COMPLEX ANALYTIC APPROACHES TO INVERSE SPECTRAL PROBLEMS

A Dissertation

by

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ABSTRACT

In this dissertation we consider methods from complex analysis to solve inverse spectral problems for Schrödinger operators on finite intervals and semi-infinite Jacobi operators.

After discussing necessary background from complex function theory and harmonic analysis, we consider Schrödinger operators on a finite interval with an L^1-potential. We prove that the potential can be uniquely recovered from one spectrum and subsets of another spectrum and point masses of the spectral measure (or norming constants) corresponding to the first spectrum. We also solve this Borg-Marchenko-type problem under some conditions on two spectra, when missing part of the second spectrum and known point masses of the spectral measure have different index sets.

In the discrete case, we consider semi-infinite Jacobi matrices with discrete spectrum. We prove that a Jacobi operator can be uniquely recovered from one spectrum and subsets of another spectrum and norming constants corresponding to the first spectrum. As a corollary, we obtain semi-infinite Jacobi analog of Marchenko's inverse spectral theorem for Schrödinger operators, i.e. a Jacobi operator can be uniquely recovered from the Weyl m-function (or the spectral measure). We also solve our Borg-Marchenko-type problem under some conditions on two spectra, when missing part of the second spectrum and known norming constants have different index sets.

To my parents

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NOMENCLATURE

$b_\lambda(z)$	Blaschke factor
$B_\Lambda(z)$	Blaschke product
$\mathbb{C}_+$	upper-half plane
$c_{00}(\mathbb{N})$	set of sequences with finitely many non-zero elements
$\mathbb{D}$	unit disk
$D^*(\Lambda)$	exterior Beurling-Malliavin density of the sequence Λ
E_Λ	exponential system of the sequence Λ
$\gamma_k(h)$	norming constant of the eigenvalue λ_k for $J_h(g)$
$\mathcal{H}(\Omega)$	space of complex analytic functions
$H^p(\mathbb{C}_+)$	Hardy space on the upper half plane
J_{max}	maximal self-adjoint Jacobi operator
J_{min}	minimal self-adjoint Jacobi operator
$J_h(g)$	self-adjoint Jacobi operator with boundary conditions h, g
$l^2(\mathbb{N})$	Hilbert space of square summable sequences
L^1_Π	space of Poisson summable functions
$m_{\alpha,\beta}$	Weyl m-function for Schrödinger operators
$m_h(z, g)$	Weyl m-function for $J_h(g)$
$\mu_{\alpha,\beta}$	spectral measure
Sf	Schwarz integral of the function f
$S\mu$	Schwarz integral of the measure μ
$p.v.$	principal value
Pf	Poisson integral of the function f
Π	Poisson measure

PW_a	Paley-Wiener space on $[-a, a]$
Qf	conjugate Poisson integral of the function f
$q(t)$	potential function of Schrödinger operators
$R(\Lambda)$	radius of completeness of Λ
$\sigma_{\alpha,\beta}$	spectrum of Schrödinger operator
$\sigma(J_h(g))$	spectrum of $J_h(g)$
$\tau_\alpha(a_n)$	norming constant of a_n for Schrödinger operators
$W(f, g)$	Wronskian
$\hat{f}$	Fourier transform of f
T^*	adjoint of the operator T
$z \to_\angle a$	non-tangential convergence

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1. INTRODUCTION

The Schrödinger (Sturm-Liouville) equation is given by

$$Lu = -u'' + qu = zu$$

on the interval $(0, \pi)$ with the boundary conditions

$$u(0)\cos\alpha - u'(0)\sin\alpha = 0$$
$$u(\pi)\cos\beta + u'(\pi)\sin\beta = 0,$$

and a real-valued potential $q \in L^1(0, \pi)$. The spectrum $\sigma_{\alpha,\beta}$ of the Schrödinger operator L corresponding to these boundary conditions defines a discrete subset of the real line, bounded from below, diverging to $+\infty$.

Direct spectral problems aim to get spectral information from the potential. In inverse spectral problems, the goal is to recover the potential from spectral information, such as the spectrum, the norming constants, the spectral measure or Weyl-Titchmarsh m-function.

The first inverse spectral result on Schrödinger operators is given by Ambarzumian [1]. He considered continuous potential with Neumann boundary conditions at both endpoints ($\alpha = \beta = \pi/2$) and showed that $q \equiv 0$ if the spectrum consists of squares of integers.

Later Borg [13] proved that an L^1-potential is uniquely recovered from two spectra corresponding to various pairs of boundary conditions and sharing the same boundary conditions at π ($\beta_1 = \beta_2$), one of which should be Dirichlet boundary condition at 0 ($\alpha_1 = 0$). Levinson [52] extended Borg's result by removing the restriction of Dirichlet boundary condition at 0.

Furthermore, Marchenko [56] observed that the spectral measure (or Weyl-Titchmarsh m-function) uniquely recovers an L^1-potential.

Another classical result is due to Hochstadt and Lieberman [45], which says that if the first half of an L^1-potential is known, one spectrum recovers the whole.

Statements of these classical results are given in Section 3.2.1.

Gesztesy, Simon and del Rio [15] generalized Levinson's theorem to three spectra, by showing two thirds of the union of three spectra is sufficient spectral data to recover an L^1-potential.

Later on, Gesztesy and Simon [31] observed that extra smoothness conditions on the potential change required spectral data to recover the potential. They proved that the knowledge of the eigenvalues can be replaced by information on the derivatives of the potential. In addition, they [31] also generalized the Hochstadt-Lieberman theorem in the sense that more than the first half of an L^1-potential and a sufficiently large subset of a spectrum recover the potential.

Afterwards, Amour, Raoux and Faupin [3, 4] proved similar results using extra information on the smoothness of the potential.

In a remarkable result, Horváth [46] characterized unique recovery of a potential in terms of completeness of an exponential system depending on given eigenvalues and known part of the potential. This observation opened a new path [5, 46, 48, 55] by connecting inverse spectral problems and completeness of exponential systems.

Moreover, Horváth and Sáfár [48] proved similar results in terms of a cosine system. The cosine system depends on subsets of eigenvalues and norming constants and their spectral data consists of these two subsets.

Recently, Makarov and Poltoratski [55] gave a version of Horváth's theorem [46] in terms of exterior Beurling-Malliavin density by combining Horváth's result and the Beurling-Malliavin theorem. In the same paper, they obtained another characterization result, which is an uncertainty version of Borg's theorem. As their spectral data, they considered a set of intervals known to include two spectra and characterized the inverse spectral problem in terms of a convergence criterion on this set of intervals.

All of these results mentioned above are discussed in Section 3.2.2.

Classical theorems of Borg, Levinson, Marchenko, Hochstadt and Lieberman led to various other inverse spectral results on Schrödinger operators (see [2, 29, 36, 37, 38, 39, 47, 57, 60, 64, 69, 70, 71, 72, 73, 74, 75, 77, 78] and references therein). These problems can be divided into two groups. In Borg-Marchenko-type spectral problems, one tries to recover the potential from spectral data. However, Hochstadt-Lieberman-type (or mixed) spectral problems recover the potential using a mixture of partial information on the potential and spectral data.

In this thesis, our interest will be on regular Schrödinger operators with summable potentials on a finite interval. However, many problems with locally summable potentials [23, 24, 25, 26, 32, 49, 50] or on various settings such as half-line [26, 29, 31, 33, 64, 69], real-line [26, 29, 32, 33, 69] or graphs [9, 10, 11, 12, 79] are solved.

Borg's, Levinson's and Hochstadt and Lieberman's theorems suggest that one spectrum gives exactly one half of the full spectral information required to recover the potential. Recalling the fact that the spectral measure is a discrete measure supported on a spectrum, the same can be said for the set of point masses of the spectral measure. As follows from Marchenko's theorem, the set of point masses of the spectral measure (or the set of norming constants) gives exactly one half of the full spectral information required to recover the potential.

These observations allow us to formulate the following question:

Inverse Problem. Do one spectrum and partial information on another spectrum and the set of point masses of the spectral measure corresponding to the first spectrum recover the potential?

This Borg-Marchenko-type problem can be seen as a combination of Levinson's and Marchenko's results. Borg's and Marchenko's theorems can be deduced by complex theoretic methods using Krein spectral shift functions and Cauchy integrals respectively. However, neither of the methods work for the problem stated above, which makes this problem interesting not only for spectral theory, but also for complex function theory.

In Chapter 3, we answer this question positively. First, we give a proof with the most common boundary conditions, Dirichlet ($u = 0$) and Neumann ($u' = 0$). Theorem 22 solves this inverse spectral problem when given part of the point masses of the spectral measure corresponding to the Dirichlet-Dirichlet spectrum matches with the missing part of the Neumann-Dirichlet spectrum, i.e. they share same index sets. In Theorem 23 and Theorem 24, we consider the non-matching index sets case with some restrictions on two spectra.

In order to deal with general boundary conditions we introduce a more general m-function in Section 4.3. With this m-function, we extend Theorem 22 in Theorem 25 to general boundary conditions. In Theorem 26 and Theorem 27 we consider the non-matching index sets case.

Jacobi operators are discrete analogs of Schrödinger operators. The Jacobi operator J in the dense subset $c_{00}(\mathbb{N})$ of the Hilbert space $l^2(\mathbb{N})$ is the operator associated with the semi-infinite Jacobi matrix

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & b_2 & \ddots \\ 0 & b_2 & a_3 & \ddots \\ 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

where $a_n \in \mathbb{R}$ and $b_n > 0$ for any $n \in \mathbb{N}$. The symmetric operator J is closable and has deficiency indices (1,1) [limit point] or (0,0) [limit circle]. In the limit point case $\bar{J}$ is self-adjoint. However, in the limit circle case non self-adjoint operator $\bar{J}$ has a self-adjoint extension $J(g)$ uniquely determined by $g \in \mathbb{R} \cup \{\infty\}$ (see Section 4.1 and [66] Section 2.6). In both cases, a rank-one perturbation of a self-adjoint Jacobi operator can be seen as a change of the boundary condition at the origin for the corresponding Jacobi difference equation (see Section 4.1 and [62] Appendix).

In the discussion of inverse spectral problems for Jacobi operators, we replace the potential function q with the sequences $\{a_n\}_{\mathbb{N}}$ and $\{b_n\}_{\mathbb{N}}$. The study of inverse problems for Jacobi operators is motivated both by pure mathematics, e.g. moment problems [65] and physical applications, such as vibrating systems [35, 59].

Early inverse spectral problems for finite Jacobi matrices appear as discrete analogs of inverse spectral problems for Schrödinger operators. Finite Jacobi matrix analogs of Borg's and Hochstadt and Lieberman's theorems were considered by Hochstadt [42, 43, 44], where the potential q is replaced by the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$. These classical theorems led to various other inverse spectral results on finite Jacobi matrices (see [6, 16, 29, 31, 34, 61, 76] and references therein), semi-infinite or infinite Jacobi matrices (see [17, 18, 19, 20, 26, 29, 31, 34, 62, 63, 68] and references therein), generalized Jacobi matrices (see [21, 22] and references therein) and matrix-valued Jacobi operators (see [14, 30] and references therein).

Silva and Weder ([62] Theorem 3.3) proved Borg's two-spectra theorem for semi-infinite Jacobi matrices with a discrete spectrum. Later on Eckhardt and Teschl ([27] Theorem 5.2) considered infinite Jacobi matrix analog of Marchenko's result with the same discreteness of the spectrum assumption. Note that discreteness of the spectrum is an extra assumption in the limit point case.

Jacobi versions of Borg's and Hochstadt and Lieberman's theorems suggest that one spectrum gives exactly one half of the full spectral information required to recover the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$. Let us recall the fact that in the case of discrete spectrum, the spectral measure is a discrete measure supported on the spectrum with the point masses given by the corresponding norming constants (see [26] page 10). As follows from Jacobi analogs of Marchenko's theorem, the set of point masses of the spectral measure (or the set of norming constants) gives exactly one half of the full spectral information required to recover the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$.

These observations allow us to reformulate our inverse spectral problem for Jacobi operators:

Inverse Problem. Do one spectrum and partial information on another spectrum and the set of norming constants corresponding to the first spectrum recover the operator?

In Chapter 4, we answer this question positively. Theorem 30 solves this inverse spectral problem when given part of the norming constants corresponding to the first discrete spectrum matches with the missing part of the second discrete spectrum, i.e. they share the same index sets. In Theorem 31 and Theorem 32 we show that information of one of the boundary conditions can be replaced by any unknown eigenvalue from the second spectrum or any unknown norming constant corresponding to the first spectrum. In Theorems 33, 34 and 35 we consider the same problems in the non-matching index sets case with some restrictions on the two spectra.

2. PRELIMINARIES

The main reference for this chapter is [58].

2.1 Complex function theory

2.1.1 Hardy spaces in the upper-half plane

Let $\mathcal{H}(\Omega)$ denote the set of all analytic functions in the complex domain Ω. Hardy spaces are subclasses of analytic functions satisfying certain growth conditions.

Definition 1. Let $1 \leq p < \infty$. The **Hardy space** for p on the upper-half plane $\mathbb{C}_+$ is defined as

$$H^p(\mathbb{C}_+) := \big\{ f \in \mathcal{H}(\mathbb{C}_+) \;\big|\; \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p dx < \infty \big\}$$

For $1 \leq p < \infty$, the Hardy space $H^p(\mathbb{C}_+)$ becomes a Banach space with the norm

$$||f||_{H^p(\mathbb{C}_+)} := \sup_{y>0} \Big(\int_{\mathbb{R}} |f(x+iy)|^p dx \Big)^{1/p}.$$

For $p = \infty$, $H^\infty(\mathbb{C}_+)$ denotes the set of all analytic and bounded functions on the upper-half plane, equipped with the sup-norm $||f||_{H^\infty(\mathbb{C}_+)} = \sup_{z \in \mathbb{C}_+} |f(z)|$. Fatou proved existence of non-tangential boundary limits of H^p functions.

Definition 2. Let $\gamma : [0,1) \to \mathbb{C}_+$ be a continuous path such that $\lim_{t \to 1} \gamma(t) = a \in \mathbb{R}$. Let us define $\Gamma_\alpha(a) := \{z + a \mid \arg z \in (\pi/2 - \alpha, \pi/2 + \alpha)\}$ for $a \in \mathbb{R}$. The real number a is defined as the **non-tangential limit** of the path γ if there exists $\alpha \in (0, \pi/2)$ such that $\gamma \subset \Gamma_\alpha(a)$ and $\lim_{t \to 1} \gamma(t) = a$. If z approaches to a non-tangentially, it is denoted by $z \to_\angle a$.

The region $\Gamma_\alpha(a)$ is called **Stolz region** at a. Even if we discuss it on the upper-half plane, it is usually defined on the unit disk as follows. Let $\theta \in [0, 2\pi)$ and $M > 0$. A Stolz region in the unit disk is defined as

$$\Big\{ z \in \mathbb{D} \;\Big|\; \frac{|z - e^{i\theta}|}{1 - |z|} < M \Big\}.$$

The non-tangential limit can be seen as existence of the limit through a Stolz region. Figure 2.1 shows some examples of Stolz regions.

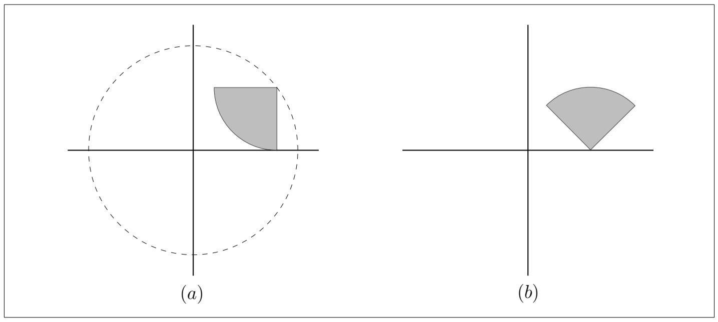

Figure 2.1: Examples of Stolz regions in the unit disk (a) and the upper-half plane (b).

Theorem 1. *Let $f(z) \in H^p(\mathbb{C}_+)$. For almost all $t \in \mathbb{R}$,*

$$f^*(t) := \lim_{z \to_\angle t} f(z) \text{ exists},$$

$$f^*(t) \in L^p(-\infty, \infty)$$

and for all $y > 0$,

$$f(x+iy) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f^*(t) dt.$$

Using Fatou's theorem we uniquely identify each function $f \in H^p(\mathbb{C}_+)$ with a function $f^* \in L^p(\mathbb{R})$. Moreover we have $||f||_{H^p(\mathbb{C}_+)} = ||f||_{L^p(\mathbb{R})}$. This allows us to extend the domain of f to $\mathbb{C}_+ \cup \mathbb{R}$ almost everywhere by letting $f(t) := f^*(t)$ for almost all $t \in \mathbb{R}$, so we can identify every $H^p(\mathbb{C}_+)$ with a closed subspace of $L^p(\mathbb{R})$. Therefore, the Hardy space $H^2(\mathbb{C}_+)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^2(\mathbb{C}_+)} = \int_{\mathbb{R}} f(t) \overline{g}(t) dt$$

for $f, g \in H^2(\mathbb{C}_+)$.

2.1.2 Inner-outer factorization

Functions from Hardy spaces can be represented in terms of products of well-understood functions, namely inner and outer functions. This representation is called inner-outer or canonical factorization. Before discussing inner and outer functions, we need to state some definitions.

A function on $\mathbb{R}$ is **Poisson-summable** if it is summable with respect to the Poisson measure Π, defined as $d\Pi := dx/(1+x^2)$. The space of Poisson-summable functions on $\mathbb{R}$ is denoted by L^1_Π.

Definition 3. The **Schwarz integral** of a Poisson-summable function f is defined as

$$Sf(z) := \frac{1}{i\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) f(t)dt,$$

where $z = x + iy$.

The Schwarz integral of a real valued Poisson-summable function is given in terms of its Poisson and conjugate Poisson integrals: $Sf = Pf + iQf$, where

$$Pf(z) := \frac{1}{\pi}\int \frac{y}{(t-x)^2+y^2} f(t)dt$$
$$Qf(z) := \frac{1}{\pi}\int \left(\frac{x-t}{(x-t)^2+y^2} + \frac{1}{1+t^2}\right) f(t)dt$$

A measure μ on $\mathbb{R}$ is Poisson-finite if $\int \frac{1}{1+t^2} d|\mu|(t) < \infty$. The Schwarz integral of a Poisson-finite measure μ, defined as

$$S\mu(z) := \frac{1}{i\pi}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t),$$

is analytic in the upper half-plane $\mathbb{C}_+$.

We discussed that any function $f \in H^\infty$ has non-tangential boundary values a.e. on $\mathbb{R}$. If absolute values of these limits are 1, then f is an inner function.

Definition 4. A bounded analytic function in $\mathbb{C}_+$ is called **inner** in $\mathbb{C}_+$ if it has non-tangential boundary values, equal to 1 in modulus, almost everywhere on $\mathbb{R}$.

Main examples of inner functions are exponential functions e^{iaz} for $a > 0$ and Blaschke products. The ratio

$$b_\lambda(z) := \frac{z - \lambda}{z - \overline{\lambda}}$$

is called a **Blaschke factor**, where $\lambda \in \mathbb{C}_+$. Let $\Lambda = \{\lambda_n\} \subset \mathbb{C}_+$ be a sequence satisfying the Blaschke condition

$$\sum_n \frac{\text{Im}\lambda_n}{1 + |\lambda_n|^2} < \infty.$$

The infinite product

$$B_\Lambda(z) := \prod \epsilon_n b_{\lambda_n}(z)$$

is called a **Blaschke product** for $\mathbb{C}_+$, where the unimodular constants ϵ_n satisfy $\epsilon_n b_{\lambda_n}(i) > 0$.

Definition 5. An analytic function in $\mathbb{C}_+$ is called **outer** in $\mathbb{C}_+$ if it is of the form e^{Sf} for $f \in L^1_\Pi$.

The **Hilbert transform** of $f \in L^1_\Pi$, denoted by $\widetilde{f}$, is defined as the singular integral

$$\widetilde{f}(x) := \frac{1}{\pi}\, p.v. \int \left[\frac{1}{x - t} + \frac{t}{1 + t^2}\right] f(t)dt.$$

It is the angular limit of $Qf = \text{Im}Sf$, hence the outer function e^{Sf} coincides with $e^{f + i\widetilde{f}}$ on $\mathbb{R}$.

Now we are ready to state canonical factorization of H^p functions.

Theorem 2. (Inner-outer factorization) *Let $f \in H^p(\mathbb{C}_+)$. Then for $z \in \mathbb{C}_+$,*

$$f(z) = I_f(z) \cdot O_f(z),$$

where the inner factor $I_f(z)$ is given by $I_f(z) = e^{i\gamma} B(z) e^{-iS\mu} e^{i\alpha z}$ such that

- $\gamma \in \mathbb{R}$,

- *$B(z) = \prod_n [e^{i\alpha_n}(z - \lambda_n)/(z - \overline{\lambda_n})]$ is a Blaschke product for $\mathbb{C}_+$, where λ_n are zeros of $f(z)$ in $\mathbb{C}_+$ and the real α_n satisfy $e^{i\alpha_n}(i - \lambda_n)/(i - \overline{\lambda_n}) \geq 0$,*

- *the singular measure μ on $\mathbb{R}$ is Poisson-finite,*

- *the mass at ∞, α is non-negative*

and the outer factor $O_f(z)$ is given by $O_f(z) = e^{S \log |f|}$.

2.1.3 Meromorphic inner functions and meromorphic Herglotz functions

We discussed that an inner function on $\mathbb{C}_+$ is a bounded analytic function on $\mathbb{C}_+$ with unit modulus a.e. on $\mathbb{R}$.

Definition 6. If an inner function extends to $\mathbb{C}$ meromorphically, it is called **meromorphic inner function**, which is usually denoted by Θ.

Meromorphic inner functions satisfy the representation $\Theta(z) = Ce^{iaz}B_\Lambda(z)$ for a unimodular constant C, a nonzero real constant a and a Blaschke product B_Λ with a discrete sequence Λ, i.e. Λ satisfies the Blaschke condition and has no finite accumulation point.

A meromorphic function is said to be **real** if it maps real numbers to real numbers on its domain.

Definition 7. A **meromorphic Herglotz function** m is a real meromorphic function with positive imaginary part on $\mathbb{C}_+$. It has negative imaginary part on $\mathbb{C}_-$ via the relation $m(\overline{z}) = \overline{m(z)}$.

There is a one-to-one correspondence between meromorphic inner functions and meromorphic Herglotz functions via equations

$$m = i\frac{1+\Theta}{1-\Theta}, \qquad \Theta = \frac{m-i}{m+i}.$$

A meromorphic Herglotz function can be described as the Schwarz integral of a positive discrete Poisson-finite measure:

$$m(z) = az + b + iS\mu,$$

where $a \geq 0$, $b \in \mathbb{R}$. The term iS is also called the **Herglotz integral** and usually denoted by H. This representation is valid even if the Herglotz function can not be extended meromorphically to $\mathbb{C}$, in which case μ may not be discrete. It is called the **Herglotz representation theorem**. Čebotarev proved a similar result.

Theorem 3 (Čebotarev [51])**.** *If the real meromorphic function m maps $\mathbb{C}_+$ onto $\mathbb{C}_+$, then its poles $\{a_k\}_{k\in\mathbb{Z}}$ are all real and simple, and it may be represented in the form*

$$m(z) = az + b + \sum_{k=N}^{M} A_k \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right), \tag{2.1}$$

where $a \geq 0$, $b \in \mathbb{R}$, $-\infty \leq N < M \leq \infty$, $A_k \geq 0$, the sum $\sum_{k=N}^{M} A_k/a_k^2$ converges and the sum in (2.1) converges normally on its domain if $N = \infty$ or $M = \infty$. Note that if $a_j = 0$ for some $N \leq j \leq M$, then the term with the index j in (2.1) is replaced by A_j/z.

Let us recall that for any infinite product (or sum) defined on an open set $\Omega \subset \mathbb{C}$, normal convergence means that the product (or the sum) converges uniformly on every compact subset of Ω.

2.2 Beurling-Malliavin theory

Let $\Lambda = \{\lambda_n\}$ be a sequence in the complex plane. One of the fundamental problems of Harmonic Analysis in the 20th century was the following question: Which conditions on Λ characterize completeness of the exponential system $E_\Lambda := \{e^{2\pi i\lambda_n x}\}_{\lambda_n\in\Lambda}$ in $L^2(0,a)$. In order to discuss this problem we need to consider maximal real number a for which E_Λ is complete in $L^2(0,a)$, i.e. the set of finite linear combinations of exponentials from E_Λ is dense in $L^2(0,a)$.

Definition 8. Let $\Lambda = \{\lambda_n\}$ be a complex sequence. The **radius of completeness** of Λ is defined as

$$R(\Lambda) := \sup\{a \mid E_\Lambda \text{ is complete in } L^2(0,a)\}.$$

Now the main goal becomes finding a formula for $R(\Lambda)$ when Λ is an arbitrary complex sequence. The problem can be reduced to the real sequences by the following observation: if Λ is a complex sequence, then E_Λ is complete in $L^2(0,a)$ if and only if $E_{\Lambda'}$ is complete in $L^2(0,a)$, where Λ' is the real sequence defined as $\lambda'_n = (\text{Re}\frac{1}{\lambda_n})^{-1}$, i.e. $R(\Lambda) = R(\Lambda')$. Note that if Λ includes purely imaginary numbers, without loss of generality we can replace Λ by $\Lambda + c$ for some $c \in \mathbb{R}$.

The Fourier transform is a useful tool to work on completeness of exponential systems.

Definition 9. Let $f \in L^2(\mathbb{R})$. The **Fourier transform** of f is defined as

$$\hat{f}(z) := \int_{\mathbb{R}} e^{-2\pi i x z} f(x) dx.$$

According to Paley-Wiener theorem, Fourier transform of a square integrable function f satisfying $\text{supp}(f) \subseteq [-a,a]$, is an entire function of exponential type at most $2\pi a$ and square integrable on the real line, i.e. $|\hat{f}(z)| \leq Ce^{2\pi a|z|}$ and $\hat{f}(x) \in L^2(\mathbb{R})$. This allows us to define Paley-Wiener spaces.

Definition 10. Let $a \in \mathbb{R}_+$. The Paley-Wiener space on $[-a,a]$ is defined as

$$PW_a := \left\{ F(z) \;\middle|\; F \text{ is entire, } |F(z)| \leq Ce^{2\pi a|z|} \text{ and } F \in L^2(\mathbb{R}) \right\}.$$

On the other hand every entire function of exponential type at most $2\pi a$ and square integrable on $\mathbb{R}$ is the Fourier transform of a square integrable function on $[-a,a]$. Therefore PW_a is the image of the space $L^2(-a,a)$ under the Fourier transform.

Let us recall that a system of exponentials E_Λ is incomplete in $L^2(0,a)$ if and only if there exists a non-zero $f \in L^2(0,a)$ such that $\langle f(x) \, , \, e^{2\pi i \lambda_n x} \rangle_{L^2(0,a)} = 0$ for every $\lambda_n \in \Lambda$, or equivalently $\hat{f}(\lambda_n) = 0$ for every $\lambda_n \in \Lambda$. Therefore using the Paley-Wiener theorem and the definition of the Fourier transform we can translate the completeness of exponential problem we stated to a complex analysis problem: E_Λ is complete in $L^2(0,2a)$ if and only if for any non-zero function $F \in PW_a$, $F(\lambda_n) \neq 0$ for some $\lambda_n \in \Lambda$. This observation allows us wlog to let Λ be discrete, i.e. Λ has no finite accumulation point, since radius of completeness of a sequence with a finite accumulation point is ∞. This follows from the identity theorem for entire functions, which implies that zero set of a non-zero entire function can not have a finite accumulation point.

At this point wlog we can restate our completeness problems as follows. How can we formulate $R(\Lambda)$ for the discrete sequence $\Lambda \subset \mathbb{R}$? A complete answer to this question was given by Beurling and Malliavin, but before stating that we need a few more definitions.

Definition 11. If $\{I_n\}_{n\in\mathbb{N}}$ is a sequence of disjoint intervals on the real line, it is called **short** if

$$\sum_{n\in\mathbb{N}} \frac{|I_n|^2}{1 + dist^2(0, I_n)} < \infty,$$

and **long** otherwise.

Definition 12. Let Λ be a sequence in $\mathbb{R}$. Then the **exterior (effective) Beurling-Malliavin density** of Λ is defined as

$$D^*(\Lambda) := \sup\{d \mid \exists \text{ long } \{I_n\} \text{ such that } \#(\Lambda \cap I_n) \geq d|I_n|,\ \forall n \in \mathbb{N}\}.$$

For a non-real sequence Λ, its exterior Beurling-Malliavin density is defined as $D^*(\Lambda) := D^*(\Lambda')$, where Λ' is a real sequence given by $\lambda'_n = (\text{Re}\frac{1}{\lambda_n})^{-1}$, if Λ has no imaginary points, and as $D^*(\Lambda) := D^*((\Lambda + c)')$ otherwise.

Now we are ready to state one of the most important results of the 20th century Harmonic Analysis.

Theorem 4 (Beurling-Malliavin [7, 8])**.** *Let Λ be a discrete sequence in $\mathbb{C}$. Then $R(\Lambda) = D^*(\Lambda)$.*

3. INVERSE SPECTRAL THEORY OF SCHRÖDINGER OPERATORS*

3.1 One-dimensional Schrödinger operator on a finite interval

As it was defined in the introduction, we consider the Schrödinger equation

$$-u''(t)+q(t)u(t)=zu(t) \tag{3.1}$$

on the interval $(0,\pi)$ associated with the boundary conditions

$$u(0)\cos\alpha-u'(0)\sin\alpha=0 \tag{3.2}$$

$$u(\pi)\cos\beta+u'(\pi)\sin\beta=0, \tag{3.3}$$

where $\alpha,\beta\in[0,\pi)$ and the potential function $q\in L^1(0,\pi)$ is real-valued.

The spectrum $\sigma_{\alpha,\beta}$ of the Schrödinger operator

$$L:u\mapsto -u''+qu$$

with $q\in L^1$ and boundary conditions (3.2), (3.3) is a discrete real sequence, bounded from below. Adding a positive constant to the potential q, shifts the spectrum by the same constant. This allows us to assume wlog $\sigma_{\alpha,\beta}\subset\mathbb{R}_+$. Note that we assume $\mathbb{N}=\{1,2,3,\dots\}$. Asymptotic behavior of the spectrum $\sigma_{\alpha,\beta}=\{a_n\}_{n\in\mathbb{N}}$, depending on the signs of α and β, is as follows:

If $\alpha\neq0$, $\beta\neq0$, then

$$a_n=(n-1)^2+\frac{2}{\pi}[\cot(\beta)+\cot(\alpha)]+\frac{1}{\pi}\int_0^\pi q(x)dx+\alpha_n \tag{3.4}$$

where $\alpha_n=o(1)$ as $n\to+\infty$.

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If $\alpha = 0$, $\beta = 0$, then

$$a_n = n^2 + \frac{1}{\pi}\int_0^\pi q(x)dx + \alpha_n \tag{3.5}$$

where $\alpha_n = o(1)$ as $n \to +\infty$.

If $\alpha \neq 0$, $\beta = 0$, then

$$a_n = \left(n - \frac{1}{2}\right)^2 + \frac{2}{\pi}\cot(\alpha) + \frac{1}{\pi}\int_0^\pi q(x)dx + \alpha_n \tag{3.6}$$

where $\alpha_n = o(1)$ as $n \to +\infty$.

If $\alpha = 0$, $\beta \neq 0$, then

$$a_n = \left(n - \frac{1}{2}\right)^2 + \frac{2}{\pi}\cot(\beta) + \frac{1}{\pi}\int_0^\pi q(x)dx + \alpha_n \tag{3.7}$$

where $\alpha_n = o(1)$ as $n \to +\infty$.

In the case $q \in L^2(0,\pi)$, the same asymptotics are valid with $\{\alpha_n\}_{n\in\mathbb{N}} \in l^2$.

One can find these results in the classical texts on Schrödinger operators, for instance [53] or [54].

Let us choose the boundary condition (3.2) and introduce two solutions $s_z(t)$ and $c_z(t)$ of (3.1) satisfying the initial conditions

$$s_z(0) = \sin(\alpha), \quad s_z'(0) = \cos(\alpha)$$
$$c_z(0) = \cos(\alpha), \quad c_z'(0) = -\sin(\alpha).$$

Definition 13. The **norming constant** τ_α, for the eigenvalue a_n is defined as

$$\tau_\alpha(a_n) := \int_0^\pi |s_{a_n}(t)|^2 dt.$$

Note that $s_z(t)$ and $c_z(t)$ are linearly independent solutions and their Wronskian satisfies

$W(c_z, s_z) = 1$, where $W(f, g) := fg' - gf'$. This allows us to represent $u_z(t)$, a solution of (3.1) with boundary conditions $u_z(\pi) = \sin\beta$, $u_z'(\pi) = -\cos\beta$, as

$$u_z(t) = c_z(t) + m_{\alpha,\beta}(z)s_z(t),$$

where

$$m_{\alpha,\beta}(z) = -\frac{W(c_z, u_z)}{W(s_z, u_z)}.$$

This is how we derive the m-function.

Definition 14. Weyl-Titchmarsh m-function with the boundary conditions (3.2), (3.3) is defined as

$$m_{\alpha,\beta}(z) := \frac{\cos(\alpha)u_z'(0) + \sin(\alpha)u_z(0)}{-\sin(\alpha)u_z'(0) + \cos(\alpha)u_z(0)},$$

where $\alpha, \beta \in [0, \pi)$.

It is well-known that Weyl m-function $m_{\alpha,\beta}$ is a meromorphic Herglotz function. Everitt [28] proved that the Weyl m-function has the asymptotic

$$m_{0,\beta}(z) = i\sqrt{z} + o(1)$$

for $\alpha = 0$, and

$$m_{\alpha,\beta}(z) = \frac{\cos\alpha}{\sin\alpha} + \frac{1}{\sin^2\alpha}\frac{i}{\sqrt{z}} + O\left(\frac{1}{|z|}\right)$$

for $\alpha \in (0, \pi)$ as z goes to infinity in the upper half plane. Asymptotics of Weyl m-function and Herglotz representation theorem imply that $m_{\alpha,\beta}$ is represented as the Herglotz integral of a discrete positive Poisson-finite measure supported on the spectrum $\sigma_{\alpha,\beta}$:

$$m_{\alpha,\beta}(z) = a + \int_{\mathbb{R}} \left[\frac{1}{t - z} - \frac{t}{1 + t^2}\right] d\mu_{\alpha,\beta}(t), \tag{3.8}$$

where $a = \mathrm{Re}(m_{\alpha,\beta}(i))$, $\sigma_{\alpha,\beta} = \{a_n\}_{n\in\mathbb{N}}$ and $\mu_{\alpha,\beta} = \sum_{n\in\mathbb{N}} \gamma_n \delta_{a_n}$. The measure $\mu_{\alpha,\beta}$ is the spectral measure of the Schrödinger operator L corresponding to the m-function $m_{\alpha,\beta}$. The point masses of the spectral measure are represented in terms of norming constants as $\gamma_n = (\tau_\alpha(a_n))^{-1}$.

Definition 15. The **spectral measure** of the Schrödinger operator L corresponding to the m-function $m_{\alpha,\beta}$ (or the boundary conditions (3.2), (3.3)) is defined as

$$\mu_{\alpha,\beta} := \sum_{n\in\mathbb{N}} \frac{\delta_{a_n}}{\tau_\alpha(a_n)},$$

where $\alpha, \beta \in [0,\pi)$ and $\sigma_{\alpha,\beta} = \{a_n\}_{n\in\mathbb{N}}$.

Since $\mu_{\alpha,\beta}$ is a Poisson-finite measure, the spectrum and the point masses of the spectral measure satisfy

$$\sum_{n\in\mathbb{N}} \frac{\gamma_n}{1+a_n^2} < \infty.$$

These properties of the m-function, the spectral measure and a detailed discussion of one dimensional Schrödinger operators appear in Chapter 9 of [66].

In order to illustrate what we have discussed so far, let us consider the free operator ($q \equiv 0$) with Dirichlet ($u = 0$) and Neumann ($u' = 0$) boundary conditions. Figure 3.1 shows the graph of Weyl m-function $m_{0,0}$ on $\mathbb{R}$, Neumann-Dirichlet spectrum σ_{ND} and Dirichlet-Dirichlet spectrum σ_{DD} for the free operator.

Example 1. The spectra, the m-function and the spectral measure for $q \equiv 0$ on $(0,\pi)$ with Dirichlet-Dirichlet, Neumann-Dirichlet and Neumann-Neumann boundary conditions are as follows.

$$\sigma_{DD} := \sigma_{0,0} = \{n^2\}_{n\in\mathbb{N}} \qquad m_{0,0} = -\sqrt{z}\cot(\sqrt{z}\pi) \qquad \mu_{0,0} = \frac{2}{\pi}\sum_{n=1}^{\infty} n^2\delta_{n^2}$$

$$\sigma_{ND} := \sigma_{\pi/2,0} = \{(n-\frac{1}{2})^2\}_{n\in\mathbb{N}} \qquad m_{\pi/2,0} = \frac{\tan(\sqrt{z}\pi)}{\sqrt{z}} \qquad \mu_{\pi/2,0} = \frac{2}{\pi}\sum_{n=1}^{\infty} \delta_{(n-1/2)^2}$$

$$\sigma_{NN} := \sigma_{\pi/2,\pi/2} = \{(n-1)^2\}_{n\in\mathbb{N}} \qquad m_{\pi/2,\pi/2} = \frac{\cot(\sqrt{z}\pi)}{\sqrt{z}} \qquad \mu_{\pi/2,\pi/2} = \frac{2}{\pi}\sum_{n=1}^{\infty} \delta_{(n-1)^2}$$

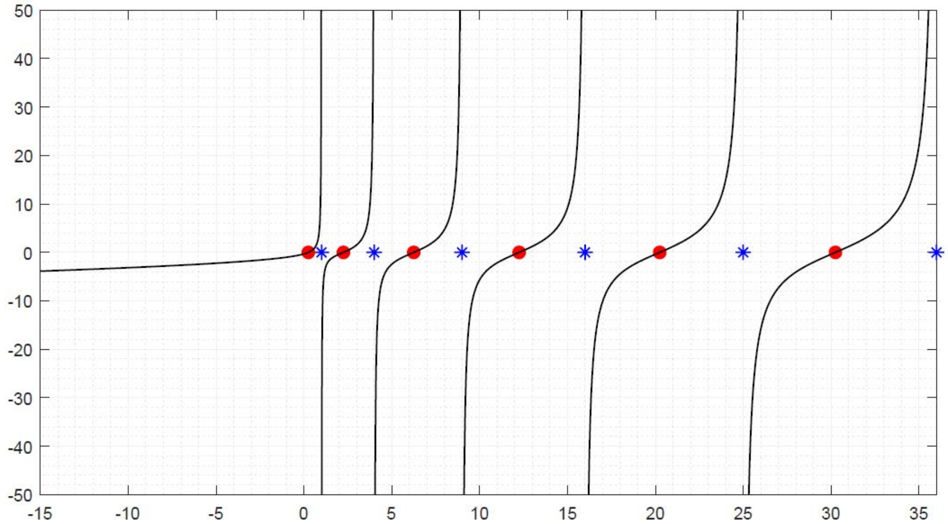

Figure 3.1: The graph of Weyl m-function $m_{0,0}$ on $\mathbb{R}$, Neumann-Dirichlet spectrum σ_{ND} (red) and Dirichlet-Dirichlet spectrum σ_{DD} (blue) for the free operator ($q \equiv 0$).

3.2 Inverse spectral theory of regular Schrödinger operators

3.2.1 Classical results

The first inverse spectral result on Schrödinger operators was given by Ambarzumian.

Theorem 5 (Ambarzumian [1], [46])**.** *Let* $q \in C[0,\pi]$ *and* $\sigma_{\pi/2,\pi/2} = \{n^2\}_{n=0}^{\infty}$. *Then* $q = 0$.

Later Borg found that in most cases two spectra is the required spectral information to recover the operator uniquely.

Theorem 6 (Borg [13], [46])**.** *Let* $q \in L^1(0,\pi)$, $\sigma_1 = \sigma_{0,\beta}$, $\sigma_2 = \sigma_{\alpha_2,\beta}$, $\sin\alpha_2 \neq 0$ *and*

$$\widetilde{\sigma}_2 = \begin{cases} \sigma_2 & \textit{if } \sin\beta = 0 \\ \sigma_2 \backslash \{a_1\} & \textit{if } \sin\beta \neq 0. \end{cases}$$

Then $\sigma_1 \cup \widetilde{\sigma}_2$ *determines the potential and no proper subset has the same property.*

A Schrödinger operator (or a potential) is said to be determined (or recovered) by its spectral data, if any other operator with the same data must have the same potential a.e. on $(0,\pi)$. Levinson extended Borg's result by removing the Dirichlet boundary condition restriction from the first spectrum.

Theorem 7 (Levinson [52], [46]). *Let* $q \in L^1(0, \pi)$ *and* $\sin(\alpha_1 - \alpha_2) \neq 0$. *Then* $\sigma_{\alpha_1,\beta}$ *and* $\sigma_{\alpha_2,\beta}$ *determine the potential.*

Marchenko showed that the spectral measure or the corresponding Weyl m-function provides sufficient spectral data to recover the potential uniquely.

Theorem 8 (Marchenko [56], [66]-Section 9.4). *Let* $q \in L^1(0, \pi)$. *Then* $\mu_{\alpha,\beta}$ *or* $m_{\alpha,\beta}$ *determines the potential.*

In the notations of Section 3.1, Marchenko's theorem says that the spectrum $\sigma_{\alpha,\beta} = \{a_n\}_{n\in\mathbb{N}}$ and the point masses $\{\gamma_n\}_{n\in\mathbb{N}}$ of the corresponding spectral measure (or the norming constants $\{\tau_\alpha(a_n)\}_{n\in\mathbb{N}}$) provide sufficient spectral data to recover the operator uniquely.

Hochstadt and Lieberman observed that one spectrum recovers the potential if the first half of it is known.

Theorem 9 (Hochstadt, Lieberman [45]). *Let* $q \in L^1(0, \pi)$. *Then q on* $(0, \pi/2)$ *and* $\sigma_{\alpha,\beta}$ *determine the potential.*

These classical theorems led to numerous results with different approaches such as

- using various spectral data (Borg-Marchenko type results),
- using mixture of partial knowledge of the potential and spectral data (Hochstadt-Lieberman type results),
- considering various smoothness classes for the potential ($q \in L^1, L^p, C^k, L^1_{loc}$),
- finding connections with exponential systems and
- changing the setting (half-line, real line, quantum graphs).

3.2.2 Some recent results in the finite interval case

For any discrete real sequence $A = \{x_n\}_{n\in\mathbb{N}}$, $x_n \to \infty$ the counting function is defined as

$$n_A(t) := \sum_{x_n \leq t} 1.$$

Gesztesy, Simon and del Rio generalized Levinson's theorem to three spectra.

Theorem 10 (del Rio, Gesztesy, Simon [15]). *Let $q \in L^1(0,\pi)$. Then $S \subset \sigma_{\alpha_1,\beta} \cup \sigma_{\alpha_2,\beta} \cup \sigma_{\alpha_3,\beta}$ satisfying*

$$n_S(t) \geq (2/3)n_{(\sigma_{\alpha_1,\beta} \cup \sigma_{\alpha_2,\beta} \cup \sigma_{\alpha_3,\beta})}(t)$$

for sufficiently large $t > 0$, determine the potential.

Gesztesy and Simon observed that the knowledge of the eigenvalues can be replaced by information on the derivatives of the potential around the midpoint of the interval.

Theorem 11 (Gesztesy, Simon [32]). *Let $q \in L^1(0,\pi)$, $\alpha, \beta \neq 0$ and $q \in C^{2k}(\pi/2 - \epsilon, \pi/2 + \epsilon)$ for some $k \in \mathbb{N}$ and $\epsilon > 0$. Then q on $(0,\pi/2)$ and $\sigma_{\alpha,\beta}$ except for $k+1$ eigenvalues determine the potential.*

In the same paper, they generalized Hochstadt-Lieberman theorem.

Theorem 12 (Gesztesy, Simon [32]). *Let $q \in L^1(0,\pi)$ and $\pi/2 < a < \pi$. Then q on $(0,a)$ and $S \subset \sigma_{\alpha,\beta}$ satisfying*

$$n_S(t) \geq 2(1 - a/\pi)n_{\sigma_{\alpha,\beta}}(t) + a/\pi - 1/2$$

for sufficiently large $t > 0$, determine the potential.

Amour, Raoux and Faupin proved similar results using extra information on smoothness of the potential.

Theorem 13 (Amour, Raoux [4]). *Let $\alpha, \beta_1, \beta_2 \neq 0$, $p \in [1,\infty)$, $q_1, q_2 \in L^1(0,\pi)$, $q_1 - q_2 \in L^p(a,\pi)$ and $\pi/2 < a < \pi$. If $q_1 = q_2$ a.e. on $(0,a)$ and $S \subset \sigma_{\alpha,\beta_1}(q_1) \cap \sigma_{\alpha,\beta_2}(q_2)$ satisfies*

$$2(1 - a/\pi)n_\sigma(t) + C \geq n_S(t) \geq 2(1 - a/\pi)n_\sigma(t) + 1/(2p) + 2a/\pi - 2$$

for a real number C and sufficiently large $t > 0$, where σ denotes either of $\sigma_{\alpha,\beta_k}(q_k)$, then $q_1 = q_2$ a.e. on $(0,\pi)$.

Theorem 14 (Amour, Faupin, Raoux [3]). *Let $\alpha, \beta_1, \beta_2 \neq 0$, $k \in \{0, 1, 2\}$, $p \in [1, \infty)$, $q_1, q_2 \in W^{k,1}(0, \pi)$, $q_1 - q_2 \in W^{k,p}(a, \pi)$ and $\pi/2 < a < \pi$. If $q_1 = q_2$ on $(0, a)$ and $S \subset \sigma_{\alpha,\beta_1}(q_1) \cap \sigma_{\alpha,\beta_2}(q_2)$ satisfying*

$$n_S(t) \geq 2(1 - a/\pi)n_\sigma(t) - k/2 + 1/(2p) + a/\pi - 3/2$$

for sufficiently large $t > 0$, where σ denotes either of $\sigma_{\alpha,\beta_k}(q_k)$, then $q_1 = q_2$ a.e. on $(0, \pi)$.

Theorem 15 (Amour, Faupin, Raoux [3]). *Let $\alpha, \beta_1, \beta_2 \neq 0$, $k \in \{0, 1, 2\}$, $p \in [1, \infty)$, $q_1, q_2 \in W^{k,1}(0, \pi)$, $q_1 - q_2 \in W^{k,p}(a, \pi)$ and $\pi/2 < a < \pi$. If $q_1 = q_2$ on $(0, a)$ and $S \subset \sigma_{\alpha,\beta_1}(q_1) \cap \sigma_{\alpha,\beta_2}(q_2)$ satisfying*

$$2(1 - a/\pi)n_\sigma(t) + C \geq n_S(t) \geq 2(1 - a/\pi)n_\sigma(t) - k/2 + 1/(2p) + 2a/\pi - 2$$

for sufficiently large $t > 0$, where σ denotes either of $\sigma_{\alpha,\beta_k}(q_k)$, then $q_1 = q_2$ a.e. on $(0, \pi)$.

Horváth proved a remarkable characterization theorem, which represents a connection between inverse spectral theory and completeness of exponential systems.

Theorem 16 (Horváth [46]). *Let $1 \leq p \leq \infty$, $q \in L^p(0, \pi)$, $0 \leq a < \pi$ and $\lambda_n \in \sigma_{\alpha,0}$. Then q on $(0, a)$ and the eigenvalues λ_n determine q if and only if the system*

$$e(\Lambda) = \{e^{\pm 2i\mu x}, e^{\pm 2i\sqrt{\lambda_n}x} : \quad n \geq 1\}$$

is complete in $L^p(a - \pi, \pi - a)$ for some $\mu \neq \pm\sqrt{\lambda_n}$.

Horváth and Sáfár proved similar results for the norming constants in terms of a cosine system. For a sequence $\Lambda = \{\lambda_1, \lambda_2, \dots\} \subset \mathbb{R}$ and a subset $S \subset \Lambda$ they considered the following cosine system:

$$C(\Lambda, S) = \{\cos(2\sqrt{\lambda_n}t) : n \in \mathbb{N}\} \cup \{t\cos(2\sqrt{\lambda_n}t) : \lambda_n \in S\}.$$

Theorem 17 (Horváth, Sáfár [48]). *Let $\beta \neq 0$, $1 \leq p \leq \infty$, $q \in L^1(0,\pi)$, $q \in L^p(a,\pi)$, $0 \leq a < \pi$ and*

$$\Lambda = \{\lambda_n : \lambda_n \in \sigma_{\alpha_n,\beta}, n \in \mathbb{N}\}$$

be a subset of eigenvalues such that $\lambda_n \not\to -\infty$ are different real numbers and $S \subset \Lambda$. Then q on $(0,a)$, Λ and $\{\tau_{\alpha_n}(\lambda_n)\}_{\lambda_n \in S}$ determine q if the system $C(\Lambda, S)$ is complete in $L^p(0,\pi - a)$.

For Dirichlet boundary condition Horváth and Sáfár obtained an optimal condition.

Theorem 18 (Horváth, Sáfár [48]). *Let us have the assumptions of Theorem 17, but $\beta = 0$. Let $\mu \neq \pm\sqrt{\lambda_n}$, $\mu \in \mathbb{R}$. Then the system $C(\Lambda, S) \cup \{\cos(2\sqrt{\mu}t)\}$ is complete in $L^p(0,\pi - a)$ if and only if q on $(0,a)$, Λ and $\{\tau_{\alpha_n}(\lambda_n)\}_{\lambda_n \in S}$ determine q.*

Makarov and Poltoratski gave a characterization theorem in terms of exterior Beurling-Malliavin density as a corollary of Horváth's result [46] (Theorem 16 above) and the Beurling-Malliavin theorem [7, 8].

If $\{I_n\}_{n\in\mathbb{N}}$ is a sequence of disjoint intervals on the real line, it is called short if

$$\sum_{n\in\mathbb{N}} \frac{|I_n|^2}{1 + dist^2(0, I_n)} < \infty$$

and long otherwise.

If Λ is a sequence of real points, its exterior (effective) Beurling-Malliavin density is defined as

$$D^*(\Lambda) = \sup\{d \mid \exists \text{ long } \{I_n\} \text{ such that } \#(\Lambda \cap I_n) \geq d|I_n|,\ \forall n \in \mathbb{N}\}.$$

For a non-real sequence its density is defined as $D^*(\Lambda) = D^*(\Lambda')$, where Λ' is a real sequence $\lambda'_n = (\mathrm{Re}\frac{1}{\lambda_n})^{-1}$, if Λ has no imaginary points, and as $D^*(\Lambda) = D^*((\Lambda + c)')$ otherwise.
For any complex sequence Λ its radius of completeness is defined as

$$R(\Lambda) = \sup\{a \mid \{e^{2\pi i\lambda z}\}_{\lambda\in\Lambda} \text{ is complete in } L^2(0,a)\}.$$

Now we are ready to state one of the fundamental results of Harmonic Analysis.

Theorem 19 (Beurling-Malliavin theorem [7, 8])**.** *Let* Λ *be a discrete sequence. Then*

$$R(\Lambda) = D^*(\Lambda).$$

Let us note that Makarov and Poltoratski considered the Schrödinger equation $Lu = -u'' + qu = z^2u$ and the m-function corresponding to this equation, which is obtained by applying the square root transform to the m-function we have discussed so far. Let us denote their m-function by $\widetilde{m}$.

Theorem 20 (Makarov, Poltoratski [55])**.** *Let* $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ *be a sequence of discrete non-zero complex numbers,* $q \in L^2(0,\pi)$ *and* $0 \leq a \leq 1$*. The following statements are equivalent:*

1. q *on* $(0,d)$ *for some* $d > a$ *and* $\{\widetilde{m}(\lambda_n)\}_{n\in\mathbb{N}}$ *determine* q*.*

2. $\pi D^*(\Lambda) \geq 1 - a$*.*

Makarov and Poltoratski's observation shows that Horváth's theorem establishes equivalence between mixed spectral problems for Schrödinger operators and the Beurling-Malliavin problem on completeness of exponentials in L^2 spaces.

In the same paper they obtained an uncertainty version of Borg's theorem.

Theorem 21 (Makarov, Poltoratski [55])**.** *Let* $\{I_n\}_{n\in\mathbb{N}}$ *be a sequence of intervals on* $\mathbb{R}$ *and* $q \in L^2(0,\pi)$*. The following statements are equivalent:*

1. *The condition* $\sigma_{DD} \cup \sigma_{ND} \subset \cup_{n\in\mathbb{N}} I_n$ *and* q *on* $(0,\epsilon)$ *for some* $\epsilon > 0$ *determine the potential* q*.*

2. *For any long sequence of intervals* $\{J_n\}_{n\in\mathbb{N}}$*,*

$$\frac{\sum_{I_n\cap J_n} \log_- |I_n|}{|J_n|} \nrightarrow 0$$

as $n \to \infty$*.*

3.3 An inverse spectral problem with mixed data

3.3.1 Matching index sets

We prove our first result, Theorem 22, by representing the Weyl-Titchmarsh m-function as an infinite product in terms of Dirichlet-Dirichlet ($\alpha = 0$, $\beta = 0$) and Neumann-Dirichlet ($\alpha = \pi/2$, $\beta = 0$) spectra. We follow the notations introduced in Example 1 for these two spectra, i.e. $\sigma_{DD} := \sigma_{0,0}$ and $\sigma_{ND} := \sigma_{\pi/2,0}$. For simplicity, let us also denote $m_{0,0}$ by m.

Lemma 1. (infinite product representation of m-function) *The m-function of a regular Schrödinger operator ($q \in L^1(0,\pi)$) for Dirichlet-Dirichlet boundary conditions ($\alpha = 0$, $\beta = 0$) has representations in terms of Dirichlet-Dirichlet and Neumann-Dirichlet spectra:*

$$m(z) = C\left(\frac{z}{b_1} - 1\right)\prod_{n\in\mathbb{N}}\left(\frac{z}{b_{n+1}} - 1\right)\left(\frac{z}{a_n} - 1\right)^{-1}, \tag{3.9}$$

and

$$m(z) = -C\prod_{n\in\mathbb{N}}\left(\frac{z}{b_n} - 1\right)\left(\frac{z}{a_n} - 1\right)^{-1}, \tag{3.10}$$

where $C > 0$, $\sigma_{DD} = \{a_n\}_{n\in\mathbb{N}}$, $\sigma_{ND} = \{b_n\}_{n\in\mathbb{N}}$ and the product converges normally on $\mathbb{C}\backslash\cup_{n\in\mathbb{N}}a_n$.

Proof. Let $m = u_z'(0)/u_z(0)$ be the Weyl m-function with boundary conditions $u(\pi) = 0$, $u'(\pi) = -1$. Since m is a meromorphic Herglotz function, $\Theta := \frac{m-i}{m+i}$ is the corresponding meromorphic inner function.

Let us define the set E in $\mathbb{R}$ as $E := \{z \in \mathbb{R} : Im\Theta(z) > 0\}$. The set E is given in terms of $\sigma_{DD} = \{a_n\}_{n\in\mathbb{N}}$ and $\sigma_{ND} = \{b_n\}_{n\in\mathbb{N}}$, namely $E = (-\infty, b_1) \cup \cup_{n\in\mathbb{N}}(a_n, b_{n+1})$.

The characteristic function of E coincides with the real part of the function $\frac{1}{i\pi}\log(i\frac{1+\Theta}{1-\Theta})$ a.e. on $\mathbb{R}$. Since m is a meromorphic Herglotz function mapping $\mathbb{R}$ to $\mathbb{R}$ a.e., $\log(m) = \log(i\frac{1+\Theta}{1-\Theta})$ is a well-defined holomorphic function on $\mathbb{C}_+$ and its imaginary part takes values 0 and π a.e. on $\mathbb{R}$. Therefore $\frac{1}{i\pi}\log(m) = \frac{1}{i\pi}\log(i\frac{1+\Theta}{1-\Theta})$ and the Schwarz integral of χ_E, S_{χ_E} differ by a purely imaginary number on a.e. $\mathbb{R}$, i.e.

$$\frac{1}{i\pi}\log\left(i\frac{1+\Theta}{1-\Theta}\right)=S_{\chi_E}+ic=P_{\chi_E}+iQ_{\chi_E}+ic,\quad c\in\mathbb{R},$$

where P and Q are Poisson and conjugate Poisson integrals of χ_E, respectively. Therefore

$$i\frac{1+\Theta}{1-\Theta}=\exp(i\pi S_{\chi_E}-\pi c)=\exp(i\pi P_{\chi_E}-\pi Q_{\chi_E}-\pi c),\quad c\in\mathbb{R}.$$

On the real line, $\exp(S_h)=\exp(h+i\widetilde{h})$ for any Poisson-summable function h, where $\widetilde{h}$ is the Hilbert transform of h. If we let $h:=\chi_E$, then

$$\widetilde{h}(x)=\frac{1}{\pi}\left[\log\left(\frac{\sqrt{1+b_1^2}}{|x-b_1|}\right)+\sum_{n\in\mathbb{N}}\log\left(\frac{|x-a_n|}{|x-b_{n+1}|}\right)+\frac{1}{2}\sum_{n\in\mathbb{N}}\log\left(\frac{1+b_{n+1}^2}{1+a_n^2}\right)\right].$$

Therefore

$$\exp(-\pi\widetilde{h}(x))=\frac{|x-b_1|}{\sqrt{1+b_1^2}}\prod_{n\in\mathbb{N}}\frac{|x-b_{n+1}|}{|x-a_n|}\prod_{n\in\mathbb{N}}\left(\frac{1+a_n^2}{1+b_{n+1}^2}\right)^{1/2}.$$

Noting that $\exp(i\pi h)$ is -1 on E and 1 on $\mathbb{R}\backslash E$, the Weyl m-function can be given in terms of σ_{DD} and σ_{ND} a.e. on $\mathbb{R}$:

$$\begin{aligned}m(x)&=i\frac{1+\Theta(x)}{1-\Theta(x)}\\&=\exp(i\pi S_{\chi_E}-\pi c)\\&=\frac{x-b_1}{\sqrt{1+b_1^2}}\prod_{n\in\mathbb{N}}\frac{x-b_{n+1}}{x-a_n}\prod_{n\in\mathbb{N}}\left(\frac{1+a_n^2}{1+b_{n+1}^2}\right)^{1/2}\exp(-\pi c)\\&=C\left(\frac{x}{b_1}-1\right)\prod_{n\in\mathbb{N}}\left(\frac{x}{b_{n+1}}-1\right)\left(\frac{x}{a_n}-1\right)^{-1}\end{aligned}$$

where $C=\exp(-\pi c)\prod_{n\in\mathbb{N}}\frac{\sqrt{1+a_n^2}}{a_n}\frac{b_n}{\sqrt{1+b_n^2}}$.

Since $m(z)$ and $C\left(\frac{z}{b_1}-1\right)\prod_{n\in\mathbb{N}}\left(\frac{z}{b_{n+1}}-1\right)\left(\frac{z}{a_n}-1\right)^{-1}$ are meromorphic functions that agree a.e. on $\mathbb{R}$, they are identical by the identity theorem for meromorphic functions. This gives the first representation (3.9). The second representation (3.10) follows from normal convergence of $\{z/b_n - 1\}_{n\in\mathbb{N}}$ to -1 in $\mathbb{C}$. □

Using this representation of the m-function, we prove our first result. At this point let us note that the eigenvalues in a spectrum are enumerated in increasing order, which is done following the asymptotics (3.4), (3.5), (3.6) and (3.7).

Theorem 22. (Inverse problem I-a) *Let $q \in L^1(0,\pi)$ and $A \subseteq \mathbb{N}$. Then $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}\setminus A}$ and $\{\gamma_n\}_{n\in A}$ determine the potential q, where $\sigma_{DD} = \{a_n\}_{n\in\mathbb{N}}$, $\sigma_{ND} = \{b_n\}_{n\in\mathbb{N}}$ are Dirichlet-Dirichlet and Neumann-Dirichlet spectra and $\{\gamma_n\}_{n\in\mathbb{N}}$ are point masses of the spectral measure $\mu_{0,0} = \sum_{n\in\mathbb{N}} \gamma_n \delta_{a_n}$.*

Proof. By representation (3.8) of the m-function as a Herglotz integral of the spectral measure, knowing γ_n means knowing $Res(m, a_n)$. Therefore, in terms of the m-function our claim says that the set of poles, $\{a_n\}_{n\in\mathbb{N}}$, the set of zeros except the index set A, $\{b_n\}_{n\in\mathbb{N}\setminus A}$, and the residues with the same index set A, $\{Res(m, a_n)\}_{n\in A}$ determine the m-function uniquely. Before starting to prove this claim let us briefly list the main steps of the proof. We will use similar ideas to prove our results in non-matching index sets case and for general boundary conditions.

Step 1: Reduce the claim to the problem of unique recovery of the infinite product

$$G(z) := -C \prod_{n\in A} \left(\frac{z}{b_n} - 1\right)\left(\frac{z}{a_n} - 1\right)^{-1}$$

from its sets of poles and residues.

Step 2: Observe that $G(z)$ is a meromorphic Herglotz function and has a representation in terms of its poles, residues and a linear polynomial $dz + e$.

Step 3: Show uniqueness of d.

Step 4: Show uniqueness of e.

Step 5: Use the representation from Step 2 to get uniqueness of the two spectra and prove the claim by Borg's theorem.

Step 1

From Lemma 1, the Weyl m-function can be represented in terms of σ_{DD} and σ_{ND},

$$m(z) = -C \prod_{n \in \mathbb{N}} \left(\frac{z}{b_n} - 1 \right) \left(\frac{z}{a_n} - 1 \right)^{-1}.$$

Note that for any $k \in A$, we know

$$Res(m, a_k) = C(b_k - a_k) \frac{a_k}{b_k} \prod_{n \in \mathbb{N}, n \neq k} \left(\frac{a_k}{b_n} - 1 \right) \left(\frac{a_k}{a_n} - 1 \right)^{-1}. \tag{3.11}$$

Let $m(z) = F(z)G(z)$, where F and G are two infinite products defined as

$$G(z) := -C \prod_{n \in A} \left(\frac{z}{b_n} - 1 \right) \left(\frac{z}{a_n} - 1 \right)^{-1}, \qquad F(z) := \prod_{n \in \mathbb{N} \setminus A} \left(\frac{z}{b_n} - 1 \right) \left(\frac{z}{a_n} - 1 \right)^{-1}$$

Also note that at any point of $\{a_n\}_{n \in A}$, the infinite product

$$F(z) = \prod_{n \in \mathbb{N} \setminus A} \left(\frac{z}{b_n} - 1 \right) \left(\frac{z}{a_n} - 1 \right)^{-1} \tag{3.12}$$

is known.

Conditions (3.11) and (3.12) imply that for any $k \in A$, we know

$$Res(G, a_k) = \frac{Res(m, a_k)}{F(a_k)},$$

i.e. we know all of the poles and residues of $G(z)$, but none of its zeros. We claim that $G(z)$ can be uniquely recovered from this data set.

Step 2

Let us observe that $arg(G(z)) = \pi - \sum_{n \in A} [arg(z - b_n) - arg(z - a_n)]$. Since zeros and poles of $G(z)$ are real and interlacing, $0 < arg(G(z)) < \pi$ for any z in the upper half plane, i.e. $G(z)$ is a meromorphic Herglotz function. Therefore by Čebotarev's theorem, see Theorem 3,

$G(z)$ has the representation

$$G(z) = dz + e + \sum_{n \in A} A_n \left(\frac{1}{a_n - z} - \frac{1}{a_n} \right), \tag{3.13}$$

where $d \geq 0$, $e \in \mathbb{R}$ and $\sum_{n \in A} A_n / a_n^2$ is absolutely convergent.

Note that $A_k = -Res(G(z), a_k)$ for any $k \in A$, which means there are only two unknowns on the right hand side of (3.13), namely constants d and e.

Step 3

Now let us show uniqueness of $G(z)$ by showing uniqueness of $dz + e$. Let $\widetilde{G}(z)$ be another infinite product sharing same properties with $G(z)$, namely:

- The infinite product $\widetilde{G}(z)$ is defined as

$$\widetilde{G}(z) := -\widetilde{C} \prod_{n \in A} \left(\frac{z}{\widetilde{b}_n} - 1 \right) \left(\frac{z}{\widetilde{a}_n} - 1 \right)^{-1},$$

  where $\widetilde{C} > 0$, the set of poles $\{\widetilde{a}_n\}_{n \in A}$ satisfies asymptotics (3.5) and the set of zeros $\{\widetilde{b}_n\}_{n \in A}$ satisfies asymptotics (3.6).

- $G(z)$ and $\widetilde{G}(z)$ share same set of poles with equivalent residues at the corresponding poles, i.e. $\widetilde{a}_k = a_k$ and $Res(\widetilde{G}, a_k) = Res(G, a_k)$ for any $k \in A$.

- By the equivalence of poles and residues of $G(z)$ and $\widetilde{G}(z)$ and Čebotarev's theorem, $\widetilde{G}(z)$ has the representation

$$\widetilde{G}(z) = \widetilde{d}z + \widetilde{e} + \sum_{n \in A} A_n \left(\frac{1}{a_n - z} - \frac{1}{a_n} \right), \tag{3.14}$$

  where $\widetilde{d} \geq 0$, $\widetilde{e} \in \mathbb{R}$.

Note that we defined $\widetilde{a}_n$ and $\widetilde{b}_n$ only for $n \in A$. Let $\widetilde{a}_n := a_n$ and $\widetilde{b}_n := b_n$ for every $n \in \mathbb{N} \backslash A$. Let us also note that $\{\widetilde{a}_n\}_{n \in \mathbb{N}}$ and $\{\widetilde{b}_n\}_{n \in \mathbb{N}}$ are interlacing sequences so that they represent two spectra of a potential function $\widetilde{q}(x)$.

Let $k \in A$ and $b_k \neq \widetilde{b}_k$. Since $G(b_k) = 0$ and $\widetilde{G}(\widetilde{b}_k) = 0$, using representations (3.13) and (3.14) we get

$$-db_k - e = \sum_{n \in A} A_n \left(\frac{1}{a_n - b_k} - \frac{1}{a_n} \right), \tag{3.15}$$

$$-\tilde{d}\tilde{b}_k - \widetilde{e} = \sum_{n \in A} A_n \left(\frac{1}{a_n - \tilde{b}_k} - \frac{1}{a_n} \right) \text{ and} \tag{3.16}$$

$$G(\tilde{b}_k) = G(\tilde{b}_k) - \widetilde{G}(\tilde{b}_k) = (d - \tilde{d})\tilde{b}_k + e - \tilde{e} \tag{3.17}$$

Replacing $e - \widetilde{e}$ by $G(\widetilde{b}_k) - (d - \widetilde{d})\widetilde{b}_k$ and taking difference of (3.15) and (3.16) we get

$$db_k - \tilde{d}\tilde{b}_k - d\tilde{b}_k + \tilde{d}\tilde{b}_k + G(\tilde{b}_k) = \sum_{n \in A} A_n \left(\frac{\widetilde{b}_k - b_k}{(a_n - \widetilde{b}_k)(a_n - b_k)} \right)$$

Dividing both sides by $\widetilde{b}_k(\widetilde{b}_k - b_k)$ we get

$$\frac{-d}{\widetilde{b}_k} + \frac{G(\tilde{b}_k)}{\widetilde{b}_k(\widetilde{b}_k - b_k)} = \sum_{n \in A} \left(\frac{A_n}{\widetilde{b}_k(a_n - \widetilde{b}_k)(a_n - b_k)} \right) \tag{3.18}$$

Note that since $\{a_n\}_{n \in A}$ satisfies asymptotics (3.5) and $\{b_n\}_{n \in A}$, $\{\widetilde{b}_n\}_{n \in A}$ satisfy asymptotics (3.6), the inequalities

$$|\widetilde{b}_k(a_n - b_k)(a_n - \widetilde{b}_k)|^{-1} \leq |\widetilde{b}_n(a_n - b_n)(a_n - \widetilde{b}_n)|^{-1} \leq 2/a_n^2 \tag{3.19}$$

are valid for any $k \in A$, for sufficiently large $n \in A$. In addition, $\sum_{n \in A} A_n / a_n^2$ is absolutely convergent. Therefore right hand side of (3.18) converges to 0 as k goes to ∞. Also note that by (3.17), left hand side of (3.18) is

$$\frac{-d}{\widetilde{b}_k} + \frac{G(\tilde{b}_k)}{\widetilde{b}_k(\widetilde{b}_k - b_k)} = \frac{-d}{\widetilde{b}_k} + \frac{G(\tilde{b}_k) - \widetilde{G}(\tilde{b}_k)}{\widetilde{b}_k(\widetilde{b}_k - b_k)} = -\frac{d}{\tilde{b}_k} + \frac{1}{\tilde{b}_k - b_k} \left[d - \tilde{d} + \frac{e - \tilde{e}}{\tilde{b}_k} \right]. \tag{3.20}$$

Now let us show $\tilde{b}_k - b_k$ converges to 0 as k goes to ∞. Recall that poles of G and $\widetilde{G}$ satisfy asymptotics

$$n^2 + \frac{1}{\pi}\int_0^\pi q(x)dx + \alpha_n \qquad \text{and} \qquad n^2 + \frac{1}{\pi}\int_0^\pi \widetilde{q}(x)dx + \widetilde{\alpha}_n$$

respectively, where $\alpha_n = o(1)$ and $\widetilde{\alpha}_n = o(1)$ as $n \to \infty$. Equivalance of poles of G and $\widetilde{G}$ imply equivalence of $\int_0^\pi q(x)dx$ and $\int_0^\pi \widetilde{q}(x)dx$. Therefore b_n and $\tilde{b}_n$ satisfy asymptotics

$$\left(n - \frac{1}{2}\right)^2 + \frac{1}{\pi}\int_0^\pi q(x)dx + \beta_n \qquad \text{and} \qquad \left(n - \frac{1}{2}\right)^2 + \frac{1}{\pi}\int_0^\pi q(x)dx + \widetilde{\beta}_n,$$

where $\beta_n = o(1)$ and $\widetilde{\beta}_n = o(1)$ as $n \to \infty$. Hence $\tilde{b}_k - b_k = o(1)$ as k goes to ∞. Therefore by (3.20), left hand side of (3.18) goes to ∞ if $d - \tilde{d} \neq 0$, so we get a contradiction unless $d = \tilde{d}$. This implies that $G(z) - \widetilde{G}(z)$ is a real constant, which is $G(0) - \widetilde{G}(0) = \widetilde{C} - C$.

<u>Step 4</u>

Now let us show $\widetilde{C} - C = 0$. Positivity of $(\widetilde{b}_k - b_n)/(\widetilde{b}_k - a_n)$ for all $n \neq k$, which follows from interlacing property of $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ and interlacing property of $\{a_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{b}_n\}_{n\in\mathbb{N}}$, implies $sgn(\widetilde{C} - C) = sgn(\widetilde{\beta}_k - \beta_k)$ for all $k \in \mathbb{N}$, i.e. $\{b_n\}_{n\in A}$ and $\{\widetilde{b}_n\}_{n\in A}$ are interlacing sequences.

Let us assume $\widetilde{C} > C$ and wlog the two spectra lie on the positive real line. This implies $\widetilde{b}_n > b_n$ for all $n \in A$. Observe that $\prod_{n\in A} \widetilde{b}_n/b_n$ is finite, since

$$\sum_{n\in A} \frac{\widetilde{b}_n - b_n}{b_n} = \sum_{n\in A} \frac{\widetilde{\beta}_n - \beta_n}{b_n} \leq \max_{n\in A}(\widetilde{\beta}_n - \beta_n) \sum_{n\in A} \frac{1}{b_n} < \infty.$$

Therefore the infinite product $H(z) := G(z)/\widetilde{G}(z)$ is represented as

$$H(z) := \frac{G(z)}{\widetilde{G}(z)} = \frac{C}{\widetilde{C}} \prod_{n\in A} \frac{z - b_n}{b_n} \frac{\tilde{b}_n}{z - \tilde{b}_n} = \frac{C}{\widetilde{C}} \prod_{n\in A} \frac{\tilde{b}_n}{b_n} \prod_{n\in A} \frac{z - b_n}{z - \tilde{b}_n}.$$

Let us denote the constant factor of $H(z)$ by $N := (C/\widetilde{C}) \prod_{n\in A} \tilde{b}_n/b_n$. Then by interlacing property of $\{b_n\}_{n\in A}$ and $\{\widetilde{b}_n\}_{n\in A}$, the infinite product $-H$ is a meromorphic Herglotz function and hence by Theorem 3 it is represented as

$$-H(z) = -N\prod_{n\in A}\frac{z-b_n}{z-\tilde{b}_n} = Dz+E+\sum_{n\in A}B_n\left(\frac{1}{z-\tilde{b}_n}+\frac{1}{\tilde{b}_n}\right), \tag{3.21}$$

where $B_k = -Res(H,\tilde{b}_k)$ and $D, E\in\mathbb{R}$.

Now let us show that $\{B_k/\tilde{b}_k\}_{k\in A}$ is summable.

$$\begin{aligned}
\left|\frac{B_k}{\widetilde{b}_k}\right| &= N\,\frac{\widetilde{b}_k-b_k}{b_k}\prod_{n\in A, n\neq k}\frac{\widetilde{b}_k-b_n}{\widetilde{b}_k-\widetilde{b}_n}\\
&\le N\,\frac{\widetilde{b}_k-b_k}{\widetilde{b}_k}\prod_{n\in A, 1\le n\le k-1}\frac{\widetilde{b}_k-b_n}{\widetilde{b}_k-\widetilde{b}_n}\\
&= N\,\frac{\widetilde{b}_k-b_k}{\widetilde{b}_k}\prod_{n\in A, 1\le n\le k-1}\left(1+\frac{\widetilde{b}_n-b_n}{\widetilde{b}_k-\widetilde{b}_n}\right)\\
&= N\,\frac{\widetilde{b}_k-b_k}{\widetilde{b}_k}\prod_{n\in A, 1\le n\le k-1}\left(1+\frac{\widetilde{\beta}_n-\beta_n}{(k-1/2)^2-(n-1/2)^2+\widetilde{\beta}_k-\widetilde{\beta}_n}\right)\\
&\le N\,\frac{\widetilde{b}_k-b_k}{\widetilde{b}_k}\prod_{n\in A, 1\le n\le k-1}\left(1+\frac{\widetilde{\beta}_n-\beta_n}{(n+1-1/2)^2-(n-1/2)^2+\widetilde{\beta}_k-\widetilde{\beta}_n}\right)\\
&\le N\,\frac{\widetilde{b}_k-b_k}{\widetilde{b}_k}\,M\prod_{n=1}^{k-1}\left(1+\frac{1}{2n}\right),
\end{aligned}$$

for sufficiently large k, where M is a real constant independent of k. Since $\widetilde{b}_k - b_k = o(1)$, $\widetilde{b}_k = O(k^2)$ and $\prod_{n=1}^{k-1}(1+1/2n) = O(\sqrt{k})$ as k goes to ∞, we get the asymptotics $B_k/\widetilde{b}_k = o(1/k^{3/2})$ as k goes to ∞ and hence $\sum_{n\in A} B_n/\tilde{b}_n$ is absolutely convergent. Then by letting z tend to $-\infty$ in (3.21) we get

$$-N = \lim_{t\to-\infty}\left(Dt+E+\sum_{n\in A}\frac{B_n}{\widetilde{b}_n}+\sum_{n\in A}\frac{B_n}{t-\widetilde{b}_n}\right)$$

and hence $D=0$ and $-N = E+\sum_{n\in A}B_n/\widetilde{b}_n$, i.e. $-H(z)$ has the representation

$$-H(z) = N - \sum_{n\in A}\frac{B_n}{z-\widetilde{b}_n}. \tag{3.22}$$

Noting that $H(b_k) = 0$ and $Res(G, a_k) = Res(\widetilde{G}, a_k)$, i.e. $H(a_k) = 1$ for all $k \in A$, we get

$$1 = H(a_k) - H(b_k) = -N + \sum_{n \in A} \frac{B_n}{a_k - \widetilde{b}_n} + N - \sum_{n \in A} \frac{B_n}{b_k - \widetilde{b}_n} = \sum_{n \in A} B_n \frac{(b_k - a_k)}{(a_k - \widetilde{b}_n)(b_k - \widetilde{b}_n)}$$

Each term of the infinite sum on the right end is positive, so by letting k go to ∞ we get the following contradiction.

$$1 = \lim_{k \to \infty} \sum_{n \in A} B_n \frac{(b_k - a_k)}{(a_k - \widetilde{b}_n)(b_k - \widetilde{b}_n)} = \sum_{n \in A} B_n \lim_{k \to \infty} \frac{(b_k - a_k)}{(a_k - \widetilde{b}_n)(b_k - \widetilde{b}_n)} = 0$$

Similar arguments give another contradiction when $\widetilde{C} < C$, so $C = \widetilde{C}$.

Step 5

Step 4 implies uniqueness of $dz + e$, i.e. uniqueness of $G(z)$ and hence uniqueness of $\{b_n\}_{n \in A}$. After unique recovery of the two spectra $\sigma_{DD} = \{a_n\}_{n \in \mathbb{N}}$ and $\sigma_{ND} = \{b_n\}_{n \in \mathbb{N}}$, the potential is uniquely determined by Borg's theorem. □

Remark 1. *If we let $A = \mathbb{N}$, Theorem 22 gives Marchenko's theorem with Dirichlet-Dirichlet, Neumann-Dirichlet boundary conditions as a corollary. By letting $A = \emptyset$, we get the statement of Borg's theorem with Dirichlet-Dirichlet, Neumann-Dirichlet boundary conditions.*

Remark 2. *Spectral data of Theorem 22 can be seen as $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N} \setminus A}$ and $\{\tau_\alpha(a_n)\}_{n \in A}$, where $\{\tau_\alpha(a_n)\}_{n \in A}$ is the set of norming constants for $\sigma_{DD} = \{a_n\}_{n \in \mathbb{N}}$.*

3.3.2 Non-matching index sets

If the known point masses of the spectral measure and unknown eigenvalues of the Neumann-Dirichlet spectrum have different index sets, one needs some control over eigenvalues of the Dirichlet-Dirichlet spectrum corresponding to known point masses and unknown part of the Neumann-Dirichlet spectrum. In this case we get a Čebotarev type representation result. Before the statement, let us clarify the notations we use. For any subsequence $\{a_{k_n}\}_{n \in \mathbb{N}} \subset \sigma_{DD}$ and

$\{b_{l_n}\}_{n\in\mathbb{N}} \subset \sigma_{ND}$, by $A_{k_n,m}$ and A_{k_n} we denote the residues at a_{k_n} of partial and infinite products, respectively, consisting of these subsequences:

$$A_{k_n,m} := Res(G_m, a_{k_n}) = \frac{a_{k_n}}{b_{l_n}}(a_{k_n} - b_{l_n}) \prod_{1\le j\le m, j\ne n} \frac{a_{k_j}}{b_{l_j}} \frac{a_{k_n} - b_{l_j}}{a_{k_n} - a_{k_j}},$$
$$A_{k_n} := Res(G, a_{k_n}) = \frac{a_{k_n}}{b_{l_n}}(a_{k_n} - b_{l_n}) \prod_{j\in\mathbb{N}, j\ne n} \frac{a_{k_j}}{b_{l_j}} \frac{a_{k_n} - b_{l_j}}{a_{k_n} - a_{k_j}},$$

where

$$G_m(z) := \prod_{n=1}^{m} \left(\frac{z}{b_{l_n}} - 1\right)\left(\frac{z}{a_{k_n}} - 1\right)^{-1}, \qquad G(z) := \prod_{n\in\mathbb{N}} \left(\frac{z}{b_{l_n}} - 1\right)\left(\frac{z}{a_{k_n}} - 1\right)^{-1}.$$

Note that these subsequences are ordered according to their indices, i.e. $a_{k_n} < a_{k_{n+1}}$ and $b_{l_n} < b_{l_{n+1}}$ for any $n \in \mathbb{N}$. This follows from the asymptotics of the spectra.

Lemma 2. (Čebotarev type representation I) *Let* $\{a_{k_n}\}_{n\in\mathbb{N}} \subset \sigma_{DD}$ *and* $\{b_{l_n}\}_{n\in\mathbb{N}} \subset \sigma_{ND}$ *satisfy following properties:*

- $\lim_{m\to\infty} \sum_{n=1}^{m} \left(|A_{k_n,m} - A_{k_n}|/a_{k_n}^2\right) < \infty$,
- $\{A_{k_n}/a_{k_n}^2\}_{n\in\mathbb{N}} \in l^1$.

Then

$$G(z) = cz^2 + dz + e + \sum_{n\in\mathbb{N}} A_{k_n}\left(\frac{1}{z - a_{k_n}} + \frac{1}{a_{k_n}}\right), \tag{3.23}$$

where c, d, e *are real numbers,* A_{k_n} *is the residue of* $G(z)$ *at the point* $z = a_{k_n}$ *and the sum converges normally on* $\mathbb{C}\setminus \cup_{n\in\mathbb{N}} a_{k_n}$.

Proof. Let $p(z)$ be the difference of $G(z)$ and the infinite sum on the right hand side of (3.23). Then, $p(z)$ is an entire function, since the infinite product and the infinite sum share the same set of poles with equivalent degrees and residues. We represent $G_m(z)$ as partial sums:

$$\prod_{n=1}^{m} \left(\frac{z}{b_{l_n}} - 1\right)\left(\frac{z}{a_{k_n}} - 1\right)^{-1} = \sum_{n=1}^{m} A_{k_n,m}\left(\frac{1}{z - a_{k_n}} + \frac{1}{a_{k_n}}\right) + 1,$$

where $A_{k_n,m} = Res(G_m, a_{k_n})$.

Let C_n be the circle with radius b_{l_n} centered at the origin. This sequence of circles satisfy following properties:

- C_n omits all the poles a_{k_n}.
- Each C_n lies inside C_{n+1}.
- The radius of C_n, b_{l_n} diverges to infinity as n goes to infinity.

Then,

$$\begin{aligned}
\max_{z\in C_t}\left|\frac{p(z)-1}{b_{l_t}^2}\right| &= \max_{z\in C_t}\left|\frac{G(z)-1-\sum_{n\in\mathbb{N}} A_{k_n}\left(\frac{1}{z-a_{k_n}}+\frac{1}{a_{k_n}}\right)}{b_{l_t}^2}\right| \\
&= \frac{1}{b_{l_t}^2}\max_{z\in C_t}\lim_{m\to\infty}\left|\sum_{n=1}^{m} A_{k_n,m}\left(\frac{1}{z-a_{k_n}}+\frac{1}{a_{k_n}}\right)-\sum_{n=1}^{m} A_{k_n}\left(\frac{1}{z-a_{k_n}}+\frac{1}{a_{k_n}}\right)\right| \\
&= \lim_{m\to\infty}\frac{1}{b_{l_t}^2}\max_{z\in C_t}\left|\sum_{n=1}^{m}(A_{k_n,m}-A_{k_n})\frac{z}{a_{k_n}(z-a_{k_n})}\right| \\
&\leq \lim_{m\to\infty}\frac{1}{b_{l_t}^2}\sum_{n=1}^{m}|A_{k_n,m}-A_{k_n}|\frac{b_{l_t}}{a_{k_n}|b_{l_t}-a_{k_n}|} \\
&= \lim_{m\to\infty}\sum_{n=1}^{m}|A_{k_n,m}-A_{k_n}|\frac{1}{a_{k_n}b_{l_t}|b_{l_t}-a_{k_n}|} \\
&\leq \lim_{m\to\infty}\sum_{n=1}^{m}|A_{k_n,m}-A_{k_n}|\frac{1}{a_{k_n}b_{l_1}|b_{l_1}-a_{k_n}|} \\
&\leq \lim_{m\to\infty} C'\sum_{n=1}^{m}\frac{|A_{k_n,m}-A_{k_n}|}{a_{k_n}^2} < \infty.
\end{aligned}$$

Note that the second inequality is a consequence of

$$\sup_{t\in\mathbb{N}}\left(b_{l_t}|b_{l_t}-a_{k_n}|\right)^{-1} \leq \left(b_{l_1}|b_{l_1}-a_{k_n}|\right)^{-1},$$

which follows from asymptotics of $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$. Therefore $|p(z)-1| \leq C''|z|^2$ on the circle C_t for any $t \in \mathbb{N}$, where C' and C'' are real numbers. By the maximum modulus theorem and the entireness of $p(z)$, we conclude that $p(z)$ is a polynomial of at most second degree. Since $G(0), G'(0)$ and $G''(0)$ are real numbers, $c, d, e \in \mathbb{R}$.

□

Using this Čebotarev type representation we prove our result in non-matching index sets case with Dirichlet-Dirichlet, Neumann-Dirichlet boundary conditions. However, we need extra information of an eigenvalue from $\{b_{l_n}\}_{n\in\mathbb{N}}$.

Theorem 23. **(Inverse problem II-a)** *Let $q \in L^1(0,\pi)$, and $\{a_{k_n}\}_{n\in\mathbb{N}} \subset \sigma_{DD}$, $\{b_{l_n}\}_{n\in\mathbb{N}} \subset \sigma_{ND}$ satisfy following properties:*

- $\lim_{m\to\infty} \sum_{n=1}^{m} \left(|A_{k_n,m} - A_{k_n}|/a_{k_n}^2\right) < \infty$,
- $\{A_{k_n}/a_{k_n}^2\}_{n\in\mathbb{N}} \in l^1$.

Then $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}} \backslash \{b_{l_n}\}_{n\in\mathbb{N}\backslash\{s\}}$ and $\{\gamma_{k_n}\}_{n\in\mathbb{N}}$ determine the potential q for any $s \in \mathbb{N}$, where $\sigma_{DD} = \{a_n\}_{n\in\mathbb{N}}$, $\sigma_{ND} = \{b_n\}_{n\in\mathbb{N}}$ are Dirichlet-Dirichlet and Neumann-Dirichlet spectra and $\{\gamma_n\}_{n\in\mathbb{N}}$ are point masses of the spectral measure $\mu_{0,0} = \sum_{n\in\mathbb{N}} \gamma_n \delta_{a_n}$.

Proof. By representation of the m-function as the Herglotz integral of the spectral measure, from γ_n, we know $Res(m, a_n)$. Therefore, in terms of the m-function our claim says that the set of poles, $\{a_n\}_{n\in\mathbb{N}}$, the set of zeros except the index set $\{l_n\}_{n\in\mathbb{N}\backslash\{s\}}$, $\{b_{l_s}\} \cup \{b_n\}_{n\in\mathbb{N}\backslash\{l_n\}_{n\in\mathbb{N}}}$, and the residues with the index set $\{k_n\}_{n\in\mathbb{N}}$, $\{Res(m, a_{k_n})\}_{n\in\mathbb{N}}$ determine the m-function uniquely.

From Lemma 1, the Weyl m-function can be represented in terms of σ_{DD} and σ_{ND},

$$m(z) = -C \prod_{n\in\mathbb{N}} \left(\frac{z}{b_n} - 1\right)\left(\frac{z}{a_n} - 1\right)^{-1}.$$

Note that for any $n \in \mathbb{N}$, we know

$$Res(m, a_{k_n}) = C(b_{k_n} - a_{k_n})\frac{a_{k_n}}{b_{k_n}} \prod_{j\in\mathbb{N}, j\neq k_n} \left(\frac{a_{k_n}}{b_j} - 1\right)\left(\frac{a_{k_n}}{a_j} - 1\right)^{-1}. \tag{3.24}$$

Let $m(z) = F(z)G(z)$, where F and G are two infinite products defined as

$$G(z) := -C \prod_{n\in\mathbb{N}} \left(\frac{z}{b_{l_n}} - 1\right) \left(\frac{z}{a_{k_n}} - 1\right)^{-1},$$
$$F(z) := \prod_{n\in\mathbb{N}\setminus\{l_n\}_{n\in\mathbb{N}}} \left(\frac{z}{b_n} - 1\right) \prod_{n\in\mathbb{N}\setminus\{k_n\}_{n\in\mathbb{N}}} \left(\frac{z}{a_n} - 1\right)^{-1}$$

Also note that $F(a_{k_n})$ is known for any $n \in \mathbb{N}$. This condition and (3.24) imply that for any $n \in \mathbb{N}$, we know

$$Res(G, a_{k_n}) = \frac{Res(m, a_{k_n})}{F(a_{k_n})}.$$

By Lemma 2, $G(z)$ has the following representation

$$G(z) = cz^2 + dz + e + \sum_{n\in\mathbb{N}} A_{k_n} \left(\frac{1}{z - a_{k_n}} + \frac{1}{a_{k_n}}\right), \tag{3.25}$$

where $A_{k_n} = Res(G, a_{k_n})$. In order to show uniqueness of $G(z)$, let us consider $\widetilde{G}(z)$ similar to the proof of Theorem 22, i.e. $\widetilde{G}(z)$ has the following properties.

- The infinite product $\widetilde{G}(z)$ is defined as

  $$\widetilde{G}(z) := -\widetilde{C} \prod_{n\in\mathbb{N}} \left(\frac{z}{\widetilde{b}_{l_n}} - 1\right) \left(\frac{z}{\widetilde{a}_{k_n}} - 1\right)^{-1},$$

  where $\widetilde{C} > 0$, the set of poles $\{\widetilde{a}_{k_n}\}_{n\in\mathbb{N}}$ satisfies asymptotics (3.5) and the set of zeros $\{\widetilde{b}_{l_n}\}_{n\in\mathbb{N}}$ satisfies asymptotics (3.6). For the given eigenvalues from $\sigma_{ND} = \{b_n\}_{n\in\mathbb{N}}$, let $\widetilde{b}_n$ be defined as b_n, i.e. $\widetilde{b}_j := b_j$ for $j \in \mathbb{N}\setminus\{l_n\}_{n\in\mathbb{N}}$. Similarly let $\widetilde{a}_j := a_j$ for $j \in \mathbb{N}\setminus\{k_n\}_{n\in\mathbb{N}}$.

- $G(z)$ and $\widetilde{G}(z)$ share same set of poles with equivalent residues at the corresponding poles, i.e. $\widetilde{a}_{k_n} = a_{k_n}$ and $Res(\widetilde{G}, a_{k_n}) = Res(G, a_{k_n})$ for any $n \in \mathbb{N}$.

- $G(z)$ and $\widetilde{G}(z)$ share one zero, namely $b_{l_s} = \tilde{b}_{l_s}$.

- By the equivalence of poles and residues of $G(z)$ and $\widetilde{G}(z)$ and Lemma 2, $\widetilde{G}(z)$ has the representation

$$\widetilde{G}(z)=\widetilde{c}z^2+\widetilde{d}z+\widetilde{e}+\sum_{n\in\mathbb{N}}A_{k_n}\left(\frac{1}{a_{k_n}-z}-\frac{1}{a_{k_n}}\right), \tag{3.26}$$

where $\widetilde{c}$, $\widetilde{d}$, $\widetilde{e}\in\mathbb{R}$.

Let $m\in\mathbb{N}\backslash\{s\}$ and $b_{l_m}\neq\widetilde{b}_{l_m}$. Since $G(b_{l_m})=0$ and $\widetilde{G}(\widetilde{b}_{l_m})=0$, using representations (3.25) and (3.26) we get

$$-cb_{l_m}^2-db_{l_m}-e=\sum_{n\in\mathbb{N}}A_{k_n}\left(\frac{1}{a_{k_n}-b_{l_m}}-\frac{1}{a_{k_n}}\right), \tag{3.27}$$

$$-\tilde{c}\tilde{b}_{l_m}^2-\tilde{d}\tilde{b}_{l_m}-\widetilde{e}=\sum_{n\in\mathbb{N}}A_{k_n}\left(\frac{1}{a_{k_n}-\widetilde{b}_{l_m}}-\frac{1}{a_{k_n}}\right)\text{ and} \tag{3.28}$$

$$G(\tilde{b}_{l_m})=G(\tilde{b}_{l_m})-\widetilde{G}(\tilde{b}_{l_m})=(c-\tilde{c})\tilde{b}_{l_m}^2+(d-\tilde{d})\tilde{b}_{l_m}+e-\tilde{e} \tag{3.29}$$

Taking difference of (3.27) and (3.28) and replacing $e-\widetilde{e}$ by $G(\tilde{b}_{l_m})-(c-\tilde{c})\tilde{b}_{l_m}^2-(d-\widetilde{d})\widetilde{b}_{l_m}$ we get

$$cb_{l_m}^2-c\tilde{b}_{l_m}^2+db_{l_m}-d\tilde{b}_{l_m}+G(\tilde{b}_{l_m})=\sum_{n\in\mathbb{N}}A_{k_n}\left(\frac{\widetilde{b}_{l_m}-b_{l_m}}{(a_{k_n}-\widetilde{b}_{l_m})(a_{k_n}-b_{l_m})}\right)$$

Dividing both sides by $\widetilde{b}_{l_m}(\widetilde{b}_{l_m}-b_{l_m})$ we get

$$\frac{-c(b_{l_m}+\tilde{b}_{l_m})}{\widetilde{b}_{l_m}}+\frac{-d}{\widetilde{b}_{l_m}}+\frac{G(\tilde{b}_{l_m})}{\widetilde{b}_{l_m}(\widetilde{b}_{l_m}-b_{l_m})}=\sum_{n\in\mathbb{N}}\left(\frac{A_{k_n}}{\widetilde{b}_{l_m}(a_{k_n}-\widetilde{b}_{l_m})(a_{k_n}-b_{l_m})}\right) \tag{3.30}$$

Note that since $\{a_n\}_{n\in\mathbb{N}}$ satisfies asymptotics (3.5) and $\{b_n\}_{n\in\mathbb{N}}$, $\{\widetilde{b}_n\}_{n\in\mathbb{N}}$ satisfy asymptotics (3.6), the inequalities

$$|\widetilde{b}_{l_m}(a_{k_n}-b_{l_m})(a_{k_n}-\widetilde{b}_{l_m})|^{-1}\leq|\widetilde{b}_{k_n}(a_{k_n}-b_{k_n})(a_{k_n}-\widetilde{b}_{k_n})|^{-1}\leq 2/a_{k_n}^2$$

are valid for any $m\in\mathbb{N}\backslash\{s\}$ and for sufficiently large $n\in\mathbb{N}$. Recall that $\widetilde{b}_{k_j}:=b_{k_j}$ if $k_j\notin\{l_n\}_{n\in\mathbb{N}}$. In addition, $\sum_{n\in\mathbb{N}}A_{k_n}/a_{k_n}^2$ is absolutely convergent. Therefore right hand side of (3.30) converges to 0 as m goes to ∞. Also note that by (3.29), left hand side of (3.30) is

$$\begin{aligned}\frac{-c(b_{l_m}+\tilde{b}_{l_m})-d}{\widetilde{b}_{l_m}}+\frac{G(\tilde{b}_{l_m})}{\widetilde{b}_{l_m}(\widetilde{b}_{l_m}-b_{l_m})}&=\frac{-c(b_{l_m}+\tilde{b}_{l_m})-d}{\widetilde{b}_{l_m}}+\frac{G(\tilde{b}_{l_m})-\widetilde{G}(\tilde{b}_{l_m})}{\widetilde{b}_{l_m}(\widetilde{b}_{l_m}-b_{l_m})}\\&=\frac{-c(b_{l_m}+\tilde{b}_{l_m})-d}{\widetilde{b}_{l_m}}+\frac{1}{\tilde{b}_{l_m}-b_{l_m}}\left[(c-\tilde{c})\tilde{b}_{l_m}+d-\tilde{d}+\frac{e-\tilde{e}}{\tilde{b}_{l_m}}\right].\end{aligned}$$

Let us observe that

$$\lim_{m\to\infty}\frac{-c(b_{l_m}+\tilde{b}_{l_m})-d}{\widetilde{b}_{l_m}}=-2c.$$

Now let us show $\tilde{b}_{l_m}-b_{l_m}$ converges to 0 as m goes to ∞. Recall that poles of G and $\widetilde{G}$ satisfy asymptotics

$$k_n^2+\frac{1}{\pi}\int_0^\pi q(x)dx+\alpha_{k_n}\qquad \textit{and}\qquad k_n^2+\frac{1}{\pi}\int_0^\pi \widetilde{q}(x)dx+\widetilde{\alpha}_{k_n}$$

respectively, where $\alpha_n=o(1)$ and $\widetilde{\alpha}_n=o(1)$ as $n\to\infty$. Equivalance of poles of G and $\widetilde{G}$ imply equivalence of $\int_0^\pi q(x)dx$ and $\int_0^\pi \widetilde{q}(x)dx$. Therefore b_{l_m} and $\tilde{b}_{l_m}$ satisfy asymptotics

$$\left(l_m-\frac{1}{2}\right)^2+\frac{1}{\pi}\int_0^\pi q(x)dx+\beta_{l_m}\qquad \textit{and}\qquad \left(l_m-\frac{1}{2}\right)^2+\frac{1}{\pi}\int_0^\pi q(x)dx+\widetilde{\beta}_{l_m},$$

where $\beta_m=o(1)$ and $\widetilde{\beta}_m=o(1)$ as $m\to\infty$. Hence $\tilde{b}_{l_m}-b_{l_m}=o(1)$ as m goes to ∞. Therefore left hand side of (3.30) goes to ∞ if $c-\tilde{c}\neq 0$ or $d-\tilde{d}\neq 0$, so we get a contradiction unless $c=\tilde{c}$ and $d=\tilde{d}$. This implies that $G(z)-\widetilde{G}(z)$ is a real constant. However, $G(z)$ and $\widetilde{G}(z)$ share the zero b_{l_s}. This implies uniqueness of $G(z)$ and hence uniqueness of $\{b_{l_n}\}_{n\in\mathbb{N}}$. After unique recovery of the two spectra σ_{DD} and σ_{ND}, the potential is uniquely determined by Borg's theorem. □

We also get the uniqueness result without knowing any point from $\{b_{l_n}\}_{n\in\mathbb{N}}$, but this requires absolute convergence of $\prod_{n\in\mathbb{N}}a_{k_n}/b_{l_n}$. By absolute convergence of $\prod_{n\in\mathbb{N}}a_{k_n}/b_{l_n}$ we mean absolute convergence of $\sum_{n\in\mathbb{N}}(a_{k_n}/b_{l_n}-1)$. Note that Limit Comparison Test implies that $\prod_{n\in\mathbb{N}}a_{k_n}/b_{l_n}$ is absolutely convergent if and only if $\prod_{n\in\mathbb{N}}b_{l_n}/a_{k_n}$ is absolutely convergent. Absolute convergence of $\prod_{n\in\mathbb{N}}a_{k_n}/b_{l_n}$ also implies the two conditions in Lemma 2, so in this case Lemma 2 can be written in the following form.

Lemma 3. **(Čebotarev type representation II)** *Let* $\{a_{k_n}\}_{n\in\mathbb{N}} \subset \sigma_{DD}$ *and* $\{b_{l_n}\}_{n\in\mathbb{N}} \subset \sigma_{ND}$ *such that* $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ *is absolutely convergent. Then*

$$G(z) = cz^2 + dz + e + \sum_{n\in\mathbb{N}} A_{k_n}\left(\frac{1}{z-a_{k_n}} + \frac{1}{a_{k_n}}\right),$$

where c, d, e *are real numbers,* A_{k_n} *is the residue of* $G(z)$ *at the point* $z = a_{k_n}$ *and the sum converges normally on* $\mathbb{C}\backslash \cup_{n\in\mathbb{N}}\, a_{k_n}$.

Proof. We will show that absolute convergence of $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ implies the two conditions in Lemma 2, but first we begin by showing that absolute convergence of $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ implies $\{1/(a_{k_n}-b_{l_n})\}_{n\in\mathbb{N}} \in l^1$. Since $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ is absolutely convergent,

$$\sum_{n\in\mathbb{N}}\left|\frac{a_{k_n}-b_{l_n}}{b_{l_n}}\right| = \sum_{n\in\mathbb{N}}\left|\frac{k_n^2-(l_n-1/2)^2+\alpha_{k_n}-\beta_{l_n}}{(l_n-1/2)^2+(1/\pi)\int_0^\pi q(x)dx+\beta_{l_n}}\right| < \infty,$$

i.e. $\{(k_n^2-l_n^2+l_n)/l_n^2\}_{n\in\mathbb{N}} \in l^1$. Note that $\lim_{n\to\infty} a_{k_n}/b_{l_n} = 1$ implies $\lim_{n\to\infty} k_n/l_n = 1$. Therefore

$$\begin{aligned}
\infty &> \sum_{n\in\mathbb{N}}\left|\frac{k_n^2-l_n^2+l_n}{l_n^2}\right| \\
&= \sum_{n\in\mathbb{N}}\frac{k_n+l_n}{l_n}\left|\frac{k_n-l_n+l_n/(k_n+l_n)}{l_n}\right| \\
&\geq \sum_{n=1}^{N}\left|\frac{k_n-l_n+l_n/(k_n+l_n)}{l_n}\right| + \sum_{n=N+1}^{\infty}\left|\frac{1/4}{l_n}\right| \\
&\geq c_1\sum_{n\in\mathbb{N}}\frac{1}{l_n}
\end{aligned}$$

where $N \in \mathbb{N}$ and $c_1 > 0$, i.e. $\{1/l_n\}_{n\in\mathbb{N}} \in l^1$ and by Limit Comparison Test $\{1/k_n\}_{n\in\mathbb{N}} \in l^1$. Therefore $\{1/(a_{k_n}-b_{l_n})\}_{n\in\mathbb{N}} \in l^1$, since $1/|a_{k_n}-b_{l_n}| \leq 1/|a_{k_n}-b_{k_n}| = O(1/k_n)$ as n goes to ∞.

The partial product G_N defined in the beginning of Section 3.3.2 can be represented as

$$G_N(z) = \sum_{n=1}^{N} \frac{A_{k_n,N}}{z - a_{k_n}} + \prod_{n=1}^{N} \frac{a_{k_n}}{b_{l_n}},$$

and hence

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{A_{k_n,N}}{a_{k_n}} = \lim_{N\to\infty} \Big[\prod_{n=1}^{N} \frac{a_{k_n}}{b_{l_n}} - G_N(0) \Big] = \prod_{n\in\mathbb{N}} \frac{a_{k_n}}{b_{l_n}} - 1 \in \mathbb{R}. \tag{3.31}$$

Since $\{1/a_{k_n}\}_{n\in\mathbb{N}} \in l^1$, existence of this limit implies $\lim_{N\to\infty} \sum_{n=1}^{N} |A_{k_n,N}/a_{k_n}^2|$ exists.

Now we are ready to prove the first assumption in Lemma 2, i.e.

$$\lim_{N\to\infty} \sum_{n=1}^{N} \left(|A_{k_n,N} - A_{k_n}|/a_{k_n}^2 \right) < \infty.$$

For $n < N$, let us define

$$P_{k_n,N} := \prod_{m=N+1}^{\infty} \frac{a_{k_m}}{b_{l_m}} \frac{a_{k_n} - b_{l_m}}{a_{k_n} - a_{k_m}}.$$

Then

$$\frac{|A_{k_n,N} - A_{k_n}|}{a_{k_n}^2} = \left| \left(\frac{A_{k_n,N}}{a_{k_n}} \right) \left(\frac{a_{k_n} - b_{l_n}}{a_{k_n}} [1 - P_{k_n,N}] \right) \left(\frac{1}{a_{k_n} - b_{l_n}} \right) \right| \tag{3.32}$$

Using (3.31), absoulte convergence of $\prod_{n\in\mathbb{N}} (a_{k_n}/b_{l_n})$ and hence absolute convergence of $\sum_{n\in\mathbb{N}} [(a_{k_n} - b_{l_n})/a_{k_n}]$ we get that the limits

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{A_{k_n,N}}{a_{k_n}} \quad \text{and} \quad \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{a_{k_n} - b_{l_n}}{a_{k_n}} [1 - P_{k_n,N}] \right) \quad \text{converge.}$$

Recall that we have also showed $\{1/(a_{k_n} - b_{l_n})\}_{n\in\mathbb{N}} \in l^1$. Therefore by (3.32) we get the first assumption in Lemma 2,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{|A_{k_n,N} - A_{k_n}|}{a_{k_n}^2} < \infty.$$

After recalling that we showed existence of $\lim_{N\to\infty} \sum_{n=1}^{N} |A_{k_n,N}/a_{k_n}^2|$, we get the second assumption in Lemma 2, i.e. $\{A_{k_n}/a_{k_n}^2\}_{n\in\mathbb{N}} \in l^1$ as follows:

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{|A_{k_n}|}{a_{k_n}^2} \le \lim_{N\to\infty}\sum_{n=1}^{N}\frac{|A_{k_n}-A_{k_n,N}|}{a_{k_n}^2} + \lim_{N\to\infty}\sum_{n=1}^{N}\frac{|A_{k_n,N}|}{a_{k_n}^2} < \infty.$$

Now using Lemma 2 we get the desired result. □

Theorem 24. (Inverse problem II-b) *Let* $q \in L^1(0,\pi)$ *and* $\{a_{k_n}\}_{n\in\mathbb{N}} \subset \sigma_{DD}$*,* $\{b_{l_n}\}_{n\in\mathbb{N}} \subset \sigma_{ND}$ *such that* $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ *is absolutely convergent. Then* $\{a_n\}_{n\in\mathbb{N}}$*,* $\{b_n\}_{n\in\mathbb{N}}\backslash\{b_{l_n}\}_{n\in\mathbb{N}}$ *and* $\{\gamma_{k_n}\}_{n\in\mathbb{N}}$ *determine the potential* q*, where* $\sigma_{DD} = \{a_n\}_{n\in\mathbb{N}}$*,* $\sigma_{ND} = \{b_n\}_{n\in\mathbb{N}}$ *are Dirichlet-Dirichlet and Neumann-Dirichlet spectra and* $\{\gamma_n\}_{n\in\mathbb{N}}$ *are point masses of the spectral measure* $\mu_{0,0} = \sum_{n\in\mathbb{N}}\gamma_n\delta_{a_n}$*.*

Proof. One can use Lemma 3 and follow the proof of Theorem 23 until the last step, i.e. showing uniqueness of the two spectra after obtaining that $G(z)-\widetilde{G}(z)$ is a real constant, so let us show $G(z)-\widetilde{G}(z)=0$. The main differences in this case are that G and $\widetilde{G}$ do not share any zero and the infinite products $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ and $\prod_{n\in\mathbb{N}}(a_{k_n}/\tilde{b}_{l_n})$ are absolutely convergent. Let us recall that the infinite products G and $\widetilde{G}$ have the following representations:

$$G(z) = cz^2 + dz - C + \sum_{n\in\mathbb{N}} A_{k_n}\left(\frac{1}{z-a_{k_n}} + \frac{1}{a_{k_n}}\right),$$
$$\widetilde{G}(z) = cz^2 + dz - \widetilde{C} + \sum_{n\in\mathbb{N}} A_{k_n}\left(\frac{1}{z-a_{k_n}} + \frac{1}{a_{k_n}}\right).$$

Therefore by taking difference of $G(z)$ and $\widetilde{G}(z)$ we get $G(z)+C = \widetilde{G}(z)+\widetilde{C}$, i.e.

$$-C\prod_{n\in\mathbb{N}}\left(\frac{z}{b_{l_n}}-1\right)\left(\frac{z}{a_{k_n}}-1\right)^{-1} + C = -\widetilde{C}\prod_{n\in\mathbb{N}}\left(\frac{z}{\tilde{b}_{l_n}}-1\right)\left(\frac{z}{a_{k_n}}-1\right)^{-1} + \widetilde{C}. \tag{3.33}$$

Note that since the infinite products $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$, $\prod_{n\in\mathbb{N}}(a_{k_n}/\tilde{b}_{l_n})$ are absolutely convergent and the two spectra $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ lie on the positive real line, the infinite products on the two sides of (3.33) are uniformly convergent on the second quadrant. Hence by letting z go to infinity on the second quadrant we get

$$-C\prod_{n\in\mathbb{N}}\frac{a_{k_n}}{b_{l_n}}+C=-\widetilde{C}\prod_{n\in\mathbb{N}}\frac{a_{k_n}}{\widetilde{b}_{l_n}}+\widetilde{C}. \tag{3.34}$$

Recall that $\prod_{n\in\mathbb{N}}\widetilde{b}_{l_n}/b_{l_n}$ is finite, since

$$\sum_{n\in\mathbb{N}}\frac{|\widetilde{b}_{l_n}-b_{l_n}|}{b_{l_n}}=\sum_{n\in\mathbb{N}}\frac{|\widetilde{\beta}_{l_n}-\beta_{l_n}|}{b_{l_n}}\leq\max_{n\in\mathbb{N}}|\widetilde{\beta}_{l_n}-\beta_{l_n}|\sum_{n\in\mathbb{N}}\frac{1}{b_{l_n}}<\infty.$$

Therefore the infinite product $H(z):=G(z)/\widetilde{G}(z)$ is represented as

$$H(z):=\frac{G(z)}{\widetilde{G}(z)}=\frac{C}{\widetilde{C}}\prod_{n\in\mathbb{N}}\frac{z-b_{l_n}}{b_{l_n}}\frac{\tilde{b}_{l_n}}{z-\tilde{b}_{l_n}}=\frac{C}{\widetilde{C}}\prod_{n\in\mathbb{N}}\frac{\tilde{b}_{l_n}}{b_{l_n}}\prod_{n\in\mathbb{N}}\frac{z-b_{l_n}}{z-\tilde{b}_{l_n}}.$$

We know that G and $\widetilde{G}$ share same poles with equivalent residues at the corresponding poles. Therefore for any $m\in\mathbb{N}$

$$1=H(a_{k_m})=\frac{C}{\widetilde{C}}\prod_{n\in\mathbb{N}}\frac{\tilde{b}_{l_n}}{b_{l_n}}\prod_{n\in\mathbb{N}}\frac{a_{k_m}-b_{l_n}}{a_{k_m}-\tilde{b}_{l_n}}. \tag{3.35}$$

Now let us find the limit of the infinite product on the right end of (3.35) as m goes to ∞. This infinite product is uniformly convergent if and only if the infinite sum

$$\sum_{n\in\mathbb{N}}\left(\frac{a_{k_m}-b_{l_n}}{a_{k_m}-\tilde{b}_{l_n}}-1\right)=\sum_{n\in\mathbb{N}}\frac{\tilde{b}_{l_n}-b_{l_n}}{a_{k_m}-\tilde{b}_{l_n}} \tag{3.36}$$

is uniformly convergent. Note that asymptotics of the two spectra imply $\tilde{b}_{l_j}-b_{l_j}=o(1)$ as j goes to infinity. Then the asymptotics of $\{a_{k_n}\}_{n\in\mathbb{N}}$, $\{b_{l_n}\}_{n\in\mathbb{N}}$ and $\{\tilde{b}_{l_n}\}_{n\in\mathbb{N}}$ together with absolute convergence of the infinite products $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$, $\prod_{n\in\mathbb{N}}(a_{k_n}/\tilde{b}_{l_n})$ imply that

$$\sum_{n\in\mathbb{N}}\left|\frac{\tilde{b}_{l_n}-b_{l_n}}{a_{k_m}-\tilde{b}_{l_n}}\right|\leq\sum_{n\in\mathbb{N}}\left|\frac{\tilde{b}_{l_n}-b_{l_n}}{a_{k_n}-\tilde{b}_{l_n}}\right|<\infty, \tag{3.37}$$

since $\{1/(a_{k_n}-\tilde{b}_{l_n})\}_{n\in\mathbb{N}}\in l^1$ as we discussed in the proof of Lemma 3. Therefore by letting

m go to ∞ in (3.35) we get $\widetilde{C}/C = \prod_{j\in\mathbb{N}} \tilde{b}_{l_j}/b_{l_j}$. If we define $\gamma := \prod_{n\in\mathbb{N}} a_{k_n}/b_{l_n}$ and $\widetilde{\gamma} := \prod_{n\in\mathbb{N}} a_{k_n}/\widetilde{b}_{l_n}$, we get $\widetilde{C}/C = \gamma/\widetilde{\gamma}$. This identity and (3.34) imply $\dfrac{\gamma-1}{\widetilde{\gamma}-1} = \dfrac{\gamma}{\widetilde{\gamma}}$ and hence $\gamma = \widetilde{\gamma}$. Therefore $C = \widetilde{C}$. This implies uniqueness of $G(z)$ and hence uniqueness of $\{b_{l_n}\}_{n\in\mathbb{N}}$. After unique recovery of the two spectra σ_{DD} and σ_{ND}, the potential is uniquely determined by Borg's theorem. □

3.3.3 General boundary conditions

As discussed in Section 3.1, the Weyl m-function for the Schrödinger equation

$$Lu = -u'' + qu = zu \tag{3.38}$$

with boundary conditions

$$u(0)\cos\alpha - u'(0)\sin\alpha = 0 \tag{3.39}$$

$$u(\pi)\cos\beta + u'(\pi)\sin\beta = 0, \tag{3.40}$$

is defined as $m_{\alpha,\beta}(z) = \dfrac{\cos(\alpha)u_z'(0) + \sin(\alpha)u_z(0)}{-\sin(\alpha)u_z'(0) + \cos(\alpha)u_z(0)}$, where $u_z(t)$ is a solution of (3.38) satisfying (3.40) and $\alpha, \beta \in [0,\pi)$. In order to prove our result with boundary conditions (3.39) and (3.40) we need to consider more general m-functions. Recall that we have defined the m-function in Section 3.1 by introducing two solutions $s_z(t)$ and $c_z(t)$ of (3.38) satisfying the initial conditions

$$s_z(0) = \sin(\alpha), \quad s_z'(0) = \cos(\alpha)$$

$$c_z(0) = \cos(\alpha), \quad c_z'(0) = -\sin(\alpha)$$

and $u_z(t)$, a solution of (3.38) with boundary conditions $u_z(\pi) = \sin\beta$, $u_z'(\pi) = -\cos\beta$. The same steps to define the m-function as in Section 3.1 can be followed if $c_z(t)$ is a linearly independent solution with $W(c_z, s_z) = 1$. Therefore we introduce two solutions $s_z(t)$ and $c_z(t)$ of (3.38) satisfying the initial conditions

$$s_z(0) = \sin(\alpha_2), \quad s'_z(0) = \cos(\alpha_2)$$
$$c_z(0) = \frac{\sin(\alpha_1)}{\sin(\alpha_1 - \alpha_2)}, \quad c'_z(0) = \frac{\cos(\alpha_1)}{\sin(\alpha_1 - \alpha_2)}$$

for $\alpha_1, \alpha_2 \in [0, \pi)$, $\sin(\alpha_1 - \alpha_2) \neq 0$ and same $u_z(t)$. Then we can define the m-function $m_{\alpha_1,\alpha_2,\beta}$.

Definition 16. **The m-function** $m_{\alpha_1,\alpha_2,\beta}$ is defined as

$$m_{\alpha_1,\alpha_2,\beta}(z) := \frac{1}{\sin(\alpha_2 - \alpha_1)} \left[\frac{-\sin(\alpha_1)u'_z(0) \ + \cos(\alpha_1)u_z(0)}{-\sin(\alpha_2)u'_z(0) + \cos(\alpha_2)u_z(0)} \right],$$

where $\alpha_1, \alpha_2, \beta \in [0, \pi)$, $\sin(\alpha_2 - \alpha_1) \neq 0$ and $u_z(t)$ is a solution of (3.38) with boundary conditions $u_z(\pi) = \sin\beta$, $u'_z(\pi) = -\cos\beta$.

Remark 3. *The m-function $m_{\alpha,\beta}$ we discussed in Section 3.1 is obtained by letting $\alpha_1 = \alpha - \pi/2$ and $\alpha_2 = \alpha$, i.e. $m_{\alpha-\frac{\pi}{2},\alpha,\beta}(z) = m_{\alpha,\beta}(z)$.*

The m-function $m_{\alpha_1,\alpha_2,\beta}(z)$ is a meromorphic Herglotz function having real zeros on $\sigma_{\alpha_1,\beta}$ and real poles on $\sigma_{\alpha_2,\beta}$, which are interlacing. It is a meromorphic Herglotz function, since $m_{0,\beta}(z) = u'_z(0)/u_z(0)$ is a meromorphic Herglotz function and $sgn[\mathrm{Im}(m_{\alpha_1,\alpha_2,\beta}(z))] = sgn[\mathrm{Im}(m_{0,\beta}(z))]$. Therefore Herglotz representation theorem implies

$$m_{\alpha_1,\alpha_2,\beta}(z) = az + b + \int \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\mu_{\alpha_1,\alpha_2,\beta}(t),$$

where $a, b \in \mathbb{R}$ and $\mu_{\alpha_1,\alpha_2,\beta}$ is a positive discrete Poisson-summable measure supported on the spectrum $\sigma_{\alpha_2,\beta}$. Let us call $\mu_{\alpha_1,\alpha_2,\beta}$ the **spectral measure** corresponding to $(\alpha_1, \alpha_2, \beta)$. Now we prove our results with general boundary conditions.

Theorem 25. **(Inverse problem I-b)** *Let $q \in L^1(0, \pi)$, $A \subset \mathbb{N}$, $\sin(\alpha_2 - \alpha_1) \neq 0$ and $\alpha_1, \alpha_2, \beta \in [0, \pi)$. Then $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}\setminus A}$ and $\{\gamma_n\}_{n\in A}$ determine the potential q, where $\sigma_{\alpha_2,\beta} = \{a_n\}_{n\in\mathbb{N}}$,*

$\sigma_{\alpha_1,\beta} = \{b_n\}_{n\in\mathbb{N}}$ *are two spectra and* $\{\gamma_n\}_{n\in\mathbb{N}}$ *are point masses of the spectral measure* $\mu_{\alpha_1,\alpha_2,\beta} = \sum_{n\in\mathbb{N}} \gamma_n \delta_{a_n}$.

Proof. Wlog let a_n and b_n be positive for all $n \in \mathbb{N}$. We follow the arguments we used in the proofs of Lemma 1 and Theorem 22, but there are two differences: asymptotics of the two spectra, depending on $\alpha_1, \alpha_2, \beta$ and hence the order relation betweeen a_n and b_n. Thus, we consider the following cases.

(i) $\underline{\alpha_1 \neq 0,\, \alpha_2 \neq 0,\, \alpha_1 > \alpha_2:}$

When $\beta \neq 0$, the two spectra $\sigma_{\alpha_2,\beta} = \{a_n\}_{n\in\mathbb{N}}$ and $\sigma_{\alpha_1,\beta} = \{b_n\}_{n\in\mathbb{N}}$ satisfy the asymptotics (3.4) and hence $a_n > b_n$ for all $n \in \mathbb{N}$. Therefore using the proof of Lemma 1, $m_{\alpha_1,\alpha_2,\beta}(z)$ can be represented as (3.10). Using this representation and Čebotarev's theorem as we discussed in the proof of Theorem 22, the meromorphic Herglotz function $G(z)$ defined as

$$G(z) := -C \prod_{n\in A} \left(\frac{z}{b_n} - 1\right)\left(\frac{z}{a_n} - 1\right)^{-1} \tag{3.41}$$

has the following representation:

$$G(z) = dz + e + \sum_{n\in A} A_n \left(\frac{1}{z - a_n} + \frac{1}{a_n}\right).$$

Only unknown constants on the right hand side are d and e. In order to show uniqueness of the linear term $dz + e$, let us introduce $\widetilde{G}(z)$ as we did in the proof of Theorem 22:

- The infinite product $\widetilde{G}$ is defined as

$$\widetilde{G}(z) := -\widetilde{C} \prod_{n\in A} \left(\frac{z}{\widetilde{b}_n} - 1\right)\left(\frac{z}{\widetilde{a}_n} - 1\right)^{-1},$$

  where $\widetilde{C} > 0$, the set of poles $\{\widetilde{a}_n\}_{n\in A}$ and the set of zeros $\{\widetilde{b}_n\}_{n\in A}$ satisfy asymptotics (3.4). Let $\widetilde{a}_k := a_k$ and $\widetilde{b}_k := b_k$ for $k \in \mathbb{N}\backslash A$.

- G and $\widetilde{G}$ share same set of poles with equivalent residues at the corresponding poles, i.e. $\widetilde{a}_k = a_k$ and $Res(\widetilde{G}, a_k) = Res(G, a_k)$ for any $k \in A$.

- By the equivalence of poles and residues of G and $\widetilde{G}$ and Čebotarev's theorem, $\widetilde{G}(z)$ has the representation
$$\widetilde{G}(z) = \widetilde{d}z + \widetilde{e} + \sum_{n\in A} A_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right),$$
where $\widetilde{d} \geq 0$, $\widetilde{e} \in \mathbb{R}$.

Therefore the difference of G and $\widetilde{G}$ is a linear polynomial, i.e.

$$G(z) - \widetilde{G}(z) = (d - \widetilde{d})z + e - \widetilde{e} \tag{3.42}$$

Note that since $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ and $\{\tilde{b}_n\}_{n\in\mathbb{N}}$ are subsets of $(0, \infty)$ and satisfy asymptotics (3.4), for any $x \in (-\infty, 0)$ we get

$$\begin{aligned} |G(x) - \widetilde{G}(x)| &\leq \left| C \prod_{n\in A} \left(\frac{x}{b_n} - 1 \right) \left(\frac{x}{a_n} - 1 \right)^{-1} \right| + \left| \widetilde{C} \prod_{n\in A} \left(\frac{x}{\widetilde{b}_n} - 1 \right) \left(\frac{x}{a_n} - 1 \right)^{-1} \right| \\ &\leq C \prod_{n\in A} \frac{a_n}{b_n} + \widetilde{C} \prod_{n\in A} \frac{a_n}{\widetilde{b}_n} < \infty. \end{aligned}$$

Convergence of the infinite product $\prod_{n\in A} a_n/b_n$ follows from the fact that

$$\sum_{n\in A} \frac{a_n - b_n}{b_n} \leq M \sum_{n\in A} \frac{1}{n^2},$$

for some $M < \infty$, since asymptotics (3.4) imply $|a_n - b_n| \leq M_1$ for some $M_1 < \infty$ independent of n and $a_n = n^2 + o(n^2)$, $b_n = n^2 + o(n^2)$ as n goes to infinity. Therefore

$$\lim_{x\to-\infty} \left| (d - \widetilde{d})x + e - \widetilde{e} \right| = \lim_{x\to-\infty} \left| G(x) - \widetilde{G}(x) \right| = \left| C \prod_{n\in A} \frac{a_n}{b_n} - \widetilde{C} \prod_{n\in A} \frac{a_n}{\widetilde{b}_n} \right| < \infty,$$

so we get a contradiction unless $d = \tilde{d}$. This implies that $G(z) - \widetilde{G}(z)$ is a real constant, which is

$G(0) - \widetilde{G}(0) = \widetilde{C} - C$. In order to show $\widetilde{C} = C$, we follow exactly the same arguments used in the proof of Theorem 22.

This gives uniqueness of $G(z)$ and hence uniqueness of $\{b_n\}_{n \in A}$. After unique recovery of the two spectra $\sigma_{\alpha_2,\beta}$ and $\sigma_{\alpha_1,\beta}$, Levinson's theorem uniquely determines the potential.

When $\beta = 0$, one can apply same arguments. The only difference appears in asymptotics of $\sigma_{\alpha_2,\beta} = \{a_n\}_{n \in \mathbb{N}}$ and $\sigma_{\alpha_1,\beta} = \{b_n\}_{n \in \mathbb{N}}$, which does not affect the result.

(ii) $\underline{\alpha_1 \neq 0, \alpha_2 = 0, \beta = 0:}$

The two spectra $\sigma_{\alpha_2,\beta} = \{a_n\}_{n \in \mathbb{N}}$ and $\sigma_{\alpha_1,\beta} = \{b_n\}_{n \in \mathbb{N}}$ satisfy the asymptotics (3.5) and (3.6) respectively. One then obtains the result by following the proofs of Lemma 1 and Theorem 22.

(iii) $\underline{\alpha_1 \neq 0, \alpha_2 = 0, \beta \neq 0:}$

The two spectra $\sigma_{\alpha_2,\beta} = \{a_n\}_{n \in \mathbb{N}}$ and $\sigma_{\alpha_1,\beta} = \{b_n\}_{n \in \mathbb{N}}$ satisfy the asymptotics (3.7) and (3.4) respectively, which is similar to the previous case.

(iv) <u>$\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_1 < \alpha_2$ or $\alpha_1 = 0, \alpha_2 \neq 0, \beta \neq 0$ or $\alpha_1 = 0, \alpha_2 \neq 0, \beta = 0:$</u>

In all of these three cases, $a_n < b_n$ for all $n \in \mathbb{N}$. Therefore using the proof of Lemma 1, $m_{\alpha_1,\alpha_2,\beta}(z)$ can be represented as

$$m_{\alpha_1,\alpha_2,\beta}(z) = C \prod_{n \in \mathbb{N}} \left(\frac{z}{b_n} - 1 \right) \left(\frac{z}{a_n} - 1 \right)^{-1}.$$

In order to represent $G(z)$ as (3.41), an extra factor is required, so we shift indices of b_n up by one inside A and let b_1 be a positive real number less than a_1, assuming wlog $1 \in A$. Then $\frac{z-b_1}{b_1} G(z)$ can be represented as (3.41). Using this representation and Čebotarev's theorem, the meromorphic Herglotz function $\frac{z-b_1}{b_1} G(z)$ has the following representation:

$$\left(\frac{z - b_1}{b_1} \right) G(z) = az + b + \sum_{n \in A} A_n \left(\frac{1}{a_n - z} - \frac{1}{a_n} \right).$$

Therefore if we introduce $\widetilde{G}(z)$ similar to the previous cases, then $\frac{z-b_1}{b_1}G(z)$ and $\frac{z-b_1}{b_1}\widetilde{G}(z)$ share the same set of poles $\{a_n\}_{n\in A}$ with the same residues $\{-A_n\}_{n\in A}$ and have the sets of zeros $\{b_n\}_{n\in A}$ and $\{b_1\}\cup\{\widetilde{b}_n\}_{n\in A\setminus\{1\}}$ respectively, so the difference of $\frac{z-b_1}{b_1}G(z)$ and $\frac{z-b_1}{b_1}\widetilde{G}(z)$ is a linear polynomial with real coefficients and hence $G(z)-\widetilde{G}(z)$ is a real constant, which is $G(0)-\widetilde{G}(0)=\widetilde{C}-C$. In order to show $\widetilde{C}=C$, we follow exactly the same arguments used in the proof of Theorem 22.

This implies uniqueness of $G(z)$ and hence uniqueness of $\{b_n\}_{n\in A}$. After unique recovery of the two spectra $\sigma_{\alpha_2,\beta}$ and $\sigma_{\alpha_1,\beta}$, Levinson's theorem uniquely determines the potential. □

Remark 4. *Theorem 25 gives Marchenko's theorem with the m-function $m_{\alpha_1,\alpha_2,\beta}$ as a corollary if we let $A=\mathbb{N}$. By letting $A=\emptyset$, we get the statement of Levinson's theorem.*

For the non-matching index sets case, let us recall the definitions of $A_{k_n,m}$ and A_{k_n}:

$$A_{k_n,m} := \frac{a_{k_n}}{b_{l_n}}(a_{k_n}-b_{k_n})\prod_{j=1,j\neq n}^{m}\frac{a_{k_j}}{b_{l_j}}\frac{a_{k_n}-b_{l_j}}{a_{k_n}-a_{k_j}},$$
$$A_{k_n} := \frac{a_{k_n}}{b_{l_n}}(a_{k_n}-b_{k_n})\prod_{j=1,j\neq n}^{\infty}\frac{a_{k_j}}{b_{l_j}}\frac{a_{k_n}-b_{l_j}}{a_{k_n}-a_{k_j}}.$$

We can prove Theorem 23 and Theorem 24 with general boundary conditions following the same proofs. However, if boundary conditions α_1 and α_2 are nonzero, then we need that eventually the two index sets $\{k_n\}_{n\in\mathbb{N}}$ and $\{l_n\}_{n\in\mathbb{N}}$ have no common element.

Theorem 26. **(Inverse problem II-c)** *Let $q\in L^1(0,\pi)$, $\sin(\alpha_2-\alpha_1)\neq 0$, $\alpha_1,\alpha_2,\beta\in[0,\pi)$ and $\{a_{k_n}\}_{n\in\mathbb{N}}\subset\sigma_{\alpha_2,\beta}$, $\{b_{l_n}\}_{n\in\mathbb{N}}\subset\sigma_{\alpha_1,\beta}$ satisfy following properties:*

- $\lim_{m\to\infty}\sum_{n=1}^{m}\left(|A_{k_n,m}-A_{k_n}|/a_{k_n}^2\right)<\infty$,
- $\{A_{k_n}/a_{k_n}^2\}_{n\in\mathbb{N}}\in l^1$.

(i) If $\alpha_1=0$ or $\alpha_2=0$, then $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}\setminus\{b_{l_n}\}_{n\in\mathbb{N}\setminus\{s\}}$ and $\{\gamma_{k_n}\}_{n\in\mathbb{N}}$ determine the potential q for any $s\in\mathbb{N}$, where $\sigma_{\alpha_2,\beta}=\{a_n\}_{n\in\mathbb{N}}$, $\sigma_{\alpha_1,\beta}=\{b_n\}_{n\in\mathbb{N}}$ are two spectra and $\{\gamma_n\}_{n\in\mathbb{N}}$ are point masses of the spectral measure $\mu_{\alpha_1,\alpha_2,\beta}=\sum_{n\in\mathbb{N}}\gamma_n\delta_{a_n}$.

(ii) If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and there exists $N \in \mathbb{N}$ such that $k_n \neq l_n$ for all $n > N$, then $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}} \backslash \{b_{l_n}\}_{n\in\mathbb{N}\backslash\{s\}}$ and $\{\gamma_{k_n}\}_{n\in\mathbb{N}}$ determine the potential q for any $s \in \mathbb{N}$, where $\sigma_{\alpha_2,\beta} = \{a_n\}_{n\in\mathbb{N}}$, $\sigma_{\alpha_1,\beta} = \{b_n\}_{n\in\mathbb{N}}$ are two spectra and $\{\gamma_n\}_{n\in\mathbb{N}}$ are point masses of the spectral measure $\mu_{\alpha_1,\alpha_2,\beta} = \sum_{n\in\mathbb{N}} \gamma_n \delta_{a_n}$.

Proof. In the proof of Theorem 23 we used the inequalities (3.19), namely

$$|\widetilde{b}_{l_m}(a_{k_n} - b_{l_m})(a_{k_n} - \widetilde{b}_{l_m})|^{-1} \leq |\widetilde{b}_{k_n}(a_{k_n} - b_{k_n})(a_{k_n} - \widetilde{b}_{k_n})|^{-1} \leq 2/a_{k_n}^2.$$

If $\alpha_1 = 0$ or $\alpha_2 = 0$, these inequalities are still valid for any $m \in \mathbb{N}\backslash\{s\}$ and for sufficiently large $n \in \mathbb{N}$. Recall that $\widetilde{b}_{k_j} := b_{k_j}$ if $k_j \notin \{l_n\}_{n\in\mathbb{N}}$.

If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and there exists $N \in \mathbb{N}$ such that $k_n \neq l_n$ for all $n > N$, we modify these inequalities as follows:

$$|\widetilde{b}_{l_m}(a_{k_n} - b_{l_m})(a_{k_n} - \widetilde{b}_{l_m})|^{-1} \leq |\widetilde{b}_{k_n+1}(a_{k_n} - b_{k_n+1})(a_{k_n} - \widetilde{b}_{k_{n+1}})|^{-1} \leq 2/a_{k_n}^2,$$

which are valid for any $m \in \mathbb{N}\backslash\{s\}$ and for sufficiently large $n \in \mathbb{N}$.

After getting these inequalities we apply proofs of Lemma 2 and Theorem 23 with the m-function $m_{\alpha_1,\alpha_2,\beta}$ and the spectral measure $\mu_{\alpha_1,\alpha_2,\beta}$ and obtain uniqueness of $\{b_{l_n}\}_{n\in\mathbb{N}}$. Even though asymptotics of the spectra may be different than Dirichlet-Dirichlet, Neumann-Dirichlet case, the same arguments can be used. After unique recovery of the two spectra $\sigma_{\alpha_2,\beta}$ and $\sigma_{\alpha_1,\beta}$, Levinson's theorem uniquely determines the potential.

□

Theorem 27. (Inverse problem II-d) *Let $q \in L^1(0,\pi)$, $\sin(\alpha_2 - \alpha_1) \neq 0$, $\alpha_1, \alpha_2, \beta \in [0,\pi)$ and $\prod_{n\in\mathbb{N}} a_{k_n}/b_{l_n}$ be absolutely convergent, where $\{a_{k_n}\}_{n\in\mathbb{N}} \subset \sigma_{\alpha_2,\beta}$, $\{b_{l_n}\}_{n\in\mathbb{N}} \subset \sigma_{\alpha_1,\beta}$.*

(i) If $\alpha_1 = 0$ or $\alpha_2 = 0$, then $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}} \backslash \{b_{l_n}\}_{n\in\mathbb{N}}$ and $\{\gamma_{k_n}\}_{n\in\mathbb{N}}$ determine the potential q, where $\sigma_{\alpha_2,\beta} = \{a_n\}_{n\in\mathbb{N}}$, $\sigma_{\alpha_1,\beta} = \{b_n\}_{n\in\mathbb{N}}$ are two spectra and $\{\gamma_n\}_{n\in\mathbb{N}}$ are point masses of the

spectral measure $\mu_{\alpha_1,\alpha_2,\beta} = \sum_{n\in\mathbb{N}} \gamma_n \delta_{a_n}$.

(ii) If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ *and there exists* $N \in \mathbb{N}$ *such that* $k_n \neq l_n$ *for all* $n > N$*, then* $\{a_n\}_{n\in\mathbb{N}}$*,* $\{b_n\}_{n\in\mathbb{N}} \backslash \{b_{l_n}\}_{n\in\mathbb{N}}$ *and* $\{\gamma_{k_n}\}_{n\in\mathbb{N}}$ *determine the potential* q*, where* $\sigma_{\alpha_2,\beta} = \{a_n\}_{n\in\mathbb{N}}$*,* $\sigma_{\alpha_1,\beta} = \{b_n\}_{n\in\mathbb{N}}$ *are two spectra and* $\{\gamma_n\}_{n\in\mathbb{N}}$ *are point masses of the spectral measure* $\mu_{\alpha_1,\alpha_2,\beta} = \sum_{n\in\mathbb{N}} \gamma_n \delta_{a_n}$.

Proof. If $\alpha_1 = 0$ or $\alpha_2 = 0$, we follow the proofs of Lemma 3 and Theorem 24 with the m-function $m_{\alpha_1,\alpha_2,\beta}$ and the spectral measure $\mu_{\alpha_1,\alpha_2,\beta}$ and obtain uniqueness of $\{b_{l_n}\}_{n\in\mathbb{N}}$. After unique recovery of the two spectra $\sigma_{\alpha_2,\beta}$ and $\sigma_{\alpha_1,\beta}$, Levinson's theorem uniquely determines the potential.

If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and there exists $N \in \mathbb{N}$ such that $k_n \neq l_n$ for all $n > N$, then the only difference appears in showing $\{1/(a_{k_n} - b_{l_n}\}_{n\in\mathbb{N}} \in l^1$, so let us show that absolute convergence of $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ implies $\{1/(a_{k_n} - b_{l_n})\}_{n\in\mathbb{N}} \in l^1$. Since $\prod_{n\in\mathbb{N}}(a_{k_n}/b_{l_n})$ is absolutely convergent,

$$\sum_{n\in\mathbb{N}} \left|\frac{a_{k_n} - b_{l_n}}{b_{l_n}}\right| = \sum_{n\in\mathbb{N}} \left|\frac{(k_n-1)^2 - (l_n-1)^2 + \gamma_1 + \alpha_{k_n} - \beta_{l_n}}{(l_n-1)^2 + \gamma_2 + (2/\pi)\int_0^\pi q(x)dx + \beta_{l_n}}\right| < \infty,$$

i.e. $\{(k_n^2 - l_n^2 - 2k_n + 2l_n)/l_n^2\}_{n\in\mathbb{N}} \in l^1$. Here $\gamma_1 = 2[\cot(\alpha_2) - \cot(\alpha_1)]/\pi$, $\gamma_2 = 2[\cot(\beta) + \cot(\alpha_1)]/\pi$ and wlog we assume $\beta \neq 0$. Note that $\lim_{n\to\infty} a_{k_n}/b_{l_n} = 1$ implies $\lim_{n\to\infty} k_n/l_n = 1$. Therefore

$$\begin{aligned}
\infty &> \sum_{n\in\mathbb{N}} \left|\frac{k_n^2 - l_n^2 - 2(k_n - l_n)}{l_n^2}\right| \\
&= \sum_{n\in\mathbb{N}} \frac{k_n + l_n - 2}{l_n}\left|\frac{k_n - l_n}{l_n}\right| \\
&\geq \sum_{n=1}^{N}\left|\frac{k_n - l_n}{l_n}\right| + \sum_{n=N+1}^{\infty}\frac{1}{l_n} \\
&\geq c_1 \sum_{n\in\mathbb{N}} \frac{1}{l_n}
\end{aligned}$$

where $N \in \mathbb{N}$ and $c_1 > 0$, so $\{1/l_n\}_{n\in\mathbb{N}} \in l^1$ and hence by Limit Comparison Test $\{1/k_n\}_{n\in\mathbb{N}} \in l^1$. Therefore $\{1/(a_{k_n} - b_{l_n})\}_{n\in\mathbb{N}} \in l^1$, since for $n > N$, $1/|a_{k_n} - b_{l_n}| \leq 1/|a_{k_n} - b_{k_n+1}| = O(1/k_n)$ as n goes to ∞. Now we apply proofs of Lemma 3 and Theorem 24 with the m-function $m_{\alpha_1,\alpha_2,\beta}$

and the spectral measure $\mu_{\alpha_1,\alpha_2,\beta}$ and obtain uniqueness of $\{b_{l_n}\}_{n\in\mathbb{N}}$. After unique recovery of the two spectra $\sigma_{\alpha_2,\beta}$ and $\sigma_{\alpha_1,\beta}$, Levinson's theorem uniquely determines the potential.

$\square$

4. INVERSE SPECTRAL THEORY OF JACOBI OPERATORS*

4.1 Semi-infinite Jacobi matrices

In this section we closely follow [67].

We consider the difference expression $\tau : l(\mathbb{N}) \to l(\mathbb{N})$

$$(\tau f)_n := b_{n-1} f_{n-1} + a_n f_n + b_n f_{n+1}, \qquad n \in \mathbb{N}\backslash\{1\} \tag{4.1}$$

$$(\tau f)_1 := a_1 f_1 + b_1 f_2 \tag{4.2}$$

where $a_n \in \mathbb{R}$, $b_n > 0$ for all $n \in \mathbb{N}$ and $l(\mathbb{N})$ is the set of complex valued sequences indexed by natural numbers. The difference expression τ is represented as the tridiagonal matrix

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 \\ b_1 & a_2 & b_2 & 0 & \ddots \\ 0 & b_2 & a_3 & b_3 & \ddots \\ 0 & 0 & b_3 & a_4 & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.3}$$

with respect to the canonical basis of $l^2(\mathbb{N})$.

Let c_z,$s_z \in l(\mathbb{N})$ be two fundamental solutions of the **Jacobi difference equation**

$$\tau u = zu \qquad u \in l(\mathbb{N}),\ z \in \mathbb{C}, \tag{4.4}$$

satisfying the initial conditions

*Submitted for publication [41].

$$s_z(1) = 0, \quad s_z(2) = 1,$$

$$c_z(1) = 1, \quad c_z(2) = 0.$$

Since c and s are linearly independent, we write any solution u of (4.4) as a linear combination of these two solutions

$$u_z(n) = \frac{W_n(u_z, s_z)}{W_n(c_z, s_z)} c(n) - \frac{W_n(u_z, c_z)}{W_n(c_z, s_z)} s(n), \tag{4.5}$$

where W is the Wronskian given by

$$W_n(f, g) = a(n)(f(n)g(n+1) - g(n)f(n+1)).$$

Note that the Wronskian of two solutions of (4.4) with the same z is constant, so the coefficients of c and s in (4.5) are constant.

If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are bounded, then the **Jacobi operator** $J : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is defined as $Jf = \tau f$. However without the boundedness condition on $\{a_n\}_{n\in\mathbb{N}}$ or $\{b_n\}_{n\in\mathbb{N}}$, the operator J is no longer defined on all of $l^2(\mathbb{N})$. Here one needs to introduce the **minimal** and **maximal operators** associated with τ as

$$J_{min} : \mathcal{D}(J_{min}) \to l^2(\mathbb{N}), \qquad J_{max} : \mathcal{D}(J_{max}) \to l^2(\mathbb{N})$$

$$f \mapsto \tau f \qquad\qquad f \mapsto \tau f,$$

where $\mathcal{D}(J_{min}) = c_{00}(\mathbb{N})$ and $\mathcal{D}(J_{max}) = \{f \in l^2(\mathbb{N}) \mid \tau f \in l^2(\mathbb{N})\}$. Green's formula implies that $J_{min}^* = J_{max}$ and

$$J_{max}^* = \overline{J_{min}} : \mathcal{D}(J_{max}^*) \to l^2(\mathbb{N})$$

$$f \mapsto \tau f,$$

where $\mathcal{D}(J^*_{max}) = \{f \in J_{max} \mid \lim_{n\to\infty} W_n(\overline{f}, g) = 0\ ,\ g \in J_{max}\}$ ([67], Section 2.6).

In order to discuss self-adjoint extensions of the minimal operator we use limit point and limit circle classifications of τ. The difference expression τ is called **limit point** ($l.p.$) if $s_{z_0} \notin l^2(\mathbb{N})$ for some $z_0 \in \mathbb{C}\backslash\mathbb{R}$ and **limit circle** ($l.c.$) otherwise.

The maximal operator J_{max} is self-adjoint if and only if τ is $l.p.$ ([67], Lemma 2.16). Therefore in the limit point case J_{max} is a self adjoint extension of the minimal Jacobi operator J_{min}.

If τ is limit circle, we define the set of boundary conditions at ∞ as

$$BC(\tau) = \{v \in \mathcal{D}(J_{max}) \mid \lim_{n\to\infty} W_n(\bar{v}, v) = 0, \lim_{n\to\infty} W_n(\bar{v}, f) \neq 0 \text{ for some } f \in \mathcal{D}(J_{max})\}.$$

Then for any $v \in BC(\tau)$, the operator

$$J_v :\ \mathcal{D}(v) \to l^2(\mathbb{N})$$
$$f \mapsto \tau f,$$

is a self-adjoint extension of J_{min}, where $\mathcal{D}(v) = \{f \in \mathcal{D}(J_{max}) \mid \lim_{n\to\infty} W_n(v, f) = 0\}$ ([67], Theorem 2.18). We parametrize self-adjoint extensions of J_{min} in the limit circle case by defining

$$v_\alpha(n) = \cos(\alpha)c_0(n) + \sin(\alpha)s_0(n), \qquad \alpha \in [0, \pi)$$

and observing that different values of α give different extensions. Then all self-adjoint extensions of J_{min} correspond to some v_α with unique $\alpha \in [0, \pi)$ ([67], Lemma 2.20). Therefore in the limit circle case, following [62] we define $J(g) := J_v$ for $g \in \mathbb{R} \cup \{\infty\}$, where $g = \cot(\alpha)$ and $\alpha \in [0, \pi)$. In the limit point case, i.e. if $\overline{J_{min}}$ is self-adjoint, we let $J(g) := \overline{J_{min}}$ for all $g \in \mathbb{R} \cup \{\infty\}$.

If τ is $l.c.$, i.e. $J_{min} \neq J^*_{min}$, then the spectrum of $J(g)$, denoted by $\sigma(J(g))$, is discrete ([67], Lemma 2.19). We assume $J(g)$ has a discrete spectrum, which is a restriction in the limit point case. Note that since the essential spectrum of a bounded Jacobi operator is always nonempty, discreteness of $\sigma(J(g))$ implies unboundedness of $J(g)$ ([67], Section 3.2).

We define the self-adjoint operator $J_h(g)$ by $J_h(g) := J(g) - h\langle \cdot, e_1\rangle e_1$ for $h \in \mathbb{R}$, where $\{e_n\}_{n\in\mathbb{N}}$ is the canonical basis in $l^2(\mathbb{N})$. It is the rank-one perturbation of $J(g)$ by h. If we consider the operator $J(\beta, g)$ defined by the difference expression

$$(\tilde{\tau} f)_n := b_{n-1} f_{n-1} + a_n f_n + b_n f_{n+1}, \qquad n \in \mathbb{N}$$

with the boundary condition

$$f_1 \cos\beta + f_0 \sin\beta = 0, \qquad \beta \in (0, \pi),$$

then $J_h(g) = J(\beta, g)$ for $h = \cot\beta$. Hence h can be seen as a boundary condition. Note that discreteness of $\sigma(J(g))$ implies discreteness of $\sigma(J_h(g))$ for any $h \in \mathbb{R}$. Moreover, $\sigma(J_{h_1}(g)) \cap \sigma(J_{h_2}(g)) = \emptyset$ if $h_1 \neq h_2$.

Definition 17. The **Weyl m-function** of $J_h(g)$ is defined as $m_h(z, g) := \langle e_1, (J_h(g) - z)^{-1} e_1\rangle$.

Weyl m-function is a meromorphic Herglotz function ([67], Section 2.1). By Neumann expansion for the resolvent

$$(J_h(g) - z)^{-1} = -\sum_{n=0}^{N-1} \frac{(J_h(g))^n}{z^{n+1}} + \frac{1}{z^N}(J_h(g))^N (J_h(g) - z)^{-1},$$

where $z \in \mathbb{C}\backslash\sigma(J_h(g))$ we get the following asymptotics of $m_h(z, g)$:

$$m_h(z, g) = -\frac{1}{z} - \frac{a_1 - h}{z^2} - \frac{(a_1 - h)^2 + b_1^2}{z^3} + O(z^{-4}), \tag{4.6}$$

as $z \to \infty$ for $\text{Im} z \geq \epsilon$, $\epsilon > 0$ ([67], Section 6.1).

Since the m-function $m_h(z, g)$ is Herglotz, if λ is an isolated eigenvalue of $J_h(g)$, then $m_h(z, g)$ has a simple pole at $z = \lambda$ ([67], Section 2.2).

Definition 18. The **norming constant** corresponding to the eigenvalue λ_k of $J_h(g)$ is defined as

$$\gamma_k(h) = \left(\sum_{n\in\mathbb{N}} |u_{\lambda_k}(n)|^2 \right)^{-1},$$

where $u_z \in l^2(\mathbb{N})$ solves (4.4).

The residue of $m_h(z, g)$ at the pole λ_k is given by $-\gamma_k(h)$ ([67], p.214).

One finds a detailed discussion of the spectral theory of Jacobi operators in [67], which we have followed so far.

4.2 Inverse spectral problems with mixed data

4.2.1 Matching index sets

We follow the enumeration introduced in [62] to enumerate the sequences of eigenvalues. Let $\{\lambda_n\}_{n\in M}$ and $\{\nu_n\}_{n\in M}$ be a pair of discrete, interlacing, infinite real sequences and $M \subset \mathbb{Z}$. Then $\lambda_n < \nu_n < \lambda_{n+1}$ for all $n \in M$, where

- If $\inf_{n\in M} \lambda_n = -\infty$ and $\sup_{n\in M} \lambda_n = \infty$, then $M := \mathbb{Z}$ and $\nu_{-1} < 0 < \lambda_1$.
- If $0 < \sup_{n\in M} \lambda_n < \infty$, then $M := \{n\}_{n=-\infty}^{n_{max}}$, $n_{max} \geq 1$ and $\nu_{-1} < 0 < \lambda_1$.
- If $\sup_{n\in M} \lambda_n \leq 0$, then $M := \{n\}_{n=-\infty}^{0}$.
- If $\inf_{n\in M} \nu_n \geq 0$, then $M := \{n\}_{n=0}^{\infty}$.
- If $-\infty < \inf_{n\in M} \nu_n < 0$, then $M := \{n\}_{n=n_{min}}^{\infty}$, $n_{min} \leq -1$ and $\nu_{-1} < 0 < \lambda_1$.

Silva and Weder gave a characterization of two spectra of $J(g)$ corresponding to different boundary conditions, if $J(g)$ has a discrete spectrum.

Theorem 28. (*[62] Theorem 3.4*) **(Characterization of two spectra)** *Given $h_1 \in \mathbb{R}$ and two infinite discrete sequences of real numbers $\{\lambda_n\}_{n\in M}$ and $\{\nu_n\}_{n\in M}$, there is a unique real number $h_2 > h_1$, a unique operator $J(g)$, and if $J_{min} \neq J_{min}^*$ also a unique $g \in \mathbb{R} \cup \{+\infty\}$, such* *that* $\{\nu_n\}_{n\in M} = \sigma(J_{h_1}(g))$ *and* $\{\lambda_n\}_{n\in M} = \sigma(J_{h_2}(g))$ *if and only if the following conditions are satisfied.*

1. $\{\lambda_n\}_{n\in M}$ *and* $\{\nu_n\}_{n\in M}$ *interlace and, if* $\{\lambda_n\}_{n\in M}$ *is bounded from below,*

$$\min_{n\in M}\{\nu_n\}_{n\in M} \;>\; \min_{n\in M}\{\lambda_n\}_{n\in M},$$

   while if $\{\lambda_n\}_{n\in M}$ *is bounded from above,*

$$\max_{n\in M}\{\nu_n\}_{n\in M} \;>\; \max_{n\in M}\{\lambda_n\}_{n\in M}.$$

2. *The following series converges*

$$\Delta := \sum_{n\in M}\gamma_n \;<\; \infty, \tag{4.7}$$

   where $\gamma_n := \nu_n - \lambda_n$*. By condition (4.7) the product* $\prod_{n\in M, n\neq k}\frac{\nu_n-\lambda_k}{\lambda_n-\lambda_k}$ *is convergent, so we can define*

$$\tau_k^{-1} := \frac{\nu_k-\lambda_k}{\Delta}\prod_{n\in M, n\neq k}\frac{\nu_n-\lambda_k}{\lambda_n-\lambda_k}, \qquad \forall k\in M. \tag{4.8}$$

3. *The sequence* $\{\tau_n\}_{n\in M}$ *is such that, for* $m = 0, 1, 2, \ldots$*, the series* $\sum_{n\in M}\frac{\lambda_n^{2m}}{\tau_n}$ *converges.*

4. *If a sequence of complex numbers* $\{\beta_n\}_{n\in M}$ *is such that the series* $\sum_{n\in M}\frac{|\beta_n|^2}{\tau_n}$ *converges and for* $m = 0, 1, 2, \ldots$, $\sum_{n\in M}\frac{\beta_n\lambda_n^m}{\tau_n} = 0$*, then* $\beta_n = 0$ *for all* $n \in M$*.*

Silva and Weder also proved that the spectral data consisting of two discrete spectra and one of the boundary conditions uniquely determine the operator $J(g)$ and the other boundary condition.

Theorem 29. (*[62] Theorem 3.3*) **(Two-spectra theorem)** *Let* $J(g)$ *be a Jacobi operator with discrete spectrum,* $h_1, h_2 \in \mathbb{R}$*,* $h_1 \neq h_2$*,* $\sigma(J_{h_1}(g)) = \{\lambda_n\}_{n\in M}$ *and* $\sigma(J_{h_2}(g)) = \{\nu_n\}_{n\in M}$*. Then*

$\{\lambda_n\}_{n\in M}$, $\{\nu_n\}_{n\in M}$ *and* h_1*(respectively* h_2*) uniquely determine the operator* $J(g)$, h_2*(respectively* h_1*) and if* $J_{min} \neq J^*_{min}$*, the boundary condition* g *at infinity.*

Using Theorem 28 and Theorem 29 we prove our main result. The spectral data consists of one spectrum, a subset of another spectrum, the norming constants of the first spectrum for the missing part of the second spectrum and the two boundary conditions.

Theorem 30. (Inverse problem III-a) *Let* $J(g)$ *be a Jacobi operator with discrete spectrum,* $\sigma(J_{h_1}(g)) = \{\lambda_n\}_{n\in M}$, $\sigma(J_{h_2}(g)) = \{\nu_n\}_{n\in M}$ *and* $A \subseteq M$*. Then* $\{\lambda_n\}_{n\in M}$, $\{\nu_n\}_{n\in M\setminus A}$, $\{\gamma_n(h_1)\}_{n\in A}$, h_1 *and* h_2 *uniquely determine the operator* $J(g)$*, and if* $J_{min} \neq J^*_{min}$*, the boundary condition* g *at infinity, where* $\{\gamma_n(h_1)\}_{n\in M}$ *are norming constants corresponding to* $J_{h_1}(g)$*.*

Proof. The Weyl m-function m_{h_1} can be represented in terms of m_{h_2}. Indeed, by the second resolvent identity and the definition of the Weyl m-function

$$\begin{aligned} m_{h_1}(z,g) - m_{h_2}(z,g) &= \big\langle (T_{h_1} - T_{h_2})e_1, e_1 \big\rangle \\ &= \big\langle (T_{h_2})((h_1 - h_2)\big\langle \cdot, e_1\big\rangle)(T_{h_1})e_1, e_1 \big\rangle \\ &= \big\langle (h_1 - h_2)\big\langle T_{h_1}e_1, e_1\big\rangle T_{h_2}e_1, e_1 \big\rangle \\ &= (h_1 - h_2)m_{h_1}(z,g)m_{h_2}(z,g), \end{aligned}$$

where $T_h = (J_h(g) - zI)^{-1}$. Therefore

$$m_{h_2}(z,g) = \frac{m_{h_1}(z,g)}{1 - (h_2 - h_1)m_{h_1}(z,g)}. \tag{4.9}$$

Since $J(g)$ has discrete spectrum and $m_h(z,g) = \dfrac{m_0(z,g)}{1 - hm_0(z,g)}$, the poles of $m_h(z,g)$ are the eigenvalues of $J_h(g)$, given by the zeros of $1 - hm_0(z,g)$ for any $h \in \mathbb{R}$. Hence

$$F(z,g) := \frac{m_{h_1}(z,g)}{m_{h_2}(z,g)} = \frac{1 - h_2 m_0(z,g)}{1 - h_1 m_0(z,g)} \tag{4.10}$$

is a meromorphic function such that the zeros of F are the eigenvalues of $J_{h_2}(g)$ and the poles of

F are the eigenvalues of $J_{h_1}(g)$. Moreover if $h_1 - h_2 > 0$, then F is a Herglotz function, since m_0 is a Herglotz function and

$$F(z, g) = 1 + \frac{-1}{\frac{h_1}{h_1 - h_2} + \frac{-1}{[h_1 - h_2] m_0(z, g)}}.$$

Let us assume $h_1 > h_2$. We consider the case $h_1 < h_2$ at the end of the proof, which will require minor changes. Since F is a meromorphic Herglotz function, by the infinite product representation of meromorphic Herglotz functions ([51], Theorem VII.1.1) and using the enumeration introduced above, F can be represented as

$$F(z, g) = C \, \frac{z - \nu_0}{z - \lambda_0} \prod_{n \in M, n \neq 0} \left(1 - \frac{z}{\nu_n}\right) \left(1 - \frac{z}{\lambda_n}\right)^{-1}, \qquad C \ > \ 0. \tag{4.11}$$

Recalling (4.7) and interlacing property of the two spectra $\{\lambda_n\}_{n \in M}$ and $\{\nu_n\}_{n \in M}$, one gets

$$\Delta = \sum_{n \in M} |\nu_n - \lambda_n| < \infty,$$

and hence

$$0 < \prod_{n \in M, n \neq 0} \frac{\nu_n}{\lambda_n} < \infty.$$

Therefore

$$\begin{aligned} \lim_{z \to \infty, \mathrm{Im} z \geq \epsilon} \frac{F(z, g)}{C} &= \lim_{z \to \infty, \mathrm{Im} z \geq \epsilon} \frac{z - \nu_0}{z - \lambda_0} \prod_{n \in M, n \neq 0} \left(1 - \frac{z}{\nu_n}\right) \left(1 - \frac{z}{\lambda_n}\right)^{-1} \\ &= \lim_{z \to \infty, \mathrm{Im} z \geq \epsilon} \frac{z - \nu_0}{z - \lambda_0} \prod_{n \in M, n \neq 0} \frac{\lambda_n}{\nu_n} \prod_{n \in M, n \neq 0} \left(1 + \frac{\nu_n - \lambda_n}{\lambda_n - z}\right) \\ &= \prod_{n \in M, n \neq 0} \frac{\lambda_n}{\nu_n}, \end{aligned}$$

for $\epsilon > 0$. By (4.6), asymptotics of the m-function $m_0(z, g)$ implies $\lim_{z \to \infty, \mathrm{Im} z \geq \epsilon} m_0(z, g) = 0$ and by the definition of $F(z,g)$, we get $\lim_{z\to\infty, \mathrm{Im} z\geq\epsilon} F(z,g) = 1$. Therefore $C = \prod_{n\in M, n\neq 0} \nu_n/\lambda_n$ and

$$F(z,g) = \prod_{n\in M} \frac{z-\nu_n}{z-\lambda_n}. \tag{4.12}$$

The residue of F at λ_k is given in terms of the norming constant $\gamma_k(h_1)$. Indeed,

$$\text{Res}(F,\lambda_k) = \text{Res}\left(\frac{m_{h_1}}{m_{h_2}},\lambda_k\right) = \text{Res}(1-(h_2-h_1)m_{h_1},\lambda_k) = \frac{-(h_1-h_2)}{\gamma_k(h_1)},$$

since $\gamma_k^{-1}(h_1) = -\text{Res}(m_{h_1},\lambda_k)$ for any $k\in M$. Recall that $\Delta = h_1 - h_2$. Therefore

$$\frac{-1}{\gamma_n(h_1)} = \text{Res}\left(\frac{F}{\Delta},\lambda_n\right), \tag{4.13}$$

i.e. the residues of $F(z,g)/\Delta$ are known at λ_n for each $n\in A$.

At this step we can restate our claim in terms of F as the set of poles, $\{\lambda_n\}_{n\in M}$, the set of zeros except the index set A, $\{\nu_n\}_{n\in M\setminus A}$, and the residues with the same index set A, $\{\text{Res}(F/\Delta,\lambda_n)\}_{n\in A}$ determine $F(z,g)$ uniquely.

Since $\{\nu_n-\lambda_n\}_{n\in M}\in l^1$, $F(z,g)$ has the representation $F = GH$, where

$$G(z,g) := \prod_{n\in A}\frac{z-\nu_n}{z-\lambda_n} \qquad \text{and} \qquad H(z,g) := \prod_{n\in M\setminus A}\frac{z-\nu_n}{z-\lambda_n}.$$

Note that for any $k\in A$, we know

$$Res\,(F/\Delta,\lambda_k) = \frac{\lambda_k-\nu_k}{\Delta}\prod_{n\in M, n\neq k}\frac{\lambda_k-\nu_n}{\lambda_k-\lambda_n}. \tag{4.14}$$

In addition, for any $k\in A$, we also know

$$H(\lambda_k,g) = \prod_{n\in M\setminus A}\frac{\lambda_k-\nu_n}{\lambda_k-\lambda_n}. \tag{4.15}$$

Conditions (4.14) and (4.15) imply that for any $k \in A$, we know

$$Res\,(G/\Delta, z = \lambda_k) = \frac{\text{Res}(F/\Delta, \lambda_k)}{H(\lambda_k)}.$$

Note that the zeros and the poles of G are real and interlacing, and hence

$$0 \;<\; arg(G(z,g)) = \sum_{n \in A} (arg(z - \nu_n) - arg(z - \lambda_n)) \;<\; \pi$$

for any z in the upper half plane, i.e. $G(z,g)$ is a meromorphic Herglotz function. Therefore by Čebotarev's theorem, G/Δ has the following representation

$$\frac{G(z,g)}{\Delta} = az + b + \sum_{n \in A} A_n \left(\frac{1}{\lambda_n - z} - \frac{1}{\lambda_n} \right), \tag{4.16}$$

where $a \geq 0$ and $b \in \mathbb{R}$. Note that $A_k = -\text{Res}(G/\Delta, a_k)$ for any $k \in A$, which means there are only two unknowns on the right hand side, namely constants a and b.

On the upper half-plane G/Δ converges to $1/\Delta$ as z goes to infinity, since

$$\sum_{n \in A} |\nu_n - \lambda_n| \leq \sum_{n \in M} |\nu_n - \lambda_n| \;<\; \infty.$$

Let $t \in \mathbb{R}$. Then

$$G(it) = \left[b + \sum_{n \in A} \left(\frac{\lambda_n A_n}{t^2 + \lambda_n^2} - \frac{A_n}{\lambda_n} \right) \right] + i \left[at + \sum_{n \in A} \frac{t A_n}{t^2 + \lambda_n^2} \right],$$

so $a = 0$ and $b = 1/\Delta + \sum_{n \in A} A_n/\lambda_n$, since $\lim_{t \to +\infty} G(it,g) = 1/\Delta$. Therefore

$$G(z,g) = \frac{1}{\Delta} + \sum_{n \in A} \frac{A_n}{\lambda_n - z} = \frac{1}{h_1 - h_2} + \sum_{n \in A} \frac{A_n}{\lambda_n - z}, \tag{4.17}$$

so the right hand side of (4.17) is known. This implies uniqueness of $G(z,g)$ and hence uniqueness of $\{\nu_n\}_{n \in A}$. After unique recovery of the two spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$, the operator J is uniquely determined by Theorem 29.

If $h_2 > h_1$, then $1/F(z,g)$ is Herglotz instead of $F(z,g)$, so we get the infinite product representation

$$\frac{1}{F(z,g)} = \prod_{n\in M} \frac{z-\lambda_n}{z-\nu_n}. \tag{4.18}$$

Note that $-F(z,g)$ is also a meromorphic Herglotz function. Therefore using similar arguments as $h_1 > h_2$ case, the function $G(z,g)$ definded as

$$G(z,g) := \frac{1}{h_1-h_2}\prod_{n\in A}\frac{z-\nu_n}{z-\lambda_n}$$

is represented as

$$G(z,g) = \frac{1}{h_1-h_2} + \sum_{n\in A}\frac{A_n}{\lambda_n - z},$$

where $A_k = -\text{Res}(G,\lambda_k)$ for any $k \in A$. This implies uniqueness of $G(z,g)$ and hence uniqueness of $\{\nu_n\}_{n\in A}$. After unique recovery of the two spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$, the operator J is uniquely determined by Theorem 29. □

Corollary 1. **(Recovery from m-function)** *Let $h \in \mathbb{R}$ and $J(g)$ be a Jacobi operator with discrete spectrum. Then the m-function $m_h(z,g)$ (or the corresponding spectral measure) and h uniquely determine the operator $J(g)$, and if $J_{min} \neq J^*_{min}$, the boundary condition g at infinity.*

Proof. If we let $h_1 := h$ and h_2 any real number less than h, then using (4.9) and (4.10) we get

$$F(z,g) = \frac{m_{h_1}(z,g)}{m_{h_2}(z,g)} = 1 - (h_2 - h_1)m_{h_1}(z,g),$$

i.e. we know the meromorphic Herglotz function F, since it depends on our spectral data and h_2. Let us observe that our spectral data and h_2 give the spectral data of Theorem 30 with $A = M$. Then by Theorem 30 we get the result. □

Remark 5. *If we let $A = \emptyset$ in Theorem 30, we get the statement of the two-spectra theorem, Theorem 29.*

In our spectral data we can replace h_1 or h_2 with any eigenvalue of $J_{h_2}(g)$ from the index set A.

Theorem 31. (Inverse problem III-b) *Let $J(g)$ be a Jacobi operator with discrete spectrum, $\sigma(J_{h_1}(g)) = \{\lambda_n\}_{n\in M}$, $\sigma(J_{h_2}(g)) = \{\nu_n\}_{n\in M}$ and $A \subseteq M$. Then $\{\lambda_n\}_{n\in M}$, $\{\nu_n\}_{n\in M\setminus A}$, $\{\gamma_n(h_1)\}_{n\in A}$, h_1 (respectively h_2) and ν_m for some $m \in A$ uniquely determine the operator $J(g)$, h_2 (respectively h_1) and if $J_{min} \neq J^*_{min}$, the boundary condition g at infinity, where $\{\gamma_n(h_1)\}_{n\in M}$ are norming constants corresponding to $J_{h_1}(g)$.*

Proof. Following the proof of Theorem 30 we get the infinite sum representation

$$\frac{G(z,g)}{\Delta} = \frac{1}{\Delta} + \sum_{n\in A} \frac{A_n}{\lambda_n - z} = \frac{1}{h_1 - h_2} + \sum_{n\in A} \frac{A_n}{\lambda_n - z}, \tag{4.19}$$

for the infinite product

$$G(z,g) := \prod_{n\in A} \frac{z-\nu_n}{z-\lambda_n}.$$

Now let us prove uniqueness of $G(z,g)$. Note that we know $\{\lambda_n\}_{n\in A}$, $\{-A_n\}_{n\in A}$ and ν_m. Let the infinite product

$$\widetilde{G}(z,g) := \prod_{n\in A} \frac{z-\widetilde{\nu}_n}{z-\lambda_n}$$

share the same set of poles $\{\lambda_n\}_{n\in A}$ and the same residues $\{-A_n\}_{n\in A}$ at the corresponding poles with $G(z,g)$. In addition assume $G(z,g)$ and $\widetilde{G}(z,g)$ have the common zero ν_m, i.e. $\widetilde{\nu}_m = \nu_m$. Let us also assume zeros and poles of $\widetilde{G}(z,g)$ satisfy asymptotic properties of Theorem 28. Then we know that $\widetilde{G}(z,g)$ has the infinite sum representation

$$\widetilde{G}(z,g) = \frac{1}{\widetilde{\Delta}} + \sum_{n\in A} \frac{A_n}{\lambda_n - z} \tag{4.20}$$

Using representations (4.19) and (4.20), the difference of $G(z,g)$ and $\widetilde{G}(z,g)$ is a real constant, which is zero since $G(\nu_m,g) = \widetilde{G}(\nu_m,g)$. This implies uniqueness of $G(z,g)$ and hence uniqueness of $\{\nu_n\}_{n\in A}$. After unique recovery of the two spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$, the operator

J is uniquely determined by Theorem 29. □

In the spectral data of Theorem 30 we can also replace h_1 or h_2 with any norming constant of $J_{h_1}(g)$ outside the index set A.

Theorem 32. (Inverse problem III-c) *Let $J(g)$ be a Jacobi operator with discrete spectrum, $\sigma(J_{h_1}(g)) = \{\lambda_n\}_{n\in M}$, $\sigma(J_{h_2}(g)) = \{\nu_n\}_{n\in M}$ and $A \subseteq M$. Then $\{\lambda_n\}_{n\in M}$, $\{\nu_n\}_{n\in M\setminus A}$, $\{\gamma_n(h_1)\}_{n\in A}$, h_1 (respectively h_2) and $\gamma_m(h_1)$ for some $m \in M\setminus A$ uniquely determine the operator $J(g)$, h_2 (respectively h_1) and if $J_{min} \neq J^*_{min}$, the boundary condition g at infinity, where $\{\gamma_n(h_1)\}_{n\in M}$ are norming constants corresponding to $J_{h_1}(g)$.*

Proof. Let us define the index set $A' := A \cup \{m\}$. Then following the proof of Theorem 30 and redefining G and H as

$$G(z,g) := \prod_{n\in A'} \frac{z-\nu_n}{z-\lambda_n}$$

and

$$H(z,g) := \prod_{n\in M\setminus A'} \frac{z-\nu_n}{z-\lambda_n}$$

we get

$$G(z,g) = \frac{1}{\Delta} + \sum_{n\in A'} \frac{A_n}{\lambda_n - z} = \frac{1}{h_1 - h_2} + \sum_{n\in A'} \frac{A_n}{\lambda_n - z}. \tag{4.21}$$

Now let us prove uniqueness of $G(z,g)$. Note that we know $\{\lambda_n\}_{n\in A'}$, $\{-A_n\}_{n\in A'}$ and ν_m. Let the infinite product

$$\widetilde{G}(z,g) := \prod_{n\in A'} \frac{z-\widetilde{\nu}_n}{z-\lambda_n}$$

share the same set of poles $\{\lambda_n\}_{n\in A'}$ and the same residues $\{-A_n\}_{n\in A'}$ at the corresponding poles with $G(z,g)$. In addition $G(z,g)$ and $\widetilde{G}(z,g)$ have the same zero ν_m, i.e. $\widetilde{\nu}_m = \nu_m$. Let us also assume zeros and poles of $\widetilde{G}(z,g)$ satisfy asymptotic properties of Theorem 28. Then we know that $\widetilde{G}(z,g)$ has the infinite sum representation

$$\widetilde{G}(z,g) = \frac{1}{\widetilde{\Delta}} + \sum_{n\in A'} \frac{A_n}{\lambda_n - z} \tag{4.22}$$

Using representations (4.21) and (4.22), the difference of $G(z,g)$ and $\widetilde{G}(z,g)$ is a real constant, which is zero since $G(\nu_m, g) = \widetilde{G}(\nu_m, g)$. This implies uniqueness of $G(z,g)$ and hence uniqueness of $\{\nu_n\}_{n\in A}$. After unique recovery of the two spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$, the operator J is uniquely determined by Theorem 29. □

4.2.2 Non-matching index sets

If the known norming constants of $J_{h_1}(g)$ and unknown eigenvalues of $J_{h_2}(g)$ have different index sets, one needs some control over eigenvalues of $J_{h_1}(g)$ corresponding to the known norming constants and unknown part of the spectrum $\sigma(J_{h_2}(g))$. In this case we get a Čebotarev type representation result. Before the statement, let us clarify the notation we use. For any subsequence $\{\lambda_{k_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_1}(g))$ and $\{\nu_{l_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_2}(g))$, by $A_{k_n,m}$ and A_{k_n} we denote the residues at λ_{k_n} of partial and infinite products, respectively, consisting of these subsequences:

$$A_{k_n,m} := \mathrm{Res}(\mathcal{G}_m, \lambda_{k_n}) = \frac{\lambda_{k_n}}{\nu_{l_n}}(\lambda_{k_n} - \nu_{l_n}) \prod_{1\le j\le m, j\ne n} \frac{\lambda_{k_j}}{\nu_{l_j}} \frac{\lambda_{k_n} - \nu_{l_j}}{\lambda_{k_n} - \lambda_{k_j}},$$
$$A_{k_n} := \mathrm{Res}(\mathcal{G}, \lambda_{k_n}) = \frac{\lambda_{k_n}}{\nu_{l_n}}(\lambda_{k_n} - \nu_{l_n}) \prod_{j\in\mathbb{N}, j\ne n} \frac{\lambda_{k_j}}{\nu_{l_j}} \frac{\lambda_{k_n} - \nu_{l_j}}{\lambda_{k_n} - \lambda_{k_j}},$$

where

$$\mathcal{G}_m(z) := \prod_{n=1}^{m} \left(\frac{z}{\nu_{l_n}} - 1\right)\left(\frac{z}{\lambda_{k_n}} - 1\right)^{-1}, \qquad \mathcal{G}(z) := \prod_{n\in\mathbb{N}} \left(\frac{z}{\nu_{l_n}} - 1\right)\left(\frac{z}{\lambda_{k_n}} - 1\right)^{-1}.$$

Note that these subsequences are ordered according to their indices, i.e. $\lambda_{k_n} < \lambda_{k_{n+1}}$ and $\nu_{l_n} < \nu_{l_{n+1}}$ for any $n \in \mathbb{N}$. This follows from the fact that the two spectra are both real and discrete. Also note that if the spectrum $\sigma(J_h(g))$ is unbounded from both sides, i.e. $\inf M = -\infty$ and $\sup M = \infty$ in the enumeration, then $\{k_n\}_n$ and $\{l_n\}_n$ should be indexed by $\mathbb{Z}$ instead of $\mathbb{N}$. However, wlog we index them by $\mathbb{N}$.

Lemma 4. (Čebotarev type representation III) *Let* $\{\lambda_{k_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_1}(g))$, $\{\nu_{l_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_2}(g))$ *such that*

- $\inf_{n\in\mathbb{N}} |\nu_{l_n} - \lambda_{k_n}| \; > \; 0$,

- $\lim_{m\to\infty} \; \sum_{n=1}^{m} \left(|A_{k_n,m} - A_{k_n}|/\lambda_{k_n}^2 \right) \; < \; \infty$ *and*

- $\{\lambda_{k_n}^{-1}\}_{n\in\mathbb{N}} \in l^1$.

Then the infinite product

$$\mathcal{G}(z) := \prod_{n\in\mathbb{N}} \left(\frac{z}{\nu_{l_n}} - 1 \right) \left(\frac{z}{\lambda_{k_n}} - 1 \right)^{-1}$$

is represented as

$$\mathcal{G}(z) = az^2 + bz + c + \sum_{n\in\mathbb{N}} A_{k_n} \left(\frac{1}{z - \lambda_{k_n}} + \frac{1}{\lambda_{k_n}} \right), \tag{4.23}$$

where a, b, c are real numbers, A_{k_n} is the residue of $\mathcal{G}(z)$ at the point $z = \lambda_{k_n}$ and the product converges normally on $\mathbb{C} \backslash \cup_{n\in\mathbb{N}} \lambda_{k_n}$.

Proof. Let $p(z)$ be the difference of $\mathcal{G}(z)$ and the infinite sum in the right hand side of (4.23). Then, $p(z)$ is an entire function, since the infinite product and the infinite sum share the same set of poles with equivalent degrees and residues. We represent partial products of $\mathcal{G}(z)$ as partial sums:

$$\prod_{n=1}^{m} \left(\frac{z}{\nu_{l_n}} - 1 \right) \left(\frac{z}{\lambda_{k_n}} - 1 \right)^{-1} = \sum_{n=1}^{m} A_{k_n,m} \left(\frac{1}{z - \lambda_{k_n}} + \frac{1}{\lambda_{k_n}} \right) + 1,$$

where $A_{k_n,m}$ is the residue of the partial product at a_{k_n}.

If $\sigma(J_h(g))$ is not bounded above, then let C_n be the circle with radius ν_{l_n} centered at the origin for $\nu_{l_n} > 0$. If $\sigma(J_h(g))$ is bounded above, then let C_n be the circle with radius $|\nu_{l_n}|$ centered at the origin for $\nu_{l_n} < 0$. This sequence of circles satisfy following properties for sufficiently large n:

- C_n omits all the poles λ_{k_n}.

- Each C_n lies inside C_{n+1}.

- The radius of C_n, $|\nu_{l_n}|$ diverges to infinity as n goes to infinity.

At this point wlog let us assume $\nu_{l_n} > 0$ for any $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\max_{z\in C_t}\left|\frac{p(z)-1}{\nu_{l_t}^2}\right| &= \max_{z\in C_t}\left|\frac{\mathcal{G}(z)-1-\sum_{n\in\mathbb{N}}A_{k_n}\left(\frac{1}{z-\lambda_{k_n}}+\frac{1}{\lambda_{k_n}}\right)}{\nu_{l_t}^2}\right| \\
&= \max_{z\in C_t}\lim_{m\to\infty}\left|\frac{\sum_{n=1}^{m}A_{k_n,m}\left(\frac{1}{z-\lambda_{k_n}}+\frac{1}{\lambda_{k_n}}\right)-\sum_{n=1}^{m}A_{k_n}\left(\frac{1}{z-\lambda_{k_n}}+\frac{1}{\lambda_{k_n}}\right)}{\nu_{l_t}^2}\right| \\
&= \lim_{m\to\infty}\frac{1}{\nu_{l_t}^2}\max_{z\in C_t}\left|\sum_{n=1}^{m}(A_{k_n,m}-A_{k_n})\frac{z}{\lambda_{k_n}(z-\lambda_{k_n})}\right| \\
&\le \lim_{m\to\infty}\frac{1}{\nu_{l_t}^2}\sum_{n=1}^{m}|A_{k_n,m}-A_{k_n}|\frac{\nu_{l_t}}{\lambda_{k_n}|\nu_{l_t}-\lambda_{k_n}|} \\
&\le \lim_{m\to\infty}\sum_{n=1}^{m}|A_{k_n,m}-A_{k_n}|\frac{1}{\nu_{l_t}\lambda_{k_n}|\nu_{l_t}-\lambda_{k_n}|} \\
&\le C'\lim_{m\to\infty}\sum_{n=1}^{m}\frac{|A_{k_n,m}-A_{k_n}|}{\lambda_{k_n}^2} \;<\; \infty,
\end{aligned}$$

where $C' \in \mathbb{R}$ is independent of n and m. The last line follows from

$$\begin{aligned}
&\max_{t\in\mathbb{N}}\left|\frac{1}{\nu_{l_t}\lambda_{k_n}(\nu_{l_t}-\lambda_{k_n})}\right| \le \max\left\{\frac{1}{|\nu_{l_1}\lambda_{k_n}(\lambda_{k_n}-\nu_{l_1})|},\frac{1}{|\nu_{k_{n+1}}\lambda_{k_n}(\nu_{k_{n+1}}-\lambda_{k_n})|}\right\} \\
&= \max\left\{\frac{1}{|\nu_{l_1}\lambda_{k_n}(\lambda_{k_n}-\nu_{l_1})|},\frac{1}{|(\lambda_{k_{n+1}}+\gamma_{k_{n+1}})\lambda_{k_n}(\lambda_{k_{n+1}}+\gamma_{k_{n+1}}-\lambda_{k_n})|}\right\} \le C'\frac{1}{\lambda_{k_n}^2}.
\end{aligned}$$

Therefore $|p(z)-1| \le C|z|^2$ on the circle C_t for any $t \in \mathbb{N}$, where C is a positive real number. By the maximum modulus theorem and the entireness of $p(z)$, we conclude that $p(z)$ is a polynomial of at most second degree. Since $\mathcal{G}(0), \mathcal{G}'(0)$ and $\mathcal{G}''(0)$ are real numbers, $a, b, c \in \mathbb{R}$. □

Using the Čebotarev type representation (4.23) we prove our inverse spectral results in non-matching index sets case with some additional convergence criterion on the two spectra. Theorems 33, 34 and 35 are non-matching index sets versions of Theorems 30, 31 and 32 respectively.

Theorem 33. (Inverse Problem IV-a) *Let $J(g)$ be a Jacobi operator with discrete spectrum, $\sigma(J_{h_1}(g)) = \{\lambda_n\}_{n\in M}$, $\sigma(J_{h_2}(g)) = \{\nu_n\}_{n\in M}$ and $\{\lambda_{k_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_1}(g))$, $\{\nu_{l_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_2}(g))$ satisfying that*

- *there exists* $N \in \mathbb{N}$ *such that* $k_n \neq l_n \; \forall \, n > N$,
- $\lim_{m\to\infty} \sum_{n=1}^{m} \left(|A_{k_n,m} - A_{k_n}|/\lambda_{k_n}^2 \right) \; < \; \infty$,
- $\{\lambda_n^{-1}\}_{n\in M} \in l^1$ *and*
- $0 \; < \; \prod_{n\in\mathbb{N}} \frac{\nu_{l_n}}{\lambda_{k_n}} \; < \; \infty$.

Then $\{\lambda_n\}_{n\in M}$, $\{\nu_n\}_{n\in M} \backslash \{\nu_{l_n}\}_{n\in\mathbb{N}}$, $\{\gamma_{k_n}(h_1)\}_{n\in\mathbb{N}}$, h_1 *and* h_2 *uniquely determine the operator* $J(g)$, *and if* $J_{min} \neq J^*_{min}$, *the boundary condition* g *at infinity, where* $\{\gamma_n(h_1)\}_{n\in M}$ *are norming constants corresponding to* $J_{h_1}(g)$.

Proof. As we discussed in the proof of Theorem 30 wlog we assume $h_1 > h_2$. Recall that in this case

$$\Delta := h_1 - h_2 = \sum_{n\in M} \nu_n - \lambda_n < \infty.$$

Let us define

$$\mathcal{F}(z,g) := \frac{1}{\Delta} \prod_{n\in M} \frac{\lambda_n}{\nu_n} \prod_{n\in M} \frac{z-\nu_n}{z-\lambda_n}. \tag{4.24}$$

Note that we assume $0 < \prod_{n\in\mathbb{N}} \nu_{l_n}/\lambda_{k_n} < \infty$ and $\{\lambda_n^{-1}\}_{n\in M} \in l^1$, which also implies $\{\nu_n^{-1}\}_{n\in M} \in l^1$. Therefore $\mathcal{F}(z,g)$ has the representation $\mathcal{F} = \mathcal{G}\mathcal{H}$, where

$$\mathcal{G}(z,g) := \frac{1}{\Delta} \prod_{n\in\mathbb{N}} \frac{\nu_{l_n} - z}{\nu_{l_n}} \frac{\lambda_{k_n}}{\lambda_{k_n} - z} = \frac{1}{\Delta} \prod_{n\in\mathbb{N}} \frac{\lambda_{k_n}}{\nu_{l_n}} \prod_{n\in\mathbb{N}} \frac{z-\nu_{l_n}}{z-\lambda_{k_n}}$$

and

$$\mathcal{H}(z,g) := \prod_{n\in M\backslash\{l_n\}} \frac{\nu_n - z}{\nu_n} \prod_{n\in M\backslash\{k_n\}} \frac{\lambda_n}{\lambda_n - z}.$$

By (4.24)

$$Res\left(\left[\prod_{n\in M} \frac{\nu_n}{\lambda_n} \right] \mathcal{G}(z,g), z = \lambda_k \right) = \frac{1}{\mathcal{H}(\lambda_k)\gamma_k(h_1)},$$

so we know the residues of the infinite product $\left[\prod_{n\in M} \nu_n/\lambda_n\right] \mathcal{G}(z,g)$ at λ_k for any $k \in \{k_n\}_{n\in\mathbb{N}}$.

This infinite product has the representation

$$\left(\prod_{n\in M}\frac{\nu_n}{\lambda_n}\right)\mathcal{G}(z,g)=\left(\frac{1}{\Delta}\prod_{n\in M}\frac{\nu_n}{\lambda_n}\prod_{n\in\mathbb{N}}\frac{\lambda_{k_n}}{\nu_{l_n}}\right)\prod_{n\in\mathbb{N}}\frac{z-\nu_{l_n}}{z-\lambda_{k_n}}=C\prod_{n\in\mathbb{N}}\frac{z-\nu_{l_n}}{z-\lambda_{k_n}}.$$

Let us observe that C is a real constant depending only on $\{\lambda_n\}_{n\in M\setminus\{k_n\}}$, $\{\nu_n\}_{n\in M\setminus\{l_n\}}$, h_1 and h_2, so we also know C. From Lemma 4 we get the Čebotarev type representation

$$\left(\prod_{n\in M}\frac{\nu_n}{\lambda_n}\right)\mathcal{G}(z,g)=C\prod_{n\in\mathbb{N}}\frac{z-\nu_{l_n}}{z-\lambda_{k_n}}=az^2+bz+c+\sum_{n\in\mathbb{N}}A_{k_n}\left(\frac{1}{z-\lambda_{k_n}}+\frac{1}{\lambda_{k_n}}\right).$$

Using similar arguments as in the proof of Theorem 30 one finds that $a=0$, $b=0$ and $c=C-\sum_{n\in\mathbb{N}}A_{k_n}/\lambda_{k_n}$ and hence

$$\left(\prod_{n\in M}\frac{\nu_n}{\lambda_n}\right)\mathcal{G}(z,g)=C+\sum_{n\in\mathbb{N}}\frac{A_{k_n}}{z-\lambda_{k_n}}. \tag{4.25}$$

The right hand side of (4.25) is known. This implies uniqueness of $\mathcal{G}(z,g)$ and hence uniqueness of $\{\nu_{l_n}\}_{n\in\mathbb{N}}$. After unique recovery of the two spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$, the operator J is uniquely determined by Theorem 29. □

Theorem 34. (Inverse Problem IV-b) *Let $J(g)$ be a Jacobi operator with discrete spectrum, $\sigma(J_{h_1}(g))=\{\lambda_n\}_{n\in M}$, $\sigma(J_{h_2}(g))=\{\nu_n\}_{n\in M}$ and $\{\lambda_{k_n}\}_{n\in\mathbb{N}}\subset\sigma(J_{h_1}(g))$, $\{\nu_{l_n}\}_{n\in\mathbb{N}}\subset\sigma(J_{h_2}(g))$ satisfying that*

- *there exists $N\in\mathbb{N}$ such that $k_n\neq l_n\ \forall\, n>N$,*
- $\displaystyle\lim_{m\to\infty}\sum_{n=1}^{m}\left(|A_{k_n,m}-A_{k_n}|/\lambda_{k_n}^2\right)<\infty,$
- *$\{\lambda_n^{-1}\}_{n\in M}\in l^1$ and*
- $\displaystyle 0<\prod_{n\in\mathbb{N}}\frac{\nu_{l_n}}{\lambda_{k_n}}<\infty.$

Then $\{\lambda_n\}_{n\in M}$, $\{\nu_n\}_{n\in M\setminus\{l_n\}_{n\in\mathbb{N}}}$, $\{\gamma_{k_n}(h_1)\}_{n\in\mathbb{N}}$, h_1(respectively h_2) and ν_m for some $m\in\{l_n\}_{n\in\mathbb{N}}$

uniquely determine the operator $J(g)$, h_2(respectively h_1) and if $J_{min} \neq J^*_{min}$, the boundary condition g at infinity, where $\{\gamma_n(h_1)\}_{n\in M}$ are norming constants corresponding to $J_{h_1}(g)$.

Proof. Following the proof of Theorem 33 we get representation

$$\left(\prod_{n\in M} \frac{\nu_n}{\lambda_n}\right) \mathcal{G}(z,g) = C + \sum_{n\in\mathbb{N}} \frac{A_{k_n}}{z-\lambda_{k_n}}, \tag{4.26}$$

for the infinite product

$$\mathcal{G}(z,g) := \frac{1}{\Delta}\prod_{n\in\mathbb{N}} \frac{\nu_{l_n}-z}{\nu_{l_n}}\frac{\lambda_{k_n}}{\lambda_{k_n}-z},$$

where A_j is the residue of $\left[\prod_{n\in M}\nu_n/\lambda_n\right]\mathcal{G}(z,g)$ at λ_j and $C\in\mathbb{R}$.

Now let us prove uniqueness of $\left[\prod_{n\in M}\nu_n/\lambda_n\right]\mathcal{G}(z,g)$. Note that we know $\{\lambda_{k_n}\}_{n\in\mathbb{N}}$, $\{-A_{k_n}\}_{n\in\mathbb{N}}$ and ν_m. Let the infinite product

$$\left(\prod_{n\in M} \frac{\widetilde{\nu}_n}{\lambda_n}\right) \widetilde{\mathcal{G}}(z,g) := \left(\prod_{n\in M} \frac{\widetilde{\nu}_n}{\lambda_n}\right)\frac{1}{\Delta}\prod_{n\in\mathbb{N}} \frac{\widetilde{\nu}_{l_n}-z}{\widetilde{\nu}_{l_n}}\frac{\lambda_{k_n}}{\lambda_{k_n}-z}$$

share the same set of poles $\{\lambda_{k_n}\}_{n\in\mathbb{N}}$ and the same residues $\{-A_{k_n}\}_{n\in\mathbb{N}}$ at the corresponding poles with $\left[\prod_{n\in M}\nu_n/\lambda_n\right]\mathcal{G}(z,g)$. In addition assume $\widetilde{\nu}_j=\nu_j$ for all $j\in M\backslash\{l_n\}_{n\in\mathbb{N}}$ and the functions $\mathcal{G}(z,g)$ and $\widetilde{\mathcal{G}}(z,g)$ have the common zero ν_m, i.e. $\widetilde{\nu}_m=\nu_m$. Let us also assume the zeros and the poles of $\widetilde{\mathcal{G}}(z,g)$ satisfy asymptotic properties of Theorem 28. Then we know that $\widetilde{\mathcal{G}}(z,g)$ has the infinite sum representation

$$\left(\prod_{n\in M} \frac{\widetilde{\nu}_n}{\lambda_n}\right) \widetilde{\mathcal{G}}(z,g) = \widetilde{C} + \sum_{n\in\mathbb{N}} \frac{A_{k_n}}{z-\lambda_{k_n}} \tag{4.27}$$

From (4.26) and (4.27), the difference of $\left[\prod_{n\in M}\nu_n/\lambda_n\right]\mathcal{G}(z,g)$ and $\left[\prod_{n\in M}\widetilde{\nu}_n/\lambda_n\right]\widetilde{\mathcal{G}}(z,g)$ is a real constant, which is zero since $\mathcal{G}(\nu_m,g)=\widetilde{\mathcal{G}}(\nu_m,g)=0$. This implies uniqueness of $\left[\prod_{n\in M}\nu_n/\lambda_n\right]\mathcal{G}(z,g)$ and hence uniqueness of $\{\nu_{l_n}\}_{n\in\mathbb{N}}$. After unique recovery of the two spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$, the operator J is uniquely determined by Theorem 29. □

Theorem 35. (Inverse Problem IV-c) *Let $J(g)$ be a Jacobi operator with discrete spectrum,*

$\sigma(J_{h_1}(g)) = \{\lambda_n\}_{n\in M}$, $\sigma(J_{h_2}(g)) = \{\nu_n\}_{n\in M}$ *and* $\{\lambda_{k_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_1}(g))$, $\{\nu_{l_n}\}_{n\in\mathbb{N}} \subset \sigma(J_{h_2}(g))$ *satisfying that*

- *there exists* $N \in \mathbb{N}$ *such that* $k_n \neq l_n \; \forall\, n > N$,
- $\lim_{m\to\infty} \sum_{n=1}^{m} \left(|A_{k_n,m} - A_{k_n}|/\lambda_{k_n}^2\right) \, < \, \infty$,
- $\{\lambda_n^{-1}\}_{n\in M} \in l^1$ *and*
- $0 \, < \, \prod_{n\in\mathbb{N}} \frac{\nu_{l_n}}{\lambda_{k_n}} \, < \, \infty.$

Then $\{\lambda_n\}_{n\in M}$, $\{\nu_n\}_{n\in M\setminus\{l_n\}_{n\in\mathbb{N}}}$, $\{\gamma_{k_n}(h_1)\}_{n\in\mathbb{N}}$, h_1*(respectively* h_2*) and* $\gamma_m(h_1)$ *for some* $m \in M\setminus\{k_n\}_{n\in\mathbb{N}}$ *uniquely determine the operator* $J(g)$, h_2*(respectively* h_1*) and if* $J_{min} \neq J^*_{min}$*, the boundary condition* g *at infinity, where* $\{\gamma_n(h_1)\}_{n\in M}$ *are norming constants corresponding to* $J_{h_1}(g)$.

Proof. Following the proof of Theorem 33 and redefining $\mathcal{G}$ and $\mathcal{H}$ as

$$\mathcal{G}(z,g) := \frac{1}{\Delta}\left(\frac{\nu_m - z}{\nu_m}\frac{\lambda_m}{\lambda_m - z}\right)\prod_{n\in\mathbb{N}}\frac{\nu_{l_n} - z}{\nu_{l_n}}\frac{\lambda_{k_n}}{\lambda_{k_n} - z}$$

and

$$\mathcal{H}(z,g) := \left(\frac{\nu_m - z}{\nu_m}\frac{\lambda_m}{\lambda_m - z}\right)^{-1}\prod_{n\in M\setminus\{l_n\}}\frac{\nu_n - z}{\nu_n}\prod_{n\in M\setminus\{k_n\}}\frac{\lambda_n}{\lambda_n - z}$$

we get

$$\left(\prod_{n\in M}\frac{\nu_n}{\lambda_n}\right)\mathcal{G}(z,g) = C + \frac{A_m}{z - \lambda_m} + \sum_{n\in\mathbb{N}}\frac{A_{k_n}}{z - \lambda_{k_n}}, \tag{4.28}$$

where A_j is the residue of $\left[\prod_{n\in M}\nu_n/\lambda_n\right]\mathcal{G}(z,g)$ at λ_j and $C \in \mathbb{R}$.

Now let us prove uniqueness of $\left[\prod_{n\in M}\nu_n/\lambda_n\right]\mathcal{G}(z,g)$. Note that we know $\{\lambda_{k_n}\}_{n\in\mathbb{N}} \cup \{\lambda_m\}$, $\{A_{k_n}\}_{n\in\mathbb{N}} \cup \{A_m\}$ and ν_m. Let the infinite product

$$\left(\prod_{n\in M}\frac{\widetilde{\nu}_n}{\lambda_n}\right)\widetilde{\mathcal{G}}(z,g) := \left(\prod_{n\in M}\frac{\widetilde{\nu}_n}{\lambda_n}\right)\frac{1}{\Delta}\left(\frac{\nu_m - z}{\nu_m}\frac{\lambda_m}{\lambda_m - z}\right)\prod_{n\in\mathbb{N}}\frac{\nu_{l_n} - z}{\nu_{l_n}}\frac{\lambda_{k_n}}{\lambda_{k_n} - z}$$

share the same set of poles $\{\lambda_{k_n}\}_{n\in\mathbb{N}} \cup \{\lambda_m\}$ and the same residues $\{A_{k_n}\}_{n\in\mathbb{N}} \cup \{A_m\}$ at the corresponding poles with $\mathcal{G}(z,g)$. In addition assume $\widetilde{\nu}_j = \nu_j$ for all $j \in M\backslash\{l_n\}_{n\in\mathbb{N}}$. Let us also assume zeros and poles of $\widetilde{\mathcal{G}}(z,g)$ satisfy asymptotic properties of Theorem 28. Then we know that $\widetilde{\mathcal{G}}(z,g)$ has the infinite sum representation

$$\left(\prod_{n\in M} \frac{\widetilde{\nu}_n}{\lambda_n}\right) \widetilde{\mathcal{G}}(z,g) = \widetilde{C} + \frac{A_m}{z-\lambda_m} + \sum_{n\in\mathbb{N}} \frac{A_{k_n}}{z-\lambda_{k_n}} \tag{4.29}$$

From (4.28) and (4.29), the difference of $\left[\prod_{n\in M} \nu_n/\lambda_n\right] \mathcal{G}(z,g)$ and $\left[\prod_{n\in M} \widetilde{\nu}_n/\lambda_n\right] \widetilde{\mathcal{G}}(z,g)$ is a real constant, which is zero since $\mathcal{G}(\nu_m, g) = \widetilde{\mathcal{G}}(\nu_m, g) = 0$. This implies uniqueness of $\left[\prod_{n\in M} \nu_n/\lambda_n\right] \mathcal{G}(z,g)$ and hence uniqueness of $\{\nu_{l_n}\}_{n\in\mathbb{N}}$. After unique recovery of the two spectra $\sigma(J_{h_1}(g))$ and $\sigma(J_{h_2}(g))$, the operator J is uniquely determined by Theorem 29. $\square$

5. SUMMARY

In this thesis we considered several versions of the following inverse spectral problem for Schrödinger and Jacobi operators in the third and fourth chapters respectively.

Inverse Spectral Problem. Do one spectrum and partial information on another spectrum and the set of norming constants (or the point masses of the spectral measure) corresponding to the first spectrum uniquely recover the operator?

We answered this question positively in the following settings:

- Schrödinger operator on $(0, \pi)$ with a real-valued L^1- potential in the matching index sets case, Theorems 22, 25.

- Schrödinger operator on $(0, \pi)$ with a real-valued L^1- potential in the non-matching index sets case, Theorems 23, 24, 26, 27.

- Semi-infinite Jacobi operator in the matching index sets case, Theorems 30, 31, 32.

- Semi-infinite Jacobi operator in the non-matching index sets case, Theorems 33, 34, 35.

In the matching index sets case our spectral data consists of $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\nu_n\}_{n\in\mathbb{N}\setminus A}$ and $\{\tau_{\alpha_1}(\lambda_n)\}_{n\in A}$ for $A \subseteq \mathbb{N}$, where $\sigma_{\alpha_1,\beta} = \{\lambda_n\}_{n\in\mathbb{N}}$, $\sigma_{\alpha_2,\beta} = \{\nu_n\}_{n\in\mathbb{N}}$ are two spectra and $\{\tau_{\alpha_1}(\lambda_n)\}_{n\in\mathbb{N}}$ is the set of norming constants corresponding to the first spectrum.

In the non-matching index sets case our spectral data consists of $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\nu_n\}_{n\in\mathbb{N}\setminus A}$ and $\{\tau_{\alpha_1}(\lambda_n)\}_{n\in B}$ for $A, B \subseteq \mathbb{N}$ with some convergence restrictions, where $\sigma_{\alpha_1,\beta} = \{\lambda_n\}_{n\in\mathbb{N}}$, $\sigma_{\alpha_2,\beta} = \{\nu_n\}_{n\in\mathbb{N}}$ are two spectra and $\{\tau_{\alpha_1}(\lambda_n)\}_{n\in\mathbb{N}}$ is the set of norming constants corresponding to the first spectrum.

The main objects we used from the spectral theory are spectral measures, norming constants (Definitions 13, 18) and Weyl m-functions (Definitions 14, 17) for Schrödinger and Jacobi operators respectively. In order to deal with general boundary conditions for Schrödinger operators, we introduced more general m-functions (Definition 16).

Our general approach was to transfer the above stated inverse spectral problem to a complex analysis problem, namely the problem of unique recovery of a certain meromorphic function from a data set consisting of the set of poles, a subset of the set of zeros and a subset of the set of residues. In order to do this we used some well-known properties and infinite product representations of Weyl m-functions (Lemma 1 and equations (4.10), (4.12)). In order to solve the uniqueness problem we obtained, we mainly used Čebotarev type representation results (Theorem 3 and Lemmas 2, 3, 4) and asymptotic properties of eigenvalues (equations (3.4), (3.5), (3.6), (3.7) and Theorem 28). Finally we obtained our inverse spectral results using two-spectra theorems for the corresponding settings (Theorems 6, 7, 29).

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