EXTENSION PHENOMENA OF INTEGRABLE HOLOMORPHIC FUNCTIONS IN REINHARDT DOMAINS OF HOLOMORPHY

A Dissertation

by

JOSEPH LAWRENCE TORRES

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Harold P. Boas
Committee Members,	Emil J. Straube
	J. Maurice Rojas
	Andreas Kronenberg
Head of Department,	Sarah Witherspoon

August 2020

Major Subject: Mathematics

ABSTRACT

Holomorphic functions of several complex variables showcase many interesting extension phenomena which have historically motivated much of the development of the discipline. The purpose of this thesis is to explore the extension phenomena of integrable holomorphic functions, an important subclass of the holomorphic functions. We give two classification theorems for two-dimensional Reinhardt L^1_h-domains of holomorphy, as well as two partial results towards classifying n-dimensional Reinhardt L^1_h-domains of holomorphy. Both classification theorems for the two-dimensional domains are geometric classifications in terms of elementary Reinhardt domains. The first gives a classification in terms of monomial inequality representations of elementary Reinhardt domains, while the second gives a classification in terms of a parameterization of such domains by points on the unit circle. While we did not achieve a complete classification of n-dimensional domains, we demonstrate that all bounded Reinhardt domains of holomorphy are themselves L^1_h-domains of holomorphy. Furthermore, while fat L^1_h-domains of holomorphy have been characterized via functional analysis in the past, we provide a geometric characterization of such domains in terms of elementary Reinhardt domains.

DEDICATION

Ad Corda Sacratissima Verbi Incarnati et Sedis Sapientiae

ACKNOWLEDGMENTS

In a letter to Robert Hooke, Sir Isaac Newton famously said, “If I have seen further it is by standing on the shoulders of Giants.” If even this great giant of scientific and mathematical thought saw the need to thank his forebears, then surely I must also give thanks to all of those who have played a role in my academic successes not least of which is this project. I first would like to thank my parents, Timothy Torres and Elizabeth Thoms, and stepparents, Bryan and Johnice Thoms and Valerie Torres, who gave me my start in life and have always encouraged me in my aspirations towards academic excellence. I also need to thank the other members of my family who have been endlessly supportive of my studies: Ham and Midge Benson, who have been constant supportive presences in my life; Hiram and Joyce Torres, who were always proud that I wanted to pursue teaching and academics; Bob and Ann Thoms, without whom I might never have thought to apply to Texas A&M and who were always able to empathize with me because of their own experiences with academia and research; my brother, William Thoms, who was always up for a phone call when I needed a pick-me-up; my sisters, Phoebe Torres and Kayli Rusk, for their constant encouragement; Christopher and Linda Benson, the latter of whom played a big role in encouraging my enthusiasm for mathematics when she taught me about subtraction and negative numbers when I was talking to her excitedly about the addition problems I was doing in 1st grade, and who didn’t discourage me from reading the glossary in my math workbook when she picked me up from school on my last day of 2nd grade; Greg and Jan Torres, for their support of my personal and educational goals; my cousin, Jim Stark, who in a conversation with me about his own time in graduate school one Christmas, excited me enough to pursue it myself; and all of my other aunts, uncles, nephews, nieces, cousins, and assorted other family members.

Next, I must thank all of my colleagues from my times as a graduate student at Texas A&M, who have pushed me towards this completion through their encouagement and their solidarity. Special thanks in particular must be given to Ola Sobieska, who started at the same time as me, and is also finishing at the same time as me. We got each other through several of our classes, and she constantly encouraged me through the difficulties of graduate school. I also need to thank the other members of my research group, Zach Mitchell and Blake Boudreaux, with whom I was able to discuss ideas and who previewed my preliminary exam presentation to give me advice. I would also like to thank Nida Obatake, Alex Ruys de Perez, David Sykes, Brian Hunter, Mahishanka Withanachchi, Konrad Wrobel, Ayo Adeniran, Burak Hatinoglu, Cesar Cobos May, and the numerous other friends I've made in this department as we progressed through this program.

I must also thank my former colleagues at the schools at which I have taught who have all been greatly encouraging of me throughout this process, especially my former department heads, Stuart Cornwell and Barbara Rourke; my fellow math teachers, Ashley Dugas, Lisa Boyer, Roxy Tate, Lynne Guarisco, and Karen Timmreck; my colleagues from other departments, especially Andy LeGoullon, Garrett and Allie Rosen, Diana Maggini, James and Susan Dunlap, Kellie Kinsland, Jon Berthelot, Lisa Baldridge, Ken Timmreck, Jamie and Kim Anson, Craig Baker, Chris Cole, Grace Krause, Christine Mendizabal, Emily Froeba, and Meredith Percy; and my former administrators who gave me a leg-up in the world of education, Paul Baker, Anne Tate, Ellen Lee, and Julie Lechich.

I must now give thanks to the many academic mentors I have had: my committee members, and especially the chair of my committee, Dr. Harold P. Boas, from whom I consider it to have been one of the greatest privileges of my life to have learned and whose weekly conversations about math and teaching, I will greatly miss; my graduate professors; Dr. Dean Baskin whose graduate student writing group helped me keep on track with all of my writing projects, my undergraduate mathematics professors, Dr. Roger Waggoner, Dr. Victor Schneider, Dr. Patricia Beaulieu, and Dr. Christina Eubanks-Turner, without whom I would assuredly not have succeeded in graduate school; my other supportive undergraduate faculty members, especially Dr. Julia Frederick, Dr. Susan Nicassio, Mrs. Nona Istre, Dr. Mark Radle, and Dr. John Meriwether; my elementary and secondary math instructors, especially Mrs. Linda Bonnette, Mrs. Barbara Hoffmann, and Mrs. Lindsay Waddell; my other elementary and secondary teachers, especially the late Mrs. Elise Hook, Mr. Yasuo Namba, and Mr. Rudy Gebauer, as well as Mrs. Suzanne Bentley-Smith, Dr. Barbara Walsh, Mrs. Sarah Kirkpatrick, Mrs. Pam Middlebrook, Ms. Billie Smith, Mr. Jeremy Grissell, Mrs. Jan Lennie, Mr. Richard Haywood, Mr. Gregory Sampson, and Mrs. Stephanie de la Cerda.

I next would like to thank my friends who have supported me through their prayers and conversation for years, among whom special mention must be made of Tim and Sarah Trosclair, Nick Trosclair, Peter Youngblood, Kyle Albarado, Ryan Carruth, Geoffrey and Emily Bain, Trey and Sara Dietz, Travis and Madeline Rooney, Brody and Rebecca Britten, Nathan and Jennifer Streger, and the Marchetti family (especially Tommy, Johnny, Teresa, and Peter). I also would like to thank my friends Jonathan and Meredith Tyler, whom I have had the great privilege of knowing starting with my time at A&M, as well as their daughter and my goddaughter, Marie Tyler, whose smile and joy light up my entire world. I would also like to thank the members of the St. Mary's Men's Chant Choir who have been sources of great happiness and consolation in my daily life here at A&M, as well as the Cathedral Choir of Our Lady of Walsingham in Houston, with whom I have had the great pleasure of singing this past year.

I would also like to thank my many pastors (former and current) and spiritual mentors who have helped me navigate the trials of adulthood, including most recently the difficulties of graduate school. Special thanks must be given to Bishop David Konderla, who got me connected to the community at St. Mary's in College Station when he was pastor, Fr. Jason Vidrine, Fr. Clinton Sensat, Fr. Michael Russo, Fr. Augustine Ariwaodo, Fr. Brian McMaster, Fr. Greg Gerhart, Fr. Charles Hough, Fr. Geoff Horton, Sr. Celestina Menin, and Dcn. Tim Maragos. Last but not least, I would be remiss if I did not thank the Triune God: Father, Son, and Holy Ghost, for His providential care for me all the days of my life and for all the gifts of nature and grace which He has given me in creating, preserving, and sanctifying me; I thank Our Lady the Seat of Wisdom whose special love for me is the fount of my devotion to the Truth – both eternal and temporary, both divine and mundane, both supernatural and natural. I give thanks for my baptismal patrons St. Joseph and St. Lawrence, and my confirmation patron, St. Augustine, as well as the many other holy men and women who have interceded to God on my behalf, most especially Ss. Joachim and Anne, St. Paul, St. John the Apostle, Ss. Martha and Mary Magdalene, St. Hubert, St. Benedict, St. Anselm, St. Bernard of Clairvaux, St. Dominic, St. Francis of Assisi, St. Albert the Great, St. Thomas Aquinas, St. Bonaventure, St. Catherine of Siena, St. John of the Cross, St. Teresa of Avila, St. Therese of Liseux, St. John Henry Newman, and St. Rose of Lima.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a dissertation committee consisting of Professor Boas and Professors Straube and Rojas of the Department of Mathematics and Professor Kronenberg of the Department of Geology and Geophysics.

All other work conducted for the dissertation was completed by the student independently.

Funding Sources

Graduate study was supported by an assistantship from Texas A&M University.

TABLE OF CONTENTS

1. INTRODUCTION

Given a domain $\Omega \subset \mathbb{C}^n$ and a function holomorphic on Ω, recall that there is a maximal Riemann domain $\mathcal{R}$ over $\mathbb{C}^n$ to which it can be extended called its *domain of existence.* If Ω is the domain of existence for a holomorphic function, then we say that it is a domain of holomorphy. Furthermore, if $\mathscr{S}$ is a family of holomorphic functions on Ω and Ω is the domain of existence for some $f \in \mathscr{S}$, then we say that Ω is an $\mathscr{S}$-domain of holomorphy. In this paper, we will be most concerned with the case that $\mathscr{S}$ is the family of L^p holomorphic functions, i.e, when $\mathscr{S} = L^p_h(\Omega)$. (See §§8-9 of Chapter II in [1].)

In Chapter 2, we give a characterization of Reinhardt L^1_h-domains of holomorphy in $\mathbb{C}^2$, which we have done in Theorem 1. This research question arose when considering the removable sets for bounded L^p_h-domains of holomorphy in the plane: while some sets are always removable for bounded holomorphic functions or for L^2 holomorphic functions in the plane (see Theorem 2 in [7]), there are no such sets for L^p holomorphic functions for $p < 2$. In other words, every bounded open subset of the plane is an L^p_h-domain of holomorphy for all $p < 2$. This resulted in the following conjecture, which remains open:

Conjecture 1. *Every bounded domain of holomorphy in $\mathbb{C}^n$ is an L^p_h-domain of holomorphy for $p < 2$.*

Since every L^p_h-domain of holomorphy is itself a domain of holomorphy, the "domain of holomorphy" hypothesis in the conjecture is necessary. We undertook to prove Conjecture 1 first for bounded Reinhardt domains of holomorphy in $\mathbb{C}^n$, which we accomplished in Proposition 4 via the geometric characterization of Reinhardt domains of holomorphy (Theorem 1.11.13 in [6]). After this, it was natural to ask whether unbounded Reinhardt domains of holomorphy exhibited the same phenomenon or not. Clearly, it will be necessary to assume that there exist nontrivial

L_h^p-functions on a given Reinhardt domain of holomorphy for there to be any hope of it being an L_h^p-domain of holomorphy. Jarnicki and Pflug showed in [4] that for fat Reinhardt domains, this is sufficient. Recall that a domain in $\mathbb{C}^n$ is *fat* provided it is the interior of its closure. This led to another conjecture:

Conjecture 2. *If Ω is a Reinhardt domain of holomorphy such that for some $p < 2$, $L_h^p(\Omega) \neq \{0\}$, then Ω is an L_h^p-domain of holomorphy.*

However, Conjecture 2 fails and the work in this paper furnishes a counterexample. In fact, Proposition 14 will yield a family of unbounded domains with nontrivial L_h^p-functions for all $p \geq 1$. However, even more than this, Propositions 15 and 16 will yield that whenever $1 \leq p < q < 2$, then there exists a domain from the family in Proposition 14 which is an L_h^p-domain of holomorphy, but is not an L_h^q-domain of holomorphy.

Since Conjecture 2 fails, we began to seek out exactly which Reinhardt domains in $\mathbb{C}^2$ are L_h^1-domains of holomorphy in the hopes of finding a characterization. This characterization is given in Theorem 1. In developing this characterization, heavy use was made of the logarithmic convexity of pseudoconvex Reinhardt domains. To this end, we introduce below the notion of logarithmic half-planes – those fat domains in $\mathbb{C}^2$ whose images under the function $\log|z|$ are half-planes. While Jarnicki and Pflug give a function-theoretic characterization of fat L_h^p-domains of holomorphy in [4], we have given a geometric characterization in terms of these logarithmic half-planes of fat L_h^p-domains of holomorphy in $\mathbb{C}^2$ in Propositions 30 and 31.

We then sought to generalize this result to higher dimensions. Towards this end, in Chapter 3, Corollary 1 gives a characterization of fat L_h^1-domains of holomorphy in $\mathbb{C}^n$ in terms of the linear span of a set of real vectors representing elementary Reinhardt domains – higher-dimensional analogs of logarithmic half-planes.

While we had originally parameterized the elementary Reinhardt domains using vectors in $\mathbb{R}^n$, it became evident that this method did not give a unique parameterization of these domains; in other words, multiple vectors could represent the same elementary Reinhardt domain. However, each elementary Reinhardt domain can be represented by a unique unit vector in $\mathbb{R}^n$. This suggested that the results concerning such domains should be stated not in terms of members of $\mathbb{R}^n$, but members of the sphere S^{n-1}. This insight led to the work in Chapter 4, which gives a simplified restatement of Theorem 1 in terms of this new parameterization. The success of this parameterization combined with the linear algebra techniques in Chapter 2 suggest a possible route for further research concerning non-fat Reinhardt L^1_h-domains of holomorphy in n dimensions.

2. L_h^1-DOMAINS OF HOLOMORPHY IN $\mathbb{C}^2$

In order to give a geometric characterization of Reinhardt L_h^1-domains of holomorphy, we first recall two results: the first gives a function-theoretic characterization of fat, Reinhardt L_h^p-domains of holomorphy found in [4], while the second gives a geometric characterization of non-fat Reinhardt domains of holomorphy in relation to their fat hulls.

Proposition 1. *If Ω is a fat, Reinhardt domain of holomorphy, and if there is a $p \in [1, \infty)$ such that $L_h^p(\Omega) \neq \{0\}$, then for all $q \in [1, \infty]$, Ω is an L_h^q-domain of holomorphy.*

Proof. First, we recall the notation of Jarnicki and Pflug from [4]. Recall that

$$L_h^{\Diamond,0}(\Omega) := \bigcap_{q \in [1,\infty]} L_h^{q,0}(\Omega) = \bigcap_{q \in [1,\infty]} L_h^q(\Omega).$$

Now, on the assumption that there exists $p \in [1, \infty)$ such that $L_h^p(\Omega) \neq 0$, it follows from Proposition 9 of [4] that Ω is an $L_h^{\Diamond,0}$-domain of holomorphy. Therefore, there exists $f \in L_h^{\Diamond,0}(\Omega)$ having Ω as its domain of existence. Fix $q \in [1, \infty]$. Now, by definition of $L_h^{\Diamond,0}(\Omega)$, $f \in L_h^q(\Omega)$. Therefore, Ω is the domain of existence of an L_h^q function and so Ω is an L_h^q-domain of holomorphy. □

Definition 1. *For all $j \in \{1, \dots, n\}$, we define $V_j \subset \mathbb{C}^n$ by $V_j := \{z_j = 0\}$. We define $V_0 \subset \mathbb{C}^n$ by $V_0 := \{z_1 \cdots z_n = 0\}$. In other words,*

$$V_0 := \bigcup_{j=1}^{n} V_j$$

Proposition 2. *If Ω is a Reinhardt domain of holomorphy in $\mathbb{C}^n$ and Ω^* is its fat hull (i.e, if $\Omega^* = \left(\overline{\Omega}\right)^\circ$), then for some $J \subset \{0, 1, \dots, n\}$, we have that:*

$$\Omega^* \setminus \Omega = \bigcup_{j \in J} (\Omega^* \cap V_j)$$

This Proposition follows directly from Theorem 1.11.13 in [6], which in effect states that the only way to construct non-fat Reinhardt domains of holomorphy is to remove one or more of the coordinate axes ($V_1, V_2, \ldots$) from the domain. With these two results in mind, we will now proceed to our characterization first of bounded, Reinhardt L^p_h-domains of holomorphy, and then of unbounded Reinhardt L^1_h-domains of holomorphy in $\mathbb{C}^2$.

2.1 Bounded Reinhardt Domains of Holomorphy

We first proceed to characterize bounded Reinhardt domains of holomorphy in $\mathbb{C}^n$ for arbitrary n. This characterization proceeds in a series of steps, which are outlined as follows: (1) we note that all bounded Reinhardt domains of holomorphy which are fat are L^p_h-domains of holomorphy (Proposition 3), and then (2) we show that for $p < 2$, the hypothesis for the domain may be relaxed (Proposition 4).

Proposition 3. *Every bounded, fat Reinhardt domain of holomorphy is an L^p_h-domain of holomorphy, for all $p \in [1, \infty]$.*

Note that this is a simple consequence of Proposition 1, since in particular $L^1_h(\Omega)$ contains all of the polynomials, if Ω is a bounded domain. Before proceeding to Proposition 4, which characterizes bounded L^p_h-domains of holomorphy, we consider the following example.

Example: Consider $\Omega := \mathbb{D}^2 \setminus V_1$, and observe that Ω^* is the bidisk. We note that by Proposition 3, the bidisk is an L^p_h-domain of holomorphy for all p. This means that for all p, there is some f_p holomorphic on the bidisk which is also L^p and which does not extend holomorphically to any boundary point of the bidisk. Now, a simple calculation shows that z_1^{-1} is L^p on the bidisk for all $p < 2$ and is not L^p for any $p \geq 2$. Hence, $g_p := f_p + z_1^{-1}$ is L^p on the bidisk for all $p < 2$ and holomorphic on Ω. Furthermore, g_p does not extend holomorphically to any boundary point of the bidisk (or else $g_p - z_1^{-1} = f_p$ would) nor to any point in $\mathbb{D}^2 \cap V_1$ or else ($g_p - f_p = z_1^{-1}$ would). The domain of definition for g_p is Ω and so Ω is an L^p_h-domain of holomorphy, for every $p < 2$.

This example is indicative of the proof that all bounded Reinhardt domains of holomorphy are also L_h^p-domains of holomorphy for $p < 2$. Furthermore, it indicates why we must take as an assumption that $p < 2$, since z_1^{-1} is not L^2 on the bidisk. Indeed, more generally, for all integers m, n, $z_1^m z_2^n$ is L^2 if and only if $m, n \geq 0$. It follows from this fact and from Lemma 1 that the only bounded Reinhardt L_h^2-domains of holomorphy are those which are fat.

This example is also consistent with the characterization of bounded L_h^2-domains of holomorphy given in Theorem 2 of [7], which states that pluripolar sets are removable sets for L_h^2 functions. Since V_j is an analytic variety, for each j, it is also a pluripolar set. Hence, a bounded Reinhardt domain of holomorphy is an L_h^2-domain of holomorphy if and only if it is fat.

Proposition 4. *Every bounded Reinhardt domain of holomorphy is an L_h^p-domain of holomorphy, for all $p \in [1, 2)$.*

Proof. Let Ω be a bounded, Reinhardt domain of holomorphy and fix $p \in [1, 2)$. First, we note that the claim follows from Proposition 3 if $\Omega = \Omega^*$. We assume now that $\Omega \subsetneq \Omega^*$. Then we let J be the indexing set guaranteed by Proposition 2. It now follows that for each $j \in J$, z_j^{-1} is holomorphic on Ω. Furthermore, since Ω is bounded, there exists a polydisk of radius $R > 0$ such that $\Omega \subset P$. Therefore, for all $p \in [1, 2)$,

$$\int_\Omega \left|z_j^{-1}\right|^p \leq 2\pi^n R^{2n-2} \cdot \int_0^R r_j^{1-p} dr_j = \frac{2\pi^n R^{2n-p}}{2-p} < \infty.$$

Hence, for each $j \in J$, $z_j^{-1} \in L_h^p(\Omega)$. Define $g \in L_h^p(\Omega)$ by $g(z) := \sum_{j \in J} z_j^{-1}$. Also, from Proposition 3, there exists an $f \in L_h^p(\Omega^*)$ such that Ω^* is the domain of definition for f. We now define $h \in L_h^p(\Omega)$ by $h := f + g$. Now, since f does not extend holomorphically to any boundary point of Ω^* and g does not extend holomorphically to any point in $\Omega^* \setminus \Omega$, it follows that h does not extend holomorphically to any boundary point of Ω. Hence, h is an L_h^p-function for which Ω is the domain of definition, and it therefore follows that Ω is an L_h^p-domain of holomorphy. $\square$

2.2 Unbounded Reinhardt Domains of Holomorphy in $\mathbb{C}^2$

We now consider the more difficult case of unbounded Reinhardt domains of holomorphy. We have no easy analog to Proposition 3. There is no guarantee on a given unbounded domain that nontrivial L^p holomorphic functions exist. Therefore, we will now invoke more explicitly the geometry of domains of holomorphy which are Reinhardt in particular.

For any domain $\Omega \subset \mathbb{C}^2$,

$$\log|\Omega| := \left\{(x,y) \in \mathbb{R}^2 : \text{for some } (z_1,z_2) \in \Omega, (e^x, e^y) = (|z_1|, |z_2|)\right\}.$$

Also, recall that every Reinhardt domain of holomorphy is logarithmically convex. In other words, for every Reinhardt domain of holomorphy Ω, we have that $\log|\Omega|$ is a convex subset of $\mathbb{R}^2$. Therefore, every Reinhardt domain of holomorphy Ω has the property that either $\Omega^* = \mathbb{C}^2$ or that $\log|\Omega|$ is the intersection of a family of half-planes in $\mathbb{R}^2$. Since for all $p \in (0,\infty)$, $L_h^p(\mathbb{C}^2) = \{0\}$, we may consider only those Reinhardt domains of holomorphy Ω with $\Omega^* \neq \mathbb{C}^2$. In order to do this more simply, we now define the notion of logarithmic half-planes and then in Proposition 5, we give a description of these logarithmic half-planes.

Definition 2. *A* ***logarithmic half-plane*** *in $\mathbb{C}^2$ is a fat Reinhardt domain $\Omega \subset \mathbb{C}^2$ such that $\log|\Omega|$ is a half-plane in $\mathbb{R}^2$.*

Proposition 5. *Ω is a logarithmic half-plane in $\mathbb{C}^2$ if and only if for some $\alpha > 0$, one of the following statements is true:*

1. *For some $x \in \mathbb{R}$, $\Omega = \{|z_2| < \alpha |z_1|^x\} =: U_\alpha^x$.*

2. *For some $x \in \mathbb{R}$, $\Omega = \{|z_2| > \alpha |z_1|^x\} =: \widetilde{U}_\alpha^x$.*

3. *$\Omega = \{|z_1| < \alpha\} =: U_\alpha$.*

4. *$\Omega = \{|z_1| > \alpha\} =: \widetilde{U}_\alpha$.*

Proof. First, suppose Ω is a logarithmic half-plane. Then $\log|\Omega|$ must be defined by an open, linear inequality in two variables. That is, $\partial \log|\Omega|$ is a line in $\mathbb{R}^2$. Hence, $\partial \log|\Omega|$ is either equal to $\{(x_1, x_2) : x_2 = mx_1 + b\}$, for some $m, b \in \mathbb{R}$, or equal to $\{(x_1, x_2) : x_1 = b\}$ for some $b \in \mathbb{R}$, where $x_j = \log|z_j|$, for $j = 1, 2$.

Now, in the first case, we have that $\partial\Omega = \left\{|z_2| = e^b \cdot |z_1|^m\right\}$, since Ω is fat. Therefore, taking $\alpha = e^b$ and $x = m$, we have that either $\Omega = U_\alpha^x$ or $\Omega = \widetilde{U}_\alpha^x$. Similarly, in the second case, we have that $\partial\Omega = \left\{|z_1| = e^b\right\}$, so taking $\alpha = e^b$, we have that either $\Omega = U_\alpha$ or $\Omega = \widetilde{U_\alpha}$. For the converse, now note by a simple computation that each domain described in statements (1)-(4) of this proposition is itself a logarithmic half-plane. □

In order to understand the main result, it is useful to analyze separately the cases of Reinhardt domains of holomorphy with (a) a fat hull which intersects precisely one of V_1 and V_2 (subsection 2.2.1), (b) a complete fat hull (subsection 2.2.2), and (c) a fat hull which is disjoint from V_0 (subsection 2.2.3). Toward this end, we will now give characterizations of complete Reinhardt domains of holomorphy (Proposition 6) and Reinhardt domains of holomorphy intersecting precisely one of V_1 and V_2 in $\mathbb{C}^2$ (Proposition 7) in terms of logarithmic half-planes.

Proposition 6. *A complete Reinhardt domain of holomorphy in $\mathbb{C}^2$ must be either $\mathbb{C}^2$ or an intersection of logarithmic half-planes of the form U_α and U_α^x, where $x \leq 0$.*

Proof. Let $\Omega \subsetneq \mathbb{C}^2$ be a complete Reinhardt domain of holomorphy. Then since Ω must be logarithmically convex, $\log|\Omega|$ must be an intersection of half-planes in $\mathbb{R}^2$. Hence, Ω must be an intersection of logarithmic half-planes.

Furthermore, since Ω is complete, it must contain the origin. Therefore, it must be an intersection of logarithmic half-planes containing the origin. Note now that the origin is not contained in any domain of the form $\widetilde{U}_\alpha^x$ or $\widetilde{U}_\alpha$. Furthermore, if $x > 0$, then $0 = \alpha \cdot 0^x$, and so the origin is not contained in U_α^x. Evidently, if $\alpha > 0$ and $x \leq 0$, then the origin is contained in U_α and U_α^x. Hence,

Ω must be an intersection of logarithmic half-planes of the form U_α^x, where $x \leq 0$, and U_α. $\square$

Proposition 7. *If Ω is a Reinhardt domain of holomorphy such that its fat hull Ω^* has nonempty intersection with exactly one of V_1 and V_2, then Ω must be contained in a logarithmic half-plane of one of the following forms: U_α^x, where $x > 0$; $\widetilde{U}_\alpha$; $\widetilde{U}_\alpha^x$, where $x > 0$; or $\widetilde{U}_\alpha^0$.*

Proof. Let Ω be a Reinhardt domain of holomorphy such that Ω^* has nonempty intersection with exactly one of V_1 or V_2. Since Ω is a Reinhardt domain of holomorphy, Ω must be logarithmically convex. But then Ω^* must also be logarithmically convex and so Ω^* is an intersection of logarithmic half-planes. Since by hypothesis Ω^* must omit the origin, at least one of these logarithmic half-planes must also omit the origin. Now, observe that for every $\alpha > 0$, $\widetilde{U}_\alpha$ omits the origin as does $\widetilde{U}_\alpha^0$. Furthermore, for every $\alpha, x > 0$, U_α^x omits the origin as does $\widetilde{U}_\alpha^x$. Furthermore, these are the only logarithmic half-planes which omit the origin and intersect exactly one of V_1 or V_2. $\square$

2.2.1 Domains with Non-Complete Fat Hull Not Disjoint from V_2

The results in this section and those following come in three flavors. (1) First, we have results which demonstrate the existence of nontrivial L_h^p-functions on certain fat Reinhardt domains of holomorphy. From Proposition 1 above, it will then follow that these domains are L_h^p-domains of holomorphy for all $p \geq 1$. (2) We will then show when certain non-fat Reinhardt domains of holomorphy are L_h^p-domains of holomorphy for specified p. The proofs of these propositions will follow a method similar to the one used in Proposition 4 — we will find an L_h^p Laurent monomial on the specified non-fat domain. (3) Finally, we have results in which we determine that certain non-fat Reinhardt domains of holomorphy are not L_h^p-domains of holomorphy. Proofs of these propositions will proceed by showing that the L_h^p monomials on the specified domains extend to a larger domain. It will then follow from Lemma 1 below that the specified domain is not an L_h^p-domain of holomorphy.

In this section, we will consider only those domains having a fat hull which intersects precisely one of V_1 and V_2. Furthermore, since $L_h^p(\Omega)$ is invariant under a permutation of the coordinates of Ω, we will consider only those Reinhardt domains of holomorphy which are disjoint from V_1 but not from V_2. By the argument in Proposition 7, we only need consider domains which are contained in logarithmic half-planes of the form $\widetilde{U}_\alpha$ (Propositions 8 and 9) or U_α^x where $x > 0$ (Propositions 10-12).

Proposition 8. *Let Ω be a Reinhardt domain of holomorphy. Also, let $\alpha, \beta > 0$ and $x \in \mathbb{R}$. If $\Omega \subset \widetilde{U}_\alpha \cap U_\beta^x$, then $L_h^p(\Omega) \neq \{0\}$, for all $p > 0$.*

Proof. To see this, let n be an integer strictly less than $-\frac{2(1+x)}{p}$. We now show that $z_1^n \in L_h^p(\Omega)$. First, note that z_1^n is holomorphic on Ω, since $\Omega \cap V_1 = \varnothing$. Now, observe:

$$\int_\Omega |z_1^n|^p \leq 4\pi^2 \int\limits_\alpha^\infty \int\limits_0^{\beta r_1^x} r_1^{1+pn} r_2 dr_2 dr_1 = 2\pi^2\beta^2 \int\limits_\alpha^\infty r_1^{1+pn+2x} dr_1.$$

Now, observe that $1 + pn + 2x < 1 - 2(1+x) + 2x = -1$, and so

$$\int_\Omega |z_1^n|^p \leq 2\pi^2\beta^2 \int\limits_\alpha^\infty r_1^{1+pn+2x} dr_1 < \infty.$$

Therefore, $z_1^n \in L_h^p(\Omega)$. □

Proposition 9. *If Ω is a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 8, then Ω is an L_h^p-domain of holomorphy for all $p \in [1, 2)$.*

Proof. If $\Omega = \Omega^*$, then this follows from Propositions 1 and 8. Now, suppose that $\Omega \neq \Omega^*$. It now follows from Proposition 2 that $\Omega = \Omega^* \setminus V_2$. Fix $p \in [1, 2)$ and let n be an integer strictly less than $-\frac{x(2-p)+2}{p}$. We now show that $z_1^n z_2^{-1} \in L_h^p(\Omega)$. First, since $\Omega \cap V_0 = \varnothing$, $z_1^n z_2^{-1}$ is holomorphic on Ω. Next, observe that

$$\int_\Omega \left|z_1^n z_2^{-1}\right|^p \leq 4\pi^2 \int\limits_\alpha^\infty \int\limits_0^{\beta r_1^x} r_1^{1+pn} r_2^{1-p} dr_2 dr_1.$$

Since $p < 2$, we have that $1 - p > -1$ and so

$$\int_\Omega \left|z_1^n z_2^{-1}\right|^p \leq \frac{4\pi^2 \beta^{2-p}}{2-p} \int_\alpha^\infty r_1^{1+pn+x(2-p)} dr_1.$$

Finally, since $pn < -x(2-p) - 2$, we have that $1 + pn + x(2-p) < -1$, and so

$$\int_\Omega \left|z_1^n z_2^{-1}\right|^p < \infty.$$

Now, let $f \in L_h^p(\Omega^*)$ have Ω^* as its domain of definition and define $g \in L_h^p(\Omega)$ by $g(z) := f(z) + z_1^n z_2^{-1}$. Note that since f does not extend holomorphically to any boundary point of Ω^* and $z_1^n z_2^{-1}$ does not extend holomorphically to V_2, it follows that Ω is the domain of definition for g, and so Ω is an L_h^p-domain of holomorphy. □

Remark: The conclusion of Proposition 9 would sometimes be false if we took $p = 2$. This follows from Proposition 2 above and from Theorem 2 in [7].

Proposition 10. *Let Ω be a Reinhardt domain of holomorphy, and let $y < x$ and $x > 0$ and $\alpha, \beta > 0$. If $\Omega \subset U_\alpha^x \cap U_\beta^y$, then $L_h^1(\Omega) \neq \{0\}$.*

Proof. Let $r = \frac{m'}{n'}$ be a rational number in $(y, x) \setminus \mathbb{Z}$. Assume without loss of generality that n' is positive. Now, let $m := -2 - m'$ and $n := -2 + n'$. Since $r \notin \mathbb{Z}$, it follows that $n' \geq 2$, so that $n \geq 0$. I now claim that $z_1^m z_2^n \in L_h^p(\Omega)$. Since $n \geq 0$ and $\Omega \cap V_1 = \varnothing$, $z_1^m z_2^n$ is holomorphic on Ω. Now, let $R = \left(\frac{\beta}{\alpha}\right)^{1/(x-y)}$ and observe that

$$\int_\Omega |z_1^m z_2^n| \leq 4\pi^2 \left(\int_0^R \int_0^{\alpha r_1^x} r_1^{1+m} r_2^{1+n} dr_2 dr_1 + \int_R^\infty \int_0^{\beta r_1^y} r_1^{1+m} r_2^{1+n} dr_2 dr_1 \right)$$

$$= \frac{4\pi^2}{2+n} \left(\alpha^{2+n} \int_0^R r_1^{1+m+x(2+n)} dr_1 + \beta^{2+n} \int_R^\infty r_1^{1+m+y(2+n)} dr_1 \right).$$

Now, note that the integral above is finite provided $1+m+x(2+n) > -1$ and $1+m+y(2+n) < -1$. But this is true if and only if $-x(2+n) < 2+m < -y(2+n)$, which in turn is true if and only if $y < \frac{-2-m}{2+n} < x$. However, $m' = -2-m$ and $n' = 2+n$, and $r = \frac{m'}{n'} \in (y, x)$. Therefore, $\int_\Omega |z_1^m z_2^n| < \infty$, and so $z_1^m z_2^n \in L_h^1(\Omega)$. $\square$

Proposition 11. *If Ω is a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 10, then Ω is an L_h^1-domain of holomorphy provided that either $\Omega = \Omega^*$ or $(y, x) \cap \mathbb{Z} \neq \varnothing$.*

Proof. If $\Omega = \Omega^*$, then this follows from Proposition 10 above and from Proposition 1. Now, suppose that $\Omega \neq \Omega^*$. It follows that $\Omega = \Omega^* \setminus V_2$. Now let $r \in (y, x) \cap \mathbb{Z}$. Then taking $m = -2-r$ and $n = -1$, it follows from the same argument as in Proposition 10 above that $z_1^m z_2^n \in L_h^1(\Omega)$. Furthermore $z_1^m z_2^n$ does not extend holomorphically to V_2.

Therefore, since Ω^* is an L_h^1-domain of holomorphy, let $f \in L_h^1(\Omega^*)$ such that Ω^* is the domain of definition for f. Now define $g \in L_h^1(\Omega)$ by $g(z_1, z_2) := f(z_1, z_2) + z_1^m z_2^n$. Now, since g does not extend holomorphically to $\partial\Omega^*$ nor to V_2, it follows that Ω is the domain of definition for g, so that Ω is an L_h^1-domain of holomorphy. $\square$

Lemma 1. *Let $f(z) = \sum_{\nu \in \mathbb{Z}^n} a_\nu z^\nu$ be a holomorphic function on a Reinhardt domain $\Omega \subset \mathbb{C}^n$. If $f \in L_h^p(\Omega)$, then $a_\nu z^\nu \in L_h^p(\Omega)$, for all $\nu \in \mathbb{Z}^n$.*

Proof. The lemma follows from the proof of Proposition 9 on p. 261 of [4]. $\square$

Proposition 12. *Let $y < x$ and $x, \alpha, \beta > 0$ and $\Omega = \left(U_\alpha^x \cap U_\beta^y\right) \setminus V_2$. If $(y, x) \cap \mathbb{Z} = \varnothing$, then Ω is not an L_h^1-domain of holomorphy and its L_h^1-envelope of holomorphy is Ω^*.*

Proof. Suppose for a contradiction that Ω is an L_h^1-domain of holomorphy. Since Ω is a Reinhardt domain, every holomorphic function on Ω has a Laurent power series representation on Ω. Now, observe from Lemma 1 that if $f(z) := \sum_{\nu \in \mathbb{Z}^2} a_\nu z^\nu \in L_h^1(\Omega)$ with Ω the domain of existence for f, then we have that $a_\nu z^\nu \in L_h^1(\Omega)$, for each $\nu \in \mathbb{Z}^2$.

Now, note that V_2 has nonempty intersection with Ω and so if Ω were an L^1_h domain of holomorphy, there would exist $m, n \in \mathbb{Z}$ with $n < 0$ such that $a_{(m,n)} \neq 0$. Hence, $z_1^m z_2^n \in L^1_h(\Omega)$. Next, let $R = \left(\frac{\beta}{\alpha}\right)^{1/(x-y)}$ and observe:

$$\int_\Omega |z_1^m z_2^n| = 4\pi^2 \left(\int_0^R \int_0^{\alpha r_1^x} r_1^{1+m} r_2^{1+n} dr_2 dr_1 + \int_R^\infty \int_0^{\beta r_1^y} r_1^{1+m} r_2^{1+n} dr_2 dr_1 \right) < \infty.$$

But this implies that $1 + n > -1$, which means that $n > -2$. Since $n < 0$ and $n \in \mathbb{Z}$, this implies that $n = -1$. Hence,

$$\int_\Omega |z_1^m z_2^n| = 4\pi^2 \left(\alpha \int_0^R r_1^{1+m+x} dr_1 + \beta \int_R^\infty r_1^{1+m+y} dr_1 \right) < \infty.$$

This now implies that $1+m+y < -1 < 1+m+x$. This is equivalent to $y < -2-m < x$. Now, since $m \in \mathbb{Z}$, this implies that $\mathbb{Z} \cap (y, x) \neq \varnothing$. But this contradicts our hypothesis. Hence, Ω is not an L^1_h-domain of holomorphy. Furthermore, we have that every L^1_h function on Ω holomorphically extends across V_2 to a holomorphic function on Ω^*. Therefore, the inclusion map $\mathcal{O}(\Omega^*) \hookrightarrow \mathcal{O}(\Omega)$ given by $f \mapsto f|_\Omega$ is surjective and so since Ω^* is an L^1_h-domain of holomorphy, we have that Ω^* is the L^1_h-envelope of holomorphy of Ω. □

Example: Note that we can now provide a counterexample to Conjecture 2. Consider $\Omega := (U_1^1 \cap U_1^2) \setminus V_2$. Observe that by Proposition 10, $L^1_h(\Omega) \supset L^1_h(\Omega^*) \neq \{0\}$. However, there is no integer in the interval $(1, 2)$, and so by Proposition 12, Ω is not an L^1_h-domain of holomorphy. In fact, any domain satisfying the hypotheses of Proposition 12 provides another counterexample.

2.2.2 Domains with Complete Fat Hull

We now turn our attention to those Reinhardt domains of holomorphy having a complete fat hull. By Proposition 6, we must consider domains which are intersections of logarithmic half-planes of the form U_α and U_α^x, where $x \leq 0$. In Propositions 13-18, we consider domains which are contained in logarithmic half-planes of type U_α, whereas in Propositions 19-27 we consider domains which are purely intersections of logarithmic half-planes of the form U_α^x for $x \leq 0$.

Proposition 13. *Let Ω be a Reinhardt domain of holomorphy and let $\alpha, \beta > 0$ and $x < 0$. If $\Omega \subset U_\alpha^x \cap U_\beta$, then for some $p \in [1, \infty)$, $L_h^p(\Omega) \neq \{0\}$.*

Proof. Let $p = 1 - 2x$. I claim that $z_1 \in L_h^p(\Omega)$. To see this, observe that

$$\int_\Omega |z_1|^p \leq (2\pi)^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1^{1+p} r_2 dr_2 dr_1 = 2\pi^2\alpha^2 \int_0^\beta r_1^{1+p+2x} dr_1 = 2\pi^2\alpha^2 \int_0^\beta r_1^2 dr_1 < \infty. \qquad \square$$

If Ω^* is a complete Reinhardt domain of holomorphy, then by Proposition 2, Ω is either Ω^*, $\Omega^* \setminus V_1$, $\Omega^* \setminus V_2$, or $\Omega^* \setminus V_0$. Since under the hypotheses of Proposition 13, $\Omega^* \cap V_2$ is bounded, but $\Omega^* \cap V_1$ is unbounded, we will analyze these cases separately: (1) in Proposition 14, we analyze the cases when $\Omega = \Omega^*$ and $\Omega = \Omega^* \setminus V_2$; (2) in Propositions 15-16, we analyze the case when $\Omega = \Omega^* \setminus V_1$; (3) in Propositions 17-18, we analyze the case when $\Omega = \Omega^* \setminus V_0$.

Proposition 14. *If Ω is a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 13, and if $\Omega = \Omega^*$ or if $\Omega = \Omega^* \setminus V_2$, then Ω is an L_h^p-domain of holomorphy for all $p \in [1, 2)$.*

Proof. Fix $p \in [1, 2)$. Note that by Proposition 1 above, Ω^* is an L_h^p-domain of holomorphy. Now suppose $\Omega = \Omega^* \setminus V_2$. Let n be a positive integer strictly greater than $-\frac{2+x(2-p)}{p}$. We now claim that $z_1^n z_2^{-1} \in L_h^p(\Omega)$. Since n is positive and $V_2 \cap \Omega = \varnothing$, $z_1^n z_2^{-1}$ is holomorphic on Ω. Observe now that:

$$\int_\Omega \left| z_1^n z_2^{-1} \right|^p \leq (2\pi)^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1^{1+pn} r_2^{1-p} dr_2 dr_1 = \frac{4\pi^2}{2-p} \alpha^{2-p} \int_0^\beta r_1^{1+pn+x(2-p)} dr_1.$$

Now, since $pn > -2 - x(2-p)$, we have that $1 + pn + x(2-p) > -1$. Therefore, $z_1^n z_2^{-1} \in L_h^p(\Omega)$. Furthermore, $z_1^n z_2^{-1}$ does not extend holomorphically to V_2. Also, since Ω^* is an L_h^p-domain of holomorphy, there exists an $f \in L_h^p(\Omega^*)$ that does not extend holomorphically to any point in $\partial\Omega^*$.

Define $g \in L^p_h(\Omega)$ by $g(z_1, z_2) := f(z_1, z_2) + z_1^n z_2^{-1}$. Now, g clearly has Ω as its domain of definition, and so Ω is an L^p_h-domain of holomorphy. □

Proposition 15. *If Ω is a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 13, and if $\Omega = \Omega^* \setminus V_1$, then Ω is an L^p_h-domain of holomorphy for all p such that $1 \leq p < 2 + 2x$. [Note that this inequality is null if $x \leq -\frac{1}{2}$.]*

Proof. I claim that $z_1^{-1} \in L^p_h(\Omega)$. Clearly, since $V_1 \cap \Omega = \varnothing$, z_1^{-1} is holomorphic on Ω. Now, observe that

$$\int_\Omega \left|z_1^{-1}\right|^p \leq (2\pi)^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1^{1-p} r_2 dr_2 dr_1 = 2\pi^2 \alpha^2 \int_0^\beta r_1^{1-p+2x} dr_1.$$

Now, note that this integral converges precisely when $1 - p + 2x > -1$ or when $p < 2 + 2x$. Now, the argument follows as in Proposition 14 above, taking $g(z_1, z_2) := f(z_1, z_2) + z_1^{-1}$. □

Proposition 16. *If $\Omega = (U^x_\alpha \cap U_\beta) \setminus V_1$, for some $\alpha, \beta > 0$ and for some $x < 0$, then for any $p \geq 2 + 2x$, Ω is not an L^p_h-domain of holomorphy, and its L^p_h-envelope of holomorphy is Ω^*. In particular, if $x \leq -\frac{1}{2}$, then Ω is not an L^p_h-domain of holomorphy for any $p \in [1, \infty]$.*

Proof. We proceed by contradiction. Let $p \geq 2 + 2x$ and suppose Ω is an L^p_h-domain of holomorphy. Since Ω is a Reinhardt domain, every holomorphic function on Ω has a Laurent power series representation on Ω. Now, observe from Lemma 1 above that if $f(z) := \sum_{\nu \in \mathbb{Z}^2} a_\nu z^\nu \in L^p_h(\Omega)$, then we have that $a_\nu z^\nu \in L^p_h(\Omega)$, for each $\nu \in \mathbb{Z}^2$.

Now, since V_1 has nonempty intersection with $\partial\Omega$ and V_2 has nonempty intersection with Ω, if Ω is the domain of existence for f, there exist $m, n \in \mathbb{Z}$ such that $m < 0 \leq n$ and $a_{(m,n)} \neq 0$. Without loss of generality, suppose that $a_{(m,n)} = 1$. Therefore, $z_1^m z_2^n \in L^p_h(\Omega)$. Hence,

$$\begin{aligned}\int_\Omega |z_1^m z_2^n|^p &= 4\pi^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1^{1+pm} r_2^{1+pn} dr_2 dr_1 \\ &= \frac{4\pi^2}{2+pn} \alpha^{2+pn} \int_0^\beta r_1^{1+pm+x(2+pn)} dr_1 < \infty.\end{aligned}$$

Now observe that since $p \geq 2 + 2x$ and m is a negative integer, we have that

$$1 + pm + x(2 + pn) \leq 1 - p + x(2 + pn) \leq 1 - 2 - 2x + 2x + pnx = -1 + pnx.$$

Finally, note that $p > 0$, $n \geq 0$, and $x < 0$, so $-1 + pnx \leq -1$. Hence,

$$\int_0^\beta r_1^{1+pm+x(2+pn)} dr_1 = \infty.$$

This is a contradiction, and so Ω is not an L^p_h-domain of holomorphy.

Now, let $f \in L^p_h(\Omega)$. Note that since Ω is a Reinhardt domain of holomorphy, we will let $\sum_{\nu \in \mathbb{Z}^2} a_\nu z^\nu$ be the Laurent series expansion of f. As in Lemma 1, $a_\nu z^\nu \in L^p_h(\Omega)$, for all ν. Since $V_2 \cap \Omega \neq \varnothing$, we have that when $\nu_2 < 0$, $a_\nu = 0$. The above argument shows furthermore that if $\nu_1 < 0$ and $\nu_2 \geq 0$, then $a_\nu = 0$. Hence, a_ν can only be nonzero if ν_1 and ν_2 are both non-negative. Therefore, f extends holomorphically to $\Omega^* \cap V_1$, and so f extends holomorphically to Ω^*. Therefore, Ω^* is contained in the L^p_h-envelope of holomorphy of Ω.

Now observe that Ω^* is an L^p_h-domain of holomorphy by Propositions 1 and 13, and so there is an $f \in L^p_h(\Omega^*)$ for which Ω^* is the domain of existence. Therefore, Ω^* is the domain of existence for $f|_\Omega$, and so Ω^* contains the L^p_h-envelope of holomorphy of Ω. Hence, Ω^* is the L^p_h-envelope of holomorphy of Ω. □

Example: Propositions 13, 15, and 16 enable one to construct further counterexamples to Conjecture 2. Moreso, if $1 \leq p_1 < p_2 < 2$, Propositions 15 and 16, then one can construct $L^{p_1}_h$-domains of holomorphy which are not $L^{p_2}_h$-domains of holomorphy. To see this, fix $x \in \left(\frac{p_1-2}{2}, \frac{p_2-2}{2}\right]$. We will let $\Omega := (U_1^x \cap U_1) \setminus V_1$. Observe that since $p_2 < 2$, $x < 0$, and so Ω^* satisfies the hypotheses of Proposition 13. Hence, since $1 \leq p_1 < 2 + 2x$, we have from Proposition 15 that Ω is an $L^{p_1}_h$-domain of holomorphy. However, since $p_2 \geq 2 + 2x$, we have from Proposition 16 that Ω is not an $L^{p_2}_h$-domain of holomorphy.

Proposition 17. *If Ω is a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 13 with $-1 < x < 0$, and if $\Omega = \Omega^* \setminus V_0$, then Ω is an L^p_h-domain of holomorphy, for all $p \in [1, 2)$.*

Proof. We claim that $z_1^{-1} z_2^{-1} \in L^p_h(\Omega)$. Since $V_0 \cap \Omega = \varnothing$, this function is clearly holomorphic on Ω. Now, note that when $p < 2$, $1 - p > -1$, and so we have

$$\int_\Omega \left|z_1^{-1} z_2^{-1}\right|^p \leq (2\pi)^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1^{1-p} r_2^{1-p} dr_2 dr_1 = \frac{4\pi^2}{2-p} \alpha^{2-p} \int_0^\beta r_1^{1-p+x(2-p)} dr_1.$$

Now, note that since $x > -1$, $1-p+x(2-p) > 1-p-(2-p) = -1$, and so $z_1^{-1} z_2^{-1} \in L^p_h(\Omega)$. From here, the proof is the same as in Proposition 14, defining $g(z_1, z_2) := f(z_1, z_2) + z_1^{-1} z_2^{-1}$. □

Proposition 18. *If $\Omega = (U^x_\alpha \cap U_\beta) \setminus V_0$, for some $\alpha, \beta > 0$ and some $x \leq -1$, then Ω is not an L^p_h-domain of holomorphy for any $p \in [1, \infty]$, and its L^p_h-envelope of holomorphy is Ω^*.*

Proof. We proceed by contradiction. Fix $p \geq 1$ and suppose Ω is an L^p_h-domain of holomorphy. Then as in the proofs of Propositions 12 and 16 above, there exist $m, n \in \mathbb{Z}$ such that $m < 0$ and $z_1^m z_2^n \in L^p_h(\Omega)$. However, by the proof of Proposition 16, since $x \leq -\frac{1}{2}$, there is no L^p_h monomial $z_1^m z_2^n$ on Ω with $m < 0$ and $n \geq 0$. Therefore, $n < 0$. Now, observe that:

$$\int_\Omega |z_1^m z_2^n|^p = 4\pi^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1^{1+pm} r_2^{1+pn} dr_2 dr_1 < \infty.$$

This implies that $1+pn > -1$ and so $-1 \geq n > \frac{-2}{p}$. Therefore, $1 \leq p < 2$, and so $-2 \leq \frac{-2}{p} < -1$. But, since $n \in \mathbb{Z}$, this implies that $n = -1$. Hence, we have:

$$\int_\Omega |z_1^m z_2^n|^p = 4\pi^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1^{1+pm} r_2^{1-p} dr_2 dr_1 = \frac{4\pi^2}{2-p} \alpha^{2-p} \int_0^\beta r_1^{1+pm+x(2-p)} dr_1 < \infty.$$

Therefore, we have that $2+pm+x(2-p) > 0$. Hence, $x(2-p) > -2-pm$. Thus, since $2-p > 0$, $x > \frac{-2-pm}{2-p}$. Thus, since $x \leq -1$, we have that $-2 + p > -2 - pm$, which yields that $1 > -m$, and so $m > -1$. But since $m \in \mathbb{Z}$, this means that $m \geq 0$ which is a contradiction. Hence, Ω is not an L_h^p-domain of holomorphy. $\square$

Remark: It is noteworthy that the hypothesis that $x > -1$ in Proposition 17 is equivalent to the domain $U_\alpha^x \cap U_\beta$ having finite volume, since the volume of this domain is given by:

$$\int_{U_\alpha^x \cap U_\beta} dV = 4\pi^2 \int_0^\beta \int_0^{\alpha r_1^x} r_1 r_2 dr_2 dr_1 = 2\pi^2\alpha^2 \int_0^\beta r_1^{1+2x} dr_1 = \frac{\pi^2}{1+x}\alpha^2\beta^{2+2x}.$$

Hence, Propositions 17 and 18 yield that for $x < 0$, $U_\alpha^x \cap U_\beta \setminus V_0$ is an L_h^p-domain of holomorphy if and only if it has finite volume.

Now, in Propositions 19-27, we analyze those domains which are intersections of logarithmic half-planes of the form U_α^x, where $x \leq 0$. In Propositions 19 and 20, we look specifically at such domains which have finite volume. Then in Propositions 21-27, we analyze such domains more generally.

Proposition 19. *Let Ω be a Reinhardt domain of holomorphy and let $\alpha, \beta > 0$, $-1 < x < 0$, and $y < -1$. If $\Omega \subset U_\alpha^x \cap U_\beta^y$, then $L_h^p(\Omega) \neq \{0\}$ for all p, and moreso, Ω has finite volume.*

Proof. Clearly, if Ω has finite volume then $1 \in L_h^p(\Omega)$, for all p. To see that Ω has finite volume, we first let $R = \left(\frac{\beta}{\alpha}\right)^{1/(x-y)}$. Now, observe that:

$$\begin{aligned} \int_\Omega dV &\leq 4\pi^2 \left(\int_0^R \int_0^{\alpha r_1^x} r_1 r_2 dr_2 dr_1 + \int_R^\infty \int_0^{\beta r_1^y} r_1 r_2 dr_2 dr_1 \right) \\ &= 2\pi^2 \left(\alpha^2 \int_0^R r_1^{1+2x} dr_1 + \beta^2 \int_R^\infty r_1^{1+2y} dr_1 \right). \end{aligned}$$

Now, since $x > -1$, $1 + 2x > -1$. Also, since $y < -1$, $1 + 2y < -1$. Therefore, both integrals above are finite, and so Ω has finite volume. $\square$

Proposition 20. *Let Ω be a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 19. If $\Omega = \Omega^*$ or $\Omega = \Omega^* \setminus V_0$, then Ω is an L^p_h-domain of holomorphy, for all $p \in [1, 2)$.*

Proof. If $\Omega = \Omega^*$, then the result follows from Proposition 1. Now, suppose that $\Omega = \Omega^* \setminus V_0$. Then, as in the proof of Proposition 17, we claim that $z_1^{-1} z_2^{-1} \in L^p_h(\Omega)$, for all $p \in [1, 2)$. As before, since $V_0 \cap \Omega = \varnothing$, $z_1^{-1} z_2^{-1}$ is holomorphic on Ω. Furthermore, if $R = \left(\frac{\beta}{\alpha}\right)^{1/(x-y)}$, then the following is clear:

$$U^x_\alpha \cap U^y_\beta = (U^x_\alpha \cap U_R) \cup \left(\widetilde{U}_R \cap U^y_\beta\right)$$

Now, it was shown in the proof of Proposition 17 that $z_1^{-1} z_2^{-1}$ is L^p on $U^x_\alpha \cap U_1$. Furthermore, since $y < -1$, $-y(2-p) - 2 > 2 - p - 2 = -p$, and so $-\frac{y(2-p)+2}{p} < -1$. Therefore, taking $n = -1$, the argument in Proposition 9 above demonstrates that $z_1^{-1} z_2^{-1}$ is L^p on $\widetilde{U}_R \cap U^y_\beta$. Hence, $z_1^{-1} z_2^{-1} \in L^p_h(U^x_\alpha \cap U^y_\beta \setminus V_0) \subset L^p_h(\Omega)$, for all $p \in [1, 2)$. □

Now, we turn our attention to the general case of domains having fat hulls which are intersections of logarithmic half-planes of the form U^x_α, for $x \leq 0$. In Proposition 21, we show that non-trivial L^1_h functions exist on such domains. Then in Propositions 22 and 23, we discuss when removing V_1 from the fat hull yields an L^1_h-domain of holomorphy. In Propositions 24 and 25, we do the same for V_2, and in Propositions 26 and 27, we do the same for V_0.

Proposition 21. *Let Ω be a Reinhardt domain of holomorphy and let $\alpha, \beta > 0$ and $y < x \leq 0$. If $\Omega \subset U^x_\alpha \cap U^y_\beta$, then $L^1_h(\Omega) \neq \{0\}$.*

Proof. Let $r = \frac{m'}{n'}$ be a rational number in $(-x, -y)$, where m', n' are taken to be positive integers. Now, let $m := 2m' - 2$ and $n := 2n' - 2$. Observe that $m, n \geq 0$. Therefore, $z_1^m z_2^n$ is holomorphic on Ω. We now claim that $z_1^m z_2^n \in L_h^1(\Omega)$. To see this, let $R = \left(\frac{\beta}{\alpha}\right)^{1/(x-y)}$ and observe that

$$\int_\Omega |z_1^m z_2^n| = 4\pi^2 \left(\int_0^R \int_0^{\alpha r_1^x} r_1^{1+m} r_2^{1+n} dr_2 dr_1 + \int_R^\infty \int_0^{\beta r_1^y} r_1^{1+m} r_2^{1+n} dr_2 dr_1 \right)$$
$$= \frac{4\pi^2}{2+n} \left(\alpha^{2+n} \int_0^R r_1^{1+m+x(2+n)} dr_1 + \beta^{2+n} \int_R^\infty r_1^{1+m+y(2+n)} dr_1 \right).$$

The above integral is finite provided that $2+m+x(2+n) > 0 > 2+m+y(2+n)$, which is true if and only if $x > -\frac{2+m}{2+n} > y$. However, since $-r = -\frac{2+m}{2+n}$ and since $r \in (-x, -y)$, the desired result holds. Hence, $z_1^m z_2^n \in L_h^1(\Omega)$. □

Proposition 22. *Let Ω be a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 21. Then if $\Omega = \Omega^* \setminus V_1$ and $\left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \{2,3,4,\ldots\} \neq \varnothing$, then Ω is an L_h^1-domain of holomorphy. (Note, that for the sake of this result, we will use the convention that $-\frac{1}{0} = \infty$.)*

Proof. Let $n' \in \left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \{2,3,4,\ldots\}$. Let $n = n' - 2$. I now claim that $z_1^{-1} z_2^n \in L_h^p(\Omega)$. First, since $n \geq 0$ and since V_1 is disjoint from Ω, this monomial is clearly holomorphic on Ω. Furthermore, by the computation in the proof of Proposition 21 above, the monomial is integrable provided that $\frac{1}{n'} = \frac{2-1}{2+n} \in (-x, -y)$. But since $n' \in \left(-\frac{1}{y}, -\frac{1}{x}\right)$, this follows easily.

Now, by Proposition 21 above and by Proposition 9 in [4], Ω^* is an L_h^1-domain of holomorphy. Hence, let $f \in L_h^1(\Omega^*)$ be a function such that Ω^* is its domain of definition, and define $g(z_1, z_2) := f(z_1, z_2) + z_1^{-1} z_2^n$. Then $g \in L_h^1(\Omega)$ and does not extend to any boundary point of Ω^* or to any point of V_1, since f does not extend to any boundary point of Ω^* and $z_1^{-1} z_2^n$ does not extend to any point in V_1. Hence, Ω is the domain of definition for g, and thus Ω is an L_h^1-domain of holomorphy. □

Proposition 23. *If $\Omega = \left(U_\alpha^x \cap U_\beta^y\right) \setminus V_1$ for some $\alpha, \beta > 0$ and some $y < x < 0$ such that $\left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \{2,3,4,\ldots\} = \varnothing$, then Ω is not an L_h^1-domain of holomorphy and its L_h^1-envelope of holomorphy is Ω^*.*

Proof. We proceed by contradiction. Suppose Ω is an L_h^1-domain of holomorphy. Then, as in the proof of Proposition 16 above, there are $m, n \in \mathbb{Z}$ such that $m < 0 \leq n$ and $z_1^m z_2^n \in L_h^1(\Omega)$. Now, by the calculation in the proof of Proposition 21 above, we can see that this is true only if $\frac{2+m}{2+n} \in (-x, -y)$. Now, note that since $m < 0$, $2 + m < 2$. However, $-x > 0$ and $2 + n > 0$, so $2 + m > 0$. Hence, $m = -1$. Therefore, $-x < \frac{1}{2+n} < -y$, and so $-\frac{1}{y} < 2 + n < -\frac{1}{x}$. Now, since $n \geq 0$, $2 + n \geq 2$. Hence, $\left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \{2, 3, 4, \ldots\} \neq \varnothing$, and this is a contradiction. Hence, Ω is not an L_h^1-domain of holomorphy. Furthermore, we have that every L_h^1-function on Ω extends across V_1 to an L_h^1 function on Ω^*. Therefore, the embedding $L_h^1(\Omega^*) \hookrightarrow L_h^1(\Omega)$ is surjective, and so since Ω^* is an L_h^1-domain of holomorphy, it is the L_h^1-envelope of holomorphy for Ω. □

Proposition 24. *Let Ω be a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 21. Then if $\Omega = \Omega^* \setminus V_2$ and $(-x, -y) \cap \{2, 3, 4, \ldots\} \neq \varnothing$, then Ω is an L_h^1-domain of holomorphy.*

Proof. Define $F : \mathbb{C}^2 \to \mathbb{C}^2$ by $F(z_1, z_2) := (z_2, z_1)$. Then note that F induces an isometric isomorphism $L_h^p\left(U_\alpha^x \cap U_\beta^y \setminus V_2\right) \cong L_h^p\left(U_{\alpha^*}^{1/x} \cap U_{\beta^*}^{1/y} \setminus V_1\right)$, where $\alpha^* = \alpha^{-1/x}$ and $\beta^* = \beta^{-1/y}$. Therefore, by Proposition 22, Ω is an L_h^1-domain of holomorphy. □

Proposition 25. *If $\Omega = \left(U_\alpha^x \cap U_\beta^y\right) \setminus V_2$ for some $\alpha, \beta > 0$ and some $y < x < 0$ such that $(-x, -y) \cap \{2, 3, 4, \ldots\} = \varnothing$, then Ω is not an L_h^1-domain of holomorphy and its L_h^1-envelope of holomorphy is Ω^*.*

Proof. As in the proof of Proposition 24, F induces an isometric isomorphism $L_h^p\left(U_\alpha^x \cap U_\beta^y \setminus V_2\right) \cong L_h^p\left(U_{\alpha^*}^{1/x} \cap U_{\beta^*}^{1/y} \setminus V_1\right)$, where $\alpha^* = \alpha^{-1/x}$ and $\beta^* = \beta^{-1/y}$. Hence, by Proposition 23, Ω is not an L_h^1-domain of holomorphy. □

Proposition 26. *Let Ω be a Reinhardt domain of holomorphy such that Ω^* satisfies the hypotheses of Proposition 21 and such that $\mathbb{Z} \cap (-x, -y) \neq \varnothing$ and $\mathbb{Z} \cap \left(-\frac{1}{y}, -\frac{1}{x}\right) \neq \varnothing$. Then if $\Omega = \Omega^* \setminus V_0$, then Ω is an L_h^1-domain of holomorphy.*

Proof. The case in which the intervals $(-x, -y)$ and $\left(-\frac{1}{y}, -\frac{1}{x}\right)$ each contain a positive integer greater than 1 follows from Propositions 21, 22, and 24 above, by considering $f_1 + f_2$ where

$\Omega^* \setminus V_j$ is the domain of existence of f_j. If either of these intervals contains 1, then both contain 1. In this case, $z_1^{-1}z_2^{-1} \in L_h^1(\Omega)$ and so is an integrable monomial on Ω which does not extend to V_0. Therefore, in this case also, Ω is an L_h^1-domain of holomorphy. □

Proposition 27. *Let $\Omega = \left(U_\alpha^x \cap U_\beta^y\right) \setminus V_0$ for some $\alpha, \beta > 0$ and some $y < x < 0$. Then if Ω does not satisfy the conditions of Proposition 26, then Ω is not an L_h^1-domain of holomorphy. Furthermore, if $\Omega^* \setminus V_1$ satisfies the conditions of Proposition 22, then it is the L_h^1-envelope of holomorphy for Ω, whereas if $\Omega^* \setminus V_2$ satisfies the conditions of Proposition 24, then it is the L_h^1-envelope of holomorphy for Ω. Otherwise, Ω^* is the L_h^1-envelope of holomorphy for Ω.*

Proof. This follows from Propositions 22-25 above. □

2.2.3 Domains with Fat Hull Disjoint from V_0

We now only have Reinhardt domains of holomorphy with fat hull disjoint from V_0 to consider. However, since these domains are always fat by Proposition 2, we need only determine when such domains have nontrivial L_h^1 functions. In Proposition 28, we give a condition for such a domain to fail to be an L_h^p-domain of holomorphy. Finally, in Proposition 29, we show that the condition given in Proposition 28 is the only way an intersection of two logarithmic half-planes which is disjoint from V_0 can fail to be an L_h^p-domain of holomorphy.

Proposition 28. *If $\Omega = \widetilde{U}_\alpha^x \cap U_\beta^x$ with $0 < \alpha < \beta$ and $x \in \mathbb{R}$, then $L_h^1(\Omega) = \{0\}$.*

Proof. Since Ω is fat, by Proposition 9 in [4], it suffices to show that there are no integrable monomials of the form $z_1^m z_2^n$, where $m, n \in \mathbb{Z}$. First, note that when $n \neq -2$,

$$\int_\Omega |z_1^m z_2^n| = 4\pi^2 \int_0^\infty \int_{\alpha r_1^x}^{\beta r_1^x} r_1^{1+m} r_2^{1+n} dr_2 dr_1 = \frac{4\pi^2}{2+n}\left(\beta^{2+n} - \alpha^{2+n}\right) \int_0^\infty r_1^{1+m+x(2+n)} dr_1.$$

But no power function is integrable on the interval $(0, \infty)$. Hence, $z_1^m z_2^n$ is not integrable if $n \neq$

-2. Now, observe that

$$\int_\Omega \left|z_1^m z_2^{-2}\right| = 4\pi^2 \int_0^\infty \int_{\alpha r_1^x}^{\beta r_1^x} r_1^{1+m} r_2^{-1} dr_2 dr_1 = 4\pi^2 \log\left(\frac{\beta}{\alpha}\right) \int_0^\infty r_1^{1+m} dr_1.$$

However, once again, no power function is integrable on $(0,\infty)$. Therefore, $z_1^m z_2^n$ is not integrable on Ω. □

Proposition 29. *Suppose $\Omega \neq \varnothing$ is not a logarithmic half-plane, but that $\Omega = H_1 \cap H_2$, where H_j is a logarithmic half-plane for $j = 1, 2$. Then if Ω does not satisfy the condition of Proposition 28 and $\Omega \cap V_0 = \varnothing$, then $L_h^1(\Omega) \neq \{0\}$. Furthermore, any Reinhardt domain of holomorphy contained in Ω is an L_h^p-domain of holomorphy, for all p.*

Proof. First note that since $\Omega \cap V_0 = \varnothing$, every monomial of the form $z_1^m z_2^n$ is holomorphic on Ω. Also, we may assume either (1) that $H_1 \cap V_0 = \varnothing$, or (2) that $H_1 \cap V_1 = \varnothing$ and $H_2 \cap V_2 = \varnothing$.

(1) Suppose that the former is true. Then by a dilation, we may suppose that $H_1 = \widetilde{U}_1^x$, for some $x < 0$. We suppose first that $H_2 = U_\alpha^y$ and $\beta = \alpha^{1/(x-y)}$. Note that $x \neq y$ since Ω does not satisfy the hypotheses of Proposition 28. Then if $y > x$, $\Omega \subset H_1 \cap \widetilde{U}_\beta$. Let $m < x - 2$ be an integer. We now show that $z_1^m z_2^{-3} \in L_h^1(\Omega)$:

$$\int_\Omega \left|z_1^m z_2^{-3}\right| \leq 4\pi^2 \int_\beta^\infty \int_{r_1^x}^\infty r_1^{1+m} r_2^{-2} dr_2 dr_1 = 4\pi^2 \int_\beta^\infty r_1^{1+m-x} dr_1.$$

Since $1 + m - x < 1 + x - 2 - x = -1$, the above integral is finite. Hence, $z_1^m z_2^{-3} \in L_h^1(\Omega)$. It now follows from Proposition 1 that Ω is an L_h^p-domain of holomorphy, for all $p \geq 1$.

Now, suppose $y < x$. Observe that $\Omega \subset H_1 \cap U_\beta$ and let $F : (z_1, z_2) \mapsto (z_2, z_1)$. Then F induces an isometric isomorphism $L_h^p(\Omega) \cong L_h^p(F(\Omega))$, for all p. Now observe that $F(H_1) = \widetilde{U}_1^{1/x}$ and $F(U_\beta) = U_\beta^0$. Therefore, $F(\Omega) \subset F(H_1 \cap U_\beta) \subset \widetilde{U}_1^{1/x} \cap U_\beta^0$. But $\frac{1}{x} < 0$. Hence, by the preceding paragraph, $L_h^1(F(\Omega)) \neq \{0\}$, and so Ω is an L_h^p-domain of holomorphy, for all p. Again, transposing coordinates yields the desired result if $H_2 = U_\alpha$ or if $H_2 = \widetilde{U}_\alpha$.

(2) Now, suppose that for $j = 1, 2$, $H_j \cap V_0 \neq \varnothing$, but that $H_j \cap V_j = \varnothing$. In this case, $H_1 = U_\alpha^x$ or $H_1 = \widetilde{U}_\alpha$, for some $\alpha > 0, x > 0$. Also, $H_2 = \widetilde{U}_\beta^y$, for some $\beta > 0, y > 0$. Suppose first that $H_1 = U_\alpha^x$. Then, if $y > x$, then Ω is bounded and so this case follows trivially. However, if $y < x$, then let (R_1, R_2) be the solution to the system:

$$\begin{cases} r_2 = \alpha r_1^x \\ r_2 = \beta r_1^y \end{cases}$$

Then $\Omega \subset \widetilde{U}_{R_1} \cap \widetilde{U}_{R_2}^0$. Now, suppose that $H_1 = \widetilde{U}_\alpha$. Then since

$$|z_1| > \alpha \text{ and } |z_2| > \beta\,|z_1|^y \implies |z_2| > \beta\alpha^y,$$

that $H_1 \cap H_2 \subset H_1 \cap \widetilde{U_{\beta^*}^0}$, where $\beta^* = \beta\alpha^y$. Hence, it suffices to show that $L_h^1(\Omega) \neq \{0\}$, if $\Omega = \widetilde{U}_\alpha \cap \widetilde{U}_\beta^0$. But in this case, $z_1^{-3} z_2^{-3} \in L_h^1(\Omega)$. To see this, observe:

$$\int_\Omega \left|z_1^{-3} z_2^{-3}\right| = 4\pi^2 \int_\alpha^\infty \int_\beta^\infty r_1^{-2} r_2^{-2} dr_2 dr_1 = \frac{4\pi^2}{\alpha\beta} < \infty.$$

We have now shown that $L_h^1(\Omega) \neq \{0\}$. It now follows from Proposition 1 that every Reinhardt domain of holomorphy contained in Ω is an L_h^p-domain of holomorphy, for all $p \in [1, \infty]$, since every such domain of holomorphy must be fat. □

2.3 A General Characterization in Terms of Logarithmic Half-Planes

Now, having analyzed separately the bounded and unbounded Reinhardt domains of holomorphy, we may state our first characterization in terms of logarithmic half-planes. As will be seen in the proof, this theorem mostly summarizes the results above, which adequately characterize the case when a domain is an intersection of two logarithmic half-planes. The main fact remaining to prove is that the conditions suffice for describing Reinhardt domains of holomorphy which are intersections of more than two logarithmic half-planes.

Theorem 1. *Suppose that $\Omega \subsetneq \mathbb{C}^2$ is a Reinhardt domain of holomorphy. Then Ω is an L^1_h-domain of holomorphy if and only if one of the following conditions holds (note that the conditions are not mutually exclusive):*

1. *Ω is bounded.*
2. *Ω is fat, is not a logarithmic half-plane, and is neither $\widetilde{U}^x_\alpha \cap U^x_\beta$ nor $\widetilde{U}_\alpha \cap U_\beta$, for any $\alpha, \beta > 0$ and $x \in \mathbb{R}$.*
3. *$\Omega^* \subset \widetilde{U}_\alpha \cap U^x_\beta$, for some $\alpha, \beta > 0$ and $x \in \mathbb{R}$.*
4. *$\Omega^* \subset U^x_\alpha \cap U^y_\beta$, where $\alpha, \beta > 0$ and $x > \max\{0, y\}$ and $(y, x) \cap \mathbb{Z} \neq \varnothing$.*
5. *$\Omega^* \subset U^x_\alpha \cap U_\beta$ for some $\alpha, \beta > 0$ and $x < 0$, and $\Omega = \Omega^* \setminus V_2$.*
6. *$\Omega^* \subset U^x_\alpha \cap U_\beta$ for some $\alpha, \beta > 0$ and $x \in \left(-\frac{1}{2}, 0\right)$ and $\Omega = \Omega^* \setminus V_1$.*
7. *$\Omega^* \subset U^x_\alpha \cap U_\beta$ for some $\alpha, \beta > 0$ and $x \in (-1, 0)$ and $\Omega = \Omega^* \setminus V_0$.*
8. *$\Omega^* \subset U^x_\alpha \cap U^y_\beta$ for some $\alpha, \beta > 0$ and $\left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \{2, 3, 4, \ldots,\} \neq \varnothing$ and $\Omega = \Omega^* \setminus V_1$.*
9. *$\Omega^* \subset U^x_\alpha \cap U^y_\beta$ for some $\alpha, \beta > 0$ and $(-x, -y) \cap \{2, 3, 4, \ldots,\} \neq \varnothing$ and $\Omega = \Omega^* \setminus V_2$.*
10. *$\Omega^* \subset U^x_\alpha \cap U^y_\beta$ for some $\alpha, \beta > 0$ and $y < x < 0$ and $(y, x) \cap \mathbb{Z} \neq \varnothing \neq \left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \mathbb{Z}$ and $\Omega = \Omega^* \setminus V_0$.*
11. *$F(\Omega)$ satisfies any of the above conditions where $F(z_1, z_2) := (z_2, z_1)$.*

This follows from Propositions 30 and 31 below.

Proposition 30. *Each of the conditions in the theorem above is sufficient for Ω to be an L_h^1-domain of holomorphy.*

Proof. Condition (1) and conditions (3)-(10) are sufficient by Propositions 4, 9, 11, 14, 15, 17, 22, 24, and 26 respectively. Furthermore, it is clear that the property of being an L_h^1-domain of holomorphy is invariant under permutations of coordinates, and so Condition (11) is sufficient. It remains to show that Condition (2) is a sufficient condition.

Since Ω is a Reinhardt domain of holomorphy, Ω is logarithmically convex. Therefore, since Ω is fat and properly contained in $\mathbb{C}^2$, there exists a nonempty family $(H_\lambda)_{\lambda\in\Lambda}$ of distinct logarithmic half-planes such that $\Omega = \bigcap_{\lambda\in\Lambda} H_\lambda$. Also, since Ω is not a logarithmic half-plane, $|\Lambda| \neq 1$.

Therefore, we first suppose that $\Omega = H_1 \cap H_2$, with $H_1 \neq H_2$. First note that if $H_1 = U_\alpha$ (resp. $\widetilde{U}_\alpha$) and $H_2 = U_\beta$ (resp. $\widetilde{U}_\beta$), then $\Omega = U_{\min\{\alpha,\beta\}}$ (resp. $\widetilde{U}_{\max\{\alpha,\beta\}}$) and so Ω would be a logarithmic half-plane contrary to our hypothesis.

Now, if there exist $\alpha, \beta > 0$ and $x, y \in \mathbb{R}$ such that $H_1 = U_\alpha^x$ and $H_2 = U_\beta^y$, then by Propositions 1, 10, and 21, if Ω is not an L_h^1-domain of holomorphy, then $x = y$. But if $x = y$, then $\Omega = U^x_{\min\{\alpha,\beta\}}$ which is contrary to our hypothesis.

Next, suppose that $H_1 = \widetilde{U}_\alpha^x$ and $H_2 = \widetilde{U}_\beta^y$. Suppose without loss of generality that $y < x$. If $y < 0$, then H_2 is a logarithmic half-plane which is disjoint from V_0. Hence, $\Omega \cap V_0 = \varnothing$ and so by Proposition 29, Ω is an L_h^1-domain of holomorphy. Now suppose that $x > y > 0$ and then observe that $F\left(\widetilde{U}_\alpha^x\right) = U_{\alpha^*}^{1/x}$, where $\alpha^* = \alpha^{-1/x}$. Also, $F\left(\widetilde{U}_\beta^y\right) = U_{\beta^*}^{1/y}$, where $\beta^* = \beta^{-1/y}$. Hence, $F(\Omega) = U_{\alpha^*}^{1/x} \cap U_{\beta^*}^{1/y}$. But by the previous paragraph, this implies that $F(\Omega)$ is an L_h^1-domain of holomorphy, and so by Condition (11) above, Ω is an L_h^1-domain of holomorphy.

Now, note that $F\left(\widetilde{U}_\alpha^0\right) = \widetilde{U}_\alpha$, and so if $x = 0$ or $y = 0$, then $F(\Omega)$ satisfies Condition (3), and so Ω satisfies Condition (11), and so is an L^1_h-domain of holomorphy.

Next, note that the case where $H_1 = U^x_\alpha$ and $H_2 = U_\beta$ follows from Propositions 4 and 13. The case where $H_1 = U^x_\alpha$ and $H_2 = \widetilde{U}_\beta$ follows from Proposition 8. When $H_1 = \widetilde{U}^x_\alpha$, we have similar results via Condition (11) above, as in the preceding paragraphs.

Finally, suppose $H_1 = \widetilde{U}^x_\alpha$ and $H_2 = U^y_\beta$. If $x < 0$, then the conclusion follows from Proposition 29. So we suppose $x \geq 0$. If $0 \leq y < x$, then Ω is bounded and so the result follows from Proposition 4 above. If $y < 0 < x$, then $F(\Omega) = U^{1/x}_{\alpha^{-1/x}} \cap U^{1/y}_{\beta^{-1/y}}$. Therefore, by Propositions 1 and 10, $F(\Omega)$ is an L^1_h-domain of holomorphy, and so Ω satisfies Condition (11) and is itself an L^1_h-domain of holomorphy. If $y < x = 0$, then the conclusion follows similarly from Propositions 1 and 8 via Condition (11). If $y > x$, then the result follows from Proposition 29 above. This completes the case where Ω is an intersection of two logarithmic half-planes.

Now, suppose $|\Lambda| > 2$. Then $\Omega \subset H_{\lambda_1} \cap H_{\lambda_2}$, for each $\lambda_1, \lambda_2 \in \Lambda$. Therefore, since $\Omega = \Omega^*$, Ω certainly is an L^1_h-domain of holomorphy, unless every pair of logarithmic half-planes is one of the exceptions given in the statement of Condition (2). Thus, there exist disjoint $\Lambda_1, \Lambda_2 \subset \Lambda$ such that $\Lambda = \Lambda_1 \cup \Lambda_2$ and such that either (a) there is an $x \in \mathbb{R}$ such that for each $\lambda \in \Lambda_1$, $H_\lambda = \widetilde{U}^x_{\alpha_\lambda}$ and for each $\lambda \in \Lambda_2$, $H_\lambda = U^x_{\beta_\lambda}$, or (b) for each $\lambda \in \Lambda_1$, $H_\lambda = \widetilde{U}_{\alpha_\lambda}$ and for each $\lambda \in \Lambda_2$, $H_\lambda = U_{\beta_\lambda}$.

Now, in case (a), observe that $\bigcap_{\lambda \in \Lambda_1} H_\lambda = \widetilde{U}^x_{\sup \alpha_\lambda}$ and $\bigcap_{\lambda \in \Lambda_2} H_\lambda = U^x_{\inf \beta_\lambda}$. [Observe that if $\sup \alpha_\lambda = \infty$, then $\Omega = \varnothing$, while if $\inf \beta_\lambda = 0$, then $\Omega \subset V_0$, and so is not an open set.] Hence, this case reduces to the case where $|\Lambda| = 2$. Finally, in case (b), observe that $\bigcap_{\lambda \in \Lambda_1} H_\lambda = \widetilde{U}_{\sup \alpha_\lambda}$ and $\bigcap_{\lambda \in \Lambda_2} H_\lambda = U_{\inf \beta_\lambda}$. Once again this reduces to the case where $|\Lambda| = 2$, and this suffices to prove that if Condition (2) holds, Ω is an L^1_h-domain of holomorphy. $\square$

Proposition 31. *It is necessary that a Reinhardt L_h^1-domain of holomorphy properly contained in $\mathbb{C}^2$ satisfy at least one of Conditions (1)-(11) from Theorem 1.*

Proof. We first note that since Ω is an L_h^1-domain of holomorphy, Ω is not a logarithmic half-plane. Therefore, since Ω is a Reinhardt domain of holomorphy, there exists a family $\{H_\lambda\}_{\lambda\in\Lambda}$ of at least two logarithmic half-planes such that $\Omega^* = \bigcap_{\lambda\in\Lambda} H_\lambda$.

We suppose first that $|\Lambda| = 2$, so that $\Omega^* = H_1 \cap H_2$. We now suppose that there exist $\alpha, \beta > 0$ and $x, y \in \mathbb{R}$ such that $H_1 = U_\alpha^x$ and $H_2 = U_\beta^y$. Since Ω is not a logarithmic half-plane, $x \neq y$, so we suppose without loss of generality that $y < x$. Now, if $\Omega = \Omega^*$, then Ω satisfies Condition (2). Suppose now that $\Omega \neq \Omega^*$. Then, if $x > 0$, $\Omega = \Omega^* \setminus V_2$, and so Proposition 12 above yields that $(y, x) \cap \mathbb{Z} \neq \varnothing$. Hence, in this case, Ω satisfies Condition (4). Now suppose that $x < 0$. Then if $\Omega = \Omega^* \setminus V_1$, then by Proposition 23 above, $\left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \{2, 3, 4, \ldots\} \neq \varnothing$ and so Ω satisfies Condition (8). Similarly, if $\Omega = \Omega^* \setminus V_2$, then by Proposition 25, Ω satisfies Condition (9), whereas if $\Omega = \Omega^* \setminus V_0$, then by Proposition 27, Ω satisfies Condition (10). Finally, if $x = 0$, then $F\left(U_\alpha^x\right) = U_\alpha$, and so $F(\Omega)$ satisfies Condition (5) if $\Omega = \Omega^* \setminus V_2$; Condition (6) if $\Omega = \Omega^* \setminus V_1$, by Proposition 16; and Condition (7) if $\Omega = \Omega^* \setminus V_0$, by Proposition 18. Hence, Ω satisfies Condition (11).

Now, suppose that for some $\alpha, \beta > 0$ and $x, y \in \mathbb{R}$, $H_1 = U_\alpha^x$ and $H_2 = \widetilde{U}_\beta^y$. Since Ω is an L_h^1-domain of holomorphy, $x \neq y$ by Proposition 28. Now, if $y < x$, then $\Omega = \Omega^*$ and so Ω satisfies Condition (2). Now, suppose that $y > x$. In this case, if $x \geq 0$, then Ω satisfies Condition (1). If $y \leq 0$ on the other hand, then Ω satisfies Condition (2). Now, suppose $0 \in (x, y)$. In this case, $F\left(H_1\right) = U_{\alpha^*}^{1/x}$ and $F\left(H_2\right) = U_{\beta^*}^{1/y}$, where $\alpha^* = \alpha^{-1/x}$ and $\beta^* = \beta^{-1/y}$. Hence, this case now reduces to the preceding paragraph.

Now, suppose that $H_1 = U_\alpha^x$ and $H_2 = U_\beta$. First, note that if $x \geq 0$, then Ω satisfies Condition (1). Now, suppose that $x < 0$. If $\Omega = \Omega^*$, then Ω satisfies Condition (2). Also, if $\Omega = \Omega^* \setminus V_2$, then

Ω satisfies Condition (5). On the other hand, if $\Omega = \Omega^* \setminus V_1$, then by Proposition 16, Ω satisfies Condition (6), whereas if $\Omega = \Omega^* \setminus V_0$, then Ω satisfies Condition (7) by Proposition 18.

If $H_1 = U_\alpha^x$ and $H_2 = \widetilde{U}_\beta$, then Ω satisfies Condition (3).

Now, suppose that $H_1 = \widetilde{U}_\alpha^x$ and $H_2 = \widetilde{U}_\beta^y$. Since Ω is not a logarithmic half-plane, $x \neq y$. Suppose without loss of generality that $y < x$. Observe now that if $y < 0$, then Ω satisfies Condition (2). Suppose now that $y = 0$. Then $F(H_1) = U_{\alpha^*}^{1/x}$, where $\alpha^* = \alpha^{-1/x}$, and $F(H_2) = \widetilde{U}_\beta$. Thus, $F(\Omega)$ satisfies Condition (3) and so Ω satisfies Condition (11). Finally, suppose $y > 0$. Then $F(H_1) = U_{\alpha^*}^{1/x}$, where $\alpha^* = \alpha^{-1/x}$, and $F(H_2) = U_{\beta^*}^{1/y}$, where $\beta^* = \beta^{-1/y}$. Now, since F has constant Jacobian, $F(\Omega)$ is an L_h^1-domain of holomorphy. Hence, by Proposition 12, $F(\Omega)$ satisfies Condition (4), and so Ω satisfies Condition (11).

Next, suppose $H_1 = \widetilde{U}_\alpha^x$ and $H_2 = U_\beta$. Then, if $x \leq 0$, Ω satisfies Condition (2). Suppose now that $x > 0$. Then $F(H_1) = U_{1/\alpha}^{1/x}$ and $F(H_2) = U_\beta^0$. Since $x > 0$, $\frac{1}{x} > 0$. Therefore, by Proposition 12, $F(\Omega)$ satisfies Condition (4). Hence, Ω satisfies Condition (11).

Now, if $H_1 = \widetilde{U}_\alpha^x$ and $H_2 = \widetilde{U}_\beta$, then Ω satisfies Condition (2). Since Ω is an L_h^1-domain of holomorphy, if $H_1 = U_\alpha$, neither $H_2 = U_\beta$ nor $H_2 = \widetilde{U}_\beta$, and for similar reason, if $H_1 = \widetilde{U}_\alpha$, $H_2 \neq \widetilde{U}_\beta$. Hence, since all possible pairs have been considered, this proves the case when $|\Lambda| = 2$.

Furthermore, an intersection of logarithmic half-planes of the form U_α (resp. $\widetilde{U}_\alpha$) which is still an open set is another logarithmic half-plane of the form U_α (resp. $\widetilde{U}_\alpha$). Hence, we can suppose there is at most one of each type in $\{H_\lambda\}_{\lambda \in \Lambda}$. Furthermore, by the arguments given in the last two paragraphs of the proof of Proposition 30, for each x, we may assume that there is at most one $\alpha_x > 0$ such that $U_{\alpha_x}^x = H_\lambda$ for some λ and at most one $\beta_x > 0$ such that $\widetilde{U}_{\beta_x}^x = H_\lambda$ for some λ.

Now, suppose $|\Lambda| > 2$. Note that if there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $(H_{\lambda_1} \cap H_{\lambda_2}) \setminus V_j$ satisfies one of Conditions (1)-(11), where $\Omega = \Omega^* \setminus V_j$, then Ω satisfies the same condition. We will now suppose that this hypothesis is not the case, seeking a contradiction.

If Ω were fat, then Ω would satisfy Condition (2), and so Ω is not fat. Then $\Omega^* \cap V_0 \neq \varnothing$ and $\Omega = \Omega^* \setminus V_j$, for some $j \in \{0, 1, 2\}$. Suppose that for some $\lambda, \mu \in \Lambda$, $H_\lambda = U_\alpha^x$ and $H_\mu = U_\beta^y$ with $0 \leq y < x$. Now, if for some $\nu \in \Lambda$, $H_\nu = U_\gamma$, then Ω satisfies Condition (1). However, if for some ν, $H_\nu = \widetilde{U}_\nu^t$, then Ω is fat contrary to hypothesis. Also, if for some ν, $H_\nu = \widetilde{U}_\gamma$, then Ω satisfies Condition (3). Finally, if $H_\nu = U_\gamma^t$, for some $t < 0$, then Ω satisfies Condition (4) since $0 \in (t, x)$.

Now, if $y < 0 < x$, then Ω satisfies Condition (4), so suppose that $y < x \leq 0$. Then if $H_\nu = \widetilde{U}_\gamma$, then Ω satisfies Condition (3). If $H_\nu = U_\gamma$ and $x = 0$, then Ω satisfies Condition (1). Now, suppose $H_\nu = \widetilde{U}_\gamma^t$. Since Ω is not fat, $t \geq 0$, and so Ω satisfies Condition (1) when $x = 0$. If $x < 0$, then $F(\Omega)$ satisfies Condition (4) when $t > 0$ and Condition (3) when $t = 0$, so that Ω satisfies Condition (11). Now, suppose $H_\nu = U_\gamma^t$ for $t > 0$, then Ω satisfies Condition (4). The case where $H_\nu = U_\gamma$ is covered more generally in the following paragraph.

We now suppose that there exists $\mu \in \Lambda$ such that $H_\mu = U_\alpha$ for some α and that for all $\lambda \neq \mu$, there exist $x_\lambda < 0, \beta_\lambda > 0$ such that $H_\lambda = U_{\beta_\lambda}^{x_\lambda}$. First observe that $\bigcap_{\lambda \neq \mu} H_\lambda$ must be an open set. If $j = 1$, then since Ω doesn't satisfy Condition (8), we must have the property that for all λ_1, λ_2, there is no positive integer strictly greater than 1 which is between $-x_{\lambda_1}^{-1}$ and $-x_{\lambda_2}^{-1}$. Therefore, $T := \{-x_\lambda^{-1} : \lambda \neq \mu\}$ is contained either in $(0, 2]$ or in $[n, n+1]$. However, Ω also does not satisfy Condition (6) and so for each $\lambda \neq \mu$, $-x_\lambda^{-1} \leq 2$. Hence, $T \subset (0, 2]$, and so $\{x_\lambda : \lambda \neq \mu\} \subset \left(-\infty, -\frac{1}{2}\right]$. Let $\beta = \inf_{\lambda \neq \mu} \beta_\lambda \cdot \alpha^{x_\lambda + 1/2}$. If $\beta = 0$, then $\bigcap_{\lambda \in \Lambda} H_\lambda = \bigcap_{\lambda \neq \mu} H_\lambda$ and so the case reduces to that dealt with two paragraphs below (based on Lemma 2). Now, if $\beta \neq 0$, let $S := U_\alpha \cap U_\beta^{-1/2}$. By construction, $S \subset \Omega^*$. Therefore, $S \setminus V_1 \subset \Omega$. Also, by Proposition

16, $S \setminus V_1$ is not an L_h^1-domain of holomorphy. Hence, every L_h^1-monomial on $S \setminus V_1$ is also an L_h^1-monomial on S. It now follows that Ω has no L_h^1-monomials that do not extend to V_1, but then Ω is not an L_h^1-domain of holomorphy contrary to hypothesis. A similar contradiction is obtained by arguing from Conditions (5) and (9) if $j = 2$, and from Conditions (7) and (10) if $j = 0$.

Now, suppose that for each $\lambda \in \Lambda$, there is some $\alpha_\lambda > 0$ and some $x_\lambda \geq 0$ such that $H_\lambda = U_{\alpha_\lambda}^{x_\lambda}$. By Lemma 3 below, we have that there exist positive β_1, β_2 and real numbers $0 \leq y_1 < y_2$ such that $y_2 - y_1 \leq 1$ and $S := U_{\beta_1}^{y_1} \cap U_{\beta_2}^{y_2} \subset \Omega^*$. Now observe that S satisfies Condition (2) and so S is an L_h^1-domain of holomorphy. It now follows that $S \setminus V_j$ is an L_h^1-domain of holomorphy. [To see this, observe that since $\Omega = \Omega^* \setminus V_j$, we have that $S \setminus V_j \subset \Omega$. Therefore, let f be a holomorphic function for which S is the domain of existence and g be a holomorphic function for which Ω is the domain of existence. By Lemma 1, there must be a monomial z^m in the Laurent series expansion of g such that $m_j < 0$ which is integrable on Ω. Since $S \subset \Omega^*$, note that z^m is also integrable on S. It is now clear that $S \setminus V_j$ is the domain of existence for $f(z) + z^m$.] Hence, by Proposition 12, $S \setminus V_j$ must satisfy Condition (4). But then (y_1, y_2) contains an integer, and so Ω satisfies Condition (4) also, which is a contradiction.

Next, suppose for each $\lambda \in \Lambda$, there is an α_λ and some $x_\lambda \leq 0$ such that $H_\lambda = U_{\alpha_\lambda}^{x_\lambda}$. By Lemma 2 below, there exist real $\alpha > 0$ and $y_1 < y_2 \leq 0$, with $(-y_2, -y_1) \cap \{2, 3, 4, \ldots\} = \varnothing$ and $S := U_\alpha^{y_1} \cap U_\alpha^{y_2}$. Since Ω is an L_h^1-domain of holomorphy, and $S \setminus V_2 \subset \Omega$, there exists an L_h^1 monomial on $S \setminus V_2$ which does not extend holomorphically to S. However, by Proposition 25, S is the L_h^1-envelope of holomorphy of $S \setminus V_2$. This is a contradiction. Similar contradictions can be derived if $\Omega = \Omega^* \setminus V_1$ or $\Omega = \Omega^* \setminus V_0$ via Propositions 23 and 27 respectively.

Now suppose for some $\lambda, \mu \in \Lambda$, $H_\lambda = \widetilde{U}_\alpha^x$ and $H_\mu = \widetilde{U}_\beta^y$. Since Ω is not fat, we may suppose that $0 \leq y < x$. If $y = 0$, then $F(\Omega)$ satisfies Condition (3), and so Ω satisfies Condition (11). Now, if $y > 0$, then $F(H_\lambda) = U_{\alpha^*}^{1/x}$ and $F(H_\mu) = U_{\beta^*}^{1/y}$, with $0 < \frac{1}{x} < \frac{1}{y}, \alpha^* = \alpha^{-1/x}$, and $\beta^* = \beta^{-1/y}$.

Since this case was dealt with above, $F(\Omega)$ satisfies one of Conditions (1)-(11). Hence, Ω satisfies one of Conditions (1)-(11) (since $F = F^{-1}$). Finally, if $H_\lambda = U_\alpha$ and $H_\mu = \widetilde{U}_\beta$, then since Ω is not fat, Ω satisfies Condition (1). This completes the proof. $\square$

Lemma 2. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of logarithmic half-planes in $\mathbb{C}^2$ such that for each $\lambda \in \Lambda$, there exists $\alpha_\lambda > 0$ and $x_\lambda < 0$ such that $H_\lambda = U_{\alpha_\lambda}^{x_\lambda}$. Furthermore, suppose that for each $\lambda, \mu \in \Lambda$, $H_\lambda \cap H_\mu$ does not satisfy Conditions (9) of Theorem 1 above. If Ω is an L_h^1-domain of holomorphy with $\Omega^* = \bigcap_{\lambda\in\Lambda} H_\lambda$ and $\Omega = \Omega^* \setminus V_2$, then there exists $\alpha > 0$ and real numbers $y_1 < y_2 \leq 0$ such that $(-y_2, -y_1) \cap \{2, 3, 4, \ldots,\} = \varnothing$ and $S := U_\alpha^{y_1} \cap U_\alpha^{y_2} \subset \Omega^*$.*

Proof. Note that $\{x_\lambda\}$ is contained in either $[-2, 0]$ or $[m, m+1]$, for some negative integer m, since for all $\lambda, \mu \in \Lambda$, $H_\lambda \cap H_\mu$ does not satisfy Condition (9). We now show that $\alpha := \inf\{\alpha_\lambda\} > 0$. To see this, suppose for contradiction that $\alpha = 0$. Note that for all $z \in \Omega \setminus V_0, \lambda \in \Lambda$, $|z_2| < \alpha_\lambda |z_1|^{x_\lambda}$. But since $\{x_\lambda\}$ is bounded and $\alpha = 0$, it now follows that $|z_2| = 0$, for all $z \in \Omega \setminus V_0$. Thus, $\Omega \subset V_0$, and thus Ω is not a subdomain of $\mathbb{C}^2$. But this is a contradiction. Hence, $\alpha > 0$.

Let y_1 be the greatest integer less than or equal to every member of $\{x_\lambda\}$ and let y_2 be the least integer greater than or equal to every member of $\{x_\lambda\}$. Note that by construction, $y_1 < y_2 \leq 0$, and $(-y_2, -y_1) \cap \{2, 3, 4, \ldots\} = \varnothing$. Define $S := U_\alpha^{y_1} \cap U_\alpha^{y_2}$. We claim that $S \subset \Omega^*$. To see this, suppose $z \in S$. If $|z_1| \leq 1$, then $|z_2| < \alpha$.

$$|z_1| \leq 1 \text{ and } x_\lambda \leq y_2 \implies \alpha_\lambda |z_1|^{x_\lambda} \geq \alpha_\lambda |z_1|^{y_2} \geq \alpha |z_1|^{y_2} > |z_2|.$$

Therefore, for each $\lambda \in \Lambda, z \in H_\lambda$, and so $z \in \Omega^*$. Therefore, $(S \cap \{|z_1| \leq 1\}) \subset \Omega^*$. On the other hand, if $|z_1| > 1$, then $|z_2| < \alpha |z_1|^{-2}$.

$$|z_1| > 1 \text{ and } x_\lambda \geq y_1 \implies \alpha_\lambda |z_1|^{x_\lambda} \geq \alpha_\lambda |z_1|^{y_1} \geq \alpha |z_1|^{y_1} > |z_2|.$$

Therefore, for each $\lambda \in \Lambda, z \in H_\lambda$, and so $z \in \Omega^*$. Therefore, $(S \cap \{|z_1| > 1\}) \subset \Omega^*$. Hence, $S \subset \Omega^*$. □

Lemma 3. *Let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a family of logarithmic half-planes in $\mathbb{C}^2$ such that for each $\lambda \in \Lambda$, there exists $\alpha_\lambda > 0$ and $x_\lambda \geq 0$ such that $H_\lambda = U_{\alpha_\lambda}^{x_\lambda}$. Furthermore, suppose that for each $\lambda, \mu \in \Lambda$, $H_\lambda \cap H_\mu$ does not satisfy Condition (4) in Theorem 1. If Ω is a non-fat L_h^1-domain of holomorphy with $\Omega^* = \bigcap_{\lambda \in \Lambda} H_\lambda$, then there exist $\beta_1, \beta_2 > 0$ and real numbers $0 \leq y_1 < y_2$ such that $y_2 - y_1 \leq 1$ and $S := U_{\beta_1}^{y_1} \cap U_{\beta_2}^{y_2} \subset \Omega^*$.*

Proof. Since no pair of logarithmic half-planes H_λ, H_μ satisfy Condition (4) in the Theorem above, there exists a non-negative integer m such that $\{x_\lambda\} \subset [m, m+1]$. We let $y_1 := \inf \{x_\lambda\}$ and $y_2 := \sup \{x_\lambda\}$. As in the Proof of Lemma 2 above, $\alpha := \inf \{\alpha_\lambda\} > 0$. The proof now proceeds similarly to the proof of Lemma 2, with $S := U_\alpha^{y_1} \cap U_{\alpha^2}^{y_2}$ when $0 < \alpha < 1$, and with $S := U_1^{y_1} \cap U_1^{y_2}$ when $\alpha \geq 1$. □

3. FAT L_h^1-DOMAINS OF HOLOMORPHY IN $\mathbb{C}^n$

Having acquired an understanding of the 2-dimensional case, it is now desirable to describe the case of Reinhardt L_h^1-domains of holomorphy in $\mathbb{C}^n$. While we have not yet solved the n-dimensional problem in general, we have found a characterization of all such domains which are fat. This is given in Theorem 2 and Corollary 1 below. We first define the analog of logarithmic half-planes in dimension n.

Definition 3. *Let $x \in \mathbb{R}^n \setminus \{0\}$ for some $n \in \mathbb{N}$ and $\alpha > 0$. Then we define the elementary Reinhardt domain U_α^x as follows:*

$$U_\alpha^x := \left\{ z \in \mathbb{C}^n : \prod_{j=1}^{n} |z_j|^{x_j} < \alpha \right\}.$$

Each of these elementary Reinhardt domains is determined in this definition by $n+1$ real parameters. However, observe that $U_\alpha^x = U_\beta^y$ if and only if there exists some positive real number r such that $\beta = \alpha^r$ and $y = rx$. Therefore, if so desired, we can assume that x is not an arbitrary n-dimensional real vector, but is a unit vector. In other words, we can take $x \in S^{n-1} \subset \mathbb{R}^n$. Therefore the family of such elementary Reinhardt domains is actually an n-dimensional family with parameter space $S^{n-1} \times \mathbb{R}_{>0}$. We will use this fact in the case of $n = 2$, to simplify considerably Theorem 1 above in Theorem 3 below. For the remainder of this chapter excluding Corollary 1, we will define Ω as follows:

$$\Omega := \bigcap_{j=1}^{n} U_{\alpha_j}^{x_j}.$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$.

Theorem 2. *Ω is an L_h^1-domain of holomorphy if and only if $x_1, \ldots, x_n$ are linearly independent.*

Proof. Together Propositions 32 and 33 below demonstrate that if $x_1, \ldots, x_n$ are linearly independent, then $L_h^1(\Omega) \neq \{0\}$, and so by Proposition 1, Ω is an L_h^1-domain of holomorphy. Conversely, Proposition 34 demonstrates that if Ω is an L^1_h-domain of holomorphy, then $x_1, \ldots, x_n$ are linearly independent. $\square$

Since the hypothesis of Theorem 2 is a statement about vectors in $\mathbb{R}^n$, we first convert the problem of finding integrable functions in subdomains of $\mathbb{C}^n$ to a problem of finding integrable functions in subdomains of $\mathbb{R}^n$ in Proposition 32 below.

Proposition 32. *Suppose* $\Omega = \bigcap_{j=1}^n U_{\alpha_j}^{x_j}$ *and* $m \in \mathbb{Z}^n$. *Then* $z^m \in L^1(\Omega)$ *if and only if*

$$\exp\left(\sum_{j=1}^n m'_j \rho_j\right) \in L^1\left(\log|\Omega|\right),$$

where $m'_j = 2 + m_j$.

Proof. This follows from

$$\int_\Omega |z^m| = (2\pi)^n \int_{|\Omega|} \prod_{j=1}^n r_j^{1+m_j} dr = (2\pi)^n \int_{\log|\Omega|} \exp\left(\sum_{j=1}^n (2+m_j)\rho_j\right) d\rho,$$

where $|\Omega|$ is the image of Ω in absolute space; i.e, if

$$|\Omega| \coloneqq \{(|z_1|, \ldots, |z_n|) \in \mathbb{R}^n : z \in \Omega\}$$

$\square$

Proposition 33. *If* $x_1, \ldots, x_n \in \mathbb{R}^n$ *are linearly independent, then* $L^1_h(\Omega) \neq \{0\}$.

Proof. First note that $\log|\Omega|$ is the intersection of n open half-spaces $H_1, \ldots, H_n \subset \mathbb{R}^n$ with the property that ∂H_j is the codimension-1 hyperplane given by the equation $x_j \cdot \rho = \log \alpha_j$. Now, observe that since $x_1, \ldots, x_n$ are linearly independent, $\bigcap_{j=1}^n \partial H_j$ is a singleton set $\{p\}$. Now, by translation we may assume that $p_j = 0$, for each j. Thus, since $x_1, \ldots, x_n$ are linearly independent vectors, the region $\log|\Omega|$ is linearly isomorphic to the space $\Omega' \coloneqq \{\sigma \in \mathbb{R}^n : \sigma_1, \ldots, \sigma_n < 0\}$.

Define $X \in \mathrm{GL}_n(\mathbb{R})$ by

$$X := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then a linear isomorphism $\log|\Omega| \cong \Omega'$ is given by $\rho \mapsto X^{-1}\rho$. For each j, define $m'_j := 2 + m_j$ and for each k, define $y_k := \sum_{j=1}^{n} m'_j x_{jk}$. Now, we have the following:

$$\int_{\log|\Omega|} \exp\left(\sum_{j=1}^{n} m'_j \rho_j\right) d\rho = \det\left(X^{-1}\right) \int_{\Omega'} \exp\left(\sum_{k=1}^{n} y_k \sigma_k\right) d\sigma.$$

Now, observe that $\exp\left(\sum_{k=1}^{n} y_k \sigma_k\right) \in L^1(\Omega')$ if and only if $y_k > 0$, for each k. Hence, from Proposition 32 we have that $z^m \in L^1(\Omega)$ if and only if every entry in $m'X$ is strictly positive. This yields that the set of integrable Laurent monomials on Ω is lattice-isomorphic to $\mathbb{N}^n$.

Now, suppose $z^m \in L^1(\Omega) \setminus \mathcal{O}(\Omega)$. Then for some $j \in \{1, 2, \ldots, n\}$, we have that $\Omega \cap V_j \neq \varnothing$, and $m_j < 0$. Note that Ω must then contain a product $A := \prod_{k=1}^{n} A_k$ of 1-dimensional complex domains such that A_j is a disk. Observe that $z^m \in L^1(A)$, and so $z_j^{m_j} \in L^1(A_j)$. But then $m_j \geq -1$. Hence, $m_j = -1$.

Let $J := \{j \in \{1, \ldots, n\} : \Omega \cap V_j \neq \varnothing\}$. From the preceding paragraph, it remains to find an $m \in \mathbb{Z}^n$ such that $z^m \in L^1(\Omega)$ such that for each $j \in J, m_j \neq -1$. This must be possible, or else the set of integrable Laurent monomials on Ω would not be lattice-isomorphic to $\mathbb{N}^n$, contrary to what has already been shown. This completes the proof. □

Remark: Note that the above proof actually demonstrates more than the statement of Proposition 33. It demonstrates that if $z^m \in L^1(\Omega)$, for some $m \in \mathbb{Z}^n$, then $L^1_h(\Omega)$ is infinite-dimensional, and has a Schauder basis which is lattice-isomorphic to $\mathbb{N}^n$. Furthermore, the proof gives a useful condition for checking whether a given monomial is integrable on a given domain of this type, namely that z^m is integrable on Ω if and only if every entry of $(2 + m_1, \ldots, 2 + m_n)\, X$ is strictly positive.

Proposition 34. *If $x_1, \ldots, x_n$ are linearly dependent, then Ω is not an L^1_h-domain of holomorphy.*

Proof. By Lemma 1, it is sufficient to demonstrate that for all $m \in \mathbb{Z}^n$, $z^m \notin L^1(\Omega)$. Therefore, by Proposition 32 above, it suffices to show that for all $m \in \mathbb{Z}^n$, $\exp\left(\sum_{j=1}^n m'_j \rho_j\right) \notin L^1(\log|\Omega|)$. But observe that $\log|\Omega|$ is linearly isomorphic to $\mathbb{R} \times S$, where S is an open subset of $\mathbb{R}^{n-1}$. However, observe that the desired conclusion follows from the fact that no exponential function is integrable on $\mathbb{R}$. □

Corollary 1. *If $\Omega \subsetneq \mathbb{C}^n$ is a fat, Reinhardt domain of holomorphy, then Ω is an L^1_h-domain of holomorphy if and only if*

$$\Omega = \bigcap_{\lambda \in \Lambda} U^{x_\lambda}_{\alpha_\lambda}$$

where $\{x_\lambda\}_{\lambda \in \Lambda}$ spans $\mathbb{R}^n$ and each α_λ is positive.

Proof. If $\Omega \subsetneq \mathbb{C}^n$ is a fat, Reinhardt domain of holomorphy, then there exist $\{x_\lambda\}_{\lambda \in \Lambda} \subset \mathbb{R}^n$ and for each $\lambda \in \Lambda$, there is an $\alpha_\lambda > 0$ such that

$$\Omega = \bigcap_{\lambda \in \Lambda} U^{x_\lambda}_{\alpha_\lambda}.$$

Suppose that $\operatorname{span}\{x_\lambda\}_{\lambda \in \Lambda} = \mathbb{R}^n$. Then there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $\left\{x_{\lambda_j}\right\}_{j=1}^n$ is linearly independent. Therefore by Proposition 33, since $\Omega \subset \bigcap_{j=1}^n U^{x_{\lambda_j}}_{\alpha_{\lambda_j}}$, there exists $m \in \mathbb{Z}^n$ such that $z^m \in L^1_h(\Omega)$. Hence, by Proposition 1, Ω is an L^1_h-domain of holomorphy.

Now, suppose that $\operatorname{span}\{x_\lambda\}_{\lambda \in \Lambda} \neq \mathbb{R}^n$. Then $\log|\Omega|$ is linearly isomorphic to $\mathbb{R} \times S$, where S is an open subset of $\mathbb{R}^{n-1}$. Therefore, by the reasoning in the proof of Proposition 34 above, Ω is not an L^1_h-domain of holomorphy. □

The logarithmic half-planes are characterized by inequalities of the form $|z_1|^a |z_2|^b < \alpha$, with $a, b \in \mathbb{R}$ not both zero and $\alpha > 0$. Since these inequalities can be scaled by positive exponents, we can assume $a^2 + b^2 = 1$, or in other words, $(a, b) \in S^1 \subset \mathbb{R}^2$. Therefore, there is a correspondence between logarithmic half-planes in $\mathbb{C}^2$ and S^1. By stereographic projection, we can parameterize S^1 by $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. This leads to the following definition.

Definition 4. *For every $\alpha > 0$ and for every $x \in \mathbb{R}_\infty$, we define W_α^x to be the logarithmic half-plane given by:*

$$W_\alpha^x = \left\{ |z_1|^a |z_2|^b < \alpha \right\},$$

where if $x = \infty$, then $(a, b) = (0, 1)$, and otherwise $(a, b) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right)$.

Note that the map $x \mapsto (a, b)$ described in the above definition is in fact a map $\mathbb{R}_\infty \to S^1 \subset \mathbb{R}^2$ which inverts stereographic projection of the unit circle onto the real line. Observe also that we have the following equation:

$$W_\alpha^x := \begin{cases} \widetilde{U}_{\alpha^*}^{x^*} & x \in (-1, 1) \\ U_\alpha & x = 1 \\ U_{\alpha^*}^{x^*} & x \in \mathbb{R}_\infty \setminus [-1, 1] \\ \widetilde{U}_{1/\alpha} & x = -1, \end{cases}$$

where $\alpha^* = \alpha^{(x^2+1)/(x^2-1)}$, and $x^* = \frac{-2x}{(x^2-1)}$. Note, therefore that this notation already has an advantage over that used in Theorem 1 since it enables us to use one notation to capture all four classifications of logarithmic half-planes from Proposition 5. The next proposition reveals another advantage of this notation: there is an easy formula for the image of W_α^x under the transposition of coordinates F.

Proposition 35. *Define $F : (z_1, z_2) \mapsto (z_2, z_1)$. Then for each $\alpha > 0$, $x \in \mathbb{R}_\infty$, $F(W_\alpha^x) =$*

$W_\alpha^{(x+1)/(x-1)}$.

Proof. Observe first that

$$\frac{2\left(\frac{x+1}{x-1}\right)}{\left(\frac{x+1}{x-1}\right)^2+1}=\frac{x^2-1}{x^2+1}$$

and that

$$\frac{\left(\frac{x+1}{x-1}\right)^2-1}{\left(\frac{x+1}{x-1}\right)^2+1}=\frac{2x}{x^2+1}.$$

The conclusion now follows from the definition. □

We now state a simpler characterization than the one given in Theorem 1 of Reinhardt L_h^1-domains of holomorphy in $\mathbb{C}^2$ in terms of the new parameterization of logarithmic half-planes. We note furthermore that it would seem that this is a theorem in simplest terms. That is to say, we could not reasonably expect it to be stated more simply, since there are four different domain geometries to be considered based on whether the given domain is fat or not, and how it fails to be fat. We prove the theorem in two parts: (1) In Proposition 37 below, we demonstrate that the conditions given in Theorem 1 imply the conditions given in Theorem 3, while (2) in Proposition 38, we demonstrate the converse. As an intermediary proof, we demonstrate in Proposition 36 below that the condition of being fat in Theorem 3 is equivalent to the corresponding condition in Theorem 1.

Theorem 3. *If $\Omega \subsetneq \mathbb{C}^2$ is a Reinhardt domain of holomorphy, then Ω is an L_h^1-domain of holomorphy if and only if there exist $x, y \in \mathbb{R}_\infty$ and $\alpha, \beta > 0$ such that $-\frac{1}{y} \neq x \neq y$ and $\Omega \subset W_\alpha^x \cap W_\beta^y$ and one of the following holds:*

1. *Ω is fat.*

2. *$\Omega = \Omega^* \setminus V_1$ and if $0 < x < y < \infty$, then the interval $\left(\frac{x^2-1}{2x}, \frac{y^2-1}{2y}\right)$ contains an integer other than 1.*

3. *$\Omega = \Omega^* \setminus V_2$ and if $-1 < \frac{1}{x} < \frac{1}{y} < 1$, then the interval $\left(\frac{2x}{x^2-1}, \frac{2y}{y^2-1}\right)$ contains an integer other than 1.*

4. $\Omega = \Omega^* \setminus V_0$ *and both*

- *if* $0 < y < x < \infty$, *then the interval* $\left(\frac{y^2-1}{2y}, \frac{x^2-1}{2x}\right)$ *contains an integer.*
- *if* $-1 < \frac{1}{x} < \frac{1}{y} < 1$, *then the interval* $\left(\frac{2x}{x^2-1}, \frac{2y}{y^2-1}\right)$ *contains an integer.*

Proposition 36. *Suppose that* $\Omega \subsetneq \mathbb{C}^2$ *is a fat, Reinhardt domain of holomorphy. Then* Ω *satisfies Condition (2) of Theorem 1 if and only if there exist* $x, y \in \mathbb{R}_\infty$ *and* $\alpha, \beta > 0$ *such that* $-\frac{1}{y} \neq x \neq y$, *and* $\Omega \subset W_\alpha^x \cap W_\beta^y$.

Proof. ($\Longrightarrow$:) Suppose first that Ω satisfies Condition (2) of Theorem 1. Since Ω is a Reinhardt domain of holomorphy properly contained in $\mathbb{C}^2$, it must be an intersection of logarithmic half-planes. Since, in addition, it is not a logarithmic half-plane, there exists $\{x_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}_\infty$ with $|\Lambda| > 1$ and for each λ, there exists $\alpha_\lambda > 0$ such that

$$\Omega = \bigcap_{\lambda\in\Lambda} W_{\alpha_\lambda}^{x_\lambda}.$$

Since Condition (2) of Theorem 1 holds, Ω is an L_h^1-domain of holomorphy. Now by Theorem 2 above, $\{G(x_\lambda)\}_{\lambda\in\Lambda}$ is a spanning set for $\mathbb{R}^2$, where $G : \mathbb{R}_\infty \to \mathbb{R}^2$ is defined by

$$G(x_\lambda) := \left(\frac{2x_\lambda}{x_\lambda^2+1}, \frac{x_\lambda^2-1}{x_\lambda^2+1}\right)$$

Therefore, choose $x, y \in \{x_\lambda\}_{\lambda\in\Lambda}$ such that $G(x)$ and $G(y)$ are linearly independent. Note that $\Omega \subset W_\alpha^x \cap W_\beta^y$ for some $\alpha, \beta > 0$. Furthermore, $x \neq y$, since $G(x) \neq G(y)$. Observe that if $x = -\frac{1}{y}$, then

$$G(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right) = \left(\frac{-2y}{1+y^2}, \frac{1-y^2}{1+y^2}\right) = -G(y).$$

But $G(y), -G(y)$ are linearly dependent. Therefore, $x \neq -\frac{1}{y}$.

($\Longleftarrow$:) Now suppose that there exist $x, y \in \mathbb{R}_\infty$ and $\alpha, \beta > 0$ such that $-\frac{1}{y} \neq x \neq y$, and

$\Omega \subset W^x_\alpha \cap W^y_\beta$. Observe that $\log|\Omega|$ is contained in a convex subset of $\mathbb{R}^2$ bounded by two intersecting lines. Therefore, Ω is not a logarithmic half-plane since $\log|\Omega|$ is not a half-plane.

Finally, observe that if Ω were to be either $\widetilde{U}_{\alpha^*} \cap U_{\beta^*}$ or $\widetilde{U}^{x^*}_{\alpha^*} \cap U^{x^*}_{\beta^*}$ for some $x^* \in \mathbb{R}$, and some $\alpha^*, \beta^* > 0$, then $\log|\Omega|$ would be a convex domain in $\mathbb{R}^2$ bounded by two parallel lines. Since this is not the case, Ω satisfies Condition (2) of Theorem 1. $\square$

Proposition 37. *If $\Omega \subsetneq \mathbb{C}^2$ is a Reinhardt L^1_h-domain of holomorphy, then there exist $x, y \in \mathbb{R}_\infty$ and $\alpha, \beta > 0$ such that $-\frac{1}{y} \neq x \neq y, \Omega \subset W^x_\alpha \cap W^y_\beta$, and one of Conditions (1)-(4) of Theorem 3 above holds.*

Proof. Since Ω is a Reinhardt L^1_h-domain of holomorphy, at least one of the conditions from Theorem 1 holds. We define the function $g : \mathbb{R} \to \mathbb{R}_\infty$ as follows:

$$g(x) = \begin{cases} \infty, & x = 0, \\ -\frac{1}{x}\left(1 + \sqrt{1 + x^2}\right), & x \neq 0. \end{cases}$$

Note that for $\alpha > 0$ and $x \in \mathbb{R}$, if $\alpha^* = \alpha^{1/\sqrt{1+x^2}}$, then $U^x_\alpha = W^{g(x)}_{\alpha^*}$. Also, observe that $g(\mathbb{R}) = \mathbb{R}_\infty \setminus [-1, 1]$, and that g is injective.

1. Suppose Ω is bounded. Then for some $\alpha, \beta > 0, \Omega \subset W^\infty_\alpha \cap W^1_\beta$. Therefore one of Conditions (1)-(4) of Theorem 3 trivially holds.

2. By Proposition 36 above, if Ω is fat, then Ω satisfies Condition (1) and the hypotheses of Theorem 3 above.

3. Suppose for some $\alpha, \beta > 0$ and $x \in \mathbb{R}$ that $\Omega^* \subset \widetilde{U}_\alpha \cap U^x_\beta$. Observe that $\widetilde{U}_\alpha = W^{-1}_{\alpha^*}$, where $\alpha^* = \alpha^{-1}$. and $U^x_\beta = W^{g(x)}_{\beta^*}$. Therefore, one of Conditions (1)-(4) of Theorem 3 holds provided $g(x) \neq \pm 1$. However, $\pm 1 \notin g(\mathbb{R})$. Therefore, one of Conditions (1)-(4) of Theorem 3 holds.

4. Suppose for some $\alpha, \beta > 0$ and for some $x, y \in \mathbb{R}$ with $x > \max\{0, y\}$ and $(y, x) \cap \mathbb{Z} \neq \varnothing$ that $\Omega^* \subset U_\alpha^x \cap U_\beta^y$. Observe that $U_\alpha^x = W_{\alpha^*}^{g(x)}$ and $U_\beta^y = W_{\beta^*}^{g(y)}$. Since g is injective and $x \neq y, g(x) \neq g(y)$. Furthermore, since $g(\mathbb{R}) \cap [-1, 1] = \varnothing, g(x) \neq -\frac{1}{g(y)}$. Therefore, if Ω is fat, then Ω satisfies Condition (1) of Theorem 3 above.

   Now observe that $\Omega^* \cap V_1 = \varnothing$. Hence, it suffices to show that if Ω is not fat, then Condition (3) of Theorem 3 holds. Suppose $\Omega = \Omega^* \setminus V_2$, and then observe that

$$\frac{2g(x)}{g^2(x) - 1} = \frac{\frac{-2\left(1+\sqrt{1+x^2}\right)}{x}}{\left(\frac{1+\sqrt{1+x^2}}{x}\right)^2 - 1} = \frac{-2x\left(1 + \sqrt{1 + x^2}\right)}{2 + 2\sqrt{1 + x^2}} = -x.$$

   It now follows since $(y, x) \cap \mathbb{Z} \neq \varnothing$, that $(-x, -y) \cap \mathbb{Z} = \left(\frac{2x}{x^2-1}, \frac{2y}{y^2-1}\right) \cap \mathbb{Z} \neq \varnothing$. Suppose $1 \in (-x, -y)$. But then $y < 0 < x$, so that the interval $(-x, -y)$ contains 0.

5. Suppose that for some $\alpha, \beta > 0$ and some $x < 0, \Omega^* \subset U_\alpha^x \cap U_\beta$ and $\Omega = \Omega^* \setminus V_2$. Note that $\Omega \subset W_{\alpha^*}^{g(x)} \cap W_\beta^1$. Since $x < 0, g(x) > 1$. Therefore, $-1 \neq g(x) \neq 1$. Therefore, Ω trivially satisfies Condition (3) of Theorem 3.

6. Suppose that for some $\alpha, \beta > 0$ and some $x \in \left(-\frac{1}{2}, 0\right), \Omega^* \subset U_\alpha^x \cap U_\beta$ and $\Omega = \Omega^* \setminus V_1$. Then note that $\Omega \subset W_{\alpha^*}^{g(x)} \cap W_\beta^1$. Since $x < 0, g(x) > 1$. Therefore, it suffices to demonstrate that $\left(0, \frac{g^2(x)-1}{2g(x)}\right) \cap (\mathbb{Z} \setminus \{1\}) \neq \varnothing$. But observe that $\frac{g^2(x)-1}{2g(x)} = -\frac{1}{x} \in (2, \infty)$. Therefore, $2 \in \left(0, \frac{g^2(x)-1}{2g(x)}\right)$, and so Ω satisfies Condition (2) of Theorem 3.

7. Suppose that for some $\alpha, \beta > 0$ and some $x \in (-1, 0)$ that $\Omega^* \subset U_\alpha^x \cap U_\beta$ and that $\Omega = \Omega^* \setminus V_0$. Now, note that $\Omega \subset W_{\alpha^*}^{g(x)} \cap W_\beta^1$. Also, $g(x) \in (1 + \sqrt{2}, \infty)$. Therefore, $0 < 1 < g(x) < \infty$. Thus, it suffices to show that $\left(0, \frac{g^2(x)-1}{2g(x)}\right) = \left(0, -\frac{1}{x}\right)$ contains an integer. But since $x \in (-1, 0)$, $-\frac{1}{x} \in (1, \infty)$, and so $1 \in \left(0, -\frac{1}{x}\right)$. Therefore, Ω satisfies Condition (4) of Theorem 3.

8. Suppose for some $\alpha, \beta > 0$ and for some $y, x \in \mathbb{R}$ with $\left(-\frac{1}{y}, -\frac{1}{x}\right) \cap \{2, 3, 4, \ldots\} \neq \varnothing$ that

$\Omega^* \subset U_\alpha^x \cap U_\beta^y$ and $\Omega = \Omega^* \setminus V_1$. Next, observe that $\Omega \subset W_{\alpha^*}^{g(x)} \cap W_{\beta^*}^{g(y)}$. Next, observe that $y < x < 0$, and so we have that $0 < g(y) < g(x) < \infty$. It remains to show then that $\left(\frac{g^2(y)-1}{2g(y)}, \frac{g^2(x)-1}{2g(x)}\right) = \left(-\frac{1}{y}, -\frac{1}{x}\right)$ contains an integer other than 1. But this is true by assumption.

9. Suppose for some $\alpha, \beta > 0$ and for some $y, x \in \mathbb{R}$ with $(-x, -y) \cap \{2, 3, 4, \ldots\} \neq \varnothing$ that $\Omega^* \subset U_\alpha^x \cap U_\beta^y$ and $\Omega = \Omega^* \setminus V_2$. First, note that if $x \geq 0$, then $g(x) < -1$, and so Ω trivially satisfies Condition (3) of Theorem 3. Now, suppose $y < x < 0$. Then we have that $1 < g(y) < g(x) < \infty$, and so $0 < \frac{1}{g(x)} < \frac{1}{g(y)} < 1$. Also, by hypothesis, $\left(\frac{2g(x)}{g^2(x)-1}, \frac{2g(y)}{g^2(y)-1}\right) = (-x, -y)$ contains a positive integer greater than 1. Hence, Ω satisfies Condition (3) of Theorem 3.

10. Suppose for some $\alpha, \beta > 0$ and for some $y < x < 0$ with $(y, x) \cap \mathbb{Z} \neq \varnothing \neq \left(-\frac{1}{y}, -\frac{1}{x}\right)$, we have that $\Omega^* \subset U_\alpha^x \cap U_\beta^y$, and $\Omega = \Omega^* \setminus V_0$. Note then that $\Omega \subset W_{\alpha^*}^{g(x)} \cap W_{\beta^*}^{g(y)}$. Furthermore, we have that $\left(\frac{g^2(y)-1}{2g(y)}, \frac{g^2(x)-1}{2g(x)}\right) = \left(-\frac{1}{y}, -\frac{1}{x}\right)$ must contain an integer and $\left(\frac{2g(x)}{g^2(x)-1}, \frac{2g(y)}{g^2(y)-1}\right) = (-x, -y)$ must contain an integer. Therefore, Ω satisfies Condition (4).

11. Observe that if $F(\Omega)$ satisfies one of Conditions (1)-(10) of Theorem 1, then $F(\Omega)$ satisfies one of Conditions (1)-(4) of Theorem 3. Also, observe that

$$\frac{\left(\frac{x+1}{x-1}\right)^2 - 1}{2\left(\frac{x+1}{x-1}\right)} = \frac{(x+1)^2 - (x-1)^2}{2(x+1)(x-1)} = \frac{4x}{2(x^2-1)} = \frac{2x}{x^2-1}.$$

Finally, let $\varphi : \mathbb{R}_\infty \to S^1$ be defined by $\varphi = (\varphi_1, \varphi_2) : x \mapsto (a, b)$ as in Definition 4. Now, observe that $-1 < \frac{1}{x} < \frac{1}{y} < 1 \iff \varphi(x), \varphi(y)$ are in the upper half-plane and $\varphi_2(x) < \varphi_2(y)$. But by Proposition 35 and Definition 4, we also have that $\varphi\left(\frac{x+1}{x-1}\right) = (\varphi_2(x), \varphi_1(x))$. Therefore, $-1 < \frac{1}{x} < \frac{1}{y} < 1 \iff \varphi\left(\frac{x+1}{x-1}\right), \varphi\left(\frac{y+1}{y-1}\right)$ are in the right-hand half-plane and $\varphi_2\left(\frac{x+1}{x-1}\right) < \varphi\left(\frac{y+1}{y-1}\right) \iff 0 < \frac{x+1}{x-1} < \frac{y+1}{y-1} < \infty$. Therefore, we have the following:

- $F(\Omega)$ satisfies Condition (1) of Theorem 3 $\iff$ Ω satisfies Condition (1) of Theorem 3.

- $F(\Omega)$ satisfies Condition (2) of Theorem 3 $\iff$ Ω satisfies Condition (3) of Theorem 3.
- $F(\Omega)$ satisfies Condition (4) of Theorem 3 $\iff$ Ω satisfies Condition (4) of Theorem 3.

Hence, Ω satisfies one of Conditions (1)-(4) of Theorem 3.

$\square$

Proposition 38. *Suppose $\Omega \subsetneq \mathbb{C}^2$ is a Reinhardt domain of holomorphy and that there exist $x, y \in \mathbb{R}_\infty$ and $\alpha, \beta > 0$ such that $-\frac{1}{y} \neq x \neq y$ and $\Omega \subset W_\alpha^x \cap W_\beta^y$ and one of Conditions (1)-(4) of Theorem 3 hold. Then Ω is an L_h^1-domain of holomorphy.*

Proof. It is suffices to show that each of Conditions (1)-(4) of Theorem (3) implies that one of Conditions (1)-(11) of Theorem (1) holds.

1. If Ω is fat, then by Proposition 36 above, Ω satisfies Condition (2) of Theorem 1.
2. Suppose Ω satisfies Condition (2) of Theorem 3. Note first that if $x \in (-\infty, 0)$, then $W_\alpha^x \cap V_1 = \varnothing$, so that $\Omega^* \setminus V_1 = \Omega^*$. Therefore, Ω is fat and so satisfies Condition (2) of Theorem 1.

   Now, suppose that $x = \infty$. Then by hypothesis $0 \neq y \neq \infty$. If $y \in (-\infty, 0)$, then as in the preceding paragraph, Ω satisfies Condition (2) of Theorem 1. If $y \in (0, 1]$, then Ω is contained in the bidisk with biradius $(\alpha \cdot \beta^{(y^2+1)/(2y)}, \alpha)$ and so is bounded. Therefore, Ω satisfies Condition (1) of Theorem (1). Finally, suppose that $y \in (1, \infty)$. Note then that by Proposition 35 above, $F(\Omega) \subset W_\alpha^1 \cap W_\beta^{(y+1)/(y-1)}$. Now, observe that since $y \in (1, \infty)$, $\frac{y+1}{y-1} \in (1, \infty)$, and so for some $y^* < 0$ and some $\alpha^*, \beta^* > 0$, $F(\Omega) \subset U_{\alpha^*} \cap U_{\beta^*}^{y^*}$ and $F(\Omega) = F(\Omega)^* \setminus V_2$. Therefore, $F(\Omega)$ satisfies Condition (5) of Theorem 1 and so Ω satisfies Condition (11) of Theorem 1.

Next, suppose that $x \in (1, \infty)$. If $y \in \mathbb{R}_\infty \setminus [0, \infty)$, then by symmetry, Ω satisfies one of Conditions (1)-(11) of Theorem 1. If $y = 0$, then note that $F(\Omega) \subset W_\alpha^{(x+1)/(x-1)} \cap W_\beta^{-1} = U_{\alpha^*}^{x^*} \cap \widetilde{U}_{\beta^*}$, for some $\alpha^*, \beta^* > 0$ and some $x^* < 0$. Therefore, $F(\Omega)$ satisfies Condition (3) of Theorem 1 and so Ω satisfies Condition (11) of Theorem 1. Now, suppose $y \in (0, 1)$. Observe that $\frac{x+1}{x-1} \in (1, \infty)$ and $\frac{y+1}{y-1} \in (-\infty, -1)$. Therefore, $F(\Omega) \subset U_{\alpha^*}^{x^*} \cap U_{\beta^*}^{y^*}$, for some $\alpha^*, \beta^* > 0$ and for $x^* < 0 < y^*$. It then follows that $F(\Omega)$ satisfies Condition (4) of Theorem 1. Next, suppose $y = 1$. It then follows that for some $\alpha^*, \beta^* > 0$ and $x^* = -\frac{2x}{x^2-1}$, that $\Omega \subset U_{\alpha^*}^{x^*} \cap U_{\beta^*}$. Also, $0 < y < x < \infty$, so $\left(0, \frac{x^2-1}{2x}\right)$ contains an integer other than 1. But in this case, the interval must contain 2. Therefore, $\left(\frac{x^2-1}{2x}\right) \in (2, \infty)$, and so $x^* \in \left(-\frac{1}{2}, 0\right)$. Therefore, Ω satisfies Condition (6) of Theorem 1. By similar reasoning, if $y \in (1, \infty)$, then Ω satisfies Condition (8) of Theorem 1.

Assume now that $x = 1$. The case where $y \in \mathbb{R}_\infty \setminus [0, 1)$ has been handled by symmetry. Therefore, we may suppose that $y \in [0, 1)$. If $y = 0$, then for some $\alpha^*, \beta^* > 0, F(\Omega) \subset \widetilde{U}_{\alpha^*} \cap U_{\beta^*}^0$. Therefore, $F(\Omega)$ satisfies Condition (3) of Theorem 1 and so Ω satisfies Condition (11) of Theorem 1. On the other hand, if $y \in (0, 1)$, then $F(\Omega) \subset U_{\alpha^*}^0 \cap U_{\beta^*}^{y^*}$, where $y^* = \frac{1-y^2}{2y}$. Now, since Ω satisfies Condition (2) of Theorem 3, we have that $(-y^*, 0) = \left(\frac{y^2-1}{2y}, 0\right)$ contains an integer other than 1. Therefore, $y^* > 1$, and so $1 \in (0, y^*)$. Hence, $F(\Omega)$ satisfies Condition (4) of Theorem 1 and so Ω satisfies Condition (11) of Theorem 1.

Now, assume that $x \in (0, 1)$. We may again by symmetry suppose that $y \in [0, 1)$. Again, if $y = 0$, then as in the preceding paragraph $F(\Omega)$ satisfies Condition (3) of Theorem 1. On the other hand, if $y \in (0, 1)$, then suppose without loss of generality that $x < y$. Now, observe that $F(\Omega) \subset U_{\alpha^*}^{x^*} \cap U_{\beta^*}^{y^*}$, where $x^* = \frac{1-x^2}{2x}$ and $y^* = \frac{1-y^2}{2y}$ and by assumption (y^*, x^*) contains an integer. Therefore, $F(\Omega)$ satisfies Condition (4) of Theorem (1). Hence, if Ω satisfies Condition (2) of Theorem 3, then Ω satisfies one of Conditions (1)-(11) of Theorem 1. Note also that the case where $x = 0$ has been covered by symmetry.

3. If Ω satisfies Condition 3 of Theorem 3, then $F(\Omega)$ satisfies Condition 2 of Theorem 3. Hence, $F(\Omega)$ satisfies one of Conditions (1)-(11) of Theorem 1. It then follows that Ω satisfies one of Conditions (1)-(11) of Theorem 1.

4. Suppose Ω satisfies Condition 4 of Theorem 3. If $x \in (-1, 0)$, then $W_\alpha^x \cap V_0 = \varnothing$, so that $\Omega^* \setminus V_0 = \Omega^*$. Therefore, Ω is fat and so satisfies Condition (2) of Theorem 1.

   Assume now that $x = -1$. Then $W_\alpha^x = \widetilde{U}_{\alpha^*}$. Also, note that by symmetry, we may assume that $y \notin (-1, 0)$. Furthermore, if $y \in [0, 1)$, then $W_\beta^y = \widetilde{U}_{\beta^*}^{y^*}$ for some non-negative y^*. Therefore, $\Omega^* \cap V_0 = \varnothing$, and so Ω is fat and satisfies Condition (2) of Theorem 1. On the other hand, if $y \in \mathbb{R}_\infty \setminus [-1, 1]$, then for some $a \in (-1, 1), b \in (0, 1]$,

$$W_\beta^y = \left\{ |z_1|^a \, |z_2|^b < \beta \right\} = U_{\beta^*}^{y^*},$$

   for some β^*, y^*. It therefore follows that Ω satisfies Condition (3) of Theorem 1 in this case.

   Suppose now that $x \in (-\infty, -1)$. Then for some $\alpha^*, x^* > 0$, $W_\alpha^x = U_{\alpha^*}^{x^*}$. Note therefore that Ω^* is disjoint from V_1 so that Ω satisfies Condition (2) of Theorem 3.

   Suppose now that $x = \infty$. Then $W_\alpha^x = U_\alpha^0$. We may suppose by symmetry that $y \notin (-\infty, 0)$. Note also that if $y \in (0, 1]$, then Ω is bounded and so satisfies Condition (1) of Theorem 1. Now, suppose that $y \in (1, \infty)$. Note then that by Proposition 35, $F(W_\beta^y) = W_\beta^{(y+1)/(y-1)}$. Therefore, $F(\Omega^*) = U_\alpha \cap U_{\beta^*}^{(1-y^2)/2y}$. Observe that $\frac{1-y^2}{2y} < 0$. Also, since $\frac{1}{x} = 0$ and $\frac{1}{y} \in (0, 1)$, we have that $\left(0, \frac{2y}{y^2-1}\right) \cap \mathbb{Z} \neq \varnothing$. Therefore,

$$\frac{2y}{y^2-1} > 1 \implies \frac{y^2-1}{2y} < 1 \implies \frac{1-y^2}{2y} > -1.$$

   Therefore, $F(\Omega)$ satisfies Condition (7) of Theorem 1, and so Ω satisfies Condition (11) of

Theorem 1.

Suppose now that $x \in (1, \infty)$. We may suppose by symmetry that $y \notin (-\infty, 0) \cup \{\infty\}$. Now assume that $y \in [0, 1)$. Then note that $\Omega^* \cap V_2 = \varnothing$, so Ω satisfies Condition (2) of Theorem 3 unless $y > 0$ and $\left(\frac{y^2-1}{2y}, \frac{x^2-1}{2x}\right) \cap \mathbb{Z} = \{1\}$. But note that if $y > 0$, then $\frac{y^2-1}{2y} < 0$ and $\frac{x^2-1}{2x} > 0$. Therefore, Ω satisfies Condition (2) of Theorem 3.

Now, suppose $y = 1$. Note then that $W_\alpha^x = U_{\alpha^*}^{x^*}$ for some $\alpha^* > 0$ and $x^* = \frac{-2x}{x^2-1} < 0$, and also that $W_\beta^y = U_\beta$. Now observe that since $0 < y < x < \infty$, we have that $\left(0, \frac{x^2-1}{2x}\right) \cap \mathbb{Z} \neq \varnothing$. Therefore, we have that

$$\frac{x^2-1}{2x} > 1 \implies \frac{-2x}{x^2-1} > -1.$$

Therefore, Ω satisfies Condition (7) of Theorem 1. Furthermore, if $y \in (1, \infty)$, then similar arguments show that Ω satisfies Condition (10) of Theorem 1.

Next, suppose that $x \in (0, 1]$. We may suppose by symmetry that $y \in [0, 1)$. Now, assume $y \in (0, 1)$. Then $\Omega^* \cap V_2 = \varnothing$. Therefore, since $\frac{x^2-1}{2x} \in (-\infty, 0]$ and $\frac{y^2-1}{2y} \in (-\infty, 0)$, there is a negative integer strictly between $\frac{x^2-1}{2x}$ and $\frac{y^2-1}{2y}$, and so Ω satisfies Condition (2) of Theorem 3. Next, suppose that $y = 0$. If $x \in (0, 1)$, then $F(\Omega) = W_\alpha^{(x+1)/(x-1)} \cap W_\beta^{-1}$, where $\frac{x+1}{x-1} \in (-\infty, -1)$, while if $x = 1$, then $F(\Omega) = W_\alpha^\infty \cap W_\beta^{-1}$. However, $W_\beta^{-1} = \widetilde{U}_{\beta^*}$, and so $F(\Omega)$ satisfies Condition (3) of Theorem 1. Finally, note that the case when $x = 0$, is now handled by symmetry. $\square$

5. CONCLUSION

In Chapter 2, we developed a geometric characterization of Reinhardt L_h^1-domains of holomorphy in $\mathbb{C}^2$ in terms of logarithmic half-planes (Theorem 1). We also gave an example of an unbounded Reinhardt domain of holomorphy which is not an L_h^1-domain of holomorphy, demonstrating the importance of the "bounded" hypothesis in Conjecture 1. We also gave an example of a family of domains which provides a counterexample for Conjecture 2. In Chapter 3, we showed that if Ω is a fat Reinhardt domain of holomorphy in $\mathbb{C}^n$, then Ω is an L_h^1-domain of holomorphy if and only if $\log|\Omega|$ is contained in a region bounded by n linearly independent codimension 1 hyperplanes (Corollary 1). In Chapter 4, we altered our perspective from a rectangular to a spherical perspective, which enabled us to greatly simplify our characterization in Theorem 1 (Theorem 3).

The results given in Chapters 2-4 also prompt further questions. First and foremost, is there a geometric characterization of Reinhardt L_h^1-domains of holomorphy in n dimensions? It seems that if this were possible, it could be found by generalizing the spherical perspective given in Chapter 4 and by using the linear-algebraic fact given in the remark following Proposition 33 in Chapter 3, namely that z^m is an integrable Laurent monomial on Ω if and only if every entry in $m'X$ is strictly positive.

Finally, Proposition 4 in Chapter 2 gives a special case of Conjecture 1 in Chapter 1. However, the method used in proving Proposition 4 seemingly cannot be used to prove the general case. I think that the geometric characterization of L_h^2-domains of holomorphy given in [7] gives a possible way towards a solution. If one could demonstrate that given any pluripolar set K and any bounded domain of holomorphy Ω, there exists an L_h^1 function on $\Omega \setminus K$ which is completely singular at every point in K, then Conjecture 1 would follow. Alternatively, the same result should follow if there is a locally L_h^1 function on $\mathbb{C}^n \setminus K$ which is completely singular at every point in K. Note that this latter problem has the advantage of depending only on K and not on Ω.

REFERENCES

[1] Klaus Fritzsche and Hans Grauert, *From holomorphic functions to complex manifolds*, Springer-Verlag, 2010.

[2] Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, AMS Chelsea, 1965.

[3] Lars Hörmander, *An introduction to complex analysis in several variables*, Elsevier Science, 1990.

[4] Marek Jarnicki and Peter Pflug, *On n-circled $\mathcal{H}^\infty$-domains of holomorphy*, Annales Polonici Mathematici **65** (1997), 253–264.

[5] ———, *Extension of holomorphic functions*, Walter de Gruyter, 2000.

[6] ———, *First steps in several complex variables: Reinhardt domains*, European Mathematical Society, 2008.

[7] Peter Pflug and Włodzimierz Zwonek, *L^2_h-domains of holomorphy and the Bergman kernel*, Studia Mathematica **151** (2002), 99–108.

[8] R. Michael Range, *Holomorphic functions and integral representations in several complex variables*, Springer-Verlag, 1986.