = P_1 + \cdots + P_M$ and a nice error base $\mathcal{E}_n$ as defined in Section 1.3.3, we define the two weight enumerators of the code following Shor and Laflamme [94]:

$$A(z) = \sum_{d=0}^{n} A_d z^d \text{ and } B(z) = \sum_{d=0}^{n} B_d z^d,$$

where the coefficients are given by

$$A_d = \frac{1}{K^2M^2} \sum_{\substack{E\in\mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(EP)\,\mathrm{tr}(E^*P)$$

and

$$B_d = \frac{1}{KM} \sum_{\substack{E\in\mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(EPE^*P)\,.$$

We can also define weight enumerators using the inner code projectors P_a. Let

$$A^{(a,b)}(z) = \sum_{d=0}^{n} A_d^{(a,b)} z^d \text{ and } B^{(a,b)}(z) = \sum_{d=0}^{n} B_d^{(a,b)} z^d,$$

where

$$A_d^{(a,b)} = \frac{1}{K^2} \sum_{\substack{E\in\mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(EP_a)\,\mathrm{tr}(E^*P_b)$$

and

$$B_d^{(a,b)} = \frac{1}{K} \sum_{\substack{E\in\mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(EP_aE^*P_b)\,.$$

Note that $A^{(a,a)}(z)$ and $B^{(a,a)}(z)$ are the weight enumerators of the quantum code associated with projector P_a. We can then write the weight enumerators for the outer code in terms of the weight enumerators for the inner codes:

Lemma 9. *The weight enumerators of $\mathcal{C}$ can be written as*

$$A(z) = \frac{1}{M^2} \sum_{a,b=1}^{M} A^{(a,b)}(z) \text{ and } B(z) = \frac{1}{M} \sum_{a,b=1}^{M} B^{(a,b)}(z)\,.$$

Proof. By linearity of the projector P we have

$$\begin{aligned} A_d &= \frac{1}{K^2M^2} \sum_{\substack{E\in\mathcal{E}_n \\ \mathrm{wt}(E)=d}} \operatorname{tr}(EP)\operatorname{tr}(E^*P) \\ &= \frac{1}{K^2M^2} \sum_{\substack{E\in\mathcal{E}_n \\ \mathrm{wt}(E)=d}} \sum_{a,b=1}^{M} \operatorname{tr}(EP_a)\operatorname{tr}(E^*P_b) \\ &= \frac{1}{M^2} \sum_{a,b=1}^{M} A_d^{(a,b)}. \end{aligned}$$

We can then rewrite the weight enumerator as

$$\begin{aligned} A(z) &= \sum_{d=0}^{n} A_d z^d \\ &= \frac{1}{M^2} \sum_{d=0}^{n} \sum_{a,b=1}^{M} A_d^{(a,b)} z^d \\ &= \frac{1}{M^2} \sum_{a,b=1}^{M} A^{(a,b)}(z)\,. \end{aligned}$$

The result for $B(z)$ follows from the same argument. □

While the weight enumerator $B(z)$ is the same as the one introduced by the authors in [73], the weight enumerator $A(z)$ is different. There the $A^{(a,b)}(z)$ weight enumerators with $a \neq b$ were ignored, causing $A(z)$ and $B(z)$ to not satisfy the MacWilliams identity. The approach presented in this paper is more natural, as it treats both the inner and outer codes as quantum codes. The following result may be found in [84, 94], which we include for completeness:

Lemma 10 ([84, 94])**.** *Let $\mathcal{C}$ be a $(\!(n, K\!:\!M)\!)_q$ hybrid code with weight distributions A_d and B_d. Then for all integers d in the range $0 \leq d \leq n$ and all $a \in [M]$ we have*

1. $0 \leq A_d \leq B_d$
2. $0 \leq A_d^{(a,a)} \leq B_d^{(a,a)}$.

Proof. For every orthogonal projector $\Pi : \mathbb{C}^{q^n} \to \mathbb{C}^{q^n}$ of rank K, we have

$$0 \leq \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(E\Pi)\,\mathrm{tr}(E^*\Pi)$$

by the non-negativity of the trace inner product. Furthermore, we can write this inequality in the form

$$\begin{aligned} 0 &\leq \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(E\Pi)\,\mathrm{tr}(E^*\Pi) \\ &= \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \left|\mathrm{tr}(E\Pi)\right|^2 \\ &= \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \left|\mathrm{tr}((\Pi E \Pi)\,\Pi)\right|^2 . \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 0 &\leq \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}((\Pi E \Pi)\,(\Pi E \Pi)^*)\,\mathrm{tr}(\Pi^*\Pi) \\ &= \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(E\Pi E^*\Pi) . \end{aligned}$$

Substituting $\Pi = P$ implies (1) and substituting $\Pi = P_a$ implies (2). □

The main utility of weight enumerators for quantum codes is that they allow for a complete characterization of the error-correction capability of the code in terms of the minimum distance of the code. In the following proposition, we prove a similar result for the weight enumerators of hybrid codes.

Proposition 11. *Let $\mathcal{C}$ be a $((n, K\!:\!M))_q$ hybrid code with weight distributions A_d and B_d. Then $\mathcal{C}$ can detect all errors in $\mathcal{E}_n$ of weight d if and only if $A_d^{(a,a)} = B_d^{(a,a)}$ for all $a \in [M]$ and $B_d^{(a,b)} = 0$ for all $a, b \in [M]$, $a \neq b$.*

Proof. Recall that an error is detectable by a code if and only if it satisfies the hybrid Knill-Laflamme conditions in Equation (2.2), and that a projector onto one of the inner codes $\mathcal{C}_a$ may be written as $P_a = \sum_{i=1}^{K} \left|c_i^{(a)}\right\rangle \left\langle c_i^{(a)}\right|$, where $\left\{\left|c_i^{(a)}\right\rangle \mid i \in [K]\right\}$ is an orthonormal basis for $\mathcal{C}_a$. Suppose that all errors of weight d are detectable by $\mathcal{C}$. Then

$$\begin{aligned} A_d^{(a,a)} &= \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(EP_a)\,\mathrm{tr}(E^* P_a) \\ &= \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \left| \sum_{i=1}^{K} \left\langle c_i^{(a)} \right| E \left| c_i^{(a)} \right\rangle \right|^2 \\ &= \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \left| \alpha_E^{(a)} \right|^2 . \end{aligned}$$

Similarly, we have

$$\begin{aligned} B_d^{(a,a)} &= \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(EP_a E^* P_a) \\ &= \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \sum_{i,j=1}^{K} \left| \left\langle c_i^{(a)} \right| E \left| c_j^{(a)} \right\rangle \right|^2 \\ &= \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \sum_{i=1}^{K} \left| \left\langle c_i^{(a)} \right| E \left| c_i^{(a)} \right\rangle \right|^2 \\ &= \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \left| \alpha_E^{(a)} \right|^2 . \end{aligned}$$

Therefore, we have that $A_d^{(a,a)} = B_d^{(a,a)}$. Additionally, if $a \neq b$, then by Equation (2.2) we have

$\left\langle c_i^{(a)} \middle| E \middle| c_j^{(b)} \right\rangle = 0$. Therefore,

$$B_d^{(a,b)} = \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \sum_{i,j=1}^{K} \left| \left\langle c_i^{(a)} \middle| E \middle| c_j^{(b)} \right\rangle \right|^2$$
$$= 0.$$

Conversely, suppose that (a) $A_d^{(a,a)} = B_d^{(a,a)}$ for all $a \in [M]$ and (b) $B_d^{(a,b)} = 0$ for all $a, b \in [M], a \neq b$. Condition (a) implies that equality holds for each E in the Cauchy-Schwarz inequality. Therefore, we have that $P_a E P_a$ and P_a must be linearly dependent, so there must be a constant $\alpha_E^{(a)} \in \mathbb{C}$ such that $P_a E P_a = \alpha_E^{(a)}$, or equivalently, $\left\langle c_i^{(a)} \middle| E \middle| c_j^{(a)} \right\rangle = \alpha_E^{(a)} \delta_{i,j}$, for all errors of weight d. Condition (b) implies that $\left\langle c_i^{(a)} \middle| E \middle| c_j^{(b)} \right\rangle = 0$ if $a \neq b$, for all errors of weight d. Putting these together, we get the hybrid Knill-Laflamme conditions, so all errors of weight d are detectable. $\square$

2.2.2 Linear Programming Bounds

One of the more useful properties of weight enumerators is that they satisfy the Macwilliams identity [94]:

$$B^{(a,b)}(z) = \frac{K}{q^n} \left(1 + \left(q^2 - 1\right) z\right)^n A^{(a,b)} \left(\frac{1 - z}{1 + \left(q^2 - 1\right) z} \right). \tag{2.4}$$

The MacWilliams identities, along with the results from Lemma 10 and Proposition 11 and the shadow inequalities for qubit codes [87] allow us to define linear programming bounds on the parameters of general hybrid codes (see [11, 22, 84] for linear programming bounds on quantum codes). Let

$$K_j(r) = \sum_{k=0}^{j} (-1)^k \left(q^2 - 1\right)^{j-k} \binom{r}{k} \binom{n-r}{j-k} \tag{2.5}$$

denote the q^2-ary Krawtchouk polynomials.

Proposition 12. *The parameters of an* $((n, K : M, d))_q$ *hybrid code must satisfy the following conditions:*

1. $A_j = \frac{1}{M^2} \sum_{a,b=1}^{M} A_j^{(a,b)}$

2. $B_j = \frac{1}{M} \sum_{a,b=1}^{M} B_j^{(a,b)}$

3. $A_0^{(a,b)} = 1$

4. $B_0^{(a,b)} = \begin{cases} 1 & \textit{if } a = b \\ 0 & \textit{if } a \neq b \end{cases}$

5. $A_j^{(a,a)} = B_j^{(a,a)}$, *for all* $0 \leq j < d$

6. $B_j^{(a,b)} = 0$, *for all* $0 \leq j < d$, $a \neq b$

7. $0 \leq A_j^{(a,a)} \leq B_j^{(a,a)}$, *for all* $0 \leq j \leq n$

8. $0 \leq A_j \leq B_j$, *for all* $0 \leq j \leq n$

9. $0 \leq B_j^{(a,b)}$, *for all* $0 \leq j \leq n$

10. $B_j^{(a,b)} = \frac{K}{q^n} \sum_{r=0}^{n} K_j(r)\, A_r^{(a,b)}$, *for all* $0 \leq j \leq n$ *(MacWilliams Identity)*

11. $0 \leq \sum_{r=0}^{n} (-1)^r\, K_j(r)\, A_r^{(a,b)}$, *for all* $0 \leq j \leq n$, *for qubit codes (Shadow Inequalities)*

Proof. Conditions 1) and 2) follow from the definition of A_j and B_j. The constraints 3) and 4) respectively result from substituting $E = I$ into the definition of $A_0^{(a,b)}$ and $B_0^{(a,b)}$.

The Knill-Laflamme error-detecting conditions of the hybrid codes shown in Proposition 11 imply the constraints 5) and 6).

The claims 7) and 8) are a consequence of Lemma 10. Essentially, these two conditions follow from the Cauchy-Schwarz inequalities when applied to the quantum and hybrid projectors, respectively.

The statement 9) is simply a consequence of the non-negativity of all $B_j^{(a,b)}$. Conditions 10) and 11) follow from the MacWilliams identities [94] and shadow inequalities [85] respectively. $\square$

Note that conditions 10) and 11) imply the MacWilliams identity and shadow inequality respectively for the outer code.

If we consider only hybrid stabilizer codes, we have that all of the weight distributions for the inner quantum codes are identical. This along with our error-detecting condition from Proposition 11 give us that a stabilizer hybrid stabilizer code can detect all errors of weight d if and only if $A_d^{(a,a)} = B_d^{(a,a)} = B_d$. Additionally, straightforward calculations give us the missing piece of the nested code condition, $A_d \leq A_d^{(a,a)}$ for all d. Thus we recover the linear programming bounds of Grassl et al. when we restrict our bounds to hybrid stabilizer codes, with the exception that we have the additional constraint of the shadow inequality for the outer code. This constraint strengthens the bounds found in Table I of [41] and rules out the possibility, for example, of $[\![10,4{:}1,3]\!]_2$, $[\![12,5{:}1,3]\!]_2$, and $[\![10,2{:}1,4]\!]_2$ hybrid stabilizer codes.

Notably missing in our linear programming bounds is part of the nested code condition found in the linear programming bounds for hybrid stabilizer codes, namely that $A_d \leq A_d^{(a,a)}$ for all d. In fact we can construct a nonadditive hybrid code that violates this condition, as shown in the example below.

Example 13. We return to our $(\!(6,2{:}2,1)\!)_2$ nonadditive hybrid code from Example 5. The weight distributions for $\mathcal{C}_a$, $\mathcal{C}_b$, and $\mathcal{C}$ are

$$\begin{aligned} A^{(a,a)} &= [1,1,0,0,15,15,0] \\ A^{(b,b)} &= [1,0,1,0,11,16,3] \\ A &= \left[1,\frac{1}{4},\frac{1}{4},0,6,\frac{31}{4},\frac{3}{4}\right], \end{aligned}$$

where the weight distributions are the coefficients of the weight enumerators. These weight distributions clearly violate the inequality $A_d \leq A_d^{(a,a)}$.

Interestingly, we were unable to find any separation between our bounds with and without the nested code condition, suggesting the possibility that any hybrid code that meets these bounds must also satisfy this additional constraint. Since this condition is satisfied by any hybrid code constructed using the CWS framework, it seems that this comparable to the situation with quantum codes, where all known nonadditive codes meeting the linear programming bounds are CWS codes. Our bounds suggest the possibility of several nonadditive hybrid codes, such as a $(\!(10,8\!:\!6,3)\!)_2$ code.

2.3 Family of Single Error Detecting Hybrid Codes

2.3.1 Binary Error Detecting Hybrid Codes

In [88], Rains et al. constructed a $(\!(5,6,2)\!)_2$ nonadditive quantum code which was later extended to several families of $(\!(n,q^{n-3}<K<q^{n-2},2)\!)_q$ nonadditive codes with n odd, see [1, 6, 35, 86, 89, 97]. Rains [86, Theorem 2] also showed that for any $(\!(n,K,2)\!)_2$ quantum code with odd n,

$$K \leq 2^{n-2}\left(1-\frac{1}{n-1}\right). \tag{2.6}$$

In particular, this disallows the existence of odd-lengthed $(\!(n,2^{n-2},2)\!)_2$ quantum codes.

Here we give a construction for a family of single error-detecting hybrid stabilizer codes such that n is odd and $KM=2^{n-2}$, so these codes have the remarkable feature in that they allow one to squeeze in an additional classical bit. The generators of these codes are similar to the generators of the family of even-length stabilizer codes with parameters $[\![n,n-2,2]\!]_q$, see [39, 86].

Theorem 14. *For n odd, there exists an $[\![n,n-3\!:\!1,2]\!]_2$ genuine hybrid code with generators*

$$\left(\begin{array}{cc} X^{\otimes n-1} & X \\ Z^{\otimes n-1} & I \\ \hdashline I^{\otimes n-1} & X \end{array}\right)$$

Proof. Recall that a number is said to have *even parity* if it has an even number of 1's in its binary expansion. Let $J\subseteq \mathbb{F}_2^{n-1}$ be the set of even integers with even parity. We define two codes $\mathcal{C}_0$ and $\mathcal{C}_1$ as follows:

$$\mathcal{C}_0=\left\{\frac{1}{2}\left(|x\rangle+|\overline{x}\rangle\right)\left(|0\rangle+|1\rangle\right)\middle| x\in J\right\},$$

$$\mathcal{C}_1 = \left\{ \frac{1}{2} \left(|x\rangle - |\overline{x}\rangle \right) \left(|0\rangle - |1\rangle \right) \middle| x \in J \right\}.$$

It is clear that the stabilizer of $\mathcal{C}_0$ is $\langle X^{\otimes n}, Z^{\otimes n-1}I, I^{\otimes n-1}X \rangle$ and that the stabilizer of $\mathcal{C}_1$ is $\langle X^{\otimes n}, Z^{\otimes n-1}I, -I^{\otimes n-1}X \rangle$. To show that our hybrid code has minimum distance 2, we note first that both $\mathcal{C}_0$ and $\mathcal{C}_1$ have minimum distance 2 when viewed as separate quantum codes. Thus we only need to look at how single-qubit Pauli errors affect the classical information. Consider two basis states $\left| c_i^{(0)} \right\rangle$ and $\left| c_j^{(1)} \right\rangle$, one from each quantum code. If $i \neq j$, it is clear that $\left\langle c_i^{(0)} \middle| E \middle| c_j^{(1)} \right\rangle = 0$ for any single-qubit Pauli error, since they will be linear combinations of disjoint sets of orthonormal basis vectors. Therefore, we can consider only the case when $i = j$.

Suppose that a single-qubit error occurs on the first $n-1$ qubits, that is $E = I_2^{\otimes \ell} \otimes E' \otimes I_2^{\otimes n-\ell-2} \otimes I_2$, for $\ell \in [n-1]$. Then since each of our basis states is separable between the first $n-1$ qubits and the last qubit, we can write

$$\begin{aligned}
\left\langle c_i^{(0)} \middle| E \middle| c_i^{(1)} \right\rangle &= \frac{1}{4} \left(\left(\langle x| + \langle \overline{x}| \right) \left(\langle 0| + \langle 1| \right) E \left(\left(|x\rangle - |\overline{x}\rangle \right) \left(|0\rangle - |1\rangle \right) \right) \right) \\
&= \frac{1}{4} \left(\left(\langle x| + \langle \overline{x}| \right) E' \left(|x\rangle - |\overline{x}\rangle \right) \right) \cdot \left(\left(\langle 0| + \langle 1| \right) \left(|0\rangle - |1\rangle \right) \right) \\
&= \frac{1}{4} \left(\left(\langle x| + \langle \overline{x}| \right) E' \left(|x\rangle - |\overline{x}\rangle \right) \right) \cdot 0 \\
&= 0.
\end{aligned}$$

Similarly, if a single-qubit error occurs on the last qubit, that is $E = I_2^{\otimes n-1} \otimes E'$, we have

$$\begin{aligned}
\left\langle c_i^{(0)} \middle| E \middle| c_i^{(1)} \right\rangle &= \frac{1}{4} \left(\left(\langle x| + \langle \overline{x}| \right) \left(\langle 0| + \langle 1| \right) E \left(\left(|x\rangle - |\overline{x}\rangle \right) \left(|0\rangle - |1\rangle \right) \right) \right) \\
&= \frac{1}{4} \left(\left(\langle x| + \langle \overline{x}| \right) \left(|x\rangle - |\overline{x}\rangle \right) \right) \cdot \left(\left(\langle 0| + \langle 1| \right) E' \left(|0\rangle - |1\rangle \right) \right) \\
&= 0 \cdot \left(\left(\langle 0| + \langle 1| \right) E' \left(|0\rangle - |1\rangle \right) \right) \\
&= 0.
\end{aligned}$$

Thus the hybrid code given by $\mathcal{C}_0 \oplus \mathcal{C}_1$ has minimum distance 2.

By the bound in Equation 2.7, for a general $(\!(n, K, 2)\!)_2$ quantum code with n odd, we have

$$K \leq 2^{n-2}\left(1 - \frac{1}{n-1}\right).$$

In particular, this precludes the possibility of an $(\!(n, 2^{n-2}, 2)\!)_2$ code for n odd. Similarly, suppose that we could construct a code in our family using an $[\![n_q, k, d]\!]_2$ quantum code and an $[n_c, m, d]_q$ classical code. Then we would have $n_q + n_c = n$, $k = n-3$, and $m = 1$, and in particular, we have an $[\![n_q, n_q + n_c - 3, 2]\!]_2$ quantum code. By the quantum Singleton bound, we must have $n_c \leq 1$, forcing us to have a $[1, 1, 2]_2$ classical code, which of course does not exist. It follows that all of the codes in our family must be genuine hybrid codes. □

An interesting question is whether or not this family of hybrid codes are optimal, by which we mean do there exist odd-length $(\!(n, 2^{n-3}\!:\!M, 2)\!)_2$ codes with $M > 2$, or $(\!(n, 2^{n-3}+1\!:\!2, 2)\!)_2$ codes? For small lengths ($n \leq 19$) this family achieves the linear programming bounds for general hybrid codes given in Section 2.2.2, and we suspect that the family is optimal for all odd n.

2.3.2 Nonbinary Error-Detecting Hybird Codes Over $\mathbb{Z}_q$

In the previous section we constructed a family of binary $[\![n, n-3\!:\!1, 2]\!]_2$ error-detecting hybrid stabilizer codes where n is odd. In this section we provide a generalization of this family to hybrid stabilizer codes over $\mathbb{Z}_q$, inspired by the non-additive nonbinary quantum codes constructed from qudit graph states by Hu et al. [50] and Looi et al. [67], as well as the family of single error-detecting codes given by Smolin, Smith, and Wehner [97].

A quantum code is a subspace of a Hilbert space that allows for the recovery of encoded quantum information even in the presence of arbitrary errors on a certain number of physical qudits. A quantum code has parameters $(\!(n, K, d)\!)_q$ if and only if it can encode a superposition of K orthogonal quantum states into the Hilbert space $(\mathbb{C}^q)^{\otimes n} \cong \mathbb{C}^{q^n}$, while protecting the quantum information against all errors ocurring on less than d physical qubits.

Most generalizations of quantum codes from the binary alphabets to the case where $q > 2$ are constructed over the finite fields $\mathbb{F}_q$, where q is a prime power, see [7, 52, 85]. In this paper, we instead follow [50, 67, 97] and construct codes over $\mathbb{Z}_q$ for reasons that will become apparent later with Proposition 16. Let $a, b \in \mathbb{Z}_q$. We define the unitary operators $X(a)$ and $Z(b)$ on $\mathbb{C}^q$ as

$$X(a)\,|x\rangle = |x + a\rangle \text{ and } Z(b)\,|x\rangle = \omega^{bx}\,|x\rangle\,,$$

where $\omega = e^{2\pi i/q}$. The operators $X(a)$ and $Z(b)$may be viewed as a generalization of the Pauli-X bit-flip error and the Pauli-Z phase error respectively. The set $\mathcal{E} = \{X(a)\,Z(b) \mid a, b \in \mathbb{Z}_q\}$ forms a nice error basis on $\mathbb{C}^q$, see [56, 57, 59], meaning any error on a single qudit may be written as a linear combination of elements from $\mathcal{E}$. Additionally, any error on $\mathbb{C}^{q^n}$ may be written as a linear combination of errors from $\mathcal{E}_n = \mathcal{E}^{\otimes n} = \{E_1 \otimes E_2 \otimes \cdots \otimes E_n \mid E_k \in \mathcal{E}, 1 \leq k \leq n\}$. By correcting errors from $\mathcal{E}_n$ we are able to deal with arbitrary errors on the n qudits that are linear combinations of those errors. The weight $\mathrm{wt}(E)$ of an error $E \in \mathcal{E}_n$ is the number of non-identity tensor components it contains.

The first good non-additive quantum code (that is a quantum code that is not a stabilizer code) was the $(\!(5, 6, 2)\!)_2$ code given by Rains et al. [88]. This code outperforms the optimal $[\![5, 2, 2]\!]_2$ stabilizer code, and was further generalized by Rains [86] into a family of odd-length non-additive codes that outperform optimal stabilizer codes. However, for an odd-length $(\!(n, K, 2)\!)$ quantum code we have the following bound:

$$K \leq 2^{n-2}\left(1 - \frac{1}{n-1}\right), \tag{2.7}$$

and many families of codes that approach this bound have been constructed. In [75], the authors gave a construction for a familiy of hybrid stabilizer codes with parameters $[\![n, n-3{:}1, 2]\!]_2$ that beat this bound. Stabilizer codes over rings are defined as defined as the $+1$-eigenspace of an abelian group generated by elements from $\mathcal{E}_n$.

Nonbinary quantum codes with similar parameters were hinted at by Rains in [86], and first given by Smolin et al. [97] as a generalization of their family of non-additive binary codes. Soon after, further families were constructed by Hu et al. [50] and Looi et al. [67] using qudit graph states. All of these families are codes over integer rings rather than finite fields, and our construction of nonbinary hybrid stabilizer codes will follow in their footsteps. The reason we choose to construct codes over $\mathbb{Z}_q$ rather than $\mathbb{F}_q$ is due to the following result of Grassl and Rötteler:

Theorem 15 ([44, Theorem 12])**.** *Let $q > 1$ be an arbitrary integer, not necessarily a prime power. Quantum MDS codes $\mathcal{C} = [\![n, n-2, 2]\!]_q$ exist for all even length n, and for all length $n \geq 2$ when the dimension q of the quantum systems is an odd integer or is divisible by 4.*

While the construction below will certainly produce a hybrid stabilizer code when $q \not\equiv 2 \mod 4$, it will not be a genuine hybrid code, as the previous theorem implies that there will be an $[\![n, n-2, 2]\!]_q$ stabilizer code that can be transformed into a hybrid code using the first construction in Proposition 6. When $q = 2$, Equation 2.7 tells us that there can be no $[\![n, n-2, 2]\!]_2$ quantum code, implying that the family given in Section 2.3.1 is indeed genuine. To the best of our knowledge there are no known $[\![n, n-2, 2]\!]_q$ codes when $q = 4r + 2$, which is why the codes using the construction below may in fact be genuine. However, since $\mathbb{F}_{4r+2}$ does not exist except when $r = 0$, we instead construct our codes over $\mathbb{Z}_q$.

Proposition 16. *Let n be odd. Then there exists an $[\![n, n-3\!:\!1, 2]\!]_{\mathbb{Z}_q}$ hybrid code.*

Proof. Let $a, b \in \mathbb{Z}_q^n$, $m \in \mathbb{Z}_q$, and ω a primitive q-th root of unity. Define the following states:

$$|\phi_{a,b}\rangle = \frac{1}{q^n} \sum_{c \in \mathbb{Z}_q^{2n}} \omega^{\sum_{i=1}^{n} (c_{2i-1} - a_i)(c_{2i} - b_i)} |c\rangle$$

$$|\psi_m\rangle = \frac{1}{\sqrt{q}} \sum_{c \in \mathbb{Z}_q} \omega^{mc} |c\rangle$$

Define the inner code $\mathcal{C}_m$ as follows:

$$\mathcal{C}_m = \left\langle |\phi_{a,b}\rangle \otimes |\psi_m\rangle \middle| a, b \in \mathbb{Z}_q^n, m \in \mathbb{Z}_q \sum_{i=1}^{n} a_i = 0, \sum_{i=1}^{n} b_i = m \right\rangle$$

The state $|\phi_{a,b}\rangle$ is the tensor product of two-qubit states of the form

$$|\phi_{a_i,b_i}\rangle = \frac{1}{q}\sum_{c\in\mathbb{Z}_q^2}\omega^{(c_1-a_i)(c_2-b_i)}|c\rangle .$$

For two of these states $|\phi_{a_i,b_i}\rangle, |\phi_{a'_i,b'_i}\rangle$ we have

$$\begin{aligned}
\langle\phi_{a_i,b_i}|\phi_{a'_i,b'_i}\rangle &= \frac{1}{q^2}\sum_{c\in\mathbb{Z}_q^2}\omega^{\left(c_1-a'_i\right)\left(c_2-b'_i\right)-(c_1-a_i)(c_2-b_i)} \\
&= \frac{\omega^{a'_ib'_i-a_ib_i}}{q^2}\sum_{c\in\mathbb{Z}_q^2}\omega^{c_1\left(b'_i-b_i\right)+c_2\left(a'_i-a_i\right)} \\
&= \frac{\omega^{a'_ib'_i-a_ib_i}}{q^2}\left(\sum_{c_1\in\mathbb{Z}_q}\omega^{c_1\left(b'_i-b_i\right)}\right)\left(\sum_{c_2\in\mathbb{Z}_q}\omega^{c_2\left(a'_i-a_i\right)}\right) \\
&= \begin{cases} 1 & \text{if } a_i = a'_i \text{ and } b_i = b'_i \\ 0 & \text{otherwise} \end{cases} .
\end{aligned}$$

Therefore for the full states $|\phi_{a,b}\rangle, |\phi_{a',b'}\rangle$ we have the same:

$$\langle\phi_{a,b}|\phi_{a',b'}\rangle = \begin{cases} 1 & \text{if } a = a' \text{ and } b = b' \\ 0 & \text{otherwise} \end{cases} .$$

Similarly, for $|\psi_m\rangle, |\psi_{m'}\rangle$ we have

$$\langle\psi_m|\psi_{m'}\rangle = \begin{cases} 1 & \text{if } m = m' \\ 0 & \text{otherwise} \end{cases} .$$

Thus all of the codewords are orthogonal to one another.

Consider two codewords $|\phi_{a,b}\rangle\otimes|\psi_m\rangle, |\phi_{a',b'}\rangle\otimes|\psi_{m'}\rangle$. Suppose that a Pauli-$X(u)$ error occurs on the first $n-1$ qudits. Without loss of generality, we can assume that the error occurred on either the first or second qudit. If $m \neq m'$, $a_i \neq a_i'$, or $b_i \neq b_i'$ for $1 < i \leq n$, then

$$(\langle \phi_{a,b}| \otimes \langle \psi_m|)\, X(u)\, (|\phi_{a',b'}\rangle \otimes |\psi_{m'}\rangle) = 0$$

by the orthogonality relations above. Therefore we can restrict our attention to the case where $m = m'$, $a_i = a_i'$, and $b_i = b_i'$ for $1 < i \leq n$). We note that these restrictions along with the requirement that the a_i and a_i' sum to 0 and b_i and b_i' sum to m and m' respectively completely determine the values of a_1 and b_1 and in particular we must have $a_1 = a_1'$ and $b_1 = b_1'$. If the error occurred on the first qudit, we have

$$\begin{aligned}
\langle \phi_{a_1,b_1}|\, X(u)\, |\phi_{a_1,b_1}\rangle &= \frac{1}{q^2}\left(\sum_{c\in\mathbb{Z}_q^2} \omega^{-(c_1-a_1)(c_2-b_1)} \langle c_1 c_2|\right)\left(\sum_{c\in\mathbb{Z}_q^2} \omega^{(c_1-a_1)(c_2-b_1)} |(c_1+u)\, c_2\rangle\right) \\
&= \frac{1}{q^2}\left(\sum_{c\in\mathbb{Z}_q^2} \omega^{-(c_1-a_1)(c_2-b_1)} \langle c_1 c_2|\right)\left(\sum_{c\in\mathbb{Z}_q^2} \omega^{(c_1-a_1-u)(c_2-b_1)} |c_1 c_2\rangle\right) \\
&= \frac{1}{q^2}\sum_{c_2\in\mathbb{Z}_q^2} \omega^{u(b_1-c_2)} \\
&= \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

A similar argument holds if the error occurs on the second qudit, thus the code can detect any single Pauli-$X(u)$ error that occurs on the first $n-1$ qudits.

Now suppose that a Pauli-$Z(v)$ error occurs on the first $n-1$ qudits. As above, we restrict our attention to the case where $a = a'$, $b = b'$, $m = m'$, and the error occurs on one of the first two qudits. If the error occurs on the first qudit we have

$$\begin{aligned}
\langle\phi_{a_1,b_1}|\, Z(v)\, |\phi_{a_1,b_1}\rangle &= \frac{1}{q^2}\sum_{c\in\mathbb{Z}_q^2}\omega^{(c_1-a_1)(c_2-b_1)-(c_1-a_1)(c_2-b_1)+vc_1} \\
&= \frac{1}{q}\sum_{c_1\in\mathbb{Z}_q}\omega^{vc_1} \\
&= \begin{cases} 1 & \text{if } v=0 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

The same argument holds if the error occurs on the second qudit, thus the code can detect any single Pauli-$Z(v)$ error that occurs on the first $n-1$ qudits.

Now suppose that a Pauli error E occurs on the last qudit. If $a \neq a'$, $b \neq b'$, or $m \neq m'$, then the orthogonality of the first $n-1$ qudits gives us

$$(\langle\phi_{a,b}| \otimes \langle\psi_m|)\, E\, (|\phi_{a',b'}\rangle \otimes |\psi_{m'}\rangle) = 0,$$

so again we only need to examine the case where the two codewords are the same.

If we have a Pauli-$X(u)$ error on the last qudit we have

$$\begin{aligned}
\langle\psi_m|\, X(u)\, |\psi_m\rangle &= \frac{1}{q}\left(\sum_{c\in\mathbb{Z}_q}\omega^{-mc}\langle c|\right)\left(\sum_{c\in\mathbb{Z}_q}\omega^{mc}|c+u\rangle\right) \\
&= \frac{1}{q}\sum_{c\in\mathbb{Z}_q}\omega^{-mu} \\
&= \omega^{-mu},
\end{aligned}$$

meaning that the error is degenerate. Note that since the value depends on the classical information m, each inner code can detect the error but the outer code (as a quantum code) cannot.

If a Pauli-$Z(v)$ error occurs on the last qudit we have

$$\begin{aligned}
\langle\psi_m| Z(v) |\psi_m\rangle &= \frac{1}{q}\left(\sum_{c\in\mathbb{Z}_q}\omega^{-mc}\langle c|\right)\left(\sum_{c\in\mathbb{Z}_q}\omega^{mc+vc}|c\rangle\right) \\
&= \frac{1}{q}\sum_{c\in\mathbb{Z}_q}\omega^{vc} \\
&= \begin{cases} 1 & \text{if } v=0 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

□

We also mention in passing that this construction can be generalized further to codes over Frobenius rings by replacing the primitive root of unity by an irreducible additive character of the additive group of the ring [72, 55].

2.4 A More General Construction

In this section we give a generalization of the construction of distance-2 hybrid stabilizer codes in Theorem 14. The main idea behind this construction is to start with a stabilizer code of length n, then extend the stabilizer code into an impure code of length $n+m$, chosing the stabilizer generators of the outer hybrid code in a nonstandard way.

Theorem 17. *Let $\mathcal{C}$ be an $[\![n,k,d]\!]_2$ stabilizer code with normalizer $N(\mathcal{S})$. Then there is an $[\![n+m,k\!:\!m,d]\!]_2$ hybrid code if there are m elements $\{t_i\}_{i\in[m]}$ from separate, independent cosets of $G_n/N(\mathcal{S})$ such that $\langle t_iN(\mathcal{S})\rangle_{i\in[m]}$ forms a group and for each coset $tN(\mathcal{S})$ we have*

$$\min \operatorname{wt}\{tN(\mathcal{S})-\mathcal{S}\} \geq d-\operatorname{wt}(\mathbf{a}_t),$$

where $t=t_1^{a_1}t_2^{a_2}\cdots t_m^{a_m}$ and $\mathbf{a}_t$ is the binary vector $a_1a_2\ldots a_m$.

Proof. Each coset corresponding to one of the elements t_i has an associated error syndrome based on whether or not its elements commute or anticommute with each stabilizer generator. We encode these commutation/anticommutation relations in an $(n-k) \times m$ binary matrix H such that h_{ji} is 0 if the elements of $t_i N(\mathcal{S})$ commute with the j-th stabilizer generator of $\mathcal{S}$ and 1 if they anticommute.

We now extend the stabilizer generators of S by m qubits by appending to the end of the j-th stabilizer the Pauli operator

$$\bigotimes_{i \in [m]} X^{h_{ji}},$$

as well as adding m new stabilizer generators that are each an X-operator on one of the appended qubits (so there is one weight-1 stabilizer generator for each appended qubit). We call the group generated by the $n-k$ appended stabilizer generators $\mathcal{S}'$ and the group generated by all $n-k+m$ stabilizer generators $\mathcal{S}_0'$, which will stabilize the outer and inner quantum codes $\mathcal{C}'$ and $\mathcal{C}_0'$ respectively. We note that the code $\mathcal{C}_0'$ is the same as the impure extension that results from the application of [22, Theorem 6] to our original code $\mathcal{C}$ (applied m times to obtain a code of length $n+m$), although here we have chosen a nonstandard choice of stabilizer generators.

Using the assumptions in the statement of the theorem, we now prove that $\mathcal{C}'$ is an $[\![n+m, k\!:\!m, d]\!]_2$ hybrid stabilizer code. From the previous paragraph we know that $\mathcal{C}'$ has length $n+m$ and can encode k logical qubits and m classical bits. Therefore, we only must show that the minimum distance of this hybrid code is d. To do this, we look at the set $N(\mathcal{S}') - \mathcal{S}_0'$ and show that it has no elements of weight less than d. Note that $N(\mathcal{S}')$ are all the Pauli operators that commute with every element of $\mathcal{S}'$. Therefore, the elements of $N(\mathcal{S})$ appended with m identity operators on the end are elements of the $N(\mathcal{S}')$. Any X-operators on the last m qubits will have no effect as they are in the inner stabilizer $\mathcal{S}_0'$, so we need only look at operators made from Z-errors and I-operators. Suppose that F is a Z-type error represented by a binary vector $\mathbf{a}_F = a_1 a_2 \dots a_m$ where a_i is either a 0 or a 1 depending on whether the i-th tensor component of the error is I or Z. It is clear that this error will commute and anticommute with the same stabilizer generators as any element from the corresponding coset $tN(\mathcal{S})$ (extended by m identity operators) with $\mathbf{a}_t = \mathbf{a}_E$. Therefore, any element of the form $E \otimes F$ with $E \in tN(\mathcal{S})$ will commute with all the stabilizer generators of $\mathcal{S}'$, and since we know from our assumption that the weight of $E \otimes F$ is at least d, we can see that any element of $N(\mathcal{S}')$ excluding those in $\mathcal{S}'_0$ must have weight at least d, finishing the proof. □

In the case of distance-3 stabilizer codes

Corollary 18. *If a non-perfect* $[\![n,k,3]\!]_2$ *stabilizer code exists, then there is an* $[\![n+1,k\!:\!1,3]\!]_2$ *hybrid stabilizer code.*

Proof. Since our quantum code is non-perfect it does not meet the quantum Hamming bound [34, 60] for distance-3 codes $3n+1 \leq 2^{n-k}$, so there are more possible syndromes than there are single-qubit errors. Therefore, there is at least one coset of $G_n/N(\mathcal{S})$ with a minimum weight of 2. The result then follows directly from the previous theorem. □

Using Theorem 17 and Corollary 18, we construct several new hybrid stabilizer codes:

Example 19. We now construct a $[\![9,3\!:\!1,3]\!]_2$ hybrid stabilizer code. Starting with Gottesman's pure 8-qubit code [37], we extend it [52, Lemma 69] to an impure $[\![9,3,3]\!]_2$ code. This code can alternatively be viewed as a subsystem code with gauge operators $IIIIIIIIX$ and $IIIIIIIIZ$. Due to the structure of the original code, each single-qubit error will have a syndrome that begins with 01, 10, or 11 for an X-, Z-, or Y-error respectively. Therefore we can choose any coset whose syndrome starts with 00 for applying Theorem 17.

$$\begin{pmatrix} X & X & X & X & X & X & X & X & I \\ Z & Z & Z & Z & Z & Z & Z & Z & I \\ X & I & X & I & Z & Y & Z & Y & I \\ X & I & Y & Z & X & I & Y & Z & I \\ X & Z & I & Y & I & Y & X & Z & X \\ \hdashline I & I & I & I & I & I & I & I & X \end{pmatrix}. \tag{2.8}$$

This code has better parameters than both the $[\![9,1\!:\!2,3]\!]_2$ code of Kremsky et al. [61] and the $[\![9,2\!:\!2,3]\!]_2$ code of Grassl et al. [41], and since it meets the linear programming bounds for hybrid stabilizer bounds it is a genuine hybrid stabilizer code.

Similarly, we can extend this code a second time and obtain a new $[\![10,3\!:\!2,3]\!]_2$ hybrid stabilizer code that is not equivalent to the one given in [41]:

$$\begin{pmatrix} X & X & X & X & X & X & X & X & I & I \\ Z & Z & Z & Z & Z & Z & Z & Z & I & I \\ X & I & X & I & Z & Y & Z & Y & I & I \\ X & I & Y & Z & X & I & Y & Z & I & X \\ X & Z & I & Y & I & Y & X & Z & X & I \\ \hdashline I & I & I & I & I & I & I & I & X & I \\ I & I & I & I & I & I & I & I & I & X \end{pmatrix}. \tag{2.9}$$

This code also meets the linear programming bounds for hybrid stabilizer codes, so it is also a genuine hybrid code. On the other hand, extending the code a third time results in a new $[\![11,3\!:\!3,3]\!]_2$ hybrid stabilizer code, but this code transmits less quantum information than the $[\![11,4\!:\!2,3]\!]_2$ code in [41], so in some sense it has worse parameters. However, we can still use Gottesman's 8-qubit code to construct a new hybrid stabilizer code with these better parameters:

$$\begin{pmatrix} X & X & X & X & X & X & X & X & I & I & I \\ Z & Z & Z & Z & Z & Z & Z & Z & I & I & I \\ X & I & X & I & Z & Y & Z & Y & X & I & Z \\ X & I & Y & Z & X & I & Y & Z & I & X & Z \\ X & Z & I & Y & I & Y & X & Z & Z & Z & X \\ \hdashline I & I & I & I & I & I & I & I & X & I & Z \\ I & I & I & I & I & I & I & I & I & X & Z \end{pmatrix}. \tag{2.10}$$

While this code was not constructed using Theorem 17, it shares some similarities to the previous two codes, as it is a stabilizer code with an appended "gadget" that turns it into a hybrid stabilizer code. In this case it is doubly interesting, as it "beats" the quantum Hamming bound in a certain sense: we might expect the code to satisfy the single-error varient of the quantum Hamming bound $3n+1 \leq 2^{n-k-m}$, where each error has a distinct syndrome, but due to the impure nature of the code some errors share syndromes and the code violates the bound (for the similar example of subsystem codes "beating" the quantum Hamming bound, see [58]). It would be interesting to see if this gadgets could be unified with the construction in Theorem 17 for a more general construction.

Example 20. We construct a new $[\![7,1\!:\!1,3]\!]_2$ hybrid stabilizer code that is not equivalent to the one given in [41] using Theorem 17. Starting with the non-perfect 6-qubit code given by Shaw et al. [91], we know automatically from Corollary 18 that a hybrid stabilizer code with the desired parameters exists.

$$\begin{pmatrix} Y & I & Z & X & X & Y & I \\ Z & X & I & I & X & Z & I \\ I & Z & X & X & X & X & I \\ I & I & I & Z & I & Z & X \\ Z & Z & Z & I & Z & I & X \\ \hdashline I & I & I & I & I & I & X \end{pmatrix}. \tag{2.11}$$

As with the similarly parametered hybid stabilizer code in [41], this code does not meet the linear programming bounds for hybrid stabilizer codes, but has parameters superior to all other known hybrid codes.

Example 21. In this example we use Theorem 17 to construct three new genuine hybrid stabilizer codes with parameters $[\![18, 11\!:\!1, 3]\!]_2$, $[\![19, 11\!:\!2, 3]\!]_2$, and $[\![20, 11\!:\!3, 3]\!]_2$ defined by the stabilizer generators given in Eq. 2.12, 2.13, and 2.14 respectively. The $[\![17, 11, 3]\!]_2$ quantum stabilizer code all three codes are built off of the corresponding entry in Grassl's online code table [40].

$$\begin{pmatrix} Y & Z & Z & I & Z & X & I & Y & X & I & Y & Z & Y & X & I & Z & Z & I \\ Z & X & Z & I & Z & Y & Z & X & Y & I & X & I & X & Y & I & Y & X & I \\ I & I & X & I & Z & Y & Z & Y & X & X & Z & Y & I & I & X & X & Y & X \\ I & I & Z & X & I & Z & I & Y & Y & Y & X & X & Z & Y & Y & X & X & X \\ Z & Z & Z & Z & X & X & I & Z & Y & Z & I & Y & I & Y & I & X & Y & X \\ I & Z & I & Z & Z & I & Y & X & Y & X & Z & Y & Z & X & Z & Z & X & I \\ \hdashline I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & X \end{pmatrix}, \tag{2.12}$$

$$
\begin{pmatrix}
Y & Z & Z & I & Z & X & I & Y & X & I & Y & Z & Y & X & I & Z & Z & I & X \\
Z & X & Z & I & Z & Y & Z & X & Y & I & X & I & X & Y & I & Y & X & I & I \\
I & I & X & I & Z & Y & Z & Y & X & X & Z & Y & I & I & X & X & Y & X & I \\
I & I & Z & X & I & Z & I & Y & Y & Y & X & X & Z & Y & Y & X & X & X & I \\
Z & Z & Z & Z & X & X & I & Z & Y & Z & I & Y & I & Y & I & X & Y & X & X \\
I & Z & I & Z & Z & I & Y & X & Y & X & Z & Y & Z & X & Z & Z & X & I & I \\
\hdashline
I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & X & I \\
I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & X
\end{pmatrix}, \qquad (2.13)
$$

$$
\begin{pmatrix}
Y & Z & Z & I & Z & X & I & Y & X & I & Y & Z & Y & X & I & Z & Z & I & X & I \\
Z & X & Z & I & Z & Y & Z & X & Y & I & X & I & X & Y & I & Y & X & I & I & X \\
I & I & X & I & Z & Y & Z & Y & X & X & Z & Y & I & I & X & X & Y & X & I & X \\
I & I & Z & X & I & Z & I & Y & Y & Y & X & X & Z & Y & Y & X & X & X & I & I \\
Z & Z & Z & Z & X & X & I & Z & Y & Z & I & Y & I & Y & I & X & Y & X & X & X \\
I & Z & I & Z & Z & I & Y & X & Y & X & Z & Y & Z & X & Z & Z & X & I & I & I \\
\hdashline
I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & X & I & I \\
I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & X & I \\
I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & X
\end{pmatrix}. \qquad (2.14)
$$

Here we note that there are nonadditive quantum codes with parameters similar to the $[\![9,3\!:\!1,3]\!]_2$ and $[\![10,3\!:\!2,3]\!]_2$ from Example 19, and we ask if the same is true for lengths 18, 19, and 20. To our knowledge the most comprehensive search of nonadditive codes was done by Rigby et al. [89], but there they limited their search of distance-3 codes to lengths of $n \leq 12$. It would be interesting if there is a gadget-like construction of nonadditive codes similar to the one for hybrid codes in Theorem 17 that allows for the construction of length 18, 19, and 20 nonadditive codes.

We will use the codes from Examples 19, 20, 21 as the building blocks of infinite families of hybrid codes in Section 2.5.

Example 22. Here we give a distance-4 hybrid stabilizer code with parameters $[\![11,2\!:\!1,4]\!]_2$, building off of the $[\![10,2,4]\!]_2$ quantum stabilizer code from Grassl's online code table [40]. This code improves the parameters of the $[\![11,1\!:\!2,4]\!]_2$ code from [41].

$$\begin{pmatrix} X & Z & I & Z & I & X & I & Z & Z & I & X \\ I & Y & I & Z & Z & Y & I & I & Z & Z & X \\ I & Z & X & I & I & Y & I & Y & Z & X & X \\ I & Z & Z & Y & I & Y & Z & I & I & Z & I \\ I & I & Z & Z & Y & Y & Z & X & Z & X & I \\ I & Z & Z & I & Z & Z & X & X & Z & Z & I \\ I & Z & I & I & I & Z & Z & Z & Y & Y & I \\ Z & Z & Z & Z & Z & Z & I & I & I & I & I \\ \hdashline I & I & I & I & I & I & I & I & I & I & X \end{pmatrix}. \tag{2.15}$$

2.5 Families of Hybrid Codes from Stabilizer Pasting

In this section, we construct two families of single-error correcting hybrid codes that can encode one or two classical bits. An infinite family of nonadditive quantum codes was constructed by Yu et al. [105] by pasting together (see [38]) the stabilizers of Gottesman's $[\![2^j, 2^j - j - 2, 3]\!]_2$ codes [37] with the non-Pauli observables of the $(\!(9,12,3)\!)_2$ and $(\!(10,24,3)\!)_2$ nonadditive CWS codes [104, 103] which function in the same role as the Pauli stabilizers in stabilizer codes.

Here, we will make use of the generators for the $[\![7,1\!:\!1,3]\!]_2$, $[\![9,3\!:\!1,3]\!]_2$, $[\![10,3\!:\!2,3]\!]_2$, and $[\![11,4\!:\!2,3]\!]_2$ hybrid stabilizer codes given by Eq. 2.11, 2.8, 2.9, and 2.10 respectively in Examples 19 and 20. Alternatively, we could make use of the $[\![7,1\!:\!1,3]\!]_2$, $[\![10,3\!:\!2,3]\!]_2$, and $[\![11,4\!:\!2,3]\!]_2$ hybrid stabilizer codes constructed by Grassl et al. [41]. Note that in each case, the generators above the dotted line define a pure $[\![n, n-5, 2]\!]_2$ quantum code.

The next theorems describe families of hybrid quantum codes. Notice that $2^{2m+5} \equiv 2^5 \pmod 3$ and $2^{2m+6} \equiv 2^6 \pmod 3$, so the length n given in Theorems 23 and 24 is well-defined.

Theorem 23. *Let m be a nonnegative integer and n a positive integer given by*

$$n = \frac{2^{2m+5} - 32}{3} + a,$$

where the parameter a is a small positive integer that is specified below. Then there exists *(a) an* $[\![n, n-2m-6:1,3]\!]_2$ *hybrid code for* $a = 7, 9$ *and*

(b) an $[\![n, n-2m-7:2,3]\!]_2$ *hybrid code for* $a = 10, 11$.

Proof. Roughly speaking, we construct our code by partitioning the first $(2^{2m+5} - 32)/3$ qubits into disjoints sets, forming a perfect code on each partition, and use one of the four small hybrid codes on the remaining last a qubits. These codes are then “glued" to one another by using stabilizer pasting. Other than a small number of degenerate errors introduced by the small hybrid code that must be handled individually, each single-qubit Pauli error has a unique syndrome, allowing for the correction of any single-qubit error.

We will now describe the code construction in more detail. We take the $n = (2^{2m+5} - 32)/3 + a$ qubits and partition them into disjoint sets

$$U_m \cup U_{m-1} \cup \cdots \cup U_1 \cup V_a,$$

where $|U_\ell| = 2^{2\ell+3}$ and $|V_a| = a$. The set U_m contains the first 2^{2m+3} qubits, U_{m-1} the next 2^{2m+1} qubits, and so forth. The final a qubits are contained in V_a.

Let $\ell \in [m]$. On the qubits in the set U_ℓ, we can construct a stabilizer code of length $2^{2\ell+3}$ with $2\ell + 5$ stabilizer generators, following Gottesmann [37]. The $2\ell + 5$ stabilizer generators are given as follows. Two of these generators are the tensor product of only Pauli-X and Z operators, which we call X_{U_ℓ} and Z_{U_ℓ} respectively. We define the other $2\ell + 3$ stabilizers by

$$\mathcal{S}_j^\ell = X^{h_j} Z^{h_{j-1}+h_1+h_{2\ell+3}},$$

for $j \in [2\ell + 3]$. Here we let h_j be the j-th row of the $(2\ell + 3) \times 2^{2\ell+3}$ matrix H_ℓ, whose i-th column is the binary representation of i, h_0 is defined to be the all-zero vector, and $X^{h_j} = X^{h_{j,0}} X^{h_{j,1}} \dots X^{h_{j,2^{2\ell+3}-1}}$, with Z^{h_j} defined similarly.

For the set V_a, let $H_j^{\mathcal{Q}}$ be the generators of the quantum stabilizer $\mathcal{S}_{\mathcal{Q}}$ of the length a hybrid code defined by the generators in (2.11), (2.8), (2.9), or (2.10), and $H_j^{\mathcal{C}}$ be the generators of the classical stabilizer $\mathcal{S}_{\mathcal{C}}$ (since the length 7 and 9 hybrid codes only have one generator in $\mathcal{S}_{\mathcal{C}}$, we can remove $H_2^{\mathcal{C}}$). The stabilizer can be pasted together as shown in (2.16), where suitable identity operators should be inserted in the blank spaces:

$$\left(\begin{array}{cccccc}
X_{U_m} & & & & & \\
Z_{U_m} & & & & & \\
S_1^m & X_{U_{m-1}} & & & & \\
S_2^m & Z_{U_{m-1}} & & & & \\
\vdots & \vdots & \ddots & & & \\
S_{2m-6}^m & S_{2m-8}^{m-1} & \cdots & & & \\
S_{2m-5}^m & S_{2m-7}^{m-1} & \cdots & X_{U_2} & & \\
S_{2m-4}^m & S_{2m-6}^{m-1} & \cdots & Z_{U_2} & & \\
S_{2m-3}^m & S_{2m-5}^{m-1} & \cdots & S_1^2 & X_{U_1} & \\
S_{2m-2}^m & S_{2m-4}^{m-1} & \cdots & S_2^2 & Z_{U_1} & \\
S_{2m-1}^m & S_{2m-3}^{m-1} & \cdots & S_3^2 & S_1^1 & H_1^{\mathcal{Q}} \\
S_{2m}^m & S_{2m-2}^{m-1} & \cdots & S_4^2 & S_2^1 & H_2^{\mathcal{Q}} \\
S_{2m+1}^m & S_{2m-1}^{m-1} & \cdots & S_5^2 & S_3^1 & H_3^{\mathcal{Q}} \\
S_{2m+2}^m & S_{2m}^{m-1} & \cdots & S_6^2 & S_4^1 & H_4^{\mathcal{Q}} \\
S_{2m+3}^m & S_{2m+1}^{m-1} & \cdots & S_7^2 & S_5^1 & H_5^{\mathcal{Q}} \\
\hdashline
 & & & & & H_1^{\mathcal{C}} \\
 & & & & & H_2^{\mathcal{C}}
\end{array}\right) \tag{2.16}$$

Suppose that we have a single-qubit Pauli error on the block U_m. Since the code is pure, the syndrome of each error will be distinct and such that the Pauli-X, Y, and Z sydromes will start with 01, 11, and 10 respectively. However, this leaves all of the syndromes starting with 00 unused, so Pauli-X, Y, and Z errors on the block U_{m-1} will have distinct syndromes starting with 0001, 0011, and 0010 respectively. Continuing on, any single-qubit Pauli error occurring on the block U_k will have a distinct syndrome starting with $2(m-k)$ zeros.

All of the syndromes of errors occurring on the block V_a start with $2m$ 0s. Here our code is not pure, but it is almost pure, with the only degenerate errors being the weight 2 errors in $\mathcal{S}_{\mathcal{C}}$. For example, when V_a has 11 qubits, it will have three weight 1 degenerate errors: X_9 (a Pauli-X error on the ninth qubit of the block), X_{10}, and Z_{11}, each with the syndrome 00001 (preceeded by $2m$

zeros). If we measure this syndrome, we apply the operator $IIIIIIIXXZ$ to the state, which maps the original encoded state to itself up to a global phase. Note, however, that while this global phase is the same for encoded states of the same inner code for a given error, it may differ for encoded states from different inner codes. In fact, this is exactly what prevents the outer code from being a distance 3 quantum code rather than a distance 3 hybrid code. The argument for when V_a has 7, 9, and 10 qubits is similar.

Since we know how to correct any single-qubit Pauli error based on its syndrome, each of the codes must have minimum distance 3. □

Theorem 24. *Let m be a nonnegative integer and n a positive integer given by*

$$n = \frac{2^{2m+6} - 64}{3} + a,$$

where the parameter a is a small positive integer that is specified below. Then there exists

(a) an $[\![n, n-2m-7 : 1, 3]\!]_2$ hybrid code for $a = 18$,

(b) an $[\![n, n-2m-8 : 2, 3]\!]_2$ hybrid code for $a = 19$, and

(c) an $[\![n, n-2m-9 : 3, 3]\!]_2$ hybrid code for $a = 20$.

Proof. The proof is the same as that for Theorem 23, except that we resize the U_ℓ blocks so that $|U_\ell| = 2^{2\ell+4}$ and the replace the codes on these blocks with Gottesman's codes with lengths $2^{2\ell+4}$, as well as using the length 18, 19, and 20 hybrid codes defined by the generators in Eq. 2.12, 2.13, and 2.14 from Example 21 on the V block of qubits. □

Here we show that these hybrid codes are better than optimal quantum stabilizer codes using a result of Yu et al. [102].

Proposition 25. *Let m be a nonnegative integer and n a positive integer given by*

$$n = \frac{2^{2m+5} - 32}{3} + a,$$

where $a \in \{7, 9, 10, 11\}$. Then there does not exist an $[\![n, n-2m-5, 3]\!]_2$ stabilizer code.

Proof. When $a = 7, 9, 10$, we have

$$\begin{aligned} n &= \frac{2^{2m+5} - 32}{3} + a \\ &= \frac{2^{2m+5} - 8}{3} + (a - 8) \\ &= \frac{8}{3}\left(4^{m+1} - 1\right) + (a - 8) \,. \end{aligned}$$

By a result of Yu et al. [102, Theorem 1], distance 3 stabilizer codes with lengths of the form

$$\frac{8}{3}\left(4^{k} - 1\right) + b,$$

where $b \in \{-1, 1, 2\}$, can exist if and only if

$$2m + 5 \geq \lceil \log_2(3n + 1) \rceil + 1.$$

But in this case we have

$$\begin{aligned} \lceil \log_2(3n + 1) \rceil + 1 &= \left\lceil \log_2\left(2^{2m+5} + 3a - 31\right) \right\rceil + 1 \\ &> \left\lceil \log_2\left(2^{2m+5} - 2^{2m+4}\right) \right\rceil + 1 \\ &= 2m + 5, \end{aligned}$$

so when $a = 7, 9, 10$, there is no distance 3 stabilizer code of length n.

When $a = 11$, a different case of [102, Theorem 1] applies, so distance 3 stabilizer codes with lengths of this form can exist if and only if

$$2m + 5 \geq \lceil \log_2(3n + 1) \rceil \,.$$

However, this gives us

$$\begin{aligned}\lceil \log_2(3n+1) \rceil &= \left\lceil \log_2\left(2^{2m+5}+2\right) \right\rceil \\ &> \left\lceil \log_2\left(2^{2m+5}\right) \right\rceil \\ &= 2m+5,\end{aligned}$$

so when $a = 11$, there is likewise no distance 3 stabilizer code of length n. □

Proposition 26. *Let m be a nonnegative integer and n a positive integer given by*

$$n = \frac{2^{2m+6}-64}{3} + a,$$

where $a \in \{18, 19, 20\}$. Then there does not exist an $[\![n, n-2m-6, 3]\!]_2$ stabilizer code.

Proof. This proof follows the same logic as the proof of Proposition 25. Note that for $a = 18, 19, 20$, we can rewrite

$$n = \frac{4^{m+3}-1}{3} - b$$

where $b \in \{1, 2, 3\}$. By [102, Theorem 1], a stabilizer code with a length in this form can exist if and only if

$$2m + 6 \geq \lceil \log_2(3n+1) \rceil + 1.$$

But this gives us

$$\begin{aligned}\lceil \log_2(3n+1) \rceil + 1 &= \left\lceil \log_2\left(2^{2m+6}-3b\right) \right\rceil + 1 \\ &= 2m+6,\end{aligned}$$

meaning that no $[\![n, n-2m-6, 3]\!]_2$ stabilizer codes exist for these values of n. □

As with our family of error-detecting hybrid codes, it would be interesting to know whether any of these codes meet the linear programming bounds from Section 2.2.2. Since none of the hybrid codes we started with meet these bounds, it is doubtful that any of the hybrid codes constructed from stabilizer pasting would also meet this bound, leaving it unclear whether or not these codes are optimal among all hybrid codes.

2.6 Conclusion

In this paper we have proven some general results about hybrid codes, showing that they can always detect more errors than comparable quantum codes. Furthermore we proved the necessity of impurity in the construction of genuine hybrid codes. Additionally, we generalized weight enumerators for hybrid stabilizer codes to nonadditive hybrid codes, allowing us to develop linear programming bounds for nonadditive hybrid codes. Finally, we have constructed several infinite families of hybrid stabilizer codes that provide an advantage over optimal stabilizer codes. Addtitionally, we generalized our family of single error-detecting codes from the binary case to the nonbinary case over the integer residue rings $\mathbb{Z}_q$. While it is known that the construction gives genuine hybrid codes when $q = 2$, the existence of quantum codes with the similar parameters when $q \equiv 0, 1, 3 \mod 4$ means the construction does not produce genuine hybrid codes in all cases. One open question is whether or not the codes given by the construction are always genuine when $q \equiv 2 \mod 4$. As the code family here is the only construction of nonbinary hybrid codes, further investigation is needed.

Both of our families of hybrid codes were inspired by the construction of nonadditive quantum codes. In hindsight this is not very surprising, as the examples of hybrid codes with small parameters given by Grassl et al. [41] were constructed using a CWS/union stabilizer construction. Most interesting is that all known good nonadditive codes with small parameters have a hybrid code with similar parameters. This would suggest that looking at larger nonadditive codes such as the quantum Goethals-Preparata code [42] or generalized concatenated quantum codes [45] might be helpful in constructing larger hybrid codes. Alternatively, it may be possible to use the existence of hybrid codes to point to where nonadditive codes may be found. For instance the existence of an $[\![11, 4{:}2, 3]\!]_2$ hybrid code suggests a nonadditive $(\!(11, K, 3)\!)_2$ code with $K < 32$ might exist, although none have been found in past searches [89].

As previously suggested by Grassl et al. [41], one possible way to construct new hybrid codes with good parameters is to start with degenerate quantum codes with good parameters. Another possible approach to constructing new hybrid stabilizer codes is to find codes such that there are few small weight errors that are in the normalizer but not in the stabilizer, and then add those small weight errors to the generating set of the stabilizer to get a degenerate code. Here, the original code becomes the outer code of the hybrid code and the degenerate code the inner code.

3. ENCODING CLASSICAL INFORMATION IN GAUGE SUBSYSTEMS OF QUANTUM CODES

The majority of the material in this chapter comes from the journal paper [77][1].

3.1 Background

Hybrid codes allow for the simultaneous transmission of both quantum and classical information across a quantum channel. Devetak and Shor [31] showed for certain small error rates that simultaneous transmission is superior to the time-sharing of a quantum channel, and subsequent work on the topic has been focused primarily on information-theoretic results [48, 49, 101]. The first examples of finite-length hybrid codes were given by Kremsky, Hsieh, and Brun [61] as a generalization of entanglement-assisted quantum codes, and later Grassl, Lu, and Zeng [41] gave multiple examples of good hybrid codes with small parameters using a codeword stabilizer construction. Surprisingly, these codes provide an advantage over optimal quantum codes regardless of the error rate.

Further examples of good hybrid codes were constructed by the authors [75, 74] (Sections 2.3, 2.4, and 2.5 in this dissertation) over the Pauli channel and by Li, Lyles, and Poon [66] over a fully correlated quantum channel where the space of errors is spanned by $I^{\otimes n}$, $X^{\otimes n}$, $Y^{\otimes n}$, and $Z^{\otimes n}$. Additional work on hybrid codes from an operator-theoretic perspective has also been done by Bény, Kempf, and Kribs [18, 19] and Majidy [70]. While few good hybrid code constructions are known, there are already multiple areas in which they can be used, including the protection of hybrid quantum memory [64] and the construction of hybrid secret sharing schemes [106]. Additionally, the work on higher rank matrical ranges by Cao et al. [25] was inspired by hybrid codes.

Previous work on hybrid codes has assumed that both the quantum and classical information

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should be protected from all errors of weight up to the same minimum distance d. In this paper we introduce hybrid codes with two separate minimum distances for quantum and classical information. Loosening this restriction on the minimum distance allows us to construct new hybrid stabilizer codes by encoding classical information in the gauge qudits of subsystem codes, making use of gauge fixing. Using this result, we show how to construct hybrid codes from classical codes using Bacon-Casaccino subsystem codes [14] including a family of good hybrid codes constructed using Bacon-Shor subsystem codes and conjecture that all hybrid stabilizer codes must satisfy a variant of the quantum Singleton bound. Finally, we also show how hybrid codes can be used to protect against faulty syndrome measurement errors and how they can lead to new constructions of quantum data-syndrome codes [10, 36].

3.1.1 Subsystem Codes

Subsystem codes (also called operator quantum error-correcting codes) are a generalization of stabilizer codes that enforce a tensor product structure on the code subspace $Q = A \otimes B$. Quantum information is encoded into subsystem A, while subsystem B, known as the gauge subsystem, is useful for fault tolerance [2] and designing improved decoding algorithms [90]. However, no information is encoded in subsystem B, so in a certain sense it is unused space.

One way to view subsystem codes is through the stabilizer formalism of the previous section. Informally, a subsystem code can be viewed as a stabilizer code where only a subset of the logical qudits are used to encode quantum information. The logical qudits containing the quantum information correspond to the K-dimensional subsystem A, while the unused gauge qudits correspond to the R-dimensional subsystem B. Similar to stabilizer codes, we write the parameters of a subsystem code as $[\![n, k, r, d]\!]_q$ where $k = \log_q(K)$ and $r = \log_q(R)$. A subsystem code has $\ell\,(n - k - r)$ mutually commuting generators S_i that generate the abelian stabilizer group $\mathcal{S}$ of the subsystem code. The KR-dimensional subspace Q is then the $+1$-eigenpace of the elements of the stabilizer group $\mathcal{S}$.

To induce the subsystem $A \otimes B$ on Q, we define the gauge group $\mathcal{G}$, which consists of those Pauli operators on Q that act as the identity on A. These include elements in $\mathcal{S}$, as well as the logical operators on the subsystem B, which are generated by ℓr pairs of gauge operators G_i^X and G_i^Z such that G_i^X and G_j^Z do not commute if $i = j$ and commute otherwise, G_i^X and G_j^X all commute, and G_i^Z and G_j^Z all commute. The gauge group is then given by

$$\mathcal{G} = \left\langle \omega, \mathcal{S}, G_i^X, G_i^Z \mid i \in [\ell r] \right\rangle.$$

Each pair G_i^X and G_i^Z corresponds to some pair of logical operators $\overline{X_i(a)}$ and $\overline{Z_i(a)}$ on the stabilizer code Q, but they are written differently to better distinguish them from the logical operators on the subsystem A, which are given by $\mathcal{L} = N(\mathcal{S})\,/\mathcal{G}$. For further details on the stabilizer formalism of subsystem codes, see Kribs and Poulin [63] and Poulin [82].

3.2 Hybrid Codes

We now would like to simultaneously transmit a classical message along with our quantum information. A hybrid code has parameters $(\!(n, K\!:\!M, d\!:\!c)\!)_q$ if and only if it can simultaneously encode a superposition of K orthogonal quantum states as well as one of M different classical messages into the Hilbert space $\mathcal{H} = \mathbb{C}^{q^n}$, while detecting all errors of weight less than d and c on the quantum and classical information respectively. The hybrid code $\mathcal{C}$ may be thought of as a collection of M orthogonal quantum codes $\mathcal{C}_m$ of dimension K, indexed by the classical message $m \in [M] = \{1, 2, \ldots, M\}$, as seen in Figure 3.1. We refer to the codes $\mathcal{C}_m$ as the inner codes and $\mathcal{C} = \{\mathcal{C}_m \mid m \in [M]\}$ as the outer code. To send a quantum state $|\varphi\rangle$ and a classical message m, we simply encode $|\varphi\rangle$ into the quantum code $\mathcal{C}_m$.

If the quantum and classical minimum distances are the same (i.e., $d = c$), we write $(\!(n, K\!:\!M, d)\!)_q$. If both the outer code and all of the inner codes are stabilizer codes, we refer to the code as a hybrid stabilizer code and write its parameters as $[\![n, k\!:\!m, d\!:\!c]\!]_q$ where $k = \log_q(K)$ and $m = \log_q(M)$.

Grassl et al.[41] presented a set of necessary and sufficient conditions for the error-correcting capabilities of hybrid codes with $d = c$ that generalize the Knill-Laflamme conditions [60] for quantum codes. Here we generalize these conditions further to allow for hybrid codes with $d \leq c$.

Theorem 27. *An $(\!(n, K\!:\!M, d\!:\!c)\!)_q$ hybrid code with $d \leq c$ can detect up to $d - 1$ errors to the*

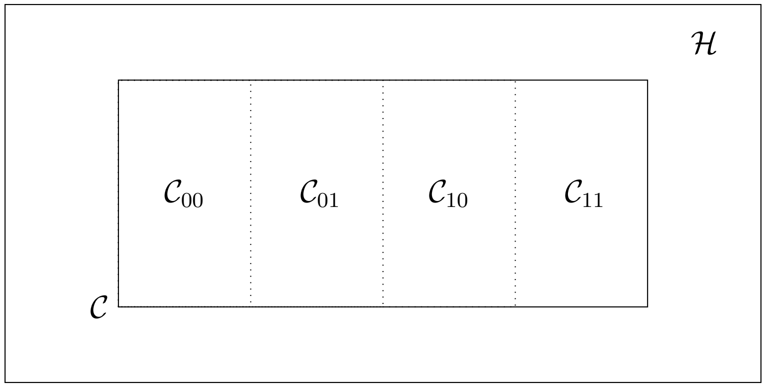

Figure 3.1: Each hybrid code $\mathcal{C}$ is a collection of orthogonal quantum codes $\mathcal{C}_i$ indexed by a classical message i, here represented as a binary string in $\{00, 01, 10, 11\}$.

quantum information and up to $c-1$ *errors to the classical information if and only if*

1. $P_a E P_a = \lambda_{E,a} P_a$*, for all* $a \in [M]$ *and all error operators* E *such that* $\mathrm{wt}(E) < d$*, and*

2. $P_a F P_b = 0$*, for all* $a, b \in [M]$*,* $a \neq b$*, and all error operators* F *such that* $\mathrm{wt}(F) < c$*.*

Proof. Suppose that (1) and (2) hold. If the weight of an error E on the system is less than d, then (1) implies that the hybrid code can detect an error on the quantum information of weight less than d, following directly from the Knill-Laflamme conditions for quantum codes [60]. Additionally, (2) implies that the image of the codes under all the errors of weight less than c are all mutually orthogonal, that is, for $a \neq b$, $P_a \perp \langle E P_b : \mathrm{wt}(E) < c \rangle$. This means that by applying a measurement based on our projectors P_a we can always detect an error to the classical information. If instead an error F with $d \leq \mathrm{wt}(F) < c$ affects the system, then we can no longer detect the error on the quantum information, but since $P_a F P_b = 0$, $a \neq b$, still holds for the error F, we can perform a measurement and detect an error to the classical information.

Now suppose that either (1) or (2) fails to hold. If (1) fails to hold, then the Knill-Laflamme conditions tell us that there is an error to the quantum information of weight less than d that the code cannot detect. If (2) fails to hold, then there is an error F of weight less than c such that for some $a \neq b$, P_a and $F P_b$ will not be orthogonal, meaning we will not be able to completely distinguish between the two of them. □

As with quantum codes, an error-correction variant of the conditions immediately follows.

Corollary 28. *An* $(\!(n, K\!:\!M, d\!:\!c)\!)_q$ *hybrid code with* $d \leq c$ *can correct up to* $\left\lfloor \frac{d-1}{2} \right\rfloor$ *errors to the quantum information and up to* $\left\lfloor \frac{c-1}{2} \right\rfloor$ *errors to the classical information if and only if*

1. $P_a E^\dagger F P_a = \lambda_{E,F,a} P_a$*, for all* $a \in [M]$ *and all error operators* E, F *such that* $\mathrm{wt}(E), \mathrm{wt}(F) \leq \left\lfloor \frac{d-1}{2} \right\rfloor$*, and*

2. $P_a E^\dagger F P_b = 0$*, for all* $a, b \in [M]$*,* $a \neq b$*, and all error operators* E, F *such that* $wt(E), \mathrm{wt}(F) \leq \left\lfloor \frac{c-1}{2} \right\rfloor$*.*

Note that when $c < d$ there is a potential problem. Consider the following: let $\mathcal{C}_a$ and $\mathcal{C}_b$ be 1-dimensional inner codes in a hybrid code with $P_i = |\psi_i\rangle \langle\psi_i|$ the projector onto $\mathcal{C}_i$, and suppose the code satisfies conditions (1) and (2) in Theorem 27. We may still have an error E with $c \leq \mathrm{wt}(E) < d$ such that $\langle\psi_a| E |\psi_b\rangle = \alpha \neq 0$. Setting up a measurement and supposing that $|\psi_b\rangle$ is sent, we get

$$\begin{aligned}(P_a + P_b) E |\psi_b\rangle &= |\psi_a\rangle \langle\psi_a| E |\psi_b\rangle + |\psi_b\rangle \langle\psi_b| E |\psi_b\rangle \\ &= \alpha |\psi_a\rangle + \lambda_{E,b} |\psi_b\rangle ,\end{aligned}$$

which is a superposition of encoded states from the two inner codes. However, as we will show here and in Section 3.3, we can still construct hybrid codes with $c < d$ by encoding the quantum and classical information using a subsystem structure on the encoding subspace.

Given a code $\mathcal{C}$ with a subsystem structure $A \otimes B$ on it, let $\{|\varphi_i\rangle | \, i \in [K]\}$ and $\{|v_i\rangle \mid i \in [M]\}$ be orthonormal bases for A and B respectively. We define the operators

$$P_{a,b} = \left(\sum_{i=1}^{K} |\varphi_i\rangle \langle\varphi_i| \right) \otimes |v_a\rangle \langle v_b| ,$$

which allows us to write the following error-detection conditions similar to those for subsystem codes [79].

Theorem 29. *Let $\mathcal{C}$ be an $(\!(n, K\!:\!M, d\!:\!c)\!)_q$ hybrid code with a subsystem structure $A \otimes B$ on it, with $\{|\varphi_i\rangle \mid i \in [K]\}$ and $\{|v_i\rangle \mid i \in [M]\}$ as orthonormal bases for A and B respectively. Let $P_a = P_{a,a}$ be the projector onto the inner code $\mathcal{C}_a$. Then $\mathcal{C}$ can detect up to $d-1$ errors to the quantum information and up to $c-1$ errors to the classical information if and only if*

1. $P_a E P_b = \lambda_{E,a,b} P_{a,b}$, *for all $a, b \in [M]$ and all E such that* $\mathrm{wt}(E) < d$, *and*

2. $P_a F P_b = 0$, *where $a \neq b$, for all $a, b \in [M]$, $a \neq b$, and all F such that* $\mathrm{wt}(F) < c$.

Proof. The proof is the same as the proof of Theorem 27, except we must now check the case when $c < d$. Here we will first project the code onto the subspace $\mathcal{C}$ using the projector

$$P = \sum_{j\in[M]} P_j = \left(\sum_{i\in[K]} |\varphi_i\rangle\langle\varphi_i|\right) \otimes \left(\sum_{j\in[M]} |v_j\rangle\langle v_j|\right),$$

measure the classical information in subsystem B in the $\{|v_a\rangle\}$ basis to determine which code was sent, and then use the recovery procedure associated with that code.

Suppose that (1) and (2) are true, and let $|\varphi_a\rangle = (|\varphi\rangle \otimes |v_a\rangle)$ be the encoded state and E the error on the encoded state. If $\mathrm{wt}(E) < c$, then

$$\begin{aligned} PE\,|\varphi_a\rangle &= \sum_{j\in[M]} P_j E P_a\,|\varphi_a\rangle \\ &= P_a E P_a\,|\varphi_a\rangle , \end{aligned}$$

by condition (2). It follows from condition (1) that

$$\begin{aligned} PE\,|\varphi_a\rangle &= P_a E P_a\,|\varphi_a\rangle \\ &\lambda_{E,a}\,|\varphi_a\rangle . \end{aligned}$$

Performing a measurement on the subsystem B will not have an effect on the encoded state and it will inform us of which code was used.

If $c \leq \text{wt}(E) < d$, then by condition (1) we have

$$\begin{aligned} PE\left|\varphi_a\right\rangle &= \sum_{j\in[M]} P_j E P_a \left|\varphi_a\right\rangle \\ &= \sum_{j\in[M]} \lambda_{E,j,a} P_{j,a} \left|\varphi_a\right\rangle \\ &= \left|\varphi\right\rangle \otimes \sum_{j\in[M]} \lambda_{E,j,a} \left|v_j\right\rangle . \end{aligned}$$

Measuring the subsystem B in the $\{|v_i\rangle\}$ basis, we get $\lambda_{E,x,a} \left(|\varphi\rangle \otimes |v_x\rangle\right) / \left|\lambda_{E,x,a}\right|$, where x may not be the original classical message. However, we are still able to detect an error to the quantum information.

The converse follows the same logic as the proof of Theorem 27, making use of the subsystem variant of the Knill-Laflamme conditions. □

Similar to Theorem 27, we immediately get the error-correction variant of Theorem 29.

Corollary 30. *Let $\mathcal{C}$ be an $(\!(n, K\!:\!M, d\!:\!c)\!)_q$ hybrid code with a subsystem structure $A \otimes B$ on it, with $\{|\varphi_i\rangle \mid i \in [K]\}$ and $\{|v_i\rangle \mid i \in [M]\}$ as orthonormal bases for A and B respectively. Let $P_a = P_{a,a}$ be the projector onto the inner code $\mathcal{C}_a$. Then $\mathcal{C}$ can correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors to the quantum information and up to $\left\lfloor \frac{c-1}{2} \right\rfloor$ errors to the classical information if and only if*

1. *$P_a E^\dagger F P_b = \lambda_{E,F,a,b} P_{a,b}$, for all $a, b \in [M]$ and all E, F such that $\text{wt}(E), \text{wt}(F) \leq \left\lfloor \frac{d-1}{2} \right\rfloor$, and*

2. *$P_a E^\dagger F P_b = 0$, for all $a, b \in [M]$, $a \neq b$, and all E, F such that $\text{wt}(E), \text{wt}(F) \leq \left\lfloor \frac{c-1}{2} \right\rfloor$.*

We leave the cases where errors to either the quantum or classical information are corrected while errors to the other are only detected for future research.

3.2.1 Genuine Hybrid Codes

Constructing hybrid codes from quantum codes is not a particularly difficult task. When $d = c$, Grassl et al. [41] gave several simple constructions of hybrid codes from quantum codes:

Proposition 31 (Grassl et al.[41]). *Hybrid codes can be constructed using the following "trivial" constructions:*

1. *Given an* $((n, KM, d))_q$ *quantum code of composite dimension* KM*, there exists a hybrid code with parameters* $((n, K\!:\!M, d))_q$.
2. *Given an* $[\![n, k\!:\!m, d]\!]_q$ *hybrid code with* $k > 0$*, there exists a hybrid code with parameters* $[\![n, k-1\!:\!m+1, d]\!]_q$.
3. *Given an* $[\![n_1, k_1, d]\!]_q$ *quantum code and an* $[n_2, m_2, d]_q$ *classical code, there exists a hybrid code with parameters* $[\![n_1 + n_2, k_1\!:\!m_2, d]\!]_q$.

We call a hybrid code with $d = c$ *genuine* if there is no code constructable using Proposition 31 with the same parameters. Grassl et al. [41] showed the first examples of genuine hybrid codes, constructing multiple small-parametered hybrid codes, while the authors constructed several infinite families of genuine hybrid stabilizer codes using stabilizer pasting [75]. Note that by calling such codes "genuine", we do not mean to imply that the hybrid codes constructed using the approaches of Proposition 31 are in any sense "fake". Hybrid codes constructed using one of these three methods are in a sense wasting a quantum resource, in that they are transmitting classical information using space that could have been used to transmit quantum information.

Similar to the case where there is a single minimum distance, we can construct trivial hybrid codes with two minimum distances using the following construction that generalizes the third construction of Proposition 31:

Proposition 32. *Given an* $((n_1, K_1, d))_q$ *quantum code and an* $(n_2, M_2, c)_q$ *classical code, there exists a hybrid code with parameters* $((n_1 + n_2, K_1\!:\!M_2, d\!:\!c))_q$.

Proof. Use the quantum code to encode the quantum information on the first n_1 physical qudits and use the classical code to encode the classical information on the remaining n_2 physical qudits. □

To generalize the first and second constructions to allow for two minimum distances, we will define a partial order on the parameters of hybrid codes to determine which codes have "better" parameters than others:

Definition 33. *Given two hybrid codes $\mathcal{C}$ and $\mathcal{C}'$ with parameters $((n, K:M, d:c))_q$ and $((n, K':M', d':c'))_q$ respectively, we say $\mathcal{C} \preceq \mathcal{C}'$ if $KM \leq K'M'$, $K \leq K'$, $d \leq d'$, and $c \leq c'$ are all true.*

Note that while we write $\mathcal{C} \preceq \mathcal{C}'$, we are only comparing the parameters of the codes and not the codes themselves.

Proposition 34. *The relation $\preceq$ defines a partial order on the set of hybrid code parameters.*

Proof. The reflexivity, antisymmetry, and transitivity of $\preceq$ all follow directly from the fact that $\leq$ is a partial order. □

If $\mathcal{C} \preceq \mathcal{C}'$, we say that $\mathcal{C}'$ has at least as good parameters as $\mathcal{C}$. Intuitively, this covers the case when $\mathcal{C}'$ has at least one parameter greater than the corresponding parameter in $\mathcal{C}$, with all other parameters being equal. For example, an $[\![8,3,3]\!]_2$ quantum code has better parameters than an $[\![8,1,3]\!]_2$ quantum code, as the former can encode two more logical qubits than the latter. We also give preference to codes that can transmit more quantum information if the total amount of information that can be transmitted by each code is the same. For example, we can compare the $[\![9,1:2,3]\!]_2$ hybrid code of Kremsky et al. [61] with the $[\![9,2:2,3]\!]_2$ code of Grassl et al. [41], with the latter having better parameters since it can encode one logical qubit more than the former. Similarly, we can compare both of these codes with the $[\![9,3:1,3]\!]_2$ code we construct in Example 19, which has better parameters than both, as we can use it to construct a $[\![9,2:2,3]\!]_2$ by using one of the logical qubits to transmit a classical bit. However, we cannot compare any of these three codes with the $[\![9,1:4,3:2]\!]_2$ code we construct in Example 44, since it transmits more total information (has a larger sum $k+m$) but has a lower classical distance.

We call a hybrid code (with $K, M > 1$, although the partial order is also defined on purely quantum and classical codes) genuine if it is a maximal element in the partially ordered set and has parameters that cannot be achieved by a code constructed using Proposition 32, and we call it a genuine hybrid stabilizer code if it satisfies the same conditions on the partially ordered set induced by $\preceq$ on the subset of hybrid stabilizer codes. Intuitively, this means that a genuine hybrid code is one in which any one parameter of the code cannot be improved without sacrificing some other parameter, with the exception that we can sacrifice one bit of classical information for one qudit of quantum information. When restricted to the case with $c = d$, we recover the original definition of genuine codes.

3.2.2 Hybrid Stabilizer Codes

For the remainder of the paper we will restrict our attention to hybrid stabilizer codes, which have a particularly nice structure. Starting with a quantum stabilizer code $\mathcal{C}_0$ with stabilizer group $\mathcal{S}_0$, we choose M translation operators t_i from different cosets of $N(\mathcal{S}_0)$ in G_n in such a way that the cosets form a group (we will always take t_1 to be the identity). The hybrid code $\mathcal{C}$ is then the union of the translated codes:

$$\mathcal{C} = \bigcup_{i \in [M]} t_i \mathcal{C}_0$$

The stabilizer generators of the inner code $\mathcal{C}_0$ can be divided into a quantum stabilizer $\mathcal{S}_{\mathcal{Q}}$ and a classical stabilizer $\mathcal{S}_{\mathcal{C}}$ such that $\mathcal{S}_0 = \langle \mathcal{S}_{\mathcal{Q}}, \mathcal{S}_{\mathcal{C}} \rangle$ [61]. The quantum stabilizer $\mathcal{S}_{\mathcal{Q}}$ is the stabilizer of the outer code $\mathcal{C}$ and is generated by those generators of $\mathcal{S}_0$ that commute with all of the translation operators t_i. The classical stabilizer $\mathcal{S}_{\mathcal{C}}$ is generated by the remaining stabilizer generators of $\mathcal{S}_0$, each of which does not commute with at least one translation operator. We can associate each of the ℓm generators g_i of $\mathcal{S}_{\mathcal{C}}$ with an operator $\overline{Z_j(\alpha^i)}$, for $i \in \{0, \ldots, \ell - 1\}$, $j \in [\lfloor m \rfloor]$, which acts on the j-th virtual qudit, as well as $\overline{Z_{\lceil m \rceil}(\alpha^i)}$ for $i \in \{0, \ldots, \ell(m - \lfloor m \rfloor) - 1\}$ if m is not a power of q. Similarly, we can associate each of the generators of the translation operators $\overline{X_j(\alpha^i)}$ for $i \in \{0, \ldots, \ell - 1\}$, $j \in [\lfloor m \rfloor]$, as well as $\overline{X_{\lceil m \rceil}(\alpha^i)}$ for $i \in \{0, \ldots, \ell(m - \lfloor m \rfloor) - 1\}$ if m is not a power of q. These operators satisfy the commutation relations from Section 1.3.3, and we can associate each classical message $\mathbf{a} \in \mathbb{F}_q^{\lceil m \rceil}$ with the translation operator $t_{\mathbf{a}} = \overline{X(\mathbf{a})} = \overline{X_1(a_1)} \cdot \overline{X_2(a_2)} \cdots \overline{X_{\lceil m \rceil}(a_{\lceil m \rceil})}$. In addition to mapping between the inner codes, these translation operators are also logical operators for the outer code $\mathcal{C}$.

The quantum and classical stabilizers are sufficient to fully define a hybrid code. The following result was originally given in the binary case by Kremsky et al. [61] and by the authors in the case of prime fields [75]. Here we generalize it to arbitrary finite fields.

Theorem 35. *Let $\mathcal{C}$ be an $[\![n,k\!:\!m,d\!:\!c]\!]_q$ hybrid stabilizer code over a finite field of characteristic p, where $q=p^\ell$, with quantum stabilizer $\mathcal{S}_\mathcal{Q}$ and classical stabilizer $\mathcal{S}_\mathcal{C}=\langle g_i \mid i\in[\ell m]\rangle$, where $g_i=\overline{Z(\mathbf{b}_i)}$, $\mathbf{b}_i\in\mathbb{F}_q^{[m]}$. Then the inner stabilizer code $t_\mathbf{a}\mathcal{C}_0$ associated with the classical message $\mathbf{a}\in\mathbb{F}_q^{[m]}$ is stabilized by*

$$\left\langle \mathcal{S}_\mathcal{Q}, \omega^{-\operatorname{tr}(\mathbf{b}_i\cdot\mathbf{a})} g_i \mid i\in[\ell m]\right\rangle,$$

where ω is a primitive p-th root of unity.

Proof. Let $|\varphi\rangle$ be an encoded state of $\mathcal{C}_0$, so that $t_\mathbf{a}|\varphi\rangle$ is an encoded state of $t_\mathbf{a}\mathcal{C}_0$. Since elements of the quantum stabilizer commute with $t_\mathbf{a}$ and stabilize $|\varphi\rangle$, they are all elements of the stabilizer of $t_\mathbf{a}\mathcal{C}_0$. In the case of $\omega^{-\operatorname{tr}(\mathbf{b_i}\cdot\mathbf{a})}g_i$, it follows from the commutation relations that $t_\mathbf{a}|\varphi\rangle$ is one of its $+1$-eigenstates, so it is also in the stabilizer of $t_\mathbf{a}\mathcal{C}_0$. □

3.3 Hybrid Codes from Subsystem Codes

In this section we show how every subsystem code leads to a hybrid code with the same quantum error-correcting properties. While the tensor structure of classical-quantum systems (see Devetak and Shor [31] and Bény et al. [19]) suggests that subsystem codes might be useful in constructing hybrid codes, it is not immediately obvious whether or not they can protect the encoded classical information from errors. The main idea behind our construction is to follow the reasoning of Theorem 29 and encode the quantum information in the subsystem A stabilized by the stabilizer group $\mathcal{S}$, and then use *gauge fixing* to encode the classical information into the subsystem B.

3.3.1 Gauge Fixing Construction

Gauge fixing is a technique that takes commuting gauge operators of the subsystem code and uses them to generate a larger stabilizer group $\mathcal{S}_0$. In essence, we are taking a subset of the gauge qudits and fixing them to certain states. Since the states are fixed, no information can be encoded on those qudits, but any errors that occur on them is now either a pure error or in the stabilizer $\mathcal{S}_0$.

Gauge fixing is well known in quantum error-correction for its use in code switching [20, 81], which allows for a way around Eastin and Knill's famous no-go theorem in fault tolerance [33]. Our construction picks a commuting set of ℓr independent gauge operators of the subsystem code, and then multiplies them by a phase, which forces the gauge qudits to change to a different fixed state. For example, in a binary subsystem code if the gauge operator G_i^Z is fixed, it means that the i-th gauge qubit will be fixed as the $+1$ eigenstate of the operator, so we have a logical $|0\rangle$ that is fixed. If instead we fix the operator $-G_i^Z$, the i-th gauge qubit will be fixed as $|1\rangle$, the -1 eigenstate of the operator.

Theorem 36. *Let $\mathcal{C}$ be an $[\![n,k,r,d]\!]_q$ subsystem code. Then there exists an $[\![n,k\!:\!r,d\!:\!c]\!]_q$ hybrid code.*

Proof. Let $\mathcal{S} = \langle S_i \rangle$ be the stabilizer group of $\mathcal{C}$, which will be the stabilizer of the hybrid code's outer code. Choose $2\ell r$ operators G_i^X and G_i^Z where $i \in [\ell r]$, so that

$$\mathcal{G} = \left\langle \omega, \mathcal{S}, G_i^X, G_i^Z \mid i \in [\ell r)] \right\rangle .$$

Without loss of generality, we will fix a gauge and let

$$\mathcal{S}_0 = \left\langle \mathcal{S}, G_i^Z \mid i \in [\ell r] \right\rangle$$

be the stabilizer of our inner stabilizer code $\mathcal{C}_0$.

The centralizer of $\mathcal{S}$ and $\mathcal{S}_0$ are given by

$$N(\mathcal{S}) = \left\langle \omega, \mathcal{S}, G_i^X, G_i^Z, \overline{X_j}, \overline{Z_j} \mid i \in [\ell r], j \in [\ell k] \right\rangle$$

and

$$N(\mathcal{S}_0) = \left\langle \omega, \mathcal{S}_0, \overline{X_i}, \overline{Z_i} \mid i \in [\ell k] \right\rangle$$

respectively. The quantum minimum distance of the hybrid code is the minimum weight of one of the logical operators on the quantum information, so it will be the identical to the minimum distance of the subsystem code, given by $d = \mathrm{wt}(N(\mathcal{S}) \setminus \mathcal{G})$.

The classical minimum distance c is given by the minimum weight of a logical operator on the classical information, so $c = \mathrm{wt}(N(\mathcal{S}) \setminus N(S_0))$. For any two elements $t_a, t_b \notin N(\mathcal{S})$, $t_a\mathcal{C}_0$ and $t_b\mathcal{C}_0$ will be orthogonal to each other if and only if t_a and t_b are in different cosets of $N(\mathcal{S}_0)$. We will use the gauge operators G_i^X to construct our translation operators as in Theorem 35. Any error element of the error group G_n may be written (modulo a global phase) as $E = RSTUV$, where $R \in \mathcal{S}$ is an element of the quantum stabilizer, and S, T, U, and V are coset representatives of the classical stabilizer $\mathcal{S}_0/\mathcal{S}$, the logical quantum operators $N(\mathcal{S}_0)\,/\mathcal{S}_0$, the logical classical or the translation operators $N(\mathcal{S})\,/N(\mathcal{S}_0)$, and the pure errors $G_n/N(\mathcal{S})$ respectively. We now have three cases to consider: (i) $\mathrm{wt}(E) < c, d$, (ii) $c \leq \mathrm{wt}(E) < d$, and (iii) $d \leq \mathrm{wt}(E) < c$:

(i) Suppose $\mathrm{wt}(E) < c \leq d$. Then $E \notin N(\mathcal{S}) \setminus \mathcal{G}$ and $E \notin N(\mathcal{S}) \setminus N(\mathcal{S}_0)$, meaning that any error is of the form RSV. If V is not the identity, then the error can be detected, but if not then the error has no effect on either the quantum or classical information.

(ii) Suppose $c \leq \mathrm{wt}(E) < d$. Then $E \notin N(\mathcal{S}) \setminus \mathcal{G}$, so any error is of the form $RSUV$. If V is not the identity then the error can be detected, but if not then the classical information may be corrupted. However, the quantum information will be preserved.

(iii) Suppose $d \leq \mathrm{wt}(E) < c$. Then $E \notin N(\mathcal{S}) \setminus N(\mathcal{S}_0)$, so any error is of the form $RSTV$. As in (ii), if V is not the identity then the error can be detected, but if not then the quantum information may be corrupted, while the classical information will be preserved.

Therefore the hybrid code is able to detect all errors in the quantum and classical information less than their respective minimum distances. □

From the proof, we can see that encoding the classical message in the phases of the classical stabilizer generators that occurs in Theorem 35 is in effect gauge fixing. The relationship between the stabilizer and gauge groups of the original subsystem code and the quantum and classical stabilizer groups and the translation operators of the hybrid code are shown in Figure 3.2.

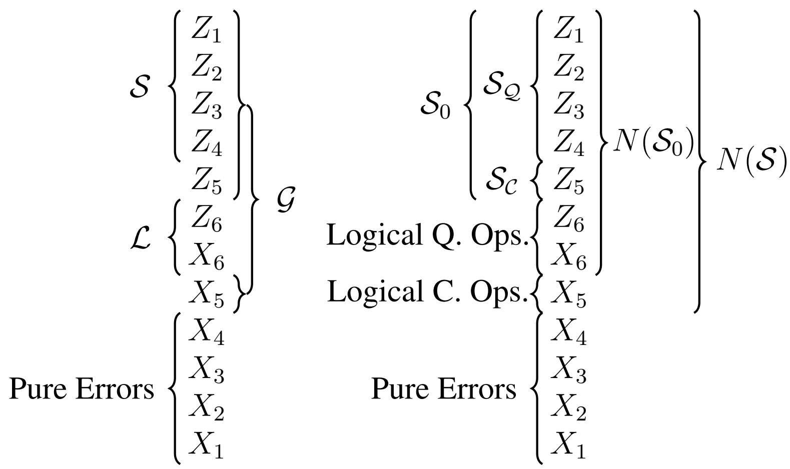

Figure 3.2: The relationship between a 6 qubit subsystem code (left) and the hybrid stabilizer code (right) derived from it, such as the one given in Example 38. In the hybrid code the translation operators are the logical classical operators and $\mathcal{S} = \mathcal{S}_\mathcal{Q}$.

Additionally, since all hybrid stabilizer codes may be written as a subsystem code, they may all be obtained using this construction. This allows us to make use of results for subsystem codes and apply them to hybrid stabilizer codes. For instance, in [58] Klappenecker and Sarvepalli showed that any $\mathbb{F}_q$-linear Clifford subsystem code satisfies the quantum Singleton bound, and it is conjectured that any subsystem code satisfies the bound [3, 58]. We extend this conjecture to hybrid stabilizer codes:

Conjecture 37. *An* $[\![n, k\!:\!m, d\!:\!c]\!]_q$ *hybrid stabilizer code satisfies the following variant of the (quantum) Singleton bound:*

$$k + m \leq n - 2\,(d-1)\,.^2$$

[2] After the publication of [77], this conjecture was partially proven by Mamindlapally and Winter in [71] for the more general case of entanglement-assisted hybrid codes, though with a single minimum distance $d = c$, using entirely information theoretic methods. In addition, the code must also satisfy the similar inequality

$$k + \frac{1}{2}m \leq n - (d-1)\,.$$

3.3.2 Examples of New Hybrid Codes

We now give several examples of new hybrid codes constructed from subsystem codes using Theorem 36.

Example 38. Using the 6-qubit subsystem code was given by Shaw et al. [91] and the construction detailed in Theorem 36, we get a $[\![6,1{:}1,3{:}2]\!]_2$ hybrid code with the following generators:

$$\left(\begin{array}{cccccc} Y & I & Z & X & X & Y \\ Z & X & I & I & X & Z \\ I & Z & X & X & X & X \\ Z & Z & Z & I & Z & I \\ \hdashline I & I & I & X & I & I \\ \hline Z & I & X & I & X & I \\ I & Z & I & I & Z & Z \\ \hline\hline I & I & I & Z & I & Z \end{array}\right).$$

Here the stabilizer generators of the subsystem code are given above the dotted line and the gauge operator G^Z is directly below it, so that the Pauli elements above the single solid line define the inner code $\mathcal{C}_0$. The logical operators on the quantum information are below the single solid line, while the logical operator on the classical information, i.e., the translation operator that takes $\mathcal{C}_0$ to $\mathcal{C}_1$ and vice versa, is given below the double solid line.

Each individual single-qubit error has a distinct syndrome, with the exception of Y_4 (the Pauli-Y on the 4th qubit), Z_4, and Z_6, which all share the same syndrome. The errors Z_4 and Z_6 each map the codeword to an orthogonal subspace, so the quantum information remains unaffected, but it is impossible to determine the classical information as there are two elements of weight 2 in the outer code's centralizer, although the presence of an error on the classical information can be detected. Since X_4 is in the stabilizer of the code, the error Y_4 may be viewed as the same as Z_4.

By using both the quantum and classical Singleton bounds, we find that there cannot be any hybrid code with equivalent parameters constructed from Proposition 32. Since the linear programming bounds for hybrid stabilizer codes [41] rule out the existence of a $[\![6,1{:}1,3]\!]_2$ code, this code is in fact genuine and saturates the bound of Conjecture 37.

Example 39. We now show how to construct a hybrid code out of Kitaev's well known $[\![18,2,3]\!]_2$ toric code [53] which can be converted into an $[\![18,2,12,3]\!]_2$ subsystem code similar to the way used by Poulin to convert Shor's 9-qubit code into a subsystem code [82]. Using the construction from Theorem 36, we can construct an $[\![18,2\!:\!12,3\!:\!2]\!]_2$ hybrid code.

$$\left(\begin{array}{cccccccccccccccccc}
X & X & I & X & X & I & X & X & I & X & X & X & X & X & X & I & I & I \\
I & X & X & I & X & X & I & X & X & I & I & I & X & X & X & X & X & X \\
Z & Z & Z & Z & Z & Z & I & I & I & Z & Z & I & Z & Z & I & Z & Z & I \\
I & I & I & Z & Z & Z & Z & Z & Z & I & Z & Z & I & Z & Z & I & Z & Z \\
\hdashline
X & I & X & I & I & I & I & I & I & X & I & I & I & I & I & X & I & I \\
X & X & I & I & I & I & I & I & I & I & X & I & I & I & I & I & X & I \\
I & I & I & X & I & X & I & I & I & X & I & I & X & I & I & I & I & I \\
I & I & I & X & X & I & I & I & I & I & X & I & I & X & I & I & I & I \\
X & X & I & X & X & I & X & X & I & I & I & I & I & I & I & I & I & I \\
I & X & X & I & X & X & I & X & X & I & I & I & I & I & I & I & I & I \\
Z & I & I & Z & I & I & I & I & I & Z & Z & I & I & I & I & I & I & I \\
I & Z & I & I & Z & I & I & I & I & I & Z & Z & I & I & I & I & I & I \\
I & I & I & Z & I & I & Z & I & I & I & I & I & Z & Z & I & I & I & I \\
I & I & I & I & Z & I & I & Z & I & I & I & I & I & Z & Z & I & I & I \\
I & I & I & I & I & I & I & I & I & Z & Z & I & Z & Z & I & Z & Z & I \\
I & I & I & I & I & I & I & I & I & I & Z & Z & I & Z & Z & I & Z & Z
\end{array}\right).$$

Using both the classical and quantum Singleton bounds together, we can see that a hybrid code with these parameters cannot be constructed from a pair of quantum and classical codes using Proposition 32. This code also saturates the bound of Conjecture 37.

All of the previous examples are of hybrid codes with $d = 3$, but the construction can be used on codes with higher minimum distances.

Example 40. Here we give an example of a $[\![12,1\!:\!1,5\!:\!4]\!]_2$ hybrid code, constructed by modifying the extended $[\![12,1,5]\!]_2$ stabilizer code from Grassl's online table of quantum codes [40] in a similar manner as in Example 19.

$$\left(\begin{array}{cccccccccccc}
X & Z & I & Z & I & X & I & Z & Z & I & I & X \\
I & Y & I & Z & Z & Y & I & I & Z & Z & I & X \\
I & Z & X & I & Z & X & I & I & I & Z & Z & X \\
I & Z & Z & Y & I & Y & Z & I & I & Z & I & I \\
I & I & Z & Z & X & X & Z & Z & I & Z & Z & I \\
I & I & Z & Z & I & I & Y & Z & Z & I & Y & I \\
I & Z & I & Z & Z & Z & Z & Y & I & Z & Y & I \\
I & I & I & Z & I & Z & I & Z & X & Z & X & I \\
I & Z & I & Z & I & I & Z & I & Z & X & X & I \\
Z & Z & Z & Z & Z & Z & I & I & I & I & I & I \\
\hdashline
I & I & I & I & I & I & I & I & I & I & I & X
\end{array}\right).$$

Since a code with these parameters cannot be constructed using Proposition 32, and no $[\![12, 2, 5]\!]_2$ code exists, this is a good code, though whether or not it is genuine depends on the existence of better codes not ruled out by bounds on hybrid codes such as linear programming bounds [41, 75].

3.4 Bacon-Casaccino Hybrid Codes

We give an explicit construction of hybrid codes using the Bacon-Casaccino family of subsystem codes. This family was introduced in the binary case by Bacon and Casaccino [14] and by Klappenecker and Sarvepalli [58] in the nonbinary case as a generalization of the Bacon-Shor subsystem codes [13, 93], and allow for the construction of subsystem codes from pairs of classical linear codes that need not be self-orthogonal. For completeness, we give the result below:

Theorem 41 (Bacon-Casaccino Codes [14, 58])**.** *For $i \in \{1, 2\}$, let $C_i \subseteq \mathbb{F}_q^{n_i}$ be an $\mathbb{F}_q$-linear code with parameters $[n_i, k_i, d_i]_q$. Then there exists a subsystem code with the parameters*

$$[\![n_1 n_2, k_1 k_2, (n_1 - k_1)(n_2 - k_2)\,, \min\{d_1, d_2\}]\!]_q\,,$$

that is pure to $d_p = \min\{d_1^\perp, d_2^\perp\}$, where $d_i^\perp$ denotes the minimum distance of $C_i^\perp$.

A subsystem code is said to be *pure to d_p* if its gauge group contains no error of weight less than d_p.

We give a brief explanation of this construction, restricting ourselves to the binary case for simplicity. Denote by P_1 and P_2 the parity-check matrices and G_1 and G_2 the generator matrices for the classical linear codes C_1 and C_2 respectively. We can use the rows of P_1 to define $n_1 - k_1$ stabilizers $S_i = \otimes_{j=1}^{n_1} Z^{(P_1)_{ij}}$ of length n_1, and the stabilizer group of the code is $\mathcal{S}_1 = \langle S_1, \ldots, S_{n_1-k_1} \rangle$, which defines a classical stabilizer code able to detect $d_1 - 1$ Pauli-X errors. Similarly, we can use P_2 to define $n_2 - k_2$ stabilizers $T_i = \otimes_{j=1}^{n_2} X^{(P_2)_{ij}}$ that generate the stabilizer group $\mathcal{S}_2$. This defines a classical stabilizer code able to detect $d_2 - 1$ Pauli-Z errors, but here the codewords are given in the Hadamard basis $\{|+\rangle, |-\rangle\}$ rather than the computational basis $\{|0\rangle, |1\rangle\}$. By classical stabilizer code, we mean a stabilizer code in which the encoded basis states are protected against noise, but a superposition of the encoded basis states are not.

To construct a quantum subsystem code out of these two classical stabilizer codes, we arrange $n_1 n_2$ qubits on an $n_1 \times n_2$ rectangular lattice. We use the stabilizers from $\mathcal{S}_1$ to operate on each column, that is each column has a copy of $\mathcal{S}_1$ acting on it, and likewise those stabilizers from $\mathcal{S}_2$ on the rows. Let $\mathcal{T}_1$ be the abelian group generated by $\mathcal{S}_1$ acting on the columns and $\mathcal{T}_2$ the abelian group generated by $\mathcal{S}_2$ acting on the rows. The group $\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle$ is nonabelian, but we can construct an abelian subgroup of $\mathcal{T}$ that commutes with every element in $\mathcal{T}$ using the following construction: take an element $S \in \mathcal{S}_1$ and a codeword $v \in C_2$, and construct an element of $\mathcal{T}_1$ where S^{v_j} acts on column j. In addition to commuting with all of the elements of $\mathcal{T}_1$, every element of this form also commutes with all of the elements of $\mathcal{T}_2$. Likewise we can construct elements in $\mathcal{T}_2$ that commute with all elements in $\mathcal{T}$. Together, these elements generate the stabilizer group $\mathcal{S}$ of the subsystem code.

Example 42. As an example we present the 9-qubit Bacon-Shor code, a subsystem code version of the original 9-qubit Shor code. Start with $C_1 = C_2$ as the length 3 repetition code with generator matrix G and parity-check matrix P given by

$$G = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Using the construction, we find that the stabilizer of the code is given by

$$\mathcal{S} = \left\langle \boxed{\begin{matrix} Z & Z & Z \\ Z & Z & Z \\ I & I & I \end{matrix}}, \boxed{\begin{matrix} I & I & I \\ Z & Z & Z \\ Z & Z & Z \end{matrix}}, \boxed{\begin{matrix} X & X & I \\ X & X & I \\ X & X & I \end{matrix}}, \boxed{\begin{matrix} I & X & X \\ I & X & X \\ I & X & X \end{matrix}} \right\rangle .$$

Here the stabilizers appear as they would on the 3×3 lattice of qubits.

We can also choose our gauge operators in such a way so that they form four anticommuting pairs $\left(G_i^Z, G_i^X\right)$:

$$\left(\boxed{\begin{matrix} Z & I & I \\ Z & I & I \\ I & I & I \end{matrix}}, \boxed{\begin{matrix} X & I & X \\ I & I & I \\ I & I & I \end{matrix}} \right), \left(\boxed{\begin{matrix} I & I & I \\ Z & I & I \\ Z & I & I \end{matrix}}, \boxed{\begin{matrix} I & I & I \\ I & I & I \\ X & I & X \end{matrix}} \right),$$
$$\left(\boxed{\begin{matrix} I & Z & I \\ I & Z & I \\ I & I & I \end{matrix}}, \boxed{\begin{matrix} I & X & X \\ I & I & I \\ I & I & I \end{matrix}} \right), \left(\boxed{\begin{matrix} I & I & I \\ I & Z & I \\ I & Z & I \end{matrix}}, \boxed{\begin{matrix} I & I & I \\ I & I & I \\ I & X & X \end{matrix}} \right).$$

Note that if we pick a gauge and look at the subspace stabilized by the abelian group $\left\langle \mathcal{S}, G_1^Z, G_2^Z, G_3^Z, G_4^Z \right\rangle$, it is the same as the original 9-qubit Shor code (up to a permutation of the qubits).

We can apply our hybrid code construction from Theorem 36 to the Bacon-Casaccino subsystem codes to construct hybrid codes out of a pair of linear codes.

Theorem 43. *For $i \in \{1, 2\}$, let $C_i \subseteq \mathbb{F}_q^{n_i}$ be an $\mathbb{F}_q$-linear code with parameters $[n_i, k_i, d_i]_q$. Then there exists a hybrid code with the parameters*

$$[\![n_1 n_2, k_1 k_2 : (n_1 - k_1)(n_2 - k_2)\, , d : c]\!]_q \, ,$$

where $d = \min\{d_1, d_2\}$, $c \geq \min\{d, \max\{d_1^\perp, d_2^\perp\}\}$, and $d_i^\perp$ denotes the minimum distance of $C_i^\perp$.

Proof. Without loss of generality, suppose that $d_2^\perp \geq d_1^\perp$. Using Theorems 36 and 41, we construct a hybrid code, gauge fixing all of the $G_i^{Z(a)}$ operators. The only thing that needs to be checked is the classical distance $c = \mathrm{wt}(N(\mathcal{S}) \setminus N(\mathcal{S}_0))$. Since all of the translation operators are tensor products of X-type operators and the identity matrix, we only need to consider the minimum distance of operators of this type. Note that it may be possible to do better than this by picking both G_i^Z and G_j^X operators that commute to be fixed. Suppose that $d_2^\perp \leq d$. Then $\mathcal{G}$ does not contain any X-type operators of weight less than $d_2^\perp$, so $N(\mathcal{S})$ also does not contain any X-type operators of weight less than $d_2^\perp$, giving us the lower bound $c \geq d_2^\perp$. If $d \leq d_2^\perp$, there may be an element of $(N(\mathcal{S}) \setminus N(\mathcal{S}_0)) \setminus \mathcal{G}$ of weight less than $d_2^\perp$, since a logical quantum X-type operator and a translation operator together might have a weight less than each operator separately. However, this weight will still be lower bounded by d. Following the same argument with $d_1^\perp \geq d_2^\perp$ gives us the lower bound on the classical minimum distance. □

Example 44. We will continue to use the 9-qubit Bacon-Shor subsystem code from Example 42 and show how to turn it into a $[\![9, 1{:}4, 3{:}2]\!]_2$ hybrid code. Gauge fix the subsystem code by letting $\langle \mathcal{S}, G_1^Z, G_2^Z, G_3^Z, G_4^Z \rangle$ be the stabilizer of the code $\mathcal{C}_0$. To send the classical binary message $m = m_1 m_2 m_3 m_4$, use $\left(G_1^X\right)^{m_1} \left(G_2^X\right)^{m_2} \left(G_3^X\right)^{m_3} \left(G_4^X\right)^{m_4}$ as the translation operator.

Similar to the previous examples, we can use the quantum and classical Singleton bounds to show that this code is superior to any hybrid code constructed using Proposition 32.

As mentioned above, this code cannot be compared to any of the other length 9 hybrid codes mentioned in this paper, but it does have the distinction of being able to transmit the conjectured maximal amount of total information, as it achieves the bound in Conjecture 37. In fact, this property is shared by all of the hybrid Bacon-Shor codes:

Corollary 45. *Hybrid codes with parameters*

$$[\![n^2, 1{:}(n-1)^2, n{:}2]\!]_2$$

can be constructed from Bacon-Shor subsystem codes, and no code constructed using Proposition 32 can have these parameters. Furthermore, these codes saturate the bound given in Conjecture

37.

3.5 Application to Faulty Syndrome Measurement Errors

In this section we show how hybrid codes can be applied to help mitigate the effects of faulty syndrome measurements, and how they can be used to inspire the construction of new quantum data-syndrome codes. In fault-tolerant quantum computing, we look to prevent errors caused by quantum gates operating on the quantum information. In some cases, such as trapped ion technologies, it may be assumed that the error rate for measurements (in the Z-basis), which are called faulty syndrome measurements, is much greater than the error rate for all other quantum gates [8]. To model this we will assume that the error rate for all non-measurements is 0, so as to focus on errors caused by faulty measurements.

In [96], Shor gave a syndrome-extraction scheme for stabilizer codes in which each stabilizer generator is measured multiple times. This effectively means that each bit of the syndrome is protected against faulty-syndrome errors with its own separate repetition code. Later, more efficient schemes for protecting against these errors were given by Ashikhmin, Lai, and Brun [8, 9, 10] and Fujiwara [36] in their formulation of quantum data-syndrome codes.

3.5.1 Correcting Faulty Syndrome Measurement Errors with Hybrid Codes

The main idea behind our scheme will be to start with a hybrid code where the logical classical information is in a known state (without loss of generality we assume that the m encoded classical bits are set to $|0^m\rangle$). As the logical classical bits are fixed to a set state, they cannot be used for transmitting information across the quantum channel. Rather, we will use the extra redundency they provide to protect against faulty syndrome measurements. Once the quantum information has been received, the decoding procedure occurs in two steps: first, we extract the syndromes of the quantum stabilizers onto the logical classical bits (in particular, we will put multiple syndromes onto the same encoded classical bit to form a parity bit for the syndrome), and then we extract both the quantum syndrome as well as the classical information as normal.

For a stabilizer code, syndrome extraction is typically done by measuring each stabilizer gen-

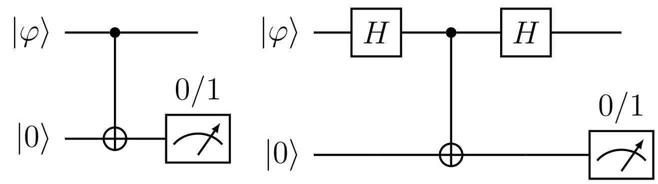

Figure 3.3: Quantum circuit for measuring Z and X operators (left and right respectively) with one ancilla qubit.

erator using an ancilla qubit, where each non-identity component of the generator is placed on the ancilla using the quantum circuits from Figure 3.3. In effect we are constructing an encoded CNOT gate whose control is the encoded redundancy qubit associated with the stabilizer generator being measured and whose target is the ancilla qubit to be measured. We would now like to construct an encoded CNOT gate whose target is instead the encoded classical bit of our hybrid code.

Define the state $|a_1 \dots a_k b_1 \dots b_m c_1 \dots c_{n-k-m}\rangle_L$ as the state generated by the abelian group

$$\left\langle (-1)^{a_1}\, \overline{Z}_1, \dots, (-1)^{a_k}\, \overline{Z}_k, (-1)^{b_1}\, g_1, \dots, (-1)^{b_m}\, g_m, (-1)^{c_1}\, s_1, \dots, (-1)^{c_{n-k-m}}\, s_{n-k-m} \right\rangle,$$

where $\overline{Z}_i$ are the logical operators on the encoded quantum information, g_i the generators of $\mathcal{S}_{\mathcal{C}}$, and s_i the generators of $\mathcal{S}_{\mathcal{Q}}$. This is an extended version of our encoded basis states, where we additionally have the added "redundancy" qubits added during the encoding process. An encoded CNOT gate with the i-th encoded redundancy qubit and j-th encoded classical qubit as its control and target respectively is given by

$$\sum_{\substack{\mathbf{a}\in\mathbb{F}_2^k \\ \mathbf{b}\in\mathbb{F}_2^m \\ \mathbf{c}\in\mathbb{F}_2^{n-k-m}}} |a_1 \dots a_k b_1 \dots b_{j-1}\, (b_j \oplus c_i)\, b_{j+1} \dots b_m c_1 \dots c_{n-k-m}\rangle \langle \mathbf{abc}| \,. \tag{3.1}$$

Example 46. We will demonstrate this scheme for protecting against faulty syndrome measurement errors with the $[\![7, 1\!:\!1, 3]\!]_2$ hybrid code from Example 20, and we will place all five error syndrome measurements and redirect them to the encoded classical qubit, which when mea-

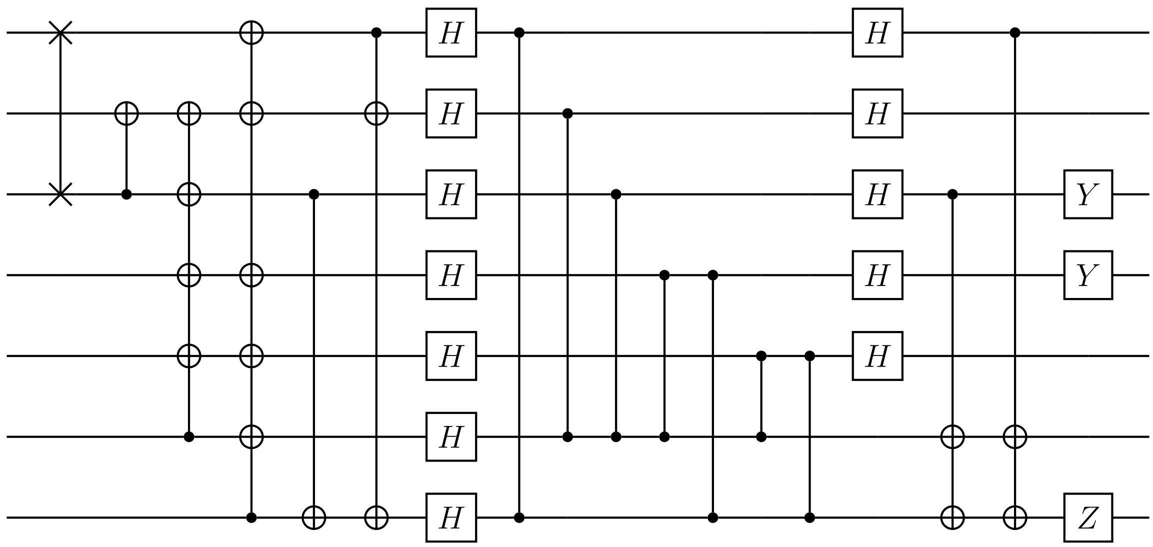

Figure 3.4: Quantum circuit for extracting the error syndrome from the first stabilizer generator of the $[\![7,1\!:\!1,3]\!]_2$ hybrid code from Example 20 and placing it on the encoded classical qubit.

sured will act as a parity bit for the error syndrome. We can get the encoded logical operators $\overline{X} = ZIXIXII$ and $\overline{Z} = IZIIZZI$ by taking the logical operators of the original $[\![6,1,3]\!]_2$ code from [91] and extending them each by a single identity operator.

The quantum circuit that performs the extraction of the error syndrome associated with the first stabilizer generator and places it on the encoded classical qubit is given in Figure 3.4. The quantum circuits implementing the other four encoded CNOT gates are included in Appendix A. These quantum circuits were decomposed using the software[3] implementing the work of Can et al. in [24].

We then proceed to extract both the error syndromes associated with the quantum stabilizer generators as well as the classical information which is acting similar to a parity bit for the error syndrome. The error syndromes of all the single-qubit Pauli errors as well as any single bit-flip on the error syndrome are given in Table 3.1. Note that with the exception of Y_7 and Z_7, all single-qubit Pauli errors have distinct syndromes of even weight, meaning that all weight-1 syndromes are free to be used by single bit-flips caused by faulty syndrome measurements. The syndrome shared

[3]Available at `https://github.com/nrenga/symplectic-arxiv18a`.

Error	Syndrome	Error	Syndrome	Error	Syndrome
X_1	110011	Y_1	010010	Z_1	100001
X_2	001010	Y_2	011011	Z_2	010001
X_3	100010	Y_3	101011	Z_3	001001
X_4	000101	Y_4	101101	Z_4	101000
X_5	000011	Y_5	111010	Z_5	111001
X_6	110101	Y_6	011101	Z_6	101000
X_7	000000	Y_7	000111	Z_7	000111
F_1	100000	F_2	010000	F_3	001000
F_4	000100	F_5	000010	F_6	000001

Table 3.1: Syndromes of single-qubit Pauli errors and faulty syndrome measurement errors for the $[\![7,1\!:\!1,3]\!]_2$ hybrid code. Here F_i represents a bit-flip error on the i-th bit of the error syndrome caused by a faulty measurement.

by Y_7 and Z_7 is a result of measuring our classical stabilizer generator from $\mathcal{S}_\mathcal{C}$, but as it is not a weight-1 syndrome there is no overlap with the syndromes used by the faulty syndrome measurement errors. Additionally, we do not need to distinguish between the two errors as applying either a Y- or a Z-operator to the final qubit is sufficient to correct either error. Therefore, this scheme can correct either one error to the quantum information or one faulty syndrome measurement.

3.5.2 New Quantum Data-Syndrome Codes

An alternate approach to protecting against faulty syndrome measurement errors is the use of quantum data-syndrome codes [10, 36]. These codes generalize Shor's syndrome extraction technique [96] by extracting combinations of the stabilizer generators instead of extracting each of the $n-k$ generators multiple times. We call an $[\![n,k,d]\!]_q$ stabilizer code an $[\![n,k,d:r]\!]_q$ quantum data-syndrome code if it can correct $\lfloor (d-1)/2 \rfloor$ errors either to the quantum information or to the extracted error syndrome by measuring $n-k+r$ stabilizer elements (i.e., r additional elements beyond the normal $n-k$).

In [36], Fujiwara showed that any $[\![n,k,3]\!]_2$ stabilizer code can be transformed into an $[\![n,k,3:1]\!]_2$ quantum data syndrome code by measuring an additional stabilizer element combining all of the $n-k$ stabilizer generators. The following question is also posed: when is it possible to have quantum data-syndrome codes with $r=0$? We give three new quantum data-syndrome codes with parameters $[\![6,1,3:0]\!]_2$, $[\![7,1,3:0]\!]_2$, and $[\![9,3,3:0]\!]_2$ inspired by our decoding scheme from Section 3.5.1.

Example 47. We start by looking at the $[\![7,1{:}1,3]\!]_2$ hybrid code we used in Example 46. The process of extracting the error syndromes of the stabilizer generators and placing it on the encoded classical qubit is equivalent to measuring a modified stabilizer element instead of our classical stabilizer generator. In our example, if we take s_i to be the syndrome of the i-th stabilizer generator S_i and s' the syndrome of the classical stabilizer generator S', then when we measure the classical stabilizer generator S' we will measure $s_1 \oplus \cdots \oplus s_5 \oplus s'$. However this is the same syndrome that would be measured if did not do the encoded CNOT gates in Figure 3.4 and Appendix A and measured the stabilizer element $S_1 \cdots S_5 S'$ instead of S'. This gives us a code with the following stabilizer generators:

$$\begin{pmatrix} Y & I & Z & X & X & Y & I \\ Z & X & I & I & X & Z & I \\ I & Z & X & X & X & X & I \\ I & I & I & Z & I & Z & X \\ Z & Z & Z & I & Z & I & X \\ Y & X & X & Z & Y & Z & X \end{pmatrix}. \tag{3.2}$$

This code has the same syndromes as the hybrid code when used in conjunctions with the encoded CNOT gates as in Example 46, which are given in Table 3.1, making it a $[\![7,1,3:0]\!]_2$ quantum data-syndrome code. This code is the same as the inner code $\mathcal{C}_0$ of the hybrid code, although with a different choice of generators. In [36], Fujiwara showed that the 7-qubit Steane code [98] could also be used as a quantum data-syndrome code with the same parameters as the code in this example, albeit with a non-standard choice of stabilizer generators. The code here is non-equivalent to the Steane code, and in particular it is impure, which appears to give us some potential advantage when constructing the codes, as there are more error syndromes available. However, since the Steane code also has $r = 0$, it appears that impurity is not necessary for the construction of good quantum data-syndrome codes.

In the same way, we can additionally construct a new $[\![9,3,3:0]\!]_2$ quantum data syndrome code from the $[\![9,3\!:\!1,3]\!]_2$ hybrid code whose stabilizer generators were given in Equation 2.8:

$$\begin{pmatrix} X & X & X & X & X & X & X & X & I \\ Z & Z & Z & Z & Z & Z & Z & Z & I \\ X & I & X & I & Z & Y & Z & Y & I \\ X & I & Y & Z & X & I & Y & Z & X \\ X & Z & I & Y & I & Y & X & Z & X \\ Z & X & X & Z & I & Y & Y & I & X \end{pmatrix}. \tag{3.3}$$

Here, we must make a modification and add an extra X-operator in the last column, but again this is still the same inner code as the hybrid code. This modification makes it so that the errors Y_9 and Z_9 do not have syndromes of weight 1, which would conflict with the syndromes designated for faulty syndrome measurement errors.

Example 48. Finally, we give a $[\![6,1,3:0]\!]_2$ quantum data-syndrome code from the $[\![6,1\!:\!1,3\!:\!2]\!]_2$ hybrid code from Example 38, using the same approach as in Example 47:

$$\begin{pmatrix} Y & I & Z & X & X & Y \\ Z & X & I & I & X & Z \\ I & Z & X & X & X & X \\ Z & Z & Z & I & Z & I \\ Y & X & X & X & Y & I \end{pmatrix}.$$

It is interesting that a hybrid code with $c < d$ is sufficient to construct a quantum data-syndrome code with a minimum distance of d. This might suggest that the classical minimum distance is less important than the existence of an unused subsystem that the hybrid code provides, meaning that subsystem codes might be a natural choice for constructing future quantum data-syndrome codes with $r = 0$.

We end this section with a brief comparison between the two strategies given for dealing with faulty syndrome extraction. The first strategy using hybrid codes has the advantage of measuring the classical stabilizer generators, which are typically low-weight and are less likely to cause faulty syndrome-extraction errors. On the other hand, they have the disadvantage of having a large overhead in the form of the encoded CNOT gates in Figure 3.4 and Appendix A, which has the potential for introducing more errors if that possibility is not ignored as we did in this section, requiring the use of fault-tolerant encoded gates. The second strategy of using quantum data-syndrome codes is the opposite: we have no need to worry about errors occurring during a preprocessing stage, but we now have higher-weight stabilizer generators to contend with. The trade-offs between these two strategies is interesting and requires more study in the future.

3.6 Conclusion

In this paper we have shown how to encode classical information in the gauge qudits of subsystem codes, allowing us to use previously unused logical qudits to transmit information. The hybrid codes that arise from this construction are allowed to have separate minimum distances for the quantum and classical information. We give several examples of good hybrid codes using this construction on subsystem codes, as well as use the Bacon-Casaccino subsystem code construction to construct hybrid stabilizer codes from a pair of classical codes and their duals, including the Bacon-Shor hybrid codes constructed from the classical repetition codes. We also conjecture that hybrid stabilizer codes must satisfy a variant of the quantum Singleton bound, which follows from a similar conjecture for subsystem codes. Finally, we give an application where hybrid codes are used to protect against faulty syndrome measurement errors and show how hybrid codes can inspire the construction of new quantum data-syndrome codes.

Previous work on hybrid codes required the construction of good families of degenerate quantum codes to construct families of genuine hybrid codes. By relating hybrid codes to the well-studied class of subsystem codes and separating the quantum and classical minimum distances of the code, it should be easier to find families of genuine hybrid codes. One important question raised by having separate minimum distances is that of bounds for hybrid codes when $c \neq d$. In Conjecture 37, the variant of the quantum Singleton bound does not put any restrictions on the classical distance, so finding bounds such as the linear programming bounds for hybrid codes [41, 75] that put restrictions on both minimum distances would allow for a better understanding of these codes. Other topics of future research include the cases where errors to either the quantum or classical information are corrected while errors to the other are only detected, as we only considered the cases where errors were either both detected or both corrected, as well as exploring the connection between good quantum data-syndrome codes and hybrid or subsystem codes.

4. WEIGHT ENUMERATORS FOR NONADDITIVE CODES

The material in this chapter comes from the paper [76][1]. In this chapter, all codes are assumed to be binary (i.e., over $\mathbb{F}_2$) and we use the notation $C_{G_n}(\mathcal{S})$ for the centralizer of the code instead of the notation $N(\mathcal{S})$ used in the rest of the paper.

4.1 Introduction

The weight enumerators of classical error-correcting codes arise in the derivation of the upper bounds for code parameters via linear programming [30]. These weight enumerators admit a combinatorial interpretation, as for additive codes they count the number of codewords of each weight in the code, and more generally for nonadditive codes they describe the distances between each pair of codewords. The weight enumerators of a code and its dual code are connected by the MacWilliams identity [69].

Shor and Laflamme defined a pair of weight enumerators for quantum codes [94] which were used by Ashikhmin and Litsyn to develop linear programming bounds for the parameters of quantum codes [11]. Rains also defined the similar unitary enumerators [84], as well as the quantum analogue of the shadow enumerators [87] which provide sharper linear programming bounds when used in conjunction with the Shor-Laflamme weight enumerators. Weight enumerators have also been used to derive linear programming bounds for other quantum code variants, such as subsystem codes [4], asymmetric quantum codes [90], hybrid codes [41], entanglement-assisted codes [65], and quantum amplitude damping codes [80].

While the Shor-Laflamme weight enumerators do not in general have a known combinatorial interpretation similar to the weight enumerators of classical codes, they do for the well-known class of stabilizer codes. The $A(z)$ Shor-Laflamme weight enumerator counts the number of elements of each weight in the stabilizer group associated with the stabilizer code, while the $B(z)$ Shor-

[1]Results reproduced with permission from "A Combinatorial Interpretation for the Shor-Laflamme Weight Enumerators of CWS Codes" by Andrew Nemec and Andreas Klappenecker, 2022. To appear in *IEEE Transaction on Information Theory*. Copyright 2022 by IEEE.

Laflamme weight enumerator does the same for elements in the centralizer of the stabilizer group [39]. These correspond to the weight enumerators of a classical self-orthogonal additive code and its dual code.

For non-stabilizer quantum codes (known as nonadditive quantum codes), little is known about the weight enumerators. The unitary enumerators may be interpreted as the binomial moments of the distance distribution of classical codes [12], and the Shor-Laflamme weight enumerator $A(z)$ can be interpreted as the two-norms of the j-body correlations of the code [51], but their remains no combinatorial interpretation of the Shor-Laflamme weight enumerator $B(z)$. In this paper, we show that for the nonadditive codeword stabilized codes [29], the Shor-Laflamme weight enumerator $B(z)$ may be interpreted as the distance enumerator of an associated nonadditive classical code, partially answering a question recently posed by Ball, Centelles, and Huber [15].

4.2 Background

For classical codes, the distance between codewords is given by the Hamming distance:

$$d_H(x, y) = |\{i \mid x_i \neq y_i\}| \,.$$

The distance distribution A of an (n, M, d) classical code C is a vector of length $(n + 1)$, where

$$A_i = \frac{1}{M} |\{(x, y) \mid x, y \in C, d_H(x, y) = i\}| \,,$$

meaning that A_i is the number of codewords at distance i from each other, normalized by the size of the code. The polynomial

$$A(z) = \sum_{i=0}^{n} A_i z^i$$

is the distance enumerator of the code. The minimum distance d of the code is the smallest index $i \neq 0$ such that A_i is non-zero.

The Hamming weight of a codeword is the distance from the all zero codeword, that is $\mathrm{wt}_H(x) = d_H(x, 0^n)$. If C is an additive code, that is a code which is closed under addition, then A counts the number of codewords of each weight, so

$$A_i = |\{x \mid x \in C, \mathrm{wt}(x) = i\}|,$$

and we call A the weight distribution and $A(z)$ the weight enumerator of the code. The weight enumerator of an additive code C is connected to the weight enumerator $B(z)$ of its dual code $C^\perp$ by the MacWilliams identity [30, 69]:

$$B(z) = \frac{(1+z)^n}{M} A\left(\frac{1-z}{1+z}\right).$$

For a nonadditive code, the MacWilliams identity may still be formally defined in the same way, although the resultant polynomial in general does not correspond to the distance enumerator of any code [68, 28].

An $((n, K, d))$ quantum code $\mathcal{C}$ on n physical qubits is a K-dimensional subspace of the Hilbert space $\mathbb{C}^{2^n}$. Let X and Z be the Pauli operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A basis for the linear operators of the Hilbert space can be given by tensor products of the Pauli operators:

$$\mathcal{E}_n = \left\{ E_1 \otimes \cdots \otimes E_n \mid E_i = X^{a_i} Z^{b_i}, a_i, b_i \in \mathbb{F}_2 \right\}.$$

Each element $E \in \mathcal{E}_n$ can be associated with a unique codeword $(a \mid b) = (a_1, \ldots, a_n \mid b_1, \ldots, b_n)$ of length $2n$. The distance between two codewords of this type is given by the symplectic distance:

$$d_s((a \mid b), (a' \mid b')) = |\{k \mid (a_k, b_k) \neq (a'_k, b'_k)\}|.$$

The symplectic weight $\mathrm{wt}(E)$ is the number of non-identity tensor components E_i make up E.

The most well studied class of quantum codes are the stabilizer codes [39]. An $[\![n,k,d]\!]$ stabilizer code is defined by its stabilizer group $\mathcal{S}$, which is generated by $n-k$ mutually commuting independent operators $S_i \in G_n$ (which does not include $-I$), where

$$G_n = \left\{ i^{\ell} E \mid \ell \in \mathbb{Z}_4, E \in \mathcal{E}_n \right\}$$

is the error group on n qubits. The stabilizer code is then the 2^k-dimensional joint $+1$-eigenspace of $\mathcal{S}$. Associated with the stabilizer group is its centralizer in $C_{G_n}(\mathcal{S})$, the group of all elements in G_n that commute with every element in $\mathcal{S}$. These are the operators that act as the logical operators on the encoded states of the code.

Shor and Laflamme [94] defined a pair of weight enumerators $A(z)$ and $B(x)$ for quantum codes in the following fashion:

$$A_i = \frac{1}{K^2} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=i}} \mathrm{tr}(EP)\,\mathrm{tr}(E^* P)$$

and

$$B_i = \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=i}} \mathrm{tr}(EPE^* P)\,,$$

where P is the orthogonal projector onto the code $\mathcal{C}$. In general, the weight enumerators of quantum codes do not seem to admit as nice a combinatorial interpretation as they do for classical codes. However, for stabilizer codes there is such an interpretation, as $A(z)$ counts the number of elements of each weight in the stabilizer group $\mathcal{S}$ and $B(z)$ counts the number of elements in the centralizer $C_{G_n}(\mathcal{S})$ (modulo the phases on the Pauli elements). Additionally, each element of the stabilizer and centralizer can be associated with a unique (up to phase) codeword of length $2n$. Let C be the code containing the set of codewords associated with $\mathcal{S}$. Then its symplectic dual $C^{\perp}$ is the code associated with $C_{G_n}(\mathcal{S})$. Additionally, since $\mathcal{S} \le C_{G_n}(\mathcal{S})$ (as $\mathcal{S}$ is Abelian), we have that $C \subseteq C^{\perp}$, that is C is self-orthogonal.

4.3 Weight Enumerators of CWS Codes

Codeword stabilized (CWS) codes were introduced by Cross et al. [29] as a framework to construct quantum codes that includes all stabilizer codes and most nonadditive codes with good parameters, for example, see [88, 104, 103]. A CWS code is comprised of two objects: a stabilizer group $\mathcal{S}$ generated by n mutually commuting independent elements of G_n and containing no scalar multiples of I, so $\mathcal{S}$ stabilizes a single stabilizer state $|\varphi\rangle$ which comprises a 1-dimensional stabilizer code, and a collection $\mathcal{T}$ of K commuting codeword operators $T_i \in G_n$ such that each T_i is from a separate coset of $G_n/\mathcal{S}$. Without loss of generality, we can choose $T_1 = I$. We note that the results of this paper also apply to the union stabilizer code construction of Grassl and Rötteler [42], as it is an alternative way to derive the CWS construction [43].

The set $\mathcal{TS} = \left\{ i^\ell T_i S_j \mid T_i \in \mathcal{T}, S_j \in \mathcal{S}, \ell \in \mathbb{Z}_4 \right\}$ plays a similar role to the centralizer of a stabilizer code, and a CWS code is a stabilizer code precisely when $\mathcal{TS}$ forms an abelian group under multiplication. We associate with this set a classical (and in general nonadditive) code C, constructed using the additive code D associated with the stabilizer group $\mathcal{S}$ and the classical codewords t_i associated with the codeword operators T_i, so that

$$C = \bigcup_{i=1}^{12} (t_i + D). \tag{4.1}$$

This code is referred to as the *union normalizer code* by Grassl and Rötteler [42]. As an example, the components describing the $((9, 12, 3))$ CWS code from [103] is given in Figure 4.1. We show now that the Shor-Laflamme weight enumerator $B(z)$ of a CWS code is the distance enumerator of this classical code C associated with the set $\mathcal{TS}$.

Theorem 49. *Let $\mathcal{C}$ be a CWS quantum code. Let C be the classical symplectic code associated with the set $\mathcal{TS}$. Then the Shor-Laflamme weight enumerator $B(z)$ of the quantum code $\mathcal{C}$ is the distance enumerator of the classical code C.*

Proof. Let P be the projector onto $\mathcal{C}$, $\mathcal{S}$ be the stabilizer group of the stabilizer state $|\varphi\rangle$, and

$\mathcal{T} = \{T_1, T_2, \ldots, T_K\}$ be the set of codeword operators. We can write P as

$$P = \sum_{i=1}^{K} T_i \left|\varphi\right\rangle \left\langle\varphi\right| T_i^*.$$

We can expand the weight enumerator as

$$\begin{aligned} B_d &= \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \mathrm{tr}(EPE^*P) \\ &= \frac{1}{K} \sum_{\substack{E \in \mathcal{E}_n \\ \mathrm{wt}(E)=d}} \sum_{i,j=1}^{K} \left|\left\langle\varphi\right| T_j^* E T_i \left|\varphi\right\rangle\right|^2. \end{aligned}$$

Since elements of the Pauli group either commute or anticommute with each other, it follows that $\left|\left\langle\varphi\right| T_j^* E T_i \left|\varphi\right\rangle\right|^2 = 1$ for all error operators $E \in T_i^* T_j \mathcal{S}$. Furthermore, if $E \notin T_i^* T_j \mathcal{S}$, we may write $E = T'F$, where $F \in \mathcal{S}$ and $T' \neq T_i^* T_j = T_i T_j^*$ is a coset representative of $T'\mathcal{S} \neq T_i^* T_j \mathcal{S}$. Then

$$\begin{aligned} \left\langle\varphi\right| T_j^* E T_i \left|\varphi\right\rangle &= \pm \left\langle\varphi\right| T_j^* T_i T' F \left|\varphi\right\rangle \\ &= \pm \left\langle\varphi\right| T_j^* T_i T' \left|\varphi\right\rangle \\ &= 0, \end{aligned}$$

as there must be a stabilizer element $s \in \mathcal{S}$ that anticommutes with $T_i^* T_j T'$, and so

$$\begin{aligned} \pm \left\langle\varphi\right| T_j^* T_i T' \left|\varphi\right\rangle &= \pm \left\langle\varphi\right| s T_j^* T_i T' \left|\varphi\right\rangle \\ &= \mp \left\langle\varphi\right| T_j^* T_i T' s \left|\varphi\right\rangle \\ &= \mp \left\langle\varphi\right| T_j^* T_i T' \left|\varphi\right\rangle, \end{aligned}$$

implying that $\langle\varphi| T_j^* T_i T' |\varphi\rangle = 0$. Therefore, we have that

$$\left|\langle\varphi| T_j^* E T_i |\varphi\rangle\right|^2 = \begin{cases} 1, & E \in T_i^* T_j \mathcal{S} \\ 0, & \text{otherwise} \end{cases}$$

Let S be the set of 2^n classical codewords associated with the elements of $\mathcal{S}$, $T = \{t_1, t_2, \ldots, t_k\}$ the set of classical codewords associated with $\mathcal{T}$, and F a set of 2^n classical codewords such that $T \subset F$ and $FS = \{0,1\}^{2n}$. Any codeword may be written as $f + s$, where $f \in F$ and $s \in S$. Given two pairs of codewords $c = (f + s, g + u), c' = (f' + s', g' + u') \in \{0,1\}^{4n}$, we define the equivalence relation $\sim$ such that $c \sim c'$ if and only if $s + u = s' + u'$, $f = f'$, and $g = g'$. It is straightforward to check that $\sim$ is indeed a reflexive, symmetric, and transitive relation, and therefore an equivalence relation.

Since S is a normal subgroup of $(FS)^2$, the quotient group $(FS)^2/S$ is isomorphic to F^2S. Denote elements of this set by $f + g + w$, where $f, g \in F$, $w \in S$. For all pairs of codewords $(f + s, g + u)$ in the same partition, $d_s(f + s, g + u) = \mathrm{wt}_s(f + g + w)$, where $w = s + u$. This means that for the classical code $C = TS$, the distance distribution

$$\begin{aligned} A_d =& \frac{1}{K2^n} \left|\{(t_i + s, t_j + u) \mid d_s(t_i + s, t_j + u) = d\}\right| \\ =& \frac{1}{K} |\{t_i + t_j + w \mid t_i, t_j \in T, w \in S, \\ & \quad \mathrm{wt}_s(t_i + t_j + w) = d\}|. \end{aligned}$$

We associate t_i, t_j, and w with the quantum operators T_i, T_j, and W. Note that $\left|\langle\varphi| T_j^* E T_i |\varphi\rangle\right|^2 = 1$ if and only if $E = T_i^* T_j W$, meaning that the weight distribution of the quantum code is identical to the distance distribution of the classical code. In the case that there are two (or more) pairs of codeword operators such that $T_i T_j = T_k T_\ell$, the operator E might be counted twice by both $\left|\langle\varphi| T_j^* E T_i |\varphi\rangle\right|^2$ and $|\langle\varphi| T_\ell^* E T_k |\varphi\rangle|^2$, but this is offset by E not being checked separately as $E = T_i^* T_j W$ and $E = T_k^* T_\ell W$.

$$
G = \left(\begin{array}{ccccccccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}\right)
$$

Figure 4.1: Generating matrix and coset representatives for the classical code associated with the $((9, 12, 3))$ CWS quantum code, with the generating matrix for the linear code D above the dashed line and the 12 codewords in T below it.

This shows that the Shor-Laflamme weight enumerator $B(z)$ of the quantum code $\mathcal{C}$ is the same as the distance enumerator of its associated classical code C. $\square$

Using the $((9, 12, 3))$ CWS code constructed by Yu et al. [103] as an example, the code has Shor-Laflamme weight enumerators

$$A(z) = 1 + \frac{2}{3}z^4 + \frac{32}{3}z^6 + \frac{64}{3}z^7 + 9z^8$$

and

$$B(z) = 1 + 68z^3 + 242z^4 + 684z^5 + 1464z^6 + 1852z^7 + 1365z^8 + 468z^9.$$

The nonadditive classical code C associated with the set $\mathcal{TS}$ is constructed using Equation 4.1, where the linear code D is generated by the 9 codewords above the dashed line in Figure 4.1, and the t_i are the 12 codewords of the set T below the dashed line. Calculating the symplectic distance between each pair of codewords, we find that the distance enumerator

$$A'(z) = 1 + 68z^3 + 242z^4 + 684z^5 + 1464z^6 + 1852z^7 + 1365z^8 + 468z^9$$

is identical to the Shor-Laflamme weight enumerator $B(z)$ of the quantum code.

One interesting observation about the $((9, 12, 3))$ code is that the distance enumerator $A'(z)$ of the associated classical code C has all integral-valued coefficients which count the number of codewords of each weight in the code, so the Shor-Laflamme weight enumerator $B(z)$ also counts the number of elements of each weight in $\mathcal{TS}$ like a Shor-Laflamme weight enumerator for stabilizer codes. This also holds true for the $((10, 24, 3))$ CWS code constructed by Yu et al. [104], but not for the $((7, 22, 2))$ CWS code constructed by Smolin et al. [97], the weight enumerator of which has nonintegral coefficients.

4.4 Conclusion

In this paper we give a combinatorial interpretation for the Shor-Laflamme weight enumerator $B(z)$ for codeword stabilized codes, by connecting the centralizer analogue to a classical code. One question that remains is whether there is a similar combinatorial interpretation for the Shor-Laflamme weight enumerators for nonadditive codes not equivalent to CWS codes. Another question is which CWS codes are similar to the $((9, 12, 3))$ and $((10, 24, 3))$ codes whose weight enumerators $B(z)$ count the number of elements in $\mathcal{TS}$, similar to the case with stabilizer codes.

5. CONCLUSION AND FUTURE DIRECTIONS

In this dissertation, we expand the theory of quantum-classical hybrid codes and nonadditive quantum code, both of which are generalizations of the well-studied class of quantum stabilizer codes. Hybrid codes encode quantum and classical information together, and as many quantum comminications protocols involve both quantum and classical information, they may potentially lead to more efficient quantum communications schemes. Nonadditive codes promise the ability to encode more quantum information than a stabilizer code of the same length, at the price of sacrificing some of the structure inherent to stabilizer codes.

We first gave some general results about hybrid codes, showing that genuine hybrid codes must be constructed out of impure quantum codes and giving linear-programming bounds for general hybrid codes. We then gave the first known families of genuine hybrid codes, including a family of single-error-detecting hybrid codes as well as a more general construction that gave multiple new hybrid codes of small length. These small-length hybrid codes were then expanded into their own infinite families of codes using a stabilizer pasting technique. As each of these constructions were inspired by constructions of nonadditive quantum codes, it would be interesting to further investigate the relationship between these two classes of codes, and to see if constructions of hybrid codes can be used to inspire new constructions of nonadditive quantum codes. One future direction would be to investigate the construction in Section 2.4 can be extended to include the $[\![11, 4{:}2, 3]\!]_2$ hybrid code in Example 19 which has a different "gadget" appended to the end. Another future direction would be to see if there are any nonadditive hybrid codes (with the amount of quantum and/or classical information not a prime power of q) that may be constructed using the CWS/union stabilizer codes [29, 42] that can encode more information than the codes in Chapter 36.

We also investigated a more general type of hybrid code where the quantum and classical information are protected to different degrees, with each having a separate minimum distance. We gave a very general construction of hybrid codes of this type that starts with quantum subsystem codes, and give as a special case the hybrid Bacon-Casaccino codes which can be constructed directly from a pair of classical linear codes. One interesting open question here are good upper bounds on the parameters of hybrid codes with two minimum distances. This construction also seems like it should be generalizable to include gauge subsystems as well as preshared entanglement (see [47, 61, 92] for partial results along this line). We also showed how hybrid codes can be used to protect against faulty syndrome measurement errors and how they can inspire new constructions of quantum data-syndrome codes. A future research direction would be to investigate how subsystem codes can improve the current quantum data-syndrome codes by requiring fewer measurements, and whether the "gadgets" in the construction in Section 2.4 can be useful for constructing fault-tolerant schemes in the vein of flag fault-tolerance [26, 27].

Finally, we give a result regarding the Shor-Laflamme weight enumerators of the nonadditive codeword stabilized codes, showing that they may be viewed as the distance enumerator of an associated classical code. This raises many new questions regarding combinatorial interpretations of the weight enumerators of non-CWS codes and we hope that it may lead to a better understanding of the structure of nonadditive quantum codes.

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APPENDIX A

QUANTUM CIRCUITS FOR EXAMPLE 46

This appendix contains the remaining four encoded CNOT operators from Example 46 in Section 3.5.1. The software used to decompose these quantum circuits can be found at `https://github.com/nrenga/symplectic-arxiv18a`.

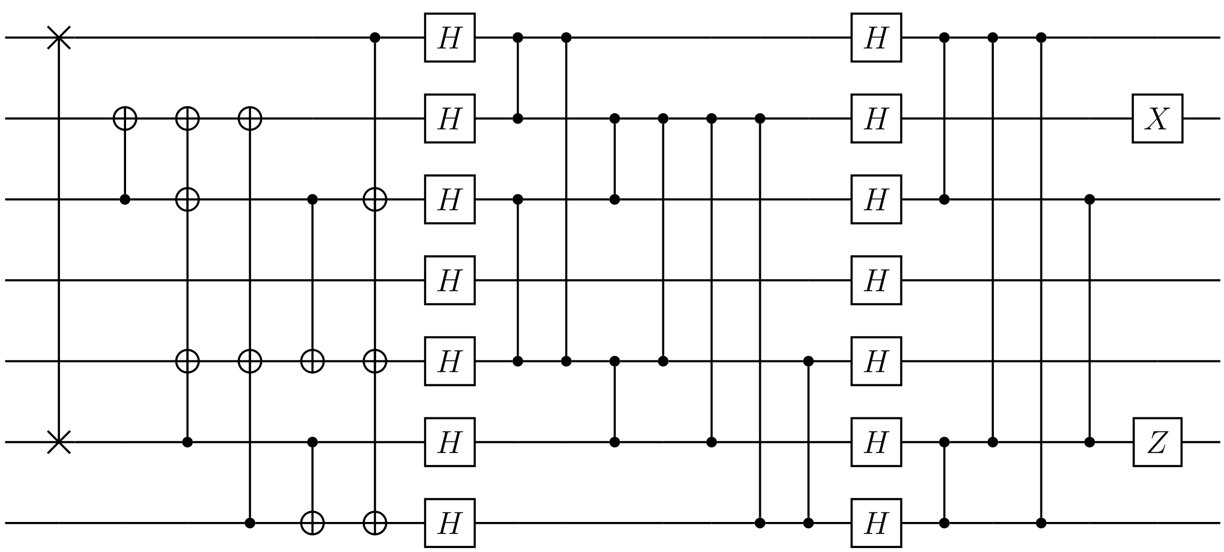

Figure A.1: Quantum circuit for extracting the error syndrome from the second stabilizer generator of the $[\![7,1:1,3]\!]_2$ hybrid code from Example 20 and placing it on the encoded classical qubit.

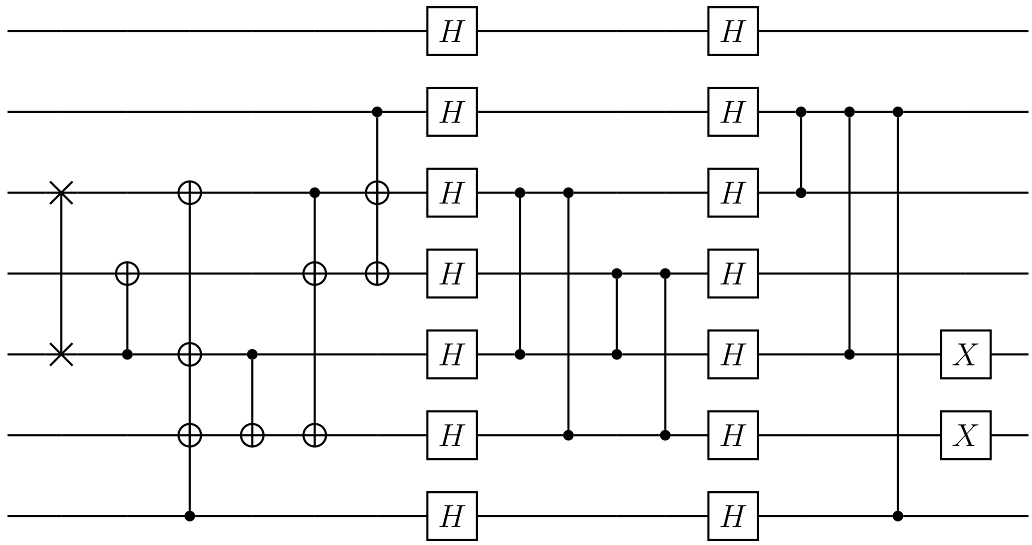

Figure A.2: Quantum circuit for extracting the error syndrome from the third stabilizer generator of the $[\![7,1:1,3]\!]_2$ hybrid code from Example 20 and placing it on the encoded classical qubit.

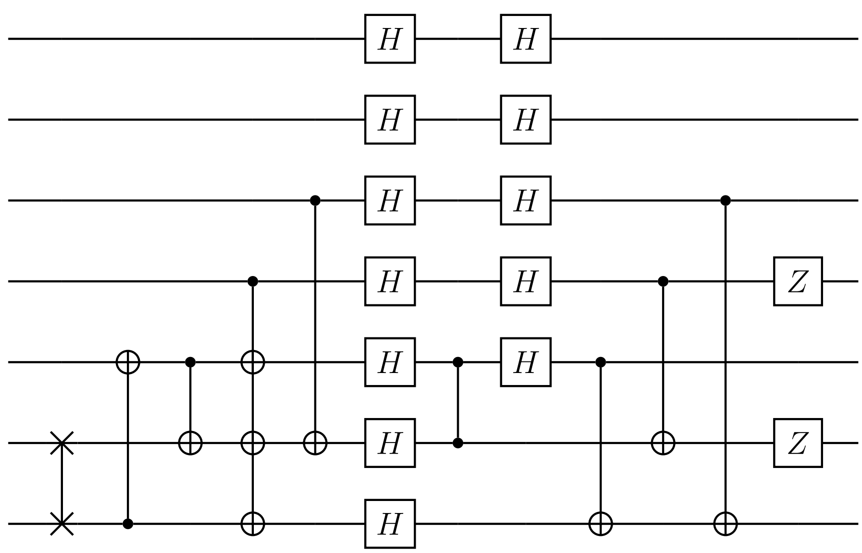

Figure A.3: Quantum circuit for extracting the error syndrome from the fourth stabilizer generator of the $[\![7,1:1,3]\!]_2$ hybrid code from Example 20 and placing it on the encoded classical qubit.

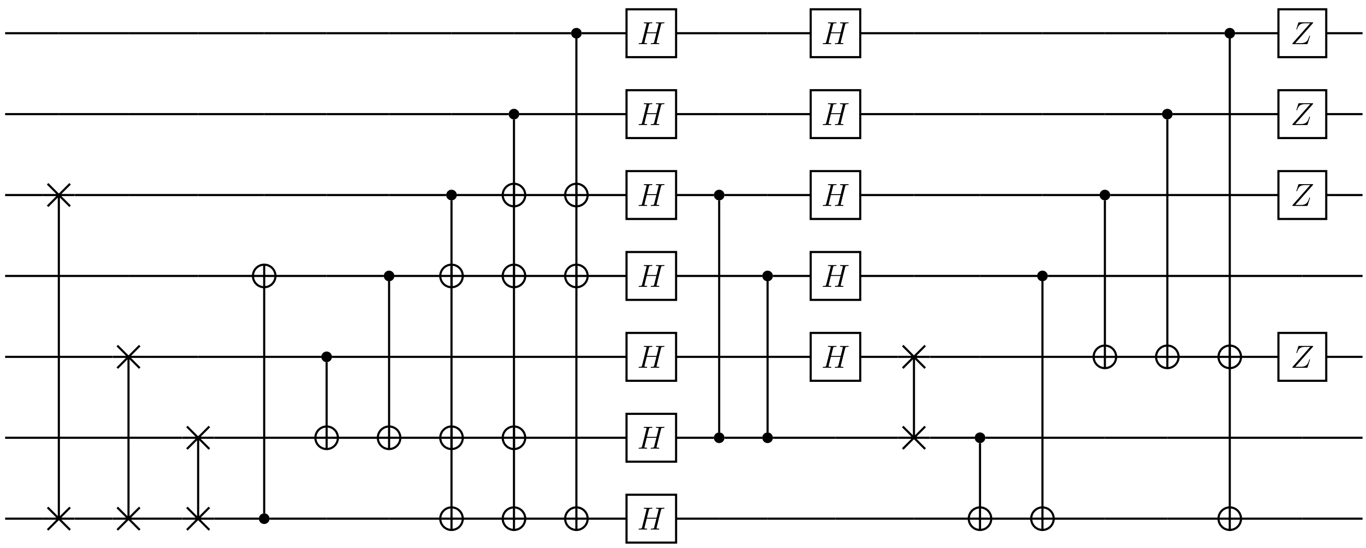

Figure A.4: Quantum circuit for extracting the error syndrome from the fifth stabilizer generator of the $[\![7,1\!:\!1,3]\!]_2$ hybrid code from Example 20 and placing it on the encoded classical qubit.