ESSAYS ON ENERGY OPERATIONS

A Dissertation

by

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ABSTRACT

The global energy landscape is going through major shifts triggered by consumer preferences, regulations, and technological development. My dissertation develops optimization models to derive insights into strategic decisions in energy operations management. In the first essay, I examine how blockchain-enabled peer-to-peer energy trading shifts electricity consumers' investment in renewable energy. Using the equilibrium model, I show that electricity consumers are always better off by participating in the virtual network, with their resulting cost savings averaging 9.7%. I also prove that blockchain is able to fully coordinate heterogeneous participants in the network to minimize the total cost in the system. The second essay addresses how a Transmission System Operator (TransCo) can optimally invest in a long-distance transmission line to allow renewable energy development by a Power Generation Company (GenCo) in a geographically remote region. Using a continuous-time, infinite-horizon, Stackelberg game between TransCo and GenCo, I show that transmission and generation act as complements with regard to the value functions for both companies. I derive the value-maximizing transmission fee charged by TransCo to GenCo for each unit of energy exported via transmission lines. I characterize a Pareto-improving cost-sharing contract through which both companies can improve the value of their investment. The third essay focuses on how to better manage a decentralized supply chain of an oil-field service company. To minimize the transportation and inventory holding costs of different members in a cross-docking supply chain, I formulate multi-period, mixed-integer programming models. I use structural properties of optimal solutions to show that different collaborations in the supply chain can generate significant cost savings for individual supply chain members, whereas the quantified cost savings exhibit significant variations depending on product weight and holding cost. I also develop a Stackelberg pricing game between an independent logistics company and oil wells seeking to lower their costs by outsourcing their operations. I provide the best response of oil wells to the price of outsourcing services and the structure of the logistics provider's optimal pricing policy.

DEDICATION

To my mother Bockki Choe, father Haekwan Lee, sister Seulki Lee,

and my fiance JiEun Moon.

Thank you for all of your support.

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Contributors

This work was supervised by a dissertation committee consisting of Professor Chelliah Sriskandarajah [advisor] and Professor Alexandar Angelus [co-advisor] and Professor Gregory R. Heim and Professor Bala Shetty of the Department of Information and Operations Management [Home Department] and Professor Ximing Wu of the Department of Agricultural Economics [Outside Department].

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1. INTRODUCTION

The global energy landscape is going through major shifts triggered by consumer preferences, regulations, and technological development (Fitschen et al. 2021). Due to the growing social preference, renewable energy has now come to represent a major source of power supply in the energy sector. For instance, renewable energy's share in United States electricity generation has nearly doubled over the last decade, from 10% in 2010 to 17.6% in 2018 (United States Energy Information Administration 2019). The last decade has also seen a significant growth of private investment in renewables, prompted by favorable energy policies and technological advancement (United States Department of Energy 2021). In particular, better access to low-cost renewable energy improves the affordability of individual ownership of renewable energy, leading more private investors and consumers to explore the opportunity of participating in renewable energy generation.

With the continued interest in renewable energy investments, there is a wide range of research needs to better understand how energy investors can invest in renewable energy. These investments involve strategic choices, because investments in general are irreversible, and there is uncertainty over the future return from the long-term investment (Dixit and Pindyck 1994). In particular, renewable energy investment can have more far-reaching consequences than other types of investments, as it will determine social, economic, and environmental outcomes for decades to come. Hence, understanding how to make the optimal investment in renewable energy bears significant social and economic implications for most public and private sectors. Yet, many questions regarding the optimal level of investment in renewable energy remain unexplored.

With this regard, my dissertation focuses on providing strategic insights related to a wide range of renewable energy investments, from small-scale residential solar panels to commercial-scale wind power plants. The first essay focuses on how peer-to-peer energy trading impacts consumer investments in residential proprietary renewable generation. I analyze how consumers can optimally invest in renewable generation capacity to minimize their electricity costs in a peer-to-peer energy trading network, and how one such decentralized power system can coordinate heterogeneous consumers, compared to a centralized power system. To capture the potential of abundant renewable energy sources and avoid the economic and social consequences of the lack of transmission lines, the second essay focuses on the transmission grid development problem. While it is the transition to clean energy that motivated my research, rapid transitions in the energy landscape have also affected conventional energy carriers, primarily those in the oil and gas industry. Stagnant demand and increasing production costs for fossil fuels prompt lower operating margins for all companies in this industry. Hence, the third essay focuses on how the oil-field service industry can improve operational efficiency, which has become imperative in order to keep companies' businesses profitable. In what follows, I briefly describe each essay in sequence.

1.1 Implications of Peer-to-Peer Trading of Electricity on Renewable Energy Investments

*Virtual microgrid*s refer to a local network of electricity consumers who are organized for peer-to-peer energy trading and its impact on consumer investment in renewable energy. Although virtual microgrids are becoming a practical reality in a number of electricity markets, little is known about how consumers in virtual microgrids can optimally invest in renewable energy, the resulting cost savings for consumers, and the ability of blockchain technologies to coordinate participants in a decentralized virtual microgrid.

In this essay, I formulate and solve an equilibrium model of investment in renewable energy by consumers in a virtual microgrid. I investigate how virtual microgrids impact consumers participating in the network, their optimal capacity investment, and pricing under equilibrium. I show that participating in virtual microgrids can generate significant cost savings for electricity consumers—9.7% on average. I also show that a virtual microgrid in equilibrium can achieve the minimum system-wide cost equivalent to that of a centralized energy distribution system such as a physical microgrid. Hence, blockchain is able to coordinate different participants in a decentralized energy trading network, giving support to the growing investment in blockchain-related projects in renewable energy and electricity markets.

Virtual microgrids have an overarching impact on the future of renewable energy investments and blockchain technology development. This research contributes by providing insights into how electricity consumers can make more cost-effective decisions pertaining to their participation in virtual microgrids. Also, electric utilities can use these results to assess the impact of virtual microgrids on their revenues. Investors can better understand the impact of a virtual microgrid on renewable energy investments. Researchers can study more complex virtual microgrid settings, as well as tackle other problems related to energy sharing.

1.2 Transmission Grid Development and Implications for Renewable Energy Investment

Limited transmission grid capacity hinders renewable energy development. Because the potentials for large-scale commercial renewable energy projects are often found far from population load centers, this essay focuses on how transmission system operators can optimally invest in a long-distance transmission line to allow renewable energy development by a power generation company in a geographically remote region.

In the presence of stochastic demands for renewable electricity on both ends of the transmission line, I formulate a continuous-time, infinite-horizon, Stackelberg game of capacity investment between TransCo and GenCo. It is shown that for any demand distributions at both regions, transmission and generation act as complements with regard to the value functions for both companies.

This essay derives the explicit expressions for optimal transmission and generation capacities installed by each company. Using this result, I characterize the value-maximizing transmission fee charged by TransCo to GenCo for each unit of energy exported from the low-demand remote region to the high-demand urban region.

1.3 Optimal Shipping, Collaboration, and Outsourcing Decisions in an Energy Supply Chain

The third essay, "On Value and Structure of Collaborations in a Hybrid Cross-docking Supply Chain," focuses on shipping decisions in a decentralized, multi-stage energy supply chain with a hybrid cross-docking facility. Hybrid cross-docking is an industry practice that allows products to be held in inventory at the cross-dock instead of being immediately shipped out.

Motivated by industry practices, I explore how individual supply chain members benefit from those collaborations and identify conditions that facilitate the realization of their benefits. I formulate multi-period, mixed-integer programming models to minimize transportation and inventory holding costs of different members in a hybrid cross-docking supply chain and establish structural properties of optimal solutions. I make use of those results to identify conditions under which hybrid cross-docking is more cost efficient than traditional cross-docking, which consolidates multiple products from suppliers for immediate fulfillment of downstream orders. I make use of this result to find that upstream collaboration results in 4.9% to 16.4% average cost savings for the cross-dock, while downstream collaboration generates 1.9% to 22.1% in average cost savings for the oil well plants, depending on product weight and holding cost.

The study explores outsourcing of operations by oil well plants in cross-docking supply chains to an independent logistics provider. I develop a Stackelberg pricing game between an independent logistics company and oil well plants seeking to lower their costs by outsourcing their operations. I identify the structure of oil well plants' best response to the price of outsourcing services and the structure of the logistics provider's optimal pricing policy.

In summary, my dissertation offers strategic insights into energy operations management problems grounded in industry practice, and the literature in economic theory, and inventory theory. Each essay derives new theoretical results and addresses timely and relevant issues for both academic researchers and practitioners in the energy sector.

The remainder of this dissertation is structured as follows. In Chapter 2, I address the impact of peer-to-peer energy trading networks on consumer investments in renewable energy. In Chapter 3, I examine the strategic interaction of transmission and generation and its impact on investments in transmission and generation capacities. Chapter 4 formulates optimization models of a decentralized supply chain in the context of the oil-field service industry. Chapter 5 offers concluding remarks and summarizes contributions.

2. VIRTUAL MICROGRIDS: IMPLICATIONS FOR BLOCKCHAIN TECHNOLOGIES FOR THE PEER-TO-PEER TRADING OF ELECTRICITY, AND RENEWABLE ENERGY INVESTMENTS

2.1 Introduction

Blockchain represents a cryptographically secure database (i.e., ledger) that achieves data security and identity authentication while eliminating the need for intermediaries (Olsen and Tomlin 2020). As such, blockchain is transforming the way we record, verify, and arrange transactions, with the focus shifting away from centralized structures (e.g., exchanges, trading platforms, energy companies) towards decentralized systems (e.g., end customers, energy consumers interacting directly). For those reasons, outside of the financial sector, the energy sector is considered to be an industry where blockchain technologies will have great potential to disrupt the functioning of existing markets and regulatory structures. This is because blockchain makes it possible to bypass prevailing market structures (and authority of electric utilities) by providing consumers with a digital platform to trade energy directly with each other (Hertz-Shargel 2019). By ensuring the security and authenticity of financial and physical flows across the electricity grid, blockchain technology enables organization of electricity consumers with individual ownership of renewable energy production (such as solar panels) into a *decentralized* network of interconnected demand and production nodes. Such a network of interconnected demands and consumer-owned, distributed, renewable energy resources is referred to as *a virtual microgrid* (Bremdal and Ilieva 2019), as it operates over the utility's existing transmission grid without needing its own transmission lines.

One example of such a microgrid is the Brooklyn Microgrid, a fully operational blockchain-enabled microgrid that allows its participants to trade the surplus of their self-produced solar energy with their neighbors (Mengelkamp et al. 2018). The Brooklyn Microgrid was launched in 2016 in the Brooklyn borough of New York City when 50 homes were connected into a virtual energy market for locally-generated renewable energy. The Brooklyn Microgrid is made possible by blockchain technology that counts and logs every unit of energy created by each home's solar panel system, and a blockchain application of 'smart contracts'[1] that make those units of energy available to the open market to be bought and sold in the local community. This process enables consumers to produce and sell power locally (instead of putting it into the central grid), and receive income from the energy thus sold. As such, a blockchain-enabled virtual microgrid, such as the Brooklyn Microgrid, represents a new business model in the energy sector with the potential to disrupt traditional processes and buyer-seller relationships in electricity markets.

In addition to the Brooklyn Microgrid, which was the first project to facilitate a blockchain-based electricity transaction, our work is also motivated by a growing number of other blockchain-related energy projects already in existence or under development and testing. Some of those projects include a residential electricity trading market in Australia, peer-to-peer trading networks among households with and without rooftop solar in Bangladesh, and a network of some 8,000 electricity consumers in Germany trading stored energy with each other (Marcus 2019).

Blockchain-enabled virtual microgrids raise a number of questions of importance to electricity consumers, renewable energy advocates, electric utilities, and blockchain technology investors. Some of those questions concern the level of optimal investment in renewable energy by each participant in a virtual microgrid, and key drivers of that investment. Others are related to the cost savings realized by each participant in a virtual microgrid and factors that maximize those savings. Yet others deal with optimal pricing and volume of energy traded. There are also questions concerning the ability of blockchain technologies to coordinate participants in a decentralized virtual microgrid, as independent decision makers, in the direction of minimizing the total cost in the system. To the best of our knowledge, those questions have not been considered in the academic literature, nor do there seem to be any references to their answers in practitioner journals.

In this chapter, we address these and other questions related to the implications of virtual mi-

[1] Smart contracts are software functions that maintain internal data about participants and their devices. They also serve as protocols "designed to facilitate, verify, or enforce the negotiation or execution of a contract" (Babich and Hilary 2020), and their execution is automatically triggered by some externally verified event (Olsen and Tomlin 2020). A smart contract might, for example, store the number of kilowatt hours of energy produced by a solar panel, and then, following a virtual trade, direct a smart meter to increment that value and the utility to read the value.

crogrids for renewable energy investments, blockchain technologies, consumer savings and peer-to-peer trading of electricity. For that purpose, we formulate and solve an equilibrium model of a decentralized virtual microgrid, in which each participating electricity consumer first decides, at the beginning of the time horizon, on the level of renewable generation capacity (such as solar energy) to invest in. Then, throughout the time horizon, he satisfies his (stochastic) demand for electricity either from his own installed generation capacity, or by means of blockchain-mediated purchases of electricity available from his peers in the virtual microgrid. If this combination of his own energy generation and purchases of electricity available in the microgrid cannot fully satisfy a consumer's demand for electricity at any time, he resorts to buying electricity from his electric utility.

Our first contribution is to explicitly determine the optimal level (i.e., capacity) of investment in renewable energy for each participant in the microgrid. We also establish how the resulting production of electricity is rationed for his own use versus for sale to other participants. Contrary to conventional wisdom, the optimal level of generation capacity for each participant in a virtual microgrid is not necessarily higher than that of an identical, single, cost-minimizing consumer without access to a microgrid. Instead, a novel insight we arrive at is that the sign of the difference between the two optimal capacity levels depends on the price of peer-to-peer electricity traded in the microgrid. Hence, it is the intra-microgrid price of traded electricity that determines the impact of blockchain-enabled virtual microgrids on the magnitude of renewable energy investments.

Our second contribution is to obtain, for each participant, the price of peer-to-peer electricity traded in a virtual microgrid that minimizes his total electricity-related costs. We derive an explicit expression for this optimal price even though the objective function for the corresponding minimization problem is not, in general, convex in the price of traded electricity. In that manner, we obtain the full set of equilibrium outcomes for this problem. We also establish that the cost-minimizing price of traded electricity is increasing in the marginal cost of investment in renewable energy, and convex in the electricity rate charged to consumers by their electricity utility.

Our third contribution is to prove that an electricity consumer is always better off within a virtual microgrid than without. The resulting cost savings to each microgrid participant average 9.7% in our numerical studies. By showing that a blockchain-enabled microgrid always provides an incentive for consumers to self-organize into a virtual market for the trading of self-generated electricity, our work helps illuminate an answer to “one of the biggest questions surrounding the future of (virtual) microgrids in the U.S” regarding “justifying the business case” (Peck 2016).

Our final contribution is to evaluate the power of blockchain to coordinate individual electricity consumers in a decentralized virtual microgrid towards minimizing the total cost in the system. We prove that, under equilibrium outcomes, the cost performance of a blockchain-enabled virtual microgrid exactly matches that of a centralized microgrid optimized for total cost. Thus, when participants in a virtual microgrid act optimally in minimizing their own individual costs, they achieve full coordination of the entire system and realize its minimum total cost.

The rest of this chapter is organized as follows. In Section 2, we present a review of related literature. In Section 3, we formulate the model. In Section 4, we derive equilibrium outcomes in a virtual microgrid. Section 5 addresses some key implications of virtual microgrids for electricity consumers and blockchain technology investors. In Section 6, we offer concluding remarks.

2.2 Literature Review

This chapter is primarily related to the literature on renewable energy systems. Reviews of this literature can be found in Parker et al. (2019) and Agrawal and Yücel (2021). Of particular relevance to our research are those papers that focus on optimal capacity of renewable energy investments in the presence of uncertainty. Hu et al. (2015) establish optimal capacity of renewable distributed generation when yield and demand are uncertain, and show that data granularity matters because coarse data may not reflect intermittency of renewable generation. Kök et al. (2018) explore the effect of pricing policies (flat vs. peak) on investment and find that flat pricing leads to increased investment in solar energy by the centralized utility, while peak pricing generates higher levels of solar investment in distributed generation. Aflaki and Netessine (2017) demonstrate that when demand for electricity is uncertain, due to the interaction between intermittency and pricing, the share of renewable capacity in the generation portfolio may be reduced by higher carbon prices and market liberalization. Angelus (2021) identifies optimal timing and capacity of a consumer's investment in distributed renewable generation under stochastically evolving demand for electricity.

All of those papers deal with investments by individual market participants, either utilities, firms, or consumers, who are *not* organized (into microgrids or otherwise), and who buy and/or sell electricity *only* to and from the central transmission grid, without having the ability to trade energy with each other. In this chapter, we consider consumers organized into a microgrid, and find that their option to trade electricity with each other fundamentally alters the level of installed renewable generation capacity. To the best of our knowledge, our research is the first to address investment in distributed renewable energy within a (virtual) microgrid platform. In addition, we also address optimal pricing within such a microgrid, and explore the resulting consequences for consumers' cost savings and renewable energy investments.

Several recent papers in the renewable energy literature have dealt with optimal pricing of electricity. Alizamir et al. (2016) consider heterogeneous investors in renewable technologies in the presence of feed-in tariffs, information diffusion, and learning, and determine optimal feed-in tariffs and the threshold structure of the optimal stopping problem for each investor. In evaluating the impact of pricing policies on renewable energy investments, Kök et al. (2018) optimize the peak and flat electricity price charged by the utility. Angelus (2021) determines optimal electricity pricing for a utility serving consumers who have the option to invest in distributed renewable power generation. Adelman and Uçkun (2019) develop asymptotically linear, social welfare-maximizing, electricity price schedules for the utility when its consumers deploy smart, price-responsive appliances. In contrast to this literature, this chapter is not concerned with optimal pricing of an utility's electricity rate or optimal feed-in-tariffs for renewable energy (since, in our model, a virtual microgrid cannot export electricity to the utility); instead, we determine optimal peer-to-peer pricing for the electricity traded *within* a virtual (decentralized) microgrid.

A number of other papers on renewable energy operations, albeit not on the topic of microgrids, is also worth mentioning. Wu and Kapuscinski (2013) find that curtailing intermittent renewable energy reduces system costs and emissions. Zhou et al. (2016) investigate negative electricity prices that can arise from excess wind energy. Sunar and Birge (2019) show that an undersupply penalty rate can result in commitment inflation by renewable generators. Babich et al. (2020) address solar panel investment under feed-in-tariff versus tax-rebate policies. Agrawal et al. (2019) provide managerial insights and policy implications for non-ownership business models for solar energy.

Because the model and results in this chapter concern peer-to-peer trading of electricity, we also contribute to the growing body of research literature on peer-to-peer markets and the sharing economy (for a recent review, see Benjaafar and Hu (2020)), in which the exclusive ownership and consumption of resources is replaced with shared use and consumption. Specifically pertinent to our work are those papers that address optimal pricing and/or capacity in peer-to-peer networks. Cachon et al. (2017), for example, study several contractual forms for a platform that offers a service via a pool of independent providers. They establish that surge pricing, commonly used in practice, although not optimal generally, achieves nearly the optimal profit for the platform. Jiang and Tian (2018) find that, when a firm strategically chooses its retail price, consumers' sharing of the firm's products with high marginal costs increases both the consumer surplus and the firms' profit, whereas their sharing of products with low marginal costs can lead to decreases in both. Benjaafar et al. (2019) introduce a model of collaborative consumption to characterize equilibrium outcomes, including ownership and usage levels, consumer surplus, and social welfare, and show that consumers always benefit from collaborative consumption. Using a closed queueing network, Benjaafar et al. (2021) obtain explicit upper and lower bounds on the optimal fleet size in a vehicle sharing system, and derive insights pertaining to the fleet's vehicle buffer capacity.

While it is the sharing economy that gives rise to the problem considered in this chapter, our model differs from the existing ones in this area in several important ways. First, a blockchain-enabled, virtual network does not require a separate marketplace intermediary (such as Uber and Lyft for transportation services, Instacart and Postmates for home deliveries, and TaskRabbit and Handy for household tasks) to provide a platform for peer-to-peer resource sharing. This eliminates the need for transaction fees, contracts with the intermediary entity, and platform-imposed pricing. Second, in contrast to existing work that distinguishes between 'owners' and 'nonowners' in the sharing economy, so that "owners are able generate income from renting their products to nonowners" (Benjaafar et al. 2019), in a virtual microgrid every participant is both a consumer *and* a producer. (As such, microgrid participants have come to be referred to as "prosumers" (Bichler et al. 2010), which is the term that we use in this chapter as well). Thus, each prosumer generates income both *from* other participants and *for* other participants in the microgrid. Third, in our model, it is not the product itself (e.g., solar panels) that is shared between microgrid participants, but rather a product (i.e., electricity) *of* that product. This distinction is important because the sharing economy literature generally assumes that an owner has only one product (e.g., a car) to share, so that network capacity is determined by each owner's decision about whether or not to participate in resource sharing (Cachon et al. 2017). By comparison, each participant in a blockchain-enabled microgrid chooses not only whether to participate in resource sharing through virtual electricity trading, but also: (1) how much renewable energy capacity to install for that participation; and, (2) how much of his self-generated electricity to use for his own demand versus to make available for sale to other participants. Finally, in a virtual microgrid, the fee for resource sharing is not determined by some marketplace intermediary, but rather by microgrid participants themselves.

2.3 Model Formulation

In this section, we describe the equilibrium model of a blockchain-enabled virtual microgrid. We formulate a continuous-time, infinite horizon, stochastic optimization model of a decentralized microgrid formed by heterogeneous prosumers, such as households, who act independently in minimizing their individual total cost of electricity. This type of decentralization is made possible by data security, identity authentication, and smart contract execution provided by blockchain.

Electricity demand (rate) for each prosumer $i \in \{1, 2, \ldots, n\}$ in the microgrid at time t is a stationary random variable X_i. To facilitate analysis, we assume that X_i is uniformly distributed on $\mathbb{I}_i = [a_i, b_i]$ with $n = 2$ and $w := b_1 - a_1 = b_2 - a_2$, so that a virtual microgrid considered in the model consists of two *non-identical* prosumers with equal interval widths for the domain of their demand probability distributions. Having two participants in a blockchain-enabled microgrid is enough to capture the dynamics of peer-to-peer electricity trading and analytically explore its impact on each prosumer's optimal capacity, price of energy traded in the microgrid, and resulting cost savings.

Most countries in the world and most states in the United States have regulated retail electricity markets for residential consumers (Prentis, 2015). In such markets in the U.S., consumers are served by their regulated electric utility, and their long-term electricity price (i.e., "the rate") is negotiated by each state's Public Utility Commission (PUC) and the utility. Following that negotiation, the PUC approves a rate that subsequently stays in place usually for a number of years. Once determined, this price is allowed to increase only at the rate of inflation. Accordingly, in our model, the utility announces a PUC-approved electricity rate p_e at the beginning of the time horizon, so that, at any subsequent time t, the electricity rate paid by the consumers for each kilowatt of energy purchased from their utility is $p_e\, e^{\eta t}$, where η is the inflation rate. We refer to p_e as *the base (electricity) rate*.

Each heterogeneous consumer invests in his own renewable energy resource (e.g., solar panels) of capacity C_i at time $t = 0$, so that, at the beginning of the time horizon, microgrid participants are involved in a simultaneous noncooperative game with complete information. We will refer to this first-stage game as *the capacity game*. Capacity C_i of a renewable energy resource represents the upper limit on the amount of electricity generated by that resource at any time t. Consumer investment in renewable energy tends to be long-term and irreversible, because once installed, a renewable energy system, such as solar panels, is difficult to resell or reinstall. For most households, this type of investment is also one-time, due to its high cost and long lifetime. Thus, once installed at the beginning of the time horizon, capacity C_i stays in place through the end of the time horizon. Let k denote the marginal cost of investment for each unit of renewable generation capacity. It follows that each Consumer i incurs investment cost kC_i at the beginning of the time horizon.

Following the installation of his renewable capacity C_i, Consumer i becomes *Prosumer* i, and he utilizes that capacity to produce electricity throughout the time horizon. At any time t, his self-generated electricity can be sold to other microgrid participants and/or used satisfy his own demand. Each prosumer can also buy electricity from other prosumers whenever his demand exceeds his own capacity. In this manner, prosumers in a virtual microgrid can trade self-produced electricity in a peer-to-peer fashion on their own virtual energy market. Hence, at each time t, every microgrid participant has to decide how much of his capacity C_i to use for his own demand, and how much electricity to sell to (or buy from) other microgrid participants. In that manner, following their investments in renewable generation, prosumers in a virtual microgrid play a repeated, simultaneous noncooperative game with complete information. This second-stage game is *the energy game*.

If, at any time t, Prosumer i cannot meet his demand for electricity either through his own generation or by means of electricity purchased from his microgrid counterpart(s), he satisfies that excess demand directly from his utility at the regulated electricity price $p_e e^{\eta t}$.

Since renewable energy is effectively costless to generate, each prosumer is willing to sell all of his self-generated electricity in excess of his demand. If, on the other hand, his electricity demand exceeds his generation capacity at any time, then, because satisfying that excess demand from the utility costs $p_e e^{\eta t}$, he is willing to buy the needed amount of electricity from other participants in the microgrid at any price below $p_e e^{\eta t}$. Let $\pi(t)$ denote the market clearing price for electricity trades in a virtual microgrid. We assume the following regarding the clearing of the energy market.

Assumption 1. *In a virtual microgrid: (i) the energy market always clears; and (ii) the market clearing price $\pi(t)$ satisfies $\pi(t) \in [0, p_e e^{\eta t}]$ at all times.*

By part (i), as long as one microgrid participant has excess electricity demand *and* another microgrid participant has excess supply, energy trades will occur. By part (ii), the resulting market clearing price for the traded energy will not exceed $p_e e^{\eta t}$ at any time t. Otherwise, if $\pi(t)$ exceeds $p_e e^{\eta t}$ at some time t, it becomes cheaper for a prosumer to purchase all of his needed electricity from the utility at time t than to buy any electricity from other microgrid participants. As this outcome is clearly suboptimal for all microgrid participants, we eliminate it from consideration.

Because different blockchain energy applications follow different protocols for arriving at the market clearing price for their energy trades[2], in this chapter we do not assume any particular mechanism for that market price formation; instead, we follow a more general approach of assuming only the existence of a market clearing mechanism and associated market clearing price.

Any energy exchange between the two prosumers in a virtual microgrid occurs over the utility's main electrical grid and does not require central intermediaries such as the utility to facilitate the trade. Motivated by the example of Brooklyn Microgrid, we assume that microgrid participants can sell their excess energy only to each other and not to their utility. By not requiring the utility to compensate prosumers for excess electricity generated in a microgrid through programs such as net-metering, we can capture the value of the energy trade enabled by a virtual microgrid. In that manner, we also avoid the utility "death spiral" that can occur when, due to net-metering purchases that erode their profits, utilities are forced to increase their electricity rates, rendering renewable energy investments even more attractive, thus further undermining utility profitability (see Satchwell et al. 2015, and Darghouth et al. 2016).

We index microgrid participants by i or j; $i, j \in \{1, 2\}$ and $i \neq j$. We use the notation:

Z_{it} := energy produced by Prosumer i at time t that is used to satisfy his own demand;

Y_{ijt} := energy produced by Prosumer i at time t that is sold to Prosumer j.

Consequently, Z_{it} and Y_{ijt} represent *energy-related* decision variables in our model. The two prosumers choose their energy-related decision variables simultaneously, and those decision variables satisfy the following feasibility constraints for each Prosumer $i \in \{1, 2\}$ at time t:

$$Z_{it}, Y_{ijt} \geq 0$$

$$Z_{it} + Y_{ijt} \leq C_1$$

$$Z_{it} + Y_{ijt} + Z_{jt} + Y_{jit} \leq \min(x_i + x_j, C_i + C_j).$$

[2]In the Brooklyn Microgrid, for example, the market clearing price is determined with buyers and sellers setting their own bids for electricity purchases and sales. Blockchain application platform PowerLedger that supports residential electricity trading market in Australia mentioned in the Introduction allows the market clearing price to be set exogenously by the operators of the platform, and for the market to be cleared in arbitrarily small time periods.

The second constraints limit the total energy produced (i.e., the sum of the energy produced used to satisfy own demand plus the energy produced that is sold to other participants in the microgrid) by each Prosumer i to the level of his generation capacity C_i. The third constraint limits the total energy produced in the microgrid to the smaller of the total demand and total capacity in the microgrid, since, as already stated, the energy generated by participants in a virtual microgrid cannot be sold outside of their virtual microgrid.

Let x_i be a realization of random variable X_i that, as already stated, represents electricity demand for Prosumer i. Let $f_i\big(x_i, x_j, Z_{it}, Y_{ijt}, Y_{jit}, t, \pi(t)\big)$ denote the cost rate to Prosumer i at time t, given energy-related decisions Z_{it}, Z_{jt}, Y_{ijt}, and Y_{jit}, and market clearing price $\pi(t)$ in the microgrid. Based on our modeling assumptions, we obtain:

$$f_i\big(x_i, x_j, Z_{it}, Y_{ijt}, Y_{jit}, t, \pi(t)\big) = p_e e^{\eta t}\big[x_i - Z_{it} - Y_{jit}\big]^+ - \pi(t)Y_{ijt} + \pi(t)\,Y_{jit}, \tag{2.1}$$

using the standard notation $[y]^+ := \max(y, 0)$. Hence, each Prosumer i's optimal energy-related decisions at any time t become the solution of the following (constrained) minimization problem:

$$\min_{Z_{it}, Y_{ijt}} f_i\big(x_i, x_j, Z_{it}, Y_{ijt}, Y_{jit}, t, \pi(t)\big) \tag{2.2}$$

$$\text{subject to:} \quad Z_{it}, Y_{ijt} \geq 0 \tag{2.3}$$

$$Z_{it} \leq \min(x_i, C_i) \tag{2.4}$$

$$Z_{it} + Y_{ijt} \leq C_i \tag{2.5}$$

$$Z_{it} + Y_{ijt} \leq \min(x_i + x_j, C_i + C_j) - Z_{jt} - Y_{jit}. \tag{2.6}$$

Let $\overline{Z}_{it}\big(x_i, x_j, C_i, C_j, Z_{jt}, Y_{jit}, t, \pi(t)\big)$ and $\overline{Y}_{ijt}\big(x_i, x_j, C_i, C_j, Z_{jt}, Y_{jit}, t, \pi(t)\big)$ denote a solution of the minimization problem given in (2.2) – (2.6). We seek a subgame-perfect, pure-strategy Nash equilibrium of this second-stage, simultaneous game played between microgrid participants at each point in the time horizon. Let $Z^*_{it}\big(x_i, x_j, C_i, C_j, t, \pi(t)\big)$ and $Y^*_{ijt}\big(x_i, x_j, C_i, C_j, t, \pi(t)\big)$, for $i, j \in \{1, 2\}$ and $i \neq j$, denote that pure-strategy equilibrium. Since such a strategy represents each prosumer's best response to the other prosumer's equilibrium decisions, then, for $i, j \in \{1, 2\}$ and $i \neq j$, we get:

$$Z^*_{it}\big(x_i,x_j,C_i,C_j,t,\pi(t)\big) = \overline{Z}_{it}\big(x_i,x_j,C_i,C_j,Z^*_{jt}(x_j,x_i,C_j,C_i,t,\pi(t)),Y^*_{jit}(x_j,x_i,C_j,C_i,t,\pi(t)),t,\pi(t)\big),$$

$$Y^*_{ijt}\big(x_i,x_j,C_i,C_j,t,\pi(t)\big) = \overline{Y}_{ijt}\big(x_i,x_j,C_i,C_j,Z^*_{jt}(x_j,x_i,C_j,C_i,t,\pi(t)),Y^*_{jit}(x_j,x_i,C_j,C_i,t,\pi(t)),t,\pi(t)\big).$$

We will refer to the decision set $\{Z^*_{it}, Y^*_{ijt}\}$ as Prosumer i's *optimal energy policy*.

Using each prosumer's optimal energy policy, define, for $i,j \in \{1,2\}$ and $i \neq j$, the following:

$$F_i\big(C_i,C_j,x_i,x_j,t,\pi(t)\big) := f_i\big(x_i,x_j,Z^*_{it}(\cdot),Y^*_{ijt}(\cdot),Y^*_{jit}(\cdot),t,\pi(t)\big).$$

For any realizations x_i and x_j of X_i and X_j, F_i is the cost rate for Prosumer i at time t under his optimal energy policy, market clearing price $\pi(t)$, and capacity investments C_i and C_j.

Let $V_i\big(C_i,C_j,\pi(\cdot)\big)$ denote the expected discounted present value of all electricity-related costs for Prosumer i when both prosumers use their optimal energy policies, given a *market clearing price function* $\pi(\cdot)$ for energy trades, and installed capacities C_i and C_j in the microgrid. Hence,

$$V_i\big(C_i,C_j,\pi(\cdot)\big) = \int_0^\infty e^{-rt}\ \mathbb{E}_{X_i,X_j}\Big[F_i\big(C_i,C_j,x_i,x_j,t,\pi(t)\big)\Big]\,dt + kC_1, \tag{2.7}$$

where r is a discount rate, and $\mathbb{E}_{X_i,X_j}$ denotes the expectation over random variables X_i and X_j. Because $\pi(\cdot)$ enters V_i as a function rather than just a state variable, V_i represents a *functional*. For the integral in (2.7) to exist, the following assumption on the clearing price function is required.

Assumption 2. *$\int_0^\infty e^{-rt}\pi(t)dt$ exists and is finite.*

Quantities C_i and C_j represent *capacity-related* decision variables in our model. For $i,j \in \{1,2\}$ and $i \neq j$, let $\overline{C}_i\big(C_j,\pi(\cdot)\big)$ be the capacity that minimizes $V_i\big(C_i,C_j,\pi(\cdot)\big)$, given C_j and $\pi(\cdot)$. Then,

$$\overline{C}_i\big(C_j,\pi(\cdot)\big) := \operatorname*{arg\,min}_{C_i \geq 0}\ V_i\big(C_i,C_j,\pi(\cdot)\big). \tag{2.8}$$

We seek a pure-strategy Nash equilibrium of this first-stage, simultaneous game played by virtual microgrid participants at the beginning of the time horizon. Let $C^*_i\big(\pi(\cdot)\big)$ denote the corresponding pure-strategy equilibrium decision for Prosumer i, as a functional of the market clearing price function π. It follows that, for $i, j \in \{1, 2\}$ and $i \neq j$,

$$C_i^*\big(\pi(\cdot)\big) = \overline{C}_i\big(C_j^*(\pi(\cdot)), \pi(\cdot)\big). \tag{2.9}$$

We refer to $C_i^*\big(\pi(\cdot)\big)$ as Prosumer i's *optimal capacity selection*. Next, we look to determine conditions on the market clearing price function that minimizes each prosumer's total electricity costs in an equilibrium. Let $V_i^*\big(\pi(\cdot)\big)$ represent Prosumer i's total expected discounted cost as a functional of $\pi(\cdot)$, under equilibrium outcomes for energy and capacity-related decisions. Thus,

$$V_i^*\big(\pi(\cdot)\big) := V_i\big(C_i^*(\pi(\cdot)), C_{-i}^*(\pi(\cdot)), \pi(\cdot)\big).$$

For each Prosumer i, let $\pi_i^*(\cdot)$ denote the market clearing price function that minimizes his total expected discounted cost of electricity under his optimal capacity selection. Thus, we obtain

$$\pi_i^*(\cdot) := \arg\min_{\pi(\cdot)} V_i^*\big(\pi(\cdot)\big). \tag{2.10}$$

Hence, $\pi_i^*(\cdot)$, referred to as *the optimal (market clearing) price function* represents a function that minimizes the functional $V_i^*\big(\pi(\cdot)\big)$. We will refer to the optimization problem given in (2.10) as *the market clearing price problem*. In the next section, in addition to solving for a prosumer's optimal energy policy and optimal capacity selection, we also provide a characterization of $\pi_i^*(\cdot)$.

2.4 Optimal Decisions in a Virtual Microgrid

We derive optimal energy policy in Section 2.4.1. In Section 2.4.2, we establish each prosumer's optimal level of renewable generation capacity. In Section 2.4.3, we characterize the market clearing price function. Section 2.4.4 numerically illustrates our results and provides additional insights.

2.4.1 Energy Policy Decisions

The following theorem specifies each prosumer's optimal energy policy and his resulting cost rate. All proofs are deferred to the Appendix.

Theorem 1. *There exists a unique, subgame-perfect Nash equilibrium of the energy game at each time t. The resulting optimal energy policy for each Prosumer i is stationary and given by*

$$Z_i^*(x_i, x_j, C_i, C_j) = \min[x_i, C_i];$$
$$Y_{ij}^*(x_i, x_j, C_i, C_j) = \min\big[(C_i - x_i)^+, (x_j - C_j)^+\big].$$

Prosumer i's cost rate under this subgame-perfect Nash equilibrium is

$$F_i\big(C_i, C_j, x_i, x_j, t, \pi(t)\big) = p_e\, e^{\eta t}\big[x_i - C_i - (C_j - x_j)^+\big]^+ - \pi(t)\min\big[(C_i - x_i)^+, (x_j - C_j)^+\big] + \pi(t)\min\big[(C_j - x_j)^+, (x_i - C_i)^+\big]. \tag{2.11}$$

Thus, in equilibrium, each prosumer in a virtual microgrid uses his installed renewable generation capacity C_i to first fulfill his own demand for electricity. Then, he makes available the remainder of that generation capacity for sale to other participants in the virtual microgrid. Those other participants also use their installed generation capacity to meet their own demand first, and rely on peer-to-peer purchases of electricity to satisfy any excess of that demand over their capacity.

In practice, this energy policy would be implemented by smart contract application of blockchain that keeps track of each unit of energy created by prosumers. This application would also include instructions to make all units of energy in excess of a prosumer's own demand available on the open market, to be bought by other microgrid participants. In that manner, the energy policy described in Theorem 1 can be executed automatically, without any (additional) external input from prosumers.

2.4.2 Renewable Generation Capacity

Having derived the optimal energy policy for each prosumer, we now proceed to determine his optimal capacity selection under that policy. The maximum benefit rate to Prosumer i from installing one unit of capacity occurs when the energy generated by that unit is used to substitute for the purchase of electricity from his utility at price $p_e\, e^{it}$. Thus, his maximum discounted benefit over the time horizon from having installed capacity C_i at $t = 0$ is $\int_0^\infty e^{-rt} p_e e^{it} C_i\, dt = \frac{p_e}{r-i} C_i$. Since the cost of investing in C_i is kC_i then, if $-\frac{p_e}{r-i} + k \geq 0$, a prosumer's cost of installing renewable generation exceeds his maximum benefit from that installation, and he will not invest in renewable energy. Our interest in this chapter, is in those settings in which a consumer does have the incentive to make an investment in renewable power generation, so that a virtual microgrid can come into existence. Consequently, going forward, we make the following assumption (using $\lambda := r - \eta$).

Assumption 3. $k < \dfrac{p_e}{\lambda}$.

For notational convenience, going forward we also define Π as follows:

$$\Pi := \int_0^\infty e^{-rt}\pi(t)dt. \tag{2.12}$$

Thus, Π represents the discounted present value of one unit of energy continuously traded in the microgrid during the time horizon. As such, Π can be interpreted as the *marginal value of energy traded* in a virtual microgrid. By Assumption 2, Π is well-defined and finite. The following proposition lays the foundation for the subsequent results in this chapter (using $w := b_1 - a_1 = b_2 - a_2$).

Proposition 1. *For each Prosumer i, the functional $V_i\big(C_i, C_j, \pi(\cdot)\big)$ can be expressed as a function $V_i(C_i, C_j, \Pi)$ of C_i, C_i and Π that is given by the following.*

(a) If $C_1 + C_2 > a_1 + a_2 + w$, then

$$V_i(C_i, C_j, \Pi) = kC_i + \frac{p_e}{6\lambda w^2}(a_i + w - C_i)^2(a_i + 3a_j - C_i - 3C_j + 4w) + \frac{\Pi}{6w^2}(C_i - C_j - a_i + a_j) \tag{2.13}$$
$$\times \Big[a_i^2 + a_j^2 + C_i^2 + 4C_iC_j + C_j^2 - 3w(C_i + C_j) + a_i(4a_j - 2C_i - 4C_j + 3w) + a_j(-4C_i - 2C_j + 3w)\Big].$$

(b) If $0 \le C_1 + C_2 \le a_1 + a_2 + w$, then

$$V_i(C_i, C_j, \Pi) = kC_i + \frac{p_e}{6\lambda w^2}\Big[-a_j^3 + 3C_iC_j^2 + C_j^3 - 3a_jC_j(2C_i + C_j - 2w) + 3a_j^2(C_i + C_j - w)$$
$$+ 3wa_i^2 + 3wC_i^2 - 3wC_j^2 - 6w^2C_i + 3w^3 - 3a_i\big[a_j^2 - 2a_jC_j + C_j^2 + 2w(C_i - w)\big]\Big] + \frac{\Pi}{6w^2}(C_i - C_j - a_i + a_j)$$
$$\times \Big[a_i^2 + a_j^2 + C_i^2 + 4C_iC_j + C_j^2 - 3w(C_i + C_j) + a_i(4a_j - 2C_i - 4C_j + 3w) - a_j(4C_i + 2C_j - 3w)\Big]. \tag{2.14}$$

The fact that a prosumer's optimal cost V_i can be expressed as a function instead of a functional greatly facilitates subsequent analysis. Proposition 1 provides explicit expressions for the cost function $V_i(C_i, C_j, \Pi)$ for two distinct regions of total capacity C_1+C_2 in the microgrid. Because $a_1+a_2+w = a_1+a_2+\frac{b_1-a_1}{2}+\frac{b_2-a_2}{2} = \frac{b_1+a_1}{2}+\frac{b_2+a_2}{2} = \mu_{X_1}+\mu_{X_2}$, the first region addressed in the proposition (i.e., $C_1+C_2 > a_1+a_2+w$) refers to the setting in which the total capacity in the microgrid exceeds its total average demand. The second region (i.e., $C_1+C_2 \leq a_1+a_2+w$) corresponds to the total capacity in the microgrid being below its total average demand. As a result, the economics driving prosumers' electricity costs are different in those two regions. Thus, it is the ordering between total capacity and total (average) demand that determines the actual form of a prosumer's expected cost function V_i. We refer to the region $C_1+C_2 > \mu_{X_1}+\mu_{X_2}$ as the *excess capacity region* and to the region $C_1+C_2 \leq \mu_{X_1}+\mu_{X_2}$ as the *excess demand region*.

In both regions of Proposition 1, the first terms in the expressions for V_i represent the up-front cost of investing in renewable energy generation of given capacity. The second terms denote the expected discounted cost of Prosumer i's purchases of electricity directly from his utility, which occur any time his demand exceeds the sum of his own generation capacity and his counterpart's excess capacity. The third terms in expressions (2.13) and (2.14) represent the expected discounted cost of peer-to-peer purchases of electricity for each Prosumer i, net of his peer-to-peer sales.

To proceed with our analysis, we first provide a structural result concerning the upper and lower bounds on the optimal level of installed renewable generation capacity for each prosumer.

Proposition 2. *The functional $C_i^*\big(\pi(\cdot)\big)$ can be expressed as a function of the marginal value of energy traded in a virtual microgrid (i.e., $C_i^*(\Pi)$). Further, $C_i^*(\Pi) \in \mathbb{I}_i$.*

Even though a virtual microgrid enables peer-to-peer trading of energy, as shown in Proposition 2, no participant in such a microgrid will find it optimal to invest in capacity that is either in excess of his maximum demand or below his minimum demand, regardless of the market clearing price function. As a consequence, in a virtual microgrid considered in this chapter, in spite of their heterogeneity, neither participant acts as only a buyer or only as a seller of electricity throughout the time horizon; instead, every participant is both a seller and a buyer of electricity (albeit at different times).

The following theorem makes use of this structural result to provide an explicit solution for optimal capacity selection for each prosumer in a virtual microgrid, under equilibrium outcomes.

Theorem 2. *There exists a unique Nash equilibrium of the capacity game in a virtual microgrid. Under that equilibrium, Prosumer i's optimal capacity selection is given by:*

$$C_i^*(\Pi) = \begin{cases} \mathcal{C}_i^A(\Pi) & \text{if} \quad \Pi > 4k - \frac{3p_e}{2\lambda}; \\ \mathcal{C}_i^B(\Pi) & \text{if} \quad \Pi \leq 4k - \frac{3p_e}{2\lambda}, \end{cases}$$

where $\mathcal{C}_i^A(\cdot)$ and $\mathcal{C}_i^B(\cdot)$, as functions of the marginal value of energy traded, are defined as:

$$\mathcal{C}_i^A(\Pi) := b_i - \frac{\sqrt{6\lambda k p_e - 4\lambda^2 k\Pi + \lambda^2\Pi^2} - \lambda\Pi}{3p_e - 2\lambda\Pi}(b_i - a_i) \tag{2.15}$$

$$\mathcal{C}_i^B(\Pi) := a_i + \frac{\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k)} + \lambda\Pi - p_e}{p_e + 2\lambda\Pi}(b_i - a_i). \tag{2.16}$$

Theorem 2 identifies two distinct regions of model parameters, each with its own distinct explicit expression for a prosumer's optimal capacity selection in a virtual microgrid. The first region, given by $\Pi > 4k - \frac{3p_e}{2\lambda}$, derives from the excess-capacity region of Proposition 1, while the second region, defined by $\Pi \leq 4k - \frac{3p_e}{2\lambda}$, derives from the excess-demand region. The former can also be interpreted as a region with a lower marginal cost of investment k (i.e., $k < \frac{\Pi}{4} + \frac{3p_e}{8\lambda}$), while the latter can be viewed as a region with a higher marginal cost of investment (i.e., $k \geq \frac{\Pi}{4} + \frac{3p_e}{8\lambda}$). Thus, in the region of lower marginal cost, the total generation capacity optimally invested in a virtual microgrid (i.e., $\mathcal{C}_1^A + \mathcal{C}_2^A$) exceeds the total average demand in the microgrid, while in the region of higher marginal cost, total optimal capacity installed in the microgrid (i.e., $\mathcal{C}_1^B + \mathcal{C}_2^B$) is below the total average demand. Figure 2.1 illustrates those relationships and the results of Theorem 2.

Having derived explicit expressions for a prosumer's optimal capacity selection, we now proceed with the corresponding sensitivity analysis. Our objective in doing so is to illuminate some key features of renewable capacity investments in a virtual microgrid and provide insight into the behavior of microgrid participants in response to changing external parameters. We begin with an analysis of the sensitivity of each prosumer's optimal capacity selection to the utility's base electricity rate p_e and marginal cost k. In what follows, all regularity properties, such as monotonicity

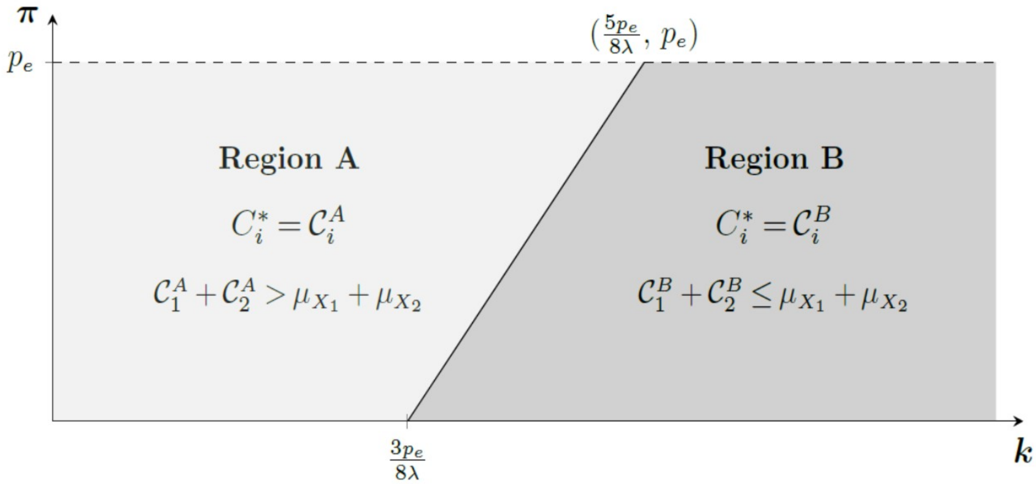

Figure 2.1: Optimality Regions for $C_i^*(\Pi)$

and convexity, are stated in their 'strict' sense.

Proposition 3. *Optimal capacity selection $C_i^*(\Pi)$ is increasing in p_e and decreasing in k.*

To provide intuition for the results of Proposition 3, consider Prosumer 1, whose each unit of capacity $C_1^*(\Pi)$, when operating at time t, is used to generate electricity for sale to Prosumer 2 and/or to satisfy his own demand. If a unit of Prosumer 1's generation capacity is used to satisfy his own demand at any time t, then that unit of capacity is saving him $p_e\, e^{\eta t}$ dollars in costs whenever it replaces a purchase from his electric utility. Hence, the marginal value of each unit of his renewable generation capacity is increasing in p_e, so that Prosumer 1's optimal capacity selection $C_1^*(\Pi)$ is increasing in p_e. The same holds true for Prosumer 2. Similarly, as k increases, the marginal cost of investment in renewable generation capacity is higher, so that $C_1^*(\Pi)$ and $C_2^*(\Pi)$ are reduced.

Next, we explore the sensitivity of Prosumer i's optimal capacity selection to the mean μ_{X_i} and standard deviation σ_{X_i} of his demand distribution. By a well-known property of uniform distributions, $\sigma_{X_i} = (b_i - a_i)^2/12$. Thus, $b_1 - a_1 = b_2 - a_2$ implies that $\sigma_{X_1} = \sigma_{X_2} = \sigma$.

Proposition 4. *Optimal capacity selection $C_i^*(\Pi)$ is increasing in μ_{X_i}, and increasing in σ for $\Pi > 4k - \frac{3p_e}{2\lambda}$, and decreasing in σ for $\Pi \leq 4k - \frac{3p_e}{2\lambda}$.*

It is not surprising that the optimal installed capacity for each participant in a virtual microgrid is increasing in his mean demand, as higher demand for electricity necessitates higher levels of generation capacity. What is perhaps more surprising is that optimal capacity selection $C_i^*(\Pi)$ is not monotonic in demand volatility, but rather increasing in demand volatility for smaller values of k (i.e., for $k < \frac{\Pi}{4} + \frac{3p_e}{2\lambda}$) and decreasing in demand volatility for larger values of k (i.e., for $k \geq \frac{\Pi}{4} + \frac{3p_e}{2\lambda}$). This type of behavior, in which optimal capacity is increasing in the volatility of the underlying demand for low values of the marginal investment cost, and decreasing in that volatility for high values of that marginal cost, was first observed in Angelus (2019) in the context of a very different, single-consumer model of investment in renewable power generation. That the identical qualitative behavior emerges, under equilibrium outcomes, in our model of a blockchain-enabled virtual microgrid, points to a deeper relationship between the volatility of each consumer's demand for electricity and the optimal level of generation capacity used to satisfy that demand.

2.4.3 Peer-to-Peer Pricing of Traded Electricity

Having the explicit solution for each prosumer's optimal level of renewable generation in a virtual microgrid, we now address the market clearing price in a virtual microgrid.

Proposition 5. *Let functions $\Omega(\cdot)$ and $\Phi(\cdot)$ be defined as follows:*

$$\Omega(\Pi) := \frac{\sqrt{6\lambda k p_e - 4\lambda^2 k\Pi + \lambda^2\Pi^2} - \lambda\Pi}{3p_e - 2\lambda\Pi}; \tag{2.17}$$

$$\Phi(\Pi) := \frac{\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\pi^2 + 2p_e\lambda(\Pi - k)} + \lambda\Pi - p_e}{p_e + 2\lambda\Pi}. \tag{2.18}$$

Then, $V_i^*(\Pi) = V_i^*\big(\pi(\cdot)\big) := V_i\big(C_i^*(\pi(\cdot)), C_{-i}^*(\pi(\cdot)), \pi(\cdot)\big)$ is given by:

$$V_i^*(\Pi) = \begin{cases} k\big[b_i - (b_i - a_i)\Omega(\Pi)\big] + \dfrac{2p_e(b_i - a_i)}{3\lambda}\Omega^3(\Pi) & \textit{if } \pi > 4\lambda k - \frac{3}{2}p_e; \\ k\big[a_i + (b_i - a_i)\Phi(\Pi)\big] + \dfrac{p_e(b_i - a_i)\big[3 - 6\Phi(\Pi) + 4\Phi^3(\Pi)\big]}{6\lambda} & \textit{if } \pi \leq 4\lambda k - \frac{3}{2}p_e. \end{cases} \tag{2.19}$$

Proposition 5 provides explicit expressions for V_i^* as a function of Π. Therefore, the market clearing price problem given in (2.10) can be rewritten as:

$$\Pi_i^* := \arg\min_{\Pi} V_i^*(\Pi), \tag{2.20}$$

since the market clearing price function $\pi(t)$ is found, by means of Proposition 2 and 5, to impact equilibrium outcomes only through its integral Π. Consequently, any price function $\pi(t)$ whose integral Π minimizes $V_i^*(\Pi)$ will be a solution to the market clearing price problem given in (2.10).

Theorem 3. *The cost function $V_i^*(\Pi)$ is unimodal, and the following hold for each Consumer i.*
(a) Let Π_i^* be as defined in (2.20). Then Π_i^* is unique, and it is given by

$$\Pi_1^* = \Pi_2^* = \begin{cases} \dfrac{p_e\left(\lambda k + \sqrt{2\lambda k p_e}\right)}{2\lambda(2p_e - \lambda k)} & \text{if } k < \frac{p_e}{2\lambda}; \\[2ex] \dfrac{p_e\left(p_e + 3\lambda k - \sqrt{2p_e(p_e - \lambda k)}\right)}{2\lambda(p_e + \lambda k)} & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases}$$

(b) Each prosumer's optimal cost $V_i^*(\Pi_i^*)$ *is given by*

$$V_i^*(\Pi_i^*) = \begin{cases} k\left(b_i - \dfrac{(b_i - a_i)}{3}\sqrt{\dfrac{2\lambda k}{p_e}}\right) & \text{if } k < \frac{p_e}{2\lambda}; \\[2ex] k\left(a_i + (b_i - a_i)\sqrt{\dfrac{1}{2} - \dfrac{\lambda k}{2p_e}}\right) + \dfrac{(b_i - a_i)\left(3p_e - (2p_e + \lambda k)\sqrt{2 - \frac{2\lambda k}{p_e}}\right)}{6\lambda} & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases}$$

(c) Each prosumer's optimal capacity level $C_i^*(\Pi_i^*)$ *is given by*

$$C_i^*(\Pi_i^*) = \begin{cases} b_i - (b_i - a_i)\sqrt{\dfrac{k\lambda}{2p_e}} & \text{if } k < \frac{p_e}{2\lambda}; \\[2ex] a_i + (b_i - a_i)\sqrt{\dfrac{1}{2} - \dfrac{k\lambda}{2p_e}} & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases}$$

By Theorem 3, the marginal value Π_i^* of the energy traded in a virtual microgrid that minimizes a prosumer's total expected electricity-related costs is identical for each participant in a virtual microgrid. Accordingly, we will denote that value as Π^* and refer to it as the *optimal marginal value of energy traded*. A key implication of Theorem 3 is that it is possible for all participants in a resource-sharing network, even when acting as independent entities, to have identical, cost-minimizing resource-sharing market clearing price functions. The importance of this result lies in the fact that, in the absence of market clearing price disparity in a virtual microgrid, there is no need for a separate market entity to mediate the price-setting process, so that prosumers can self-organize into virtual microgrids on their own. Further, having all microgrid participants agree on their preferred market clearing price function facilitates the design of blockchain protocols for a virtual microgrid because, under equilibrium outcomes, it is not necessary for a blockchain application to keep track of the different buy/sell prices at all times in the system, or to manage real-time engagement of prosumers to in price-setting mechanisms such as continuous energy auctions.

Theorem 3 also provides explicit expressions for the optimal marginal value of energy traded Π_i^*: one for the low values of the marginal cost of investment and another for the high values of that marginal cost. We explicitly characterize the value of Π that minimizes each prosumer's cost function $V_i^*(\Pi)$ even though that cost function is, in general, not convex in Π. (In reality, neither of the two segments that define $V_i^*(\Pi)$ is convex on its own domain). The proof of Theorem 3 relies on a technical result pertaining to a characterization of unimodal functions by means of functional composition (see Lemma 6 in the Appendix).

By providing explicit expressions for optimal capacity level $C_i^*(\Pi_i^*)$ and the optimal marginal value of energy traded Π_i^* associated with each microgrid participant in our model, Theorem 3 completes the characterization of equilibrium outcomes in a blockchain-enabled virtual microgrid.

We next analyze the sensitivity of optimal peer-to-peer electricity pricing, capacity, and each prosumer's optimal cost in a virtual microgrid to some key model parameters.

Proposition 6. *(a) The optimal marginal value of energy traded Π^* is increasing in k, decreasing in p_e for $\lambda k < p_e \leq \frac{3(\sqrt{3}-1)}{2}\lambda k$, and increasing in p_e for $p_e > \frac{3(\sqrt{3}-1)}{2}\lambda k$;*
(b) A prosumer's optimal cost $V_i^*(\Pi_i^*)$ is increasing in k, p_e, μ_{X_i}, and σ;
(c) A prosumer's optimal capacity level $C_i^*(\Pi_i^*)$ is decreasing in k, increasing in p_e, increasing in μ_i, increasing in σ if $k < \frac{p_e}{2\lambda}$, and decreasing in σ if $k \geq \frac{p_e}{2\lambda}$.

To acquire insight into the behavior described in Proposition 6(a), note that increasing k raises the marginal cost of investment in renewable energy. When p_e is fixed, the marginal benefit of this investment increases only with the market clearing price function $\pi(t)$. Consequently, since the optimal marginal value of energy traded Π^* seeks to minimize the total cost to a prosumer, then, as k increases, Π^* must increase as well. The effect of increasing the base rate p_e, on the other hand, is twofold: first, a higher base rate results in higher fees paid by each prosumer for his purchases of electricity from the utility; however, because a unit of renewable energy capacity also replaces, at times, a unit of energy purchased from the utility, a higher base electricity rate also increases the marginal benefit of that capacity. The impact of those two opposing effects on the optimal marginal value of energy traded is such that, for a narrow range of smaller values of p_e (compared to λk) defined by $\left(\lambda k, \frac{3(\sqrt{3}-1)}{2}\lambda k\right]$, the optimal marginal value of energy traded Π^* is decreasing in p_e. For all other values of p_e, Π^* is increasing in p_e. As a result, Π^* is convex in p_e.

By part (b) of Proposition 6, increasing the base electricity rate p_e increases the total cost of electricity for each participant in a virtual microgrid. This follows directly from the expression for his cost rate given in (2.11). Further, increasing mean demand increases his expected cost of satisfying that demand, due to the increased (expected) amount of purchased electricity both from other microgrid participants and his electric utility. Increasing mean demand raises the cost of his investment in renewable energy because, by Proposition 4, optimal capacity selection $C_i^*(\Pi)$ is increasing in μ_{X_i}. As a result, a prosumer's optimal cost $V_i^*(\Pi_i^*)$ is increasing in his average demand. This optimal cost is also increasing in σ, because higher demand volatility leads to increased likelihood of the prosumer's demand exceeding his installed renewable generation capacity and his having to resort to purchasing electricity either from his peers in the microgrid or his utility.

Because k represents the marginal cost of investment in renewable energy generation, it is not surprising that, by part (c) of Proposition 6, optimal capacity level $C_i^*(\Pi^*)$ is decreasing in that marginal cost. Increasing p_e, on the other hand, raises the marginal value of that capacity, so that $C_i^*(\Pi^*)$ is increasing in p_e. Also, $C_i^*(\Pi^*)$ is increasing in demand volatility for low values of marginal cost k, and increasing in demand volatility for high values of k. The intuition for this behavior derives from considerations similar to those for the newsvendor model, in which the optimal order quantity is decreasing in σ for low values of the critical ratio (i.e., high values of the marginal cost), and increasing in σ for high values of the critical ratio (i.e., low values of the marginal cost).

2.4.4 Equilibrium Outcomes Quantified

We now quantify equilibrium outcomes in a blockchain-enabled virtual microgrid. We discretize time into hours, and assume that participants in the microgrid have identical ranges of demand, so that $\mathbb{I}_1 = \mathbb{I}_2 := [a, b]$. We make use of the mean and standard deviation of the hourly residential household energy consumption data in San Francisco for 2013 (US DoE, 2014) to obtain estimates of a and b. The resulting domain of the uniform distribution of a consumer's electricity demand is $[1.15\ kWh, 10.57\ kWh]$. The annual discount rate used is 10%.

To facilitate arriving at insights from our numerical studies, we assume a particular form of the market clearing price function, namely $\pi(t) = \pi e^{\eta t}$, where $\pi := \pi(0)$ is referred to as the *base energy exchange price*. It then follows from the definition of Π that the optimal market clearing price function $\pi^*(t)$ is given by $\pi^*(t) = \pi^* e^{\eta t}$, where $\pi^* = \lambda \Pi^*$. Table 2.1 displays the optimal base energy exchange price π^* as a function of base electricity rate p_e and marginal cost k.

Marg. Cost k (in \$/kW)	Base Electricity Rate p_e (in cents/kWh)										
	10	12	14	16	18	20	22	24	26	28	30
1000	1.57	1.67	1.77	1.86	1.95	2.03	2.11	2.19	2.26	2.33	2.40
2000	2.55	2.68	2.80	2.92	3.03	3.14	3.24	3.35	3.45	3.54	3.64
3000	3.53	3.64	3.76	3.89	4.01	4.14	4.26	4.37	4.49	4.60	4.71
4000	4.58	4.64	4.74	4.86	4.98	5.10	5.23	5.35	5.47	5.60	5.71
5000	5.68	5.71	5.76	5.85	5.95	6.07	6.19	6.32	6.44	6.57	6.69
6000	6.71	6.82	6.85	6.89	6.96	7.06	7.17	7.28	7.41	7.53	7.65
7000	7.68	7.85	7.96	7.99	8.02	8.08	8.17	8.27	8.38	8.50	8.62
8000	8.68	8.83	8.99	9.10	9.13	9.15	9.21	9.28	9.38	9.48	9.60

Table 2.1: Optimal Base Energy Exchange Price π^* (in cents/kWh)

As predicted by Proposition 6, the optimal base energy exchange price π^* shown in Table 2.1 is increasing in both k and p_e. (Because $p_e > \frac{3(\sqrt{3}-1)}{2}\lambda k$ for all parameter choices in Table 2.1, those monotonicity observations are also implied by Proposition 6.) At the same time, π^* appears to be increasing in k much faster than in p_e for the values shown in the table. The optimal base energy exchange price can also be observed to be relatively small, on average, compared to the base electricity rate, as π^* averages only about 31% of p_e across all entries in Table 2.1.

Table 2.2 displays each prosumer's optimal capacity level $C_i^*(\Pi^*)$ as a function of base electricity rate p_e and marginal cost k. (Since prosumers are identical in this numerical study, so are their optimal capacity levels.) In accordance with Proposition 6, that optimal capacity level is found to be decreasing in the marginal cost of investment, and increasing in the base electricity rate. Further, except for a few entries in the bottom right corner, the values in this table exceed the prosumer's average demand for electricity, even at the very high end of the range of that marginal cost.

Marg. Cost k	Base Electricity Rate p_e (in cents/kWh)										
(in $/kW)	10	12	14	16	18	20	22	24	26	28	30
1000	8.32	8.52	8.67	8.79	8.89	8.98	9.05	9.12	9.17	9.23	9.27
2000	7.39	7.66	7.88	8.05	8.20	8.32	8.42	8.52	8.60	8.67	8.73
3000	6.67	7.01	7.28	7.49	7.66	7.81	7.94	8.05	8.15	8.24	8.32
4000	6.07	6.46	6.77	7.01	7.22	7.39	7.54	7.66	7.78	7.88	7.97
5000	5.51	5.98	6.32	6.59	6.82	7.01	7.18	7.32	7.45	7.56	7.66
6000	4.89	5.51	5.91	6.21	6.46	6.67	6.85	7.01	7.15	7.28	7.39
7000	4.14	5.00	5.51	5.86	6.13	6.36	6.56	6.73	6.88	7.01	7.13
8000	3.11	4.41	5.08	5.51	5.83	6.07	6.28	6.46	6.62	6.77	6.89

Table 2.2: Optimal Capacity Level $C_i^*(\pi^*)$ (in kW)

In summary, the set of optimal energy policies $\left\{Z_i^*, Y_{ij}^*\right\}$ and capacity levels $\left\{C_i^*\right\}$, together with the optimal marginal value of energy traded Π^*, represent equilibrium outcomes for participants in the blockchain-enabled virtual microgrid considered in this chapter. Our derivation of explicit expressions for those equilibrium outcomes makes it possible not only to conduct sensitivity analysis on those outcomes, but also to obtain their numerical values and examine their behavior.

2.5 Implications of Virtual Microgrids

We now analyze some key ramifications of blockchain-enabled virtual microgrids. In Section 2.5.1, we compare the expected cost for a prosumer in a virtual microgrid to that of a single consumer, without access to a microgrid, who seeks to minimize his total electricity-related costs of electricity by investing in renewable energy generation. In Section 2.5.2, we evaluate the implications of blockchain-enabled virtual microgrids for the magnitude of renewable energy investments. In Section 2.5.3, we explore how well blockchain technology coordinates the participants of a decentralized virtual microgrid relative to a fully centralized microgrid optimized for total cost.

2.5.1 Cost Savings

To evaluate the implications of a blockchain-enabled microgrid for electricity consumers' cost savings, we now contrast the results obtained in the previous section to those derived herein for a single, cost-minimizing, consumer without access to a virtual microgrid. Without a virtual microgrid, an electricity consumer can still invest in renewable power generation; however, any energy produced by his installed renewable energy capacity in excess of his demand can no longer be sold to another consumer for profit (or to his utility, under the model assumptions), nor can he purchase electricity from anyone other than his electric utility.

In analyzing a single, cost-minimizing consumer, we assume that his (random) demand X for electricity at any time t is stationary and uniformly distributed on $\mathbb{I} := [a,\, b]$. Just like the participants in a blockchain-enabled virtual microgrid, at time $t = 0$, this consumer invests in his own distributed renewable energy generation by deciding on capacity c to install at that time. His marginal cost of investment is also k, so that his total cost of investment at time $t = 0$ is kc.

Following the installation of his renewable generation capacity, this consumer uses that capacity to satisfy his demand for electricity and only resorts to purchasing electricity from the regulated utility when his demand X exceeds his generation capacity c. Therefore, his cost rate at any time t is $p_e\, e^{\eta t}(X - c)^+$. Let $v(c)$ denote the total discounted expected value of all electricity costs incurred by this consumer from $t = 0$ through the end of the time horizon. Thus, we have:

$$v(c) = \int_0^\infty e^{-rt}\, \mathbb{E}_X \big[p_e\, e^{\eta t}(X - c)^+\big]\, dt + kc = p_e \int_0^\infty \int_a^b e^{-\lambda t}\, \mathbb{E}_X \big[(X - c)^+\big]\, dx\, dt + kc. \quad (2.21)$$

This consumer's optimal generation capacity c^* is then given simply as:

$$c^* := \arg\min_{c \geq 0} v(c).$$

We will refer to this optimization problem as *the single consumer's capacity problem*, and to

$v^* := v(c^*)$ as *the single consumer's minimum cost.*

Proposition 7. *For the single consumer's capacity problem, the following hold:*

$$c^* = b - \frac{\lambda k}{p_e}(b-a) \quad \textit{and} \quad v^* = bk - \frac{\lambda k^2(b-a)}{2p_e}.$$

It follows from Proposition 7 that a single consumer's optimal capacity is concave increasing in base electricity rate p_e and decreasing linearly in k, the marginal cost of investment. His minimum cost is concave increasing in each of p_e and k.

To render the comparison between a virtual microgrid participant and a single, cost-minimizing consumer meaningful, we assume that: (a) the participants in a virtual microgrid have identical ranges of their probability distributions of demand, so that $\mathbb{I}_1 = \mathbb{I}_2$; and (b) the range of demand for microgrid participants is identical to that of a single, cost-minimizing consumer without a microgrid, so that $\mathbb{I}_1 = \mathbb{I}_2 = \mathbb{I} = [a, b]$. Consequently, we drop the subscript i from the variables used to describe optimal capacity and optimal cost for each prosumer in a virtual microgrid.

The following key theorem shows that an electricity prosumer always benefits from engaging in peer-to-peer energy trading made possible in a blockchain-enabled virtual microgrid.

Theorem 4. *For any* $\Pi \in \left[0, \frac{p_e}{\lambda}\right)$, $V^*(\Pi) < v^*$.

Theorem 4 guarantees that an electricity prosumer is always better off with a virtual microgrid than without one. This is true regardless of the market clearing price function in the microgrid, so that a blockchain-enabled microgrid need not have the optimal marginal value of energy traded for its constituents to realize cost savings from participating. This theorem also helps explain (and justify) the growing interest in virtual microgrids such as the Brooklyn Microgrid, as each prosumer is better off with it than without, no matter what price he buys and sells electricity for.

The question that naturally arises on the heels of Theorem 4 concerns the exact amount of cost savings generated by a consumer joining a virtual microgrid. To quantify those cost savings and provide meaningful comparisons, we again consider a virtual microgrid with identical prosumers. We assume the same model parameters as those used in Tables 1 and 2 of Section 4.3.

Table 2.3 displays cost savings for a prosumer in a virtual microgrid under equilibrium outcomes relative to the minimum cost of a single, cost-minimizing prosumer without access to a microgrid. Thus, the values shown in Table 2.3 represent the ratio $\frac{v^* - V^*(\Pi^*)}{v^*}$, evaluated across a range of the base electricity rates (from 10 cents/kWh to 30 cents/kWh, in increments of two cents) and marginal costs of investment in renewable generation (from \$1000/kW to \$8000/kW, in increments of \$1000).

The percentage cost savings displayed in the table appear to be concave in both k and p_e. Those cost savings are also at their highest when k and p_e are either both low or both high, and they peak at about 11.1%, regardless of the combination of values used for the marginal cost and base electricity rate. While those savings are not insignificant (averaging 9.7% across the entries in Table 2.3), they also exhibit considerable variation, as they can reach as low as 1.4%.

k	Base Electricity Rate p_e (in cents/kWh)									
(in \$/kW)	10	12	14	16	18	20	22	24	26	28
1000	9.6%	9.1%	8.7%	8.3%	8.0%	7.7%	7.4%	7.2%	7.0%	6.8%
2000	11.0%	10.8%	10.5%	10.2%	9.9%	9.6%	9.3%	9.1%	8.9%	8.7%
3000	11.0%	11.1%	11.1%	10.9%	10.8%	10.6%	10.4%	10.2	10.0%	9.8%
4000	10.1%	10.8%	11.1%	11.1%	11.1	11.0%	10.9%	10.8%	10.6%	10.5%
5000	8.5%	9.9%	10.6%	10.9%	11.1%	11.1%	11.1%	11.1%	11.0%	10.9%
6000	6.3%	8.5%	9.7%	10.4%	10.8%	11.0%	11.1%	11.1%	11.1%	11.1%
7000	3.9%	6.7%	8.5%	9.6%	10.2%	10.7%	10.9%	11.0%	11.1%	11.1%
8000	1.4%	4.7%	7.0%	8.5%	9.5%	10.1%	10.5%	10.8%	11.0%	11.1%

Table 2.3: Consumer Cost Savings Generated by a Virtual Microgrid (in %)

In summary, under equilibrium outcomes, a blockchain-enabled virtual microgrid can generate significant cost savings for an electricity prosumer. At the same time, the exact amount of those savings very much depends on the characteristics of the underlying microgrid electricity market, such as the marginal cost of renewable energy investment and base electricity rate. As a result, even though a prosumer is always better off with a virtual microgrid than without, when analyzing an investment in a virtual microgrid, care should be taken to accurately capture the details of the particular electricity market being considered for the investment.

2.5.2 Size of Renewable Energy Investments

In a virtual microgrid with identical consumers, for any market clearing price function $\pi(t)$, each consumer will install the optimal capacity selection $C^*(\Pi)$. From the perspective of renewable energy investments, a key question becomes whether a consumer would invest more or less in renewable energy generation with a virtual microgrid than without. To answer that question, we compare optimal capacity of a participant in a virtual microgrid under equilibrium outcomes to that of a single, cost-minimizing consumer who invests in renewable energy generation.

It may seem intuitive that a consumer's optimal capacity investment with a virtual microgrid would be higher than without it, because, with a microgrid, each consumer's excess capacity can be used to generate and sell electricity to other microgrid participants. As a result, the marginal value of each unit of renewable generation capacity would appear to be higher with a virtual microgrid, so that the resulting optimal capacity would then necessarily also be higher for any market clearing price function in the microgrid. As it turns out, the reality of the relative sizes of renewable energy investments with and without a virtual microgrid turns out to be more complicated. The following theorem provides an explicit, "threshold-type" resolution of this issue.

Theorem 5. *Let $C^*(\Pi)$ and c^* be as established in Theorem 2 and Proposition 7, respectively. Then, there exist a threshold marginal value $\overline{\Pi}$ of energy traded in a microgrid such that $C^*(\Pi) < c^*$ whenever $\Pi < \overline{\Pi}$, and $C^*(\Pi) \geq c^*$ whenever $\Pi \geq \overline{\Pi}$. Further,*

(a) If $k < \frac{p_e}{2\lambda}$, then $\overline{\Pi} = \frac{p_e(2p_e - 3\lambda k)}{2\lambda(p_e - \lambda k)}$ and $C^*(\Pi^*) < c^*$;

(b) If $k \geq \frac{p_e}{2\lambda}$, then $\overline{\Pi} = \frac{p_e(p_e - \lambda k)}{2\lambda^2 k}$ and $C^*(\Pi^*) \geq c^*$.

By Theorem 5, there exists a *threshold* marginal value of energy traded in a virtual microgrid such that the optimal capacity selection for each participant in the microgrid is higher than the optimal capacity for a single, cost-minimizing consumer without a microgrid, whenever the *actual* marginal value of energy traded exceeds that threshold; otherwise, the single, cost-minimizing consumer optimally invests in a higher level of renewable energy capacity. Therefore, the "marginal value" intuition described above regarding each consumer's optimal renewable energy investments in a virtual microgrid does not (fully) capture the complexity of the underlying capacity decision.

As it turns out, the energy trading option in a virtual microgrid plays a significant role in this decision. This is because, with peer-to-peer energy trading, the capacity decision for each participant in the microgrid involves a unique tradeoff between: (1) buying more capacity, which can be used to satisfy own demand *and* sell electricity to other participants, while incurring a higher investment cost; and (2) buying less capacity, incurring a smaller investment cost, and relying (more) on energy purchases from other participants to meet own demand for electricity.

Hence, when the marginal value of energy traded in a virtual microgrid is small, then in equilibrium, using energy purchases from other participants to satisfy own demand starts to substitute, for each prosumer, for his investing in renewable energy capacity. In other words, at low electricity trading prices in a virtual microgrid, each prosumer starts to rely more on energy purchases rather than his own electricity generation to meet his demand. As a result, his optimal capacity investment ends up being lower than that of a single, cost-minimizing consumer, since the latter, by definition, does not have access to low-priced energy purchases from prosumers in a microgrid.

Similarly, when the marginal value of energy traded in the microgrid is high, then the above tradeoff shifts in the other direction, and each participant in the microgrid finds it more beneficial, in equilibrium, to invest more in renewable generation capacity. This is because, with more generation capacity, he can not only meet more of his own demand with his own generation, but also generate significant revenue from selling electricity to other participants in the microgrid at prevailing higher prices. Hence, with higher electricity trading prices in a virtual microgrid implied by a higher marginal value of energy traded, each prosumer's optimal capacity investment ends up being higher than that of a single, cost-minimizing consumer, as the latter cannot sell his excess electricity.

Theorem 5 also provides explicit expressions for the marginal energy value threshold $\overline{\Pi}$, whose representation depends on whether the marginal cost of investment k is low (i.e., less than $\frac{p_e}{2\lambda}$), or high (i.e., greater than $\frac{p_e}{2\lambda}$). The proof of Theorem 5 establishes that this threshold value is always in the interval $(0, p_e)$, so that this "threshold-type" characteristic of renewable energy investments is present, in all blockchain-enabled virtual microgrids, regardless of the particular features of the underlying electricity market. When marginal cost k is low, each prosumer's optimal capacity level $C^*(\Pi^*)$ is smaller than the single-consumer's optimal capacity c^* because, in that case, $\Pi^* < \overline{\Pi}$. Similarly, when marginal cost k is high, each prosumer's optimal capacity level $C^*(\Pi^*)$ is higher than the single-consumer's optimal capacity c^* since, then, $\Pi^* > \overline{\Pi}$.

In summary, participating in a virtual microgrid and being able to trade electricity with its constituents fundamentally alters the magnitude of a prosumer's investment in renewable energy. This observation, in turn, becomes an important point to keep in mind for those electricity consumers considering joining a virtual microgrid, as that decision can impact their level of capital needed for renewable energy investment and their subsequent cost savings generated. We are not aware of any mention in the research literature or media coverage of blockchain-enabled energy projects (such as the Brooklyn Microgrid) of the fact that consumers who join a virtual microgrid may not want to install the same amount of solar panel capacity, for example, as those who do not.

2.5.3 Coordination Power of Blockchain

Blockchain represents, to the best of our knowledge, the only technology capable of facilitating organization of electricity consumers into decentralized virtual microgrids by coordinating their financial and energy flows for the purpose of peer-to-peer trading of electricity.

In contrast to the decentralized, blockchain-enabled, virtual microgrid analyzed in this chapter, a traditional, physical microgrid comprises a network of energy production and demand nodes, with its own transmission lines, that operates in parallel to the central transmission grid. A number of different designs and algorithms have been used to successfully operate microgrid controllers that centrally manage all operational aspects of such microgrids (Su et al. 2013, Khan et al. 2016, Askarzadeh 2017). Those algorithms generally aim to minimize the total cost in the system. While a virtual microgrid does not require the infrastructure investment needed for a traditional, centrally-controlled microgrid, a key question regarding the use of blockchain technologies to operate a virtual microgrid concerns its power to coordinate the relevant decision makers (i.e., participants in a microgrid) towards not only optimizing their individual cost, but also reducing the total cost in the system, much like centralized controller would do for a traditional microgrid. From the perspective of widespread adoption of virtual microgrids by residential and other consumers, providing insight into this issue is an important task that has remained outstanding in the literature.

For that purpose, in this section we compare the cost performance of a blockchain-enabled virtual microgrid under equilibrium outcomes to the cost performance of a fully centralized microgrid optimized for total cost. Since our focus in this section is on coordination performance only, we do not consider infrastructure costs (i.e., transmission and distribution lines) needed to set up a traditional microgrid; instead, we only analyze direct electricity-related costs, including the cost of investment in renewable power generation incurred by the constituents of such a microgrid.

We retain the assumptions regarding consumer i ($i \in \{1, 2\}$) and his demand for electricity $X_i \sim U[a_i, b_i]$ at time t. In a traditional microgrid, a central agent invests in renewable generation capacity $\mathcal{C}$ at time $t = 0$, and this generation capacity subsequently serves the demand from both consumers. When the demand in such a centralized microgrid (i.e., $X_1 + X_2$) exceeds the installed generation capacity, the microgrid draws power from its electric utility to satisfy the resulting excess demand. Therefore, at any time t, the cost rate of the traditional centralized microgrid is $p_e e^{\eta t}(x_1 + x_2 - \mathcal{C})^+$, where, as usual, x_1 and x_2 denote realizations of random demands X_1 and X_2.

Let $\mathcal{F}(\mathcal{C})$ be the expected discounted present value of all electricity-related costs in a centralized microgrid over the time horizon, including the investment in capacity $\mathcal{C}$ at time $t = 0$. Thus,

$$\mathcal{F}(\mathcal{C}) = \int_0^\infty e^{-\lambda t}\, \mathbb{E}_{X_1, X_2}\left[p_e(x_1 + x_2 - \mathcal{C})^+\right] dt + k\mathcal{C}. \tag{2.22}$$

The central agent chooses the capacity to minimize the expected total cost $\mathcal{F}(\mathcal{C})$. Thus, in a centralized microgrid, we define *the optimal (centralized) capacity* $\mathcal{C}^*$ as

$$\mathcal{C}^* = \arg\min_{\mathcal{C}} \mathcal{F}(\mathcal{C}).$$

Also, we will refer to $\mathcal{F}^* := \mathcal{F}(\mathcal{C}^*)$ as *the minimum (centralized) cost*. The following proposition derives optimal centralized capacity and minimum centralized cost.

Proposition 8. *In a centralized microgrid optimized for total cost, the following hold.*

(a) The optimal centralized capacity is given by

$$\mathcal{C}^* = \begin{cases} a_1 + a_2 + 2w - w\sqrt{\dfrac{2\lambda k}{p_e}} & \textit{if } \ k < \frac{p_e}{2\lambda}; \\ a_1 + a_2 + w\sqrt{2\left(1 - \dfrac{\lambda k}{p_e}\right)} & \textit{if } \ k \geq \frac{p_e}{2\lambda}. \end{cases}$$

(b) The minimum centralized cost is given by

$$\mathcal{F}^* = \begin{cases} k\left(b_1 + b_2 - \dfrac{2w}{3}\sqrt{\dfrac{2\lambda k}{p_e}}\right) & \textit{if } \ k < \frac{p_e}{2\lambda}; \\ k(a_1 + a_2) + \dfrac{w\left(3p_e - 2(p_e - \lambda k)\sqrt{2 - \dfrac{2\lambda k}{p_e}}\right)}{3\lambda} & \textit{if } \ k \geq \frac{p_e}{2\lambda}. \end{cases}$$

Having established analytical expressions for the minimum cost in a traditional, centralized microgrid, we now compare that minimum cost to the total cost in a blockchain-enabled virtual microgrid under equilibrium outcomes. Let $V^* := V_1^*(\Pi_1^*) + V_2^*(\Pi_2^*)$ denote that total cost.

Theorem 6. *For any positive base electricity rate p_e, $\mathcal{F}^* = V^*$.*

By Theorem 6, a virtual microgrid with a decentralized coordination of prosumers by means of blockchain technology achieves the same total minimum cost in the system as a microgrid with a fully centralized control designed to minimize total system costs. At the same time, a blockchain-enabled virtual microgrid does not require expensive infrastructure of central control devices needed to run a centralized microgrid. Further, as already mentioned, a centralized microgrid generally also requires its own transmission and distribution network, whereas a blockchain-enabled microgrid, such as the Brooklyn Microgrid, provides a virtual marketplace for energy trading that runs on the existing transmission grid.

In summary, blockchain can achieve the same degree of coordination of prosumers in a decentralized virtual microgrid (referred to as "virtual horizontal coordination" in Babich and Hilary (2020)) as a centralized microgrid controller set up to minimize total costs in the system. Our finding that a blockchain-enabled microgrid achieves complete virtual horizontal coordination is of significance for future adoption of virtual microgrids in electricity markets worldwide.

2.6 Concluding Remarks

Blockchain-related energy projects are emerging as a major area of application of blockchain technologies, as those technologies are making it possible for individual consumers and producers of renewable energy to bypass existing markets and self-organize into decentralized virtual networks for the purpose of trading self-generated electricity. In this chapter, we provided an analytical model of a decentralized microgrid that captures the dynamics of virtual trading of self-generated electricity and associated peer-to-peer financial and energy flows enabled by blockchain technologies.

For the resulting two-stage, simultaneous noncooperative game, we characterized equilibrium outcomes, including each consumer's optimal energy policy and capacity of installed renewable generation. We also derived the marginal value of energy traded in a virtual microgrid that minimizes each participant's total electricity costs. That value was shown to be increasing in the marginal cost of investment, and convex in the base electricity rate. We found that each prosumer's optimal cost was increasing in all key parameters of the model: base electricity rate, marginal cost of investment, and mean and standard deviation of his demand distribution. His optimal capacity level, on the other hand, is uniformly increasing in all model parameters but the standard deviation of demand.

Some key results, from the perspective of electricity prosumers, concern the implications of virtual microgrids for their cost savings and renewable energy investments. In that regard, we established that each of heterogeneous, cost-minimizing prosumers is always better off within a virtual microgrid than without, regardless of the market clearing price function in the microgrid. A prosumer's expected cost savings, under equilibrium outcomes, were found to average 9.7% in our numerical studies. We also determined that the impact of virtual microgrids on the magnitude of renewable energy investments can be characterized with a threshold structure; that is, there exists a threshold marginal value of energy traded such that the optimal capacity for each prosumer in the microgrid is higher than that of the single, cost-minimizing consumer without a microgrid whenever the actual marginal value of energy traded in the microgrid exceeds that threshold.

With regard to future development and investment into blockchain-related energy projects, our key result is that, in equilibrium, a blockchain-enabled microgrid fully coordinates all the participants of a decentralized virtual microgrid in minimizing the total electricity-related cost in the system. A virtual microgrid therefore achieves the same total cost of electricity for its participants as a fully centralized, traditional microgrid that additionally requires expensive investment in control devices and other expensive equipment. Further, the optimal energy policy derived in this chapter for each participant in a virtual microgrid is easily executable by smart contracts application of blockchain, which facilitates the realization of participant cost savings.

From the perspective of sustainability and community engagement with local renewable energy projects such as the Brooklyn Microgrid, our results provide support for the belief, held by most Brooklyn Microgrid participants, that they are "laying the groundwork" for a future in which communities produce their own electricity, and retain profits from that electricity while moving towards increased renewable energy use and away from fossil fuels (Peck 2016).

We conclude by pointing out some caveats and potential directions for future research. First, we assumed a regulated electricity market in which there is no uncertainty in the price charged for electricity purchased from the central transmission grid. It would be of interest to also consider unregulated electricity markets in which competition would induce price uncertainty, or at least nonstationarity, so that at times it may be more beneficial for a microgrid participant to buy power from the central transmission grid rather than from other prosumers in his virtual microgrid. In that case, one might expect a lower optimal energy exchange price and potentially even greater cost savings for a microgrid participant due to his option to choose, at any time, the lower of the two prices for the purchase of needed electricity.

Second, there may exist circumstances (such as different forms of the demand probability distribution) under which heterogeneous prosumers would not all arrive at the same cost-minimizing marginal value of energy traded in the microgrid. In such settings, it might be necessary to develop an auction-type mechanism for the determination of the peer-to-peer electricity trading price. The resulting analysis would represent a worthwhile extension of our research.

Third, while we assume that electricity generated by a microgrid participant cannot be stored in any way, an important development in electricity markets has been an increased availability of residential battery storage. With battery storage, a prosumer would not necessarily be motivated to always sell all of his excess electricity to other microgrid participants, but would likely want to store some of it for his own future use. With battery storage, the set of equilibrium outcomes would expand to include the optimal capacity of storage installed by each consumer. This would result in increased complexity of the corresponding minimization problem, while potentially enhancing cost savings available to microgrid participants. Research into consumer investment in renewable energy under storage availability would be of great academic and societal interest.

3. TRANSMISSION GRID DEVELOPMENT AND IMPLICATIONS FOR RENEWABLE ENERGY INVESTMENT

3.1 Introduction

Renewable energy has now come to represent an indispensable source of electricity generation. Due to ambitious political targets to reach the deep de-carbonization of electric grids, wind and solar energy installations have grown rapidly over the last decade. In Europe, wind energy installations grew by 10 GW in 2009 and 2010 alone. The cumulative photovoltaic capacity over the European countries reached 29 GW in 2010 (Schaber et al. 2012). At the same time, the growing renewable generation capacity demands an increasing electricity transmission capacity on the current grid infrastructure. Procuring sufficient transmission capacity is considered one of the most urgent tasks to support the sustainable transition of a carbon-dependent economy to a low-carbon economy. Borrowing one wind energy industry stakeholder's comment, "the list of top three challenges for wind industry [are] . . . transmission, transmission, and transmission." (Fischlein et al. 2013)

Currently, transmission capacity expansion planning is administered by federal and state jurisdiction - regional transmission organizations or independent system operators in the United States. Traditionally, the installation of a transmission grid is centrally planned and developed as a response to an increase in market demand in the region. Over the next decade, long-term goals of developing and operating new transmission grids should support the projected growth of renewable electricity generation. Because renewable generation is intermittent and time/space-dependent, high penetration of renewable energy would increase the frequency of over-generation or under-generation compared to the transmission capacity or demand experienced at load centers. This increased imbalance is costly for both energy producers and transmission grid operators alike. For instance, excess electricity is often sold at a negative wholesale price to relieve congestion in transmission grids (Zarnikau 2011, Zhou et al. 2016). In the western part of Texas, wind developers commonly experience a negative wholesale electricity price due to insufficient transmission capability (Zarnikau 2011). Southern California Edison reported that transmission constraints caused the wind energy loss of about 15 MW for 6% – 8% of the time as of 2010 (Rogers et al. 2010).

The limited transmission capability in the current grids has been impeding renewable energy development for more technical and economic constraints. First, the majority of commercial renewable energy projects are developed in rural regions remote from load centers where the current grid infrastructures are built around. Also, because many renewable energy projects have been driven by non-utility independent power producers, political and meteorological factors draw those projects to be developed in regions remote from the load centers (United States Department of Energy 2019). Over the next decade, more renewable energy will be developed in remote offshore. Global Wind Energy Council (GWEC) forecasts that offshore wind capacity alone will surge to 234 GW by 2030 (Pullen and Sawyer 2011). Second, planning and constructing a transmission grid generally takes a much longer time than developing a renewable power plant. For instance, while developing a wind farm takes only about one year, developing a fully operational transmission grid can take at least five years or longer (Fischlein et al. 2013). Without the well-ahead planning to develop sufficient transmission capacity over the next decades, significant social and environmental values generated via renewable energy development can be lost due to the lack of transmission lines.

Developing these long-distance grids has some important differences from merely upgrading the current transmission infrastructure, which has been considered by many jurisdictions, including the U.S. and several European countries. First, renewable projects can be deployed in regions with higher meteorological potential when dedicated transmission lines are available because the development of a renewable energy project is less constrained by the current grid infrastructure. Also, transmission grid planners can avoid expensive upgrades to the current transmission infrastructure. For instance, the cost of upgrading onshore transmission grids in New England to support wind energy deployment was estimated at 1 $ billion. Extended transmission grids will generate savings by reducing the upgrades and repairs needed with the old transmission grids. Second, developing transmission grids can generate a potential of higher revenue for energy producers. Higher transmission capacity implies these energy producers can transmit more energy generated to remote high population centers, and more potential demand fulfilled with renewable energy. Thus, the economic benefit of expanding the transmission network in the United States is estimated at $ 1.5 trillion under a high renewable penetration scenario (Krishnan et al. 2013). Yet, little is known about the strategic questions related to a long-distance transmission grid.

These qualitative arguments motivate me to consider how we can capture the potential of abundant renewable energy sources and avoid economic and social consequences of increasing fossil fuel consumption. In this chapter, we study how a transmission company (TransCo) can optimally invest in a long-distance transmission line, that connects a low-demand remote region to a high-demand urban region, to allow renewable energy development by a power generation company (GenCo) in a geographically remote region. We formulate and solve a continuous-time, infinite-horizon, Stackelberg game between TransCo and GenCo in the presence of stochastic demand for renewable electricity on both ends of the transmission line. We show that transmission and generation act as complements with regard to the value functions for both companies. We determine explicit expressions for optimal transmission and generation capacities installed by each company. We use this result to derive the value-maximizing transmission fee charged by TransCo to GenCo for each unit of energy exported from the low-demand, remote region to the high-demand urban region. Finally, we characterize a Pareto-improving cost-sharing contract through which both companies improve the value of their investment. We identify key drivers of those investments and quantify their size across a range of model parameters.

The remaining of this chapter is organized as follows. In Section 3.2, we review the literature. In Section 3.3, we formulate our model. In Section 3.4 we characterize the structural properties and equilibrium outcomes of the solutions to optimal generation and transmission capacity selection. Section 3.5 explores strategic interaction between transmission and generation, In Section 3.6, we provide concluding remarks.

3.2 Literature Review

This chapter is primarily related to the literature on renewable energy operations. Hu et al. (2015) establish the optimal level of renewable generation capacity under intermittent supply and stochastic demand and show that data granularity matters because coarse data may not reflect intermittency of renewable generation. Kök et al. (2018) study the effects of utilities' different pricing policies (peak vs. flat) on renewable energy investment and find that flat pricing can lead to a higher investment in renewable generation from the utility, while peak pricing leads to increased investment in distributed generation. Aflaki and Netessine (2017) show that higher carbon prices and market liberalization may reduce the share of renewable capacity in the generation portfolio due to the interaction between generation intermittency and pricing. Angelus (2021) identifies optimal timing and capacity of investment in a distributed renewable generation for a consumer whose demand for electricity is stochastically evolving. In contrast to this literature, our focus is on those settings where the return on the investment in renewable energy (through selling electricity to market participants) is constrained by the available transmission capacity.

Several recent papers in the renewable energy literature have dealt with optimal pricing of electricity. Alizamir et al. (2016) determine the optimal feed-in-tariff under information diffusion and the structure of the optimal investment timing for heterogeneous consumers. Kök et al. (2018) determine the optimal peak and flat electricity price. Adelman and Uçkun (2019) develop asymptotically linear, social welfare-maximizing, electricity price schedules for a utility that serves price-responsive strategic consumers. Al-Gwaiz et al. (2017) demonstrate that curtailing renewable generation can increase market competition. Angelus (2021) determines optimal electricity pricing for a utility that serves a consumer who has the option to invest in distributed renewable generation. In contrast to these papers, our research is not concerned with determining the optimal pricing of a utility's electricity pricing. We instead address how the regulated electricity price can change the investments by individual market participants (renewable power companies), which also impacts the optimal level of transmission grid capacity investment.

It is worth mentioning a number of other papers that deal with operations management problems in renewable energy systems. Wu and Kapuscinski (2013) find that curtailing intermittent renewable energy reduces system costs and emissions. Sunar and Birge (2019) show that an undersupply penalty rate can result in commitment inflation by renewable generators. Babich et al. (2020) address how subsidy designs (feed-in-tariff versus tax-rebate) impacts solar panel investment. Agrawal and Yücel (2021) provide managerial insights and policy implications for non-ownership business models for solar energy.

Because our research is concerned with optimizing the transmission capacity investment and impacts for renewable energy thereof, we also contribute to the literature about transmission expansion planning. For general topics in transmission expansion planning, we refer the interested readers to a comprehensive review by Hemmati et al. (2013). Only a few research studies deal with the problems related to expanding transmission infrastructure for renewable energy delivery. Denholm and Sioshansi (2009) establish that energy storage systems reduce transmission costs when they are closely located to wind farms. Qi et al. (2015) determine the optimal level of transmission capacities with energy storage systems and show that the marginal values of storage capacity decrease faster than that of transmission capacity. In contrast to these papers, we first solve for the investment in generation capacity from a renewable power producer, and then determine the optimal level of transmission capacity investment by the transmission company. By separating the problem of investing in transmission grids from the renewable energy investment problem, we address the impact of transmission capacity on renewable energy investment and its strategic implications for the stakeholders in renewable energy and transmission grid infrastructures. Also, different from most previous transmission planning models that minimize the cost of operation and investment, our model provides insights for a new business model for transmission grids by accounting for the revenue generated from electricity transmission grid usage.

3.3 Model Formulation

We consider a setting with two geographically separate regions, a sparsely-populated rural Region 1 and a densely-populated urban Region 2. Each region has its own intrinsic (and uncertain) demand for renewable energy. Region 1 is abundant in renewable energy resources such as solar or wind, while sources of renewable energy in Region 2 are either not existent or not possible to develop due to zoning and other restrictions.

We formulate a continuous-time, infinite-horizon model with two decision makers: a transmission infrastructure company and a renewable power generation company. Transmission companies are generally state-regulated transmission system operators, while renewable power generation companies are typically private, profit-maximizing business entities.

Prior to the start of the time horizon, we assume there are no power transmission lines between Region 1 and Region 2, and no installed renewable power generation in Region 1. At the beginning of the time horizon, the transmission infrastructure company moves first, and decides on the level of transmission (grid) capacity to be installed between Regions 1 and 2. This transmission capacity represents the maximum power (in MW) that can be transmitted over that line at any time t. Knowing the level of transmission grid capacity T to be installed, the renewable power generation company moves next and chooses how much renewable power generation capacity G to develop in Region 2. This generation capacity represents the maximum power (in MW) that can be generated by installed renewable energy sources at any time t. Thus, T and G represent the decision variables in our model.

As linear costs for renewable energy-related investments are standard in the literature (Hu et al. 2015, Aflaki and Netessine 2017, Kök et al. 2018), we assume that the cost to the transmission company to install transmission capacity T is $k_T T$, where k_T is referred to as the *marginal cost of transmission capacity*. Similarly, the cost to the power generation company to develop renewable generation capacity G is $k_G G$, where k_G is the *marginal cost of renewable generation capacity*.

3.3.1 Generation Company's Profit Maximization Problem

Following the investments in transmission and renewable power generation capacity at the beginning of the time horizon, the energy company uses its renewable generation capacity to produce electricity, and generate revenues by selling that electricity both locally to Region 1 and, over the installed transmission line, to Region 2. We assume that the generation company has a locked-in feed-in tariff or a long-term price contract for any renewable electricity it generated over the time horizon. As described in Alizamir et al. (2016), feed-in-tariffs represent commonly used instruments that attract investments in renewable energy by offering long-term guaranteed purchase agreements to producers of renewable energy to sell their electricity into the grid. Thus, for every unit of electricity (expressed in MWh) sold to either Region 1 or Region 2 at any point during the time horizon, the renewable energy company receives p dollars in revenue. Since renewable energy is effectively costless to generate, the variable cost of electricity generation for the generation company is assumed to be zero. Finally, for every unit of energy sold to Region 2 over the installed transmission line, the transmission company imposes a charge s (in \$/MWh) on the generation company.

Let random variables X_1 and X_2 denote demand for electricity at any time t in Regions 1 and 2, respectively. We assume that X_1 and X_2 have continuous probability densities and use x_1 and x_2 to denote particular realizations of X_1 and X_2. As demand is never unbounded in practice, X_1 and X_2 are assumed to have finite support; thus, $X_1 \sim (0, b_1)$ and $X_2 \sim (0, b_2)$ for some b_1 and b_2.

Let $\mathcal{R}_G$ be the revenue rate for the generation company at any time t. Then, for any given level of transmission capacity T and generation capacity G, and the realizations x_1 and x_2 at time t, that revenue rate at time t is given by

$$\mathcal{R}_G(G, T, x_1, x_2) = p \min[x_1, G] + (p - s) \min\big[(G - x_1)^+, x_2, T\big], \tag{3.1}$$

where we use the standard notation $x^+ = \max(x, 0)$. The first term represents the generation company revenue from selling electricity locally to Region 1, while the second term denotes the revenue from its selling any leftover energy over the installed transmission line to Region 2. Due to the transmission fee s, the unit electricity revenue from satisfying the local demand in Region 1 is higher than from selling electricity exported to Region 2. Therefore, the generation company first seeks to sell as much electricity as possible to Region 1 before sending any excess of its generated power over the transmission line to Region 2. Further, the volume of electricity sold to Region 2 is constrained by the installed transmission capacity between the two regions, by the demand in Region 2, and by the amount of power $(G - x_1)^+$ left over after satisfying the (instantaneous) demand x_1 in Region 1.

Let $\Pi_G(G, T)$ represent the generation company's total expected discounted profit over the time horizon, as a function of generation capacity G and transmission capacity T. Let λ denote the continuous-time discount rate. We then obtain that:

$$\Pi_G(G, T) = \int_0^\infty e^{-\lambda t}\ \mathbb{E}_{X_1,X_2}\Big[\mathcal{R}_G(G, T, x_1, x_2)\Big]dt - k_G G, \tag{3.2}$$

where the expectation operator $\mathbb{E}_{X_1,X_2}$ indicates that the expectation is taken over X_1 and X_2.

Because the transmission company decides on the capacity of its transmission investment before the generation company decides on the capacity of its renewable generation investment, the problem for the generation company is to choose, for any given transmission capacity T, its best response function $\overline{G}(T)$ that maximizes $\Pi_G(G, T)$. Thus,

$$\overline{G}(T) := \arg\max_{G \geq 0}\ \Pi_G(G, T). \tag{3.3}$$

We refer to this optimization problem as the *generation capacity selection problem*. Let $\overline{\Pi}_G(T)$, referred to as the *generation company's best expected profit*, be its expected profit under its best response function $\overline{G}(T)$. Thus,

$$\overline{\Pi}_G(T) := \Pi_G\big(\overline{G}(T), T\big). \tag{3.4}$$

3.3.2 Transmission Company's Value Maximization Problem

As a state-regulated system operator, the transmission company seeks to maximize the overall value of its investment in transmission line of capacity T between Region 1 and Region 2. This value consists of both pecuniary and non-pecuniary benefits. The former represents the expected discounted value of the revenue stream from transmission fees paid by the generation company for any electricity sold to Region 2 over the time horizon, net of the investment cost. The corresponding revenue rate to the transmission company, for any given T and G, can therefore be expressed as:

$$s\ \min\big[(G - x_1)^+, x_2, T\big],$$

at any time t. The non-pecuniary benefits to the transmission company derive from providing consumers with renewable energy, which is deemed to be of societal benefit. Following the approach of Babich et al. (2020) in capturing societal benefit to government organizations from renewable energy generation, we assign benefit v to each unit of renewable energy delivered via the installed transmission line and used to satisfy the demand of electricity consumers in Region 2. Thus, the total *benefit rate* to the transmission company at any time during the time horizon, as a function of T, and given the generation company's best response $\overline{G}(T)$, can therefore be expressed as:

$$\mathcal{B}_{T}(T, x_1, x_2) = (s + v)\, \min\big[(\overline{G}(T) - x_1)^+, x_2, T\big]. \tag{3.5}$$

The resulting value function for the transmission company is obtained as:

$$\mathcal{V}_{T}(T) = \int_0^\infty e^{-\lambda t}\, \mathbb{E}_{X_1,X_2}\Big[\, \mathcal{V}_T\big(T, x_1, x_2\big)\Big] dt - k_{T} T. \tag{3.6}$$

The transmission company's objective is to maximize its value function $\mathcal{V}_{T}$ by choosing the optimal transmission capacity T to install. Let T^* denote that *optimal transmission capacity*. Thus,

$$T^* := \underset{T \geq 0}{\arg\max}\ \mathcal{V}_{T}(T), \tag{3.7}$$

which we refer to as *the transmission capacity problem*. We also define the following:

$$\mathcal{V}_{T}^* := \mathcal{V}_{T}(T^*) \qquad \text{referred to as } \textit{the transmission company's optimal value}; \tag{3.8}$$

$$G^* := \overline{G}(T^*) \qquad \text{referred to as } \textit{the optimal generation capacity}; \tag{3.9}$$

$$\Pi_{G}^* := \overline{\Pi}_{G}(T^*) \qquad \text{referred to as } \textit{the generation company's optimal profit}. \tag{3.10}$$

Thus, T^* and G^* represent the values of optimal controls for the problem, while $\mathcal{V}_{T}^*$ and Π_{G}^* refer to the values of the objective functions under those optimal controls for the two entities involved in the power transmission and generation investment problem considered in this chapter.

3.4 Optimality Results

3.4.1 Bounds and Structural Properties

We begin our analysis of the transmission capacity and generation capacity problems formulated in the previous section by providing some bounds on optimal decisions and structural results for those problems. For that purpose, we first introduce some conditions on model parameters.

Because those condition hold in effectively all practical settings relevant to our model, they are non-restrictive.

The revenue from one unit of capacity owned by the generation company and used to produce electricity for sale to Region 2 continuously throughout the time horizon is $\int_0^\infty e^{-\lambda t}(p-s)dt = \frac{(p-s)}{\lambda}$. Because this equation represents the maximum revenue that can be received when a unit of capacity in Region 1 is used to satisfy demand in Region 2, then if $\frac{(p-s)}{\lambda} \geq k_G$, the power generation company would never invest in any renewable energy capacity intended for sale to Region 2. Because our focus is on those settings in which the development of renewable energy sources in Region 1 is intended to (profitably) serve the demand in Region 2, we make the following assumption.

Assumption 4. $k_G < \dfrac{p-s}{\lambda}$.

Similarly, if one unit of transmission capacity is used continuously during the time horizon to export energy from Region 1 to Region 2, the net present value of the benefits received by the transmission company is $\int_0^\infty e^{-\lambda t}(s+v)dt = \frac{(s+v)}{\lambda}$. Consequently, if this maximum discounted benefit received by the transmission company from investing in that one unit of transmission capacity were smaller than the cost of installing that unit, the company would not find it worthwhile to install any transmission grid capacity between Region 1 and Region 2. To avoid such settings that are not relevant to our analysis, we assume the following restriction on model parameters.

Assumption 5. $k_T < \dfrac{s+v}{\lambda}$.

The following lemmas establish upper bounds on optimal decisions for the problem. They derive from the finite support of X_1 and X_2. All proofs are deferred to the Appendix.

Lemma 1. $\overline{G}(T) \leq b_1 + T$.

Consequently, for any given installed transmission capacity T, the generation company will never find it optimal to develop renewable energy capacity in excess of the sum of the maximum demand in its local market in Region 1 and transmission capacity T into Region 2, regardless of the feed-in-tariff rate p. This is due to the fact that any electricity generation in excess of $b_1 + T$ has no place to go, and can therefore produce no revenue for the power generation company.

Lemma 2. $T^* \leq \min[G^*, b_2]$.

Hence, it is never optimal for the transmission company to install any transmission capacity in excess of either the power generation capacity developed in Region 1 or the maximum demand in Region 2. The bounds on optimal decisions described in Lemmas 1 and 2 facilitate also the derivation of our subsequent results regarding optimal values of transmission and generation capacities.

One of the challenges inherent in coordinating supporting investments in transmission and generation capacities concerns the issue of their complementarity. In particular, if the two investments are substitutes, joint investment can be expected to reduce profits for both companies, while if the two investments are complements, joint investment would enhance the profitability of each. The economists' view on this issue is that transmission and generation are both complements and substitutes (see, for example, Joskow (2002)). This is because they act as complements when transmission investment is needed to enable export sales from a generation investment, and as substitutes when transmission capacity starts to substitute for generation capacity at a demand location since it enables purchases of imported electricity. The following theorem resolves this dilemma, in the context of our model of independent investments in renewable power generation at a remote location of low demand and transmission capacity from that location to a (distant) location of high demand.

Proposition 9. *For any G and T such that $T \leq \min[G, b_2]$, G and T are complements with regard to both Π_G and $\mathcal{V}_T$.*

Thus, in the domain of decision variables in which, by Lemma 2, optimal decisions are located, transmission capacity and generation capacity act as complements for both entities in the investment game considered in this chapter. This result holds for any (continuous) probability densities of X_1 and X_2, as well as any level of electricity and transmission pricing. This finding that transmission capacity and generation capacity are complementary in the domain of interest motivates the need to better coordinate the two investments, as each one enhances the profitability of the other.

3.4.2 Characterization of Optimal Generation Capacity

To facilitate our analysis going forward, we assume that X_i for $i \in \{1,2\}$ is uniformly distributed on the interval $\mathbb{I}_i = [0, b_i]$ at any time t. Also, in most practical settings relevant to our model, the demand in remote regions with renewable energy sources is usually at least an order of magnitude smaller than the demand of the closest urban center, whose demand those sources are intended to serve. Thus, to reflect that practical reality that the demand in remote, sparsely-populated Region 1 is significantly smaller than in urban, densely populated Region 2, we assume that $b_2 \geq nb_1$ for some integer $n > 1$. Our first step in characterizing optimal transmission and generation capacities for the problem considered in this chapter involves the derivation of an explicit expression for the generation company's profit function Π_G.

Theorem 7. *The generation company's best expected profit* $\Pi_G(T)$ *is given as follows:*

(a) If $G \in \big[T,\, b_1\big)$*, then*

$$\Pi_G(G,T) = \frac{p}{\lambda b_1}\left[\frac{G^2}{2} + G(b_1 - G)\right] + \frac{p-s}{\lambda b_1 b_2}\left[(b_2 - T)(G-T)T + \frac{b_2 T^2}{2} + \frac{(G-T)T^2}{2} - \frac{T^3}{6}\right] - k_G G. \tag{3.11}$$

(b) If $G \in \big[\max(T, b_1),\, b_1 + T\big]$*, then*

$$\Pi_G(G,T) = \frac{pb_1}{2\lambda} - \frac{p-s}{6\lambda b_1 b_2}\Big[b_1^3 + 3b_1^2(b_2 - G) + 3b_1 G(G - 2b_2) + (3b_2 - G - 2T)(G - T)^2\Big] - k_G G. \tag{3.12}$$

(c) For any $G \in \big[\max(T, b_1),\, b_1 + T\big]$, $\frac{\partial^2 \Pi_G(G,T)}{\partial G^2} < 0$.

Theorem 7 provides explicit expressions for the generation company's profit function $\Pi_G(G,T)$ when $G \in \big[T,\, b_1\big]$ or $G \in \big[\max(T, b_1),\, b_1 + T\big]$. This characterization will suffice for our purposes because, by Lemmas 1 and 2, optimal transmission capacity T^* and optimal generation capacity G^* are never found in the regions where $G < T$ or $G > b_1 + T$.

Next, we define k_0 as the *threshold cost of generation investment* as:

$$k_0 := \left(1 - \frac{b_1}{2b_2}\right)\frac{p-s}{\lambda}, \tag{3.13}$$

and the *threshold transmission capacity* T_0 as:

$$T_0 := b_2 - \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}. \tag{3.14}$$

Since $b_2 \geq 2b_1$ and $k_G < \dfrac{p-s}{\lambda}$ by Assumption 4, threshold transmission capacity T_0 is well-defined. Having introduced those definitions, we are now in a position to characterize the generation company's best response function $\overline{G}(T)$.

Theorem 8. *Define functions* $\mathcal{G}_\alpha, \mathcal{G}_\beta : [0, b_2] \to \mathbb{R}^+$ *as*

$$\mathcal{G}_\alpha(T) := b_1\left(1 - \frac{k_G\lambda}{p}\right) + \frac{(p-s)(2b_2 - T)T}{2pb_2};$$

$$\mathcal{G}_\beta(T) := b_1 + b_2 - \sqrt{(b_2 - T)^2 + \frac{2b_1 b_2 k_G \lambda}{p-s}}.$$

Then, the generation company's best response function $\overline{G}(T)$ *is given as follows:*

(a) If $k_G \leq k_0$ *and* $T \leq T_0$*, or if* $k_G > k_0$*, then* $\overline{G}(T) = \mathcal{G}_\alpha(T)$*;*

(b) If $k_G \leq k_0$ *and* $T > T_0$*, then* $\overline{G}(T) = \mathcal{G}_\beta(T)$*.*

Thus, Theorem 8 obtains two distinct expressions for the generation company's best response function based on the size of the marginal cost of generation capacity k_G and the level of transmission capacity T. The first expression, $\overline{G}(T) = \mathcal{G}_\alpha(T)$ applies either when the marginal cost of power generation investment is high (i.e., $k_G \geq k_0$), or when that marginal cost and transmission capacity T are both low (i.e., $k_G < k_0$ and $T < T_0$). The second expression applies when that marginal cost of investment is low and the transmission capacity is high (i.e., $k_G < k_0$ and $T \geq T_0$).

The following proposition now provides the explicit expression of total expected discounted profit for GenCo following the installment of the optimal generation capacity.

Proposition 10. *Define functions* $\overline{\Pi}_\alpha, \overline{\Pi}_\beta : [0, b_2] \to \mathbb{R}^+$ *as:*

$\overline{\Pi}_{\alpha}(T) =$

$$\frac{12b_1b_2(p-k_G\lambda)\left[b_1b_2(p-k_G\lambda)+(p-s)(2b_2-T)T\right]-(p-s)T^2\left[12b_2^2s+4b_2(p-3s)T-3(p-s)T^2\right]}{24b_1b_2^2\lambda p}; \tag{3.15}$$

$$\overline{\Pi}_{\beta}(T) = \frac{1}{6b_1b_2\lambda}\left\{3b_1^2b_2(p-2k_G\lambda)+2(p-s)(b_2-T)^2\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-(b_2-T)\right)\right.$$
$$\left.-b_1\left[3(p-s)T^2+6b_2^2k_G\lambda-b_2\left(4k_G\lambda\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}+6(p-s)T\right)\right]\right\}. \tag{3.16}$$

The generation company's best expected profit $\overline{\Pi}_G(T)$, for any $T \in [0, b_2]$ is given as follows:
(a) If $k_G \leq k_0$ and $T \leq T_0$, or $k_G > k_0$, then $\overline{\Pi}_G(T) = \overline{\Pi}_{\alpha}(T)$;
(b) If $k_G \leq k_0$ and $T > T_0$, then $\overline{\Pi}_G(T) = \overline{\Pi}_{\beta}(T)$.

We now proceed with the corresponding sensitivity analysis. Our objective is to shed light on some key features of renewable generation capacity investments under a transmission grid and provide insights into the behavior of the generation company in response to varying external parameters. In what follows, all regularity properties, such as monotonicity, are stated in their 'strict' sense.

Proposition 11. *For any $T \in [0, b_2]$, the generation's company best expected profit $\overline{\Pi}_G(T)$ is: (a) increasing in T; and (b) is increasing in p, and decreasing in k_G and s.*

Part (a) of Proposition 11 shows that the profit for generation company (GenCo) is increasing in the transmission capacity. Consider GenCo who installed a best response of generation capacity below the maximum demand of the load center 1. Because GenCo sells electricity first to the load center 1, at any time, the excess electricity can be sold to the load center 2 via the transmission grid. With more transmission capacity, GenCo can sell more electricity to the load center 2. For GenCo, whose installed capacity exceeds the maximum demand of the load center 1, because his generation always exceeds the demand of the load center 1, at any time, the revenue increases in the transmission capacity.

To provide intuition for the results of part (b), consider GenCo, for whom each unit of capacity $C^*(T)$ is used to satisfy the demand of the load center 1 and possibly, to generate electricity transmitted for sale to the load center 2. If a unit of GenCo's generation capacity is used to satisfy the load center 1 demand, then that unit of capacity is generating him a discounted revenue of $\frac{p}{\lambda}$; if his unit capacity is used to satisfy the load center 2 demand, then the unit of capacity is generating him a discounted revenue of $\frac{p-s}{\lambda}$ after accounting for the unit transmission cost s. Hence, the marginal value of each unit of generation capacity is increasing in p, so that GenCo's optimal profit function Π_G^* is increasing in p. Similarly, as k_G or s increases, the profit for GenCo is reduced.

3.4.3 Characterization of Optimal Transmission Capacity

Having elaborated the optimal generation capacity selection, we now address the explicit expression of the total expected discounted profit for transmission company (TransCo).

Proposition 12. *Let*

$$\mathcal{V}_{\alpha,T}(T) := -k_T T + \frac{(s+v)T}{12 b_1 b_2^2 \lambda p} \times \left[6 b_1 b_2 (p - k_G \lambda)(2b_2 - T) - T \left[4b_2(2p-3s)T - 6b_2^2(p-2s) - 3(p-s)T^2 \right] \right]. \tag{3.17}$$

$$\mathcal{V}_{\beta,T}(T) := -k_T T + \frac{s+v}{6 b_1 b_2 \lambda (p-s)} \times \left[2\sqrt{\frac{2 b_1 b_2 k_G \lambda}{p-s} + (b_2 - T)^2} \left[(p-s)(b_2-T)^2 - b_1 b_2 k_G \lambda \right] - (p-s)\left[2b_2^3 - 6b_2^2 T + (3b_1 - 2T)T^2 - 6b_2 T(b_1 - T) \right] \right]. \tag{3.18}$$

The profit function for TransCo is given as follows:

(a) If i) $k_G \geq \bar{k}$ or ii) $k_G < \bar{k}$ and $T < T_0(b_1, b_2)$, then $\mathcal{V}_T(T) = \mathcal{V}_{\alpha,T}$;

(b) If $k_G < \bar{k}$ and $T \geq T_0(b_1, b_2)$, then $\mathcal{V}_T(T) = \mathcal{V}_{\beta,T}$.

Proposition 12 provides two distinct explicit expressions for $\Pi_T(T)$ based on the results of Theorem 8. The first expression derives from the high marginal cost of an energy generation investment (i.e., $k_G \geq \bar{k}$) or low marginal investment cost *and* low transmission capacity (i.e., $k_G <$

$\bar{k}$, $T < T_0(b_1, b_2)$). The second expression draws on the parameter region of the low marginal cost of the energy generation investment *and* high transmission capacity ($k_G < \bar{k}$ and $T \geq T_0(b_1, b_2)$). Moving forward, we assume that $k_T < \frac{(p-k_G\lambda)(s+v)}{p\lambda}$. The following theorem characterizes the optimal transmission capacity selection based on Proposition 12.

Theorem 9. *If $\frac{b_1}{b_2} < \frac{2p-3s}{3p-3s}$ then,*

(a) $\mathcal{V}_{\alpha,T}(T)$ is either unimodal (with a unique maximizer) or strictly increasing in T;

(b) $\mathcal{V}_{\beta,T}$ is either unimodal (with a unique maximizer) or strictly increasing or strictly decreasing in T.

Further, if $\frac{b_1}{b_2} < \frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2pk_T\lambda}{(p-s)(s+v)}}$ then there exists a unique optimal transmission capacity, and it is the maximizer of $\mathcal{V}_{\beta,T}(T)$. Also,

(c) for $k_G < \bar{k}$, $\mathcal{V}_{\alpha,T}$ is strictly increasing in T;

(d) $\mathcal{V}_{\beta,T}$ is either unimodal (with a unique maximizer) or strictly increasing in T.

Note by Proposition 12, one cannot derive a closed-form solution of (3.17) and (3.18) for the optimal level of investment in transmission capacity T^*. However, we can use (3.17) and (3.18) to characterize the optimal transmission capacity. Theorem 9 shows that there exists a unique transmission capacity that maximizes the total value for TransCo. If $\frac{b_1}{b_2} < \frac{2p-3s}{3p-3s}$, then T^* is the unique maximizer of $\mathcal{V}_{\alpha,T}(T)$ or $\mathcal{V}_{\beta,T}(T)$. In addition, if $\frac{b_1}{b_2} < \frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2pk_T\lambda}{(p-s)(s+v)}}$, then $T^* = \arg\max_{T\geq 0} \mathcal{V}_{\beta,T}(T)$. This result lays foundations on understanding a wide range of policies for transmission grid expansion planning in practice. It turns out that one key factor driving the optimality of transmission capacity installed is the relative size of renewable electricity demands at both ends of transmission lines. In general, if there is a potential of huge renewable electricity demands to be captured, TransCo would be better off providing sufficient transmission capacities for GenCo to capture that demand.

3.5 Synergistic Effect of Transmission Grid for Renewable Energy Investment

3.5.1 Pareto Improving Contract Design

Having established that transmission and generation act as complements with regard to the value functions for both TransCo (i.e., transmission company) and GenCo (i.e., generation company), the question that naturally arises concern if GenCo can coordinate with TransCo to mutually increase both companies' value functions. In other words, is there a Pareto-improving contract design for GenCo and TransCo?

In this section, we address when the coordination between GenCo and TransCo can mutually increase both companies' value functions, and structural properties of the contract. Suppose GenCo shares the cost of transmission line investment. Let $r \in [0, 1]$ be the fraction of the total transmission investment cost GenCo pays under the contract. Thus, GenCo's cost for transmission line investment is $rk_T T$. The Transmission line installment cost for TransCo is then $(1 - r)k_T T$. The expected discounted profit for GenCo is now given as:

$$\Pi_{G}(r, G, T) = \int_0^{\infty} e^{-\lambda t}\, \mathbb{E}_{X_1,X_2}[\mathcal{R}_{G}(G, x_1, x_2, T)]dt - k_{G}G - rk_T T. \tag{3.19}$$

Let $\widetilde{G}(r, T)$ and $\widetilde{\Pi}_{G}(r, T)$ be the best response for GenCo, and GenCo's discounted profit under the cost sharing, respectively. By (3.19), $\widetilde{G}(r, T) = \overline{G}(T)$ and $\widetilde{\Pi}_{G}(r, T) = \overline{\Pi}_{G}(T) - rk_{T}T$. Going forward, we will explicitly include r as a variable for the function $\widetilde{\mathcal{V}}_{T}$. The total expected discounted profit for TransCo under the cost sharing contract now becomes:

$$\widetilde{\mathcal{V}}_{T}(r, T) = \int_0^{\infty} e^{-\lambda t}\, \mathbb{E}_{X_1,X_2}[\mathcal{B}_T\big(\overline{G}(r, T), T\big)]dt - (1 - r)k_{T}T, \tag{3.20}$$

Let $\widehat{T}$ be the optimal transmission capacity for TransCO under the cost sharing contract. Then,

$$\widehat{T}(r) = \underset{T \geq 0}{\arg\max}\, \widetilde{\mathcal{V}}_{T}(r, T). \tag{3.21}$$

Finally, let $\widehat{G}$ be the optimal generation capacity for GenCo, and $\widehat{\Pi}_{G}(T)$ represent GenCo's total expected discounted profit under the optimal generation capacity.

$$\widehat{G}(r) = \overline{G}\big(\widehat{T}(r)\big). \tag{3.22}$$

and,

$$\widehat{\Pi}_{G}(r) := \overline{\Pi}_{G}\big(\widehat{T}(r)\big) - rk_{T}\widehat{T}(r). \tag{3.23}$$

We assume throughout this section that $\widehat{T}(0) < f_3(b_1, b_2)$. We first provide the structural analysis based on the above cost sharing contract. The following proposition establishes how the cost sharing contract impacts the optimal transmission capacity.

Proposition 13. *For any $r > 0$, (a) $\dfrac{\partial \widehat{T}(r)}{\partial r} > 0$; (b) $\dfrac{\partial \widetilde{\mathcal{V}}_{T}\big(r, \widehat{T}(r)\big)}{\partial r} > 0$.*

Proposition 13 shows that tcost sharing between GenCo and TransCo increases the optimal investment level for the transmission capacity. It is also established that TransCo would be always better off with the cost sharing contract. These results show that the cost sharing, through GenCo partially paying for the cost of the transmission grid installment, strictly increases the profit for TransCo and TransCo's optimal investment level of the transmission grid capacity. We make use of this structural result to provide necessary and sufficient conditions under which the cost sharing contract improves the expected profit for GenCo in the following Theorem.

Theorem 10. *Let*

$$g_1(r) := \left.\frac{\partial \overline{\Pi}_{G}(T)}{\partial T}\right|_{\widehat{T}(r)}; \tag{3.24}$$

$$g_2(r) := -\left.\frac{\partial^2 \widetilde{\mathcal{V}}_{T}(T)}{\partial T^2}\right|_{\widehat{T}(r)}. \tag{3.25}$$

If $k_T^2 < \dfrac{g_1(0)g_2(0)}{\widehat{T}(0)^2}$, then there exists $\widehat{r} > 0$ such that both GenCo and TransCO are better off.

Theorem 10 shows that under a sufficiently small cost-sharing factor r, the contract mutually increases the total profit (or value) generated by TransCo and GenCo. One direct implication of Theorem 10 for transmission grid investment is the existence of a Pareto-improving cost sharing contract that will simultaneously increase the total value of TransCo and the total profit for GenCo. Considering the majority of renewable energy development has been impeded by the lack of transmission grid capacity, the potential Pareto-improving contract implies that transmission grid planners should start exploring their options to initiate discussions with independent renewable energy power producers from an early stage of transmission grid planning.

3.5.2 Sensitivity Analysis

We now analyze the sensitivity of the optimal level of investment for the transmission capacity and the generation capacity to some key model parameters.

Proposition 14. *(a) $\widehat{T}$ is increasing in p, decreasing in k_T and k_G; (b) $\widehat{C}$ is decreasing in k_T.*

By Proposition 14, increasing electricity price p raises the optimal level of transmission capacity investment. Under a higher electricity price, GenCo would invest in higher generation capacity, ceteris parabus, resulting in a higher marginal revenue for each unit of transmission capacity installed. When p is fixed, the marginal benefit of investment in transmission capacity decreases in the marginal costs of investment k_G, and k_T. Part (b) shows that the optimal generation capacity decreases in the marginal cost of investment in transmission capacity, following from the complementary nature of transmission and generation in our problem.

Next, we show how the optimal level of capacity investment is determined vis-à-vis a given transmission fee s and the social benefit v. In the following results, we include s as a variable for the optimal level of transmission capacity investment $\widehat{T}(s)$.

Theorem 11. *Define*

$$v^* := \frac{b_1 b_2 (p - k_G \lambda) + p\widehat{T}(0)\big(b_2 - \widehat{T}(0)\big)}{\big(2b_2 - \widehat{T}(0)\big)\widehat{T}(0)}.$$

For all $v \geq v^*$, $\widehat{T}(s)$ is decreasing in s. For all $v < v^*$, there exists $s^* \in (0, p)$ such that $\widehat{T}(s)$ is increasing in s if $s \leq s^*$ and decreasing in s if $s > s^*$.

It turns out there are two different scenarios to determine the optimal level of investment in transmission lines. If the social benefit of renewable energy transmission (v) is low, the optimal transmission capacity investment decreases monotonically in the transmission fee. If there exists sufficient social benefit in developing renewable energy, then there exists a unique transmission fee (s^*) that will result in the highest level of investment in transmission lines.

Going forward, we define:

$$\mathcal{W}(s) := \mathcal{V}_T(s, \widehat{T}(s)). \tag{3.26}$$

Hence, the TransCo's optimal total value is a function that explicitly includes the transmission fee s as a variable. The following theorem addresses the structural properties of TransCo's total value as a function of a given transmission fee. Similar to the findings in Theorem 11, we find a "threshold-type" resolution that characterizes the impact of the transmission fee on the optimal total value.

Theorem 12. *Let*

$$\bar{v} := \frac{6b_1b_2(p - k_G\lambda)\big(2b_2 - \widehat{T}(0)\big) + p\widehat{T}(0)\big(6b_2^2 - 8b_2\widehat{T}(0) + 3\widehat{T}(0)^2\big)}{3\widehat{T}(0)\big(2b_2 - \widehat{T}(0)\big)^2}.$$

For all $v \geq \bar{v}$, $\mathcal{W}(s)$ is decreasing in s. For all $v < \bar{v}$, $\mathcal{W}(s)$ is unimodal in s.

Theorem 12 shows the complexity of the optimal pricing policy for transmission with consideration of the social benefit accrued from renewable energy development. By Theorem 12, TransCo's total value decreases in the transmission fee under a high social benefit. While it is a high social benefit of renewable energy that drives the minimum transmission fee, if the perceived social value of renewable energy is low, then there exists a unique transmission fee that maximizes the total value for TransCo. Given that the marginal value of a transmission fee is not self-evident for most transmission companies in practice, this result lays foundations to understand why there exist different types of transmission pricing in practice. While current open access to transmission systems requires transmission pricing based on many dimensions, our result suggests that the social benefit generated from renewable energy development should be accounted for with a higher priority in transmission pricing.

Theorem 12 also shows that pricing transmission in practice depends upon the potential benefits from renewable electricity development. For instance, based on Theorem 12, if the perceived value of renewable electricity production is high, then the minimum transmission fee leads not only to the maximum value generated for TransCo, but also the highest level of investment in transmission capacity that extends the remote renewable energy producers' access to transmission systems.

3.5.3 Optimal Generation Capacity and Transmission Capacity Quantified

We now make use of our results to quantify equilibrium outcomes. We use the maximum demand for electricity $b_1 = 20MW$ for Region 1, and $b_2 = 200MW$ for Region 2. The social benefit from the unit generation of renewable electricity is \$ $30/MWh$. Annual discount rate is 10%. Values shown in Table 3.1 represent the optimal level of generation capacity investment for GenCo, evaluated across a range of feed-in-tariff rates (from 100 \$/MWh to 600 \$/MWh, in increments of 100 \$/MWh) and marginal costs of investment in renewable generation capacity (from 500,000 \$/MWh to 1,200,000 \$/MWh, in increments of 100,000 \$/MWh).

Marg. Gen. Cost	Feed-in-tariff rate p (in \$/MWh)					
k_G (in \$/MW)	100	200	300	400	500	600
500,000	136.33	146.21	153.53	159.43	164.33	168.47
600,000	135.68	145.86	153.28	159.22	164.15	168.31
700,000	135.02	145.51	153.02	159.01	163.97	168.15
800,000	134.36	145.15	152.77	158.81	163.79	167.99
900,000	133.69	144.80	152.51	158.6	163.61	167.83
1,000,000	133.02	144.44	152.25	158.39	163.43	167.67
1,100,000	132.34	144.08	151.99	158.18	163.25	167.51
1,200,000	131.66	143.71	151.73	157.96	163.06	167.34

Table 3.1: Optimal Generation Capacity Quantified (in MW)

Table 3.1 shows that the optimal generation capacity level is increasing in the feed-in-tariff rate p and decreasing in the marginal cost of investment in renewable generation capacity k_G. The values (i.e., level of optimal generation capacity) in this table well exceed the average electricity demand from both regions, even at the very low feed-in-tariff rate or the very high end of the marginal cost of investment in generation capacity.

The values in Table 3.2 represent the level of optimal transmission capacity. It is shown that the optimal transmission capacity level is increasing in the feed-in-tariff rate p and decreasing in the marginal cost of investment in renewable generation capacity k_G. Similar to what we observe from optimal generation capacity, values in this table shows that optimal transmission capacity well exceeds the average electricity demand from both regions.

Marg. Gen. Cost	Feed-in-tariff rate p (in $/MWh)					
k_G (in $/MW)	100	200	300	400	500	600
500,000	119.22	127.84	134.73	140.42	145.19	149.25
600,000	119.13	127.81	134.72	140.41	145.18	149.24
700,000	119.04	127.78	134.7	140.39	145.17	149.24
800,000	118.93	127.74	134.68	140.38	145.16	149.23
900,000	118.81	127.7	134.66	140.36	145.15	149.22
1,000,000	118.67	127.65	134.63	140.35	145.13	149.21
1,100,000	118.53	127.61	134.6	140.33	145.12	149.2
1,200,000	118.38	127.55	134.57	140.31	145.11	149.18

Table 3.2: Optimal Transmission Capacity Quantified (in MW)

To summarize, our derivation of explicit expressions of the total profit (or value) function for TransCo and GenCo allows us to conduct sensitivity analysis on equilibrium outcomes, and to evaluate behaviors of both companies. It is also shown that new transmission lines can generate a significant increase in investment for renewable energy development by providing access to a large demand for renewable electricity in remote, urban centers.

3.6 Discussion

Limited transmission lines hinder the potential development of large shares of renewable energy and impede the transition from fossil fuel dependency to renewable energy supply. To capture the potential of abundant renewable energy sources and avoid the economic and social consequences of fossil fuel consumption, we study a continuous-time, infinite-horizon non-cooperative game between a transmission grid developer (TransCo) and a renewable electricity generation company (GenCo). Based on practical settings grounded in renewable energy development practice, we consider a long-distance transmission line that connects a remote, renewable energy-rich, rural region to an urban, high-population center.

In this chapter, we analyzed a two-stage game of capacity investment between TransCo and GenCo and characterized the resulting equilibrium outcomes, including the capacity of installed renewable generation, and the capacity of the transmission line. GenCo's optimal profit was shown to be increasing in the installed transmission capacity, and the feed-in-tariff rate, while decreasing in the marginal cost of generation capacity investment and the transmission fee. We derived GenCo's optimal investment in generation capacity based on the installed transmission capacity and other key parameters of the model: feed-in tariff, transmission fee, and the marginal cost of generation capacity investment. Although the resulting problem for TransCo's capacity investment was found to have no exact solution, we proved the total value function for TransCo is unimodal in the installed transmission capacity. It is also shown that there exists a unique solution for the transmission capacity investment that maximizes the total value for TransCo.

With regard to the future development and investment in transmission lines and renewable energy, one of the industry challenges inherent in coordinating investments in transmission and generation capacities concerns the issue of their complementarity. We found that investment in transmission capacity and generation capacity act as complements for both TransCo and GenCo in the investment game, for any probability densities of regional demand, as well as any level of electricity and transmission fee. As these two investments are complements, joint investments can enhance the profitability of both companies. To capture the potential benefits of coordination, we consider a simple cost-sharing contract, where GenCo pays a fraction of the total cost of transmission line investments. We proved the existence of a Pareto-improving contract that mutually increased the profitability of TransCo and GenCo. To the best of our knowledge, this is the first time that the results about synergistic effects of transmission capacity and generation capacity have been established in the literature, despite the acknowledged importance of transmission capacity for the future of renewable energy development and global climate goals. From the perspectives of renewable energy developers and transmission grid operators, the analyses in this chapter provide support to a joint effort by TransCo and GenCo on transmission grid investments to achieve mutually increased benefits for both entities.

We conclude this chapter by discussing potential avenues for future research. One can extend the research in this chapter by including some other technologies that support the integration of growing renewable generation capacity in lieu of costly investment in transmission grids. Energy storage, for instance, can be used as a major buffer to reduce transmission congestion by storing over-generated energy for later use in times of supply shortage and decreasing the variability of energy generation. The value of energy storage systems for transmission networks has been extensively reported in Electric Advisory Committee (EAC) reports. Although not competing directly with long-distance transmission grids as considered in this chapter, it will be of interest how an interconnected residential-scale distributed generation system impacts the profitability of renewable energy producers and transmission grid developers. For instance, the deployment of a microgrid system, an interconnected distributed generation system that can generate and distribute energy off the main utility's grid, would reduce transmission capacity requirements because some energy can be consumed at the site of generation without any transmission. Although these two technologies are by no means substitutes for long-distance transmission lines, extending the analysis in this chapter can be worth investigating and will enrich our understanding of the transmission grid investment for renewable energy development. Another potential avenue of research lies in transmission network expansion. It would be of interest how the economic and social values of transmission lines change as more nodes are connected to the network. The optimal design and pricing in a transmission network that maximizes the profitability of the transmission company and minimizes the transmission congestion are of significant importance for the future of renewable energy development as well as for the reaching global climate goals.

4. HYBRID CROSS-DOCKING OPERATIONS IN AN ENERGY SUPPLY CHAIN

4.1 Introduction

Cross-docking is an industry practice that utilizes a facility (also referred to as "the cross-dock") to receive products from suppliers and sort those products into respective groupings corresponding to the downstream supply chain members' requirements (Vogt 2004). With *traditional cross-docking*, inbound product boxes and pallets from suppliers are sorted at the cross-dock for immediate outbound shipment to plants (Gürbüz et al. 2007). As traditional cross-docking does not allow for any holding of inventory at the cross-dock facility, this supply chain practice has been most successfully used for high-volume, short leadtime products in steady demand contexts, such as perishable consumer goods. Walmart, for instance, delivers 85% of its merchandise to its stores using traditional cross-docking to reduce the time it takes to get that merchandise to store shelves and minimize the resulting costs of inventory holding (Ruffa 2008).

By comparison, the supply chains for some industries in the energy sector are characterized by long leadtimes from suppliers, and low-volume, slow-moving, non-perishable products. One such industry is the oilfield service business that provides equipment and services needed to construct and maintain oil wells. Because "oilfield services companies have established themselves as the heavy lifters of the oil and gas industry" (Marcel et al. 2016), the global oilfield services market size reached $267 billion in 2019 and is projected to exceed $346 billion by 2027 (Fortune Business Insights 2020).

The particular features of the supply chain for oilfield service companies have given rise to a novel supply chain practice referred to as *hybrid cross-docking*. In contrast to traditional cross-docking, hybrid cross-docking allows for (a certain degree of) inventory holding at the cross-dock in order to accommodate slow-moving products with long lead times and demand variability. Accordingly, *the hybrid cross-dock* refers to a cross-dock facility with sufficient storage space to allow incoming products to be held in inventory from one period to another. Products arriving at the hybrid cross-dock in each period are then blended with products held in inventory to complete outbound shipments to oil-well plants. Hence, while traditional cross-docks with high-volume, steady-demand products are focused on managing transportation costs only, managers of a hybrid cross-dock are compelled to also take into account their inventory holding costs.

Our research into hybrid cross-docking supply chains emerged from a collaboration with a major oilfield service company (OSC) that is a key provider of products and services required for the drilling and maintenance of oil wells in the Middle East. OSC faces challenges in managing its operations due to three key factors: (1) decentralized nature of its supply chain, in which different business entities operate at different stages in the system; (2) high costs of transportation and inventory management at the cross-dock facility, due to product variety and demand forecasts that vary from one period to another; and (3) evolving supply chain structures, due to upstream and downstream collaborations and outsourcing in the system.

In this chapter, we seek to address those challenges and take a step in the direction of providing an understanding of how to better manage decentralized, hybrid cross-docking supply chains by characterizing optimal shipping, collaboration, and outsourcing decisions. In spite of the growing importance of hybrid cross-docking in the oilfield services industry and certain other industries as well (Kulwiec 2004), little is presently known about how to manage inventory and make transportation decisions in hybrid cross-docking supply chains, the type of collaborations their inventory flexibility allows, and the cost savings potentially realized by those collaborations.

Consequently, our first objective in this chapter is to determine conditions under which it is optimal for the cross-dock to hold inventory. Thus, we seek to establish when a hybrid cross-dock results in lower total costs for an oilfield service company (OSC) relative to the traditional (i.e., pure) cross-dock. In that manner, we aim to provide a roadmap for those managers seeking to realize cost savings from hybrid cross-docking facilities. Our second objective pertains to two types of collaborations currently being evaluated in the industry partner's supply chain. In particular, through "upstream collaboration," suppliers from geographically proximate regions group their product deliveries into joint shipments to the cross-dock facility. By means of "downstream collaboration" on the other hand, multiple oil well plants consolidate their individual shipments from the cross-dock facility at an intermediate downstream warehouse. To assess benefits generated by these collaborations, we provide structural results and quantify the resulting cost savings. While joint transportation and inventory sharing has been studied in the freight transportation literature (Cruijssen et al. 2007, Krajewska et al. 2008), we am not aware of any research on collaborations in cross-docking supply chains.

The third objective is related to the fact that oil well plants, as customers of oilfield service companies, have increasingly been turning to independent, third-party logistics providers in an effort to outsource their transportation and inventory management and thus reduce their costs. Due to the impact of such outsourcing on cross-docking operations and the structure of the resulting cross-docking supply chain, it is of importance to our OSC industry partner (as well as to the oil wells it serves) to understand conditions under which oil well managers find it optimal to outsource their operations to an independent logistics provider. Although outsourcing of operations by downstream supply chain members has been considered in the literature on joint transportation and inventory replenishment (Cetinkaya and Lee 2000, Teo and Shu 2004, Song et al. 2008), the conditions under which it is profitable for individual oil well owners to outsource their operations have not been well understood, especially in the context of cross-docking operations. Thus, we aim to provide an analysis of such conditions and derive relevant managerial insights.

To achieve our first two objectives, we formulate multi-period, mixed-integer programming models to arrive at optimal shipping policies under different supply chain structures. We make use of those results to obtain corresponding optimal costs and compare those across the supply chain structures of interest. Consequently, our first contribution is to identify structural properties of optimal solutions for an individual supply chain member's transportation and inventory management problems. These structural results make it possible for us to determine conditions under which a cross-dock facility should optimally operate as a hybrid cross-dock rather than as a traditional one. Our second contribution is to quantify the value of collaborations in a cross-docking supply chain. We find that upstream collaboration results in 4.9% to 16.4% in average cost savings for the cross-dock, while downstream collaboration results in 1.9% to 22.1% in average cost savings for the oil well plants, depending on product weight and unit holding costs.

To assess the impact of outsourcing on cross-docking operations, we formulate and solve a Stackelberg pricing game in which a third-party logistics company first sets an outsourcing price (per unit weight of product) for its services. Subsequently, oil well plants decide whether or not to outsource their operations to the logistics company. Based on the optimal response of each plant to the outsourcing price set by a logistics provider, that provider determines its profit-maximizing price for its outsourcing services. Accordingly, our third contribution is to identify the structure of the optimal outsourcing decision for each plant as a function of the outsourcing price, and establish the structure of the logistics provider's optimal pricing policy under both deterministic and stochastic demand forecasts.

The remainder of the chapter is organized as follows. Section 4.2 presents the literature review. In Section 4.3, we formulate optimization problems in a cross-docking supply chain with and without collaborations. Section 4.4 provides structural properties of solutions to those problems. Section 4.5 quantifies the impact of collaboration on individual supply chain members. In Section 4.6, we derive equilibrium outcomes for the outsourcing/pricing game played between plants and a third-party logistics company, and quantify those outcomes under deterministic demand forecasts. Section 4.7 generalizes those results to stochastic demand forecasts. In Section 4.8, we provide concluding remarks.

4.2 Literature Review

Our research is related to three streams of research. The first of those pertains to the capacitated lot-sizing problem. The literature in this stream generally assumes deterministic demand and deals with either multiple products or multiple stages in the system. In the *single-stage*, multi-product, capacitated lot-sizing literature,

Yano and Newman (2001) reduce a two-product lot-sizing problem to an equivalent single-product problem with aggregated demands. Anily and Tzur (2005) prove that a multi-product lot-sizing problem is polynomially solvable if transportation and inventory holding cost are constant. Federgruen et al. (2007) consider a capacitated multi-product lot-sizing problem and arrive at asymptotically optimal solutions. In the *single-product*, multi-stage capacitated lot-sizing literature, Florian et al. (1980) prove that a lot-sizing problem in a serial system with stationary capacity is NP-hard. Van Hoesel et al. (2005) show that the multi-stage lot-sizing problem is solvable in polynomial time if transportation and inventory cost functions are linear. Zhang et al. (2012) propose a heuristic solution algorithm for a multi-stage, lot-sizing problem with intermediate demand. Zhao and Zhang (2020) establish that capacitated multi-stage lot-sizing problems with intermediate demand are NP-hard. Research in this chapter builds on this research to provide analyses of *multi-product, multi-stage* capacitated lot-sizing problems. In our model, the intermediate stage is the cross-dock facility that faces demand from plants (i.e., oil wells). Besides being NP-hard (Zhao and Zhang 2020), multi-product, multi-stage problems addressed in our chapter pose an additional challenge in the form of "sorting" - a regrouping of incoming products at intermediate stages into outgoing shipments. (Sorting is not present in either single-stage or single-product lot-sizing problems, as the need for sorting arises from having multiple products and at least one intermediate stage where those products need to be regrouped).

Work	**Number of Products**	**Supply Chain System**	**Operations at the Distribution Center**
Chen et al. (1994)	Single vehicle (batch) with time-varying capacities	Single-product	Inventory only
Anily and Tzur (2005)	Multiple-products	Single-Stage	Inventory Mngmt.
Van Hoesel et al. (2005)	Single-product	Two Stages	Inventory Mngmt. & Distr.
Federgruen et al. (2007)	Multiple-products	Single-Stage	Inventory Mngmt.
Zhang et al. (2012)	Single-product	Multiple-Stages	Inventory Mngmt. & Distr.
Zhao and Zhang (2020)	Single-product	Multiple-Stages	Inventory Mngmt. & Distr.
Levi and Shi (2013)	Uncapacitated single vehicle	single-product	Inventory only
Our chapter	Multiple-products	Three Stages	Inventory Mngmt. & Sorting & Distr.

Table 4.1: Comparison of Capacitated Lot-Sizing Research

The second related literature stream concerns joint replenishment and (capacitated) transportation problems. Of particular relevance to our work are the papers that deal with joint inventory and transportation decisions under deterministic demands. Those papers generally consider single-product, two-stage systems consisting of a warehouse and multiple plants. Federgruen and Zheng (1992) propose a transportation and inventory holding policy that obtains a near-optimal system-wide cost. Chen et al. (2001) establish that the joint replenishment problem in such a distribution system is NP-hard. Gürbüz et al. (2007) analyze a hybrid policy for a joint inventory and transportation problem in which the warehouse monitors the inventory position at plants and replenishes their inventory based on an order-up-to inventory policy. Using linear programming relaxation, Levi et al. (2008) derive a near-optimal policy guaranteed to achieve cost within 80% of the optimal. our work contributes to this literature by addressing joint inventory replenishment and transportation problems in a multi-product, three-stage, hybrid-cross docking distribution system, and under both decentralized and collaborative settings.

To the best of our knowledge, the only paper to address transportation decisions in a hybrid cross-docking supply chain is Jones et al. (2017). They consider a centralized hybrid-cross docking supply chain under full container loads and focus on complexity analysis and heuristic solution algorithms. In contrast, to better reflect current practice in the oilfield services industry, we analyze a decentralized, hybrid cross-docking supply chain with suppliers, cross-dock, and plants acting as independent decision makers. We allow both partial and full container loads, so that each decision maker has to decide not only on the number of containers (also, trucks) to use, but also how much product to have in each container (also, truck). Further, our focus is on structural properties of optimal transportation decisions and cost savings that arise from upstream and downstream collaborations rather than on heuristic solutions. The scope of our study also includes outsourcing decisions by downstream supply chain members and the resulting optimal pricing by a logistics provider.

The third literature stream of relevance to our research is concerned with using pricing contracts to reduce costs in a supply chain. Cachon (2003) provides a comprehensive review of this literature. A number of papers consider pricing contracts in single-period, stochastic demand, newsvendor-type settings. Lariviere and Porteus (2001) explore the efficiency of a pricing contract between a price-setting supplier and a newsvendor plant who determines a stocking level. Dong and Rudi (2004) show that a price-setting supplier benefits from lower demand correlation when inventory can be transshipped among multiple plants. Chen et al. (2001) establish that, with non-identical plants, nonlinear quantity discount contracts cannot achieve full supply chain coordination. Bernstein and Federgruen (2005) extend the study of pricing contracts to competing plants with demand uncertainty. Bernstein et al. (2006) show that interdependencies among supply chain members make it possible to achieve full supply chain coordination. In our model, we address a simple pricing contract between plants and an independent logistics provider in a multi-period, multi-product, multi-stage system. While our basic model assumes deterministic demand forecasts, we extend the structural results to settings with stochastic demand forecasts.

4.3 Model Formulation

Motivated by the supply chain of our industry partner, we consider a three-stage cross-docking supply chain consisting of suppliers, a cross-dock facility, and plants (i.e., oil well facilities). Each oil well facility determines its monthly demand forecast for products for the entire planning horizon (consisting of three to six months) and places the corresponding orders with the cross-dock in advance of the planning horizon. Each oil well is responsible for picking up their ordered products at the cross-dock in each period, transporting those products to its own storage facility, and storing them until they are needed. Each oil well minimizes its own associated transportation and inventory holding costs by determining the optimal amount of each product to pick up from the cross-dock in each period and the optimal number of trucks to use to transport those products from the cross-dock to its storage facility (hereon, we will refer to the oil well (plants) supplied by the cross-dock facility as “plants”).

Based on the orders received from the plants, the OSC procures products from overseas suppliers whose deliveries arrive in each period to a cross-dock facility that the OSC owns and operates in Dubai’s Free-Trade-Zone (see Figure 4.1). In particular, the OSC decides on the amount of each product and the number of containers to be used for the shipping of each product from each overseas supplier in each period. In the current practice, products are typically single-sourced (i.e., one supplier per product). Products that arrive at the cross-dock facility are either sorted for immediate pick-up by the plant or they are placed in inventory for pick-up in a later period. The cross-dock seeks to minimize the total cost of transporting products from the suppliers and holding (some of) them in inventory while satisfying the oil wells' demands. In this supply chain, backlogging is not allowed, as the required products are generally indispensable for the proper functioning of oil wells.

Because the plants submit their orders for the entire planning horizon to the cross-dock prior to the beginning of that planning horizon, those demand forecasts can be viewed as deterministic. (Given the NP-hard nature of the capacitated lot-sizing problem (Florian et al. 1980), it is also standard in the literature to work with deterministic demand models (Yano and Newman 2001, Anily and Tzur 2005, Federgruen et al. 2007). In our study, capacity constraints take the form of the maximum weight of each container shipped to the cross dock, as well as the maximum weight of products on each truck used by the plants to pick up orders from the cross-dock. Given the duration of each period (i.e., a month), and the proximity of Middle East oil wells to the cross-dock facility in Dubai, the transportation from the cross-dock to the plants occurs effectively within the same period.

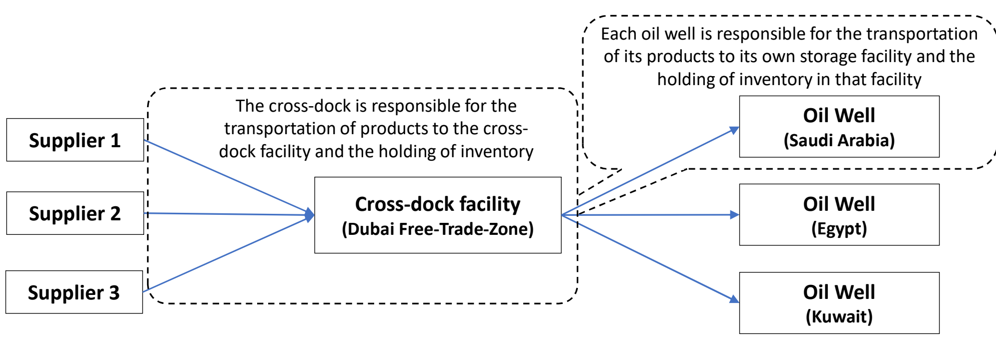

Figure 4.1: Oilfield Service Cross-Docking in a Decentralized Supply Chain

Section 4.3.1 formulates the cross-dock's optimization problem in a decentralized supply chain. Section 4.3.2 presents the cross-dock's optimization problem under upstream collaboration. In Section 4.3.3, we formulate a plant's optimization problem in a decentralized supply chain. Section 4.3.4 presents a plant's optimization problem under downstream collaboration. In what follows, we use $\lfloor\ \rfloor$ to denote the floor function (e.g., $\lfloor 1.5 \rfloor = 1$), and $\lceil\ \rceil$ to denote the ceiling function (e.g., $\lceil 1.5 \rceil = 2$).

4.3.1 Cross-Docking Operations in a Decentralized Supply Chain

The total cost for the cross dock incurred over the planning horizon consists of the cost of shipping products from overseas suppliers to the cross-dock facility and the cost of holding products in inventory until they are picked up by the plants. The cross-dock seeks to minimize those costs by choosing the quantity of each product to be delivered by overseas suppliers in each period and the number of containers in which to ship those products. Because the cross-dock receives orders from the plants for each period in the planning horizon before the beginning of that horizon, the cross-dock's orders for delivery in each period t can be also be made prior to the beginning of the planning horizon. The following summarizes our notation for the cross-dock's cost minimization problem.

PARAMETERS:

- T Number of periods in the planning horizon;
- Q Number of different product types demanded by the plants
- d_{it} Aggregate amount of product i demanded by the plants from the cross-dock in period t;
- h_{cd} Cost of holding a unit weight of product in inventory at the cross-dock for a single period;
- w_i Weight of a product i;
- c_{cd} Cost of shipping one container from a supplier site to the cross-dock facility;
- W Capacity (in weight) of a shipping container;
- $D_{i\tau}$ Cumulative demand for product i from period 1 through period τ: $D_{i\tau} = \sum_{t=1}^{\tau} d_{it}$.

DECISION VARI-

ABLES:

n_{it} Number of containers with product i delivered to cross-dock at the beginning of period t;

y_{it} Quantity of product i (in units) delivered to cross-dock at the beginning of period t.

Because each product is single-sourced, in a decentralized supply chain the cost-minimization problem for the cross-dock is separable in the products. Consequently, to provide an understanding of the structural properties of optimal solutions for the cross-dock's cost-minimization problem, it suffices to consider only a single product. For that reason, in this section, we drop index i from the notation. Let $\mathcal{C}_{CD}(\mathbf{n},\mathbf{y})$ denote the cross-dock's total shipment and inventory holding costs over the planning horizon incurred by shipping container vector $\mathbf{n} = \{n_1, n_2, \ldots, n_T\}$ and product quantity vector $\mathbf{y} = \{y_1, y_2, \ldots, y_T\}$. We refer to the decision set $(\mathbf{n},\mathbf{y})$ as *a shipping plan*.

Let I_t denote the inventory level of product at the cross-dock at the end of period t associated with shipping plan $(\mathbf{n},\mathbf{y})$. The objective function of the cross-dock is given by the sum of inventory holding and shipping costs over the planning horizon. (Due to the short length of that horizon, we forgo any discounting). Hence,

$$\mathcal{C}_{CD}(\mathbf{n},\mathbf{y}) = \sum_{t=1}^{T}(h_{cd} w I_t + c_{cd} n_t). \tag{4.1}$$

The following mixed-integer program represents the cross-dock's single-product cost minimization problem in a decentralized supply chain.

<u>Cross-dock's Cost Minimization in a Decentralized Supply Chain</u>:

$$
\begin{aligned}
\min_{\mathbf{n},\mathbf{y}} \quad & \mathcal{C}_{CD}(\mathbf{n},\mathbf{y}) && && (4.2)\\
\text{s.t.} \quad & I_t = I_{t-1} + y_t - d_t && t = 1,\ldots,T; && (4.3)\\
& Wn_t \geq wy_t && t = 1,\ldots,T; && (4.4)\\
& I_0 = 0 && &&\\
& I_t \geq 0 \text{ and integer} && t = 1,\ldots,T; &&\\
& n_t \text{ integer} && t = 1,\ldots,T. &&
\end{aligned}
$$

Expressions (4.3) represent inventory balance equations given that demands from plants have to be satisfied in every period, while (4.4) denotes the relevant capacity constraints. The latter ensure that in any period, the total weight of the product shipped to the cross-dock facility does not exceed the total capacity of the containers shipped.

4.3.2 Cross-Docking Operations under Upstream Collaboration

Because additional facilities are increasingly being built around oil wells in the Middle East, ordering products to construct and maintain those facilities and holding them in inventory has been incurring significant transportation and inventory holding costs for the focal company. For that reason, the OSC has been looking into supply chain strategies to reduce those costs. One such strategy, referred to as *upstream collaboration*, involves having suppliers in geographically proximate regions share containers to ship products from their overseas locations to the cross-dock.

To provide insight into cost savings enabled by upstream collaboration, we formulate a cost minimization problem for the cross-dock with Q inbound products, one from each supplier, in which geographically-proximate suppliers share containers to ship their products.

Let $\mathcal{C}_{uc}(\mathbf{n}_c, \mathbf{y})$ denote the cross dock's total cost over the planning horizon under upstream collaboration incurred by shipping container vector $\mathbf{n}_c = \{n_1, n_2, \ldots, n_T\}$ and product quantity matrix $\mathbf{y} = \{\mathbf{y_1}, \ldots, \mathbf{y_Q}\}$, where $\mathbf{y_i} := \{y_{i1}, y_{i2}, \ldots, y_{i,T}\}^T$. (Under upstream collaboration, suppliers share shipping containers, but each supplier loads his own ordered amount of product into each shared container). Let I_{it} denote the inventory level of product i at the cross-dock facility at the end of period t associated with shipping plan $(\mathbf{n}_c, \mathbf{y})$. Under upstream collaboration, the objective cost function $\mathcal{C}_{uc}$ for the cross-dock can therefore be written as a sum of inventory holding and shipping costs over the planning horizon as follows:

$$\mathcal{C}_{uc}(\mathbf{n}_c, \mathbf{y}) = \sum_{t=1}^{T}\sum_{i=1}^{Q} h_{cd}\, w_i I_{it} + \sum_{t=1}^{T} c_{cd} n_t.$$

The following formulation represents the resulting cross-dock's cost minimization problem under upstream collaboration.

<u>Cross-dock's Cost Minimization under Upstream Collaboration</u>:

$$\begin{aligned}
\min_{\mathbf{n}_c, \mathbf{y}} \quad & \mathcal{C}_{uc}(\mathbf{n}_c, \mathbf{y}) && && (4.5)\\
\text{s.t.} \quad & I_{it} = I_{i,t-1} + y_{it} - d_{it} && i = 1, \ldots, Q; \; t = 1, \ldots, T, && (4.6)\\
& W n_t \geq \sum_{i=1}^{Q} w_i y_{it} && t = 1, \ldots, T; && (4.7)\\
& I_{i0} = 0 && i = 1, \ldots, Q; &&\\
& I_{it}, y_{it} \geq 0 && i = 1, \ldots, Q; \; t = 1, \ldots, T; &&\\
& n_t \geq 0 \text{ and integer} && t = 1, \ldots, T. &&
\end{aligned}$$

While inventory balance equations given in (4.6) are identical to those in a decentralized supply chain, the capacity constraints given in (4.7) are different. This is because, under upstream collaboration, two suppliers share containers to (jointly) ship their individual products.

4.3.3 Plant's Operations in a Decentralized Supply Chain

Given its demand forecast, each plant determines its order schedule to minimize its total costs over the planning horizon. Those costs include transportation costs for trucks used to transport products from the cross-dock to the oil well facility and inventory holding costs at the site for any products ordered in excess of the demand in each period. A plant seeks to minimize those costs by choosing the number of trucks to use from the cross-dock to its site, and the quantity of each product to transport on those trucks. Similar to the cross-dock, each plant develops his own demand forecast for all T periods in the planning horizon, and then uses that demand forecast to make those decisions. In what follows, we make use of the following additional notation.

PARAMETERS:

M Number of plants;

$q_{it}^{(m)}$ Demand for product i (in units) forecast by plant m for period t;

h Cost of holding a unit weight of product in inventory at a plant's site;

w_i Weight of product i;

c_d Per truck transportation cost (same for all plants);

U Capacity (in weight) of a single truck;

$\Phi_{i\tau}$ Total demand for product i (in units) observed by all plants from period 1 through period τ; $\Phi_{i\tau} = \sum_{j=1}^{M} \sum_{t=1}^{\tau} q_{it}^{(j)}$.

DECISION VARIABLES:

$x_{it}^{(m)}$ Quantity of product i (in units) picked up by plant m at the beginning of period t;

$v_t^{(m)}$ Number of trucks used by plant m transport products at the beginning of period t.

Let $\mathcal{C}_R^{(m)}(\mathbf{v}^{(m)}, \mathbf{x}^{(m)})$ denote the total transportation and inventory holding cost for plant m incurred over the planning horizon by truck transportation vector $\mathbf{v}^{(m)} = \{v_1^{(m)}, \ldots, v_T^{(m)}\}$ and product quantity matrix $\mathbf{x}^{(m)} = \left\{\mathbf{x}_1^{(m)}, \ldots, \mathbf{x}_Q^{(m)}\right\}$, where $\mathbf{x}_i := \left\{x_{1t}^{(m)}, \ldots, x_{1T}^{(m)}\right\}^T$. Let $\phi_{it}^{(m)}$ denote the inventory level of product i (in units) at plant m at the end of period t associated with shipping plan $(\mathbf{v}^{(m)}, \mathbf{x}^{(m)})$. In a decentralized supply chain, $\mathcal{C}_R^{(m)}$ is given by:

$$\mathcal{C}_R^{(m)}\left(\mathbf{v}^{(m)}, \mathbf{x}^{(m)}\right) = \sum_{t=1}^{T} \sum_{i=1}^{Q} h w_i \phi_{it}^{(m)} + \sum_{t=1}^{T} c_d v_t^{(m)}. \tag{4.8}$$

The first term is the total inventory holding cost, and the second term is the total cost of truck transport from the cross-dock to plant m's location. The following formulation represents plant m's cost minimization problem in a decentralized supply chain.

Plant's Cost Minimization in a Decentralized Supply Chain:

$$\min_{\mathbf{v}^{(m)},\mathbf{x}^{(m)}} \quad \mathcal{C}_R^{(m)}\left(\mathbf{v}^{(m)},\mathbf{x}^{(m)}\right) \tag{4.9}$$

$$\text{s.t.} \quad \phi_{it}^{(m)} = \phi_{i,t-1}^{(m)} + x_{it}^{(m)} - q_{it}^{(m)} \qquad i=1,\ldots,Q;\ \ t=1,\ldots,T; \tag{4.10}$$

$$Uv_t^{(m)} \geq \sum_{i=1}^{Q} w_i x_{it}^{(m)} \qquad t=1,\ldots,T; \tag{4.11}$$

$$\phi_{i0}^{(m)} = 0 \qquad i=1,\ldots,Q;$$

$$\phi_{it}^{(m)},\ x_{it}^{(m)} \geq 0 \qquad i=1,\ldots,Q;\ \ t=1,\ldots,T;$$

$$v_t^{(m)} \geq 0 \text{ and integer} \qquad t=1,\ldots,T.$$

Expression (4.10) represents the inventory balance equations given that a plant's demand has to be satisfied in every period, while capacity constraints given in (4.11) require that the combined capacity of the trucks used by the plant is sufficient to hold the desired product quantities.

4.3.4 Plant's Operations under Downstream Collaboration

Because each oil well facility is responsible for the transport of its products from the cross-dock and for holding them in inventory until they are needed, those facilities have been exploring ways to reduce the resulting costs. In particular, managers of geographically proximate oil wells are considering consolidating their order pick-ups from the cross-dock. In addition to sharing trucks to consolidate shipments from the cross-dock to their facilities, those plants are also looking into operating a joint warehouse to store their products, thus consolidating inventory and eliminating the need to hold inventory at their own (usually more expensive) locations. We refer to this type of collaboration among downstream members of a hybrid cross-docking supply chain as *downstream collaboration*.

With downstream collaboration, geographically proximate plants seek to minimize the joint total cost of transportation and inventory holding. Let x_{it} denote the amount of product to be delivered to the joint warehouse at the beginning of period t; v_t represent the number of trucks used in each period t, and ϕ_{it} denote the associated inventory of product i left at the joint warehouse at the end of period t. Let $\mathcal{C}_{R_{dc}}(\mathbf{v}, \mathbf{x})$ represent the collaborating plants' total cost over the planning horizon incurred by truck transportation vector $\mathbf{v} = \{v_1, v_2, \ldots, v_T\}$ and product quantity matrix $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_Q\}$, where $\mathbf{x_i} := \{x_{i1}, \ldots, x_{i,T}\}^T$. (Collaborating plants share trucks, but each plant has his own order quantity loaded on each truck to meet his own demand.) we thus obtain

$$\mathcal{C}_{R_{dc}}(\mathbf{v}, \mathbf{x}) = \sum_{t=1}^{T} \sum_{i=1}^{Q} h\, w_i\, \phi_{it} + \sum_{t=1}^{T} c_d\, v_t. \tag{4.12}$$

The objective function for the collaborating plants $\mathcal{C}_{R_{dc}}$ consists of the inventory holding cost at the shared warehouse $\sum_{t=1}^{T} \sum_{i=1}^{Q} h\, w_i\, \phi_{it}$, and transportation cost $\sum_{t=1}^{T} c_d\, v_t$ associated with the use of shared number of trucks v_t in each period t.

The following formulation represents the cost minimization problem faced by m plants who participate in downstream collaboration.

<u>Plants' Cost Minimization under Downstream Collaboration</u>

$$\begin{aligned}
\min_{\mathbf{v},\mathbf{x}} \quad & \mathcal{C}_{R_{dc}}(\mathbf{v}, \mathbf{x}) & & \\
\text{s.t.} \quad & \phi_{it} = \phi_{i,t-1} + x_{it} - \sum_{j=1}^{m} q_{it}^{(j)} & i = 1, \ldots Q;\ \ t = 1, \ldots, T; & \quad (4.13)\\
& U v_t \geq \sum_{i=1}^{Q} w_i x_{it} & t = 1, \ldots, T; & \quad (4.14)\\
& \phi_{i0} = 0 & i = 1, \ldots Q; & \\
& \phi_{it}, x_{it} \geq 0 & i = 1, \ldots Q;\ \ t = 1, \ldots, T; & \\
& v_t \geq 0 \text{ and integer} & t = 1, \ldots, T. &
\end{aligned}$$

Constraints (4.13) – (4.14) ensure that the plants collaborate in both inventory holding and transportation of products, and that the number of trucks used in each period is sufficient for the weight of the products carried in each truck. The inventory balance equations in (4.13) differ from those given in (4.10) for a decentralized supply chain in that plants' shipments are stored in the shared warehouse, rather than in individual plants' facilities.

4.4 Optimal Shipping Plans and Cost Savings

In what follows, Section 4.4.1 describes a cross-dock's optimal shipping plan in a decentralized supply chain. In Section 4.4.2, we analyze a cross-dock's optimization problem under upstream collaboration. Section 4.4.3 identifies structural properties of an individual plant's optimal transportation and inventory policies in a decentralized supply chain. In Section 4.4.4, we characterize cost savings for plants involved in downstream collaboration.

4.4.1 Cross-Dock's Optimal Shipping Plan in a Decentralized Supply Chain

To facilitate the derivation of structural properties of the optimal shipping plan for a cross-dock in decentralized supply chain, define $\sigma_{cd} := \frac{h_{cd}}{c_{cd}}$. Thus, σ_{cd} represents the size of the unit product holding cost relative to each container's shipping cost. We refer to σ_{cd} as the *cross-dock's relative cost*.

Let I_t^* denote the inventory held at the cross-dock in period t under the optimal shipping plan $(\mathbf{n}^*, \mathbf{y}^*)$ in a decentralized supply chain. Define $\iota^* := \sum_{t=1}^{T} I_t^*$ and $N_T^* = \sum_{t=1}^{T} n_t^*$. Thus, ι^* denote the sum of inventory amounts held at the cross-dock, while N_T^* is the sum of all product shipments into the cross dock during the planning horizon, under the optimal shipping plan. The following lemma characterizes N_T^* and ι^*. All proofs are deferred to the Online Appendix.

Lemma 3. *N_T^* and ι^* are functions of (only) σ_{cd}.*

Consequently, N_t^* and ι^* are functions of the cross-dock's relative cost only. This result makes it possible to characterize the corresponding functional dependence on σ_{cd}.

Lemma 4. *The following properties hold for $N_T^*(\sigma_{cd})$:*

(a) $\left\lceil \frac{wD_T}{W} \right\rceil \leq N_T^*(\sigma_{cd}) \leq \sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil$ for all σ_{cd}.

(b) $N_T^*(\sigma_{cd})$ is a piece-wise constant, monotonically increasing function of σ_{cd}, and is continuous everywhere except at $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - \left\lceil \frac{wD_T}{W} \right\rceil$ points.

Thus, the number of total containers shipped during the planning horizon increases in the cross-dock's relative cost. The upper bound $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil$ for N_T^* is the number of containers shipped to the cross-dock under traditional cross-docking (i.e., without any inventory holding), whereas the lower bound, $\left\lceil \frac{wD_T}{W} \right\rceil$, represents the number of containers shipped when the cross-dock places all orders for delivery in the first period. Going forward, we make use of constants ι_0 and σ_0 defined as follows:

$$\iota_0 := \sum_{t=1}^{T-1} \left(\left\lceil \frac{wD_t}{W} \right\rceil \frac{W}{w} - D_t \right) \quad \text{and} \quad \sigma_0 = \frac{\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - \left\lceil \frac{wD_T}{W} \right\rceil + 1}{w\,\iota_0}, \tag{4.15}$$

where D_t denotes cumulative demand faced by the cross-dock for periods 1 through t.

Theorem 13. *If* $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil = \left\lceil \frac{wD_T}{W} \right\rceil$, *then* $\iota^*(\sigma_{cd}) = 0$ *for all* σ_{cd}. *Otherwise,*

(a) if $\sigma_{cd} < \frac{1}{W}$, *then* $\iota^*(\sigma_{cd}) = \iota_0$ *and* $N_T^*(\sigma_{cd}) = \left\lceil \frac{wD_T}{W} \right\rceil$;

(b) if $\frac{1}{W} \leq \sigma_{cd} < \sigma_0$, *then* $\iota^*(\sigma_{cd})$ *is a piece-wise constant and monotonically decreasing function of* σ_{cd}, *and is continuous except at* $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil = \left\lceil \frac{wD_T}{W} \right\rceil$ *points;*

(c) If $\sigma_{cd} \geq \sigma_0$, *then* $\iota^*(\sigma_{cd}) = 0$ *and* $N_T^*(\sigma_{cd}) = \sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil$.

Theorem 13 identifies specific conditions under which it is optimal for the cross-dock not to hold inventory during the problem horizon. First, if $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil = \left\lceil \frac{wD_T}{W} \right\rceil$, then it follows from Lemma 4 that $N_T^*(\sigma_{cd}) = \sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil = \left\lceil \frac{wD_T}{W} \right\rceil$ for all σ_{cd}. In that case, it is optimal for the cross-dock facility not to hold inventory during the planning horizon, regardless of the value of σ_{cd}. Second, even if condition $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil = \left\lceil \frac{wD_T}{W} \right\rceil$ is not satisfied, by part (c), there exists a threshold value σ_0 of the relative unit cost such that it is optimal for the cross-dock not to hold any inventory during the planning horizon for all $\sigma_{cd} \geq \sigma_0$. Under either of those conditions, a cross-dock facility should optimally operate as a traditional cross-docking facility, because all arriving products to the cross-dock, under the optimal shipping plan, are immediately routed to the outbound trucks destined for the plants. Thus, the first contribution of Theorem 13 is to provide explicit and quantifiable guidelines to managers regarding when to use traditional cross-docking and when to resort to the use of hybrid cross-docking operations.

Part (a) shows that, when the cross-dock's unit relative cost is low enough, all orders from the plants for the entire planning horizon are shipped to the cross-dock in the first period. In that case, the unit holding cost is sufficiently small relative to the unit shipping cost that it is optimal to save on the shipping costs by sending all orders together, in a single shipment, to save on shipping costs (at the expense of the comparatively smaller inventory holding costs). In that case, the total number of units held in inventory is the highest, and implementing the optimal shipping plan clearly requires a hybrid cross-dock rather than a traditional one.

Part (b) establishes a region of the unit relative cost over which the total inventory held at the cross-dock is piece-wise constant and monotonically decreasing in σ_{cd}. This result comes about because the number of total containers shipped $N_T^*(\sigma_{cd})$ over the planning horizon is increasing in σ_{cd} by Lemma 4. The larger the number of containers used to fulfill the same plant's demand, the less inventory (in excess of each period's demand) is shipped to the cross-dock in any period t. Consequently, $\iota^*(\sigma_{cd})$ is decreasing as more containers are used to fulfill the same plants' demand.

In summary, it is the cross-dock's relative cost that determines what type of cross-dock facility is optimal to have in a supply chain. When that relative cost is high (i.e., greater than σ_0), a traditional cross-dock is sufficient to implement the optimal shipping plan. For all smaller values of the relative cost, optimal shipping plan results in inventory carry-over from one period to another and it becomes necessary to develop hybrid cross-docking functionality. These findings can thus be used to help guide managers in their choice of the type of the cross-dock facility to have in their supply chains.

Having characterized the optimal shipping plan for a hybrid cross-dock in a decentralized supply chain, we now analyze its total optimal cost as well. Let $\mathbf{f}(\sigma_{cd}, c_{cd})$ be the cross-dock's optimal total cost over the planning horizon as a function of σ_{cd} and c_{cd}. We can write $\mathbf{f}(\sigma_{cd}, c_{cd})$ as

$$\mathbf{f}(\sigma_{cd}, c_{cd}) = c_{cd} \left[\sum_{t=1}^{T} \left(\sigma_{cd} w I_t^*(\sigma_{cd}) + n_t^*(\sigma_{cd}) \right) \right].$$

The following theorem describes some useful regularity properties of $\mathbf{f}(\sigma_{cd},\, c_{cd})$.

Theorem 14. *Optimal total cost* $\mathbf{f}(\sigma_{cd},\, c_{cd})$*, is a piecewise-linear and increasing function of* σ_{cd}*. Further, if* $\sigma_{cd} \geq \sigma_0$*, then* $\frac{\partial \mathbf{f}}{\partial \sigma_{cd}} = 0$ *everywhere except at* $\sum_{t=1}^{T} \lceil \frac{wd_t}{W} \rceil - \lceil \frac{wD_T}{W} \rceil$ *points.*

The optimal total cost for the cross-dock is a piecewise-linear, increasing function of σ_{cd}. If the unit holding cost is sufficiently large relative to the container shipping cost, then that optimal total cost is piecewise-constant, with discontinuities occurring at (a small number of) well-defined points. For sufficiently large values of the relative cost σ_{cd}, the total inventory under the optimal shipping policy becomes zero by Theorem 13, and the total optimal cost becomes independent of σ_{cd}, as the OSC no longer finds it optimal to keep any inventory at the cross-dock facility. At that point, a traditional cross-dock suffices for the implementation of the optimal shipping plan.

4.4.2 Cross-dock's Cost Savings under Upstream Collaboration

Next, we seek to characterize cross-dock's savings under upstream collaboration. Let $\lambda_{it} = w_i\, d_{it}$ and $\Lambda_{i\tau} = \sum_{t=1}^{\tau} \lambda_{it}$ for $i = 1, \ldots, Q$, and $t = 1, \ldots, T$. Thus, λ_{it} represents the total weight of product i demanded at the cross-dock in period t, while $\Lambda_{i\tau}$ represents the cumulative weight of product i demanded at the cross-dock from period 1 through τ.

Let $\mathcal{C}^*_{CD}$ denote the cross-dock's optimal cost in a decentralized supply chain and $\mathcal{C}^*_{uc}$ denote its optimal cost under upstream collaboration. Let $\Delta_{CD} := \mathcal{C}^*_{CD} - \mathcal{C}^*_{uc}$. Thus, Δ_{CD} represents the cross-dock's maximum cost savings over the planning horizon from upstream collaboration relative to to the cost under its optimal shipping plan in decentralized supply chain.

Theorem 15. *For each* $i = 1, \ldots, Q$*, define* σ_i *as*

$$\sigma_i := \frac{\sum_{t=1}^{T} \lceil \frac{\lambda_{it}}{W} \rceil - \lceil \frac{\Lambda_{iT}}{W} \rceil + 1}{\sum_{t=1}^{T-1} \left(\lceil \frac{\Lambda_{it}}{W} \rceil W - \Lambda_{it} \right)}.$$

If $\sigma_{cd} \geq \max\limits_{i} \sigma_i$*, then*

$$\Delta_{CD} = c_{cd} \left\{ \sum_{i=1}^{Q} \sum_{t=1}^{T} \left\lceil \frac{\lambda_{it}}{W} \right\rceil - \sum_{t=1}^{T} \left\lceil \sum_{i=1}^{Q} \frac{\lambda_{it}}{W} \right\rceil \right\}.$$

Theorem 15 provides an explicit expression for the cross-dock's benefit from upstream collaboration for sufficiently high unit relative cost. For high enough values of σ_{cd}, the benefit from upstream collaboration can be expressed as the difference in transportation costs for the supply chain with and without upstream collaboration. If the unit inventory holding cost is high relative to the transportation cost per unit weight, then by using more containers to ship products from suppliers, the cross-dock is able to optimally not hold any inventory during the planning horizon. Consequently, the resulting cost savings from upstream collaboration reduce to the savings from the transportation costs alone. In Section 4.5, we quantify the benefits of upstream collaboration when it is optimal for the cross-dock to operate in the hybrid mode and carry inventory from one period to another.

4.4.3 Plant's Optimal Shipping Plan in a Decentralized Supply Chain

In characterizing plant's optimal shipping policy in a decentralized supply chain, we first define $\sigma_r := \frac{h}{c_d}$. Hence, σ_r represents the relative size of the unit product holding cost at the plant warehouse relative to unit truck cost. Accordingly, we refer to σ_r as *plants' relative cost*. Let $V_\tau^{(m)}$ denote the total number of trucks used to transport products from the cross-dock to plant m from period 1 through period τ; thus, $V_\tau^{(m)} = \sum_{t=1}^{\tau} v_t^{(m)}$. The following theorem provides some structural properties of the optimal transportation policy for a plant's problem in a decentralized supply chain.

Theorem 16. *Let* $\mathbf{V}^{(\mathbf{m})^*} = \left\{V_1^{(m)^*}, \ldots, V_T^{(m)^*}\right\}$ *represent the optimal shipping vector for plant* m.

(a) If $\sigma_r < \frac{1}{U}$, *then* $V_\tau^{(m)^*} = \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}^{(m)}}{U} \right\rceil$ *for any* $\tau \leq T$;

(b) If $\sigma_r \geq \frac{1}{U}$, *then* $V_\tau^{(m)^*} \leq \sum_{t=1}^{\tau} \left\lceil \frac{\sum_{i=1}^{Q} w_i \, q_{it}^{(m)}}{U} \right\rceil$ *for any* $\tau \leq T$.

When the per-truck transportation cost is higher than the cost of holding one truck-load of product inventory for a month, a plant uses the smallest possible number of trucks to get his products from the cross-dock facility. Part (b) of Theorem 16 provides an explicit expression for an upper bound of the number of trucks shipped to the plant over the planning horizon in all other cases.

4.4.4 Plant's Cost Savings under Downstream Collaboration

To provide insight into plants' cost savings under downstream collaboration, let $\mathcal{C}_{R_{dc}}^{(m)*}$ denote the optimal cost when m plants collaborate in their transportation and warehousing operations. Let $\mathcal{C}_{R}^{(j)*}$ be plant j's optimal cost in a decentralized supply chain. The following result lays the foundation for understanding the impact of shipment consolidation on the plants' total cost.

Theorem 17. *For each $m = 1, \ldots, M$, define ϑ_m as*

$$\vartheta_m := \frac{\sum_{t=1}^{T} \left\lceil \sum_{i=1}^{Q} \frac{w_i\, q_{it}^{(m)}}{U} \right\rceil - \left\lceil \frac{\sum_{i=1}^{Q} w_i\, \Phi_{iT}^{(m)}}{U} \right\rceil + 1}{\sum_{\tau=1}^{T-1} \left(\left\lceil \frac{\sum_{i=1}^{Q} w_i\, \Phi_{i\tau}^{(m)}}{U} \right\rceil U - \sum_{i=1}^{Q} w_i\, \Phi_{i\tau}^{(m)} \right)}.$$

If $\sigma_r \geq \max_m \vartheta_m$ and

$$\sum_{t=1}^{T} \sum_{m=1}^{M} \left\lceil \frac{\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U} \right\rceil = \sum_{t=1}^{T} \left\lceil \frac{\sum_{m=1}^{M} \sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U} \right\rceil,$$

then $\mathcal{C}_{R_{dc}}^{(m)*} = \sum_{j=1}^{m} \mathcal{C}_{R}^{(j)*}$. More generally, $\mathcal{C}_{R_{dc}}^{(m)*} \leq \sum_{j=1}^{m} \mathcal{C}_{R}^{(j)*}$ for any $m \in \{1, 2, \ldots, M\}$.

While it is not surprising that participating plants' optimal costs under downstream collaboration never exceed their optimal costs in a decentralized system, Theorem 17 also derives the conditions under which collaborating plants' total cost is not impacted by their downstream collaboration. In particular, if downstream collaboration in a cross-docking supply chain does not affect the optimal total number of trucks used for shipping products from the cross-dock facility, the benefits of that collaboration disappear. In that situation, considering the costs involved in setting up a collaboration among plants in practice (coordination costs, administrative costs, etc), plants will generally be better off without such collaboration.

4.5 Value of Collaboration Benefits

To provide additional managerial insight regarding collaborations in a cross-docking supply chain, in this section we quantify the cost savings generated from upstream and downstream collaborations. In what follows, we use a planning horizon of $T = 6$ months because, in the oilfield service industry, order planning and scheduling is generally carried out three to six months out, based on point forecasts of future demands. For all numerical studies that follow, parameter values were provided by the OSC. In particular, the weight capacity W of each container is $W = 27,760$kg and the transportation cost c_{cd} is \$2,359 per container. In Sections 4.5.1 and 4.5.2 we quantify benefits from upstream and downstream collaboration, respectively.

4.5.1 Benefit to the Cross-Dock from Upstream Collaboration

In evaluating the benefit to the cross-dock from upstream collaboration, we define the value of that upstream collaboration VC_{CD}, as

$$VC_{CD} := \frac{\mathcal{C}^*_{CD} - \mathcal{C}^*_{uc}}{\mathcal{C}^*_{CD}} \times 100,$$

where, as already stated, $\mathcal{C}^*_{CD}$ denotes the cross-dock's optimal cost in a decentralized supply chain, while $\mathcal{C}^*_{uc}$ is the cross-dock's optimal cost under upstream collaboration.

Thus, VC_{CD} denotes the cost savings for the cross-dock from upstream collaboration relative to the decentralized supply chain. In our first numerical study, we consider two geographically proximate suppliers, each providing a single product, and vary the cross-dock's unit inventory holding cost h_{cd} (expressed in dollars per kilogram per month) and the ratio r of the two product weights: $r = \frac{w_2}{w_1}$. This study explores 18 sets of model parameters: six values of the unit inventory holding cost ($h_{cd} \in \{0.05, 0.10, 0.15, 0.20, 0.25, 0.30\}$) $\times$ three values of the product weight ratio ($r \in \{0.5, 1.0, 1.5\}$).

OSC also provided us with a (confidential) data set with its monthly demand forecasts by product type. The OSC's actual shipping plan for each planning horizon is developed using those monthly demand forecasts for each six-month period. At the same time, those six-month forecasts change from one six-month period to another. Hence, to capture the benefit to the cross-dock from upstream collaboration under varying demand forecasts, for each of the 18 model parameter sets described above, we simulated 100 sets of six-month forecasts by means of a normal distribution whose mean for each month is the historical average for that month, and whose standard deviation is the historical standard deviation for that month. For each of those 18 parameter sets, and for each simulated six-month demand forecast, we determine $\mathcal{C}^*_{CD}$ and $\mathcal{C}^*_{uc}$, and then use their values to calculate VC_{CD}. Then, for each parameter set, we derive $AverageVC_{CD}$ by averaging VC_{CD} over those 100 simulations. Figure 4.2 displays $AverageVC_{CD}$ as a function of h_{cd} for each value of r.

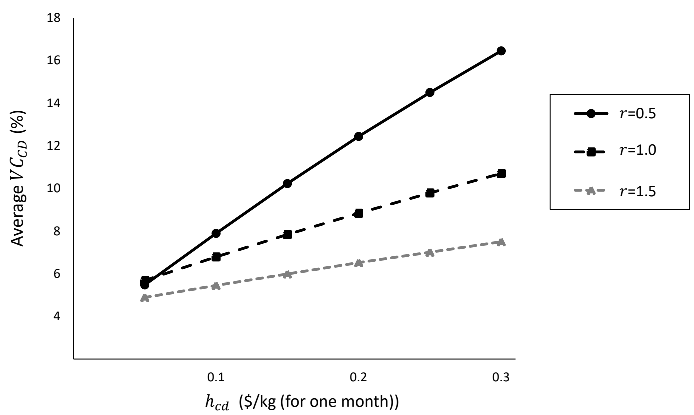

Figure 4.2: Upstream Collaboration

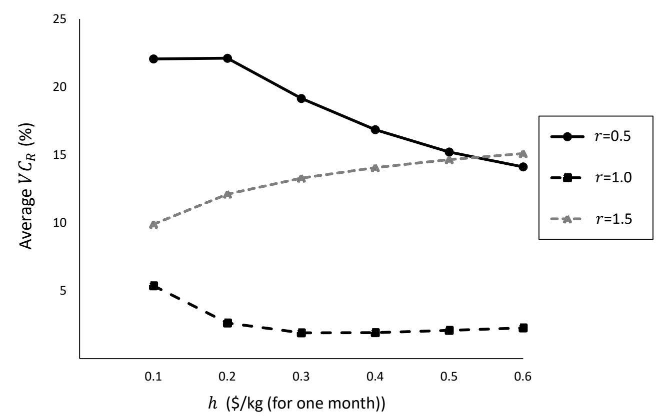

Figure 4.3: Downstream Collaboration

In Figure 4.2, average percentage cost savings for the cross-dock from upstream collaboration range from 5.4% to 16.4% for $r = 0.5$; from 5.7% to 10.7% for $r = 1.0$; and from 4.9% to 7.5% for $r = 1.5$. Those average percentage cost savings are also found to be increasing and concave in h_{cd}, and decreasing in r (except for $h_{cd} = 0.05$). The slope of $AverageVC_{CD}$ in Figure 4.2 can be observed to be decreasing in r; thus, the higher the product weight ratio, the slower average cost savings for the cross-dock increase as a function of its unit inventory holding cost. As a consequence, while there is little variability in average cost savings from unit holding cost variation for the high product weight ratio ($\sim 1.6\%$), there is significant variability in those savings for the low product weight ratio ($\sim 11\%$). This weight ratio-driven variability reveals that upstream collaboration is more impactful for lower product weight ratios, but only at higher unit holding costs.

In summary, the cost savings for the cross-dock from upstream collaboration can more than triple in size depending on the ratio of product weights and their holding cost at the cross-dock facility. Our studies indicate that upstream collaboration is most beneficial for low weight-ratio, high holding-cost products, while at lower holding costs those benefits are both small and effectively independent of product weight ratio. At the same time, at lower values of the holding cost, there is effectively no impact of product weight ratio on the realized cost savings. As far as we know, these findings are new to both industry and the research literature. It is also worth noting that the benefits realized from upstream collaboration would be significantly reduced if the OSC's cross-dock facility were traditional rather than hybrid; it is because of its inventory holding functionality that a hybrid cross-dock facility allows for the degree of cost savings from upstream collaboration evidenced in Figure 4.2.

4.5.2 Benefit to the Plants from Downstream Collaboration

In evaluating the benefits to plants from their downstream collaboration in cross-docking supply chain, we define the value of that downstream collaboration as

$$VC_{R} := \frac{\mathcal{C}_{R}^{*} - \mathcal{C}_{R_{dc}}^{*}}{\mathcal{C}_{R}^{*}} \times 100,$$

where, as already defined, $\mathcal{C}_{R}^{*}$ represents the total optimal cost for those plants in a decentralized supply chain, while $\mathcal{C}_{R}^{*}$ denotes their combined optimal cost under downstream collaboration. Hence, VC_{R} represents their combined cost savings from participating in downstream collaboration relative to their combined optimal cost in a decentralized supply chain.

Our second numerical study is structurally similar to the first one. We consider two geographically proximate plants that face demands for two types of products, and we explore 18 model parameter sets: six values of the unit inventory holding cost ($h \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$) $\times$ three values of the product weight ratio ($r \in \{0.5, 1.0, 1.5\}$). The confidential data file from OSC also includes each plant's monthly demand forecasts. While plants' decisions for each six-month planning horizon are made using those demand forecasts, the forecasts vary from one six-month planning horizon to another. Therefore, we capture the benefit from downstream collaboration under the practical reality of varying demand forecasts, again simulated 100 sets of six-month forecasts by means of the normal distribution whose mean for each month is the historical average of each plant's demand for that month, and whose standard deviation is the historical standard deviation for that month. For each of those 18 parameter sets, and for each simulated six-month demand forecast, we determined $\mathcal{C}_R^*$ and $\mathcal{C}_{R_{dc}}^*$, and then used those values to calculate VC_R. Then, for each model parameter set, we arrived at $AverageVC_R$ by averaging VC_R over those 100 simulations. Figure 4.3 shows $AverageVC_R$ as a function of the plants' unit inventory holding cost h, for each value of r.

As displayed in Figure 4.3, average cost savings for collaborating plants can be either decreasing, increasing, or concave in their unit inventory holding cost depending on product weight. In particular, for $r = 0.5$, they start at 22.1$\%$ for $h = 0.1$ and drop to 14.1$\%$ at $h = 0.6$; and, for $r = 1.5kg$, they start at 9.9$\%$ for $h = 0.1$ and increase to 15.1$\%$ at $h = 0.6$. For $r = 1.0$, average cost savings first decrease before reaching the minimum of 1.9$\%$ at $h = 0.3$; then, they increase up to 2.3$\%$ for $h = 0.6$. Thus, in contrast to the cost savings from upstream collaboration, the unit holding cost does not have a monotonic impact on the benefit realized by plants from their downstream collaboration. Further, the highest (average) cost savings in Figure 4.3 occur at the lowest level of the unit inventory holding cost. While these observations reveal some fundamental differences in the two types of collaborations considered in this chapter, the greatest cost savings for both types of collaborations are realized at the lowest product weight ratio, though at different unit holding costs.

In summary, when it comes to evaluating the benefits from each type of collaboration in a hybrid cross-docking supply chain, it becomes important for managers to carefully evaluate the realized cost savings based on relevant optimization models, such as the ones analyzed in this chapter. Interestingly, what appears to be conventional wisdom in the oilfield service industry is that, under high unit inventory costs, downstream collaboration will always generate significant cost reductions. This industry wisdom, however, appears to be of limited value: as shown in Figure 4.3, when $r = 1.0$, the savings from downstream collaboration are quite low, staying below 3% for most values of the unit inventory holding cost, and even falling below 2% at times. This is because the magnitude of cost savings from downstream collaboration very much depends on product weight ratio. In most practical settings, such small cost savings of 3$\%$ or less are typically not worth the cost of investment in setting up a downstream collaboration and the ongoing administrative cost of running it.

4.6 Outsourcing Downstream Operations

Because oil companies have historically been very focused on oil exploration and production, the development of their capabilities in other areas such as logistics operations has remained limited. As a result, in order to reduce their high costs of transportation and inventory management, oil well facilities have been looking to outsource their operations to third-party logistics companies. Because outsourcing of operations to independent logistics providers has financial and operational impact on the cross-dock and the oil wells it serves, it is of importance to all parties concerned to acquire a better understanding of that impact. For that reason, in this section we explore the outsourcing of downstream operations by plants in a hybrid cross-docking supply chain.

we consider a simple pricing contract as such contract is commonly offered by independent logistics providers to the oil wells in the Middle East. We analyze how such a contract impacts each plant's decision to outsource their operations. Formally, we develop and solve a Stackelberg pricing game between a logistics provider and plants under deterministic demand. At the beginning of the planning horizon, a logistics provider moves first by announcing price p to be charged for each unit weight of outsourced products. Subsequently, each plant makes a one-time decision whether to outsource its operations to the logistics provider for the duration of the entire planning horizon. The resulting best response function for each plant is analyzed in Section 4.6.1. Based on each plant's best response, the logistics provider determines his optimal price p^*, whose structure is derived in Section 4.6.2. In Section 4.6.3, we quantify that optimal price and discuss managerial insights.

4.6.1 Plants' Best Response

Let χ_m represent plant m's decision regarding outsourcing operations to an independent logistics provider at the beginning of the planning horizon: $\chi_m = 1$ if the plant outsources, and $\chi_m = 0$ otherwise. The plant's outsourcing cost is then given by $p\,\chi_m \sum_{t=1}^{T} \sum_{i=1}^{Q} w_i q_{it}^{(m)}$, where

$\sum_{t=1}^{T}\sum_{i=1}^{Q} w_i q_{it}^{(m)}$ is the total weight of the plant's demand over the planning horizon.

Let $\Delta^{(m)}(p,\chi_m)$ the net benefit to plant m from outsourcing operations to the logistics provider, as function of the outsourcing price p and the decision to outsource χ_m. Using the notation from Section 4.3.3, $\Delta^{(m)}(p,\chi_m)$ can be written as

$$\Delta^{(m)}(p,\chi_m) = \chi_m \left[\mathcal{C}_R^{(m)*} - p\sum_{t=1}^{T}\sum_{i=1}^{Q} w_i q_{it}^{(m)} \right], \tag{4.16}$$

where $\mathcal{C}_R^{(m)*}$ is plant m's optimal cost without collaboration (i.e., in a decentralized system). Define

$$\chi_m^*(p) := \underset{\chi_m\in\{0,1\}}{\arg\max}\, \Delta^{(m)}\big(p,\chi_m\big).$$

Thus, $\chi_m^*(p)$ represents the plant's best response to the logistics provider's outsourcing price p. In what follows, for each $m = 1, 2, \ldots, M$, define

$$p^{(m)} := \frac{\mathcal{C}_R^{(m)*}}{\sum_{t=1}^{T}\sum_{i=1}^{Q} w_i q_{it}^{(m)}}. \tag{4.17}$$

Theorem 18. *(a) For any outsourcing price p, the optimal outsourcing decision $\chi_m^*(p)$ of plant m is*

$$\chi_m^*(p) = \begin{cases} 1 & \textit{if} \quad p \le p^{(m)} \\ 0 & \quad \textit{otherwise,} \end{cases}$$

where $p^{(m)}$ is as defined in (4.17).
(b) Suppose that M plants are ordered so that $p^{(1)} \ge p^{(2)} \ge \cdots \ge p^{(M)}$. Then, one of the following holds true: (i) if $p \le p^{(M)}$, the number of outsourcing plants is M; or (ii) if $p^{(m)} \ge p > p^{(m+1)}$ for some $m \in \{1, 2, \ldots, M-1\}$, the number of outsourcing plants is m; or (iii) no plant outsources.

Hence, each plant's optimal outsourcing policy is a threshold policy: plant m outsources if and only if the outsourcing price p is below his optimal threshold price $p^{(m)}$. Further, that threshold price $p^{(m)}$ increases in the plant optimal cost without collaboration $\mathcal{C}_R^{(m)*}$, and decreases in the total weight of the plant's demand over the planning horizon, given by $\sum_{t=1}^{T}\sum_{i=1}^{Q} w_i q_{it}^{(m)}$. Going forward, without loss in generality, we relabel plants in a cross-docking supply chain as prescribed in

Theorem 18 (b). Accordingly, if plant m finds it optimal to outsource operations to an independent logistics provider, then plants 1 through $m-1$ will find it optimal to do so as well.

4.6.2 The Logistics Provider's Optimal Outsourcing Price

A logistics provider seeks to maximize profit from its outsourcing services over the planning horizon. As each plant develops its demand forecast for all T periods at the beginning of the planning horizon, those forecasts are also available to the logistics provider for all outsourcing plants. Knowing those demand forecasts, the first step in that profit maximization is to determine the logistics provider's optimal shipping plan for any given outsourcing price p.

Let $x_{it}^{(lp)}$ be the quantity of product i transported from the cross-dock facility to the logistics provider's warehouse at the beginning of period t. Let $v_t^{(lp)}$ be the number of trucks used by the logistics provider to transport products in period t, and $\phi_{it}^{(lp)}$ be the inventory level of product i at the logistics provider's warehouse at the end of period t. Define truck shipping vector $\mathbf{v}_{lp} = \{v_1^{(lp)}, v_2^{(lp)}, \ldots, v_T^{(lp)}\}$ and product quantity matrix $\mathbf{x}_{lp} = \{\mathbf{x_1}^{(lp)}, \ldots, \mathbf{x_Q}^{(lp)}\}$, where $\mathbf{x_i}^{(lp)} := \{x_{i1}^{(lp)}, x_{12}^{(lp)}, \ldots, x_{yT}^{(lp)}\}^T$. Let h_{l_p} denote the unit inventory holding cost at the logistics provider's warehouse, and c_{l_p} denote the per truck transportation cost faced by the logistics provider. We make the following assumption.

Assumption 6. $h_{l_p} \in (0, h)$ *and* $c_{l_p} \in (0, c_d)$.

Thus, a logistics provider faces lower unit operating costs than a plant. Because logistics providers specialize in the transportation and warehousing of their customers' inventory, it stands to reason that due to such specialization and economies of scale, the resulting unit operating costs would be lower than those of an oil well whose main focus is on exploration and development of oil reserves. (A logistics provider can, for example, choose his warehouse in a low-cost location, while an oil well generally keeps its inventory on-site).

Let $\Pi_{lp}(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p)$ represent the logistics provider's profit over the planning horizon. Then,

$$\Pi_{lp}(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p) := \underbrace{p \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{Q} w_i \chi_m^*(p)\, q_{it}^{(m)}}_{\text{Revenue from plants' outsourcing}} - \underbrace{\left(\sum_{t=1}^{T} \sum_{i=1}^{Q} h_{lp} w_i \phi_{it}^{(lp)} + \sum_{t=1}^{T} c_{lp} v_t^{(lp)} \right)}_{\text{Inventory holding and transportation cost}}$$

$$= p \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{Q} w_i \chi_m^*(p)\, q_{it}^{(m)} - \mathcal{C}_{lp}(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p),$$

where, as before, $q_{it}^{(m)}$, represents demand for product i observed by plant m in period t, while

$$\mathcal{C}_{lp}(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p) = \sum_{t=1}^{T} \sum_{i=1}^{Q} h_{lp} w_i \phi_{it}^{(lp)} + \sum_{t=1}^{T} c_{lp} v_t^{(lp)} \tag{4.18}$$

denotes the total cost for the logistics provider. Given the outsourcing price p and corresponding $\chi_m^*(p)$ as each plant's best response function, the logistics provider first seeks to maximize $\Pi_{lp}(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p)$ by deciding on his shipping and transportation schedules, $\mathbf{v}_{lp}$ and $\mathbf{x}_{lp}$, for each price p.

For any given outsourcing price p, the logistics provider has no influence over his revenue from plants' outsourcing, but rather only over his total inventory and transportation cost $\mathcal{C}_{lp}(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p)$. Hence, given outsourcing price p, the logistics provider actually faces a cost-minimization problem given in the following formulation.

<u>Logistics Provider's Cost Minimization Problem</u>:

$$\min_{\mathbf{v}_{lp}, \mathbf{x}_{lp}} \quad \mathcal{C}_{lp}(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p) \tag{4.19}$$

$$\text{s.t.} \quad \phi_{it}^{(lp)} = \phi_{i,t-1}^{(lp)} + x_{it}^{(lp)} - \sum_{m=1}^{M} \chi_m^*(p) q_{it}^{(m)}, \;\; i = 1, \ldots, Q; \;\; t = 1, \ldots, T; \tag{4.20}$$

$$U v_t^{(lp)} \geq \sum_{i=1}^{Q} w_i x_{it}^{(lp)}, \;\; t = 1, \ldots, T; \tag{4.21}$$

$$v_t^{(lp)} \geq 0 \quad \text{and} \quad \text{integer}, \;\; t = 1, \ldots, T;$$

$$\phi_{i0}^{(lp)} = 0, \;\; i = 1, \ldots, Q;$$

$$\phi_{it}^{(lp)}, \, x_{it}^{(lp)} \geq 0, \;\; i = 1, \ldots, Q; \;\; t = 1, \ldots, T.$$

Equations (4.20) represent the inventory balance equations for each product i given that the total demand for all plants served by the logistics provider has to be satisfied in each period t. Equations (4.21) ensure that the combined capacity of all trucks used by the logistics provider to transport products is sufficient for the required amount of each product in each period.

Let $(\mathbf{v}_{lp}^*(p),\ \mathbf{x}_{lp}^*(p))$ represent a solution to the above logistics provider's minimization problem as a function of outsourcing price p. Let $\mathcal{C}_{lp}^*(p)$ denote the logistics provider's resulting minimum cost obtained for any given p. Thus, $\mathcal{C}_{lp}^*(p) := \mathcal{C}_{lp}\big(\mathbf{v}_{lp}^*(p), \mathbf{x}_{lp}^*(p), p\big)$.

Define $\Pi_{lp}^*(p)$ and p^* as follows:

$$\Pi_{lp}^*(p) = p \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{Q} w_i \chi_m^*(p)\, q_{it}^{(m)} - \mathcal{C}_{lp}^*(p) \tag{4.22}$$

$$p^* = \operatorname*{arg\,max}_{p \geq 0}\ \Pi_{lp}^*(p). \tag{4.23}$$

Thus, p^* represents the outsourcing price that maximizes the logistics provider's total profit over the planning horizon. The following theorem provides a characterization of p^*.

Theorem 19. *(a)* $\Pi_{lp}^*(p^{(1)}) > 0$;
(b) There exists an integer $m \in \{1, 2, \ldots, M\}$ *such that* $p^* = p^{(m)}$;
(c) For any $m \in \{1, 2, \ldots, M\}$ *such that* m *plants find it optimal to outsource their operations to the logistics provider,* $\mathcal{C}_{lp}^*\big(p^{(m)}\big) < \sum_{j=1}^{m} \mathcal{C}_R^{(j)*}$.

Part (a) of Theorem 19 implies that the logistics provider can always make a profit by pricing his outsourcing services at $p^{(1)}$. By part (b), the logistics provider's optimal outsourcing price is the threshold price $p^{(m)}$ for some plant m. This result significantly reduces the computational effort of finding the optimal outsourcing price. This is because, instead of searching in a continuous interval for that price, it suffices to compare the logistics provider's profit at only M discrete prices, and choose the price that yields the highest profit. By part (c), the logistics provider's optimal cost is smaller than the outsourcing plants' optimal cost in a decentralized supply chain.

4.6.3 Outsourcing Price Quantified

Next, we numerically solve for p^* and discuss implications. Let $\Pi_{lp}(n)$ denote the logistics provider's profit when the outsourcing price is p_n (i.e., the number of outsourcing plants is n). We illustrate the behavior of $\Pi_{lp}(n)$ with two numerical examples. We consider a cross-docking supply chain with four geographically proximate plants (i.e., $M = 4$) and two products (i.e., $Q = 2$). We use the planning horizon of six months, $h_{l_p} = 0.03\$/kg$, $c_d = \$2,359$ per each truck, $w_1 = 300kg$ and $w_2 = 450kg$. The only difference in the two numerical examples shown in Figures 4.4 and 4.5 is in the values of demand forecasts used by each plant. While actual values of those demand forecasts are confidential, in Figure 4.4: (i) plant 1's demands are the same as his demands in Figure 4.5; (ii) plants 2 and 3 face identical six-dimensional vectors of demands for each product with lower mean and variability across the planning horizon than they do in Figure 4.5; (iii) plant 4 faces constant demand across the planning horizon. In Figure 4.5, each plant faces the same six-dimensional vector of monthly demands.

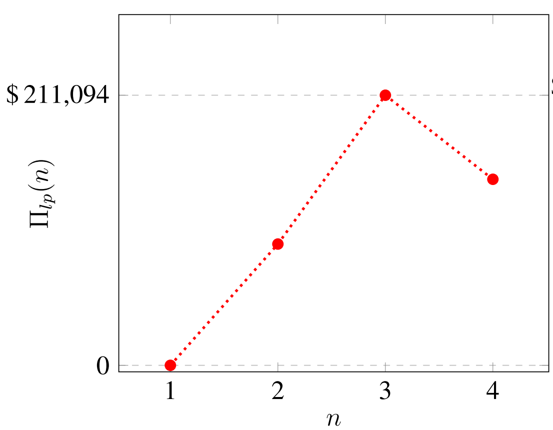

Figure 4.4: Scen. 1

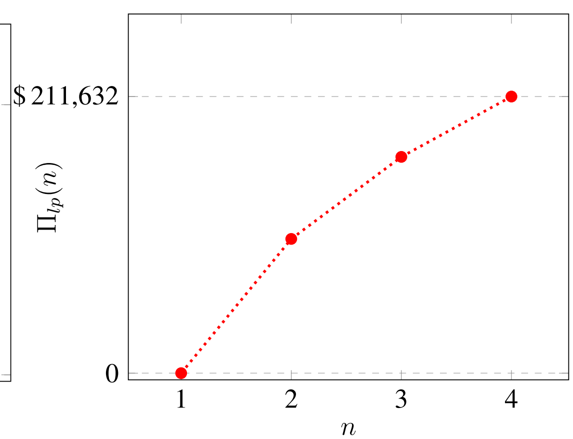

Figure 4.5: Scen. 2

In Figure 4.4, the logistics provider's profit is highest when $p^* = p^{(3)}$, which is the price at which three plants optimally outsource their operations to the logistics provider. In this numerical example, $p^* = \$0.8$/kg, with $\Pi_{lp}(3) = \$211,094$. The profit function $\Pi_{lp}(n)$ can be observed to be quasi-concave in the number of outsourcing plants, as this function is first increasing and then decreasing in n.

By comparison, in Figure 4.5, the optimal outsourcing price is given by $p^* = p^{(4)}$, as it is optimal for all four plants to outsource their operations. The profit function $\Pi_{lp}(n)$ is shown to be monotonically increasing in the number of outsourcing plants. Thus, $p^* = \$0.5$/kg, and $\Pi_{lp}(4) = \$211,362$. Further, $\Pi_{lp}(n)$ exhibits diminishing returns, so that the contribution of each additional plant to the logistics provider's bottom line is decreasing. This is in contrast to Figure 4.4 in which the third outsourcing plant contributes more to $\Pi_{lp}(n)$ than does the second. Another key implication of these observations is that the structure of $\Pi_{lp}(n)$ depends in a fundamental way on the underlying demand forecasts over the planning horizon for each plant. As a result, any structural properties of $\Pi_{lp}(n)$ are intrinsically tied to the actual values of the demand forecasts developed by the plants.

4.7 Outsourcing Downstream Operations under Demand Uncertainty

In realistic setting, the optimization of outsourcing price involves some degree of demand uncertainty. Thus, we now explore how plants can determine whether to outsource or manage their own operation without collaboration in a stochastic demand environment. We examine how demand uncertainty impacts the optimal outsourcing price and under what conditions plants outsource. We base our discussion of demand uncertainty on the outsourcing and pricing problem, but the approach we use in this section applies to other problems considered in this paper.

At $t = 0$, before demand realization, individual plants decide whether to outsource or manage their operations without collaboration. Following the practice of the focal company, all decisions are made before the planning horizon of three months ($T = 3$, a quarter of year).

We assume that plant's demand $q_{it}^{(m)}$ is a random variable with a discrete probability distribution. We use S to represent the total number of possible demand realizations during the planning horizon, and refer s to represent a given scenario of demand realization, $s = 1, 2, \ldots, S$. For computational tractability, we assume $I = 2$ throughout Section 4.7.

4.7.1 Plant's Best Response under Demand Uncertainty

Let $\rho^{(sm)}$ be the occurrence probability of scenario s with demand $q_{it}^{(sm)}$ of plant m during the planning horizon T. Also, Let $\phi_{it}^{(sm)}$ represent the inventory level of product i at the plant site in scenario s. Also, $v_t^{(m)}$ be the number of the trucks shipped to plants. Let $\mathcal{C}_{R_z}^{(m)}(\mathbf{v}_Z, \mathbf{x}_Z)$ be the plant's expected total cost where $\mathbf{v}_Z$ and $\mathbf{x}_Z$ represent a truck ship vector and a shipment quantity vector, respectively. Then, $\mathcal{C}_{R_z}^{(m)}(\mathbf{v}_Z, \mathbf{x}_Z)$ is given by

$$\mathcal{C}_{R_z}^{(m)}(\mathbf{v}_Z, \mathbf{x}_Z) = \sum_{s=1}^{S}\sum_{t=1}^{T}\sum_{i=1}^{2} \rho^{(sm)} h w_i \phi_{it}^{(sm)} + \sum_{t=1}^{T} c_d v_t^{(m)}.$$

The first term represents the expected inventory holding cost over all the total S scenarios. Because the plant's choice of the shipment quantity vector $\mathbf{x}_Z$ is made before the beginning of the planning horizon, the inventory level depends on realized demand. The second term corresponds to the transportation cost. The number of trucks to ship is determined before demand realization, the transportation cost is fixed for all demand scenario s. The plant's objective is to minimize the total expected cost by solving the following optimization problem.

Plant's Problem under Demand Uncertainty :

$$\min \quad \mathcal{C}_{R_z}^{(m)}(\mathbf{v}_Z, \mathbf{x}_Z) \tag{4.24}$$

$$\text{s.t.} \quad \phi_{it}^{(sm)} = \phi_{i,t-1}^{(sm)} + x_{it}^{(m)} - q_{it}^{(sm)}, \quad i = 1, 2, \ t = 1, 2, \ldots, T, \ s = 1, 2, \ldots, S; \tag{4.25}$$

$$U v_t^{(m)} \geq \sum_{i=1}^{2} w_i x_{it}^{(m)}, \quad t = 1, 2, \ldots, T; \tag{4.26}$$

$$v_t^{(m)} \geq 0 \quad \text{and} \quad \text{integer}, \quad t = 1, 2, \ldots, T; \tag{4.27}$$

$$\phi_{i0}^{(sm)} = 0, \quad i = 1, 2, \ s = 1, 2, \ldots, S \tag{4.28}$$

$$\phi_{it}^{(sm)} \geq 0, \quad i = 1, 2, \ t = 1, 2, \ldots, T, \ s = 1, 2, \ldots, S. \tag{4.29}$$

$$x_{it}^{(m)} \geq 0, \quad i = 1, 2, \ t = 1, 2, \ldots, T, \tag{4.30}$$

Constraint (4.25) represents the inventory balance equation under scenario s. The size of order for product i, $x_{it}^{(m)}$, is fixed through all demand scenarios, since the plant has to plan $x_{it}^{(m)}$ before demand is realized. Constraint (4.26) ensures that the trailer truck capacity is sufficient to deliver the products to the plants.

To facilitate the analysis, let $\mathcal{C}_{R_z}^{(m)*}$ be the optimal objective value (the cost for plant m) obtained from plant's Problem under demand uncertainty and $\Delta_z^{(m)}\big(p\big)$ be the cost difference for plant m with and without outsourcing under a outsourcing price p. Also, let $\chi_{m_Z}(p)$ represents the plant's decision of whether to outsource to the logistics provider under demand uncertainty. Then, $\Delta_z^{(m)}\big(p\big)$, as a function of χ_{m_Z}, is given by

$$\Delta_z^{(m)}\big(p,\, \chi_{m_Z}(p)\big) = \chi_{m_Z}(p)\left[\mathcal{C}_{R_z}^{(m)*} - p\sum_{t=1}^{T}\sum_{i=1}^{2}\sum_{s=1}^{S} w_i \rho^{(sm)} q_{it}^{(sm)}\right] \tag{4.31}$$

Define

$$\chi_{m_Z}^*(p) := \underset{\chi_{m_Z}\in\{0,1\}}{\arg\max}\ \Delta_z^{(m)}\big(p,\, \chi_{m_Z}(p)\big) \tag{4.32}$$

Thus, $\chi_{m_Z}^*(p)$ represents the plant's best response that maximizes its expected cost savings. The following Lemma provides the structure of the best response for individual plants.

Theorem 20. *For any* $m = 1, 2, \ldots, M$*, define*

$$p_Z^{(m)} := \frac{\mathcal{C}_{R_z}^{(m)*}}{\sum_{t=1}^{T}\sum_{i=1}^{2}\sum_{s=1}^{S} w_i \rho^{(sm)} q_{it}^{(sm)}}.$$

For any outsourcing price p*, the plant's best response is given by*

$$\chi_{m_Z}^*(p) = \begin{cases} 1, & \textit{if} \quad p \le p_Z^{(m)}; \\ 0, & \textit{otherwise}. \end{cases}$$

Further, suppose $p_Z^{(1)} \ge p_Z^{(2)} \ge \ldots \ge p_Z^{(M-1)} \ge p_Z^{(M)}$*. If* $p_Z^{(k+1)} < p \le p_Z^{(k)}$*, then the number of plants outsourcing is* k *for some* $k \in \{1, 2, ..., M-1\}$*.*

By Theorem 20, the optimal outsourcing decision of plant m is given as a threshold-type policy, similar to what we have observed in Theorem 18. For any outsourcing price lower than the threshold $p_Z^{(m)}$, the plant's expected total cost would be lower if it outsources its operations to the logistics provider. Using this structural results, Theorem 20 provides the number of outsourcing plants for a given outsourcing price. If $p > p_Z^1$ then there is no plant that outsourced to the logistics provider; if $p \leq p_Z^{(M)}$, then all plants outsourced. Going forward, we will refer to $p_Z^{(m)}$ as plant m's *stochastic threshold outsourcing price* because it concerns the optimization of the plant's cost under stochastic demand.

4.7.2 Optimal Outsourcing Price under Demand Uncertainty

Having established the plants' best response under demand uncertainty, we now turn to the optimal outsourcing price. Let D be a random matrix of all plants' demand. For instance, for a supply chain with four plants that requires two products over the planning horizon of three months, D is a $4 \times 3 \times 2$ matrix. With multiple plants, the logistics provider needs to consider exponential number of sample paths for demand realization. To solve this stochastic discrete optimization problem with the exponential sample paths, we use a Monte-Carlo simulation-based approach known as sample average approximation (SAA). In this procedure, we solve the sample average function, which approximates the expected objective function of the logistics provider, using a random sample generated from the random matrix D. The procedure - from generating a random sample to solving the sample average optimization problem - is repeated until a specified precision is achieved. Let κ represents the sample size. For each iteration, independently and identically distributed (i.i.d) κ realizations of the random demand are generated and used to solve the optimization problem. We use v to refer to the replication number - the number of total iterations until a stopping criterion is reached.

Having sufficient replication number ensures that there is a guarantee that an optimal solution for the true problem is produced. One can view such a procedure as Bernoulli trials with probability of success $P = P(\kappa)$. Here, the success means that a calculated optimal solution with κ samples is an optimal solution to the true problem. It follows that this probability $P(\kappa)$ tends to 1 as $\kappa \to \infty$

(Kleywegt et al. 2002). However, for a finite κ, the probability P can be small. The probability of producing an optimal solution of the true problem at least once in v replication is $1-(1-P)^v$, which tends to 1 as v increases.

To determine the replication number, let $\pi_s(p)$ be the objective value of the logistics provider's pricing problem for a sample path $s = 1, 2, \dots, \kappa$ within each replication. Let $\pi(p) = \frac{1}{\kappa}\sum_{s=1}^{\kappa}\pi_s(p)$. Thus, $\pi(p)$ represents the sample average function. Then, the optimal outsourcing price to this replication is given by $p^* = \arg\max_{p\geq 0}\pi(p)$. Also, the optimal objective value for this replication is given by $\pi^* = \pi(p^*)$. Let $\pi^{(j)*}$ be the optimal objective value of the j^{th} replication. Also, let $\overline{\pi}^{(v)} = \frac{\pi^{(1)*}+\pi^{(2)*}+\ldots+\pi^{(v)*}}{v}$. Then, $\overline{\pi}^{(v)}$ represents the average of the optimal objective values over all v replications. Let V_k^v be the sample variance over v replications, each with the sample size k. The $100(1-\alpha)$ percent confidence interval for the expected objective value $\mathbb{E}[\Pi^*]$ is then given by $\overline{\pi}^{(v)} \pm t_{v-1,1-\alpha/2}\sqrt{\frac{V_k^v}{v}}$.

If we keep increasing replications until $\frac{t_{v-1,1-\alpha/2}\sqrt{\frac{V_k^v}{v}}}{\overline{\pi}^{(v)}} \leq \lambda$, then $\overline{\pi}^{(v)}$ has a relative error at most $\frac{\lambda}{1-\lambda}$ with a probability of $1-\alpha$ (Law 2015). To determine the size of the replication number v for the given sample size κ that ensure an estimate of $\mathbb{E}[\Pi^*]$ with a relative error of λ and a confidence interval of $100(1-\alpha)$ percent, we use the following procedure (Law 2015).

- **Step 1:** Set $v_0 = 30$ random replications (each with $\kappa = 100$ sample paths), and set $v = v_0$.
- **Step 2:** Calculate $\overline{\pi}^{(v)}$ and $t_{v-1,1-\alpha/2}\sqrt{\frac{V_k^v}{v}}$ over v replications.
- **Step 3:** If $\frac{t_{v-1,1-\alpha/2}\sqrt{\frac{V_k^v}{v}}}{\overline{\pi}^{(v)}} < \frac{\lambda}{1+\lambda}$, then use $\overline{\pi}^{(v)}$ as the point estimate for Π^* and stop. Otherwise, replace v by $v+1$, proceed with another iteration of simulation, and go to Step 2.

It is standard from the literature that $\lambda < 0.05$ is a sufficiently small relative error (Law 2015, Geismar et al. 2020). To achieve $\lambda < 0.05$, we use the replication number $v = 30$ with the sample size $\kappa = 100$. We consider three different supply chains that is defined based on the number of plants ($M = 2, 3, 4$). The planning horizon for supply chains is three months. For each supply chain, we start by varying the level of demand uncertainty. We randomly generate demand data uniformly distributed and centered around a given mean demand μ, within the interval $[(1-\zeta/2)\mu,\ (1+\zeta/2)\mu]$. ζ represents a *scale parameter* for our study which we use to vary the level of demand uncertainty. We use three levels of scale parameter ($\zeta = 10\%, 20\%, 40\%$) for each supply chain.

Table 4.6 provides the plants' optimal decisions and the outsourcing price p^* and the plants' optimal decisions $\chi^*_{mz}(p^*)$ averaged over 30 replications. It shows that the outsourcing price decreases as the number of plants increases; by lowering the outsourcing price, more plants would outsource their operations, leading to a higher profit. For a same level of demand uncertainty, more plants are outsourced if the outsourcing price is lower. For the scale parameter of $\zeta = 10\%$, the outsourcing price is reduced by 12.4%, from $0.121\,\$/kg$ to $0.106\,\$/kg$, when the number of plants increases from 2 to 4. With a lower outsourcing price, more plants outsource their operations. For the scale parameter of 10%, in 23 among 30 replications (76%), it is optimal for plant 3 to outsource at the outsourcing price of $0.11\$/kg$. At the same level of demand uncertainty, it is optimal for the plant to outsource 28 among 30 replications (93%) at the outsourcing price $0.106\,\$/kg$.

At a higher level of demand uncertainty plants would have to pay a higher price for outsourcing. However, even at a higher outsourcing price, it may be profitable for plants to outsource, if they face high levels of demand uncertainty. For instance, under the scale parameter of 10%, plant 3 outsources in 23 among 30 replications at the outsourcing price of $\$0.11$, and outsources in all 30 replications at $0.128. One possible explanation is that the high variability in individual plants' demand can be reduced by aggregating demand. Thus, in aggregate, the logistics provider face a lower demand variability than the sum of demand variabilities individual plants face. Hence, more plants would outsource even at a higher outsourcing price.

4.8 Conclusion

In this chapter we analyzed a cross-docking supply chain encountered in the oilfield service industry of the energy sector. Due to long lead times from suppliers and low-volume, slow-moving non-perishable products, this industry has pioneered the use of hybrid cross-docking that, in contrast to traditional cross-docking, allows products to be held at the cross-dock facility from one

		ζ=10%		ζ=20%		ζ=40%	
M	plant (m)	Plant m Outsourcing	Outsourcing Price ($/kg)	Plant m Outsourcing	Outsourcing Price ($/kg)	Plant m Outsourcing	Outsourcing Price ($/kg)
2	plant 1	1	0.121	1	0.130	1	0.145
	plant 2	1		1		1	
3	plant 1	1	0.110	1	0.128	1	0.141
	plant 2	1		1		1	
	plant 3	0.76		1		1	
4	plant 1	1	0.106	1	0.117	1	0.129
	plant 2	1		1		1	
	plant 3	1		1		1	
	plant 4	0.93		0.97		0.97	

Table 4.6: Plants' Outsourcing under Demand Uncertainty

period to another. Further, the decentralized nature of cross-docking supply chains in the oil-field service industry has allowed supply chain members to reduce their operational costs by participating in different types of collaborations, such as upstream collaboration between the cross-dock and its suppliers and downstream collaboration among plants.

To provide a better understanding of hybrid cross-docking and arrive at optimal transportation and inventory policies for managing hybrid cross-docking facilities and the oil wells served by those facilities, we formulated and analyzed multi-period, multi-product, multi-stage optimization models in the context of both decentralized cross-docking supply chains and those with coordinated upstream or downstream shipping decisions. We identified structural properties of optimal transportation and inventory policies for those models. We made use of our findings to provide some practical guidelines to managers, including specific conditions under which the optimal shipping policy calls for using a hybrid cross-dock as opposed to a traditional cross-docking facility.

Our results made it possible to quantify savings from upstream and downstream collaborations in a cross-docking supply chain. Cost savings from upstream collaboration were found to be monotonically increasing in the unit holding cost at the cross-dock facility and decreasing in product weight, while ranging from 5.4% to to 16.4%. Cost savings from downstream collaboration exhibited less regularity but more variability, as they ranged from 1.9% to 22.1%. This observed variability highlights the importance of using appropriate analytic models in practice to evaluate the level of enabled cost-savings before investing in such collaborations, as the resulting cost savings may not always justify initial investment in and ongoing administrative costs of collaborations.

To address the growing need of oil wells to reduce costs by outsourcing operations to independent logistics providers and explore the implication of that practice for cross-docking supply chains, we developed and solved a Stackelberg pricing game between a logistics provider and plants. Assuming deterministic demand forecasts, we derived each plant's best response function to the logistics provider's outsourcing price, and then established the structure of the provider's profit-maximizing pricing strategy. These results are extended to stochastic demand forecasts in spite of the NP-hard nature of the original deterministic game. We found that the logistics provider can always find a price that results in (strictly) positive average profits, and that his profit-maximizing price can be found by a simple search among a small number of discrete price thresholds defined explicitly in this chapter.

There are a number of ways to extend the research in this chapter. One such direction is to analyze full supply chain integration and quantify the resulting benefits in the context of a multi-period, multi-stage, multi-product supply chain. Even though full supply chain integration is uncommon in the oil-field service industry, there are other industries in which complete supply chain coordination is plausible. For those industries, optimal transportation and inventory policies and the cost savings thus generated may be of considerable interest. Further, while in this chapter we analyze a simple pricing contract between a logistics provider and plants in a cross-docking supply chain, another worthwhile generalization of our work would involve an analysis of more sophisticated contracts between logistics providers and plants, such as quantity discount contracts or quantity flexibility contracts.

5. SUMMARY AND CONCLUSIONS

Motivated by the rapid transitions in the global energy landscape, my dissertation explores operations management problems faced by energy companies, investors, consumers, and policymakers. In the first essay, I examine how electricity consumers can optimally invest in the renewable energy generation capacity to participate in a blockchain-enabled peer-to-peer energy trading network, also referred to as a virtual microgrid. To capture the potential benefit of a resource-sharing network for individual electricity consumers, I explore how consumers can optimally invest in renewable energy in a virtual microgrid and key drivers of that investment. Using a simultaneous two-stage noncooperative game between consumers in a virtual microgrid, I show that each heterogeneous, cost-minimizing consumer is always better off within a virtual microgrid than without, regardless of the market-clearing price. Using equilibrium outcomes, I show that electricity consumers are always better off in a virtual microgrid, with the average cost savings of 9.7%.

At a system-level, I show that blockchain is able to coordinate heterogeneous, independent consumers in a decentralized virtual network with regard to minimizing the total cost of the system. Compared to a centralized system, I show that blockchain can fully coordinate all participants of a decentralized resource-sharing network, such that it can achieve the same total cost as a fully centralized network, that is optimized for minimizing the total cost.

The first essay contributes to knowledge by showing that blockchain-related energy projects can generate significant cost savings for each participant in a decentralized virtual network as well as achieving the full coordination of all participants in minimizing the total cost in the system. From the perspective of sustainability and community engagement with energy systems, these results provide support to a community-based energy sharing networks, where local participants can produce their own electricity and retain profits, while our society moves toward increased renewable energy use and away from fossil fuels. Such results also open several avenues for future research, including the performance of energy sharing networks for unregulated electricity markets, sustainable business models for one such a network.

Theoretical contributions of the first essay are as follows. First, I present new results to the literature on renewable energy systems by addressing investments in renewable energy within a resource-sharing network. Second, the model and analysis used in this essay extend those used in the emerging stream of research in the sharing economy. While previous research in this stream generally concerns that an owner has only one product (e.g., solar panels) to share, the model studied in this essay concerns sharing a product (e.g., electricity) of that product.

Motivated by the continuing increase of renewable energy development and the lack of transmission lines thereof, the second essay focuses on the optimal transmission grid investment and its implications for renewable energy development. This essay contributes to transmission expansion planning research by providing strategic insights into the interaction of generation and transmission and their implications for the optimal level of investments in each capacity. Grounded in industry practices, Transmission system operator (TransCo) invests in transmission lines, while a generation company (GenCo) separately invests in renewable power generation at a remote location of low demand that can be sold to the location of high demand only via transmission lines. For the resulting two-stage Stackelberg capacity investment game, I derive the equilibrium outcomes, including the capacity of installed renewable generation, and the capacity of transmission lines.

Theoretically, the second essay has two contributions to the academic literature. First, in the context of long-distance transmission development, I show that transmission and generation capacity act as complements for any probability densities of regional demand, in contrast to the traditional economists' view on the substitutability of transmission and generation (Joskow 2002). Because transmission investment is needed to enable export sales from generation investments, joint investment can be expected to enhance the profitability for both companies. Second, with regard to the future development and joint investment in the transmission grid, this essay contributes to the stream of literature on transmission expansion planning by proving the existence of a Pareto-improving contract that mutually increases the profitability for TransCo and GenCo.

The third essay devises an optimization framework to analyze a decentralized, hybrid cross-docking supply chain encountered in an oil-field service industry. Due to long lead times from suppliers and low-volume, slow-moving non-perishable products, this industry has pioneered the use of hybrid cross-docking that allows products to be held at the cross-dock from one period to another. In order to analyze decentralized supply chains and those with coordinated upstream or downstream shipping decisions, this essay formulates multi-period, multi-product, and multi-stage optimization models.

With regard to decentralized cross-docking supply chain, this study presents new structural properties of optimal shipping and inventory holding policies, and determines conditions under which a cross-dock facility should optimally operate as a hybrid cross-dock rather than as a traditional one. I also use the structural results to quantify the resulting cost savings from upstream and downstream collaboration for individual supply chain members, that range from 1.9%-22.1%, depending on product weight and holding cost.

For the growing need for outsourcing operations to independent logistics providers, this study finds that the logistics providers can always find an outsourcing price that results in strictly positive average profits. I also establish that profit-maximizing outsourcing prices can be found by a simple search among a small number of discrete price thresholds. The explicit structure of these price thresholds in this essay can be used by managers in the oil field service industry to seek improvement in their cost performance by strategically outsourcing to independent logistics providers.

From a modeling perspective, the third essay has two contributions to the literature. First, in the literature on economic lot-sizing, multi-product, multi-stage capacitated lot-sizing considered in this chapter is shown to be NP-hard (Zhao and Zhang 2020). Because the intermediate stage in our model acts as a cross-docking facility that performs "sorting" of multi-products, there are additional complexities in the problem. The results in this chapter contribute to better understanding the impact of these added layers of complexities by providing structural properties of optimal solutions. Second, I address joint inventory replenishment and transportation problems in a multi-product, three-stage, hybrid cross-docking distribution system, under both decentralized and collaborative settings.

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APPENDIX A

SUPPLEMENT TO CHAPTER 2

Proof of Theorem 1. Consider the minimization problem given in (2.2). It follows from (2.1) that $\frac{\partial f_i}{\partial Z_{it}} = -p_e e^{\eta t} < -\pi(t) = \frac{\partial f_i}{\partial Y_{ijt}}$ by Assumption 1, as long as $x_i - Z_{it} - Y_{jit} \geq 0$. Thus, to fulfill his demand, Prosumer i would first increase Z_{it} as much as possible before resorting to using any of Y_{jit}. By (2.4), (2.5), and (2.6), we get

$$\overline{Z}_{it} \leq \min\left[(x_i - Y_{jit})^+,\ \min(x_i, C_i),\ C_i - Y_{ijt},\ \min(x_i{+}x_j, C_i{+}C_j) - Z_{jt} - Y_{jit} - Y_{ijt}\right].$$

By (2.3) and (2.5), $\mathbb{Y}_i := [0,\ C_i]$ represent the strategy space of Y_{ijt}. Define

$$\begin{aligned} \mathcal{Z}_i(y) &:= \min\left[(x_i - y)^+,\ \min(x_i, C_i),\ C_i - y,\ \min(x_i{+}x_j, C_i{+}C_j) - Z_{jt} - Y_{jit} - y\right] \\ \mathcal{Y}_{ijt} &:= \arg\max_{y \in \mathbb{Y}_i} \mathcal{Z}_i(y). \end{aligned}$$

Hence, $\mathcal{Y}_{ijt}$ represents a maximizer of $\mathcal{Z}_i$ over the feasible set $\mathbb{Y}_i$. Then, because Prosumer i would always increase Z_{it} at the expense of Y_{ijt} to fulfill his demand, we obtain

$$\overline{Z}_{it} = \mathcal{Z}_i(\mathcal{Y}_{ijt}). \tag{A.1}$$

Using (2.5) and (2.6), we then get

$$\overline{Y}_{ijt} = \min\left[C_i - \overline{Z}_{it},\ \min(x_1 + x_2, C_1 + C_2) - Z_{jt} - Y_{jit} - \overline{Z}_{it}\right]. \tag{A.2}$$

We now analyze the different cases based on the realizations of $\min(x_i{+}x_j, C_i{+}C_j)$, $\min(x_i, C_i)$, and $\min(x_j, C_j)$. There are six such cases to consider, because neither (i) $C_i{+}C_j > x_i{+}x_j,\ x_i < C_i,\ x_j < C_j$; nor ii) $C_i + C_j < x_i + x_j,\ x_i > C_i,\ x_j > C_j$ is feasible.

Case 1. $C_1 + C_2 \geq x_1 + x_2$, $C_1 \geq x_1$, $C_2 \geq x_2$. For a pure-strategy Nash equilibrium, we use (A.1) to get

$$Z_{1t}^* \leq \min\big[(x_1 - Y_{21,t}^*)^+,\, x_1,\, C_1 - Y_{12,t}^*,\, x_1 + x_2 - Z_{2t}^* - Y_{21,t}^* - Y_{12,t}^*\big],$$
$$Z_{2t}^* \leq \min\big[(x_2 - Y_{12,t}^*)^+,\, x_2,\, C_2 - Y_{21,t}^*,\, x_2 + x_1 - Z_{1t}^* - Y_{12,t}^* - Y_{21,t}^*\big]. \tag{A.3}$$

Thus, we obtain $Z_{1t}^* \leq x_1,\, Z_{2t}^* \leq x_2$. Hence, based on the definition and properties of the best response functions in (A.1), (A.2), and (A.3), if there exists a unique pure-strategy $\{\widetilde{Z}_{1t}, \widetilde{Z}_{2t}, \widetilde{Y}_{12,t}, \widetilde{Y}_{21,t}\}$ such that

$$(x_1 - \widetilde{Y}_{21,t})^+,\, C_1 - \widetilde{Y}_{12,t},\, x_1 + x_2 - \widetilde{Z}_{2t} - \widetilde{Y}_{21,t} - \widetilde{Y}_{12,t} \geq x_1,$$
$$(x_2 - \widetilde{Y}_{12,t})^+,\, C_2 - \widetilde{Y}_{21,t},\, x_1 + x_2 - \widetilde{Z}_{1t} - \widetilde{Y}_{12,t} - \widetilde{Y}_{21,t} \geq x_2,$$
$$\widetilde{Y}_{12,t} = \min\left[C_1 - \widetilde{Z}_{1t},\, x_1 + x_2 - \widetilde{Z}_{2t} - Y_{21,t} - \widetilde{Z}_{1t}\right], \quad \widetilde{Y}_{12,t} = \min\left[C_2 - \widetilde{Z}_{2t},\, x_1 + x_2 - \widetilde{Z}_{1t} - Y_{12,t} - \widetilde{Z}_{2t}\right], \tag{A.4}$$

then neither prosumer has the incentive to deviate from the given strategy set, and we can conclude that the given strategy set is a unique Nash equilibrium. Because (A.4) holds only when $Z_{1t} = x_1,\, Y_{12,t} = 0,\, Z_{2t} = x_2,\, Y_{21,t} = 0$, the unique pure-strategy Nash equilibrium is $Z_{1t}^* = x_1,\, Y_{12,t}^* = 0,\, Z_{2t}^* = x_2,\, Y_{21,t}^* = 0$.

Case 2. $C_1 + C_2 \geq x_1 + x_2$, $C_1 \geq x_1$, $C_2 < x_2$: We use (A.1) to get

$$Z_{1t}^* \leq \min\big[(x_1 - Y_{21,t}^*)^+,\, x_1,\, C_1 - Y_{12,t}^*,\, x_1 + x_2 - Z_{2t}^* - Y_{21,t}^* - Y_{12,t}^*\big],$$
$$Z_{2t}^* \leq \min\big[(x_2 - Y_{12,t}^*)^+,\, C_2,\, C_2 - Y_{21,t}^*,\, x_2 + x_1 - Z_{1t}^* - Y_{12,t}^* - Y_{21,t}^*\big]. \tag{A.5}$$

Thus, we obtain $Z_{1t}^* \leq x_1,\, Z_{2t}^* \leq C_2$. Hence, based on the definition and properties of the best response functions in (A.1), (A.2), and (A.5), if there exists a unique pure-strategy $\{\widetilde{Z}_{1t}, \widetilde{Z}_{2t}, \widetilde{Y}_{12,t}, \widetilde{Y}_{21,t}\}$ such that

$$(x_1-\widetilde{Y}_{21,t})^+,\ C_1-\widetilde{Y}_{12,t},\ x_1+x_2-\widetilde{Z}_{2t}-\widetilde{Y}_{21,t}-\widetilde{Y}_{12,t} \geq x_1,$$
$$(x_2-\widetilde{Y}_{12,t})^+,\ C_2-\widetilde{Y}_{21,t},\ x_1+x_2-\widetilde{Z}_{1t}-\widetilde{Y}_{12,t}-\widetilde{Y}_{21,t} \geq C_2,$$
$$\widetilde{Y}_{12,t} = \min\left[C_1-\widetilde{Z}_{1t},\ x_1+x_2-\widetilde{Z}_{2t}-Y_{21,t}-\widetilde{Z}_{1t}\right],\ \ \widetilde{Y}_{12,t} = \min\left[C_2-\widetilde{Z}_{2t},\ x_1+x_2-\widetilde{Z}_{1t}-Y_{12,t}-\widetilde{Z}_{2t}\right], \tag{A.6}$$

then neither prosumer has the incentive to deviate from the given strategy set, and we can conclude that the given strategy set is a unique Nash equilibrium. Because (A.6) holds only when $Z_{1t} = x_1,\ Y_{12,t} = x_2 - C_2,\ Z_{2t} = C_2,\ Y_{21,t} = 0$, the unique Nash equilibrium is $Z^*_{1t} = x_1,\ Y^*_{12,t} = x_2 - C_2,\ Z^*_{2t} = C_2,\ Y^*_{21,t} = 0$.

Case 3. $C_1 + C_2 \geq x_1 + x_2$, $C_1 < x_1$, $C_2 \geq x_2$. We use (A.1) to get

$$Z^*_{1t} \leq \min\big[(x_1 - Y^*_{21,t})^+,\ C_1,\ C_1 - Y^*_{12,t},\ x_1 + x_2 - Z^*_{2t} - Y^*_{21,t} - Y^*_{12,t}\big],$$
$$Z^*_{2t} \leq \min\big[(x_2 - Y^*_{12,t})^+,\ x_2,\ C_2 - Y^*_{21,t},\ x_2 + x_1 - Z^*_{1t} - Y^*_{12,t} - Y^*_{21,t}\big]. \tag{A.7}$$

Thus, we obtain $Z^*_{1t} \leq C_1,\ Z^*_{2t} \leq x_2$. Using the definition and properties of the best response functions in (A.2), (A.1) and (A.7), if there exists a unique pure-strategy set $\{\widetilde{Z}_{1t}, \widetilde{Z}_{2t}, \widetilde{Y}_{12,t}, \widetilde{Y}_{21,t}\}$ such that

$$(x_1-\widetilde{Y}_{21,t})^+,\ C_1-\widetilde{Y}_{12,t},\ x_1+x_2-\widetilde{Z}_{2t}-\widetilde{Y}_{21,t}-\widetilde{Y}_{12,t} \geq C_1,\quad (x_2-\widetilde{Y}_{12,t})^+,$$
$$C_2-\widetilde{Y}_{21,t},\ x_1+x_2-\widetilde{Z}_{1t}-\widetilde{Y}_{12,t}-\widetilde{Y}_{21,t} \geq x_2,$$
$$\widetilde{Y}_{12,t} = \min\left[C_1-\widetilde{Z}_{1t},\ x_1+x_2-\widetilde{Z}_{2t}-Y_{21,t}-\widetilde{Z}_{1t}\right],\ \ \widetilde{Y}_{12,t} = \min\left[C_2-\widetilde{Z}_{2t},\ x_1+x_2-\widetilde{Z}_{1t}-Y_{12,t}-\widetilde{Z}_{2t}\right], \tag{A.8}$$

then neither prosumer has the incentive to deviate from the given strategy set, and we can conclude that the given strategy set is a unique Nash equilibrium. Since (A.8) holds only when $Z_{1t} = C_1,\ Y_{12,t} = 0,\ Z_{2t} = x_2,\ Y_{21,t} = x_1 - C_1$, the unique pure-strategy Nash equilibrium is $Z^*_{1t} = C_1,\ Y^*_{12,t}=0,\ Z^*_{2t}=x_2,\ Y^*_{21,t}=x_1 - C_1$.

Case 4. $C_1 + C_2 < x_1 + x_2$, $C_1 < x_1$, $C_2 > x_2$. We use (A.1) to get

$$Z^*_{1t} \le \min\big[(x_1 - Y^*_{21,t})^+,\, C_1,\, C_1 - Y^*_{12,t},\, C_1 + C_2 - Z^*_{2t} - Y^*_{21,t} - Y^*_{12,t}\big],$$
$$Z^*_{2t} \le \min\big[(x_2 - Y^*_{12,t})^+,\, x_2,\, C_2 - Y^*_{21,t},\, C_1 + C_2 - Z^*_{1t} - Y^*_{12,t} - Y^*_{21,t}\big]. \tag{A.9}$$

Therefore, we obtain $Z^*_{1t} \le C_1,\ Z^*_{2t} \le x_2$. Based on the definition and properties of the best response functions in (A.1), (A.2), and (A.9), if there exists a unique pure-strategy $\{\widetilde{Z}_{1t}, \widetilde{Z}_{2t}, \widetilde{Y}_{12,t}, \widetilde{Y}_{21,t}\}$ such that

$$(x_1 - \widetilde{Y}_{21,t})^+,\, C_1 - \widetilde{Y}_{12,t},\, C_1 + C_2 - \widetilde{Z}_{2t} - \widetilde{Y}_{21,t} - \widetilde{Y}_{12,t} \ge C_1, \quad (x_2 - \widetilde{Y}_{12,t})^+,$$
$$C_2 - \widetilde{Y}_{21,t},\, C_1 + C_2 - \widetilde{Z}_{1t} - \widetilde{Y}_{12,t} - \widetilde{Y}_{21,t} \ge x_2,$$
$$\widetilde{Y}_{12,t} = \min\left[C_1 - \widetilde{Z}_{1t},\, C_1 + C_2 - \widetilde{Z}_{2t} - Y_{21,t} - \widetilde{Z}_{1t}\right],\ \ \widetilde{Y}_{12,t} = \min\left[C_2 - \widetilde{Z}_{2t},\, C_1 + C_2 - \widetilde{Z}_{1t} - Y_{12,t} - \widetilde{Z}_{2t}\right], \tag{A.10}$$

then neither prosumer has the incentive to deviate from the given strategy set, and we can conclude that the given strategy set is a unique Nash equilibrium. Since (A.10) holds only when $Z_{1t} = C_1,\ Y_{12,t} = 0,\ Z_{2t} = x_2,\ Y_{21,t} = C_2 - x_2$, we obtain the unique Nash equilibrium $Z^*_{1t} = C_1,\ Y^*_{12,t} = 0,\ Z^*_{2t} = x_2,\ Y^*_{21,t} = C_2 - x_2$.

Case 5. $C_1 + C_2 < x_1 + x_2$, $C_1 > x_1$, $C_2 < x_2$. We use (A.1) to get

$$Z^*_{1t} \le \min\big[(x_1 - Y^*_{21,t})^+,\, x_1,\, C_1 - Y^*_{12,t},\, C_1 + C_2 - Z^*_{2t} - Y^*_{21,t} - Y^*_{12,t}\big],$$
$$Z^*_{2t} \le \min\big[(x_2 - Y^*_{12,t})^+,\, C_2,\, C_2 - Y^*_{21,t},\, C_1 + C_2 - Z^*_{1t} - Y^*_{12,t} - Y^*_{21,t}\big]. \tag{A.11}$$

Thus, we obtain $Z^*_{1t} \le x_1,\ Z^*_{2t} \le C_2$. Hence, using the definition and properties of the best response functions in (A.1), (A.2), and (A.11), if there exists a unique pure-strategy $\{\widetilde{Z}_{1t}, \widetilde{Z}_{2t}, \widetilde{Y}_{12,t}, \widetilde{Y}_{21,t}\}$ such that

$$(x_1-\widetilde{Y}_{21,t})^+,\ C_1-\widetilde{Y}_{12,t},\ C_1+C_2-\widetilde{Z}_{2t}-\widetilde{Y}_{21,t}-\widetilde{Y}_{12,t} \geq x_1, \quad (x_2-\widetilde{Y}_{12,t})^+,$$

$$C_2-\widetilde{Y}_{21,t},\ C_1+C_2-\widetilde{Z}_{1t}-\widetilde{Y}_{12,t}-\widetilde{Y}_{21,t} \geq C_2,$$

$$\widetilde{Y}_{12,t} = \min\left[C_1-\widetilde{Z}_{1t},\ C_1+C_2-\widetilde{Z}_{2t}-Y_{21,t}-\widetilde{Z}_{1t}\right],\ \ \widetilde{Y}_{12,t} = \min\left[C_2-\widetilde{Z}_{2t},\ C_1+C_2-\widetilde{Z}_{1t}-Y_{12,t}-\widetilde{Z}_{2t}\right], \tag{A.12}$$

then neither prosumer has the incentive to deviate from the given strategy set, and we can conclude that the given strategy set is a unique Nash equilibrium. Since (A.12) holds only when $Z_{1t} = x_1,\ Y_{12,t} = C_1-x_1,\ Z_{2t} = C_2,\ Y_{21,t} = 0$, we obtain the unique Nash equilibrium $Z^*_{1t} = x_1,\ Y^*_{12,t} = C_1 - x_1,\ Z^*_{2t} = C_2,\ Y^*_{21,t} = 0$.

Case 6. $C_1 + C_2 < x_1 + x_2$, $C_1 < x_1$, $C_2 < x_2$. We use (A.1) to get

$$\begin{aligned} Z^*_{1t} &\leq \min\big[(x_1 - Y^*_{21,t})^+,\ C_1,\ C_1 - Y^*_{12,t},\ C_1 + C_2 - Z^*_{2t} - Y^*_{21,t} - Y^*_{12,t}\big], \\ Z^*_{2t} &\leq \min\big[(x_2 - Y^*_{12,t})^+,\ C_2,\ C_2 - Y^*_{21,t},\ C_1 + C_2 - Z^*_{1t} - Y^*_{12,t} - Y^*_{21,t}\big]. \end{aligned} \tag{A.13}$$

Thus, we get $Z^*_{1t} \leq C_1,\ Z^*_{2t} \leq C_2$. Hence, using the definition and properties of the best response functions in (A.1), (A.2), and (A.13), if there exists a unique pure-strategy $\{\widetilde{Z}_{1t}, \widetilde{Z}_{2t}, \widetilde{Y}_{12,t}, \widetilde{Y}_{21,t}\}$ such that

$$(x_1-\widetilde{Y}_{21,t})^+,\ C_1-\widetilde{Y}_{12,t},\ C_1+C_2-\widetilde{Z}_{2t}-\widetilde{Y}_{21,t}-\widetilde{Y}_{12,t} \geq C_1, \quad (x_2-\widetilde{Y}_{12,t})^+,$$

$$C_2-\widetilde{Y}_{21,t},\ C_1+C_2-\widetilde{Z}_{1t}-\widetilde{Y}_{12,t}-\widetilde{Y}_{21,t} \geq C_2,$$

$$\widetilde{Y}_{12,t} = \min\left[C_1-\widetilde{Z}_{1t},\ C_1+C_2-\widetilde{Z}_{2t}-Y_{21,t}-\widetilde{Z}_{1t}\right],\ \ \widetilde{Y}_{12,t} = \min\left[C_2-\widetilde{Z}_{2t},\ C_1+C_2-\widetilde{Z}_{1t}-Y_{12,t}-\widetilde{Z}_{2t}\right], \tag{A.14}$$

then neither prosumer has the incentive to deviate from the given strategy set, and we can conclude that the given strategy set is a unique Nash equilibrium. Because (A.14) holds only when $Z_{1t} = C_1,\ Y_{12,t} = 0,\ Z_{2t} = C_2,\ Y_{21,t} = 0$, we obtain the unique Nash equilibrium $Z^*_{1t} = C_1,\ Y^*_{12,t} = 0,\ Z^*_{2t} = C_2,\ Y^*_{21,t} = 0$.

The results for the above six cases can be written as $Z_1^* = \min(x_1, C_1)$, $Z_2^* = \min(x_2, C_2)$, $Y_{12}^* = \min[(C_1 - x_1)^+, (x_2 - C_2)^+]$ and $Y_{21}^* = \min[(C_2 - x_2)^+, (x_1 - C_1)^+]$. It then directly follows that

$$F_i\big(C_i, C_j, x_i, x_j, t, \pi(t)\big) = p_e\, e^{\eta t}\big[x_i - C_i - (C_j - x_j)^+\big]^+ - \pi(t)\min\big[(C_i - x_i)^+, (x_j - C_j)^+\big]$$
$$+ \pi(t)\min\big[(C_j - x_j)^+, (x_i - C_i)^+\big].$$

Proof of Proposition 1. It follows from expressions (2.7) and (2.11) that

$$V_i(C_i, C_j, \pi(\cdot)) = \frac{1}{w^2}\Bigg\{\int_0^\infty \int_{a_i}^{b_i}\int_{a_j}^{b_j} p_e\, e^{-(r-\eta)t}\big[x_i - C_i - (C_j - x_j)^+\big]^+ dx_j dx_i dt$$
$$+ \int_0^\infty \int_{a_i}^{b_i}\int_{a_j}^{b_j} \pi(t)\, e^{-rt}\Big[\min\big[(C_j - x_j)^+, (x_i - C_i)^+\big] - \min\big[(C_i - x_i)^+,\ (x_j - C_j)^+\big]\Big] dx_j dx_i dt\Bigg\} + kC_i.$$

Integrating the above expression over t, using $\lambda = r - \eta$ and $\Pi = \int_0^\infty e^{-rt}\pi(t)dt$, yields

$$V_i(C_i, C_j, \pi(\cdot)) = \frac{1}{w^2}\Bigg\{\int_0^\infty \int_{a_i}^{b_i}\int_{a_j}^{b_j} p_e\, e^{-\lambda t}\big[x_i - C_i - (C_j - x_j)^+\big]^+ dx_j dx_i dt$$
$$+ \int_0^\infty \int_{a_i}^{b_i}\int_{a_j}^{b_j} \pi(t)\, e^{-rt}\Big[\min\big[(C_j - x_j)^+, (x_i - C_i)^+\big] - \min\big[(C_i - x_i)^+,\ (x_j - C_j)^+\big]\Big] dx_j dx_i dt\Bigg\} + kC_i.$$
$$= \frac{1}{\lambda w^2}\int_{a_i}^{b_i}\int_{a_j}^{b_j} p_e\big[x_i - C_i - (C_j - x_j)^+\big]^+ dx_j dx_i$$
$$+ \frac{1}{w^2}\int_{a_i}^{b_i}\int_{a_j}^{b_j} \Pi\Big[\min\big[(C_j - x_j)^+,\ (x_i - C_i)^+\big] - \min\big[(C_i - x_i)^+,\ (x_j - C_j)^+\big]\Big] dx_j dx_i + kC_i.$$

(A.15)

It follows from (A.15) that the functional $V_i\big(C_i, C_j, \pi(\cdot)\big)$ can be expressed as a function $V_i(C_i, C_j, \Pi)$ of C_i, C_i and Π. Further, the first term in (A.15) is then given by

$$\frac{1}{\lambda w^2}\int_{a_j}^{b_j}\int_{a_i}^{b_i} p_e\big[x_i - C_i - (C_j - x_j)^+\big]^+ dx_i dx_j$$
$$= \frac{p_e}{\lambda w^2}\left[\int_{a_i}^{b_i}\int_{a_j}^{C_j}\big[x_i - C_i - C_j + x_j\big]^+ dx_j dx_i + \int_{a_i}^{b_i}\int_{C_j}^{b_j}\big[x_i - C_i\big]^+ dx_j dx_i\right].$$

Let $I_i(C_i, C_j) = \int_{a_i}^{b_i}\int_{a_j}^{C_j}\big[x_i - C_i - C_j + x_j\big]^+ dx_j dx_i$ and $I_j(C_i, C_j) = \int_{a_i}^{b_i}\int_{C_j}^{b_j}\big[x_i - C_i\big]^+ dx_j dx_i$. Then,

$$I_i = \int_{C_i}^{b_i}\int_{\max[a_j, C_i + C_j - x_i]}^{C_j} (x_i - C_i - C_j + x_j) dx_j dx_i$$
$$= \int_{C_i}^{b_i} (x_i - C_i - C_j)\big[C_j - \max(a_j, C_i + C_j - x_i)\big] dx_i + \int_{C_i}^{b_i}\left[\frac{x_j^2}{2}\Big|_{\max[a_j, C_i + C_j - x_i]}^{C_j}\right] dx_i.$$

Thus, we obtain

$$I_i = \int_{C_i}^{\min[b_i, C_i + C_j - a_j]}\left[(x_i - C_i - C_j)(x_i - C_i) + \frac{C_j^2 - (C_i + C_j - x_i)^2}{2}\right] dx_i$$
$$+ \int_{\min[b_i, C_i + C_j - a_j]}^{b_i}\left[(x_i - C_i - C_j)(C_j - a_j) + \left(\frac{C_j^2}{2} - \frac{a_j^2}{2}\right)\right] dx_i.$$

We now evaluate the integrals in the above expression to arrive at

$$I_1 = \frac{x_i^3}{6}\Big|_{C_i}^{\min[b_i, C_i + C_j - a_j]} - (C_j + 2C_i)\,\frac{x_i^2}{2}\Big|_{C_i}^{\min(b_i, C_i + C_j - a_j)} + C_i(C_i + C_j)\big[\min(b_i, C_i + C_j - a_j) - C_i\big]$$
$$- \frac{C_i^2}{2}\big[\min(b_i, C_i + C_j - a_j) - C_i\big] + (C_i + C_j)\,\frac{x_i^2}{4}\Big|_{C_i}^{\min[b_i, C_i + C_j - a_j]}$$
$$+ (C_j - a_j)\,\frac{x_i^2}{2}\Big|_{\min(b_i, C_i + C_j - a_j)}^{b_i} - (C_j - a_j)\left(C_i + \frac{C_j}{2} - \frac{a_j}{2}\right)\big[b_i - \min(b_i, C_i + C_j - a_j)\big].$$

The above expression for I_1 depends on the value of $\min(b_i, C_i + C_j - a_j) = b_i$. Accordingly, we distinguish two cases: $C_i + C_j > b_i + a_j$ and $C_i + C_j \le b_i + a_j$. For $V_j(C_j, C_i, \Pi)$, we also distinguish separate cases for $C_i + C_j > b_j + a_i$ and $C_i + C_j > b_j + a_i$. Since $b_i - a_i = b_j - a_j = w$, then $b_i + a_j = b_j + a_i = a_i + a_j + w$.

Case 1: $C_i + C_j > a_i + a_j + w$. We solve for V_i. The first term in (A.15) can be expanded as:

$$\frac{1}{\lambda w^2}\int_{a_j}^{b_j}\int_{a_i}^{b_i} p_e\big[x_i - C_i - (C_j - x_j)^+\big]^+ dx_i dx_j$$
$$= \int_{a_i}^{a_i+w}\int_{a_j}^{C_j}\frac{p_e}{\lambda w^2}[x_i - C_i - C_j + x_j]^+ dx_j dx_i + \int_{a_i}^{a_i+w}\int_{C_j}^{a_2+w}\frac{p_e}{\lambda w^2}[x_i - C_i]^+ dx_j dx_i.$$
$$= \int_{C_i}^{a_i+w}\int_{C_i+C_j-x_i}^{C_j}\frac{p_e}{\lambda w^2}(x_i - C_i - C_j + x_j) dx_j dx_i + \int_{C_i}^{a_i+w}\int_{C_j}^{a_j+w}\frac{p_e}{\lambda w^2}(x_i - C_i) dx_j dx_i$$
$$= \frac{p_e}{6\lambda w^2}(a_i + w - C_i)^2(a_i + 3a_j - C_i - 3C_j + 4w) \tag{A.16}$$

For the remaining terms in (A.15), observe that $\min\big[(C_i - x_i)^+,\, (x_j - C_j)^+\big] = 0$ if either $C_j - x_j \le 0$ or $x_i - C_i \le 0$. Consequently, the remaining terms in (A.15) become

$$-\frac{\Pi}{w^2}\int_{a_i}^{b_i}\int_{a_j}^{b_j}\min\big[(C_i - x_i)^+,\, (x_j - C_j)^+\big] dx_j dx_i + \frac{\Pi}{w^2}\int_{a_i}^{b_i}\int_{a_j}^{b_j}\min\big[(C_j - x_j)^+,\, (x_i - C_i)^+\big] dx_j dx_i$$
$$= -\frac{\Pi}{w^2}\int_{a_i}^{C_i}\int_{C_j}^{a_j+w}\min[C_i - x_i,\, x_j - C_j] dx_j dx_i + \frac{\Pi}{w^2}\int_{C_i}^{a_i+w}\int_{a_j}^{C_j}\min[C_j - x_j,\, x_i - C_i] dx_j dx_i.$$
$$= -\frac{\Pi}{w^2}\left[\int_{C_j}^{a_j+w}\int_{a_i}^{C_i+C_j-x_j}(x_j - C_j) dx_i dx_j + \int_{C_j}^{a_j+w}\int_{C_i+C_j-x_j}^{C_i}(C_i - x_i) dx_i dx_j\right]$$
$$+ \frac{\Pi}{w^2}\left[\int_{C_i}^{a_i+w}\int_{a_j}^{C_i+C_j-x_i}(x_i - C_i) dx_j dx_i + \int_{C_i}^{a_i+w}\int_{C_i+C_j-x_i}^{C_j}(C_j - x_j) dx_j dx_i\right]$$
$$= \frac{\Pi}{6w^2}\Big[(C_i - C_j - a_i + a_j) \tag{A.17}$$
$$\times\left[a_i^2 + a_j^2 + C_i^2 + 4C_iC_j + C_j^2 - 3w(C_i + C_j) + a_i(4a_j - 2C_i - 4C_j + 3w) + a_j(-4C_i - 2C_j + 3w)\right]\Big].$$

Combining (A.16) and (A.17), we get

$$V_i(C_i, C_j, \Pi) = kC_i + \frac{1}{6\lambda w^2}\Big[p_e(a_i{+}w{-}C_i)^2(a_i{+}3a_j{-}C_i{-}3C_j{+}4w) + \lambda\Pi(C_i{-}C_j{-}a_i{+}a_j)$$

$$\times \left[a_i^2 + a_j^2 + C_i^2 + 4C_iC_j + C_j^2 - 3w(C_i + C_j) + a_i(4a_j - 2C_i - 4C_j + 3w) + a_j(-4C_i - 2C_j + 3w)\right]\Big]. \tag{A.18}$$

Case 2: $C_1 + C_2 \leq a_1 + a_2 + w$. The first term in (A.15) can be expanded as

$$\begin{aligned}
&\frac{1}{\lambda w^2}\int_{a_j}^{b_j}\int_{a_i}^{b_i} p_e\big[x_i - C_i - (C_j - x_j)^+\big]^+ dx_i dx_j \\
&= \int_{a_i}^{a_i+w}\int_{a_j}^{C_j} \frac{p_e}{\lambda w^2}[x_i - C_i - C_j + x_j]^+ dx_j dx_i + \int_{a_i}^{a_i+w}\int_{C_j}^{a_j+w} \frac{p_e}{\lambda w^2}[x_i - C_i]^+ dx_j dx_i \\
&= \int_{a_j}^{C_j}\int_{C_i+C_j-x_j}^{a_i+w} \frac{p_e}{\lambda w^2}(x_i - C_i - C_j + x_j) dx_i dx_j + \int_{C_i}^{a_i+w}\int_{C_j}^{a_j+w} \frac{p_e}{\lambda w^2}(x_i - C_i) dx_j dx_i \\
&= \frac{p_e}{6\lambda w^2}\Big[- a_j^3 + 3C_iC_j^2 + C_j^3 - 3a_jC_j(2C_i + C_j - 2w) + 3a_j^2(C_i + C_j - w) \\
&\quad + 3wa_i^2 + 3wC_i^2 - 3wC_j^2 - 6w^jC_i + 3w^3 - 3a_1\big[a_j^2 - 2a_jC_j + C_j^2 + 2w(C_i - w)\big]\Big]
\end{aligned} \tag{A.19}$$

Similarly, we expand the remaining terms in (A.15) to get

$$\begin{aligned}
&-\frac{\Pi}{w^2}\int_{a_i}^{b_i}\int_{a_j}^{b_j} \min\big[(C_i - x_i)^+,\, (x_j - C_j)^+\big] dx_j dx_i + \frac{\Pi}{w^2}\int_{a_i}^{b_i}\int_{a_j}^{b_j} \min\big[(C_j - x_j)^+,\, (x_i - C_i)^+\big] dx_j dx_i \\
&= -\frac{\Pi}{w^2}\int_{a_i}^{C_i}\int_{C_j}^{a_j+w} \min[C_i - x_i,\, x_j - C_j] dx_j dx_i + \frac{\Pi}{w^2}\int_{C_i}^{a_i+w}\int_{a_j}^{C_j} \min[C_j - x_j,\, x_i - C_i] dx_j dx_i. \\
&= -\frac{\Pi}{w^2}\left[\int_{a_i}^{C_i}\int_{a_j}^{C_i+C_j-x_i}(x_j - C_j) dx_j dx_i + \int_{a_i}^{C_i}\int_{C_i+C_j-x_i}^{a_j+w}(C_i - x_i) dx_j dx_i\right] \\
&\qquad + \frac{\Pi}{w^2}\left[\int_{a_j}^{C_j}\int_{C_i+C_j-x_j}^{a_i+w}(C_j - x_j) dx_i dx_j + \int_{a_j}^{C_j}\int_{C_i}^{C_i+C_j-x_j}(x_i - C_i) dx_i dx_j\right],
\end{aligned}$$

so that the remaining terms in (A.15) become

$$\frac{\Pi}{6w^2}\Big[(C_i - C_j - a_i + a_j) \tag{A.20}$$
$$\times \left[a_i^2 + a_j^2 + C_i^2 + 4C_iC_j + C_j^2 - 3w(C_i + C_j) + a_i(4a_j - 2C_i - 4C_j + 3w) + a_j(-4C_i - 2C_j + 3w)\right]\Big].$$

Combining (A.19) and (A.20), we get

$$V_1(C_i, C_j, \Pi) = kC_i + \frac{1}{6\lambda w^2}\Big\{p_e\Big[-a_j^3 + 3C_iC_j^2 + C_j^3 - 3a_jC_j(2C_i + C_j - 2w) + 3a_j^2(C_i + C_j - w)$$
$$+ 3wa_i^2 + 3wC_i^2 - 3wC_j^2 - 6w^2C_i + 3w^3 - 3a_i\big[a_j^2 - 2a_jC_j + C_j^2 + 2w(C_i - w)\big]\Big] \tag{A.21}$$
$$+\lambda\Pi(C_i - C_j - a_i + a_j)$$
$$\times\Big[a_i^2 + a_j^2 + C_i^2 + 4C_iC_j + C_j^2 - 3w(C_i + C_j) + a_i(4a_j - 2C_i - 4C_j + 3w) + a_j(-4C_i - 2C_j + 3w)\Big]\Big\}. \ \square$$

In what follows, we make use of the following Lemma.

Lemma 5. *For any marginal value of energy traded* $\Pi \in \big[0, \frac{p_e}{\lambda}\big)$, $C_2^*(\Pi) = C_1^*(\Pi) + a_2 - a_1$.

Proof. Fix Π. Define $\widehat{X} := X_1 - a_1 + a_2$, so that $\widehat{X}$ is a random variable uniformly distributed on $\mathbb{I}_2 = [a_2, b_2]$. Let $\hat{x}$ be a realization of $\widehat{X}$. Let $C_2^*(\Pi)$ be the optimal capacity selection for Prosumer 2. Then,

$$
\begin{aligned}
&V_1\big(C_1, C_2^*(\Pi), \pi(\cdot)\big) = \int_0^\infty e^{-rt}\,\mathbb{E}_{X_1,X_2}\big[F_1(C_1, C_2^*(\Pi), x_1, x_2, t, \pi(t))\big]dt + kC_1 \\
&= \frac{1}{w^2}\left\{\int_0^\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2} p_e\, e^{-(r-\eta)t}\big[x_1 - C_1 - (C_2^*(\Pi) - x_2)^+\big]^+ dx_2 dx_1 dt\right. \\
&+\int_0^\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2} \pi(t)\, e^{-rt}\Big[\min\big[(C_2^*(\Pi) - x_2)^+, (x_1 - C_1)^+\big] \\
&\left.-\min\big[(C_1 - x_1)^+,\ (x_2 - C_2^*(\Pi))^+\big]\Big] dx_2 dx_1 dt\right\} + kC_1 \\
&= kC_1 + \frac{1}{w^2}\left\{\int_0^\infty \int_{a_2}^{b_2}\int_{a_2}^{b_2} p_e\, e^{-(r-\eta)t}\Big[\hat{x} - a_2 + a_1 - C_1 - \big[C_2^*(\Pi) - x_2\big]^+\Big]^+ dx_2\, d\hat{x}\, dt\right. \\
&+\int_0^\infty \int_{a_2}^{b_2}\int_{a_2}^{b_2} \pi(t)\, e^{-rt}\Big[\min\big[(C_2^*(\Pi)-x_2)^+, (\hat{x}-a_2+a_1-C_1)^+\big] \\
&\left.-\min\big[(C_1-\hat{x}+a_2-a_1)^+, \big(x_2-C_2^*(\Pi)\big)^+\big]\Big] dx_2 d\hat{x} dt\right\} \\
&= \int_0^\infty e^{-rt}\,\mathbb{E}_{\widehat{X},X_2}\big[F_1(C_1 + a_2 - a_1, C_2^*(\Pi), x_1, x_2, t, \pi(t)\big]dt + kC_1. \qquad \text{(A.22)}
\end{aligned}
$$

It follows from (2.8) and (2.9) that $C_1^* = \arg\min_{C_1} V(C_1, C_2^*)$. Using (A.22) we thus obtain that

$$
\begin{aligned}
C_1^* &= \arg\min_{C_1}\left\{\int_0^\infty e^{-rt}\,\mathbb{E}_{\widetilde{X},X_2}[F_1(C_1 + a_2 - a_1, C_2^*, x_1, x_2)]dt + kC_1\right\} \\
&= \arg\min_{C_1}\left\{\int_0^\infty e^{-rt}\,\mathbb{E}_{\widetilde{X},X_2}[F_1(C_1 + a_2 - a_1, C_2^*, x_1, x_2)]dt + k(C_1 + a_2 - a_1)\right\}, \qquad \text{(A.23)}
\end{aligned}
$$

where the last equality holds since $k(a_2 - a_1)$ is constant. Expression (A.23) thus represents the capacity selection problem of a Prosumer with demand $\widetilde{X}$, who invests in capacity $C_1+a_2-a_1$ to minimize his expected cost, given the marginal value of energy traded Π. Since that consumer's electricity demand $\widetilde{X}$ and the Prosumer 2's demand X_2 are identical by definition of $\widetilde{X}$, his optimal capacity must be $C_2^*(\Pi)$. Since his optimal capacity investment from (A.23) is given by $C_1^*(\Pi)+a_2-a_1$, it follows that $C_1^*(\Pi)+a_2-a_1=C_2^*(\Pi)$. $\square$

Proof of Proposition 2. Because, by Theorem 1, for each Prosumer i, the functional $V_i\big(C_i, C_j, \pi(\cdot)\big)$

can be expressed as a function $V_i(C_i, C_j, \Pi)$ of C_i, C_i and Π, then it follows from expressions (2.8) and (2.9 that the functional $C_i^*\big(\pi(\cdot)\big)$ can be also expressed as a function of the marginal value of energy traded in a virtual microgrid (i.e., $C_i^*(\Pi)$). Next, fix Π. We first show that $C_i^*(\Pi) \leq b_i$. Let $\tilde{b}_1 = b_1 + \epsilon$ for some $\epsilon > 0$. Suppose $C_1^*(\Pi) = \tilde{b}_1$. We use Lemma 5 to obtain that

$$C_2^*(\Pi) = C_1^*(\Pi) + a_2 - a_1 = \tilde{b}_1 + b_2 - b_1 = \tilde{b}_2 > b_2,$$

since $\tilde{b} > b_1$. It follows from (2.11) that

$$\begin{aligned} &F_1\big(C_1^*(\Pi), C_2^*(\Pi), x_1, x_2, t, \Pi\big) = F_1\big(\tilde{b}_1, \tilde{b}_2, x_1, x_2, t, \Pi\big) \\ &= p_e e^{it}\big[x_2 - \tilde{b}_2 - \tilde{b}_1 + x_1\big]^+ + \min \pi(t) e^{\eta t}\big[\tilde{b}_1 - x_1,\, (x_2 - \tilde{b}_2)^+\big] = 0, \end{aligned}$$

since $C_1^*(\Pi) = \tilde{b}_1 > b_1 \geq x_1$ for every $x_1 \in \mathbb{I}_1$, and $C_2^*(\Pi) = \tilde{b}_2 > b_2 \geq x_2$ for every $x_2 \in \mathbb{I}_2$. It then follows from (2.11) that $V_1\big(C_1^*(\Pi), C_2^*(\Pi)\big) = V_1(\tilde{b}_1, \tilde{b}_2, \Pi) = k\tilde{b}_1$.

Let $\widetilde{C}_1 = b_1$, so that, by Lemma 5, $\widetilde{C}_2 = \widetilde{C}_1 + a_2 - a_1 = b_2$. Using (2.11), we then obtain $V_1(\widetilde{C}_1, \widetilde{C}_2, \Pi) = V(b_1, b_2, \Pi) = kb_1$. Consequently,

$$V_1\big(C_1^*(\Pi), C_2^*(\Pi), \Pi\big) - V_1\big(\widetilde{C}_1, \widetilde{C}_2, \Pi\big) = k(\tilde{b}_1 - b_1) = k\epsilon > 0.$$

Since $V_1\big(C_1^*(\Pi), C_2^*(\Pi), \Pi\big) > V_1\big(\widetilde{C}_1, \widetilde{C}_2, \Pi\big)$, we obtain a contradiction of our assumption that $C_1^*(\Pi) = \tilde{b}_1 > b_1$ is the optimal capacity investment for consumer 1. It follows that $C_1^*(\Pi) \leq b_1$.

Next, we show $C_i^*(\Pi) \geq a$. Let $\tilde{a}_1 = a_1 - \epsilon$ for some $\epsilon > 0$. Assume $C_1^*(\Pi) = \tilde{a}_1$. Then, by Lemma 5,

$$C_2^*(\Pi) = C_1^*(\Pi) + a_2 - a_1 = \tilde{a}_1 + a_2 - a_1 = \tilde{a}_2 < a_2,$$

since $\tilde{a}_1 < a_1$. It follows from (2.11) that

$$F_1\big(C_1^*(\Pi), C_2^*(\Pi), x_1, x_2, t, \Pi\big) = F_1(\tilde{a}_1, \tilde{a}_2, x_1, x_2, t, \Pi) = p_e(x_1 - \tilde{a}_1),$$

since $C_1^*(\Pi) = \tilde{a}_1 < x_1$ for every $x_1 \in \mathbb{I}_1$, and $C_2^*(\Pi) = \tilde{a}_2 < x_2$ for every $x_2 \in \mathbb{I}_2$. We use (2.11) to get

$$V_1\big(C_1^*(\Pi), C_2^*(\Pi), \Pi\big) = V_1(\tilde{a}_1, \tilde{a}_2, \Pi) = \int_0^\infty e^{-\lambda t}\, \mathbb{E}\big[F_1(\tilde{a}_1, \tilde{a}_2, x_1, x_2, t, \Pi)\big]\, dt + k\tilde{a}_1$$
$$= \frac{p_e}{\lambda w^2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} p_e(x_1 - \tilde{a}_1) dx_2 dx_1 + k\tilde{a}_1 = \frac{p_e}{\lambda w} \int_{a_1}^{b_1} (x_1 - \tilde{a}_1)\, dx_1 + k\tilde{a}_1$$
$$= \frac{p_e(b_1 + a_1)}{2\lambda} - \frac{p_e \tilde{a}_1}{\lambda} + k\tilde{a}_1.$$

Define $\widetilde{C}_1 := a_1$, so that $\widetilde{C}_2 = \widetilde{C}_1 + a_2 - a_1 = a_2$. Using identical steps to those above, we get

$$V_1(\widetilde{C}_1, \widetilde{C}_2, \Pi) = V_1(a_1, a_2, \Pi) = \frac{p_e(b_1 + a_1)}{2\lambda} - \frac{p_e a_1}{\lambda} + k a_1.$$

It follows that

$$V_1\big(C_1^*(\Pi), C_2^*(\Pi), \Pi\big) - V_1\big(\widetilde{C}_1, \widetilde{C}_2, \Pi\big) = \frac{p_e(a_1 - \tilde{a}_1)}{\lambda} + k(\tilde{a}_1 - a_1) = (a_1 - \tilde{a}_1)\left(\frac{p_e}{\lambda} - k\right) > 0.$$

Since $V_1\big(C_1^*(\Pi), C_2^*(\Pi), \Pi\big) > V_1\big(\widetilde{C}_1, \widetilde{C}_2, \Pi\big)$, we obtain a contradiction with our assumption that $C_1^*(\Pi) = \tilde{a}_1$ is the optimal capacity investment for consumer 1. It follows that $C_1^*(\Pi) \geq a_1$.
□

Proof of Theorem 2. We consider two cases: $C_1 + C_2 > a_1 + a_2 + w$ and $C_1 + C_2 \leq a_1 + a_2 + w$.

Case 1: $C_1 + C_2 > a_1 + a_2 + w$. To simplify notation, for $i\,, j \in 1, 2$ and $i \neq j$, we make use of $b_i = a_i + w$. Using expression (2.13), we differentiate V_i with respect to C_i to obtain

$$\frac{\partial V_i}{\partial C_i} = k + \frac{1}{\lambda w^2}\Big[\frac{p_e}{2}(C_i - a_i + w)(a_i + 2a_j - C_i - 2C_j + 3w) \tag{A.24}$$
$$+ \frac{\lambda\Pi}{2}\big[a_i^2 - a_j^2 - 2a_j C_i + C_i^2 + 2a_j C_j + 2C_i C_j - C_j^2 - 2C_i w + 2a_i(a_j - C_i - C_j + w)\big]\Big] = 0. \tag{A.25}$$

Further,

$$\frac{\partial^2 V_i}{\partial C_i^2} = \frac{(p_e - \lambda\Pi)(a_i + w - C_i + a_j + w - C_j) + \lambda\Pi w}{\lambda w^2}. \tag{A.26}$$

Since $a_i + w - C_i > 0$ for $i \in 1, 2$, then for any $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$, $\frac{\partial^2 V_i}{\partial C_i^2} > 0$. Therefore, we use (A.24) to get

$$\overline{C}_i(C_j, \Pi) = a_i + a_j + 2w - C_j + \frac{\lambda\Pi w - \sqrt{(C_j - a_j)^2(p_e^2 - 3p_e\lambda\Pi + 2(\lambda\Pi)^2) - 2w(p_e - \lambda\Pi)^2(C_j - a_j) + \mathcal{W}}}{p_e - \lambda\Pi}, \tag{A.27}$$

where $\mathcal{W} = 2a_j(p_e - \lambda\Pi)w^2(p_e^2 - p_e\lambda\Pi + (\lambda\Pi)^2 + 2k\lambda(p_e - \lambda\Pi))$. Because $C_i^*(\Pi) = \overline{C}_i(\overline{C}_j(C_i^*(\Pi)), \Pi)$ by (2.9), solving for $C_i^*(\Pi)$ we get

$$C_i^*(\Pi) = a_i + w - \frac{\sqrt{6\lambda k p_e - 4\lambda k \lambda\Pi + (\lambda\Pi)^2} - \lambda\Pi}{3p_e - 2\lambda\Pi} w. \tag{A.28}$$

Since $\lambda\Pi \leq p_e$, the term under the square root in expression (A.28) is positive. Thus, $C_1^*(\Pi)$ and $C_2^*(\Pi)$ exist and are well-defined. We now determine conditions for $C_1^*(\Pi) + C_2^*(\Pi) > a_1 + a_2 + w$ to hold. Using (A.28), we can rewrite $C_1^*(\Pi) + C_2^*(\Pi) > a_1 + a_2 + w$ as

$$w\left(\frac{3}{2}p_e - \sqrt{6\lambda k p_e - 4\lambda^2 k\Pi + \lambda^2\Pi^2}\right) > 0.$$

Since $w > 0$, for $C_1^*(\Pi) + C_2^*(\Pi) > a_1 + a_2 + w$ to hold, it is sufficient (and necessary) that $\frac{3}{2}p_e - \sqrt{6\lambda k p_e - 4\lambda^2 k\Pi + \lambda^2\Pi^2} > 0$. Because $\frac{3}{2}p_e + \sqrt{6\lambda k p_e - 4\lambda^2 k\Pi + \lambda^2\Pi^2} > 0$, this condition translates into

$$\left[\frac{3}{2}p_e - \sqrt{6\lambda k p_e - 4\lambda k \lambda\Pi + \lambda^2\Pi^2}\right] \times \left[\frac{3}{2}p_e + \sqrt{6\lambda k p_e - 4\lambda^2 k\Pi + \lambda^2\Pi^2}\right] = \frac{9}{4}p_e^2 - 6\lambda k p_e + 4\lambda^2 k\Pi - \lambda^2\Pi^2 > 0.$$

Let $\zeta_1(\Pi) := \frac{9}{4}p_e^2 - 6\lambda k p_e + 4\lambda^2 k\Pi - \lambda^2\Pi^2$. Then, we have

$$\zeta_1(\Pi) = (\lambda\Pi - 4k\lambda + \frac{3}{2}p_e)(\frac{3}{2}p_e - \lambda\Pi).$$

Since $\lambda\Pi \leq p_e$ by assumption, then $\zeta_1(\Pi) > 0$ if and only if $\Pi > 4k - \frac{3p_e}{2\lambda}$.

Case 2: $C_1 + C_2 \leq a_1 + a_2 + w$. Using (2.14), we differentiate V_i with respect to C_i for to obtain

$$\frac{\partial V_i}{\partial C_i} = k + \frac{1}{\lambda w^2}\Big[\frac{p_e}{2}\big[a_j^2 - 2a_jC_j + C_j^2 - 2w(a_i - C_i - w)\big] \tag{A.29}$$

$$+ \frac{\lambda\Pi}{2}\big[a_i^2 - a_j^2 - 2a_jC_i + C_i^2 + 2a_jC_j + 2C_iC_j - C_j^2 - 2C_iw + 2a_i(a_j - C_i - C_j + w)\big]\Big] = 0. \tag{A.30}$$

Differentiating further, we obtain

$$\frac{\partial^2 V_i}{\partial C_i^2} = \frac{w(p_e - \lambda\Pi) + (-a_i - a_j + C_i + C_j)\lambda\Pi}{\lambda w^2}.$$

Since $\lambda\Pi \in [0, p_e]$, then $\frac{\partial^2 V_i}{\partial C_i^2} \geq 0$ for all $C_i \in \mathbb{I}_i$ for $i \in 1, 2$. Using (A.29), for $i, j \in 1, 2$ and $i \neq j$ we get

$$\overline{C}_i(C_j, \Pi) = a_i + a_j - C_j + \frac{\sqrt{2w(p_e - \lambda\Pi)(C_2 - a_2) - (C_2 - a_2)^2(p_e - 2\lambda\Pi)\lambda\Pi + w^2(p_e^2 + \lambda^2\Pi(\Pi - 2k))} + \lambda\Pi - p_e}{\lambda\Pi}.$$

Hence, because $C_i^*(\Pi) = \overline{C}_i(\overline{C}_j(C_i^*(\Pi)), \Pi)$ by (2.9), solving for $C_i^*(\Pi)$ we get

$$C_i^*(\Pi) = a_i + \frac{w\left[\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k)} + \lambda\Pi - p_e\right]}{p_e + 2\lambda\Pi}. \tag{A.31}$$

Next, we show that the term under the square root in (A.31) is positive. Since

$$3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k) = -(4\lambda\Pi + 2p_e)\lambda k + 3p_e^2 + 2p_e\lambda\Pi + \lambda^2\Pi^2,$$

and $-(4\lambda\Pi + 2p_e) < 0$, it follows that for any $\lambda k < p_e$

$$-(4\lambda\Pi + 2p_e)\lambda k + 3p_e^2 + 2p_e\lambda\Pi + \lambda^2\Pi^2 > -(4\lambda\Pi + 2p_e)\lambda k + 3p_e^2 + 2p_e\lambda\Pi + \lambda^2\Pi^2\big|_{\lambda k = p_e} = (p_e - \lambda\Pi)^2 \geq 0.$$

Next, we specify conditions under which $C_1^*(\Pi) + C_2^*(\Pi) \leq a_1 + a_2 + w$ should hold true.

Using (A.31), after algebraic manipulations, we rewrite $C_1^*(\Pi) + C_2^*(\Pi) \leq a_1 + a_2 + w$ as

$$\frac{3}{4}p_e^2 + \Pi(\Pi - 4k\lambda) + 2p_e(\Pi - k\lambda) \leq 0. \tag{A.32}$$

Define $\zeta_2(\Pi) := \frac{3}{4}p_e^2 + \lambda^2\Pi(\Pi - 4k) + 2p_e\lambda(\Pi - k)$. Then, $\zeta_2(\Pi) = \left(\lambda\Pi + \frac{p_e}{2}\right)\left(\lambda\Pi - 4k\lambda + \frac{3}{2}p_e\right)$. If $\lambda\Pi \leq 4k\lambda - \frac{3}{2}p_e$, then $\zeta_2 \leq 0$. Therefore, we conclude that, if $\Pi \leq 4k - \frac{3p_e}{2\lambda}$, then $C_1^*(\Pi) + C_2^*(\Pi) \leq a_1 + a_2 + w$.

We next show that the expressions for $C_i^*(\Pi)$ in (A.31) are strictly positive by showing that

$$\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k)} + \lambda\Pi - p_e > 0.$$

Since $\lambda\Pi \leq p_e$, we obtain that $\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k)} - \lambda\Pi + p_e > 0$. Hence, because

$$\begin{aligned}
&\left[\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k\lambda)} + \lambda\Pi - p_e\right] \\
&\times \left[\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k)} - \lambda\Pi + p_e\right] \\
&= 2(p_e + 2\lambda\Pi)(p_e - \lambda k) > 0,
\end{aligned}$$

it follows that $\sqrt{3p_e^2 - 4\lambda^2\Pi k + \lambda^2\Pi^2 + 2p_e\lambda(\Pi - k)} + \lambda\Pi - p_e > 0$. Thus, $C_i^*(\Pi)$ is strictly positive.

Next, we establish that $C_i^*(\Pi)$ is continuous in Π. Let $\mathcal{C}_i^A(\Pi)$ and $\mathcal{C}_i^B(\Pi)$ be as established in (A.28), (A.31). Then, $\mathcal{C}_i^A(4k - \frac{3p_e}{2\lambda}) = \mathcal{C}_i^B(4k - \frac{3p_e}{2\lambda}) = \frac{b_i + a_i}{2}$ which completes the proof. □.

Proof of Proposition 3. Based on Theorem 2, we distinguish two cases.

Case 1: $\Pi > 4k - \frac{3p_e}{2\lambda}$. Then, $C_i^*(\Pi) = \mathcal{C}_i^A(\Pi)$. We first show that $\frac{\partial \mathcal{C}_i^A}{\partial p_e} > 0$. By Theorem 2, we get

$$\frac{\partial \mathcal{C}_i^A}{\partial p_e} = \frac{3(b_i - a_i)\left(\lambda^2\Pi^2 + 3k\lambda p_e - 2k\lambda^2\Pi - \lambda\Pi\sqrt{\lambda^2\Pi^2 + 6kp_e\lambda - 4k\lambda^2\Pi}\right)}{(3p_e - 2\lambda\Pi)^2\sqrt{6kp_e\lambda - 4k\lambda^2\Pi + \lambda^2\Pi^2}}.$$

Let $\gamma(p_e) := \lambda^2\Pi^2 + 3k\lambda p_e - 2k\lambda^2\Pi - \lambda\Pi\sqrt{\lambda^2\Pi^2 + 6kp_e\lambda - 4k\lambda^2\Pi}$. Then, to show that $\frac{\partial \mathcal{C}_i^A}{\partial p_e} > 0$, it suffices to show that $\gamma(p_e)>0$. Since $\lambda\Pi\leq p_e$ by assumption, then $3k\lambda p_e-2k\lambda^2\Pi>0$. Thus, we obtain $\lambda^2\Pi^2+3k\lambda p_e-2k\lambda^2\Pi+\lambda\Pi\sqrt{\lambda^2\Pi^2+6kp_e\lambda-4k\lambda^2\Pi}>0$. Hence, to show that $\gamma(p_e)>0$, it suffices to show that

$$\gamma(p_e)\times\left[\lambda^2\Pi^2+3k\lambda p_e-2k\lambda^2\Pi+\lambda\Pi\sqrt{\lambda^2\Pi^2+6kp_e\lambda-4k\lambda^2\Pi}\right]=k^2\lambda^2(3p_e-2\lambda\Pi)^2>0.$$

Since $\lambda\Pi\leq p_e$, then $k^2\lambda^2(3p_e-2\lambda\Pi)^2>0$. Thus, we conclude that $\frac{\partial\,\mathcal{C}_1^A}{\partial p_e}>0$. Finally, by Theorem 2,

$$\frac{\partial\mathcal{C}_i^A}{\partial k}=-\frac{\lambda(b_i-a_i)}{\sqrt{6\lambda kp_e-4\lambda^2k\Pi+\lambda^2\Pi^2}}<0.$$

Case 2: $\Pi\leq 4k-\frac{3p_e}{2\lambda}$. Then, $C_i^*(\Pi)=\mathcal{C}_i^B(\Pi)$. We first show that $\frac{\partial\mathcal{C}_i^B}{\partial p_e}>0$. Using Theorem 2, we get

$$\frac{\partial\mathcal{C}_i^B}{\partial p_e}=\frac{k\lambda(p_e+2\lambda\Pi)+\lambda\Pi(5p_e+\lambda\Pi)-3\lambda\Pi\sqrt{3p_e^2-4\lambda^2k\Pi+\lambda^2\Pi^2+2p_e\lambda(\Pi-k)}}{(p_e+2\lambda\Pi)^2\sqrt{3p_e^2-4\lambda^2k\Pi+\lambda^2\Pi^2+2p_e\lambda(\Pi-k)}}. \tag{A.33}$$

Define $\Lambda(\Pi):=k\lambda(p_e+2\lambda\Pi)+\lambda\Pi(5p_e+\lambda\Pi)-3\lambda\Pi\sqrt{3p_e^2-4\lambda^2k\Pi+\lambda^2\Pi^2+2p_e\lambda(\Pi-k)}$. It follows from (A.33) that $\frac{\partial\mathcal{C}_i^B}{\partial p_e}>0$ if and only if $\Lambda(\Pi)>0$. Observe that $k\lambda(p_e+2\lambda\Pi)+\lambda\Pi(5p_e+\lambda\Pi)+3\lambda\Pi\sqrt{3p_e^2-4\lambda^2k\Pi+\lambda^2\Pi^2+2p_e\lambda(\Pi-k)}>0$ for any case. Thus, to show $\Lambda(\Pi)>0$, it suffices to show

$$\begin{aligned}&\Lambda(\Pi)\times\left[k\lambda(p_e+2\lambda\Pi)+\lambda\Pi(5p_e+\lambda\Pi)+3\lambda\Pi\sqrt{3p_e^2-4\lambda^2k\Pi+\lambda^2\Pi^2+2p_e(\lambda\Pi-k\lambda)}\right]\\&\quad=(p_e+2\lambda\Pi)^2(k^2\lambda^2+10k\lambda^2\Pi-2\lambda^2\Pi^2)>0.\end{aligned}$$

Let $\phi(\Pi):=(p_e+2\lambda\Pi)^2(k^2\lambda^2+10k\lambda^2\Pi-2\lambda^2\Pi^2)$. Since $\lambda\Pi\leq 4\lambda k-\frac{3}{2}p_e$ then $10k\lambda^2\Pi-2\lambda^2\Pi^2=2\lambda\Pi(5k\lambda-\lambda\Pi)>0$. Thus, we obtain that $\phi(\Pi)>0$ and $\frac{\partial\mathcal{C}_i^B}{\partial p_e}>0$. Finally, by Theorem 2,

$$\frac{\partial\mathcal{C}_i^B}{\partial k}=-\frac{\lambda(b_i-a_i)}{\sqrt{3p_e^2-4\lambda^2k\Pi+\lambda^2\Pi^2+2p_e(\lambda\Pi-\lambda k)}}<0.\quad\square$$

Proof of Proposition 4. We first re-parameterize $\mathcal{C}_i^A$ and $\mathcal{C}_i^B$ with μ_{X_i}, σ. Since $\mu_{X_i} = (a_i + b_i)/2$, and $\sigma_i = (b_i - a_i)^2/12$, then it follows that $a_i = \mu_{X_i} - \sqrt{3}\sigma$ and $b_i = \mu_{X_i} + \sqrt{3}\sigma$. We thus obtain

$$\mathcal{C}_i^A(\mu_{X_i}, \sigma) := \mu_{X_i} + \sqrt{3}\sigma - \frac{\sqrt{6\lambda k p_e - 4\lambda^2 k\Pi + \lambda^2\Pi^2} - \lambda\Pi}{3p_e - 2\lambda\Pi}\sqrt{12}\sigma, \tag{A.34}$$

$$\mathcal{C}_i^B(\mu_{X_i}, \sigma) := \mu_{X_i} - \sqrt{3}\sigma + \frac{\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)} + \lambda\Pi - p_e}{p_e + 2\lambda\Pi}\sqrt{12}\sigma. \tag{A.35}$$

It follows that $\frac{\partial \mathcal{C}_i^A}{\partial \mu_{X_i}} = \frac{\partial \mathcal{C}_i^B}{\partial \mu_{X_i}} = 1 > 0$. Next, we show that $\frac{\partial \mathcal{C}_i^A}{\partial \sigma} > 0$. Using (A.34) we get

$$\frac{\partial \mathcal{C}_i^A}{\partial \sigma} = \frac{\sqrt{3}(3p_e - 2\sqrt{\lambda^2\Pi^2 + 6\lambda k p_e - 4\lambda^2 k\Pi})}{3p_e - 2\pi}.$$

We show that $3p_e - 2\sqrt{\lambda^2\Pi^2 + 6\lambda k p_e - 4\lambda^2 k\Pi} > 0$. Since $3p_e + 2\sqrt{\lambda^2\Pi^2 + 6\lambda k p_e - 4\lambda^2 k\Pi} > 0$, to show that $3p_e - 2\sqrt{\lambda^2\Pi^2 + 6\lambda k p_e - 4\lambda^2 k\Pi} > 0$, it suffices to establish that

$$\left[3p_e - 2\sqrt{\lambda^2\Pi^2 + 6\lambda k p_e - 4\lambda^2 k\Pi}\right] \times \left[3p_e + 2\sqrt{\lambda^2\Pi^2 + 6\lambda k p_e - 4\lambda^2 k\Pi}\right]$$
$$= (3p_e - 2\lambda\Pi)(3p_e + 2\lambda\Pi - 8\lambda k) > 0.$$

Since $\lambda\Pi > 4\lambda k - \frac{3}{2}p_e$, then $3p_e + 2\lambda\Pi - 8\lambda k > 0$. Thus, $(3p_e - 2\lambda\Pi)(3p_e + 2\lambda\Pi - 8\lambda k) > 0$, and $\frac{\partial \mathcal{C}_i^A}{\partial \sigma} > 0$.

Finally, we show that $\frac{\partial \mathcal{C}_i^A}{\partial \sigma} < 0$. Using (A.35) we get

$$\frac{\partial \mathcal{C}_i^B}{\partial \sigma} = \frac{\sqrt{3}\left(-3p_e + 2\sqrt{3p_e^2 + \lambda\Pi(\lambda\Pi - 4\lambda k) + 2p_e(\lambda\Pi - \lambda k)}\right)}{p_e + 2\lambda\Pi}.$$

Since $3p_e + 2\sqrt{3p_e^2 + \lambda\Pi(\lambda\Pi - 4\lambda k) + 2p_e(\lambda\Pi - \lambda k)} > 0$, to show that

$-3p_e + 2\sqrt{3p_e^2 + \lambda\Pi(\lambda\Pi - v4\lambda k) + 2p_e(\lambda\Pi - \lambda k)} < 0$, it suffices to show that

$$\begin{aligned}&\left[-3p_e + 2\sqrt{3p_e^2 + \lambda\Pi(\lambda\Pi - 4\lambda k) + 2p_e(\lambda\Pi - \lambda k)}\right]\\&\times\left[3p_e + 2\sqrt{3p_e^2 + \lambda\Pi(\lambda\Pi - 4\lambda k) + 2p_e(\lambda\Pi - \lambda k)}\right]\\&= (p_e + 2\lambda\Pi)(3p_e + 2\lambda\Pi - 8\lambda k) < 0.\end{aligned}$$

Since $\lambda\Pi \le 4\lambda k - \frac{3}{2}p_e$, then $3p_e + 2\lambda\Pi - 8\lambda k \le 0$. Thus, we get $(p_e + 2\lambda\Pi)(3p_e + 2\lambda\Pi - 8\lambda k) < 0$ and $\frac{\partial \mathcal{C}_i^B}{\partial \sigma} < 0$. □

Proof of Proposition 5. Based on Theorem 2, we distinguish two cases.

Case 1: $\Pi > 4k - \frac{3p_e}{2\lambda}$. Then, $C_i^*(\Pi) = \mathcal{C}_i^A(\Pi)$, and using (2.13) we get

$$\begin{aligned}V_1^*(\Pi) = kC_1^* + \frac{1}{6\lambda w^2}\Big\{&p_e(a_1 + w - C_1^*)^2(a_1 + 3a_2 - C_1^* - 3C_2^* + 4w) + \lambda\Pi\Big[(C_1^* - C_2^* - a_1 + a_2)\\&\times (a_1^2 + a_2^2 + (C_1^*)^2 + 4C_1^*C_2^* + (C_2^*)^2 - 3w(C_1^* + C_2^*) + a_1(4a_2 - 2C_1^* - 4C_2^* + 3w) + a_2(-4C_1^* - 2C_2^* + 3w)\Big]\Big\}.\end{aligned}$$

By Lemma 5, $C_2^*(\Pi) = C_1^*(\Pi) + a_2 - a_1$. We use this equivalence in the above expression to get

$$\begin{aligned}V_1^*(\Pi) &= kC_1^* + \frac{1}{6\lambda w^2}\left[p_e(a_1 + w - C_1^*)^2(a_1 + 3a_2 - C_1^* - 3C_2^* + 4w)\right]\\&= k\big[a_1 + w - \Omega(\Pi)w\big] + \frac{2p_e\big[\Omega(\Pi)w\big]^3}{3\lambda w^2} = k\big[b_1 - (b_1 - a_1)\Omega(\Pi)\big] + \frac{2p_e(b_1 - a_1)}{3\lambda}\Omega^3(\Pi).\end{aligned}$$

Using similar steps, we obtain that

$$V_2^*(\Pi) = k\big[b_2 - (b_2 - a_2)\Omega(\Pi)\big] + \frac{2p_e(b_2 - a_2)}{3\lambda}\Omega^3(\Pi).$$

Case 2: $\Pi \le 4k - \frac{3p_e}{2\lambda}$. Then, $C_i^*(\Pi) = \mathcal{C}_i^B(\Pi)$, and using (2.14) we get

$$V_1^*(\Pi) = kC_1^* + \frac{1}{6\lambda w^2}\Big\{p_e\Big[-a_2^3 + 3C_1^*(C_2^*)^2 + (C_2^*)^3 - 3a_2C_2^*(2C_1^* + C_2^* - 2w) + 3a_2^2(C_1^* + C_2^* - w)$$
$$+3wa_1^2+3w(C_1^*)^2-3w(C_2^*)^2-6w^2C_1^*+3w^3-3a_1\big[a_2^2-2a_2C_2^*+(C_2^*)^2+2w(C_1^*-w)\big]\Big]$$
$$+\lambda\Pi(C_1^*-C_2^*-a_1+a_2)$$
$$\times\Big[a_1^2+a_2^2+(C_1^*)^2+4C_1^*C_2^*+(C_2^*)^2-3w(C_1^*+C_2^*)+a_1(4a_2-2C_1^*-4C_2^*+3w)+a_2(-4C_1^*-2C_2^*+3w)\Big]\Big\}.$$

Using again the identity $C_2^*(\Pi) = C_1^*(\Pi) + a_2 - a_1$, the above expression can be reduced to

$$\begin{aligned}
V_1^*(\Pi) &= kC_1^* + \frac{p_e}{6\lambda w^2}\Big\{-a_2^3 + 3C_1^*(C_2^*)^2 + (C_2^*)^3 - 3a_2C_2^*(2C_1^* + C_2^* - 2w) + 3a_2^2(C_1^* + C_2^* - w) \\
&+ 3wa_1^2 + 3w(C_1^*)^2 - 3w(C_2^*)^2 - 6w^2C_1^* + 3w^3 - 3a_1\big[a_2^2 - 2a_2C_2^* + (C_2^*)^2 + 2w(C_1^* - w)\big]\Big\} \\
&= k\big[a_1+w\Phi(\Pi)\big]+\frac{p_e\big[3-6\Phi(\Pi)+4\Phi^3(\Pi)\big]}{6\lambda}w \\
&= k\big[a_1+(b_1-a_1)\Phi(\Pi)\big]+\frac{p_e(b_1-a_1)\big[3-6\Phi(\Pi)+4\Phi^3(\Pi)\big]}{6\lambda}.
\end{aligned}$$

Using similar steps, we obtain

$$V_2^*(\Pi) = k\big[a_2+(b_2-a_2)\Phi(\Pi)\big]+\frac{p_e(b_2-a_2)\big[3-6\Phi(\Pi)+4\Phi^3(\Pi)\big]}{6\lambda}. \quad \square$$

In what follows, we make use of a technical result concerning the preservation of unimodal functions under functional composition. As far as we know, this result is new to the literature.

Lemma 6. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, monotone function and $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, unimodal function with a finite minimizer (maximizer) y^*. Let f be defined as $f = h \circ g$. Let $x^* := g^{-1}(y^*)$. Then, the following hold:*

(a) f is unimodal;

(b) x^* is a minimizer (maximizer) of f;

(c) If y^* is the unique minimizer (maximizer) of h, then x^* is the unique minimizer (maximizer) of f.

Proof. Because g is monotone (and in the paper we use all the regularity properties in their 'strict' sense) then g^{-1} is well-defined. Further, because g is continuous and monotone, then g^{-1} is continuous and monotone as well. Since y^* is finite by assumption, then because g^{-1} is continuous, x^* is also finite.

If g is increasing then $g(x)<y^*$ for any $x<x^*$, and $g(x)>y^*$ for any $x>x^*$. If g is decreasing, then $g(x) > y^*$ for any $x < x^*$, and $g(x) < y^*$ for any $x > x^*$.

Suppose first that h has a finite minimizer y^*. Then, because h is unimodal, for any $y_1 < y_2 \leq y^*$, we have that $h(y_1) \geq h(y_2) \geq h(y^*)$. Also, for any $y_1 > y_2 > y^*$, we have $h(y_1) \geq h(y_2) \geq h(y^*)$.

Suppose first that g is increasing. Let x^* be as defined in the Lemma. Then, for any $x_1 < x_2 < x^*$, we have that $g(x_1) < g(x_2) < g(x^*) = y^*$. Therefore, $h\big(g(x_1)\big) \geq h\big(g(x_2)\big) \geq h\big(g(x^*)\big) = h(y^*)$ as established above because h is unimodal. Thus, for any $x_1 < x_2 < x^*$, $f(x_1) \geq f(x_2) \geq f(x^*)$. Similarly, for any $x_1 > x_2 > x^*$, we have that $g(x_1) > g(x_2) > g(x^*) = y^*$. It follows that $h\big(g(x_1)\big) \geq h\big(g(x_2)\big) \geq h\big(g(x^*)\big) = h(y^*)$; therefore, for any $x_1 > x_2 > x^*$, $f(x_1) \geq f(x_2) \geq f(x^*)$. It follows that f is unimodal, with a minimizer x^*.

Suppose next that g is decreasing. Then, for any $x_1 < x_2 < x^*$, we have that $g(x_1) > g(x_2) > g(x^*) = y^*$. Therefore, $h\big(g(x_1)\big) \geq h\big(g(x_2)\big) \geq h\big(g(x^*)\big) = h(y^*)$ as established above because h is unimodal. Thus, for any $x_1 < x_2 < x^*$, $f(x_1) \geq f(x_2) \geq f(x^*)$. Similarly, for any $x_1 > x_2 > x^*$, we have that $g(x_1) < g(x_2) < g(x^*) = y^*$. It follows that $h\big(g(x_1)\big) \geq h\big(g(x_2)\big) \geq h\big(g(x^*)\big) = h(y^*)$; therefore, for any $x_1 > x_2 > x^*$, $f(x_1) \geq f(x_2) \geq f(x^*)$. It follows that f is unimodal, with a minimizer x^*. This completes the proof of parts (a) and (b).

To prove part (c), let y^* be the unique minimizer of h. Assume x^* is not a unique minimizer of f. Let $\tilde{x}$, $\tilde{x} \neq x^*$ be another minimizer of f. Then, since f is unimodal, we have $f(\tilde{x}) = f(x^*)$. Hence, $h\big(g(\tilde{x})\big) = h\big(g(x^*)\big) = h(y^*)$. Since h is unimodal by assumption, it follows that $g(\tilde{x})$ is another minimizer of h. Further, since g is monotone and $\tilde{x} \neq x^*$, we get that $g(\tilde{x}) \neq y^*$. This implies that h does not have a unique minimizer, which contradicts the original assumption; hence, y^* is the unique minimizer. The proof when h has a unique finite maximizer follows the same steps. □

For the proofs of Theorems 3 and 4, we make use of the following Lemma.

Lemma 7. *For any* $\Pi \in \left[0, \frac{p_e}{\lambda}\right)$*, the optimal capacity selection* $C_i^*(\Pi)$ *is increasing in* Π.

Proof. Based on Theorem 2, we distinguish two cases.

Case 1. $\Pi > 4k - \frac{3p_e}{2\lambda}$. Then, $C_i^*(\Pi) = \mathcal{C}_i^A(\Pi)$. We use Theorem 2 to get

$$\frac{\partial \mathcal{C}_i^A}{\partial \Pi} = \frac{(b_i - a_i)\left(4k\lambda^2\Pi - 3p_e\lambda\Pi - 6k\lambda p_e + 3p_e\sqrt{\lambda^2\Pi^2 + 6kp_e\lambda - 4k\lambda^2\Pi}\right)}{(3p_e - 2\lambda\Pi)^2\sqrt{6kp_e\lambda - 4k\lambda^2\Pi + \lambda^2\Pi^2}}. \tag{A.36}$$

Define $r(\Pi) := 4k\lambda^2\Pi - 3p_e\pi - 6k\lambda p_e + 3p_e\sqrt{\lambda^2\Pi^2 + 6kp_e\lambda - 4k\lambda^2\Pi}$. To show $\frac{\partial \mathcal{C}_i^A}{\partial \pi} > 0$, it suffices to show that $r(\Pi) > 0$. Since $\lambda\Pi \le p_e$, and $k\lambda < p_e$ at any case, it follows that $-4k\lambda^2\Pi + 3p_e\lambda\Pi + 6k\lambda p_e > 0$. Thus, we obtain $-4k\lambda^2\Pi + 3p_e\lambda\Pi + 6k\lambda p_e + 3p_e\sqrt{\lambda^2\Pi^2 + 6kp_e\lambda - 4k\lambda^2\Pi} > 0$. Hence, because

$$\begin{aligned} r(\Pi) \times & \left[-4k\lambda^2\Pi + 3p_e\lambda\Pi + 6k\lambda p_e + 3p_e\sqrt{\lambda^2\Pi^2 + 6kp_e\lambda - 4k\lambda^2\Pi}\right] \\ & = 4k\lambda(3p_e - 2\lambda\Pi)^2(p_e - k\lambda) > 0. \end{aligned}$$

we obtain $r(\Pi) > 0$ and conclude that $\frac{\partial \mathcal{C}_i^A}{\partial \Pi} > 0$.

Case 2. $\Pi \le 4k - \frac{3p_e}{2\lambda}$. Then, $C_i^*(\Pi) = \mathcal{C}_i^B(\Pi)$. We use Theorem 2 to get

$$\frac{\partial \mathcal{C}_i^B}{\partial \Pi} = \frac{(b_i - a_i)\left(3p_e\sqrt{3p_e^2 - 4\lambda^2k\Pi + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)} + 2k\lambda(p_e + 2\lambda\Pi) - p_e(5p_e + \lambda\Pi)\right)}{(p_e + 2\lambda\Pi)^2\sqrt{3p_e^2 - 4\lambda^2k\Pi + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)}}. \tag{A.37}$$

Let $R(\Pi) := 3p_e\sqrt{3p_e^2 - 4\lambda^2\Pi k + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)} + 2k\lambda(p_e + 2\lambda\Pi) - p_e(5p_e + \lambda\Pi)$. It suffices to show that $R(\Pi) > 0$ to show $\frac{\partial \mathcal{C}_i^B}{\partial \Pi} > 0$. Since $\lambda\Pi \le p_e$ and $k\lambda < p_e$, we obtain that for any Π, $3p_e\sqrt{3p_e^2 - 4\lambda^2\Pi k + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)} - 2k\lambda(p_e + 2\lambda\Pi) + p_e(5p_e + \lambda\Pi) > 0$. Because

$$R(\Pi)\times\left[3p_e\sqrt{3p_e^2-4\lambda^2\Pi k+\lambda^2\Pi^2+2p_e(\lambda\Pi-k\lambda)}-2k\lambda(p_e+2\lambda\Pi)+p_e(5p_e+\lambda\Pi)\right]$$
$$=2(p_e+2\lambda\Pi)^2(p_e^2+k\lambda p_e-2k^2\lambda^2)>0,$$

we obtain $R(\Pi)>0$ and $\frac{\partial\mathcal{C}_i^B}{\partial\Pi}>0$. □

Proof of Theorem 3. Determining the minimizer of $V_i^*(\Pi)$ requires multiple steps due the fact that V_i^*, as given in Proposition 5, is defined by two distinct segments neither of which is convex in Π. In what follows, Let $\Omega,\Phi:\left[0,\frac{p_e}{\lambda}\right)\to\mathbb{R}^+$ be as defined in Proposition 5. Define functions $\mathcal{V}_i^A$ and $\mathcal{V}_i^B$ as

$$\mathcal{V}_i^A(\Omega):=k\big[b_i-(b_i-a_i)\Omega\big]+\frac{2p_e(b_i-a_i)}{3\lambda}\Omega^3 \tag{A.38}$$
$$\mathcal{V}_i^B(\Phi):=\frac{p_e(b_i-a_i)(4\Phi^3-6\Phi+3)}{6\lambda}+(a_i+(b_i-a_i)\Phi)k. \tag{A.39}$$

Following Theorem 2, we now distinguish two cases.

Case 1: $\Pi>4k-\frac{3p_e}{2\lambda}$. Then, $C_i^*=\mathcal{C}_i^A$. By Proposition 5, we then have $V_i^*(\Pi)=\mathcal{V}_i^A\big(\Omega(\Pi)\big)$. Since $\Omega(\Pi)>0$ for any $\Pi\in\left[0,\frac{p_e}{\lambda}\right]$, we get

$$\frac{\partial^2V_i^*}{\partial\Omega^2}=\frac{\partial^2\mathcal{V}_i^A}{\partial\Omega^2}=\frac{4(b_i-a_i)p_e}{\lambda}\Omega>0.$$

Thus, for any $\Pi\in\left[0,\frac{p_e}{\lambda}\right]$, $\mathcal{V}_i^A$ is convex, and thus unimodal in Ω. Let Ω_i^* be defined by

$$\left.\frac{\partial\mathcal{V}_i^A}{\partial\Omega}\right|_{\Omega=\Omega_i^*}=\frac{2(b_i-a_i)p_e}{\lambda}(\Omega_i^*)^2-k(b_i-a_i)=0, \tag{A.40}$$

which leads to

$$\Omega_i^*=\sqrt{\frac{k\lambda}{2p_e}}.$$

We now verify that the remaining assumptions of Lemma 6 hold for functions $\mathcal{V}_i^A$ and Ω. First, using the definition of Ω, it follows from Theorem 2, that $\mathcal{C}_i^A=b_i-\Omega w$. Therefore,

$$\frac{\partial \mathcal{C}_i^A}{\partial \Pi} = \frac{\partial}{\partial \Pi}\left(b_i - \Omega w\right) = -w\frac{\partial \Omega}{\partial \Pi} > 0.$$

By Lemma 7, optimal capacity $C_i^*(\Pi, p_e)$ is increasing Π. Therefore, $\frac{\partial \mathcal{C}_i^A}{\partial \Pi} > 0$. It follows that $\frac{\partial \Omega}{\partial \Pi} < 0$ since $w > 0$. Thus, Ω is monotone in Π. It is straightforward that Ω is continuous in π.

Since $\lambda\Pi > 4\lambda k - \frac{3}{2}p_e$ by assumption, then $\Omega(\Pi)$ is defined on the following disjoint intervals:

$$(k, \Pi) \in \left(0, \frac{3p_e}{8\lambda}\right) \times \left[0, \frac{p_e}{\lambda}\right]; \quad \text{and} \quad (k, \Pi) \in \left(\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda}\right) \times \left(4k - \frac{3p_e}{2\lambda}, \frac{p_e}{\lambda}\right].$$

Since $\frac{\partial \Omega}{\partial \Pi} < 0$, then $\Omega(\Pi) \in \left[\Omega\left(\frac{p_e}{\lambda}\right), \Omega(0)\right]$ for $k \in \left(0, \frac{3p_e}{8\lambda}\right)$, and $\Omega(\Pi) \in \left[\Omega\left(\frac{p_e}{\lambda}\right), \Omega\left(4k - \frac{3p_e}{2\lambda}\right)\right]$ for $k \in \left[\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda}\right)$. Our objective is to prove that Ω_i^* is the unique minimizer of $\mathcal{V}_i^A$ when $k \in \left(0, \frac{p_e}{2\lambda}\right)$. For that purpose, we will first show that $\Omega\left(\frac{p_e}{\lambda}\right) \leq \Omega_i^* \leq \Omega(0)$ when $k \in \left(0, \frac{3p_e}{8\lambda}\right)$, and that $\Omega\left(\frac{p_e}{\lambda}\right) \leq \Omega_i^* \leq \Omega\left(4k - \frac{3p_e}{2\lambda}\right)$ when $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$.

We begin by showing that $\Omega\left(\frac{p_e}{\lambda}\right) \leq \Omega_i^* \leq \Omega(0)$ when $k \in \left(0, \frac{3p_e}{8\lambda}\right)$. We have that

$$\Omega(0) = \left.\frac{\sqrt{6kp_e\lambda - 4k\lambda^2\Pi + \lambda^2\Pi^2} - \lambda\Pi}{3p_e - 2\lambda\Pi}\right|_{\Pi=0} = \sqrt{\frac{2k\lambda}{3p_e}},$$
$$\Omega\left(\frac{p_e}{\lambda}\right) = \left.\frac{\sqrt{6kp_e\lambda - 4k\lambda^2\Pi + \lambda^2\Pi^2} - \lambda\Pi}{3p_e - 2\lambda\Pi}\right|_{\Pi=\frac{p_e}{\lambda}} = \sqrt{1 + \frac{2k\lambda}{p_e}} - 1.$$

For any $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$, we have

$$\Omega(0) - \Omega_i^* = \sqrt{\frac{2k\lambda}{3p_e}} - \sqrt{\frac{k\lambda}{2p_e}} > 0.$$

Thus, we conclude that $\Omega_i^* < \Omega(0)$. Further,

$$\Omega_i^* - \Omega\left(\frac{p_e}{\lambda}\right) = 1 + \sqrt{\frac{k\lambda}{2p_e}} - \sqrt{1 + \frac{2k\lambda}{p_e}}.$$

Since $1 + \sqrt{\frac{k\lambda}{2p_e}} + \sqrt{1 + \frac{2k\lambda}{p_e}} > 0$, to show that $\Omega_i^* - \Omega(p_e) > 0$, it suffices to show that

$$\left(\Omega_i^* - \Omega\left(\frac{p_e}{\lambda}\right)\right)\left[1 + \sqrt{\frac{k\lambda}{2p_e}} + \sqrt{1 + \frac{2k\lambda}{p_e}}\right] = \sqrt{\frac{2k\lambda}{p_e}} - \frac{3k\lambda}{2p_e} > 0. \tag{A.41}$$

Since $k < \frac{3p_e}{8\lambda}$, then

$$\sqrt{\frac{2k\lambda}{p_e}} - \frac{3k\lambda}{2p_e} = \sqrt{\frac{k\lambda}{p_e}}\left[\sqrt{2} - \frac{3}{2}\sqrt{\frac{k\lambda}{p_e}}\right] > 0.$$

It follows that $\Omega_i^* - \Omega\left(\frac{p_e}{\lambda}\right) > 0$. We now turn to the case of $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$ and $\Pi \in [4k - \frac{3p_e}{2\lambda}, \frac{p_e}{\lambda}]$. First, it follows from (A.41) that, for any $\Pi \in [4k - \frac{3p_e}{2\lambda}, \frac{p_e}{\lambda}]$, $\Omega(\Pi) > \Omega\left(\frac{p_e}{\lambda}\right)$ since $k < \frac{5p_e}{8\lambda}$. Next, consider the difference $\Omega\left(4k - \frac{3p_e}{2\lambda}\right) - \Omega_i^*$ We have that

$$\Omega\left(4k - \frac{3p_e}{2\lambda}\right) - \Omega_i^* = \frac{1}{2} - \sqrt{\frac{k\lambda}{2p_e}} = \sqrt{\frac{1}{2}}\left[\sqrt{\frac{1}{2}} - \sqrt{\frac{k\lambda}{p_e}}\right]$$

Thus, $\Omega\left(4k - \frac{3p_e}{2\lambda}\right) > \Omega_i^*$ when $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$. To conclude our proof that Ω_i^* is the unique minimizer of $\mathcal{V}_i^A$, it remains to show that $\mathcal{V}_i^B\left(\Phi\left(4k - \frac{3p_e}{2\lambda}\right)\right) \geq \mathcal{V}_i^A\left(\Omega_i^*\right)$ when $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$.

For that purpose, we make use of expression (A.38) and (A.39) to get

$$\begin{aligned}
\mathcal{V}_i^B\left(\Phi\left(4k - \frac{3p_e}{2\lambda}\right)\right) &= \frac{k(b_i + a_i)}{2} + \frac{p_e(b_i - a_i)}{12\lambda}, \\
\mathcal{V}_i^A\left(\Omega_i^*\right) &= k\left(b_i - \frac{\sqrt{2}(b_i - a_i)}{3}\sqrt{\frac{k\lambda}{p_e}}\right).
\end{aligned}$$

Define $\rho_i(k) := \mathcal{V}_i^B\left(\Phi\left(4\lambda k - \frac{3}{2}p_e\right)\right) - \mathcal{V}_i^A\left(\Omega_i^*\right)$. Using the above, we therefore obtain

$$\rho_i(k) = \frac{(b_i - a_i)\left(4\sqrt{\frac{2}{p_e}}\,(k\lambda)^{3/2} - 6k\lambda + p_e\right)}{12\lambda}. \tag{A.42}$$

We now show $\rho_i(k) \geq 0$ for any $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$. We get

$$\frac{\partial \rho_i}{\partial k} = \frac{b_i - a_i}{2}\left(\sqrt{\frac{2k\lambda}{p_e}} - 1\right). \tag{A.43}$$

Further,

$$\frac{\partial^2 \rho_i}{\partial k^2} = \frac{\sqrt{\lambda}(b_i - a_i)}{2\sqrt{2kp_e}} > 0.$$

Thus, ρ_i is convex in k. By (A.43), $\arg\min_k \rho_i(k) = \frac{p_e}{2\lambda}$. Using (A.42), we get $\rho_i\left(\frac{p_e}{2\lambda}\right) = 0$. Since $\rho_i(k) \geq \rho_i\left(\frac{p_e}{2\lambda}\right)$, then $\rho_i(k) \geq 0$. Hence, $\mathcal{V}_i^B\left(\Phi\left(4k - \frac{3p_e}{2\lambda}\right)\right) \geq \mathcal{V}_i^A\left(\Omega_i^*\right)$ for any $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$.

Therefore, Ω_i^* is the unique minimizer of $\mathcal{V}_i^A$ when $k < \frac{p_e}{2\lambda}$. Thus, when $k < \frac{p_e}{2\lambda}$, the required conditions of Lemma 6 are satisfied by functions $\mathcal{V}_i^A$ and Ω, and the unique minimizer Ω_i^* of $\mathcal{V}_i^A$.

Case 2: $\Pi \leq 4k - \frac{3p_e}{2\lambda}$. Then, $C_i^* = \mathcal{C}_i^B$, and $V_i^*(\Pi) = \mathcal{V}_i^B\big(\Omega(\Pi)\big)$. As $\Phi(\Pi) > 0$ for any $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$, then

$$\frac{\partial^2 \mathcal{V}_i^B}{\partial \Phi^2} = \frac{4p_e(b_i - a_i)\Phi}{\lambda} > 0$$

Hence, for any $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$, $\mathcal{V}_i^B\left(\Phi\right)$ is convex and thus unimodal in Φ. Let Φ_i^* be defined by

$$\left.\frac{\partial \mathcal{V}_i^B}{\partial \Phi}\right|_{\Phi=\Phi_i^*} = w\left(k + \frac{p_e(2(\Phi_i^*)^2 - 1)}{\lambda}\right) = 0,$$

which leads to

$$\Phi_i^* = \sqrt{\frac{p_e - k\lambda}{2p_e}}.$$

We now verify that the remaining assumptions of Lemma 6 hold for functions $\mathcal{V}_i^B$ and Φ. First, using the definition of Φ, it follows from Theorem 2, that $\mathcal{C}_i^B = a_i + w\Phi$. Therefore,

$$\frac{\partial \mathcal{C}_i^B}{\partial \Pi} = \frac{\partial}{\partial \Pi}\left(a_i + w\Phi\right) = w\frac{\partial \Phi}{\partial \Pi} > 0.$$

By Lemma 7, optimal capacity $C_i^*(\Pi, p_e)$ is increasing Π. Therefore, $\frac{\partial \mathcal{C}_i^B}{\partial \Pi} > 0$, and it follows that $\frac{\partial \Phi}{\partial \Pi} > 0$ since $w > 0$. Thus, Φ is monotone in Π. It is also straightforward that Φ is continuous in Π. Next we show that $\mathcal{V}_i^B$ has a unique minimizer.

Because $\Pi \leq 4k - \frac{3p_e}{2\lambda}$ by assumption, $\Phi(\Pi)$ is defined on the following disjoint intervals:

$$(k, \Pi) \in \left(\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda}\right) \times \left[0, 4k - \frac{3p_e}{2\lambda}\right]; \quad \text{and} \quad (k, \Pi) \in \left(\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda}\right) \times \left[0, \frac{p_e}{\lambda}\right].$$

Since $\frac{\partial \Phi}{\partial \Pi} > 0$, then $\Phi(\Pi) \in [\Phi(0), \Phi\left(4k - \frac{3p_e}{2\lambda}\right)]$ for $k \in [\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda})$ and $\Phi(\Pi) \in \left[\Phi(0), \Phi\left(\frac{p_e}{\lambda}\right)\right]$ for $k \in [\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda})$. Next, we establish that Φ_i^* is the unique minimizer of $\mathcal{V}_i^B$ when $k \in \left[\frac{p_e}{2\lambda}, \frac{p_e}{\lambda}\right)$. For that purpose, it suffices to show that $\Phi(0) \leq \Phi_i^* \leq \Phi\left(4k - \frac{3p_e}{2\lambda}\right)$ when $k \in [\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda})$, and that $\Phi(0) \leq \Phi_i^* \leq \Phi\left(\frac{p_e}{\lambda}\right)$ when $k \in \left[\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda}\right)$. We first show that $\Phi(0) \leq \Phi_i^* \leq \Phi\left(\frac{p_e}{\lambda}\right)$ when $k \in \left[\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda}\right)$.

$$\Phi(0) = \left.\frac{\sqrt{3p_e^2 - 4\lambda\Pi k + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)} + \lambda\Pi - p_e}{p_e + 2\lambda\Pi}\right|_{\Pi=0} = \sqrt{3 - \frac{2k\lambda}{p_e}} - 1,$$

$$\Phi\left(p_e\right) = \left.\frac{\sqrt{3p_e^2 - 4\lambda^2\Pi k + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)} + \lambda\Pi - p_e}{p_e + 2\lambda\Pi}\right|_{\Pi=\frac{p_e}{\lambda}} = \sqrt{\frac{2(p_e - k\lambda)}{3p_e}}.$$

We thus obtain that $\Phi\left(\frac{p_e}{\lambda}\right) - \Phi_i^* = \sqrt{\frac{p_e - k\lambda}{p_e}}\left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right) > 0$. Next, observe that

$$\Phi_i^* - \Phi(0) = \sqrt{\frac{1}{2} - \frac{k\lambda}{2p_e}} - \sqrt{3 - \frac{2k\lambda}{p_e}} + 1.$$

Because $k\lambda < p_e$, it follows that $\sqrt{\frac{1}{2} - \frac{k\lambda}{2p_e}} + \sqrt{3 - \frac{2k\lambda}{p_e}} + 1 > 0$. Consequently, to show that $\sqrt{\frac{1}{2} - \frac{k\lambda}{2p_e}} - \sqrt{3 - \frac{2k\lambda}{p_e}} + 1 > 0$, it suffices to show that

$$\begin{aligned}
&\left[\sqrt{\frac{1}{2} - \frac{k\lambda}{2p_e}} - \sqrt{3 - \frac{2k\lambda}{p_e}} + 1\right] \times \left[\sqrt{\frac{1}{2} - \frac{k\lambda}{2p_e}} + \sqrt{3 - \frac{2k\lambda}{p_e}} + 1\right] \\
&= \frac{1}{4}\left(\sqrt{1 - \frac{k\lambda}{p_e}}\right)\left(4\sqrt{2} - 6\sqrt{1 - \frac{k\lambda}{p_e}}\right).
\end{aligned}$$

Since $k > \frac{p_e}{9\lambda}$ then $4\sqrt{2} - 6\sqrt{1 - \frac{k\lambda}{p_e}} > 0$. It follows that $\Phi_i^* > \Phi(0)$ in case of $k \in [\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda})$. Consequently, we obtain $\Phi(0) \leq \Phi_i^* \leq \Phi\left(\frac{p_e}{\lambda}\right)$ for $k \in [\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda})$.

We next examine that $\Phi(0) \leq \Phi_i^* \leq \Phi\left(4k - \frac{3p_e}{2\lambda}\right)$ in the case of $k \in [\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda})$. From previous results, it follows that $\Phi_i^* > \Phi(0)$ for $k \in [\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda})$. Also, since

$$\Phi\left(4k - \frac{3p_e}{2\lambda}\right) = \left.\frac{\sqrt{3p_e^2 - 4\lambda^2 k\Pi + \lambda^2\Pi^2 + 2p_e(\lambda\Pi - k\lambda)} + \lambda\Pi - p_e}{p_e + 2\lambda\Pi}\right|_{\Pi=4k-\frac{3p_e}{2\lambda}} = \frac{1}{2},$$

then,

$$\Phi\left(4k - \frac{3p_e}{2\lambda}\right) - \Phi_i^* = \frac{1}{2} - \sqrt{\frac{1}{2} - \frac{k\lambda}{2p_e}} > 0. \tag{A.44}$$

From (A.44), we conclude that if $k \in [\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda})$ then $\Phi\left(4k - \frac{3p_e}{2\lambda}\right) \geq \Phi_i^*$. To conclude that Φ_i^* is the unique minimizer of $\mathcal{V}_i^B$ when $k \geq \frac{p_e}{2\lambda}$, it now suffices to show that $\mathcal{V}_i^B(\Phi_i^*) \leq \mathcal{V}_i^A\left(\Omega\left(4k - \frac{3p_e}{2\lambda}\right)\right)$.

For that purpose, we use expressions (A.38) and (A.39) to obtain

$$\mathcal{V}_i^B(\Phi_i^*) = \frac{k(b_i - a_i)}{3}\sqrt{2 - \frac{2k\lambda}{p_e}} + \frac{p_e(b_i - a_i)}{6\lambda}\left(3 - 2\sqrt{2 - \frac{2k\lambda}{p_e}}\right),$$

$$\mathcal{V}_i^A\left(\Omega\left(4k - \frac{3p_e}{2\lambda}\right)\right) = \frac{k(b_i + a_i)}{2} + \frac{p_e(b_i - a_i)}{12\lambda}.$$

Define $\vartheta_i(k) := \mathcal{V}_i^A\left(\Omega\left(4k - \frac{3p_e}{2\lambda}\right)\right) - \mathcal{V}_i^B(\Phi_i^*)$. Consequently, we obtain

$$\vartheta_i(k) = \frac{(b_i - a_i)\left(6k\lambda - 4k\lambda\sqrt{2 - \frac{2k\lambda}{p_e}} + p_e\left(4\sqrt{2 - \frac{2k\lambda}{p_e}} - 5\right)\right)}{12\lambda}. \tag{A.45}$$

We now show $\vartheta_i(k) \geq 0$ for any $k \in \left[\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda}\right)$. We have that

$$\frac{\partial \vartheta_i}{\partial k} = \frac{(b_i - a_i)\left(\sqrt{p_e(p_e - k\lambda)} - \sqrt{2}(p_e - k\lambda)\right)}{2\sqrt{p_e(p_e - k\lambda)}}; \quad \frac{\partial^2 \vartheta_i}{\partial k^2} = \frac{(b_i - a_i)\lambda}{2\sqrt{2p_e(p_e - k\lambda)}} > 0. \tag{A.46}$$

Thus, $\vartheta_i(k)$ is convex in k. Using (A.46) we obtain $\arg\min_k \vartheta_i(k) = \frac{p_e}{2\lambda}$. We also use (A.45) to obtain $\vartheta_i\left(\frac{p_e}{2\lambda}\right) = 0$. Since $\vartheta_i(k) \geq \vartheta_i\left(\frac{p_e}{2\lambda}\right)$, then we obtain $\vartheta_i(k) \geq 0$. Consequently, we conclude that $\mathcal{V}_i^B(\Phi_i^*) \leq \mathcal{V}_i^A\left(\Omega\left(4k - \frac{3p_e}{2\lambda}\right)\right)$ for any $k \in \left[\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda}\right)$.

Therefore, Φ_i^* is the unique minimizer of $\mathcal{V}_i^B$ when $k \geq \frac{p_e}{2\lambda}$. Thus, when $k \geq \frac{p_e}{2\lambda}$, the required conditions of Lemma 6 are satisfied by functions $\mathcal{V}_i^B$ and Φ, and the unique minimizer Φ_i^* of $\mathcal{V}_i^B$.

If $k < \frac{p_e}{2\lambda}$, then, for any $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$, $V_i^*(\Pi) \geq \mathcal{V}_i^A(\Omega_i^*)$; if $k \geq \frac{p_e}{2\lambda}$, then, for any $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$, $V_i^*(\Pi) \geq \mathcal{V}_i^B(\Phi_i^*)$. Consequently all the required conditions of Lemma 6 hold. We can therefore now apply Lemma 6 to conclude V_i^* is unimodal and that its unique minimizer Π^* is given by

$$\Pi_i^* = \Omega^{-1}\left(\Omega_i^*\right) = \frac{p_e\left(\lambda k + \sqrt{2\lambda k p_e}\right)}{2\lambda(2p_e - \lambda k)}$$

when $k < \frac{p_e}{2\lambda}$, and by

$$\Pi_i^* = \Phi^{-1}(\Phi_i^*) = \frac{p_e\left(p_e + 3\lambda k - \sqrt{2p_e(p_e - \lambda k)}\right)}{2\lambda(p_e + \lambda k)}$$

when $k \geq \frac{p_e}{2\lambda}$. To obtain the consumer's optimal cost $V_i^*(\Pi^*)$, we substitute the above expressions for Π^* into expression (2.15). To obtain the consumer's optimal capacity level, we substitute the above expressions for Π^* into expressions (2.15) and (2.16), and simplifying the resulting equations. □

Proof of Proposition 6. To prove part (a), for $k \in \left(0, \frac{p_e}{\lambda}\right)$ and $p_e \in (0, \infty)$ define

$$P_1(k, p_e) := \frac{p_e(k\lambda + \sqrt{2k\lambda p_e})}{2\lambda(2p_e - k\lambda)}; \tag{A.47}$$

$$P_2(k, p_e) := \frac{p_e\left(p_e + 3k\lambda - \sqrt{2p_e(p_e - k\lambda)}\right)}{2\lambda(p_e + k\lambda)}. \tag{A.48}$$

It follows from Theorem 3 that $\Pi^* = P_1(k, p_e)$ for $k < \frac{p_e}{2\lambda}$, and $\Pi^* = P_2(k, p_e)$ for $k \geq \frac{p_e}{2\lambda}$. We first show that $\frac{\partial \Pi^*}{\partial k} > 0$ for any $k \in \left(0, \frac{p_e}{\lambda}\right)$. If $k < \frac{p_e}{2\lambda}$, then since $\Pi^* = P_1(p_e, k)$, we use (A.47) to get

$$\frac{\partial \Pi^*}{\partial k} = \frac{\partial P_1(p_e, k)}{\partial k} = \frac{p_e^2(4\sqrt{p_e k\lambda} + \sqrt{2}k\lambda + 2\sqrt{2}p_e)}{4\sqrt{p_e k\lambda}(2p_e - k\lambda)^2} > 0,$$

since $p_e > k\lambda$ by assumption. If $k \geq \frac{p_e}{2\lambda}$, then $\Pi^* = P_2(p_e, k)$, and we use (A.48) to get

$$\frac{\partial \Pi^*}{\partial k} = \frac{\partial P_2(p_e, k)}{\partial k} = \frac{p_e^2\left(4\sqrt{p_e(p_e - k\lambda)} - \sqrt{2}k\lambda + 3\sqrt{2}p_e\right)}{4\sqrt{p_e(p_e - k\lambda)}(k\lambda + p_e)^2} > 0,$$

since $-\sqrt{2}k\lambda + 3\sqrt{2}p_e > 0$. Thus, $\frac{\partial \Pi^*}{\partial k} > 0$ for any $k \geq \frac{p_e}{2\lambda}$.

We next examine $\frac{\partial \Pi^*}{\partial p_e}$. If $k < \frac{p_e}{2\lambda}$, then since $\Pi^* = P_1(p_e, k)$, we use (A.47) to get

$$\frac{\partial \Pi^*}{\partial p_e} = \frac{2\sqrt{2}p_e\sqrt{p_e k\lambda} - 3\sqrt{2}k\lambda\sqrt{p_e k\lambda} - 2k^2\lambda^2}{4\lambda(2p_e - k\lambda)^2}$$
$$= \frac{2\sqrt{2}\sqrt{p_e k\lambda}(p_e - 2k\lambda) + \sqrt{2}(k\lambda)^{\frac{3}{2}}(\sqrt{p_e} - \sqrt{2k\lambda})}{4\lambda(2p_e - k\lambda)^2} > 0,$$

since $p_e > 2k\lambda$. If $k \geq \frac{p_e}{2\lambda}$, then since $\Pi^* = P_2(p_e, k)$, we again use (A.48) to get

$$\frac{\partial \Pi^*}{\partial p_e} = \frac{\partial P_2(p_e, k)}{\partial p_e} = \frac{\sqrt{p_e(p_e - k\lambda)}(6k^2\lambda^2 + 4p_e k\lambda + 2p_e^2) + \sqrt{2}(3p_e k^2\lambda^2 - 3p_e^2 k\lambda - 2p_e^3)}{4\lambda\sqrt{p_e(p_e - k\lambda)}(k\lambda + p_e)^2}.$$

For notational convenience, define

$$\mathcal{D}(k) := \sqrt{p_e(p_e - k\lambda)}(6k^2\lambda^2 + 4p_e k\lambda + 2p_e^2) + \sqrt{2}(3p_e k\lambda(k\lambda - p_e) - 2p_e^3).$$

Thus, $\frac{\partial \Pi^*}{\partial p_e} > 0$ if and only if $\mathcal{D}(k) > 0$. Now, observe that, since $3p_e k\lambda(k\lambda - p_e) \leq 0$, then $-\sqrt{p_e(p_e - k\lambda)}(6k^2\lambda^2 + 4p_e k\lambda + 2p_e^2) + \sqrt{2}(3p_e k\lambda(k\lambda - p_e) - 2p_e^3) < 0$. Thus, if

$$\mathcal{D}(k) \times \left[-\sqrt{p_e(p_e - k\lambda)}(6K^2\lambda^2 + 4p_e k\lambda + 2p_e^2) + \sqrt{2}(3p_e k\lambda(k\lambda - p_e) - 2p_e^3)\right]$$
$$= 2p_e(p_e + k\lambda)^2(p_e - 2k\lambda)(2p_e^2 + 6p_e\lambda k - 9\lambda^2 k^2) < 0,$$

then $\mathcal{D}(k) > 0$. Note that $2p_e(p_e + k\lambda)^2(p_e - 2k\lambda) \leq 0$ since $k \geq \frac{p_e}{2\lambda}$. Also,

$$2p_e^2 + 6p_e\lambda k - 9\lambda^2 k^2 = -9\lambda^2(k - \frac{(1-\sqrt{3})p_e}{3\lambda})(k - \frac{(1+\sqrt{3})p_e}{3\lambda}).$$

Since $k \in [\frac{p_e}{2\lambda}, \frac{p_e}{\lambda})$, then $k < \frac{(1+\sqrt{3})p_e}{3\lambda}$ implies $\mathcal{D}(k) > 0$, and $k > \frac{(1+\sqrt{3})p_e}{3\lambda}$ implies $\mathcal{D}(k) < 0$. Consequently, $\frac{\partial \Pi^*}{\partial p_e} \leq 0$ if $p_e \leq \frac{3(\sqrt{3}-1)}{2}\lambda k$ and $\frac{\partial \Pi^*}{\partial p_e} > 0$ if $p_e > \frac{3(\sqrt{3}-1)}{2}\lambda k$.

To prove part (b), we use $a_i = \mu_{X_i} - \sqrt{3}\sigma$ and $b_i = \mu_{X_i} + \sqrt{3}\sigma$, to obtain, by Theorem 3, the following:

$$V_i^*(\Pi_i^*) = \begin{cases} k\left(\mu_{X_i} + \left(\sqrt{3} - 2\sqrt{\frac{2\lambda k}{3p_e}}\right)\sigma\right) & \text{if } k < \frac{p_e}{2\lambda}; \\ k\left(\mu_{X_i} - \sqrt{3}\sigma + \sqrt{6(1-\frac{\lambda k}{p_e})\sigma}\right) + \frac{\left(3p_e - (2p_e+\lambda k)\sqrt{2-\frac{2\lambda k}{p_e}}\right)\sigma}{6\lambda} & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases} \tag{A.49}$$

We first show that $\frac{\partial V_i^*(\Pi_i^*)}{\partial p_e} > 0$. If $k < \frac{p_e}{2\lambda}$, then it follows from (A.49) that

$$\frac{\partial V_i^*(\Pi_i^*)}{\partial p_e} = \sqrt{\frac{2\lambda}{3}}\left(\frac{k}{p_e}\right)^{3/2}\sigma > 0.$$

If $k \geq \frac{p_e}{2\lambda}$ then

$$\frac{\partial V_i^*(\Pi_i^*)}{\partial p_e} = \frac{\left(\sqrt{2}\lambda k(p_e+\lambda k) + p_e^2(3\sqrt{1-\frac{\lambda k}{p_e}} - 2\sqrt{2})\right)\sigma}{\lambda p_e^2\sqrt{3\left(1-\frac{\lambda k}{p_e}\right)}}.$$

It also follows that, if $k \geq \frac{p_e}{2\lambda}$, then

$$\frac{\partial^2 V_i^*(\Pi_i^*)}{\partial k\,\partial p_e} = \frac{1}{p_e^2}\sqrt{\frac{3\lambda k\sigma}{2-\frac{2\lambda k}{p_e}}} > 0. \tag{A.50}$$

It thus follows from (A.50) that, for any $k \geq \frac{p_e}{2\lambda}$, $\frac{\partial V_i^*(\Pi_i^*)}{\partial p_e} \geq \left.\frac{\partial V_i^*(\Pi_i^*)}{\partial p_e}\right|_{k=\frac{p_e}{2\lambda}} = \frac{\sigma}{2\sqrt{3}\lambda}$. Consequently, we obtain that $\frac{\partial V_i^*(\Pi_i^*)}{\partial p_e} > 0$. We also obtain from (A.49) that, for any $k \in \left(0, \frac{p_e}{\lambda}\right)$, $\frac{\partial V_i^*(\Pi_i^*)}{\partial \mu_{X_i}} = k$.

Next, we show that $\frac{\partial V_i^*(\Pi_i^*)}{\partial \sigma} > 0$. From expression (A.49) we obtain

$$\frac{\partial V_i^*(\Pi_i^*)}{\partial \sigma} = \begin{cases} \frac{k\left(3 - 2\sqrt{2}\sqrt{\frac{\lambda k}{p_e}}\right)}{\sqrt{3}} & \text{if } k < \frac{p_e}{2\lambda}; \\ \frac{(p_e - \lambda k)\left(3 - 2\sqrt{2-\frac{2\lambda k}{p_e}}\right)}{\sqrt{3}\lambda} & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases}$$

It therefore follows that $\frac{\partial V_i^*(\Pi_i^*)}{\partial \sigma} > 0$. Finally, we show that $\frac{\partial V_i^*(\Pi_i^*)}{\partial k} > 0$. Again by (A.49) ,

$$\frac{\partial V_i^*(\Pi_i^*)}{\partial k} = \begin{cases} \mu_{X_i} + \sqrt{3}\sigma\left(1 - \sqrt{\dfrac{2\lambda k}{p_e}}\right) & \text{if } k < \frac{p_e}{2\lambda}; \\ \dfrac{\sqrt{6}\sigma\sqrt{p_e(p_e - \lambda k)} + (\mu_{X_i} - \sqrt{3}\sigma)p_e}{p_e} & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases}$$

Since $k < \frac{p_e}{2\lambda}$ then $1 - \sqrt{\frac{2\lambda k}{p_e}} > 0$, then $\frac{\partial V_i^*(\Pi_i^*)}{\partial k} > 0$ if $k < \frac{p_e}{2\lambda}$. If $k \geq \frac{p_e}{2\lambda}$, on the other hand, then it follows from $a_i = \mu_{X_i} - \sqrt{3}\sigma > 0$ that $\frac{\partial V_i^*(\Pi_i^*)}{\partial k} > 0$.

To prove part (c), we use Theorem 3 to reparameterize $C_i^*(\Pi_i^*)$ using μ_{X_i}, and σ as follows:

$$C_i^*(\Pi_i^*) = \begin{cases} \mu_{X_i} + \left(\sqrt{3} - \sqrt{\dfrac{6k\lambda}{p_e}}\right)\sigma, & \text{if } k < \frac{p_e}{2\lambda}, \\ \mu_{X_i} - \left(\sqrt{3} - \sqrt{6 - \frac{6k\lambda}{p_e}}\right)\sigma. & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases} \tag{A.51}$$

Thus, $C_i^*(\Pi_i^*)$ is increasing in μ_{X_i}. We next show that $\frac{\partial C_i^*(\Pi_i^*)}{\partial p_e} > 0$. Using (A.51), we get

$$\frac{\partial C_i^*(\Pi_i^*)}{\partial p_e} = \begin{cases} \sqrt{\dfrac{3k\lambda}{2p_e^3}}\sigma, & \text{if } \ k < \frac{p_e}{2\lambda}; \\ \dfrac{\sqrt{3}\sigma}{p_e\sqrt{\dfrac{2p_e}{k\lambda}\left(\dfrac{p_e}{k\lambda} - 1\right)}}, & \text{if } \ k \geq \frac{p_e}{2\lambda}. \end{cases}$$

It follows that $\frac{\partial C_i^*(\Pi_i^*)}{\partial p_e} > 0$ for any $p_e > 0$. Next, if $k < \frac{p_e}{2\lambda}$, then we use (A.51) to get

$$\frac{\partial C_i^*(\Pi_i^*)}{\partial \sigma} = \sqrt{3} - \sqrt{\frac{6k\lambda}{p_e}}.$$

Since $k < \frac{p_e}{2\lambda}$, then $\frac{\partial C_i^*(\Pi_i^*)}{\partial \sigma} > 0$. If $k \geq \frac{p_e}{2\lambda}$, then

$$\frac{\partial C_i^*(\Pi_i^*)}{\partial \sigma} = -\sqrt{3} + \sqrt{6 - \frac{6k\lambda}{p_e}}.$$

Since $k \geq \frac{p_e}{2\lambda}$, then $\frac{\partial C_i^*(\Pi_i^*)}{\partial \sigma} \leq 0$. We now show that $\frac{\partial C_i^*(\Pi_i^*)}{\partial k} < 0$. Using (A.51) we get

$$\frac{\partial C_i^*(\Pi_i^*)}{\partial k} = \begin{cases} -\sqrt{\dfrac{3\lambda}{2p_e k}}\sigma, & \text{if } k < \frac{p_e}{2\lambda}; \\ -\dfrac{\sqrt{3}\lambda\sigma}{p_e\sqrt{2-\dfrac{2k\lambda}{p_e}}}, & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases}$$

From the above, we obtain $\frac{\partial C_i^*(\Pi_i^*)}{\partial k} < 0$ for any $k > 0$. $\square$

Proof of Proposition 7. Observe from (2.21) that, if $c \geq b$, then $v(c) = kc$ and $v(c)$ is increasing in c. Therefore, to minimize the total cost of installation and electricity related costs following the installation, an electricity consumer would never install any generation capacity in excess of b. It follows that $c^* < b$.

Using $c^* < b$, we solve for $v(c)$ given in (2.21) to obtain

$$v(c) = \frac{p_e}{b-a}\int_0^\infty e^{-\lambda t}\int_a^b \big[(X-c)^+\big]dx\,dt + kc = \begin{cases} \dfrac{p_e(b-c)^2}{2\lambda(b-a)} + kc, & \text{if } a \leq c < b \\ \dfrac{p_e(b+a-2c)}{2\lambda} + kc, & \text{if } c < a. \end{cases} \tag{A.52}$$

If $c < a$, then, by (A.52), $\frac{\partial v(c)}{\partial c} = -\frac{p_e}{\lambda} + k < 0$. Thus, if $c < a$, the total cost is decreasing in c. Hence, optimal capacity is never less than a. Since $a \leq c^* \leq b$, then $\frac{\partial^2}{\partial c^2}[v(c)] = \frac{p_e}{\lambda(b-a)} > 0$. Solving $\frac{\partial}{\partial c}\left[\frac{p_e(b+a-2c)}{2\lambda} + kc\right] = 0$ yields the desired expression of c^*. The expression for v^* follows from substituting c^* into (2.21). $\square$

Proof of Theorem 4 Following Theorem 2, we again distinguish two cases.

Case 1: $\Pi > 4k - \frac{3p_e}{2\lambda}$. Then, $C_i^* = \mathcal{C}_i^A$. As $\Pi > 4k - \frac{3p_e}{2\lambda}$, we use the following disjoint intervals in the proof: (a) $k \in \left(0, \frac{3p_e}{8\lambda}\right)$, $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$; (b) $k \in \left[\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda}\right)$, $\Pi \in \left(4k - \frac{3p_e}{2\lambda}, \frac{p_e}{\lambda}\right]$. Using Propositions 5 and 7, define

$$\Delta_1(\Omega) := v^* - V^*(\Pi) = k(b-a)\Omega(\Pi) - \frac{2p_e(b-a)\Omega(\Pi)^3}{3\lambda} - \frac{k^2\lambda(b-a)}{2p_e},$$

where $\Omega(\Pi)$ is as defined in expression (2.17) of Proposition 5. Observe that

$$\frac{\partial^2 \Delta_1}{\partial\,\Omega^2} = \frac{\partial^2}{\partial\,\Omega^2}\left[k(b-a)\Omega - \frac{2p_e(b-a)\Omega^3}{3\lambda} - \frac{k^2\lambda(b-a)}{2p_e}\right] = -\frac{4(b-a)p_e}{\lambda}\Omega < 0.$$

Thus, $\Delta_1(\Omega)$ is concave in Ω. It also follows from Theorem 2 that $\mathcal{C}_i^A = b_i - \Omega w$. Hence,

$$\frac{\partial \mathcal{C}_i^A}{\partial \Pi} = \frac{\partial}{\partial \Pi}\left(b_i - \Omega w\right) = -w\frac{\partial \Omega}{\partial \pi} > 0.$$

By Lemma 7, $C_i^*(\Pi, p_e)$ is increasing Π. Therefore, $\frac{\partial \mathcal{C}_i^A}{\partial \Pi} > 0$. It follows that $\frac{\partial \Omega}{\partial \Pi} < 0$. Next, since $\Delta_1(\Omega)$ is concave in Ω, and $\frac{\partial\,\Omega}{\partial\,\Pi} < 0$ on $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$, then for any $k \in (0, \frac{3p_e}{8\lambda})$ and $\Pi \in \left[0, \frac{p_e}{\lambda}\right]$,

$$\Delta_1\big(\Omega(\Pi)\big) \geq \min\left[\Delta_1\big(\Omega(0)\big), \Delta_1\big(\Omega\left(\frac{p_e}{\lambda}\right)\big)\right]. \tag{A.53}$$

Similarly, for any $k \in [\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda})$ and $\Pi \in [4k - \frac{3p_e}{2\lambda}, \frac{p_e}{\lambda}]$,

$$\Delta_1\big(\Omega(\Pi)\big) \geq \min\left[\Delta_1\left(\Omega\left(4k\lambda - \frac{3}{2}p_e\right)\right), \Delta_1\big(\Omega\left(\frac{p_e}{\lambda}\right)\big)\right]. \tag{A.54}$$

We now show that $\Delta_1\big(\Omega(0)\big) > 0$ for $k \in (0, \frac{3p_e}{8\lambda})$. We have

$$\Delta_1\big(\Omega(0)\big) = v^*(0) - V^*(0) = -\frac{k^2\lambda(b-a)}{2p_e} + k(b-a)\sqrt{\frac{2k\lambda}{3p_e}} - \frac{2p_e(b-a)}{3\lambda}\left(\sqrt{\frac{2k\lambda}{3p_e}}\right)^3 \tag{A.55}$$

$$= (b-a)k\sqrt{\frac{k\lambda}{p_e}}\left[\frac{5}{9}\sqrt{\frac{2}{3}} - \sqrt{\frac{k\lambda}{4p_e}}\right]. \tag{A.56}$$

Since $k\lambda < \frac{3}{8}p_e$, then $\frac{5}{9}\sqrt{\frac{2}{3}} - \sqrt{\frac{k\lambda}{4p_e}} > \frac{5}{9}\sqrt{\frac{2}{3}} - \sqrt{\frac{3}{32}} > 0$. Therefore, we obtain $\Omega(0) > 0$. Next, we show that $\Delta_1\big(\Omega\left(\frac{p_e}{\lambda}\right)\big) > 0$ for $k \in (0, \frac{5p_e}{8\lambda})$. We get

$$\Delta_1\big(\Omega\left(\frac{p_e}{\lambda}\right)\big) = v^* - V^*\left(\frac{p_e}{\lambda}\right) = k(b-a)\left[\sqrt{1+\frac{2k\lambda}{p_e}} - 1 - \frac{2p_e}{3\lambda}(\sqrt{1+\frac{2k\lambda}{p_e}} - 1)^3 - \frac{k\lambda}{2p_e}\right]. \tag{A.57}$$

Define $\theta := \dfrac{k\lambda}{p_e}$, so that $\theta \in (0, \frac{5}{8})$. Also define

$$G(\theta) := \sqrt{1+2\theta} - 1 - \frac{2}{3\theta}(\sqrt{1+2\theta}-1)^3 - \frac{\theta}{2}. \tag{A.58}$$

To show that $\Delta_1\big(\Omega\left(\frac{p_e}{\lambda}\right)\big) > 0$, it suffices to show that $G(\theta) > 0$ for any $\theta \in (0, \frac{5}{8})$. Observe that

$$\begin{aligned}
\frac{\partial^2 G(\theta)}{\partial \theta^2} &= \frac{\partial^2}{\partial \theta^2}\left[\sqrt{1+2\theta} - 1 - \frac{\theta}{2} - \frac{2\theta}{3}(\sqrt{1+2\theta}-1)^3\right] \\
&= -\frac{1}{(2\theta+1)^{3/2}} + \frac{2\left(\sqrt{2\theta+1}-1\right)^2}{\theta(2\theta+1)^{3/2}} - \frac{4\left(\sqrt{2\theta+1}-1\right)}{\theta(2\theta+1)} \\
&\quad - \frac{4\left(\sqrt{2\theta+1}-1\right)^3}{3\theta^3} + \frac{4\left(\sqrt{2\theta+1}-1\right)^2}{\theta^2\sqrt{2\theta+1}} \\
&= \frac{2(\sqrt{2\theta+1}-1)^2 - \theta}{\theta(2\theta+1)^{\frac{3}{2}}} \\
&\quad - \frac{4(\sqrt{2\theta+1}-1)\left[(\sqrt{2\theta+1}-1)^2(2\theta+1) + 3\theta^2 - 3\theta(\sqrt{2\theta+1}-1)\sqrt{2\theta+1}\right]}{3\theta^3(2\theta+1)} \\
&= \frac{3\theta+4-4\sqrt{2\theta+1}}{\theta(2\theta+1)^{\frac{3}{2}}} - \frac{4(\sqrt{2\theta+1}-1)\left[(4\theta+2)(\theta+1-\sqrt{2\theta+1}) - 3\theta(\theta+1-\sqrt{2\theta+1})\right]}{3\theta^3(2\theta+1)} \\
&= \frac{3\theta+4-4\sqrt{2\theta+1}}{\theta(2\theta+1)^{\frac{3}{2}}} - \frac{4(\sqrt{2\theta+1}-1)(\theta+2)(\theta+1-\sqrt{2\theta+1})}{3\theta^3(2\theta+1)}
\end{aligned} \tag{A.59}$$

We now show that $\frac{\partial^2 G(\theta)}{\partial \theta^2} < 0$. Based on (A.59), it suffices to show that $3\theta+4-4\sqrt{2\theta+1} < 0$ and $-(\theta+1-\sqrt{2\theta+1}) < 0$ for $\theta \in (0, \frac{5}{8})$. First, $3\theta+4-4\sqrt{2\theta+1} < 0$ since $3\theta+4+4\sqrt{2\theta+1} > 0$ and

$$(3\theta+4-4\sqrt{2\theta+1})(3\theta+4+4\sqrt{2\theta+1}) = 9\theta^2 - 8\theta < 0,$$

for $\theta < \frac{5}{8}$. Next, $-(\theta+1-\sqrt{2\theta+1}) < 0$, since $\sqrt{2\theta+1}+\theta+1 > 0$, and

$$-(\theta+1-\sqrt{2\theta+1})(\sqrt{2\theta+1}+\theta+1) = -\theta^2 < 0.$$

Thus, $\frac{\partial^2 G(\theta)}{\partial \theta^2} < 0$ for $\theta \in (0, \frac{3}{8})$. Since $G(\theta)$ is concave in θ, then, for any $\theta \in (0, \frac{3}{8})$,

$$G(\theta) > \min\left[\lim_{\theta \to 5/8} G(\theta), \lim_{\theta \to 0} G(\theta)\right]. \tag{A.60}$$

It follows from (A.58) that

$$\lim_{\theta\to 0^+}\left[\sqrt{1+2\theta}-1-\frac{\theta}{2}-\frac{2}{3\theta}\left(\sqrt{1+2\theta}-1\right)^3\right]=$$

$$\lim_{\theta\to 0^+}\left[\sqrt{1+2\theta}-1-\frac{\theta}{2}-\frac{2}{3}\left(\sqrt{\frac{1}{\theta}+2}-\sqrt{\frac{1}{\theta}}\right)^2\left(\sqrt{1+2\theta}-1\right)\right]=0. \tag{A.61}$$

Further,

$$\lim_{\theta\to \frac{5}{8}^+}\sqrt{1+2\theta}-1-\frac{\theta}{2}-\frac{2}{3\theta}\left(\sqrt{1+2\theta}-1\right)^3=\frac{13}{240}>0. \tag{A.62}$$

Thus, (A.61) and (A.62) imply that $G(\theta)>0$. Consequently, we conclude that $\Delta_1\big(\Omega\left(\frac{p_e}{\lambda}\right)\big)>0$.

Finally, we show that $\Delta_1\big(\Omega\left(4k-\frac{3p_e}{2\lambda}\right)\big)>0$ for $k\in[\frac{3p_e}{8\lambda},\frac{5p_e}{8\lambda})$. We obtain

$$\Delta_1\left(\Omega\left(4k-\frac{3p_e}{2\lambda}\right)\right)=v^*-V^*\left(\Omega\left(4k-\frac{3p_e}{2\lambda}\right)\right)=\frac{(b-a)\left(6kp_e\lambda-6k^2\lambda^2-p_e^2\right)}{12p_e\lambda}.$$

To show that $\Delta_1\big(\Omega\left(4k-\frac{3p_e}{2\lambda}\right)\big)>0$, it suffices to show that $6kp_e\lambda-6k^2\lambda^2-p_e^2>0$ for $k\in[\frac{3p_e}{8\lambda},\frac{5p_e}{8\lambda})$. Let $\Lambda:=k\lambda$ and define $\varrho(\Lambda):=6kp_e\lambda-6k^2\lambda^2-p_e^2=6\Lambda p_e-6\Lambda^2-p_e^2$. Then, since $\frac{\partial^2\varrho}{\partial\Lambda^2}<0$, we get

$$\varrho(\Lambda)>\min\left[\varrho\left(\frac{3}{8}p_e\right),\lim_{\Lambda\to\frac{5}{8}p_e}\varrho(\Lambda)\right].$$

Observe that $\varrho\left(\frac{3}{8}p_e\right)=\varrho\left(\frac{5}{8}p_e\right)=\frac{13}{32}p_e^2>0$. Thus, $\varrho(\Lambda)>0$. It follows that $\Delta_1\left(\Omega\left(4k-\frac{3p_e}{2\lambda}\right)\right)>0$ for $k\in[\frac{3p_e}{8\lambda},\frac{5p_e}{8\lambda})$. Consequently, we can conclude from (A.53) and (A.54) that $\Delta_1\big(\Omega(\Pi)\big)>0$.

Case 2: $\Pi\leq 4k-\frac{3p_e}{2\lambda}$. Then, $C_i^*=\mathcal{C}_i^B$. Since $\Pi\leq 4k-\frac{3p_e}{2\lambda}$ we make use of the following disjoint intervals in the proof: (a) $k\in[\frac{3p_e}{8\lambda},\frac{5p_e}{8\lambda}),\ \Pi\in[0,4k-\frac{3p_e}{2\lambda}]$; (b) $k\in[\frac{5p_e}{8\lambda},\frac{p_e}{\lambda}),\ \Pi\in\left[0,\frac{p_e}{\lambda}\right]$. Using Theorem 2 and Proposition 5, we can write

$$V^*(\Pi)=k\mathcal{C}^B(\Pi)-\frac{p_e\left[\big(a+b-2\mathcal{C}^B(\Pi)\big)^3-4\big(b-\mathcal{C}^B(\Pi)\big)^3\right]}{6\lambda(b-a)^2}. \tag{A.63}$$

Using expression (A.63), this time we define $\Delta_2\big(\mathcal{C}^B\big)$ as

$$\Delta_2\big(\mathcal{C}^B\big) := v^* - V^*(\Pi) = bk - \frac{k^2\lambda(b-a)}{2p_e} + \frac{p_e\left[\big(a+b-2\mathcal{C}^B(\Pi)\big)^3 - 4\big(b-\mathcal{C}^B(\Pi)\big)^3\right]}{6\lambda(b-a)^2} - k\mathcal{C}^B(\Pi).$$

Observe that

$$\frac{\partial^2\Delta_2}{\partial(\mathcal{C}^B)^2} = \frac{4p_e(a-\mathcal{C}^B)}{\lambda(b-a)^2} < 0.$$

Thus, $\Delta_2(\mathcal{C}^B)$ is concave in $\mathcal{C}^B$. By Lemma 7, $\frac{\partial\mathcal{C}^B}{\partial\Pi} > 0$. Hence, for any $k \in [\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda})$ and $\Pi \in \big[0, \frac{p_e}{\lambda}\big]$,

$$\Delta_2\big(\mathcal{C}^B(\Pi)\big) \geq \min\left[\Delta_2\big(\mathcal{C}^B(0)\big), \Delta_2\Big(\mathcal{C}^B\left(\frac{p_e}{\lambda}\right)\Big)\right]. \tag{A.64}$$

Similarly, for any $k \in [\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda})$ and $\Pi \in \big[0, 4k - \frac{3p_e}{2\lambda}\big]$,

$$\Delta_2\big(\mathcal{C}^B(\Pi)\big) \geq \min\left[\Delta_2\big(\mathcal{C}^B(0)\big),\ \Delta_2\left(\mathcal{C}^B\left(4k - \frac{3p_e}{2\lambda}\right)\right)\right]. \tag{A.65}$$

We first show that $\Delta_2\big(\mathcal{C}^B\left(\frac{p_e}{\lambda}\right)\big) > 0$ for $k \in [\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda})$. We get that

$$\Delta_2\Big(\mathcal{C}^B\left(\frac{p_e}{\lambda}\right)\Big) = v^* - V^*(p_e) = \frac{(b-a)(p_e-k\lambda)\left(10\sqrt{6}\sqrt{p_e(p_e-k\lambda)} + 27k\lambda - 27p_e\right)}{54p_e\lambda}.$$

To show that $\Delta_2\big(\mathcal{C}^B\left(\frac{p_e}{\lambda}\right)\big) > 0$, it suffices to show that $10\sqrt{6}\sqrt{p_e(p_e-k\lambda)} + 27k\lambda - 27p_e > 0$ for $k \in [\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda})$. Define $g(\Lambda) := 10\sqrt{6}\sqrt{p_e(p_e-\Lambda)} + 27\Lambda - 27p_e$, where, as in Case 1, $\Lambda = k\lambda$.

Since $\frac{\partial^2 g}{\partial\Lambda^2} = -\sqrt{\frac{75p_e}{2(p_e-\Lambda)^3}} < 0$, then $g(\Lambda)$ is concave in Λ. It follows that, for any $\Lambda \in [\frac{5}{8}p_e, p_e)$,

$$g(\Lambda) > \min\left[g\left(\frac{5}{8}p_e\right), \lim_{\Lambda\to p_e} g(\Lambda)\right].$$

Observe that $g(\frac{5}{8}p_e) = \frac{39}{8}p_e > 0$. Also, $g(p_e) = 0$. It follows that $\Delta_2\big(\mathcal{C}^B\left(\frac{p_e}{\lambda}\right)\big) > 0$ for $k \in [\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda})$.

Next, we show that $\Delta_2\big(\mathcal{C}^B(0)\big) > 0$ for $k \in [\frac{3p_e}{8\lambda}, \frac{p_e}{\lambda})$. We get

$$\Delta_2\big(\mathcal{C}^B(0)\big) = v^* - V^*(0)$$
$$= \frac{(b-a)\left[31p_e^2 - k\lambda\left(3k\lambda - 2\sqrt{p_e(3p_e-2k\lambda)}\right) - 6p_e\left(3\sqrt{p_e(3p_e-2k\lambda)} + 2k\lambda\right)\right]}{6p_e\lambda}.$$

To show $\left[31p_e^2 - k\lambda\left(3k\lambda - 2\sqrt{p_e(3p_e-2k\lambda)}\right) - 6p_e\left(3\sqrt{p_e(3p_e-2k\lambda)} + 2k\lambda\right)\right] > 0$ for $k \in [\frac{3p_e}{8\lambda}, \frac{p_e}{\lambda})$, define

$$G(\Lambda) := 31p_e^2 - \Lambda\left(3\Lambda - 2\sqrt{p_e(3p_e-2\Lambda)}\right) - 6p_e\left(3\sqrt{p_e(3p_e-2\Lambda)} + 2\Lambda\right)$$
$$= 31p_e^2 - 3\Lambda^2 - 12p_e\Lambda + (2\Lambda - 18p_e)\sqrt{p_e(3p_e-2\Lambda)}. \tag{A.66}$$

Since $p_e > \Lambda$ then $31p_e^2 - 3\Lambda^2 - 12p_e\Lambda > 0$ and $18p_e - 2\Lambda > 0$. Thus, we obtain $31p_e^2 - 3\Lambda^2 - 12p_e\Lambda + (18p_e - 2\Lambda)\sqrt{p_e(3p_e-2\Lambda)} > 0$. Hence, to show that $G(\Lambda) > 0$, it suffices to show that

$$G(\Lambda)\left[31p_e^2 - 3\Lambda^2 - 12p_e\Lambda + (18p_e - 2\Lambda)\sqrt{p_e(3p_e-2\Lambda)}\right] = (p_e - \Lambda)^2\left(9\Lambda^2 + 98p_e\Lambda - 11p_e^2\right) > 0.$$

Define $\varrho(\Lambda) \coloneqq 9\Lambda^2 + 98p_e\Lambda - 11p_e^2$. Since $\frac{\partial\varrho}{\partial\Lambda} = 18\Lambda + 98p_e > 0$ then, for any $\Lambda \in [\frac{3}{8}p_e, p_e)$, $\varrho(\Lambda) \geq \varrho(\frac{3}{8}p_e) = \frac{43225}{4096}p_e^4 > 0$. Thus, $G(\Lambda) > 0$, and we conclude that $\Delta_2\big(\mathcal{C}^B(0)\big) > 0$ for $k \in [\frac{3p_e}{8\lambda}, \frac{p_e}{\lambda})$.

Finally, we show that $\Delta_2\big(\mathcal{C}^B\left(4k - \frac{3p_e}{2\lambda}\right)\big) > 0$ for $k \in [\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda})$. Note that $\Delta_2\big(\mathcal{C}^B\left(4k - \frac{3p_e}{2\lambda}\right)\big) = \Delta_1\big(\Omega\left(4k - \frac{3p_e}{2\lambda}\right)\big)$, since $\mathcal{C}^A\left(4k - \frac{3p_e}{2\lambda}\right) = \mathcal{C}^B\left(4k - \frac{3p_e}{2\lambda}\right) = \frac{b+a}{2}$. From our analysis of Case 1, we get $\Delta_2\big(\mathcal{C}^B\left(4k - \frac{3p_e}{2\lambda}\right)\big) > 0$ for $k \in [\frac{3p_e}{8\lambda}, \frac{5p_e}{8\lambda})$. Thus, based on (A.64) and (A.65), we get $\Delta_2\big(\mathcal{C}^B(\Pi)\big) > 0$. □

Proof of Theorem 5. Since consumers are identical, in what follows we drop the subscript i when denoting each consumer's optimal capacity in a virtual microgrid.

Case 1: $\Pi > 4k - \frac{3p_e}{2\lambda}$. By Theorem 2, $C^*(\Pi) = \mathcal{C}^A(\Pi)$, where $\mathcal{C}^A(\Pi)$ is as established in (2.15). We use Theorem 2 and Lemma 7 to derive $\overline{\pi}$ such that $\mathcal{C}^A(\overline{\Pi}) = c^*$. From those we obtain

$$b - \frac{\sqrt{6\lambda k p_e - 4\lambda^2 k\overline{\Pi} + \lambda^2 \overline{\pi}^2} - \lambda\overline{\Pi}}{3p_e - 2\lambda\Pi}(b-a) = b - \frac{\lambda k}{p_e}(b-a).$$

It follows from the above that $\overline{\Pi}$ is uniquely given by $\overline{\Pi} = \frac{p_e(2p_e - 3k\lambda)}{2\lambda(p_e - k\lambda)} = \frac{p_e\left(p_e - \frac{3}{2}k\lambda\right)}{2\lambda(p_e - k\lambda)}$ when $\Pi > 4k - \frac{3}{2}\frac{p_e}{\lambda}$. Since $\Pi > 4k - \frac{3p_e}{2\lambda}$, we consider separately $k \in \left(0, \frac{3p_e}{8\lambda}\right)$ and $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$.

Assume $k \in \left(0, \frac{3p_e}{8\lambda}\right)$. We show that $\lambda\overline{\Pi} \in (0, p_e)$. Since $p_e > \frac{8}{3}k\lambda$ then $\overline{\Pi} = \frac{p_e(2p_e - 3k\lambda)}{2\lambda(p_e - k\lambda)} > 0$. Also,

$$p_e - \lambda\overline{\Pi} = p_e - \frac{p_e(2p_e - 3k\lambda)}{2(p_e - k\lambda)} = \frac{k\lambda p_e}{2(p_e - k\lambda)} > 0,$$

since $p_e > k\lambda$. Therefore, $\lambda\overline{\Pi} \in (0, p_e)$ when $k < \frac{3p_e}{8\lambda}$. Finally, since $k < \frac{3p_e}{8\lambda}$, then $4\lambda k - \frac{3}{2}p_e < 0$. Since $\overline{\Pi} > 0$, then $\lambda\overline{\Pi} > 4\lambda k - \frac{3}{2}p_e$ is indeed satisfied for $k < \frac{p_e}{2\lambda}$.

Assume $k \in \left[\frac{3p_e}{8\lambda}, \frac{p_e}{2\lambda}\right)$. It is straightforward that $\lambda\overline{\Pi} \in (0, p_e)$. Further,

$$\lambda\overline{\Pi} - 4k\lambda + \frac{3}{2}p_e = \frac{(5p_e - 4k\lambda)(p_e - 2k\lambda)}{2(p_e - k\lambda)} > 0,$$

since $k < \frac{p_e}{2\lambda}$. Since $C_i^*(\Pi)$ is increasing in Π by Proposition 3, it follows that, when $k < \frac{p_e}{2\lambda}$, $c^* > C^*(\Pi)$ for $\Pi < \frac{p_e(2p_e - 3\lambda k)}{2\lambda(p_e - \lambda k)}$, and $c^* \leq C^*(\Pi)$ for $\Pi \geq \frac{p_e(2p_e - 3\lambda k)}{2\lambda(p_e - \lambda k)}$

Case 2: $\Pi \leq 4k - \frac{3p_e}{2\lambda}$. By Theorem 2, $C^*(\Pi) = \mathcal{C}^B(\Pi)$, where $\mathcal{C}^B(\Pi)$ is as established in (2.16). We use Theorem 2 and Lemma 7 to derive $\overline{\Pi}$ such that $\mathcal{C}^B(\overline{\Pi}) = c^*$. We obtain

$$a + \frac{\sqrt{3p_e^2 - 4\lambda^2 k\overline{\Pi} + \lambda^2\overline{\Pi}^2 + 2p_e(\lambda\overline{\Pi} - k\lambda)} + \lambda\overline{\Pi} - p_e}{p_e + 2\lambda\overline{\Pi}}(b-a) = b - \frac{\lambda k}{p_e}(b-a).$$

It follows that $\overline{\Pi}$ is uniquely given by $\overline{\Pi} = \frac{p_e(p_e - k\lambda)}{2k\lambda^2}$. Further, $\overline{\Pi} > 0$. Since $\Pi \leq 4k - \frac{3p_e}{2\lambda}$, we separately consider $k \in \left[\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda}\right)$ and $k \in \left[\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda}\right)$. Observe that

$$p_e - \lambda\overline{\Pi} = p_e - \frac{p_e(p_e - k\lambda)}{2k\lambda} = -\frac{p_e(p_e - 3k\lambda)}{2K\lambda} > 0,$$

for either $k \in \left[\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda}\right)$ or $k \in \left[\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda}\right)$. Thus, we conclude that $\lambda\overline{\Pi} \in (0, p_e)$.

We now verify that $\overline{\Pi}$ indeed satisfies $\overline{\Pi} \leq 4k - \frac{3p_e}{2\lambda}$ for $k \geq \frac{p_e}{2\lambda}$. We get $4k\lambda - \frac{3}{2}p_e - \lambda\overline{\Pi} =$

$\frac{9k^2\lambda^2-(p_e+k\lambda)^2}{2k\lambda}$.

Observe that if $p_e \leq 2k\lambda$, then $4k\lambda - \frac{3}{2}p_e \geq \lambda\overline{\Pi}$. Therefore, this condition is satisfied for both $k \in \left[\frac{5p_e}{8\lambda}, \frac{p_e}{\lambda}\right)$ and $k \in \left[\frac{p_e}{2\lambda}, \frac{5p_e}{8\lambda}\right)$. Finally, because $C_i^*(\Pi)$ is increasing in Π by Proposition 3, it follows that, when $k \geq \frac{p_e}{2\lambda}$, $c^* > C^*(\Pi)$ for $\Pi < \frac{p_e(p_e-\lambda k)}{2\lambda^2 k}$, and $c^* \leq C^*(\Pi)$ for $\Pi \geq \frac{p_e(p_e-\lambda k)}{2\lambda^2 k}$.

To complete the proof, we make use of Theorem 3 and Lemma 7 to get

$$C^*(\Pi^*) - c^* = \begin{cases} (b-a)\sqrt{\dfrac{\lambda k}{p_e}}\left(\sqrt{\dfrac{\lambda k}{p_e}} - \sqrt{\dfrac{1}{2}}\right) & \text{if } k < \frac{p_e}{2\lambda}; \\ (b-a)\sqrt{1-\dfrac{\lambda k}{p_e}}\left(\sqrt{\dfrac{1}{2}} - \sqrt{1-\dfrac{\lambda k}{p_e}}\right) & \text{if } k \geq \frac{p_e}{2\lambda}. \end{cases}$$

It follows that $C^*(\Pi^*) - c^* < 0$ if $k < \frac{p_e}{2\lambda}$ and $C^*(\Pi^*) - c^* \geq 0$ if $k \geq \frac{p_e}{2\lambda}$. □

Proof of Proposition 8. We first prove that, in a centralized microgrid, optimized for total cost, $\mathcal{C}^* \in [a_1 + a_2, b_1 + b_2]$. For that purpose, we make use of (2.22) to get

$$\mathcal{F}(\mathcal{C}) = \frac{1}{\lambda}\,\mathbb{E}_{X_1,X_2}\left[p_e(x_1 + x_2 - \mathcal{C})^+\right] + k\mathcal{C}. \tag{A.67}$$

Let $f_{X_i}(x_i)$ denote the probability density of X_i. Thus,

$$f_{X_1}(x_1) = \begin{cases} \dfrac{1}{w} & \text{if } x \in [a_1, b_1] \\ 0 & \text{otherwise;} \end{cases} \qquad f_{X_2}(x_2) = \begin{cases} \dfrac{1}{w} & \text{if } x \in [a_2, b_2] \\ 0 & \text{otherwise.} \end{cases}$$

Define $Z := X_1 + X_2$. Let z be a realization of Z, and f_Z be the probability density of Z. Then,

$$\mathcal{F}(\mathcal{C}) = \frac{1}{\lambda}\,\mathbb{E}_Z\left[p_e(z - \mathcal{C})^+\right] + k\mathcal{C}$$
$$f_Z(z) = \int f_{X_1}(z - x_2) f_{X_2}(x_2)\, dx_2.$$

Since $f_{X_2}(x_2) = \frac{1}{w}$ on $[a_2, b_2]$ and is zero otherwise, we get,

$$f_Z(z) = \int_{a_2}^{b_2} \frac{1}{w} f_{X_1}(z - x_2)\, dx_2. \tag{A.68}$$

Note that $f_{X_1}(z-x_2)=0$ unless $a_1 \leq z - x_2 \leq b_1$ (in other words, $z - b_1 \leq x_2 \leq z - a_1$). Also, $z \in [a_1+a_2, b_1+b_2]$ (i.e., $z \in [a_1+a_2, a_1+a_2+2w]$). Thus, if $a_1+a_2 \leq z \leq a_1+a_2+w$, then, by (A.68),

$$f_Z(z) = \int_{a_2}^{z-a_1} \frac{1}{w^2} dx_2 = \frac{z-(a_1+a_2)}{w^2}. \tag{A.69}$$

If $a_1 + a_2 + w \leq z \leq a_1 + a_2 + w$, then we use (A.68) to get

$$f_Z(z) = \int_{z-b_1}^{b_2} \frac{1}{w^2} dx_2 = \int_{z-(a_1+w)}^{a_2+w} \frac{1}{w^2} dx_2 = \frac{a_1+a_2+2w-z}{w^2}. \tag{A.70}$$

Thus, we obtain

$$f_Z(z) = \begin{cases} \dfrac{z-(a_1+a_2)}{w^2}, & \text{if } z \in [a_1+a_2,\ a_1+a_2+w]; \\ \dfrac{a_1+a_2+2w-z}{w^2}, & \text{if } z \in [a_1+a_2+w,\ a_1+a_2+2w]. \end{cases} \tag{A.71}$$

We now show that $\mathcal{C}^* \geq a_1+a_2$. Assume that $\mathcal{C}^* < a_1+a_2$. Then, we use (A.71) to get

$$\begin{aligned}
\mathcal{F}(\mathcal{C}^*) &= \frac{1}{\lambda} \int_{a_1+a_2}^{a_1+a_2+2w} p_e(z-\mathcal{C}^*) f_Z(z) dz + k\mathcal{C}^* \\
&= \frac{1}{\lambda} \left[\int_{a_1+a_2}^{a_1+a_2+w} p_e(z - C_0^*) \left(\frac{z-(a_1+a_2)}{w^2} \right) dz + \int_{a_1+a_2+w}^{a_1+a_2+2w} p_e(z-\mathcal{C}^*) \left(\frac{a_1+a_2+2w-z}{w^2} \right) dz \right] + k\mathcal{C}^* \\
&= \left(k - \frac{p_e}{\lambda} \right) \mathcal{C}^* + \frac{p_e(a_1+a_2+w)}{\lambda} \\
\mathcal{F}(a_1+a_2) &= \frac{1}{\lambda} \int_{a_1+a_2}^{a_1+a_2+2w} p_e\big[z-(a_1+a_2)\big] f_Z(z) dz + k(a_1+a_2) \\
&= \left(k - \frac{p_e}{\lambda} \right) (a_1+a_2) + \frac{p_e(a_1+a_2+w)}{\lambda}.
\end{aligned}$$

It follows that

$$\mathcal{F}(\mathcal{C}^*) - \mathcal{F}(a_1+a_2) = \left(k - \frac{p_e}{\lambda} \right) \big[\mathcal{C}^* - (a_1+a_2)\big].$$

Since $\frac{p_e}{\lambda} > k$ and $\mathcal{C}^* < a_1+a_2$ we get $\mathcal{F}(\mathcal{C}^*) > \mathcal{F}(a_1+a_2)$ This contradicts the assumption that $\mathcal{C}^*$ is the optimal capacity defined by $\mathcal{C}^* = \arg\min_{\mathcal{C}} \mathcal{F}(\mathcal{C})$. Thus, we conclude that $\mathcal{C}^* \geq a_1 + a_2$.

Next, we show that $\mathcal{C}^* \leq b_1 + b_2$. Assume $\mathcal{C}^* > b_1 + b_2$. Then, $p_e(x_1 + x_2 - \mathcal{C}^*)^+ = 0$. Hence,

$\mathcal{F}(\mathcal{C}^*) = k\mathcal{C}^*$. Similarly, $\mathcal{F}(b_1 + b_2) = k(b_1 + b_2)$.

Thus, $\mathcal{F}(\mathcal{C}^*) - \mathcal{F}(b_1 + b_2) = k(\mathcal{C}^* - (b_1 + b_2)) > 0$. Since $\mathcal{F}(b_1 + b_2) < \mathcal{F}(\mathcal{C}^*)$ contradicts the assumption that $\mathcal{C}^*$ is the optimal capacity defined by $\mathcal{C}^* = \arg\min_{\mathcal{C}} \mathcal{F}(\mathcal{C})$. Thus, we conclude $\mathcal{C}^* \leq b_1 + b_2$.

Because, as just established, $\mathcal{C}^* \in [a_1 + a_2, b_1 + b_2]$, we now get

$$\mathcal{F}(\mathcal{C}) = \frac{1}{\lambda}\left[\int_{a_1+a_2}^{b_1+b_2} p_e(z - \mathcal{C})^+ f_Z(z)dz\right] + k\mathcal{C}. \tag{A.72}$$

Since

$$f_Z(z) = \begin{cases} \dfrac{z - (a_1 + a_2)}{w^2} & \text{if } z \in [a_1 + a_2,\ a_1 + a_2 + w]; \\ \\ \dfrac{a_1 + a_2 + 2w - z}{w^2} & \text{if } z \in [a_1 + a_2 + w,\ a_1 + a_2 + 2w], \end{cases}$$

we consider separately $\mathcal{C} \leq a_1 + a_2 + w$ and $\mathcal{C} > a_1 + a_2 + w$ when deriving $\mathcal{C}^*$.

Case 1: $\mathcal{C} > a_1 + a_2 + w$. Using (A.72) we obtain

$$\mathcal{F}(\mathcal{C}) = \frac{p_e}{\lambda}\int_{\mathcal{C}}^{a_1+a_2+2w} (z - \mathcal{C}) \cdot \frac{a_1 + a_2 + 2w - z}{w^2} dz + k\mathcal{C} = \frac{p_e(a_1 + a_2 + 2w - \mathcal{C})^3}{6\lambda} + k\mathcal{C}. \tag{A.73}$$

We then make use of (A.73) to obtain

$$\frac{\partial \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}} = -\frac{p_e(a_1 + a_2 + 2w - \mathcal{C})^2}{2\lambda w^2} + k; \quad \frac{\partial^2 \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}^2} = \frac{p_e(a_1 + a_2 + 2w - \mathcal{C})}{\lambda w^2}. \tag{A.74}$$

Since $\mathcal{C} \leq a_1 + a_2 + 2w$, we obtain $\frac{\partial^2 \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}^2} > 0$ from (A.74). Setting $\frac{\partial \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}} = 0$, we get

$$\mathcal{C}^* = a_1 + a_2 + 2w - w\sqrt{\frac{2\lambda k}{p_e}}. \tag{A.75}$$

We now check $\mathcal{C} > a_1 + a_2 + w$. Using (A.75), we can rewrite $\mathcal{C}^* > a_1 + a_2 + w$ as $w\sqrt{\frac{2\lambda k}{p_e}} < w$. Therefore, $\mathcal{C}^* > a_1 + a_2 + w$ if and only if $k < \frac{p_e}{2\lambda}$. Then, by (A.73) and (A.75),

$$\mathcal{F}(\mathcal{C}^*) = k(a_1 + a_2 + 2w) - w\frac{2\sqrt{2\lambda}k^{3/2}}{3\sqrt{p_e}} = k\left(b_1 + b_2 - \frac{2w}{3}\sqrt{\frac{2\lambda k}{p_e}}\right).$$

Case 2: $\mathcal{C} \leq a_1 + a_2 + w$. Using (A.72) we obtain

$$\begin{aligned}\mathcal{F}(\mathcal{C}) &= \frac{p_e}{\lambda w^2}\left[\int_{\mathcal{C}}^{a_1+a_2+w}(z-\mathcal{C})(z-a_1-a_2)dz + \int_{a_1+a_2+w}^{a_1+a_2+2w}(z-\mathcal{C})(a_1+a_2+2w-z)dz\right] + k\mathcal{C} \\ &= \frac{p_e}{6\lambda}\left[3a_1 + 3a_2 - 3\mathcal{C} + 4w - \frac{(a_1+a_2-\mathcal{C}-2w)(a_1+a_2+w-\mathcal{C})^2}{w^2}\right] + k\mathcal{C}. \qquad \text{(A.76)}\end{aligned}$$

From (A.76), we then obtain

$$\frac{\partial \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}} = \frac{p_e\left(a_1^2 + a_2^2 + 2a_1(a_2 - \mathcal{C}) - 2a_2\mathcal{C} - \mathcal{C}^2 - 2w^2\right)}{2\lambda w^2} + k \quad \text{and} \quad \frac{\partial^2 \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}^2} = \frac{p_e(\mathcal{C} - a_1 - a_2)}{\lambda w^2}.$$

Since $\mathcal{C} \geq a_1 + a_2$, we obtain $\frac{\partial^2 \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}^2} > 0$. Therefore, setting $\frac{\partial \mathcal{F}(\mathcal{C})}{\partial \mathcal{C}} = 0$ yields

$$\mathcal{C}^* = a_1 + a_2 + w\sqrt{2\left(1 - \frac{\lambda k}{p_e}\right)}. \qquad \text{(A.77)}$$

We now check that $\mathcal{C} \leq a_1 + a_2 + w$. Using (A.77), we rewrite $\mathcal{C}^* \leq a_1 + a_2 + w$ as

$$w\sqrt{2\left(1 - \frac{\lambda k}{p_e}\right)} \leq w. \qquad \text{(A.78)}$$

It follows from (A.78) that $\mathcal{C}^* \leq a_1 + a_2 + w$ if and only if $k \geq \frac{p_e}{2\lambda}$. Then, by (A.76) and (A.77),

$$\begin{aligned}\mathcal{F}(\mathcal{C}^*) &= k(a_1 + a_2) + \frac{w\left(3p_e^2 - 2\sqrt{2p_e(p_e - \lambda k)}(p_e - \lambda k)\right)}{3\lambda p_e} \\ &= k(a_1 + a_2) + \frac{w\left(3p_e - 2(p_e - \lambda k)\sqrt{2 - \frac{2\lambda k}{p_e}}\right)}{3\lambda}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 6. The result follows directly from Proposition 8 and Theorem 3. □

APPENDIX B

SUPPLEMENT TO CHAPTER 3

Proof of Lemma 1: Suppose $\overline{G}(T)=b_1+T+\epsilon$, where $\epsilon>0$. As $x_1\leq b_1$ by assumption, we get $\big(\overline{G}(T)-x_1\big)^+=\overline{G}(T)-x_1>T$. It then follows from (3.1) and (3.2) that $\mathcal{R}_G(G,x_1,x_2,T)=px_1+(p-s)\min[x_2,T]$. Hence,

$$\Pi_G\big(\overline{G}(T),T\big)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{X_1,X_2}\Big[px_1+(p-s)\min\big[x_2,T\big]\Big]dt-k_G\overline{G}(T).$$

Let $\widetilde{G}=b_1+T$. Using (3.1) and (3.2) again, we get

$$\Pi_G(\widetilde{G},T)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{X_1,X_2}\Big[px_1+(p-s)\min\big[x_2,T\big]\Big]dt-k_G\widetilde{G}.$$

It follows that $\Pi_G(\widetilde{G},T)-\Pi_G\big(\overline{G}(T),T\big)=k_G\big(\overline{G}(T)-\widetilde{G}\big)=k_G\epsilon>0$. Hence, since $\Pi_G\big(\overline{G}(T),T\big)<\Pi_G(\widetilde{G},T)$, we get a contradiction to our assumption that $\overline{G}(T)$ is the optimal generation capacity. Thus, $\overline{G}(T)\leq b_1+T$. $\square$

Proof of Lemma 2: We first show that $T^*\leq\overline{G}(T^*)=G^*$. Suppose $T^*=G^*+\epsilon$ for some $\epsilon>0$. Then, because $T^*\geq G^*\geq(G^*-x_1)^+$, using (3.5) and (3.6) we get

$$\mathcal{V}_T(G^*+\epsilon)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{X_1,X_2}\Big[s\min\big[(G^*-x_1)^+,x_2\big]\Big]dt-k_T\,(G^*+\epsilon).$$

Using (3.5) and (3.6) we also obtain that

$$\mathcal{V}_T(G^*)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}_{X_1,X_2}\Big[s\min\big[(G^*-x_1)^+,x_2\big]\Big]dt-k_TG^*.$$

It follows that $\mathcal{V}_T(G^*)-\mathcal{V}_T(G^*+\epsilon)=k_T\epsilon>0$. Therefore, because $\mathcal{V}_T(G^*+\epsilon)<\mathcal{V}_T(G^*)$, the optimal tranmission capacity cannot exceed G^*. Thus, $T^*\leq G^*$.

We next show that $T^*\leq b_2$. Suppose $T^*=b_2+\epsilon$ for some $\epsilon>0$. Because $T^*>b_2\geq x_2$, we get

$$\mathcal{V}_T(b_2+\epsilon)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}_X[s\min[(G^*-x_1)^+,x_2]]dt-k_T(b_2+\epsilon).$$

Similarly, we get

$$\mathcal{V}_T(b_2) = \int_0^{\infty} e^{-\lambda t}\, \mathbb{E}_X[s \min[(G^* - x_1)^+, x_2]] dt - k_T b_2.$$

It follows that $\mathcal{V}_T(b_2) - \mathcal{V}_T(b_2 + \epsilon) = k_T \epsilon > 0$. Thus, because $\mathcal{V}_T(b_2 + \epsilon) < \mathcal{V}_T(b_2)$, the optimal transmission capacity cannot exceed b_2. Hence $T^* \leq b_2$. □

Proof of Proposition 9: Let $F_{X_i}(x) = \int_0^x f_{X_i}(x_i) dx_i$. To prove part (a), we show that $\frac{\partial^2}{\partial T \partial G} \Pi_G(G,T) > 0$.

$$\begin{aligned}
\frac{\partial^2\, \Pi_G(G,T)}{\partial T \partial G} &= \frac{\partial^2}{\partial T \partial G} \left[\frac{1}{\lambda}\, \mathbb{E}_{X_1,X_2}\, \mathcal{R}_G(G, x_1, x_2, T) - k_G G \right] \\
&= \frac{\partial^2}{\partial T \partial G} \left[\frac{1}{\lambda}\, \mathbb{E}_{X_1,X_2} \left[(p-s) \min[(G - x_1)^+, x_2, T] \right] \right] \\
&= \frac{\partial^2}{\partial T \partial G} \left[\frac{p-s}{\lambda} \int_0^{b_1} \int_0^{b_2} \min\Big[(G - x_1)^+, x_2, T \Big] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \right].
\end{aligned}$$

To evaluate the above integral, we distinguish the separate cases for $G < b_1$ and $G \geq b_1$.

If $G < b_1$, then we get

$$\begin{aligned}
&\frac{\partial^2\, \Pi_G(G,T)}{\partial T \partial G} \left[\frac{p-s}{\lambda} \int_0^{b_1} \int_0^{b_2} \min[(G - x_1)^+, x_2, T] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \right] \\
&= \frac{\partial^2}{\partial T \partial G} \left[\frac{p-s}{\lambda} \int_0^{b_1} \int_0^{b_2} \min[(G - x_1)^+, x_2, T] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \right] \\
&= \frac{\partial^2}{\partial T \partial G} \left[\frac{p-s}{\lambda} \left\{ \int_0^{G} \int_0^{\min[G - x_1, T]} x_2 f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \right.\right. \\
&\quad \left.\left. + \int_0^{G} \int_{\min[G - x_1, T]}^{b_2} \min[G - x_1, T] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \right\} \right] \\
&= \frac{\partial^2}{\partial T \partial G} \left[\frac{p-s}{\lambda} \left\{ \int_0^{G-T} \int_0^{T} x_2 f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 + \int_{G-T}^{G} \int_0^{G - x_1} x_2 f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \right.\right. \\
&\qquad \left.\left. + \int_0^{G-T} \int_T^{b_2} T f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 + \int_{G-T}^{G} \int_{G - x_1}^{b_2} (G - x_1) f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \right\} \right]
\end{aligned} \tag{B.1}$$

We now apply the Leibniz integral rule to (B.1) to obtain

$$\frac{\partial^2\, \Pi_G(G,T)}{\partial T \partial G} = \frac{p-s}{\lambda} f_{X_1}(G - T) \left[F_{X_2}(b_2) - F_{X_2}(T) \right] > 0.$$

Thus, for any $G < b_1$ and T such that $T \leq \min[G, b_2]$, $\frac{\partial^2 \Pi_G(G,T)}{\partial T \partial G} > 0$, so that G and T are complements.

If $G \geq b_1$ on the other hand, we get

$$
\begin{aligned}
&\frac{\partial^2}{\partial T \partial G}\left[\frac{p-s}{\lambda}\int_0^{b_1}\int_0^{b_2}\min[(G-x_1)^+, x_2, T] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1\right] \\
&= \frac{\partial^2}{\partial T \partial G}\left[\frac{p-s}{\lambda}\int_0^{b_1}\int_0^{b_2}\min[(G-x_1)^+, x_2, T] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1\right] \\
&= \frac{\partial^2}{\partial T \partial G}\left[\frac{p-s}{\lambda}\left\{\int_0^{b_1}\int_0^{\min[G-x_1,T]} x_2 f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1\right.\right. \\
&\quad \left.\left. + \int_0^{b_1}\int_{\min[G-x_1,T]}^{b_2} \min[G-x_1, T] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1\right\}\right] \\
&= \frac{\partial^2}{\partial T \partial G}\left[\frac{p-s}{\lambda}\left\{\int_0^{G-T}\int_0^{T} x_2 f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 + \int_{G-T}^{b_1}\int_0^{G-x_1} x_2 f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1\right.\right. \\
&\qquad \left.\left. + \int_0^{G-T}\int_T^{b_2} T f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 + \int_{G-T}^{b_1}\int_{G-x_1}^{b_2} (G-x_1) f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1\right\}\right]
\end{aligned} \tag{B.2}
$$

We now apply Leibniz integral rule to (B.2) to obtain

$$
\frac{\partial^2 \Pi_G(G,T)}{\partial T \partial G} = \frac{p-s}{\lambda} f_{X_1}(G-T)\left[F_{X_2}(b_2) - F_{X_2}(T)\right].
$$

Hence, for any $G \geq b_1$ and T such that $T \leq \min[G, b_2]$, $\frac{\partial^2 \Pi_G(G,T)}{\partial T \partial G} > 0$, so that G and T are complements.

To show that G and T are complements with regard $\mathcal{V}_T$, it suffices to show that $\frac{\partial^2 \mathcal{V}_T}{\partial G \partial T} > 0$. We get

$$
\begin{aligned}
\frac{\partial^2 \mathcal{V}_T(G,T)}{\partial G \partial T} &= \frac{\partial^2}{\partial T \partial G}\left[\frac{s+v}{\lambda}\int_0^{b_1}\int_0^{b_2}\min\Big[(G-x_1)^+, x_2, T\Big] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 - k_T T\right] \\
&= \frac{s+v}{\lambda}\frac{\partial^2}{\partial T \partial G}\int_0^{b_1}\int_0^{b_2}\min\Big[(G-x_1)^+, x_2, T\Big] f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1.
\end{aligned} \tag{B.3}
$$

Following the exact same steps as in the proof of part (a), we obtain $\frac{\partial^2 \mathcal{V}_T(G,T)}{\partial G \partial T} > 0$, for all G and T such that $T \leq \min[G, b2]$, which completes the proof. □

Proof of Theorem 7: We use (3.2) to get

$$\Pi_G(G,T) = \frac{1}{\lambda}\,\mathbb{E}_{X_1,X_2}\,\mathcal{R}_G(G,x_1,x_2,T) - k_G G$$
$$= \frac{1}{\lambda}\,\mathbb{E}_{X_1,X_2}\Big[p\min[x_1,G] + (p-s)\min\big[(G-x_1)^+,x_2,T\big]\Big] - k_G G. \tag{B.4}$$

We distinguish separate cases for $G < b_1$ and $G \geq b_1$.

If $G < b_1$, then the first term in (B.4) is given by

$$\frac{1}{\lambda}\,\mathbb{E}_{X_1,X_2}\,p\min[x_1,G] = \frac{1}{\lambda}\int_0^{b_1}\frac{p}{b_1}\min[x_1,G]dx_1 = \frac{p}{\lambda(b_1)}\left[\frac{G^2}{2} + G(b_1-G)\right].$$

The second term in (B.4) is

$$\frac{1}{\lambda}\,\mathbb{E}_{X_1,X_2}(p-s)\min\big[(G-x_1)^+,x_2,T\big] = \frac{p-s}{\lambda b_1 b_2}\left\{\int_0^{b_1}\int_0^{b_2}\min\big[(G-x_1)^+,x_2,T\big]dx_2dx_1\right\}$$
$$= \frac{p-s}{\lambda b_1 b_2}\left\{\int_0^{G}\int_0^{b_2}\min[G-x_1,x_2,T]dx_2dx_1\right\}$$
$$= \frac{p-s}{\lambda b_1 b_2}\left\{\int_0^{G}\int_0^{\min[G-x_1,T]}x_2dx_2dx_1 + \int_0^{G}\int_{\min[G-x_1,T]}^{b_2}\min[G-x_1,T]dx_2dx_1\right\}.$$

For notational convenience, define

$$I_1 = \int_0^{G}\int_0^{\min[G-x_1,T]}x_2dx_2dx_1 \quad \text{and} \quad I_2 = \int_0^{G}\int_{\min[G-x_1,T]}^{b_2}\min[G-x_1,T]dx_2dx_1.$$

Then,

$$I_1 = \int_0^{G}\int_0^{\min[G-x_1,T]}x_2\,dx_2dx_1 = \int_0^{G}\frac{x_2^2}{2}\bigg|_0^{\min[G-x_1,T]}dx_1 = \int_0^{\max[G-T,0]}\frac{T^2}{2}dx_1 + \int_{\max[G-T,0]}^{G}\frac{(G-x_1)^2}{2}dx_1 \tag{B.5}$$

and

$$I_2 = \int_0^G \min[G-x_1, T]\Big(b_2 - \min[G-x_1, T]\Big)dx_1$$
$$= \int_0^{\max[G-T,0]} T(b_2-T)dx_1 + \int_{\max[G-T,0]}^{G} (G-x_1)(b_2-G+x_1)dx_1. \tag{B.6}$$

If $G \geq b_1$, then first term in (B.4) is given as

$$\frac{1}{\lambda}\,\mathbb{E}_{X_1,X_2}\, p\min[x_1, G] = \frac{1}{\lambda}\int_0^{b_1} \frac{p}{b_1} G dx_1 = \frac{pG}{\lambda}.$$

The second term in (B.4) is

$$\frac{1}{\lambda}\,\mathbb{E}_{X_1,X_2}(p-s)\min[(G-x_1)^+, x_2, T] = \frac{p-s}{\lambda b_1 b_2}\left\{\int_0^{b_1}\int_0^{b_2} \min[G-x_1, x_2, T]dx_2 dx_1\right\}$$
$$= \frac{p-s}{\lambda b_1 b_2}\left\{\int_0^{b_1}\int_0^{\min[G-x_1,T]} x_2 dx_2 dx_1 + \int_0^{b_1}\int_{\min[G-x_1,T]}^{b_2} \min[G-x_1, T]dx_2 dx_1\right\}.$$

This time, for notational convenience, define

$$I_3 = \int_0^{b_1}\int_0^{\min[G-x_1,T]} x_2 dx_2 dx_1 \quad \text{and} \quad I_4 = \int_0^{b_1}\int_{\min[G-x_1,T]}^{b_2} \min[G-x_1, T]dx_2 dx_1.$$

Then, we obtain that

$$I_3 = \int_0^{b_1}\int_0^{\min[G-x_1,T]} x_2 dx_2 dx_1 = \int_0^{b_1} \frac{x_2^2}{2}\bigg|_0^{\min[G-x_1,T]} dx_1$$
$$= \int_0^{\max[G-T,0]} \frac{T^2}{2} dx_1 + \int_{\max[G-T,0]}^{b_1} \frac{(G-x_1)^2}{2} dx_1, \tag{B.7}$$

since $b_1 \geq \max[G-T, 0]$ by Lemma 1. Also

$$I_4 = \int_0^{b_1} \min[G-x_1, T](b_2 - \min[G-x_1, T])dx_1$$
$$= \int_0^{\max[G-T,0]} T(b_2-T)dx_1 + \int_{\max[G-T,0]}^{b_1} (G-x_1)(b_2-G+x_1)dx_1. \tag{B.8}$$

Because expressions for I_1, I_2, I_3, and I_4 depend on the ordering of G and T, to proceed we distinguish the following two cases, based on Lemma 1.

Case 1: $T \leq G < b_1$. By (B.5) and (B.6) we obtain

$$\begin{aligned}\Pi_G(G,T) &= \frac{p}{\lambda b_1}\int_0^{b_1} \min[x_1, G]dx_1 \\ &\quad + \frac{p-s}{\lambda b_1 b_2}\left\{\int_0^{\max[G-T,0]} T(b_2-T)dx_1 + \int_{\max[G-T,0]}^{G} (G-x_1)(b_2-G+x_1)dx_1\right\} - k_G G \\ &= \frac{p}{\lambda b_1}\left[\frac{G^2}{2} + G(b_1-G)\right] + \frac{p-s}{\lambda b_1 b_2}\left[(b_2-T)(G-T)T + \frac{b_2T^2}{2} + \frac{(G-T)T^2}{2} - \frac{T^3}{6}\right] - k_G G,\end{aligned}$$

which completes the proof of part (a). It then follows that, for $T \leq G < b_1$,

$$\frac{\partial \Pi_G(G,T)}{\partial G} = \frac{p(b_1-G)}{b_1\lambda} + \frac{(p-s)\left(2(b_2-T)T+T^2\right)}{2b_1b_2\lambda} - k_G. \tag{B.9}$$

Thus, we get $\frac{\partial^2\Pi_G(G,T)}{\partial G^2} = -\frac{p}{\lambda b_1}G < 0$.

Case 2: $\max[b_1, T] \leq G < b_1 + T$. By (B.7) and (B.8) we obtain

$$\begin{aligned}\Pi_G(G,T) &= \frac{p}{\lambda b_1}\int_0^{b_1} x dx_1 + \frac{p-s}{\lambda b_1 b_2}\left[\int_0^{\max[G-T,0]} \frac{T^2}{2}dx_1 + \int_{\max[G-T,0]}^{b_1} \frac{(G-x_1)^2}{2}dx_1\right. \qquad \text{(B.10)} \\ &\quad \left. + \int_0^{\max[G-T,0]} T(b_2-T)dx_1 + \int_{\max[G-T,0]}^{b_1} (G-x_1)(b_2-G+x_1)dx_1\right] - k_G G \\ &= \frac{pb_1}{2\lambda} - \frac{p-s}{6\lambda b_1 b_2}\left[b_1^3 + 3b_1^2(b_2-G) + 3b_1G(G-2b_2) + (3b_2-G-2T)(G-T)^2\right] - k_G G, \qquad \text{(B.11)}\end{aligned}$$

which completes the proof of part (b). It then also follows that, for $\max[b_1, T] \leq G < b_1 + T$,

$$\frac{\partial \Pi_G(G,T)}{\partial G} = \frac{(p-s)(b_1+2b_2-G-T)(b_1-G+T)}{2b_1b_2\lambda} - k_G, \quad \text{if } G \in [b_1, b_1+T]. \tag{B.12}$$

Hence, $\frac{\partial^2\Pi_G(G,T)}{\partial G^2} = -\frac{p-s}{\lambda b_1 b_1}(b_1+b_2-G) < 0$. To complete the proof of part (c), we show that $\frac{\partial \Pi_G(G,T)}{\partial G}$ is continuous everywhere on (T, b_1+T). If $T \geq b_1$ then it is always the case that $\max[b_1, T] \leq G < b_1+T$. In that case, $\frac{\partial^2\Pi_G(G,T)}{\partial G^2} < 0$ as already established in Case 2 above.

If $T < b_1$, then it follows from (B.9) that $\frac{\partial \Pi_G(G,T)}{\partial G}$ is continuous everywhere on (T, b_1), and from

(B.12) that $\frac{\partial \Pi_G(G,T)}{\partial G}$ is continuous everywhere on $\left[\max[b_1,T], b_1+T\right]$. We again make use of (B.9) and (B.12) to obtain

$$\lim_{G\to b_1^-} \frac{\partial \Pi_G(G,T)}{\partial G} = \lim_{G\to b_1^+} \frac{\partial \Pi_G(G,T)}{\partial G} = \left.\frac{\partial \Pi_G(G,T)}{\partial G}\right|_{G=b_1} = \frac{(p-s)(2b_2-T)T}{2b_1b_2\lambda} - k_G.$$

Hence, $\Pi_G(G,T)$ is a smooth function for any $G \in [T, b_1+T)$, which completes the proof. □

In what follows, we make use of the following lemma.

Lemma 8. *(a)* $T_A(k_0) = T_0(k_0) = T_B(k_0) = b_1$*; (b) If* $k_G \le k_0$*, then* $T_0(k_G) \le b_1$ *and* $T_0 \le T_A$*; (c) If* $k_G > k_0$*, then* $T_A(k_G) < b_1$ *and* $T_0 > T_A(k_G)$*; (d) If* $k_G \le k_0$*, then* $T_B(k_G) \ge b_1$ *and* $T_0(k_G) \le T_B(k_G)$*; (e) If* $k_G \le \frac{b_1(p-s)}{2b_2\lambda}$*, then* $T_B(k_G) \ge b_2$*, and* $T_B(k_G) < b_2$ *otherwise.*

Proof: Part (a) follows directly from the definitions of $T_0(k_G)$, $T_A(k_G)$, and $T_B(k_G)$. To prove part (b), we use the definition of T_0 to obtain

$$\frac{\partial T_0(k_G)}{\partial k_G} = \frac{b_1b_2\lambda}{(p-s)\sqrt{b_2\left(b_2 - \frac{2b_1k_G\lambda}{p-s}\right)}} > 0.$$

Hence, for any $k_G \le k_0$, $T_0(k_G) \le T(k_0) = b_1$, by part (a). Further, for any $k_G > k_0$, $T_0(k_G) > T(k_0) = b_1$. To prove part (c), we get

$$\frac{\partial T_A(k_G)}{\partial k_G} = -\frac{b_1b_2\lambda}{\sqrt{2b_1b_2(p-k_G\lambda)(p-s)+b_2^2s^2}} < 0.$$

It follows that, for any $k_G > k_0$, $T_A(k_G) < T_A(k_0) = b_1$. Since $k_G > k_0$ implies $T_0(k_G) > T_A(k_0) = b_1$, the result follows. To prove part (d) we use definition of T_B to obtain

$$\frac{\partial T_B(k_G)}{\partial k_G} = -\frac{b_2\lambda}{p-s} < 0.$$

Hence, for any $k_G \le k_0$, $T_B(k_G) \ge T_B(k_0) = b_1$, by part (a). By part (b), for any $k_G \le k_0$, $T_B(k_G) \ge T(k_0) = b_1 \ge T_0(k_G)$. Also, because $\frac{\partial T_B(k_G)}{\partial k_G} < 0$, and $T_B\left(\frac{b_1(p-s)}{2b_2\lambda}\right) = b_2$ by definition, we obtain part (e).
□

Proof of Theorem 8: Based on Theorem 7, we distinguish the following two cases:

Case 1: $G \in \big[T,\, b_1\big)$. Using (3.11) we obtain

$$\frac{\partial \Pi_G(G,T)}{\partial G} = \frac{p}{\lambda b_1}(b_1 - G) + \frac{p-s}{2\lambda b_1 b_2}(2b_2 T - T)T - k_G. \tag{B.13}$$

Since $\frac{\partial^2 \Pi_G(G,T)}{\partial G^2} < 0$ by Theorem 7(c), the generation's company best response function is obtained by solving $\frac{\partial \Pi_G(G,T)}{\partial G} = 0$. Thus, using (B.13), we get $\overline{G}(T) = b_1\Big(1 - \frac{k_G \lambda}{p}\Big) + \frac{(p-s)(2b_2 - T)T}{2pb_2}$. It follows that $b_1 > \overline{G}(T)$ is equivalent to $T \leq T_0(k_G)$ and $\overline{G}(T) \geq T$ is equivalent to $T < T_A(k_G)$, where $T_0(k_G)$ is as defined in (3.14). Hence, Case 1 occurs when $T < \min\big[T_A(k_G), T_0(k_G)\big]$.

Case 2: $G \in \big[\max(T, b_1),\, b_1 + T\big]$. We use (3.12) to obtain

$$\frac{\partial \Pi_G(G,T)}{\partial G} = \frac{p-s}{2\lambda b_1 b_2}(b_1 + 2b_2 - G - T)(b_1 + T - G) - k_G. \tag{B.14}$$

Because $\frac{\partial^2 \Pi_G(G,T)}{\partial G^2} < 0$ by Theorem 7(c), the generation's company best response function is again obtained by solving $\frac{\partial \Pi_G(G,T)}{\partial G} = 0$. We make use of (B.14) to get $\overline{G}(T) = b_1 + b_2 - \sqrt{(b_2 - T)^2 + \frac{2b_1 b_2 k_G \lambda}{p-s}}$. It follows that $\overline{G}(T) \geq b_1$ is equivalent $T \geq T_0(k_G)$, and $\overline{G}(T) \geq T$ is equivalent to $T \leq T_B(k_G)$. We also obtain

$$b_1 + T - \overline{G}(T) = \sqrt{(b_2 - T)^2 + \frac{2b_1 b_2 (p - k_G \lambda)}{p-s}} - (b_2 - T) > 0.$$

Thus, $\overline{G}(T) < b_1 + T$ by Assumption 4, and Case 2 occurs when $T_0(k_G) \leq T \leq T_B(k_G)$. Hence, if $T < \min[T_A(k_G), T_0(k_G)]$ then $\overline{G}(T) = \mathcal{G}_\alpha$; if $T_0(k_G) \leq T \leq T_B(k_G)$ then $\overline{G}(T) = \mathcal{G}_\beta$.

We now show that if $k_G \leq k_0$ and $T \leq T_0$, or if $k_G > k_0$, then $\overline{G}(T) = \mathcal{G}_\alpha(T)$; if $k_G \leq k_0$ and $T > T_0$, then $\overline{G}(T) = \mathcal{G}_\beta(T)$. By Proposition 2, we obtain $\overline{G}(T^*) \geq T^*$. Hence, for the remainder of this proof, we only consider T such that $T \leq T_A(k_G)$ and $T \leq T_B(k_G)$, which derives from $\mathcal{G}_\alpha(T) \geq T$ and $\mathcal{G}_\beta(T) \geq T$, respectively.

If $k_G > k_0$, then $T_A(k_G) < T_0(k_G)$ by Lemma 8 Part (c). Therefore, for any $T < T_A(k_G)$, if $k_G > k_0$ then $T < T_A(k_G) < T_0(k_G)$. Hence, $\overline{G}(T) = \mathcal{G}_\alpha$ because $T < \min[T_A(k_G), T_0(k_G)]$.

If $k_G \leq k_0$, then $T_0(k_G) \leq T_A(k_G)$ by Lemma 8 Part (b) and $T_0(k_G) \leq T_B(k_G)$ by Lemma 8 Part (d). Therefore, for any $k_G \leq k_0$, if $T < T_0(k_G)$, then $\overline{G}(T) = \mathcal{G}_\alpha$ because $T < \min[T_A(k_G), T_0(k_G)]$;

Otherwise, $\overline{G}(T) = \mathcal{G}_\beta$. $\square$

Proof of Proposition 10: Following Theorem 8, we distinguish two cases.

Case 1: $\overline{G}(T) = \mathcal{G}_\alpha(T)$. We use (3.11) to get

$$\overline{\Pi}_G(T) = \frac{p}{\lambda b_1}\left[\frac{\overline{G}^2(T)}{2} + \overline{G}(T)\big(b_1 - \overline{G}(T)\big)\right] + \frac{p-s}{\lambda b_1 b_2}\left[(b_2 - T)\big(\overline{G}(T) - T\big)T + \frac{b_2 T^2}{2} + \frac{\big(\overline{G}(T) - T\big)T^2}{2} - \frac{T^3}{6}\right] - k_G \overline{G}(T).$$

We use $\overline{G}(T) = \mathcal{G}_\alpha(T)$ in the above expression to get (3.15).

Case 2: $\overline{G}(T) = \mathcal{G}_\beta(T)$. We use (3.12) to get

$$\overline{\Pi}_G(T) = \frac{pb_1}{2\lambda} + \frac{(p-s)}{6\lambda b_1 b_2}\Big[b_1^3 + 3b_1^2\big(b_2 - \overline{G}(T)\big) + 3b_1\overline{G}(T)\big(\overline{G}(T) - 2b_2\big) + (3b_2 - \overline{G}(T) - 2T)\big(\overline{G}(T) - T\big)^2\Big] - k_G\overline{G}(T).$$

We use $\overline{G}(T) = \mathcal{G}_\beta(T)$ in the above expression to get (3.16). $\square$

Proposition 15. *(a) If $k_G > k_0$ then $T^* < T_A$; (b) if $k_G \leq k_0$ then $T^* < T_B$;*

Proof of Proposition 15: We distinguish two cases:

Case 1: $\overline{G}(T) = \mathcal{G}_\alpha$. In this case, by Theorem 8, $\overline{G}(T) \in [T, b_1]$. Thus, $\mathcal{G}_\alpha(T^*) > T^*$ and $\mathcal{G}_\alpha(T^*) < b_1$. It follows directly from $\mathcal{G}_\alpha(T^*) > T^*$ that $T^* < T_A$. Similarly, it follows from $\mathcal{G}_\alpha(T^*) < b_1$ that $T^* < T_0$. By Lemma 8, $T_A < T_0$ if $k_G > k_0$, and $T_A \geq T_0$ otherwise. Thus, if $k_G > k_0$ then $T^* < T_A$. If $k_G \leq k_0$ then $T^* < T_0 \leq T_B$, where the last inequality follows from Lemma 8.

Case 2: $\overline{G}(T) = \mathcal{G}_\beta$. In this case, by Theorem 8, $\overline{G}(T) \geq \max[b_1, T]$. Therefore, $\mathcal{G}_\beta(T^*) \geq T^*$ and $\mathcal{G}_\beta(T^*) \geq b_1$. After algebraic manipulation, we rewrite $\mathcal{G}_\beta(T^*) \geq b_1$ as $T^* \geq T_0$. Also, $\mathcal{G}_\beta(T^*) \geq T^*$ is equivalent to $T^* \leq T_B$ after algebraic manipulation. By Lemma 8, $T_0 \leq T_B$ only if $k_G \leq k_0$ and $T_0 > T_B$, otherwise. Thus, we obtain that if $k_G \leq k_0$ then $T^* \leq T_B$. $\square$

Proof of Proposition 11: To prove part (a), following Theorem 8, we distinguish two cases.

Case 1: $\overline{G}(T) = \mathcal{G}_\alpha(T)$. In this case, by Theorem 8, either $k_G \leq k_0$ and $T \leq T_0$, or $k_G > k_0$. Further, if $k_G > k_0$, then for any $(k_G, T) \in \mathbb{S}$, $T < T_A$. By (3.15), we have that

$$\frac{\partial \overline{\Pi}_G(T)}{\partial T} = \frac{(p-s)(b_2-T)}{2b_1b_2^2\lambda p}\Big[2b_1b_2(p-k_G\lambda)-2b_2sT-(p-s)T^2\Big]$$

Since $T \le b_2$ by Lemma 2, to show $\frac{\partial \overline{\Pi}_G(T)}{\partial T} > 0$, it suffices to show that $2b_1b_2(p-k_G\lambda)-2b_2sT-(p-s)T^2 > 0$. Define $g_1(T,k_G) := 2b_1b_2(p-k_G\lambda)-2b_2sT-(p-s)T^2$. Because $\frac{\partial g_1(T,k_G)}{\partial T} = -2(b_2s+(p-s)T) < 0$ and $\frac{\partial g_1(T,k_G)}{\partial k_G} = -2b_1b_2\lambda < 0$, for any $T \le T_0$ and $k_G \le k_0$, we get

$$g_1(T,k_G) \ge g_1(T_0,k_G) \ge g_1(T_0,k_0) = 2\left(b_1b_2p + b_2p\left(-b_2+\sqrt{b_2\left(b_2-\frac{2b_1k_0\lambda}{p-s}\right)}\right)\right) = 0.$$

Thus, $\frac{\partial \overline{\Pi}_G}{\partial T} > 0$ if $k_G \le k_0$ and $T \le T_0$. For $T \le T_A$ and $k_G > k_0$, because $\frac{\partial g_1(T,k_G)}{\partial T} < 0$, we obtain $g_1(T,k_G) \ge g_1(T_A,k_G) = 0$. Thus, $\frac{\partial \overline{\Pi}_G}{\partial T} > 0$ if $T \le T_A$ and $k_G > k_0$.

Case 2: $\overline{G}(T) = \mathcal{G}_\beta(T)$. Then, by Theorem 8, $k_G \le k_0$ and $T > T_0$. Further, since $(k_G,T) \in \mathbb{S}$, then $T \le \min[T_B, b_2]$ when $k_G \le k_0$. We now show that $\mathcal{G}_\beta(T)$ is increasing in T for any $T \le \min[T_B, b_2]$. By (3.16),

$$\frac{\partial \overline{\Pi}_G(T)}{\partial T} = \frac{b_2-T}{b_1b_2\lambda\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}}\left[(p-s)(b_1+b_2-T)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-(p-s)(b_2-T)^2-2b_1b_2k_G\lambda\right].$$

Define

$$g_2(T,k_G) := \left[(p-s)(b_1+b_2-T)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-(p-s)(b_2-T)^2-2b_1b_2k_G\lambda\right],$$

To show that $\frac{\partial \overline{\Pi}_G(T)}{\partial T} > 0$, it suffices to show that $g_2(T,k_G) > 0$. By definition, we get

$$\frac{\partial g_2(T,k_G)}{\partial T} = \frac{-2b_1b_2k_G\lambda-(p-s)b_1(b_2-T)-2(p-s)(b_2-T)^2+2(p-s)(b_2-T)\sqrt{(b_2-T)^2+\frac{2b_1b_2k_G\lambda}{p-s}}}{\sqrt{(b_2-T)^2+\frac{2b_1b_2k_G}{p-s}}}.$$

Define

$$G_2(T) := -2b_1b_2k_G\lambda - (p-s)b_1(b_2-T) - 2(p-s)(b_2-T)^2 + 2(p-s)(b_2-T)\sqrt{(b_2-T)^2 + \frac{2b_1b_2k_G\lambda}{p-s}}.$$

Then, to show that $\frac{\partial g_2}{\partial T} < 0$ it suffices to show that $G_2(T) < 0$. Because $p > s$, we obtain, for any $T < b_2$, $-2b_1b_2k_G\lambda - (p-s)b_1(b_2-T) - 2(p-s)(b_2-T)^2 - 2(p-s)(b_2-T)\sqrt{(b_2-T)^2 + \frac{2b_1b_2k_G\lambda}{p-s}} < 0$. Therefore, for any $T < b_2$, because

$$G_2(T) \times \left[-2b_1b_2k_G\lambda - (p-s)b_1(b_2-T) - 2(p-s)(b_2-T)^2 - 2(p-s)(b_2-T)\sqrt{(b_2-T)^2 + \frac{2b_1b_2k_G\lambda}{p-s}}\right]$$
$$= b_1\left[b_1(2b_2k\lambda + (b_2-T)(p-s)\right]^2 + 4(b_2-T)^3(p-s)^2) > 0,$$

we obtain that $G_2(T) < 0$. Since $G_2(T) < 0$, then $\frac{\partial g_2}{\partial T} < 0$ for any $T < \min[b_2, T_B]$. To show that $g_2(T, k_G) \geq 0$ for any $T < \min[b_2, T_B]$, based on Lemma 8, we now distinguish two cases: $\frac{b_1(p-s)}{2b_2\lambda} < k_G \leq k_0$ and $k_G \leq \frac{b_1(p-s)}{2b_2\lambda}$.

If $\frac{b_1(p-s)}{2b_2\lambda} < k_G \leq k_0$, then by Lemma 8, $T < \min[b_2, T_B]$ is equivalent to $T < T_B$. Because $\frac{\partial g_2}{\partial T} < 0$, we obtain for any $T < T_B$ that $g_2(T, k_G) > g_2\big(T_B, k_G\big) = 0$.

If $k_G \leq \frac{b_1(p-s)}{2b_2\lambda}$, then by Lemma 8, $T < \min[b_2, T_B]$ is equivalent to $T < b_2$. We use $\frac{\partial g_2}{\partial T} < 0$ to obtain that for any $T < b_2$,

$$g_2(T, k_G) > g_2\big(b_2, k_G\big) = -2b_1b_2k_G\lambda + \sqrt{2}b_1(p-s)\sqrt{\frac{2b_1b_2\lambda}{p-s}}.$$

Because $\frac{\partial}{\partial k_G} g_2\big(b_2, k_G\big) = -\frac{b_1^3b_2^2\lambda^2}{2\sqrt{2}(p-s)\left(\frac{b_1b_2k_G\lambda}{p-s}\right)^{\frac{3}{2}}} < 0$, we obtain that for any $0 \leq k_G \leq \frac{b_1(p-s)}{2b_2\lambda}$, $g_2\big(b_2, k_G\big) \geq \min\left[g_2\big(b_2, 0\big), g_2\big(b_2, \frac{b_1(p-s)}{2b_2\lambda}\big)\right]$. Because $g_2\big(b_2, 0\big) = g_2\big(b_2, \frac{b_1(p-s)}{2b_2\lambda}\big) = 0$, we get $g_2\big(T, k_G\big) \geq 0$ for any $T < b_2$. Consequently, $\frac{\partial}{\partial T}\mathcal{G}_\beta(T) > 0$ for any $T \leq \min[T_B, b_2]$. This completes the proof of part (a).

To prove part (b), we distinguish two cases based on Theorem 8.

Case 1: $k_G \leq k_0$ and $T \leq T_0$, or $k_G > k_0$ in which case $T < T_A$, since $(k_G, T) \in \mathbb{S}$. By Lemma 8, these conditions are equivalent to $T < \min\left[T_0, T_A\right]$. Also, for any k_G, $\min\left[T_0, T_A\right] \leq b_1$ by Lemma 8. By (3.15),

$$\frac{\partial \overline{\Pi}(T)}{\partial p} = \frac{12b_1^2b_2^2(p^2-k_G^2\lambda^2)+12b_1b_2(p^2-k_G\lambda s)(2b_2-T)T+T^2\left[3(p^2-s^2)T^2-4b_2(p^2-3s^2)T-12b_2^2s^2\right]}{24b_1b_2^2\lambda p},$$

$$\frac{\partial^2 \overline{\Pi}(T)}{\partial T \partial p} = \frac{12(b_2-T)}{24b_1b_2^2\lambda p} \times \left[2b_1b_2(p^2-k_G\lambda s)-T(2b_2s^2+(p^2-s^2)T)\right].$$

Thus, to show $\frac{\partial^2 \overline{\Pi}(T)}{\partial T \partial p} > 0$ for any $T \leq b_2$, it suffices to show $\left[2b_1b_2(p^2-k_G\lambda s)-T(2b_2s^2+(p^2-s^2)T)\right] > 0$. Let

$$g_3(T,k_G) := 2b_1b_2(p^2-k_G\lambda s)-T(2b_2s^2+(p^2-s^2)T).$$

Then, because $\frac{\partial g_3(T,k_G)}{\partial T} = -2(p^2-s^2)T-2b_2s^2 < 0$, and $\frac{\partial g_3(b_1,k_G)}{\partial k_G} = -2b_1b_2\lambda s < 0$, we obtain that, for any $T \leq \min[T_0,T_A] \leq b_1$ and $k_G < \bar{k}$,

$$g_3(T,k_G) > g_3\left(b_1,\bar{k}\right) = b_1(p-s)\left(2b_2p-b_1(p+s)\right) > 0.$$

Therefore, we obtain $\frac{\partial^2 \overline{\Pi}(T)}{\partial T \partial p} > 0$. Thus, for any $T \geq 0$,

$$\frac{\partial \overline{\Pi}(T)}{\partial p} \geq \frac{\partial \overline{\Pi}(0)}{\partial p} = \frac{b_1(p^2-k_G^2\lambda^2)}{2\lambda p^2} > 0.$$

Thus, $\frac{\partial \overline{\Pi}(T)}{\partial p} > 0$ for Case 1. We now show $\frac{\partial \overline{\Pi}_G}{\partial k_G} < 0$ for Case 1. We use (3.15) to get

$$\frac{\partial \overline{\Pi}_G}{\partial k_G} = -\frac{24b_1^2b_2^2\lambda(p-k_G\lambda)+12b_1b_2\lambda(p-s)(2b_2-T)T}{24b_1b_2^2\lambda p}. \tag{B.15}$$

Because $p > s$ and $p > k_G\lambda$ and by Assumption 4, then for any $T < b_2$, we obtain by (B.15) that $\frac{\partial \overline{\Pi}_G}{\partial k_G} < 0$.

Finally, we show $\frac{\partial \overline{\Pi}_G}{\partial s} < 0$. We use (3.15) to get

$$\frac{\partial \overline{\Pi}_G}{\partial s} = -\frac{T}{24b_1b_2^2\lambda p} \times \left[6b_1b_2(p-k_G\lambda)(b_2-T)+6b_2^2\left(b_1(p-k_G\lambda)-sT\right)+(p-s)T\left(6b_2^2-8b_2T+3T^2\right)+4sb_2T^2\right].$$

Thus, to show that $\frac{\partial \overline{\Pi}_G}{\partial s} < 0$ for any $T \leq \min[T_0,T_A] \leq b_1$, it suffices to show that $6b_1b_2(p-k_G\lambda)(b_2-$

$T) + 6b_2^2\left(b_1(p-k_G\lambda)-sT\right) > 0$ and $(p-s)T\left(6b_2^2-8b_2T+3T^2\right)+4sb_2T^2 > 0$. Because $p > k_G\lambda$ and $p > s$, for any $T \le \min\left[T_0, T_A\right] \le b_1$, we get $6b_1b_2(p-k_G\lambda)(b_2-T)+6b_2^2\left(b_1(p-k_G\lambda)-sT\right) > 0$. Also, for any $T \le b_2$, because $\frac{\partial}{\partial T}\left[6b_2^2-8b_2T+3T^2\right] = -8b_2+6T < 0$, we obtain that, for any $T \le b_1$,

$$6b_2^2-8b_2T+3T^2 \ge \left(6b_2^2-8b_2T+3T^2\right)\Big|_{T=b_1} = 6b_2^2-8b_2b_1+3b_1^2 > 0,$$

because $b_2 \ge 2b_1$ by assumption. Thus, $\frac{\partial\overline{\Pi}_G}{\partial s} < 0$ for Case 1.

Case 2: $k_G \le k_0$ and $T_0 < T \le \min[T_B, b_2]$. Using (3.16), we get

$$\frac{\partial\overline{\Pi}_G}{\partial p} = \frac{1}{6b_1b_2\lambda(p-s)^2\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}} \times \Bigg[b_1^2b_2\left(-4b_2k_G^2\lambda^2+3(p-s)^2\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right)$$
$$-2(p-s)^2(b_2-T)^3\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-(b_2-T)\right)$$
$$+b_1(p-s)\left(2b_2k_G\lambda(b_2-T)^2+3(p-s)(2b_2T-T^2)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right)\Bigg]$$

$$\frac{\partial^2\overline{\Pi}_G}{\partial T\partial p} = \frac{(b_2-T)}{b_1b_2\lambda(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}}\times$$
$$\left[(b_2-T)(p-s)\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-(b_2-T)\right)+b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-b_1b_2k_G\lambda\right]. \tag{B.16}$$

We first prove that $\frac{\partial^2\overline{\Pi}_G}{\partial T\partial p} > 0$. Since $p > s$ by Assumption 4, then for any $T \le b_2$, we have that

$$(b_2-T)(p-s)\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-(b_2-T)\right) \ge 0.$$

Thus, to show $\frac{\partial^2\overline{\Pi}_G}{\partial T\partial p} > 0$, it suffices to show that $b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-b_1b_2k_G\lambda > 0$. Because $p > s$, we get that $b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}+b_1b_2k_G\lambda > 0$. Consequently, to show that $b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-b_1b_2k_G\lambda > 0$, it suffices to show that

$$\left[b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-b_1b_2k_G\lambda\right] \times \left[b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}+b_1b_2k_G\lambda\right]$$
$$= b_1^2\big((p-s)((p-2)(b_2-T)^2+2b_1b_2k_G\lambda)-b_2^2k_G^2\lambda^2\big) > 0.$$

Define $h_2(T,k_G) := b_1^2\left((p-s)((p-2)(b_2-T)^2+2b_1b_2k_G\lambda)-b_2^2k_G^2\lambda^2\right)$. Then, for any $T \le b_2$, we obtain that $\frac{\partial h_2(T,k_G)}{\partial T} = -2b_1^2(p-s)^2(b_2-T) < 0$. To show that $h_2(T,k_G) \ge 0$ for any $T \le \min[b_2,T_B]$, we now distinguish the case of $\frac{b_1(p-s)}{2b_2\lambda} < k_G \le k_0$ from the case of $k_G \le \frac{b_1(p-s)}{2b_2\lambda}$ based on Lemma 8.

If $\frac{b_1(p-s)}{2b_2\lambda} < k_G \le k_0$, then by Lemma 8, $T \le \min[b_2,T_B] = T_B$. Because $\frac{\partial h_2}{\partial T} < 0$, then for any $T \le T_B$, $h_2(T,k_G) \ge h_2\big(T_B,k_G\big) = \frac{b_1^3(p-s)(4b_2k_G\lambda+b_1(p-s))}{4} > 0$.

If $k_G \le \frac{b_1(p-s)}{2b_2\lambda}$, then by Lemma 8, $T \le \min[b_2,T_B] = b_2$. Because $\frac{\partial}{\partial T}h_2 < 0$, then for any $T \le b_2$, $h_2(T,k_G) \ge h_2\big(b_2,k_G\big) = b_1^2\left(2b_1b_2k_G\lambda(p-s)-b_2^2\lambda^2\right)$. We obtain that for any $0 \le k_G \le \frac{b_1(p-s)}{2b_2\lambda}$, $\frac{\partial h_2\left(b_2,k_G\right)}{\partial T} = 2b_1^3b_2\lambda\big(b_1(p-s)-b_2k_G\lambda\big) > 0$. Consequently, we obtain that, for any $k_G \le 0$, $h_2\big(b_2,k_G\big) \ge h_2(b_2,0) = 0$. Hence, for any $T \le \min[b_2,T_B]$ and $k_G \le k_0$, we obtain that

$$h_2(T,k_G) \ge 0. \tag{B.17}$$

Therefore, for Case 2, we conclude that

$$b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}-b_1b_2k_G\lambda > 0,$$

from which it follows that $\frac{\partial^2\overline{\Pi}_G}{\partial T\partial p} > 0$. Therefore, for any $T > T_0$,

$$\begin{aligned}&\frac{\partial\overline{\Pi}_G(T)}{\partial p} > \frac{\partial\overline{\Pi}_G(T_0)}{\partial p}\\ &= \frac{2b_1k_G\lambda\left(-3b_2+2\sqrt{b_2(b_2-\frac{2b_1k_G\lambda}{p-s})}\right)+2b_2\left(b_2-\sqrt{b_2(b_2-\frac{2b_1k_G\lambda}{p-s})}\right)+3b_1^2(2k_G\lambda+p-s)}{6b_1\lambda(p-s)}.\end{aligned}$$

Define

$$h_3(p) := 2b_1k_G\lambda\left(-3b_2+2\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}\right)+2b_2\left(b_2-\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}\right)+3b_1^2(2k_G\lambda+p-s). \tag{B.18}$$

Then, to show $\frac{\partial\overline{\Pi}_G(T_0)}{\partial p} > 0$, it suffices to show that $h_3(p) > 0$. By differentiating (B.18), we get

$$\frac{\partial^2 h_3(p)}{\partial p^2} = -\frac{6b_1^2b_2k_G^2\lambda^2}{\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}(p-s)^3} < 0.$$

Thus, $h_3(p)$ is concave in p. Since $k_G < k_0$ implies $p > \frac{2b_2k_G\lambda}{2b_2-b_1} + s$, then concavity of h_3 implies that

$$h_3(p) > \min\left[h_3\left(\frac{2b_2k_G\lambda}{2b_2-b_1}+s\right),\ \lim_{p\to\infty} h_3(p)\right]. \tag{B.19}$$

Further, we have that

$$h_3\left(\frac{2b_2k_G\lambda}{2b_2-b_1}+s\right) = \frac{2b_1^2k_G\lambda(6b_2-b_1)}{2b_2-b_1} > 0,$$

and

$$\lim_{p\to\infty} h_3(p) = \lim_{p\to\infty}\left[2b_1k_G\lambda\left(-3b_2+2\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}\right)+2b_2\left(b_2-\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}\right)+3b_1^2(2k_G\lambda+p-s)\right]$$
$$= \lim_{p\to\infty}\left[3b_1^2(2k_G\lambda+p-s)-2b_1b_2k_G\lambda\right] = \infty > 0.$$

It then follows from (B.19) that $h_3(p) > 0$. Hence $\frac{\partial\overline{\Pi}_G}{\partial p} > 0$. Next, we show $\frac{\partial\overline{\Pi}_G}{\partial k_G} < 0$. We use (3.16) to get

$$\frac{\partial\overline{\Pi}_G}{\partial k_G} = \frac{b_1\left(2b_2k_G\lambda-(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right)+(p-s)\left((b_2-T)^2-b_2\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right)\right)}{(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}}.$$

Define

$$\Gamma(T) := b_1\left(2b_2k_G\lambda-(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right)+(p-s)\left((b_2-T)^2-b_2\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right)\right).$$

Thus, to show that $\frac{\partial\overline{\Pi}_G}{\partial k_G} < 0$, it suffices to show that $\Gamma(T) < 0$. We obtain that

$$\frac{d\Gamma}{dT} = \frac{(p-s)(b_2-T)}{\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}} \times \left[b_1+b_2-2\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right]. \tag{B.20}$$

We now show that for $T \in \left[T_0,\ \min(T_B, b_2)\right]$, and any $k_G \le k_0$, $\Gamma(T)$ is decreasing in T by showing that $g_3(T) := b_1+b_2-2\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2} < 0$ for $T \in \left[T_0,\, \min(T_B, b_2)\right]$. First, for any $T \le b_2$, we get

$$\frac{dg_3(T)}{dT} = \frac{\partial}{\partial T}\left[b_1+b_2-2\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right] = -\frac{2(b_2-T)}{\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}} < 0.$$

Further, it follows from the definition of T_0 that $g_3(T_0) = b_1 - b_2 < 0$. Therefore, since $g_3(T_0) < 0$ and $g_3(T)$ is decreasing in T, it follows that $g_3(T) < 0$ for any $T_0 \le T \le b_2$. Thus, by (B.20), $\Gamma(T)$ is decreasing in T for $T \in \left[T_0,\, \min(T_B, b_2)\right]$. Hence, $\Gamma(T) < \Gamma(T_0) := -b_1b_2(p-s) < 0$, and we conclude that $\frac{\partial \overline{\Pi}_G}{\partial k_G} < 0$.

Finally, we show $\frac{\partial \overline{\Pi}_G}{\partial s} < 0$. We use (3.16) to get

$$\frac{\partial \overline{\Pi}_G}{\partial s} = \frac{1}{6b_1b_2\lambda} \times \left[b_1b_2\left(\frac{4b_1b_2k_G^2\lambda^2}{(p-s)^2\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}} - 6T\right) + \frac{2b_1b_2k_G\lambda(b_2-T)^2}{(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}}\right.$$
$$\left. +3b_1T^2 - 2(b_2-T)^2\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2} - (b_2-T)\right)\right].$$

$$\frac{\partial^2 \overline{\Pi}_G}{\partial T \partial s} = \frac{b_2-T}{b_1b_2\lambda(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}} \times \left[b_1b_2k_G\lambda - b_1(p-s)\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right)\right.$$
$$\left. -(p-s)(b_2-T)\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2} - (b_2-T)\right)\right]. \tag{B.21}$$

Since $T \le b_2$, then $-(p-s)(b_2-T)\left(\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2} - (b_2-T)\right) < 0$. Therefore, by (B.21), to show $\frac{\partial^2 \overline{\Pi}_G}{\partial T \partial s} < 0$ it suffices to show that $b_1b_2k_G\lambda - b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2} < 0$. Since $p > s$, we get $-b_1b_2k_G\lambda - b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2} < 0$. Observe that

$$\left[-b_1b_2k_G\lambda - b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right] \times \left[b_1b_2k_G\lambda - b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\right]$$
$$= b_1^2\left((p-s)((p-s)(b_2-T)^2 + 2b_1b_2k_G\lambda) - b_2^2k_G^2\lambda^2\right)$$
$$= h_2(T, k_G) \ge 0,$$

by (B.17). It follows that $b_1b_2k_G\lambda - b_1(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2} \le 0$. Thus, $\frac{\partial^2 \overline{\Pi}_G}{\partial T \partial s} < 0$.

Consequently, for any $T > T_0$,

$$\frac{\partial \overline{\Pi}_G(T)}{\partial s} < \frac{\partial \overline{\Pi}_G(T_0)}{\partial s} = \frac{b_1 k_G \lambda \left(3b_2 - 2\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right) - 3b_1^2 b_2 k_G \lambda - b_2(p-s)\left(b_2^2 - \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right)}{3b_1 \lambda (p-s)}.$$

Define

$$h_4\big(k_G\big) := b_1 k_G \lambda \left(3b_2 - 2\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right) - 3b_1^2 b_2 k_G \lambda - b_2(p-s)\left(b_2^2 - \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right). \tag{B.22}$$

Then, to show $\frac{\partial \overline{\Pi}_G(T_0)}{\partial s}$, it suffices to show that $h_4\left(k_G\right) < 0$. For any $k_G \le k_0$, we use (B.22) to get

$$\frac{\partial^2 h_4(k_G)}{\partial k_G^2} = \frac{3b_1^2 b_2 \lambda^2}{(p-s)\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}} > 0.$$

It follows from $\frac{\partial^2 h_4(k_G)}{\partial k_G^2} h_4(k_G) > 0$ that, for any $k_G \in [0, k_0]$, $h_4(k_G) \le \max\big[h_4\left(0\right), h_4\left(k_0\right)\big]$. By (B.22),

$$h_4(0) = 0; \quad \text{and} \quad h_4(k_0) = -\frac{b_1^2(3b_2 - b_1)(p-s)}{2b_2} < 0.$$

Consequently, we conclude $\frac{\partial \overline{\Pi}_G(T_0)}{\partial s} < 0$. It follows that $\frac{\partial \overline{\Pi}_G(T)}{\partial s} < 0$, which completes the proof. □

Proof of Proposition 12: Based on Theorem 7, we distinguish the following two cases:

Case 1: $G \in \big[T,\, b_1\big)$. We use expressions (3.5) and (3.6) to obtain

$$\begin{aligned}
\mathcal{V}_T(T) &= \frac{s}{\lambda b_1 b_2}\left\{\int_0^{\max[\overline{G}(T)-T,0]} T(b_2 - T)dx_1 + \int_{\max[\overline{G}(T)-T,0]}^{\overline{G}(T)} (\overline{G}(T) - x_1)(b_2 - \overline{G}(T) + x_1)dx_1\right\} - k_G\overline{G}(T) \\
&= \frac{p}{\lambda b_1}\left[\frac{\overline{G}(T)^2}{2} + \overline{G}(T)(b_1 - \overline{G}(T))\right] \\
&\quad + \frac{p-s}{\lambda b_1 b_2}\left[(b_2 - T)(\overline{G}(T) - T)T + \frac{b_2 T^2}{2} + \frac{(\overline{G}(T) - T)T^2}{2} - \frac{T^3}{6}\right] - k_T T.
\end{aligned}$$

Then, we substitute $\overline{G}(T) = G_\alpha(T)$ given in Theorem 8 to obtain expression (3.17). To show that $G \in \big[T,\, b_1\big)$ requires $k_G \le k_0$ and $T \le T_0$, or $k_G > k_0$, we follow the same steps as in the proof of Theorem 8.

Case 2: $G \in \big[\max(T, b_1),\ b_1 + T\big]$. We again use expressions (3.5) and (3.6) to obtain

$$\begin{aligned}
\mathcal{V}_T(T) &= \frac{p}{\lambda b_1}\int_0^{b_1} x dx_1 + \frac{p-s}{\lambda b_1 b_2}\left[\int_0^{\max[\overline{G}(T)-T,0]} \frac{T^2}{2} dx_1 + \int_{\max[\overline{G}(T)-T,0]}^{b_1} \frac{(\overline{G}(T)-x_1)^2}{2} dx_1\right.\\
&\left.+ \int_0^{\max[\overline{G}(T)-T,0]} T(b_2-T)dx_1 + \int_{\max[\overline{G}(T)-T,0]}^{b_1} (\overline{G}(T)-x_1)(b_2-\overline{G}(T)+x_1)dx_1\right] - k_G\overline{G}(T)\\
&= \frac{pb_1}{2\lambda} - \frac{p-s}{6\lambda b_1 b_2}\Big[b_1^3 + 3b_1^2(b_2-\overline{G}(T)) + 3b_1\overline{G}(T)(\overline{G}(T)-2b_2) + (3b_2-\overline{G}(T)-2T)(\overline{G}(T)-T)^2\Big] - k_T T.
\end{aligned}$$

Then, we substitute $\overline{G}(T) = G_\beta(T)$ given in Theorem 8 to obtain (3.18). To show that $G \in \big[\max(T, b_1),\ b_1 + T\big]$ requires $k_G \le k_0$ and $T > T_0$, we follow the same steps as in the proof of Theorem 8. □

Proof of Theorem 9: Following Proposition 12, we distinguish the following two cases.

Case 1: $k_G \le k_0$ and $T \le T_0$, or $k_G > k_0$. We use (3.17) to get

$$\frac{\partial \mathcal{V}_T(T)}{\partial T} = -k_T + s(b_2 - T)\frac{b_1 b_2(p - k_G\lambda) + T(b_2(p-2s) - (p-s)T)}{b_1 b_2^2 p\lambda} \tag{B.23}$$

$$\frac{\partial^3 \mathcal{V}_T}{\partial T^3} = \frac{2(s+v)(-2b_2 p + 3b_2 s + 3(p-s)T)}{b_1 b_2^s \lambda p} \tag{B.24}$$

$$\frac{\partial^4 \mathcal{V}_T}{\partial T^4} = \frac{6(p-s)(s+v)}{b_1 b_2^2 \lambda p} > 0.$$

If $k_G \le k_0$, then, by Lemma 8, $T \le T_0 \le b_1$. If $k_G > k_0$, then by Lemma 8, $T \le T_A \le b_1$. We first show that, for any $T \le b_1$, $\mathcal{V}_T(T)$ is strictly increasing or unimodal in T.

Since $\frac{\partial^4 \mathcal{V}_T(T)}{\partial T^4} > 0$, then, for any $T \le b_1$, $\frac{\partial^3 \mathcal{V}_T(T)}{\partial T^3} \le \frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=b_1}$. We use (B.24) to obtain that $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=b_1} = \frac{2s(3(p-s)b_1 - 2b_2 p + 3b_2 s)}{b_1 b_2^2 \lambda p}$. Because $b_1 < \frac{b_2(2p-3s)}{3(p-s)}$, we obtain $\frac{\partial^3 \mathcal{V}_T(T)}{\partial T^3} \le \frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=b_1} < 0$.

Because $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T) < 0$, then, for any $T \le b_1$, $\frac{\partial}{\partial T}\mathcal{V}_T(T)$ is a concave function of T. By (B.23) we get that $\frac{\partial}{\partial T}\mathcal{V}_T(T)\big|_{T=0} = \frac{(p-k_G\lambda)(s+v)}{p\lambda} - k_T > 0$. Consequently, because $\frac{\partial}{\partial T}\mathcal{V}_T(T)$ is concave and $\frac{\partial}{\partial T}\mathcal{V}_T(T)\big|_{T=0} > 0$, we obtain that, $T \ge 0$, $\mathcal{V}_T(T)$ is strictly increasing in T or unimodal in T in Case 1.

Case 2: $k_G < k_0$ and $T \ge T_0$. We use (3.18) to get

$$\frac{\partial}{\partial T}\mathcal{V}_T(T) = -k_T + \frac{(s+v)(b_2-T)}{3b_1b_2\lambda(p-s)}\times$$
$$\left[-2(p-s)\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}+\frac{b_1b_2k_G\lambda-(p-s)(b_2-T)^2}{\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}}+3(p-s)(b_1+b_2-T)\right]. \tag{B.25}$$

Also,

$$\frac{\partial^4}{\partial T^4}\mathcal{V}_T(T) = \frac{12b_1b_2k_G^2\lambda(s+v)\left(3b_1b_2k_G\lambda-(p-s)(b_2-T)^2\right)}{\sqrt{\frac{2b_1b_2k_G\lambda}{p-s}+(b_2-T)^2}\left[2b_1b_2k_G\lambda+(p-s)(b_2-T)^2\right]^3} \tag{B.26}$$

We now show that if $k_G < k_0$ and $T \geq T_0$, then $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)$ is a) strictly decreasing, or b) unimodal with a unique minimum, or c) strictly increasing in T.

By (B.26), we get $\frac{\partial^4}{\partial T^4}\mathcal{V}_T(T)\Big|_{T=b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}}} = 0$. Also, $\frac{\partial}{\partial T}\left[3b_1b_2k_G\lambda-(p-s)(b_2-T)^2\right] = 2(p-s)(b_2-T) > 0$. Thus, by (B.26), $\frac{\partial^4}{\partial T^4}\mathcal{V}_T(T) > 0$ for any $T > b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}}$ and $\frac{\partial^4}{\partial T^4}\mathcal{V}_T(T) \leq 0$ for any $T \leq b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}}$.

By definition of T_0 given in (3.14), we obtain that $b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}} < T_0$ if $b_1 > \frac{b_2(p-s)}{5k_G\lambda}$ and $b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}} > T_0$ if $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$. Therefore, if $b_1 > \frac{b_2(p-s)}{5k_G\lambda}$, then for any $T \geq T_0$, $\frac{\partial^4}{\partial T^4}\mathcal{V}_T(T) > 0$ because $T \geq T_0 > b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}}$. Thus, $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)$ is strictly increasing in T.

If $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$, then $b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}} > T_0$. Therefore, if $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$, then for any $T \geq T_0$, because $\frac{\partial^4}{\partial T^4}\mathcal{V}_T(T) < 0$ for $T < b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}}$ and $\frac{\partial^4}{\partial T^4}\mathcal{V}_T(T) > 0$ for $T > b_2-\sqrt{\frac{3b_1b_2k_G\lambda}{p-s}}$. Therefore, $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)$ is strictly decreasing, or unimodal (with a unique minimum) in T.

We now show that $\frac{\partial^2}{\partial T^2}\mathcal{V}_T(T)$ is either a) strictly decreasing, b) unimodal with a unique minimum, or c) strictly increasing in T. If $b_1 \geq \frac{b_2(p-s)}{5k_G\lambda}$ then because $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)$ is strictly increasing in T, $\frac{\partial^2}{\partial T^2}\mathcal{V}_T(T)$ is either a) strictly decreasing, b) unimodal with a unique minimum, or c) strictly increasing.

When $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$, we use (3.18) to get

$$\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=T_0} =$$

$$\frac{2(s+v)}{b_1 b_2^4 \lambda (p-s)^2} \times \left[b_2^3 (p-s)^2 - (3b_1^2 k_G^2 \lambda^2 + b_1 b_2 k_G \lambda (p-s) + b_2^2 (p-s)^2) \sqrt{b_2 \left(b_2 - \frac{2b_1 k_G \lambda}{p-s} \right)} \right] \tag{B.27}$$

We now show that for any $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$, $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=T_0} < 0$. By (B.27), to show $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=T_0} < 0$, it suffices to show that

$b_2^3(p-s)^2 - (3b_1^2 k_G^2 \lambda^2 + b_1 b_2 k_G \lambda (p-s) + b_2^2 (p-s)^2)\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)} < 0.$

Define

$$\eta(b_1) := \left[b_2^3 (p-s)^2 - (3b_1^2 k_G^2 \lambda^2 + b_1 b_2 k_G \lambda (p-s) + b_2^2 (p-s)^2) \sqrt{b_2 \left(b_2 - \frac{2b_1 k_G \lambda}{p-s} \right)} \right] \tag{B.28}$$

For any $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$, we use (B.28) to obtain that

$$\frac{\partial}{\partial b_1}\eta(b_1) = \frac{3b_1 b_2 k_G^2 \lambda^2}{(p-s)\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}} \times [5b_1 k_G \lambda - b_2(p-s)] < 0. \tag{B.29}$$

Consequently, because $\frac{\partial}{\partial b_1}\eta(b_1) < 0$, we obtain, for any $b_1 \geq 0$, $\eta(b_1) \leq \eta(0) = 0$.

Therefore, for any $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$, $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=T_0} < 0$. Thus, if $b_1 < \frac{b_2(p-s)}{5k_G\lambda}$, then $\frac{\partial^2}{\partial T^2}\mathcal{V}_T(T)$ is either strictly decreasing in T, or unimodal (U-shape) in T because, for any $T \geq T_0$, $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)$ is strictly decreasing or unimodal in T and $\frac{\partial^3}{\partial T^3}\mathcal{V}_T(T)\Big|_{T=T_0} < 0$.

Because $\frac{\partial^2 \mathcal{V}_T(T)}{\partial T^2}$ is either a) strictly decreasing in T, or b) unimodal (U-shape), or c) strictly increasing in T, to prove that $\frac{\partial \mathcal{V}_T(T)}{\partial T}$ is either unimodal (U-shape) or decreasing in T, we now show that for any $T \geq T_0$, $\frac{\partial^2 \mathcal{V}_T(T)}{\partial T^2}\Big|_{T=T_0} < 0$. By (3.18) we obtain

$$\frac{\partial^2 \mathcal{V}_T(T)}{\partial T^2}\Big|_{T=T_0} =$$

$$-\frac{s+v}{b_1 b_2^2 \lambda (p-s)^2} \times \left[b_1 b_2 (p-s)(2k_G\lambda + p - s) + 2b_1^2 k_G \lambda^2 + 2b_2 \left(-b_2 + \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)} \right) \right]. \tag{B.30}$$

By (B.30) to show $\left.\frac{\partial^2 \mathcal{V}_T(T)}{\partial T^2}\right|_{T=T_0} < 0$, it suffices to show that $\left[b_1 b_2(p-s)(2k_G\lambda + p - s) + 2b_1^2 k_G \lambda^2 + 2b_2\left(-b_2 + \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right)\right] > 0$. Define

$$\kappa(k_G) := b_1 b_2(p-s)(2k_G\lambda + p - s) + 2b_2\left(-b_2 + \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right). \tag{B.31}$$

Because $\kappa(k_G) \leq b_1 b_2(p-s)(2k_G\lambda + p - s) + 2b_1^2 k_G \lambda^2 + 2b_2\left(-b_2 + \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right)$, it suffices to show that $\kappa(k_G) > 0$ to prove $\left.\frac{\partial^2 \mathcal{V}_T(T)}{\partial^2 T}\right|_{T=T_0} < 0$. By (B.31) we get

$$\frac{\partial \kappa}{\partial k_G} = 2b_1 b_2 \lambda (p-s)\left(1 - \frac{b_2}{\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}}\right) < 0. \tag{B.32}$$

Consequently, because $\frac{\partial \kappa}{\partial k_G} < 0$, it follows that for any $k_G \leq k_0$, we obtain that $\kappa(k_G) \geq \kappa(k_0) = b_1(b_2 - b_1)(p-s)^2 > 0$. Consequently, $\kappa(k_G) > 0$ and we conclude that $\left.\frac{\partial^2 \mathcal{V}_T(T)}{\partial T^2}\right|_{T=T_0} < 0$. Therefore, $\frac{\partial \mathcal{V}_T(T)}{\partial T}$ is either either unimodal (U-shape) or decreasing in T.

Because $\frac{\partial \mathcal{V}_T(T)}{\partial T}$ is either unimodal (U-shape) or decreasing in T, and,

$$\left.\frac{\partial}{\partial T}\mathcal{V}_T(T)\right|_{T=b_2} = -k_T < 0, \tag{B.33}$$

we conclude that for any $T < b_2$, $\mathcal{V}_T(T)$ is either a) strictly decreasing in T, or b) unimodal (inverted U-shape), or c) strictly increasing in T.

Finally, we show that if $\frac{b_1}{b_2} < \frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2pk_T\lambda}{(p-s)(s+v)}}$ then $V_{\alpha,T}$ is strictly increasing in T, and $V_{\beta,T}$ is either unimodal (inverted U-shape) or strictly increasing in T. We use (3.14), (3.17), and (3.18) to obtain that

$$\begin{aligned}\left.\frac{\partial}{\partial T}\mathcal{V}_{\alpha,T}(T)\right|_{T=T_0} = &\frac{\left(b_2 - \sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}\right)(s+v)}{b_1 \lambda} + \frac{(p + k_G\lambda)(s+v)\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}}{b_2 \lambda p} \\ &- \frac{2k_G(s+v) + (p-s)k_T}{p-s}\end{aligned} \tag{B.34}$$

$$\frac{\partial}{\partial T}\mathcal{V}_{\beta,T}(T)\bigg|_{T=T_0} = \frac{\left(b_2-\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}\right)(s+v)}{b_1\lambda} + \frac{(p+k_G\lambda-s)(s+v)\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}}{b_2\lambda(p-s)} - \frac{2k_G(s+v)+(p-s)k_T}{p-s} \tag{B.35}$$

By (B.37) and (B.35), we obtain that

$$\frac{\partial}{\partial T}\mathcal{V}_{\beta,T}(T)\bigg|_{T=T_0} - \frac{\partial}{\partial T}\mathcal{V}_{\alpha,T}(T)\bigg|_{T=T_0} = \frac{k_G s(s+v)\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}}{b_2p(p-s)} > 0. \tag{B.36}$$

Also, by Lemma 8 we obtain that in case 1, $T \le T_0 \le b_1$. Then, because $\mathcal{V}_{\alpha,T}$ strictly increasing or unimodal in T, it suffices to ensure $\frac{\partial}{\partial T}\mathcal{V}_{\alpha,T}(T)\big|_{T=b_1} > 0$ to derive the condition under which $\mathcal{V}_{\alpha,T}$ is always increasing in T. We use (3.17) to obtain

$$\frac{\partial}{\partial T}\mathcal{V}_{\alpha,T}(T)\bigg|_{T=b_1} = (b_2-b_1)(s+v)\frac{(b_2(2(p-s)-k_G\lambda)-b_1(p-s))}{b_2^2\lambda p} - k_T \tag{B.37}$$

Using (B.37), we obtain that for any $k < k_0$, if $k_T < (b_2-b_1)(s+v)\frac{(b_2(2(p-s)-k_G\lambda)-b_1(p-s))}{b_2^2\lambda p}\Big|_{k=k_0} = \left(1-\frac{b_1}{2b_2}\right)\left(1-\frac{b_1}{b_2}\right)\frac{(p-s)(s+v)}{\lambda p}$, then $\frac{\partial}{\partial T}\mathcal{V}_{\alpha,T}(T)\big|_{T=b_1} > 0$. Therefore, if $k_T < \left(1-\frac{b_1}{2b_2}\right)\left(1-\frac{b_1}{b_2}\right)\frac{(p-s)(s+v)}{\lambda p}$ (equivalent to $\frac{b_1}{b_2} < \frac{3}{2} - \sqrt{\frac{1}{4}+\frac{2pk_T\lambda}{(p-s)(s+v)}}$), then for any $T < T_0$, $V_{\alpha,T}$ is strictly increasing in T. By (B.36), $V_{\beta,T}$ is either unimodal (inverted U-shape) or strictly increasing in T. □

Proof of Proposition 13

<u>Case 1</u>: $k_G \le k_0$ and $T < T_0$. We prove that $\mathcal{V}_{\alpha,T}(T)$ is strictly increasing in T. Because $\mathcal{V}_{\alpha,T}(T)$ is either unimodal or increasing in T by Theorem 9, to show that $\mathcal{V}_{\alpha,T}(T)$ is strictly increasing in T, it suffices to show that $\frac{\partial}{\partial T}\mathcal{V}_T(T)\big|_{T=T_0} > 0$. Define

$$\Phi(k_G) := \frac{\partial}{\partial T}\mathcal{V}_T(T)\bigg|_{T=T_0} = -k_T + (s+v)\left[\frac{b_2-\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}}{b_1\lambda} + \frac{(p+k_G\lambda)\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}}{b_2p\lambda} - \frac{2k_G}{p-s}\right]. \tag{B.38}$$

Then

$$\frac{\partial}{\partial k_G}\Phi(k_G) = -\frac{(s+v)\left(3b_1k_G\lambda + b_1p - 2b_2p + b_2s + 2p\sqrt{b_2\left(b_2 - \frac{2b_1k_G\lambda}{p-s}\right)}\right)}{p(p-s)\sqrt{b_2\left(b_2 - \frac{2b_1k_G\lambda}{p-s}\right)}} \tag{B.39}$$

Define $\underline{\Phi}(k_G) := 3b_1k_G\lambda + b_1p - 2b_2p + b_2s + 2p\sqrt{b_2\left(b_2 - \frac{2b_1k_G\lambda}{p-s}\right)}$. Because $\frac{\partial^2}{\partial k_G{}^2}\underline{\Phi}(k_G) = -\frac{2b_1^2b_2^2p\lambda^2}{(p-s)^2\left(\sqrt{b_2\left(b_2-\frac{2b_1k_G\lambda}{p-s}\right)}\right)^{3/2}} < 0$, we obtain that, for any $k_G \in [0, k_0)$, $\underline{\Phi}(k_G) \geq \min[\underline{\Phi}(0), \underline{\Phi}(k_0)]$. By definition, we obtain $\underline{\Phi}(0) = b_1p + b_2s > 0$ and $\underline{\Phi}(k_0) = b_1(2p-3s) - \frac{3b_1^2(p-s)}{2b_2} + b_2s$. Because $\frac{\partial}{\partial b_1}\underline{\Phi}(k_0) = 2p - 3s - \frac{3b_1(p-s)}{b_2} > 0 \left(\because b_1 < \frac{b_2(2p-3s)}{3(p-s)}\right)$ and $\underline{\Phi}(k_0)\big|_{b_1=0} = b_2s > 0$ we obtain that $\underline{\Phi}(k_0) > 0$. Consequently, $\underline{\Phi}(k_G) > 0$. Thus, we use (B.39) to obtain that $\frac{\partial}{\partial k_G}\Phi(k_G) < 0$.

Since $\frac{\partial}{\partial k_G}\Phi(k_G) < 0$, we use (B.38) to obtain

$$\Phi(k_G) \geq \Phi(k_0) = \frac{(b_2 - b_1)(2b_2 - b_1)(p-s)(s+v)}{2b_2^2p\lambda} - k_T. \tag{B.40}$$

It follows from the assumption $\frac{b_1}{b_2} < \frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2pk_T\lambda}{(p-s)(s+v)}}$, we obtain that $k_T < \frac{(b_2-b_1)(2b_2-b_1)(p-s)(s+v)}{2b_2^2p\lambda}$. Consequently, $\frac{\partial}{\partial T}\mathcal{V}_{\alpha,T}(T)\big|_{T=T_0} > 0$ and we conclude that $\mathcal{V}_{\alpha,T}(T)$ is strictly increasing in T.

Case 2: $k_G \leq k_0$ and $T \geq T_0$. By Theorem 9, for any $T_0 \leq T \leq \min[b_2, T_B]$, to show that $\mathcal{V}_{\beta,T}(T)$ is unimodal or strictly increasing in T, it suffices to show that that $\frac{\partial \mathcal{V}_{\beta,T}(T)}{\partial T}\Big|_{T=T_0} > 0$. We use (3.17) and (3.18) to obtain

$$\frac{\partial \mathcal{V}_{\beta,T}(T)}{\partial T}\bigg|_{T=T_0} - \frac{\partial \mathcal{V}_{\alpha,T}(T)}{\partial T}\bigg|_{T=T_0} = \frac{k_G\sqrt{b_2\left(b_2 - \frac{2b_1k_G\lambda}{p-s}\right)}s(s+v)}{b_2p(p-s)} > 0.$$

and $\frac{\partial \mathcal{V}_{\alpha,T}(T)}{\partial T}\Big|_{T=T_0} > 0$, we obtain that $\frac{\partial \mathcal{V}_{\beta,T}(T)}{\partial T}\Big|_{T=T_0} > 0$. Therefore, $\mathcal{V}_{\beta,T}(T)$ is unimodal or strictly increasing in T. □

Proof of Proposition 13

Proof of item (a). Since because $\Pi_{\beta,T}$ is unimodal then $\frac{\partial \widetilde{\Pi}_T(r,T)}{\partial T}\Big|_{T=\widehat{T}} = 0$. Thus,

$$\frac{\partial}{\partial r}\left(\frac{\partial \widetilde{\Pi}_T(r,T)}{\partial T}\bigg|_{T=\widehat{T}}\right) = \frac{\partial^2 \widetilde{\Pi}_T}{\partial T^2}\bigg|_{T=\widehat{T}} \times \frac{\partial \widehat{T}}{\partial r} + \frac{\partial^2 \widetilde{\Pi}_T}{\partial r \partial T}\bigg|_{T=\widehat{T}} = 0. \tag{B.41}$$

By (B.41) we then get

$$\frac{\partial \widehat{T}}{\partial r} = -\frac{\left.\frac{\partial^2 \widetilde{\Pi}_T}{\partial T \partial r}\right|_{T=\widehat{T}}}{\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}}}. \tag{B.42}$$

Since $\Pi_{\beta,T}$ is unimodal, it follows that $\left.\frac{\partial^2 \widetilde{\Pi}_T}{\partial T^2}\right|_{T=\widehat{T}} < 0$. Also, $\left.\frac{\partial^2 \widetilde{\Pi}_T}{\partial T \partial r}\right|_{T=\widehat{T}} = k_T$. Consequently, we use (B.42) to obtain that $\frac{\partial \widehat{T}}{\partial r} > 0$.

Proof of item (b). For $r_1 < r_2$, $\widetilde{\Pi}_T(r_2, \widehat{T}(r_2)) - \widetilde{\Pi}_T(r_1, \widehat{T}(r_1)) \geq (r_2 - r_1) k_T \widehat{T}(r_1) > 0$. Consequently, we get $\dfrac{\widetilde{\Pi}_T(r_2, \widehat{T}(r_2)) - \widetilde{\Pi}_T(r_1, \widehat{T}(r_1))}{r_2 - r_1} > 0$. □

Proof of Theorem 10

To show $\left.\frac{\partial \widehat{\Pi}_G(r)}{\partial r}\right|_{r=0} > 0$ we use (3.23) to obtain

$$\left.\frac{\partial \widehat{\Pi}_G(r)}{\partial r}\right|_{r=0} = \left.\frac{\partial \overline{\Pi}_G}{\partial T}\right|_{T=\widehat{T}} \times \left.\frac{\partial \widehat{T}}{\partial r}\right|_{r=0} - k_T \widehat{T}. \tag{B.43}$$

Because $\frac{\partial \widehat{T}}{\partial r} = -\frac{\left.\frac{\partial^2 \Pi_T}{\partial T \partial r}\right|_{T=\widehat{T}}}{\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}}}$ we use (B.43) to get

$$\left.\frac{\partial \widehat{\Pi}_G(r)}{\partial r}\right|_{r=0} = g_1(\widehat{T}) \times \frac{g_2(\widehat{T})}{k_T} - k_T \widehat{T}. \tag{B.44}$$

By Propositions 11, $\frac{\partial \overline{\Pi}_G}{\partial T} > 0$. Also, $\frac{\partial \widehat{T}}{\partial r} > 0$ by Proposition 13. Consequently, if $k_T^2 < \dfrac{g_1(\widehat{T}) g_2(\widehat{T})}{\widehat{T}^2}$, then $\left.\dfrac{\partial \widehat{\Pi}_G(r)}{\partial r}\right|_{r=0} > 0$. □

Proof of Proposition 14

Proof of Item (a). We first show that $\frac{\partial \widehat{T}}{\partial p} < 0$. Since because $\Pi_{\beta,T}$ is unimodal then $\left.\frac{\partial \Pi_T(p,T)}{\partial T}\right|_{T=\widehat{T}} = 0$. Thus,

$$\frac{\partial}{\partial p}\left(\left.\frac{\partial \Pi_T(p,T)}{\partial T}\right|_{T=\widehat{T}}\right) = \left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} \times \frac{\partial \widehat{T}}{\partial p} + \left.\frac{\partial^2 \Pi_T}{\partial p \partial T}\right|_{T=\widehat{T}} = 0. \tag{B.45}$$

Consequently,

$$\frac{\partial \widehat{T}}{\partial p} = -\frac{\left.\frac{\partial^2 \Pi_T}{\partial T \partial p}\right|_{T=\widehat{T}}}{\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}}}. \tag{B.46}$$

Because $\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} < 0$ to show that $\frac{\partial \widehat{T}}{\partial p} < 0$ it suffices to show that $\left.\frac{\partial^2 \Pi_T}{\partial T \partial p}\right|_{T=\widehat{T}} > 0$. We use Proposition 12 to obtain that

$$\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial p} = \frac{(b_2 - T)(b_1 b_2 \lambda + s(2b_2 - T)T)(s+v)}{b_1 b_2^2 \lambda p^2} > 0; \tag{B.47}$$

$$\frac{\partial^2 \Pi_{\beta,T}}{\partial T \partial p} = \frac{b_1 b_2 k_G^2 \lambda (b_2 - T)(s+v)}{(p-s)^2 \sqrt{\frac{2b_1 b_2 k_G \lambda}{p-s} + (b_2 - T)^2}\,(2b_1 b_2 k_G \lambda + (p-s)(b_2 - T)^2)} > 0. \tag{B.48}$$

We next show that $\frac{\partial \widehat{T}}{\partial k_G} > 0$. Since because $\Pi_{\beta,T}$ is unimodal then $\left.\frac{\partial \Pi_T(p,T)}{\partial T}\right|_{T=\widehat{T}} = 0$. Thus,

$$\frac{\partial}{\partial p}\left(\left.\frac{\partial \Pi_T(p,T)}{\partial T}\right|_{T=\widehat{T}}\right) = \left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} \times \frac{\partial \widehat{T}}{\partial p} + \left.\frac{\partial^2 \Pi_T}{\partial p \partial T}\right|_{T=\widehat{T}} = 0. \tag{B.49}$$

Consequently,

$$\frac{\partial \widehat{T}}{\partial k_G} = -\frac{\left.\frac{\partial^2 \Pi_T}{\partial T \partial k_G}\right|_{T=\widehat{T}}}{\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}}}. \tag{B.50}$$

Because $\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} < 0$ to show that $\frac{\partial \widehat{T}}{\partial k_G} > 0$ it suffices to show that $\left.\frac{\partial^2 \Pi_T}{\partial T \partial k_G}\right|_{T=\widehat{T}} < 0$. We use Proposition 12 to obtain that

$$\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial k_G} = -\frac{(b_2 - T)(s+v)}{b_2 p} < 0; \tag{B.51}$$

$$\frac{\partial^2 \Pi_{\beta,T}}{\partial T \partial k_G} = -\frac{b_1 b_2 k_G \lambda (b_2 - T)(s+v)}{(p-s) \sqrt{\frac{2b_1 b_2 k_G \lambda}{p-s} + (b_2 - T)^2}\,(2b_1 b_2 k_G \lambda + (p-s)(b_2 - T)^2)} < 0. \tag{B.52}$$

Finally, we show that $\frac{\partial \widehat{T}}{\partial k_T} > 0$. Since because $\Pi_{\beta,T}$ is unimodal then $\left.\frac{\partial \Pi_T(p,T)}{\partial T}\right|_{T=\widehat{T}} = 0$. Thus,

$$\frac{\partial}{\partial p}\left(\left.\frac{\partial \Pi_T(p,T)}{\partial T}\right|_{T=\widehat{T}}\right) = \left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} \times \frac{\partial \widehat{T}}{\partial p} + \left.\frac{\partial^2 \Pi_T}{\partial p \partial T}\right|_{T=\widehat{T}} = 0. \tag{B.53}$$

Consequently,

$$\frac{\partial \widehat{T}}{\partial k_T} = -\frac{\left.\frac{\partial^2 \Pi_T}{\partial T \partial k_G}\right|_{T=\widehat{T}}}{\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}}}. \tag{B.54}$$

Because $\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} < 0$ to show that $\frac{\partial \widehat{T}}{\partial k_T} > 0$ it suffices to show that $\left.\frac{\partial^2 \Pi_T}{\partial T \partial k_T}\right|_{T=\widehat{T}} < 0$. We use Proposition 12 to obtain that

$$\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial k_T} = \frac{\partial^2 \Pi_{\beta,T}}{\partial T \partial k_T} = -1. \tag{B.55}$$

Proof of Item (b). Because

$$\frac{\partial \widehat{G}}{\partial k_T} = \left.\frac{\partial \overline{G}(T)}{\partial k_T}\right|_{T=\widehat{T}} = \left.\frac{\partial \overline{G}(T)}{\partial T}\right|_{T=\widehat{T}} \times \frac{\partial \widehat{T}}{\partial k_T}, \tag{B.56}$$

and $\left.\frac{\partial \overline{G}(T)}{\partial T}\right|_{T=\widehat{T}} > 0$ by Proposition 11, we obtain that $\frac{\partial \widehat{G}}{\partial k_T} < 0$. □

Proof of Theorem 11

Since $\left.\frac{\partial \Pi_T(s,T)}{\partial T}\right|_{T=\widehat{T}} = 0$,

$$\frac{\partial}{\partial s}\left(\left.\frac{\partial \Pi_T(s,T)}{\partial T}\right|_{T=\widehat{T}}\right) = \left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} \times \frac{\partial \widehat{T}}{\partial s} + \left.\frac{\partial^2 \Pi_T}{\partial s \partial T}\right|_{T=\widehat{T}} = 0. \tag{B.57}$$

Using (B.57) we get

$$\frac{\partial \widehat{T}}{\partial s} = -\frac{\left.\frac{\partial^2 \Pi_T}{\partial T \partial s}\right|_{T=\widehat{T}}}{\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}}}. \tag{B.58}$$

Because $\left.\frac{\partial^2 \Pi_T}{\partial T^2}\right|_{T=\widehat{T}} < 0$ to show that $\widehat{T}(s)$ is unimodal in s, it suffices to show that $\left.\frac{\partial^2 \Pi_T}{\partial T \partial s}\right|_{T=\widehat{T}(s)} > 0$ for $s < \bar{s}$ and $\left.\frac{\partial^2 \Pi_T}{\partial T \partial s}\right|_{T=\widehat{T}(s)} < 0$ for $s \geq \bar{s}$. For $\Pi_T(T) = \Pi_{\alpha,T}(T)$, we use Proposition 12 to obtain that

$$\frac{\partial}{\partial s}\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial s} = -\frac{2(b_2 - T)T(2b_2 - T)}{b_1 b_2^2 \lambda p}. \tag{B.59}$$

By (B.59), for any $T = \widehat{T}(s)$ it follows that $\left.\frac{\partial}{\partial s}\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial s}\right|_{T=\widehat{T}} < 0$. Also,

$$\Gamma(v) = \left.\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial s}\right|_{s=0,\, T=\widehat{T}(0)} = \frac{(b_2-T)\big(b_1 b_2 (p-k_G\lambda)+p(b_2-T)T\big)}{b_1 b_2^2 \lambda p} - \frac{v(b_2-T)T(2b_2-T)}{b_1 b_2^2 \lambda p}. \tag{B.60}$$

By (B.62), $v^* > 0$ because $\frac{(b_2-T)\left(b_1 b_2(p-k_G\lambda)+p(b_2-T)T\right)}{b_1 b_2^2 \lambda p} > 0$ and $\frac{v(b_2-T)T(2b_2-T)}{b_1 b_2^2 \lambda p} > 0$. For all $v \geq v^*$, it follows from the expression (B.62) that $\left.\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial s}\right|_{s=0} \leq 0$. Since $\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial s}$ is strictly decreasing in s, we conclude that for any $v \geq v^*$, $\widehat{T}(s)$ is strictly decreasing in s when $\Pi_T(\widehat{T}(s)) = \Pi_{\alpha,T}(\widehat{T}(s))$.

For all $v < v^*$, $\left.\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial s}\right|_{s=0} > 0$. Since $\frac{\partial^2 \Pi_{\alpha,T}}{\partial T \partial s}$ is strictly decreasing in s, we conclude that $\widehat{T}(s)$ is strictly increasing or unimodal in s if $\Pi_T(\widehat{T}(s)) = \Pi_{\alpha,T}(\widehat{T}(s))$.

We now show that for any $v < v^*$, if $\Pi_T(\widehat{T}(s)) = \Pi_{\alpha,T}(\widehat{T}(s))$, then $\widehat{T}(s)$ is unimodal in $s \in [0,p)$. If $s = 0$, because v is positive, then $\mathcal{R}_T \geq 0$ for any $T \geq 0$. Thus, $\widehat{T}(0) \geq 0$. If $s = p$, then RC won't make any profit by selling electricity to the distant load center. Consequently, RC would never use the transmission grid. Because RC would not use the transmission grid, TC has no incentive to invest in any transmission capacity; $\widehat{T}(p) = 0$. Consequently, it follows that $\widehat{T}(s)$ cannot be strictly increasing in s. Thus, we conclude that for $\Pi_T(\widehat{T}(s)) = \Pi_{\alpha,T}(\widehat{T}(s))$, $\widehat{T}(s)$ is unimodal in s.

For $\Pi_T(T) = \Pi_{\beta,T}(T)$, we use Proposition 12 to obtain that

$$\frac{\partial}{\partial s}\frac{\partial^2 \Pi_{\beta,T}}{\partial T \partial s} = -\frac{b_1 b_2 k_G^2 \lambda (b_2-T)\left[b_1 b_2 k_G \lambda (4p-s) + (p-s)(b_2-T)^2(2p+s)\right]}{(p-s)^3 \sqrt{\frac{2b_1 b_2 k_G \lambda}{p-s} + (b_2-T)^2}\,\left(2b_1 b_2 k_G \lambda + (p-s)(b_2-T)^2\right)^2}. \tag{B.61}$$

By (B.61), it follows that $\left.\frac{\partial}{\partial s}\frac{\partial^2 \Pi_{\beta,T}}{\partial T \partial s}\right|_{T=\widehat{T}} < 0$. Consequently, $\left.\frac{\partial^2 \Pi_T}{\partial T \partial s}\right|_{T=\widehat{T}}$ is strictly decreasing in s. Thus, for $\Pi_T(T) = \Pi_{\beta,T}(T)$, $\widehat{T}(s)$ is strictly increasing or unimodal or decreasing in s.

We now show that if $\Pi_T(T) = \Pi_{\beta,T}(T)$, then $\widehat{T}(s)$ is strictly increasing or unimodal in s. By Theorem 8 and Proposition 12, if $\Pi_T(T) = \Pi_{\alpha,T}(T)$ then $\widehat{T}(s) < f_2(b_1, b_2)$; if $\Pi_T(T) = \Pi_{\beta,T}(T)$ then $\widehat{T}(s) \geq f_2(b_1, b_2)$.

Because $\frac{\partial f_2(b_1,b_2)}{\partial s} = \frac{b_1 b_2 k_G \lambda}{\sqrt{b_2\left(b_2 - \frac{2b_1 k_G \lambda}{p-s}\right)}(p-s)^2} > 0$, $f_2(b_1, b_2)$ is strictly increasing in s. Consequently, because $\widehat{T}(s)$ is unimodal in s for $\Pi_T(T) = \Pi_{\alpha,T}(T)$, and $f_2(b_1, b_2)$ is strictly increasing in s, $\widehat{T}(s)$ should be strictly increasing or unimodal in s for $\Pi_T(T) = \Pi_{\beta,T}(T)$ (otherwise, $\widehat{T}(s) < f_2(b_1, b_2)$ and $\Pi_T(\widehat{T}(s)) = \Pi_{\alpha,T}(\widehat{T}(s))$ for any s, because $f_2(b_1, b_2)$ is strictly increasing in s). $\square$

Proof of Theorem 12

By envelope theorem we obtain

$$\frac{\partial \mathcal{W}(s)}{\partial s} = \left.\frac{\partial \Pi_T(s,T)}{\partial T}\right|_{T=\widehat{T}(s)} \cdot \frac{\partial \widehat{T}(s)}{\partial s} + \left.\frac{\partial \Pi_T(s,T)}{\partial s}\right|_{T=\widehat{T}}$$
$$= \left.\frac{\partial \Pi_T(s,T)}{\partial s}\right|_{T=\widehat{T}(s)}. \quad \text{(B.62)}$$

Using (3.17), (3.18) we get

$$\frac{\partial^2 \Pi_{\alpha,T}(s,T)}{\partial s^2} = -\frac{T^2 \left(2b_2 - T\right)^2}{6b_1 b_2^2 \lambda p} < 0;$$
$$\frac{\partial^2 \Pi_{\beta,T}(s,T)}{\partial s^2} = -\frac{b_1 b_2 k_G^2 \lambda \left[(p-s)(b_2-T)^2(2p+s+3v) + b_1 b_2 k_G \lambda (4p+s+5v)\right]}{(p-s)^4 \left(2b_1 b_2 k_G \lambda + (p-s)(b_2-T)^2\right) \sqrt{\frac{2b_1 b_2 k_G \lambda}{p-s} + (b_2-T)^2}} < 0.$$

Hence, for any $\widehat{T}(s)$, $\left.\frac{\partial^2 \Pi_T(s,T)}{\partial s^2}\right|_{T=\widehat{T}(s)} < 0$. Also, we use (3.17) to obtain that

$$\left.\frac{\partial \Pi_{\alpha,T}(s,T)}{\partial s}\right|_{s=0,\,T=\widehat{T}(0)}$$
$$= \frac{6b_1 b_2 (p - k_G \lambda)(2b_2 - \widehat{T}(0)) + \widehat{T}(0)\left[6b_2^2(p-2v) + 3\widehat{T}(0)^2(p-v) + 4b_2 \widehat{T}(0)(3v-2p)\right]}{12 b_1 b_2^2 \lambda p}. \quad \text{(B.63)}$$

By (B.63), if $v \geq \bar{v}$, then $\left.\frac{\partial \Pi_{\alpha,T}(s,T)}{\partial s}\right|_{s=0,T=\widehat{T}(0)} < 0$. Because $\frac{\partial^2 \Pi_T(s,\widehat{T}(s))}{\partial s^2} < 0$, we use (B.62) to get that $\mathcal{W}(s)$ is decreasing in s if $v \geq \bar{v}$ and unimodal in s otherwise. □

APPENDIX C

SUPPLEMENT TO CHAPTER 4

Proof of Lemma 3: By definition, for h_{cd} and c_{cd},

$$\left(\iota^*\left(\frac{h_{cd}}{c_{cd}}\right), N_T^*\left(\frac{h_{cd}}{c_{cd}}\right)\right) = \underset{(\iota, N_T)\geq 0}{\arg\min}\left(h_{cd}w\iota + c_{cd}N_T\right).$$

If $\frac{h_{u_1}}{c_{u_1}} = \frac{h_{u_2}}{c_{u_2}}$, then

$$\left(\iota^*\left(\frac{h_{u_1}}{c_{u_1}}\right), N_T^*\left(\frac{h_{u_1}}{c_{u_1}}\right)\right) = \underset{(\iota, N_T)\geq 0}{\arg\min}\left(h_{u_1}w\iota + c_{u_1}N_T\right) = \underset{(\iota, N_T)\geq 0}{\arg\min}\frac{h_{u_1}}{h_{u_2}}\left(h_{u_2}w\iota + c_{u_2}N_T\right).$$

Because $\frac{h_{u_1}}{h_{u_2}}$ is constant,

$$\underset{(\iota, N_T)\geq 0}{\arg\min}\frac{h_{u_1}}{h_{u_2}}\left(h_{u_2}w\iota + c_{u_2}N_T\right) = \underset{(\iota, N_T)\geq 0}{\arg\min}\left(h_{u_2}w\iota + c_{u_2}N_T\right) = \left(\iota^*\left(\frac{h_{u_2}}{c_{u_2}}\right), N_T^*\left(\frac{h_{u_2}}{c_{u_2}}\right)\right).$$

Therefore, if $\frac{h_{u_1}}{c_{u_1}} = \frac{h_{u_2}}{c_{u_2}}$, then $\left(\iota^*\left(\frac{h_{u_1}}{c_{u_1}}\right), N_T^*\left(\frac{h_{u_1}}{c_{u_1}}\right)\right) = \left(\iota^*\left(\frac{h_{u_2}}{c_{u_2}}\right), N_T^*\left(\frac{h_{u_2}}{c_{u_2}}\right)\right)$. □

We make use of the following intermediate results presented below as Lemmas 9 and 10.

Lemma 9. *For the cross-dock problem (CD-P),*

(a) $I_{t-1}^*\left(y_t^* \bmod \frac{W}{w}\right) = 0$ *for* $t = 1, 2 \ldots, T$;

(b) $y_t^* \leq \left\lceil \frac{wd_t}{W} \right\rceil \frac{W}{w}$ *for* $t = 1, 2 \ldots, T$.

Proof: Lemma 9(a) is Lemma 1 (Property 1) from Anily and Tzur (2005) (This result also appears in Lemma 1 of Anily and Tzur (2006) and Proposition 2.3 in Jin and Muriel (2009)). Part (a) states that if the initial inventory at the beginning of t (i.e., I_{t-1}^*) is positive, then the shipping quantity in that period consists of full containers only. Otherwise, the cross-dock can could reduce the total costs by delaying the shipment of some of the items that are carried from the previous period. Lemma 9(b) is Lemma 1 (Property 3) from Anily and Tzur (2005). Part (b) states that the maximum shipment size at any period never exceeds the total capacity of the minimum number of containers required to cover the demand in that period. Otherwise, the cross-dock would reduce the total cost by delaying that shipment of at least one container. □

Lemma 10. *For the cross-dock problem (CD-P),*

(a) $\lceil \frac{wD_\tau}{W} \rceil \leq N_\tau^* \leq \sum_{t=1}^{\tau} \lceil \frac{wd_t}{W} \rceil$ *for* $\tau = 1, 2, \ldots, T$.

(b) If $\sigma_{cd} < \frac{1}{W}$*, then* $N_T^* = \left\lceil \frac{wD_T}{W} \right\rceil$.

Proof: $N_\tau^* \leq \sum_{t=1}^{\tau} \lceil \frac{wd_t}{W} \rceil$, because otherwise there is at least one container whose entire shipment is for future periods $\tau+1,\ \tau+2,\ \ldots, T$. This contradicts Lemma 9(b). Also, the total number of containers shipped by τ should be greater than or equal to $\lceil \frac{wD_\tau}{W} \rceil$ because the demand D_τ cannot be satisfied otherwise. Thus, we obtain Part (a).

To show Part (b), note that $N_T \geq \left\lceil \frac{wD_T}{W} \right\rceil$ for $T = 1, 2, \ldots, T$ by Part (a). Assume $N_T^* = \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil + a$ when $h_{cd} < \frac{c_{cd}}{W}$, where a is a positive integer. Let $\iota = \sum_{t=1}^{T} I_t^*$ denote the total inventory over the planning horizon associated with $N_T^* = \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil + a$. Let $\iota' = \sum_{t=1}^{T} I_t'$ be the minimum inventory over the planning horizon if the total number of containers shipped is $N_T = \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil$. Then, $\iota' - \iota \leq \frac{aW}{w}$ because the difference cannot exceed the total capacity of a containers. If $N_T^* = \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil + a$, then the total cost over the planning horizon is $h_{cd}\, w\, \iota + c_{cd} \left\{ \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil + a \right\}$. Then, $h_{cd}\, w\, \iota + c_{cd} \left\{ \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil + a \right\} \geq h_{cd}\, w\, (\iota' - \frac{aW}{w}) + c_{cd} \left\{ \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil + a \right\} > h_{cd}\, w\, \iota' + c_d \left\{ \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil \right\}$ (The last inequality follows from $h_{cd} < \frac{c_{cd}}{W}$). Thus, N_T^* is not optimal which contradicts the assumption. Therefore, $N_T^* = \sum_{t=1}^{T} \left\lceil \frac{wD_T}{W} \right\rceil$ when $h_{cd} < \frac{c_{cd}}{W}$. □

Proof of Lemma 4: For Problem CD-P, it follows directly Lemma 10 that $\left\lceil \frac{wD_T}{W} \right\rceil \leq N_T^* \leq \sum_{t=1}^{T} \lceil \frac{wd_t}{W} \rceil$. Also, N_τ^* is decreasing in the container shipping cost c_{cd}, and is increasing in the unit holding cost h_{cd}. Hence, N_T^* is increasing in $\sigma_{cd} = \frac{h_{cd}}{c_{cd}}$. Because N_T^* is integer, $N_T(\sigma_{cd})$ is a piece-wise constant, increasing function of σ_{cd}. Because $\left\lceil \frac{wD_T}{W} \right\rceil \leq N_T^* \leq \sum_{t=1}^{T} \lceil \frac{wd_t}{W} \rceil$, there are $\sum_{t=1}^{T} \lceil \frac{wd_t}{W} \rceil - \left\lceil \frac{wD_T}{W} \right\rceil$ discontinuities. □

Proof of Theorem 13: By Lemma 10(b), if $\sigma_{cd} < \frac{1}{W}$, then $N_T^*(\sigma_{cd}) = \left\lceil \frac{wD_T}{W} \right\rceil$. The total inventory $\iota^*(\sigma_{cd}) = \iota_0$ follows from $N_T^*(\sigma_{cd}) = \left\lceil \frac{wD_T}{W} \right\rceil$, which proves part (a).

To prove part (b), define set $\mathcal{I} = \left\{ 1, 2, \ldots, \sum_{t=1}^{T} \lceil \frac{wd_t}{W} \rceil - \left\lceil \frac{wD_T}{W} \right\rceil + 1 \right\}$. For $i \in \mathcal{I}$, define set $\Sigma^{[i]} = \left\{ \sigma_{cd} \in \{0, \infty\} \,|N_T(\sigma_{cd}) = \left\lceil \frac{wD_T}{W} \right\rceil - 1 + i \right\}$. Then, for any $i, j \in \mathcal{I}$ $(i \neq j)$, $\Sigma^{[i]} \cap \Sigma^{[j]} = \emptyset$. Also, $\Sigma^{[1]} \cup \Sigma^{[2]} \cup \Sigma^{[3]} \cup \ldots \cup \Sigma^{[|\mathcal{I}|]} = [0, \infty)$ by Lemma 4 ($|\mathcal{I}|$ refers to the cardinality of a set). We establish the following:

[1] For any $i \in \mathcal{I}$, if $\sigma_1,\ \sigma_2 \in \Sigma^{[i]}$ then $\iota^*(\sigma_1) = \iota^*(\sigma_2)$.

[2] For any $i = 1, 2, \ldots, |\mathcal{I}| - 1$, if $\sigma_1 \in \Sigma^{[i]}$ and $\sigma_2 \in \Sigma^{[i+1]}$, then $\iota^*(\sigma_1) > \iota^*(\sigma_2)$

To prove [1], for $\sigma_1, \sigma_2 \in \Sigma^{[i]}$, $(\sigma_1 \neq \sigma_2)$, suppose $\iota^*(\sigma_1) > \iota^*(\sigma_2)$. By definition, $N_T(\sigma_1) = N_T(\sigma_2)$. Thus, $\mathbf{f}(\sigma_1) > \mathbf{f}(\sigma_2)$. Because $\mathbf{f}(\sigma_1) = h_{cd}\iota^*(\sigma_1) + c_{cd}N_T(\sigma_1) > h_{cd}\iota^*(\sigma_2) + c_{cd}N_T(\sigma_2) = h_{cd}\iota^*(\sigma_2) + c_{cd}N_T(\sigma_1)$. This contradicts the assumption that $\iota^*(\sigma_1)$ is an optimal solution for $\sigma_{cd} = \sigma_1$. Suppose now $\iota^*(\sigma_1) < \iota^*(\sigma_2)$. Because $N_T(\sigma_1) = N_T(\sigma_2)$, we obtain that $\mathbf{f}(\sigma_1) < \mathbf{f}(\sigma_2)$. Because $\mathbf{f}(\sigma_2) = h_{cd}\iota^*(\sigma_2) + c_{cd}N_T(\sigma_2) > h_{cd}\iota^*(\sigma_1) + c_{cd}N_T(\sigma_1) = h_{cd}\iota^*(\sigma_1) + c_{cd}N_T(\sigma_2)$. This contradicts the assumption that $\iota^*(\sigma_2)$ is an optimal solution for $\sigma_{cd} = \sigma_2$.

For [2], suppose $\iota^*(\sigma_1) < \iota^*(\sigma_2)$. By definition, if $\sigma_1 \in \Sigma^{[i]}$ and $\sigma_2 \in \Sigma^{[i+1]}$ then $N_T(\sigma_1) < N_T(\sigma_2)$. Therefore, $\iota^*(\sigma_1)$ is feasible for $\sigma_{cd} = \sigma_2$. Thus, because $\mathbf{f}(\sigma_2) = h_{cd}\iota^*(\sigma_2) + c_{cd}N_T(\sigma_2) > h_{cd}\iota^*(\sigma_1) + c_{cd}N_T(\sigma_2)$, it contradicts the assumption that $\iota^*(\sigma_2)$ is an optimal solution for $\sigma_{cd} = \sigma_2$.

Because $N_T(\sigma_{cd})$ is an integer and $N_T(\sigma_{cd}) \in \left[\left\lceil \frac{wD_T}{W} \right\rceil, \sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil\right]$, $N_T^*(\sigma_{cd})$ is discontinuous at $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - \left\lceil \frac{wD_T}{W} \right\rceil$ points. Therefore, by [1], [2], $\iota^*(\sigma_{cd})$ is continuous in $\sigma_{cd} \geq 0$ almost everywhere, except at $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - \left\lceil \frac{wD_T}{W} \right\rceil$ points, which completes the proof of part (b).

To prove part (c), note that $N_T(\sigma_{cd})$ is an integer and $N_T(\sigma_{cd}) \in \left[\left\lceil \frac{wD_T}{W} \right\rceil, \sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil\right]$. The total number of containers $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil$, which is an upper bound of $N_T(\sigma_{cd})$, are shipped only if the corresponding total cost $\mathcal{C}_{CD}$ is less than that of shipping one less container (i.e., $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - 1$) and holding some inventory over the planning horizon. If the total cost $\mathcal{C}_{CD}$ associated with $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil$ containers shipped is always less than $\mathcal{C}_{CD}$ associated with $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - 1$ containers shipped, then

$$\underbrace{\left\{\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil\right\} c_{cd}}_{(i)} \leq \underbrace{\left\{\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - 1\right\} c_{cd}}_{(ii)}$$
$$+ \underbrace{\sum_{t=1}^{T-1} \left(\left\lceil \frac{wD_t}{W} \right\rceil \frac{W}{w} - D_t \right) h_{cd}\, w - \left(\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - \left\lceil \frac{wD_T}{W} \right\rceil \right) c_{cd}}_{(iii)}. \tag{C.1}$$

Teerm (i) in (C.1) is the total cost over the planning horizon when the total number of containers shipped is $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil$. Term (i) does not include any inventory holding cost. The term (ii) represents the total transportation cost when $\sum_{t=1}^{T} \left\lceil \frac{wd_t}{W} \right\rceil - 1$ containers are shipped to the cross-dock. Finally, the term (iii), $\sum_{t=1}^{T-1} \left(\left\lceil \frac{wD_t}{W} \right\rceil \frac{W}{w} - D_t \right)$ is the total inventory when the number of total containers shipped is $\left\lceil \frac{wD_T}{W} \right\rceil$.

The next term $\left(\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-\left\lceil\frac{wD_T}{W}\right\rceil\right)c_{cd}$ refers to maximum value of inventory reduced by shipping $\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil$ instead of shipping $\left\lceil\frac{wD_T}{W}\right\rceil$ containers. Accordingly, the sum of equations (ii), (iii) represent the total transportation and inventory holding cost, when total containers shipped is $\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-1$. We can rewrite (C.1) as

$$\left(\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-\left\lceil\frac{wD_T}{W}\right\rceil+1\right)c_{cd}\le h_{cd}\,w\,\iota_0.$$

Using $\sigma_{cd}=\frac{h_{cd}}{c_{cd}}$, we obtain $\sigma_{cd}\ge\dfrac{\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-\left\lceil\frac{wD_T}{W}\right\rceil+1}{w\,\iota_0}$. □

Proof of Theorem 14: By definition, $\mathbf{f}(\sigma_{cd})=h_{cd}w\iota^*(\sigma_{cd})+c_{cd}N_t^*(\sigma_{cd})=c_{cd}\big(\sigma_{cd}w\iota^*(\sigma_{cd})+N_t^*(\sigma_{cd})\big)$. By Lemma 4(b) and Theorem 13(b), both N_T^* and ι^* are piece-wise constant functions of σ_{cd}, we obtain $\frac{\partial N_T^*}{\partial\sigma_{cd}}=0$ and $\frac{\partial\iota^*}{\partial\sigma_{cd}}=0$ almost everywhere except at $\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-\left\lceil\frac{wD_T}{W}\right\rceil$ points. Therefore,

$$\frac{\partial\mathbf{f}(\sigma_{cd})}{\partial\sigma_{cd}}=c_{cd}\sum_{t=1}^{T}w\,I_t^*=c_{cd}\,w\,\iota^*(\sigma_{cd})+c_{cd}\left(\sigma_{cd}w\frac{\partial\iota^*}{\partial\sigma_{cd}}+\frac{\partial N_T^*}{\partial\sigma_{cd}}\right)=c_{cd}\,w\,\iota^*(\sigma_{cd}).\qquad\text{(C.2)}$$

almost everywhere. By (C.2), $\mathbf{f}(\sigma_{cd})$ is a piecewise-linear and increasing function of σ_{cd} because $\iota^*(\sigma_{cd})$ is a piece-wise constant decreasing function of σ_{cd} by Theorem 13(b). Further, if $\sigma_{cd}\ge\sigma_{cd}^*$, then by Theorem 13(c), $\iota^*(\sigma_{cd})=0$. It follows that $\frac{\partial\mathbf{f}}{\partial\sigma_{cd}}=0$ everywhere except at $\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-\left\lceil\frac{wD_T}{W}\right\rceil$ points.

Proof of Theorem 15: In proving this, we make use of Theorem 13(c) and (4.15). For a supply chain with upstream collaboration, by replacing d_t with $\sum_{i=1}^{Q}d_{it}$ and D_T with $\sum_{i=1}^{Q}D_{iT}$ for $\frac{\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-\left\lceil\frac{wD_T}{W}\right\rceil+1}{w\,\iota_0}$ using Theorem 13(c) and (4.15), we obtain that, if

$$\sigma_{cd}\ge\frac{\sum_{t=1}^{T}\left\lceil\sum_{i=1}^{Q}\frac{wd_{it}}{W}\right\rceil-\left\lceil\sum_{i=1}^{Q}\frac{wD_{iT}}{W}\right\rceil+1}{w\sum_{t=1}^{T-1}\left(\left\lceil\sum_{i=1}^{Q}\frac{wD_{it}}{W}\right\rceil\frac{W}{w}-\sum_{i=1}^{Q}D_{it}\right)},$$

then the total inventory for the cross-dock is zero, and the number of total containers shipped is $N_T^*=\sum_{t=1}^{T}\left\lceil\sum_{i=1}^{Q}\frac{wd_{it}}{W}\right\rceil$. Thus, we define σ_{uc} as

$$\sigma_{uc}:=\frac{\sum_{t=1}^{T}\left\lceil\sum_{i=1}^{Q}\frac{\lambda_{it}}{W}\right\rceil-\left\lceil\sum_{i=1}^{Q}\frac{\Lambda_{iT}}{W}\right\rceil+1}{\sum_{t=1}^{T-1}\left(\left\lceil\sum_{i=1}^{Q}\frac{\Lambda_{it}}{W}\right\rceil W-\sum_{i=1}^{Q}\Lambda_{it}\right)}.$$

For a supply chain without collaboration, by replacing d_t with d_{it} and D_T with D_{iT} in $\frac{\sum_{t=1}^{T}\left\lceil\frac{wd_t}{W}\right\rceil-\left\lceil\frac{wD_T}{W}\right\rceil+1}{w\,\iota_0}$

using Theorem 13(c) and (4.15), we obtain that, if $\sigma_{cd} \geq \max_i \sigma_i$, then the total inventory at the cross-dock is zero and the number of total containers shipped is $N_T^* = \sum_{i=1}^{Q} \sum_{t=1}^{T} \left\lceil \frac{wd_{it}}{W} \right\rceil$. Next, it is straightforward to show that $\min_i \sigma_i \leq \sigma_{uc} \leq \max_i \sigma_i$.

Hence, if $\sigma_{cd} \geq \max_i \sigma_i$, then the cross-dock's cost difference with and without upstream collaboration is exactly equal to the transportation cost difference with and without upstream collaboration. Thus, we obtain $\Delta_{CD} := \mathcal{C}_{CD}^* - \mathcal{C}_{CDuc}^* = \left\{ \sum_{i=1}^{Q} \sum_{t=1}^{T} \left\lceil \frac{wd_{it}}{W} \right\rceil - \sum_{t=1}^{T} \left\lceil \sum_{i=1}^{Q} \frac{wd_{it}}{W} \right\rceil \right\} c_{cd}$. $\square$

Proof of Theorem 16: To prove part (a), observe that the total amount of shipment required at the plant by τ is $\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}$. It follows that the total number of trucks shipped by τ is at least $\left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil$ to fulfill $\Phi_{i\tau}$ by τ without backlogging. We now show that if $h < \frac{c_d}{U}$, $V_\tau = \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil$ for $\tau = 1, 2, \ldots, T$. Suppose there is at least one period τ, in which $\left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil + a$ trucks were shipped by τ, where a is a positive integer. Let $\varphi(\tau)$ denotes the total amount of inventory by τ (i.e., $\varphi(\tau) = \sum_{i=1}^{Q} \sum_{t=1}^{\tau} w_i \, \phi_{it}$) for the case $V_\tau = \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil + a$. Also, let $\varphi(\tau)'$ denote the total amount of inventory by τ for the case $V_\tau = \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil$. By definition, $\varphi(\tau)' - \varphi(\tau) \leq a\,U$. Then,

$$h\varphi(\tau)+c_d \left\{ \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil + a \right\} \geq h(\varphi(\tau)'-aU)+c_d \left\{ \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil + a \right\} > h\varphi(\tau)'+c_d \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil ,$$

where the last inequality holds because $h < \frac{c_d}{U}$. Thus, $V_\tau = \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil + a$ is not optimal as it obtains a higher cost relative to $V_\tau = \left\lceil \frac{\sum_{i=1}^{Q} w_i \, \Phi_{i\tau}}{U} \right\rceil$ for $\tau = 1, \ldots, T$. This completes the proof of part (a).

To prove part (b), observe that, since $V_\tau \geq \left\lceil \frac{\sum_{i=1}^{Q} w_i \Phi_{i\tau}}{U} \right\rceil$ from Part (a), then it follows that $\sum_{i=1}^{Q} w_i \phi_{it} < U$, for any inventory matrix $\boldsymbol{\phi} = \{\phi_{it}\}$ associated with the optimal shipping plan. Suppose there exists $v_\tau > \left\lceil \frac{\sum_{i=1}^{Q} w_i \, q_{i\tau}}{U} \right\rceil$ in the optimal solution. Since $\left\lceil \frac{\sum_{i=1}^{Q} w_i \, q_{i\tau}}{U} \right\rceil U - \sum_{i=1}^{Q} q_{i\tau} \geq 0$, then

$$\sum_{i=1}^{Q} x_{i\tau} - \left(\sum_{i=1}^{Q} q_{i\tau} - \sum_{i=1}^{Q} \phi_{i,\tau-1} \right)^{+} > \left(\sum_{i=1}^{Q} x_{i\tau} \right) mod\, U,$$

if $(\sum_{i=1}^{Q} x_{i\tau})\, mod\, U > 0$. Then, v_τ is not optimal since the entire shipment of at least one vehicle can be delayed and reduce inventory holding cost in τ. If $(\sum_{i=1}^{Q} x_{i\tau})\, mod\, U = 0$, it follows that $\sum_{i=1}^{Q} \phi_{i\tau} = U\, v_\tau - \sum_{i=1}^{Q} q_{it} > U$. This contradicts $\sum_{i=1}^{Q} w_i \phi_{it} < U$. Thus, v_τ is not optimal. Therefore, $v_\tau \leq \left\lceil \frac{\sum_{i=1}^{Q} w_i \, q_{it}}{U} \right\rceil$ and $V_\tau = \sum_{t=1}^{\tau} v_t \leq \sum_{t=1}^{\tau} \left\lceil \frac{\sum_{i=1}^{Q} w_i \, q_{it}}{U} \right\rceil$ at the optimality. $\square$

Proof of Theorem 17:

For a supply chain with downstream collaboration, by replacing d_t with $\sum_{m=1}^{M}\sum_{i=1}^{Q} q_{it}^{(m)}$ and D_T with $\sum_{m=1}^{M}\sum_{i=1}^{Q}\Phi_{iT}^{(m)}$ using Theorem 13(c) and (4.15), we obtain that, if

$$\sigma_r \geq \frac{\sum_{t=1}^{T}\left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil - \left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, \Phi_{iT}^{(m)}}{U}\right\rceil + 1}{\sum_{\tau=1}^{T-1}\left(\left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, \Phi_{i\tau}^{(m)}}{U}\right\rceil U - \sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, \Phi_{i\tau}^{(m)}\right)}, \tag{C.3}$$

then the total inventory for plants is zero and the number of total containers shipped to plants is $\sum_{t=1}^{T}\left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil$. Thus, we define ϑ_{dc} as

$$\vartheta_{dc} := \frac{\sum_{t=1}^{T}\left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil - \left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, \Phi_{iT}^{(m)}}{U}\right\rceil + 1}{\sum_{\tau=1}^{T-1}\left(\left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, \Phi_{i\tau}^{(m)}}{U}\right\rceil U - \sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, \Phi_{i\tau}^{(m)}\right)}.$$

For a supply chain without downstream collaboration, we first replace d_t with $\sum_{i=1}^{M} q_{it}^{(m)}$ and D_T with $\sum_{i=1}^{Q}\Phi_{iT}^{(m)}$, $m \in \{1,\ldots,M\}$ using Theorem 13(c) and (4.15). Therefore, we obtain that, if $\sigma_r \geq \max_{m} \vartheta_m$ plants hold no inventory over the planning horizon and the number of total containers shipped to each plant m is $\sum_{t=1}^{T}\left\lceil \frac{\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil$. It is straightforward that $\min_{i}\ \vartheta_m \leq \vartheta_{dc} \leq \max_{i}\ \vartheta_m$.

Hence, we obtain that if $\sigma_r \geq \max_i\ \vartheta_m$, then the plants' cost difference with and without downstream collaboration is identical to the difference in the transportation cost for those plants with and without downstream collaboration. Therefore,

$$\mathcal{C}_R^{(m)*} - \sum_{j=1}^{m}\mathcal{C}_R^{(j)*} = \left\{\sum_{t=1}^{T}\sum_{m=1}^{M}\left\lceil \frac{\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil - \sum_{t=1}^{T}\left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil\right\} c_d.$$

Therefore, if it is true that

$$\sum_{t=1}^{T}\sum_{m=1}^{M}\left\lceil \frac{\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil = \sum_{t=1}^{T}\left\lceil \frac{\sum_{m=1}^{M}\sum_{i=1}^{Q} w_i\, q_{it}^{(m)}}{U}\right\rceil,$$

then $\mathcal{C}_{R_{dc}}^{(m)*} = \sum_{j=1}^{m}\mathcal{C}_R^{(j)*}$.

Let $x_{it}^{(j)*}$ and $v_t^{(j)*}$ ($j = 1, 2, \ldots, m$) be the optimal solutions for Problem R-P. Then, because $\sum_{j=1}^{m} x_{it}^{(j)*}$ and $\sum_{j=1}^{m} v_t^{(j)*}$ are feasible solutions for Problem R-P$_{dc}$, we obtain $\mathcal{C}_{R_{dc}}^{(m)*} \leq \sum_{j=1}^{m}\mathcal{C}_R^{(j)*}$. □

Proof of Theorem 18: If $p > p^{(m)}$, then it follows from the definition of $p^{(m)}$ given in (4.17) and the definition of $\Delta^{(m)}$ given in (4.16) that $\Delta^{(m)}(p,1) < 0$. Because $\Delta^{(m)}(p,1) < 0$, the net benefit to plant m from outsourcing is negative at any price $p > p^{(m)}$. Consequently, it is optimal for the plant not to outsource, so that $\chi_m^*(p) = 0$ for any $p > p^{(m)}$. If $p \leq p^{(m)}$, then it again follows from (4.16) that $\Delta^{(m)}(p,1) \geq 0$, and the net benefit to plant m for outsourcing is positive. Thus, it is optimal for the plant to outsource, so that $\chi_m^*(p) = 1$ for any $p \leq p^{(m)}$. This establishes part (a). If M plants are ordered so that $p^{(1)} \geq p^{(2)} \geq \cdots \geq p^{(M)}$, then part (b) follows directly from part (a). □

Proof of Theorem 19: Let $p = p^{(1)}$. Then, by Theorem 18, only plant 1 outsources to the logistics provider. Let $\mathcal{C}_R^{(1)*}$ be plant 1's optimal cost in a decentralized supply chain under deterministic demand and without outsourcing. Then, by (4.8),

$$\mathcal{C}_R^{(1)*} := \mathcal{C}_R^{(1)}\left(\mathbf{v}^{(1)*}, \mathbf{x}^{(1)*}\right) = \sum_{t=1}^{T}\sum_{i=1}^{Q} hw_i\phi_{it}^{(1)*} + \sum_{t=1}^{T} c_d v_t^{(1)*}, \tag{C.4}$$

where $\mathbf{v}^{(1)*}$ and $\mathbf{x}^{(1)*}$ are optimal shipment vector and transportation quantity matrix for plant 1, while quantities $\{\phi_{it}^{(1)*}\}$ are inventory levels of product i in period t associated with $\mathbf{v}^{(1)*}$ and $\mathbf{x}^{(1)*}$.

Let $\mathcal{C}_{lp}^*(p^{(1)})$ denote the logistics provider's optimal cost when outsourcing price is $p^{(1)}$. Then, by (4.18), it follows from Theorem 6 that

$$\mathcal{C}_{lp}^*(p^{(1)}) := \mathcal{C}_{lp}(\mathbf{v}_{lp}^*, \mathbf{x}_{lp}^*, p^{(1)}) = \sum_{t=1}^{T}\sum_{i=1}^{Q} h_{lp}w_i\phi_{it}^{(lp)*} + \sum_{t=1}^{T} c_{lp} v_t^{(lp)*},$$

where $\mathbf{v}_{lp}^*$ and $\mathbf{x}_{lp}^*$ are optimal shipment vector and transportation quantity matrix for the logistics provider at outsourcing price $p^{(1)}$, while quantities $\{\phi_{it}^{(lp)*}\}$ are inventory levels of product i in period t associated with $\mathbf{v}_{lp}^*$ and $\mathbf{x}_{lp}^*$. Next observe that with $m = 1$, the inventory balance equations for the logistics provider's cost minimization problem (Problem LP-P) given in (4.20) become

$$\phi_{it}^{(lp)*} = \phi_{i,t-1}^{(lp)*} + x_{it}^{(lp)*} - q_{it}^{(1)},$$

for $i = 1, \ldots, Q$ and $t = 1, \ldots, T$. Those inventory balance equations are identical to inventory balance equations for plant 1's cost minimization problem in a decentralized supply chain (Problem R-P) given in (4.10). As all other constraints for Problem LP-P are identical to their counterpart constraints for Problem R-P, the decisions $\mathbf{v}^{(1)*}$ and $\mathbf{x}^{(1)*}$, optimal for plant 1 in a decentralized supply chain, are feasible for the logistics provider under outsourcing price $p^{(1)}$. It follows that

$$
\begin{aligned}
\mathcal{C}^*_{lp}\big(p^{(1)}\big) &= \mathcal{C}_{lp}\big(\mathbf{v}^*_{lp}, \mathbf{x}^*_{lp}, p^{(1)}\big) && \big[\text{definition of } \mathcal{C}^*_{lp}\big(p^{(1)}\big)\big] \\
&\le \mathcal{C}_{lp}\big(\mathbf{v}^{(1)*}, \mathbf{x}^{(1)*}, p^{(1)}\big) && \big[\big(\mathbf{v}^*_{lp}, \mathbf{x}^*_{lp}\big) \text{ are optimal; } \big(\mathbf{v}^{(1)*}, \mathbf{x}^{(1)*}\big) \text{ are feasible for the provider}\big] \\
&= \sum_{t=1}^{T}\sum_{i=1}^{Q} h_{lp} w_i \phi^{(1)*}_{it} + \sum_{t=1}^{T} c_{lp} v^{(1)*}_t && \big[\text{definition of } \mathcal{C}_{lp}\big(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p^{(1)}\big) \text{ given in (4.18)}\big] \\
&< \sum_{t=1}^{T}\sum_{i=1}^{Q} h w_i \phi^{(1)*}_{it} + \sum_{t=1}^{T} c_d v^{(1)*}_t && \big[h_{lp} < h \text{ and } c_{lp} < c_d \text{ by Assumption 6}\big] \\
&= \mathcal{C}^{(1)*}_R. && \big[\text{definition of } \mathcal{C}^{(1)*}_R \text{ given in (C.4) }\big]
\end{aligned}
$$

Thus, $\mathcal{C}^*_{lp}(p^{(1)}) < \mathcal{C}^{(1)*}_R$. Also, $p^{(1)} \sum_{t=1}^{T} \sum_{i=1}^{Q} w_i q^{(1)}_{it} = \mathcal{C}^{(1)*}_R$ by definition of $p^{(m)}$ given in (4.17). It follows that $\Pi^*_{lp}\left(p^{(1)}\right) = p^{(1)} \sum_{t=1}^{T} \sum_{i=1}^{Q} w_i\, q^{(1)}_{it} - \mathcal{C}^*_{lp}\left(p^{(1)}\right) > 0$, which completes the proof of part (a).

To prove part (b), let p be some outsourcing price charged by the logistics provider such that $\Pi^*_{lp}(p) > 0$. Then by part (a), $p^{(1)} \ge p$ for otherwise no plant would outsource and the logistics provider's profit would be zero. Let $m \in \{1, 2, \ldots, M\}$ be such that $p^{(m)} \ge p > p^{(m+1)}$. Then by Theorem 18, the number of plants outsourcing is m and $\mathcal{C}^*_{lp}(p) = \mathcal{C}^*_{lp}(p^{(m)})$. Thus, for any $p^{(m)} \ge p > p^{(m+1)}$, $\mathcal{C}^*_{lp}(p)$ is constant. Therefore, by (4.22), we obtain that, for any $p^{(m)} \ge p > p^{(m+1)}$,

$$
\Pi^*_{lp}\big(p^{(m)}\big) - \Pi^*_{lp}(p) = \Big(p^{(m)} - p\Big) \sum_{j=1}^{m}\sum_{t=1}^{T}\sum_{i=1}^{Q} w_i q^{(j)}_{it} > 0.
$$

It follows that $p^* = p^{(m)}$ for some $m \in \{1, 2, \ldots, M\}$.

To prove (c), suppose, by part (a), that $p^* = p^{(m)}$ for some $m \in \{1, \ldots, M\}$. Let $\mathcal{C}^{(m)*}_{R_{dc}}$ be the total optimal cost of plants 1 through m under downstream collaboration. Then, it follows from (4.12) that

$$
\mathcal{C}^{(m)*}_{R_{dc}} = \mathcal{C}^{(m)}_{R_{dc}}(\mathbf{v}^*, \mathbf{x}^*) = \sum_{t=1}^{T}\sum_{i=1}^{Q} h\, w_i\, \phi^*_{it} + \sum_{t=1}^{T} c_d\, v^*_t, \tag{C.5}
$$

where $\mathbf{v}^*$ and $\mathbf{x}^*$ are optimal shipment vector and transportation quantity matrix for plants 1 through m under downstream collaboration, while quantities $\{\phi^*_{it}\}$ denote inventory levels of product i in each period t associated with $\mathbf{v}^*$ and $\mathbf{x}^*$. It also follows from the formulation of the plants' problem under downstream collaboration (Problem R-P$_{dc}$) given in Section 4.3.4 that the resulting inventory balance equations stated in (4.13) now become

$$\phi_{it}^{*} = \phi_{i,t-1}^{*} + x_{it} - \sum_{j=1}^{m} q_{it}^{(j)}, \tag{C.6}$$

for $i = 1, 2 \ldots, Q$ and $t = 1, \ldots, T$.

Let $\mathcal{C}_{lp}^{*}\big(p^{(m)}\big)$ denote the logistics provider's optimal cost when the outsourcing price is $p^{(m)}$. Then, by (4.18), it follows from Theorem 6 that

$$\mathcal{C}_{lp}^{*}\big(p^{(m)}\big) := \mathcal{C}_{lp}\big(\mathbf{v}_{lp}^{*}, \mathbf{x}_{lp}^{*}, p^{(m)}\big) = \sum_{t=1}^{T}\sum_{i=1}^{Q} h_{l_p} w_i \phi_{it}^{(lp)*} + \sum_{t=1}^{T} c_{l_p} v_t^{(lp)*},$$

where, this time, $\mathbf{v}_{lp}^{*}$ and $\mathbf{x}_{lp}^{*}$ denote optimal shipment vector and transportation quantity matrix for the logistics provider at outsourcing price $p^{(m)}$, while quantities $\{\phi_{it}^{(lp)*}\}$ are inventory levels of product i in period t associated with $\mathbf{v}_{lp}^{*}$ and $\mathbf{x}_{lp}^{*}$. Further, with m outsourcing plants, the inventory balance equations for the logistics provider's cost minimization problem (Problem LP-P) given in (4.20) become

$$\phi_{it}^{(lp)*} = \phi_{i,t-1}^{(lp)*} + x_{it}^{(lp)*} - \sum_{j=1}^{m} q_{it}^{(j)} \tag{C.7}$$

Note that the inventory balance equations for the logistics provider with m outsourcing plants given in (C.7) are structurally identical to the inventory balance equations given in (C.6) for m collaborating plants. Therefore, because all other constraints for Problem LP-P are identical to their counterpart constraints for Problem R-P$_{dc}$, it follows that the decisions $\mathbf{v}^{*}$ and $\mathbf{x}^{*}$ optimal for m plants under downstream collaborations are feasible for the logistics provider under outsourcing price $p^{(m)}$.

Consequently, we have that

$$
\begin{aligned}
\mathcal{C}^*_{lp}\big(p^{(m)}\big) &= \mathcal{C}_{lp}\big(\mathbf{v}^*_{lp}, \mathbf{x}^*_{lp}, p^{(m)}\big) && \big[\text{definition of } \mathcal{C}^*_{lp}\big(p^{(m)}\big)\big] \\
&\le \mathcal{C}_{lp}\big(\mathbf{v}^*, \mathbf{x}^*, p^{(m)}\big) && \big[\big(\mathbf{v}^*_{lp}, \mathbf{x}^*_{lp}\big) \text{ are optimal; } \big(\mathbf{v}^*, \mathbf{x}^*\big) \text{ are feasible for the provider}\big] \\
&= \sum_{t=1}^{T}\sum_{i=1}^{Q} h_{l_p} w_i \phi^*_{it} + \sum_{t=1}^{T} c_{l_p} v^*_t && \big[\text{definition of } \mathcal{C}_{lp}\big(\mathbf{v}_{lp}, \mathbf{x}_{lp}, p^{(m)}\big) \text{ given in (4.18)}\big] \\
&< \sum_{t=1}^{T}\sum_{i=1}^{Q} h w_i \phi^{(1)*}_{it} + \sum_{t=1}^{T} c_d v^{(1)*}_t && \big[h_{l_p} < h \text{ and } c_{l_p} < c_d \text{ by Assumption 6}\big] \\
&= \mathcal{C}^{(m)*}_{R_{dc}}. && \big[\text{expression (C.5) }\big]
\end{aligned}
$$

Hence, $\mathcal{C}^*_{lp}(p^{(m)}) < \mathcal{C}^{(m)*}_{R_{dc}}$. It then follows from Theorem 17 that $\mathcal{C}^*_{lp}(p^{(m)}) < \mathcal{C}^{(m)*}_{R_{dc}} \le \sum_{j=1}^{(m)} \mathcal{C}^{(j)*}_{R}$, where $\mathcal{C}^{(j)*}_{R_{dc}}$ is plant j's optimal cost in a fully decentralized supply chain. □

Proof of Theorem 20

If $p > p^{(m)}_Z$ then by (4.31), $\Delta^{(m)}_Z < 0$. Consequently, the total cost saving from outsourcing is negative for the plant m, thus, it is optimal for the plant not to outsource. By definition given in (4.32), $\chi^*_{m_Z} = 0$. If $p \le p^{(m)}_Z$, then by (4.31), $\Delta^{(m)}_Z \ge 0$ (i.e., total cost saving is positive). Thus, it is optimal for the plant to outsource. By definition given in (4.32), $\chi^*_{m_Z} = 0$. □