Mathematics for Business and Social Sciences

Kathryn Bollinger and Vanessa Coffelt

Texas A & M University

About the Authors

Senior/Lead Authors

Kathryn Bollinger, Texas A&M University

Kathryn Bollinger received a Masters of Mathematics degree from Texas A&M University in 1998. She is currently a Instructional Associate Professor in the Mathematics Department at Texas A&M University.

Vanessa Coffelt, Texas A&M University

Vanessa Coffelt received a Masters of Mathematics degree from Kansas State University in 2005. She is currently a Senior Lecturer in the Mathematics Department at Texas A&M University.

Contributing Authors

Angela J. Allen, Texas A&M University

Patrick J. Orchard, Texas A&M University

Acknowledgements

We also acknowledge the support of the following people:

Eric Chuu, Texas A&M University

Adam Deaton, Texas A&M University

Bruce Herbert, Texas A&M University

Greg Klein, Bryan, TX

Samantha Shields, Texas A&M University

Jessica Tripode, Texas A&M University

Jennifer Whitfield, Texas A&M University

A special thank you to all those who allowed themselves to be photographed:

Blake, John, Reid, Haley, Blake, Connor, Gavin, Karsten, Reid, and Mark.

Funding for the creation of this book was provided by Texas A&M University Provost's Office through the following grants: (1) Open Education Resource Grant, Widening Access for all Students in Math 140 and Math 142 with Low-or No-cost Open Education Resources, and (2) Enhancing the Design of Gateway Experiences (EDGE) Grant, Increasing Student Success in Gateway Courses for the Applied Calculus Sequence.

Suggested citation: Bollinger, K. and Coffelt, V. (2020) Mathematics for Business and Social Sciences. Available electronically from https://hdl.handle.net/1969.1/188687.

Cover Design: Kathryn Bollinger and Vanessa Coffelt

Cover and Chapter Images: © Texas A&M University. Available at https://brandguide.tamu.edu/visual-style/photo-video/photo-repository.html.

Chapter Four Image: © Erica Tillman.

Work Cited

Chapter 1 is an adaptation of the following works:

- Applied Finite Mathematics by Rupinder Sekhon, and is used under a CC-BY 3.0 License. Available electronically from http://cnx.org/content/col10613/1.5/.
- Linear Algebra with Applications by W.Keith Nicholson and Lyryx Learning Team, and is used under a CC-BY-NC-SA 4.0 License. Available electronically from https://lyryx.com/linear-algebra-applications/.
- College Algebra by Jay Abramson, and is used under a CC-BY 4.0 License. Available electronically from https://openstax.org/details/books/college-algebra.

Chapter 2 is an adaptation of the following works:

- Precalculus by Carl Stitz and Jeff Zeager, 3rd corrected edition, and is used under a CC-BY-NC-SA 3.0 License. Available electronically from https://www.stitz-zeager.com/.
- College Algebra by Jay Abramson, and is used under a CC-BY 4.0 License. Available electronically from https://openstax.org/details/books/college-algebra.
- Business Precalculus by David Lippman, Version 0.1, and is used under a CC-BY-SA 3.0 License. Available electronically from http://www.opentextbookstore.com/busprecalc/busprecalc.pdf.
- Applied Finite Mathematics by Rupinder Sekhon, and is used under a CC-BY 3.0 License. Available electronically from http://cnx.org/content/col10613/1.5/.
- Pre-Calculus by Amy Givler Chapman, Meagan Hearold, and Jessica Libertini, Version 1.0, and is used under a CC-BY-NC 3.0 License. Available electronically from https://open.umn.edu/opentextbooks/textbooks/apex-precalculus.
- Linear Algebra with Applications by W.Keith Nicholson and Lyryx Learning Team, and is used under a CC-BY-NC-SA 4.0 License. Available electronically from https://lyryx.com/linear-algebra-applications/.
- Elementary Algebra by Lynn Marecek and MaryAnne Anthony-Smith, and is used under a CC-BY 4.0 License. Available electronically from https://openstax.org/details/books/elementary-algebra.
- Intermediate Algebra by Lynn Marecek, and is used under a CC-BY 4.0 License. Available electronically from https://openstax.org/details/books/intermediate-algebra.

Chapter 3 is an adaptation of the following work:

- Applied Finite Mathematics by Rupinder Sekhon, and is used under a CC-BY 3.0 License. Available electronically from http://cnx.org/content/col10613/1.5/.

Chapter 4 is an adaptation of the following works:

- Statistics Using Technology by Kathryn Kozak (Probability Distributions), 2nd ed., and is used under a CC-BY-SA 4.0 License. Available electronically from https://libraryguides.nau.edu/c.php?g=665640.
- College Mathematics for Everyday Life by Maxie Inigo, Jennifer Jameson, Kathryn Kozak, Maya Lanzetta, and Kim Sonier, 2nd ed., and is used under a CC-BY-SA 4.0 License. Available electronically from https://libraryguides.nau.edu/c.php?g=665640.
- Math in Society by David Lippman, Version 2.4, and is used under a CC-BY-SA 3.0 License. Available electronically from http://www.opentextbookstore.com/mathinsociety/.
- Applied Finite Mathematics by Rupinder Sekhon, and is used under a CC-BY 3.0 License. Available electronically from http://cnx.org/content/col10613/1.5/.

About this Textbook

In the text's electronic format, the reader is able to navigate throughout the book using hyperlinks. Hyperlinks can be found in the table of contents, figure and table references, section exercises, and bold example references. In the text's pdf format, bookmarks allow for navigation.

Each chapter of the text is formatted in the following manner.

- Each chapter begins with a list of pre-requisite skills the authors believe will help readers fully understand the topics presented in the chapter.
- Each section begins with a list of the learning objectives the reader will be able to demonstrate upon completion of the section.
- Definitions, processes, and theorems are highlighted using shaded boxes for easy identification.
- At the end of most subsections, a try-it exists for the reader to check their understanding of the topic discussed.
- The following icons are used:
  - denotes pre-requisite skills, also called just-in-time topics
  - denotes discovery based on work in the previous example
  - N denotes reminders from the authors to the reader
  - ! denotes common mistakes made
- Each section ends with a comprehensive set of exercises.
- Each chapter ends with a chapter review.

How to Use Each Chapter for Practice and Review

Just-In-Time

As an introduction to each chapter, readers are provided with a list of topics they will need prior knowledge of as they progress through the chapter. Additional support for these topics can be found in Appendix A of the text.

Try-Its

After most subsections a Try-It provides readers with the opportunity to determine whether or not they can apply the covered concepts to a new problem. The answers to all section Try-Its can be found at the end of the section and before the section exercises begin.

Exercises

Section exercises are broken down into four practice categories: Basic Skills, Intermediate Skills, Mastery, and Communication.

- Basic Skills Practice – These exercises focus on one step or the building blocks of a concept or learning objective. Basic Skills exercises are subdivided based on the learning objectives for the section.

- Intermediate Skills Practice – These exercises focus on situations that include a larger variety of real numbers, as well as multi-step problems. Intermediate Skills exercises are subdivided based on the learning objectives for the section.

- Mastery Practice – These exercises require critical thinking and are not subdivided by learning objective. If readers are able to complete these exercises without use of reference materials, it is assumed they have mastered the learning objectives for the section.

- Communication Practice – These exercises focus on readers explaining concepts in their own words.

Readers are encouraged to begin their practice in a section with the Intermediate Skills Practice. If a reader is struggling with a group of exercises in the Intermediate Skills Practice, then they should go to the corresponding group of exercises in the Basic Skills Practice before moving on to the next group of exercises in the Intermediate Skills Practice. While not every problem in an exercise group needs to be completed by the reader, answers to all exercises (both odd and even) are provided in the back of the text as Appendix B.

Once a reader feels as though they have a solid grasp of the Intermediate Skills Practice, they can move to the Mastery Practice to confirm.

Chapter Reviews

At the end of each chapter, readers are provided with a chapter review organized by performance indicator questions. Each performance indicator question is followed by at least one problem for readers to complete.

Appendix A

Appendix A is an algebra review for readers missing some of the foundational skills necessary to complete the material covered in the rest of the textbook. Appendix A is broken into five sections: Number Sense, Introduction to Algebra, Introduction to Algebraic Expressions, Factoring, and Solving Quadratic Equations. Each section includes Skills Practice exercises for each subsection, whose answers are also included in Appendix B.

Appendix B

Appendix B contains the answers to all exercises in the text.

Contents

Exercise Answers

Chapter 1

1. Matrices

In this chapter we are going to discuss matrices and matrix operations.

The authors recommend all students are comfortable with the topics below prior to beginning this chapter. If you would like additional support on any of these topics, please refer to the Appendix.

A.1 - Simplifying Fractions

A.1 - Decimals

A.1 - Properties of Real Numbers

A.2 - Using Variables and Algebraic Symbols

A.2 - Simplifying Expressions

A.2 - Translating an English Phrase to an Algebraic Expression or Equation

A.2 - Solving Linear Equations with One Variable

A.2 - Using Problem-Solving Strategies

1.1 Basic Matrix Operations

© Photo by Erica Tillman, 2019

A vendor sells hot dogs and corn dogs at three different locations. His total sales (in hundreds) for January and February from the three locations are given in **Table 1.1** below.

	January		February	
	Hot Dogs	Corn Dogs	Hot Dogs	Corn Dogs
Location I	10	8	8	7
Location II	8	6	6	7
Location III	6	4	6	5

Table 1.1: Hot Dog and Corn Dog Sales

When large groups of data are given in tabular form we can use matrices to better organize, combine, and represent the data.

Learning Objectives:
In this section, you will learn basic matrix algebra techniques. Upon completion you will be able to:

- State the dimensions of a matrix.
- Identify specific elements of a matrix given the general notation.
- Use the properties of matrices and matrix operations to determine the dimensions of a resulting matrix.
- Demonstrate addition, subtraction, scalar multiplication, and transposition of matrices.
- Create a matrix from a given scenario.
- Apply matrix algebra to real-world scenarios.

Definition

A **matrix** is often referred to by its **size** (or **dimensions**). A matrix of size $\boldsymbol{m \times n}$ has m rows and n columns. Matrix entries are defined first by row number and then by column number. For example, to locate the **entry in matrix $\boldsymbol{A}$**, identified as $\boldsymbol{a_{ij}}$, we look for the entry in row i, column j. In matrix A, shown below, the entry in row 2, column 3 is a_{23}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A **square matrix** is a matrix with dimensions $\boldsymbol{n \times n}$, meaning that it has the same number of rows as columns. The 3×3 matrix above, A, is an example of a square matrix.

A **row matrix** is a matrix consisting of one row with dimensions $\boldsymbol{1 \times n}$.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix}$$

A **column matrix** is a matrix consisting of one column with dimensions $\boldsymbol{m \times 1}$.

$$C = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$$

■

■ **Example 1** Given matrices A and B below,

a. What are the dimensions of matrix A and matrix B?

b. State the values of entries a_{31}, a_{22}, b_{13}, and b_{42}?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & -2 & 7 \\ 22 & 6 & -3 & 11 \\ 5 & 8 & 9 & 22 \end{bmatrix}$$

Solution:

a. The dimensions of A are 3×3, because there are three rows and three columns. The dimensions of B are 3×4, because there are 3 rows and 4 columns.

b. Entry a_{31} is the value in row 3, column 1 of matrix A, which is 3. The entry a_{22} is the value in row 2, column 2 of matrix A, which is 4. The entry b_{13} is the value in row 1, column 3 of matrix B, which is -2. The entry b_{42} is the value in row 4, column 2 of matrix B, which does not exist, as matrix B only has three rows.

■

Remember, for matrix dimensions and entry identification, the row number comes first, then the column number.

Try It # 1:

Given matrix $C = \begin{bmatrix} 1 & -11 \\ 2 & 0 \\ 3 & 7 \\ 4 & 8 \\ 5 & -20 \end{bmatrix}$, state the values of entries c_{21}, c_{52}, and c_{13}, if they exist.

Adding and Subtracting Matrices

We add or subtract matrices by adding or subtracting corresponding entries. In order to do this, the entries must 'correspond.' Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. The result of adding or subtracting matrices is a matrix of the same size. We can add or subtract a 3×3 matrix and another 3×3 matrix, resulting in a 3×3 matrix. However, we cannot add or subtract a 2×3 matrix and a 3×3 matrix, because some entries in one matrix will not have a corresponding entry in the other matrix.

Definition

Given two matrices, A and B, of the *same size*, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices. The result will be a matrix having the same size as A and B.

$$\boldsymbol{A + B = C}$$

such that

$$a_{ij} + b_{ij} = c_{ij}$$

Definition

Given two matrices, A and B, of the *same size*, the matrix obtained by subtracting the corresponding entries of the two matrices is called the **difference** of the two matrices. The result will be a matrix having the same size as A and B.

$$\boldsymbol{A - B = C}$$

such that

$$a_{ij} - b_{ij} = c_{ij}$$

Consider the sum of A and B, given matrices A and B below.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Notice both A and B are 2×2 matrices. Because the matrices are the same size, we can add corresponding entries and obtain a resulting matrix of the same size, 2×2. For instance, the entry in row 1, column 1 of the matrix A, a_{11}, will be added to the entry in row 1, column 1 of matrix B, b_{11}. We will continue this pattern until all entries have been added.

$$\begin{aligned} A+B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \end{aligned}$$

■ **Example 2** Given matrices A and B below, compute the sum of A and B, if possible.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$$

Solution:
Both A and B are 2×2 matrices. When the matrices are the same size, we can add corresponding entries. In this case, the resulting matrix will be a 2×2 matrix.

$$\begin{aligned} A+B &= \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4+5 & 1+9 \\ 3+0 & 2+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 10 \\ 3 & 9 \end{bmatrix} \end{aligned}$$

■

Consider the difference $A-B$, given matrices A and B below.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 9 & -8 & 7 \\ 6 & 5 & -4 \end{bmatrix}$$

Both A and B are 2×3 matrices. Because the matrices are the same size, we can subtract corresponding entries and the resulting matrix will be a 2×3 matrix.

$$\begin{aligned} A-B &= \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} 9 & -8 & 7 \\ 6 & 5 & -4 \end{bmatrix} \\ &= \begin{bmatrix} a-9 & b-(-8) & c-7 \\ d-6 & e-5 & f-(-4) \end{bmatrix} \\ &= \begin{bmatrix} a-9 & b+8 & c-7 \\ d-6 & e-5 & f+4 \end{bmatrix} \end{aligned}$$

Be careful that you can differentiate your letter b's from your number 6's. If you are not careful, in the previous resulting matrix the entry in row 1, column 2 may be improperly written as 14 (6+8, not b+8).

■ **Example 3** Given matrices A and B below, compute the difference, $A-B$, if possible.

$$A = \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix}$$

Solution:
Notice both A and B are 3×3 matrices, so we can subtract the corresponding entries, and the resulting matrix will be a 3×3 matrix.

$$\begin{aligned} A-B &= \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2-6 & -10-10 & -2+2 \\ 14-0 & 12+12 & 10+4 \\ 4+5 & -2-2 & 2+2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -20 & 0 \\ 14 & 24 & 14 \\ 9 & -4 & 4 \end{bmatrix} \end{aligned}$$

We can verify our results of adding or subtracting matrices by using the TI-84 calculator, as long as all matrix entries are numerical (i.e. not variables).

Place both matrix A and B into your calculator, as seen in **Figures 1.1.2**, **1.1.3**, and **1.1.4**.

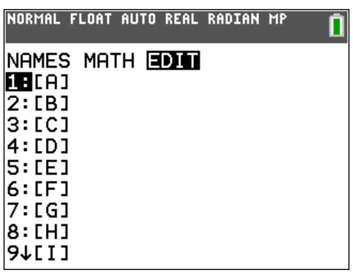


Figure 1.1.2: Calculator screenshot showing matrix menus including EDIT.

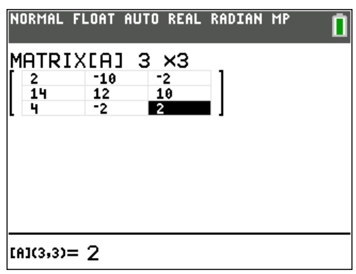


Figure 1.1.3: Calculator screenshot of the input of matrix A.

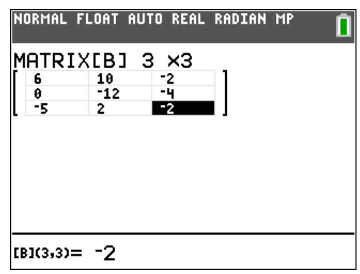


Figure 1.1.4: Calculator screenshot of the input of matrix B.

Next, return to the Home Screen and call and subtract the matrices, as seen in **Figure 1.1.5**

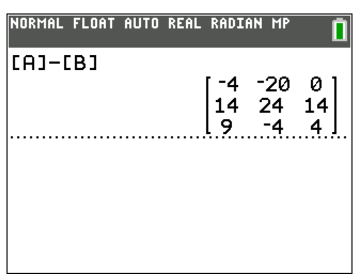


Figure 1.1.5: Calculator screenshot of the subtraction of matrix B from matrix A, $A-B$.

■

Addition works in a similar way. For example, we could add A and B from the example above and get the following result. (See **Figure 1.1.6**.)

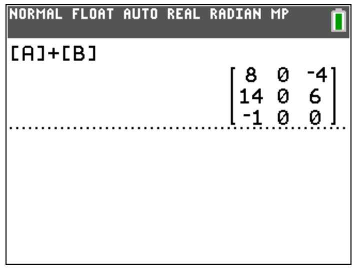


Figure 1.1.6: Calculator screenshot of the the addition of matrix A and matrix B, $A+B$.

N *Remember we cannot use the calculator if one or more of the matrices contains variable entries.*

Similar to addition of real numbers, certain properties hold true for the addition of matrices. A summary of matrix addition properties is given in **Theorem 1.1** below.

Theorem 1.1 Properties of Matrix Addition

- **Commutative Property:** For all $m \times n$ matrices, $A+B=B+A$.
- **Associative Property:** For all $m \times n$ matrices, $(A+B)+C=A+(B+C)=A+B+C$.
- **Identity Property:** If $0_{m \times n}$ is the $m \times n$ matrix whose entries are all 0, then $0_{m \times n}$ is called the ***m×n* additive identity** for all $m \times n$ matrices A and

$$A+0_{m \times n}=0_{m \times n}+A=A$$

.
- **Inverse Property:** For every given $m \times n$ matrix A, there is a unique matrix denoted $-A$, called the **additive inverse of *A***. The additive inverse of A has the same dimensions as A, but all entries of $-A$ are the opposite sign of the corresponding entries of A, such that

$$A+(-A)=(-A)+A=0_{m \times n}$$

.

As with real numbers, matrix subtraction does not hold the same properties as matrix addition. We will leave it to the reader to explore these properties, some of which can be found in Try It #2.

Try It # 2:
Given matrices A and B below,

$$A = \begin{bmatrix} 2 & d \\ 1 & 0 \\ e & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} f & -2 \\ 1 & c \\ -4 & 3 \end{bmatrix}$$

a. State the dimensions of both A and B.
b. Compute and compare $A+B$ and $B+A$.
c. Determine and compare $A-B$ and $B-A$.
d. Evaluate and compare $(A+B)+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $A+\left(B+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\right)$.
e. Evaluate and compare $(A-B)+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $A-\left(B+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\right)$.
f. Find and compare $A+(-A)$ and $(-A)+A$.

Multiplying a Matrix by a Scalar

In the real numbers, the additive inverse of a number, n, is $-1\cdot n$. Similarly, the additive inverse, of a matrix A is $-A$, which can be thought of as $-1\cdot A$.

Definition

If A is any matrix and k is any number, the **scalar multiple kA** is the matrix obtained from multiplying each entry of A by k. This is also known as the *scalar product*. The resulting matrix, kA, is the same size as matrix A. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple kA is

$$\begin{aligned} kA &= k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} k\cdot a_{11} & k\cdot a_{12} \\ k\cdot a_{21} & k\cdot a_{22} \end{bmatrix}. \end{aligned}$$

Scalar multiplication is distributive. For matrices A and B, with scalars k, c, and d,

$$k(A+B) = kA+kB$$

and

$$(c+d)A = cA+dA$$

Consider the scalar product $8A$, given

$$A = \begin{bmatrix} c & d \\ -2 & \frac{3}{7} \end{bmatrix}$$

To determine $8A$, multiply each entry in A by the scalar 8.

$$\begin{aligned} 8A &= 8\begin{bmatrix} c & d \\ -2 & \frac{3}{7} \end{bmatrix} \\ &= \begin{bmatrix} 8\cdot c & 8\cdot d \\ 8\cdot(-2) & 8\cdot\left(\frac{3}{7}\right) \end{bmatrix} \\ &= \begin{bmatrix} 8c & 8d \\ -16 & \frac{24}{7} \end{bmatrix} \end{aligned}$$

■ **Example 4** If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$, compute

a. $5A$

b. $\frac{1}{2}B$

c. $3A - 2B$

Solution:

a. $5A = 5\begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5\cdot 3 & 5\cdot(-1) & 5\cdot 4 \\ 5\cdot 2 & 5\cdot 0 & 5\cdot 1 \end{bmatrix} = \begin{bmatrix} 15 & -5 & 20 \\ 10 & 0 & 5 \end{bmatrix}$

b. $\frac{1}{2}B = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cdot 1 & \frac{1}{2}\cdot 2 & \frac{1}{2}\cdot(-1) \\ \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 3 & \frac{1}{2}\cdot 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{3}{2} & 1 \end{bmatrix}$ or $\begin{bmatrix} 0.5 & 1 & -0.5 \\ 0 & 1.5 & 1 \end{bmatrix}$

c.
$$\begin{aligned} 3A - 2B &= 3\begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3\cdot 3 & 3\cdot(-1) & 3\cdot 4 \\ 3\cdot 2 & 3\cdot 0 & 3\cdot 1 \end{bmatrix} - \begin{bmatrix} 2\cdot 1 & 2\cdot 2 & 2\cdot(-1) \\ 2\cdot 0 & 2\cdot 3 & 2\cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & -3 & 12 \\ 6 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -2 \\ 0 & 6 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 9-2 & -3-4 & 12-(-2) \\ 6-0 & 0-6 & 3-4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -7 & 14 \\ 6 & -6 & -1 \end{bmatrix} \end{aligned}$$

In the solution to part **c**, *we worked the problem by multiplying matrix B by 2 and subtracting the result from matrix $3A$. However, it is also true that $3A - 2B = 3A + (-2)B$. Thus, we could have multiplied matrix B by -2 and then added the result to matrix $3A$ for the same final matrix. In other words, you can either subtract the positive scalar multiple or add the negative scalar multiple.*

Because we are only working with matrices containing numerical values in this example, we can verify our results using the calculator. To multiply by a scalar you multiply the scalar and the matrix on the Home Screen, once the matrix has been entered into the calculator. (See Figures **1.1.7**, **1.1.8**, and **1.1.9**.)

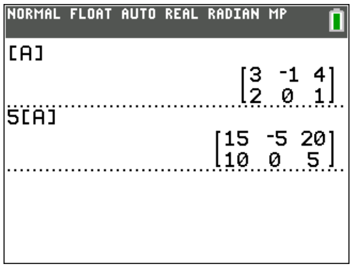


Figure 1.1.7: Calculator screenshot showing the product $5A$ and the resulting matrix.

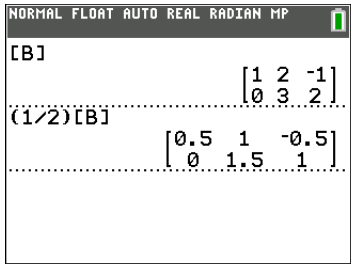


Figure 1.1.8: Calculator screenshot showing the product $1/2B$ and the resulting matrix.

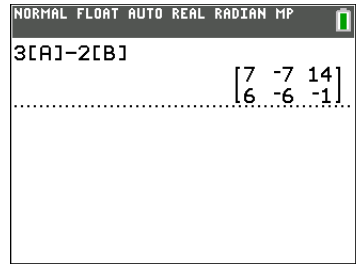


Figure 1.1.9: Calculator screenshot showing the difference $3A-2B$ and the resulting matrix.

■

When subtracting a matrix product, make sure to keep track of positives and negatives. To avoid careless errors it is the authors' recommendation to write out each individual step.

Try It # 3:

Given $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$,

compute each of the following.

a. $-4A+9B$

b. $\frac{1}{2}A-6B$

We have just discussed multiplication of a matrix by a scalar. Multiplying a matrix by another matrix is more complicated and will be discussed in the next section.

Transposing a Matrix

Sometimes it is necessary to reorganize the entries of an $m \times n$ matrix to form a new matrix of size $n \times m$, where the entries in each row now form each column. The following definition is made with such applications in mind.

Definition

If A is an $m \times n$ matrix, the **transpose** of A, written as $\boldsymbol{A^T}$, is the $n \times m$ matrix whose columns are just the rows of A in the same order.

In other words, the first column of A^T is the first row of A (that is, it consists of the entries of row 1 in order). Similarly, the second column of A^T is the second row of A, and so on. ■

■ **Example 5** Consider matrices A, B, C, and D given below.

a. State the size of each matrix.

b. State the transpose of each matrix.

c. State the size of each transpose.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ e & 4 \\ 5 & f \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution:

$$A = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} c_1 \\ \left[\begin{array}{c} 1 \\ {\color{blue}3} \\ {\color{red}2} \end{array}\right] \end{array}$$

a. A has 3 rows and 1 column, so it is 3×1 matrix.

b. $A^T = \begin{array}{c} \\ r_1 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc} 1 & {\color{blue}3} & {\color{red}2} \end{array}\right] \end{array}$

c. A^T has 1 row and 3 columns, so it is 1×3 matrix.

$$B = \begin{array}{c} \\ r_1 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc} 5 & {\color{blue}2} & {\color{red}6} \end{array}\right] \end{array}$$

a. B has 1 row and 3 columns, so it is 1×3 matrix.

b. $B^T = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} c_1 \\ \left[\begin{array}{c} 5 \\ {\color{blue}2} \\ {\color{red}6} \end{array}\right] \end{array}$

c. B^T has 3 rows and 1 column, so it is 3×1 matrix.

$$C = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ e & 4 \\ 5 & f \end{bmatrix} \end{array}$$

a. C has 3 rows and 2 columns, so it is 3×2 matrix.

b. $$C^T = \begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & e & 5 \\ 2 & 4 & f \end{bmatrix} \end{array}$$

c. C^T has 2 rows and 3 columns, so it is 2×3 matrix.

$$D = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix} \end{array}$$

a. D has 3 rows and 3 columns, so it is 3×3 matrix.

b. $$D^T = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix} \end{array}$$

c. D^T has 3 rows and 3 columns, so it is 3×3 matrix.

Due to the fact that we have numerical values for A, B, and D, we can use the calculator to verify these transpositions. Again, begin by entering each matrix in your calculator, as seen in **Figures 1.1.10**, **1.1.11**, and **1.1.12**.

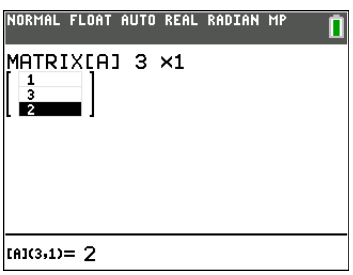


Figure 1.1.10: Calculator screenshot showing matrix A.

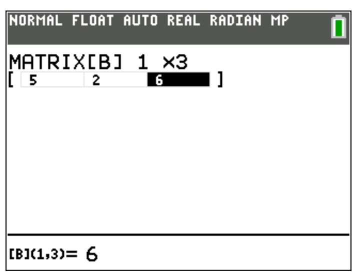


Figure 1.1.11: Calculator screenshot showing matrix B.

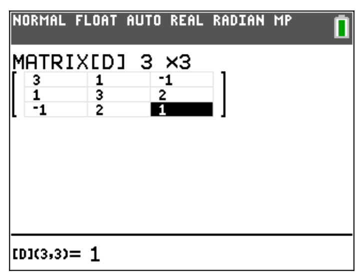


Figure 1.1.12: Calculator screenshot showing matrix D.

To transpose a matrix you will need the Matrix MATH menu. This menu is displayed below. (See **Figure 1.1.13**)

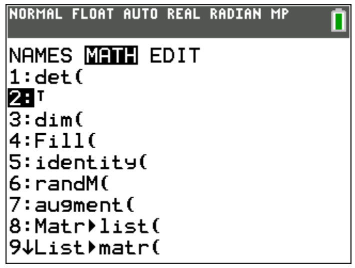


Figure 1.1.13: Calculator screenshot showing matrix menus including MATH.

You will enter the operation on the Home Screen as [matrix name]^T. (See **Figures 1.1.14**, **1.1.15**, and **1.1.16**.)

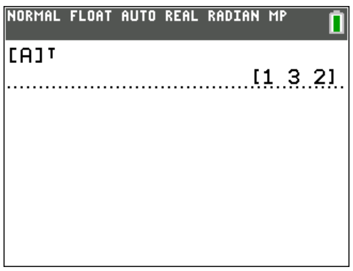


Figure 1.1.14: Calculator screenshot showing the transpose of matrix A.

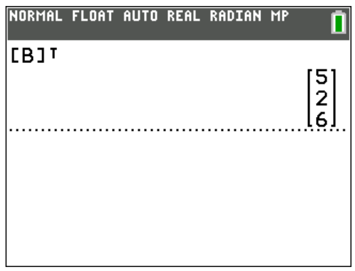


Figure 1.1.15: Calculator screenshot showing the transpose of matrix B.

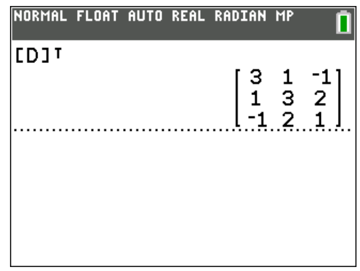


Figure 1.1.16: Calculator screenshot showing the transpose of matrix D.

■

The following theorem contains some useful properties of matrix transpositions.

Theorem 1.2 Properties of Matrix Transposition

Let A and B denote matrices of the same size, and let k denote a scalar.

1. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.
2. $\left(A^T\right)^T = A$
3. $(kA)^T = kA^T$
4. $(A+B)^T = A^T + B^T$

Try It # 4:
State the transpose of matrix C and its dimensions.

$$C = \begin{bmatrix} 2 & 1 & 7 & d \\ 6 & 0 & f & \frac{2}{3} \\ -5 & 8 & \frac{1}{2} & -11 \end{bmatrix}$$

Matrix Equality

Two points (x_1, y_1) and (x_2, y_2) in the plane are equal if and only if they have the same coordinates, that is $x_1 = x_2$ and $y_1 = y_2$. A similar property is true for matrices.

Definition

Two matrices, A and B, are called **equal** (written $A = B$) if and only if:

1. They are the same size.
2. Corresponding entries are equal.

In other words, $A = B$ means $a_{ij} = b_{ij}$ for all i and j. ▪

▪ **Example 6** Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, discuss the possibility that

a. $A = B$
b. $B = C$
c. $A = C$

Solution:

a. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix, while $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$ is a 2×3 matrix. Because A and B are not the same size, it is impossible that $A = B$.

b. Similar to part **a**, it is impossible that $B = C$ as B is a 2×3 matrix and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ is a 2×2 matrix.

c. It is possible that $A = C$ considering both A and C are 2×2 matrices. If $A = C$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

For this to be true, tthe corresponding entries must be equal. In other words, the following must be true:

$$a = 1, \ b = 0, \ c = -1, \ \text{and } d = 2$$

▪

Try It # 5:
Given matrices A and B, determine the values of x, y, and z such that $A = B$.

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & (x-2) \\ y & (y-z) \end{bmatrix}$$

Applying Basic Matrix Operations

All previously discussed matrix operations can be combined into a single algebraic matrix expression or matrix equation.

Solving Equations Involving Basic Matrix Operations

Example 7 Determine the values of a, b, and c, if the following is true.

$$\begin{bmatrix} a & 15 \\ b+6 & -2 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 24 & -32 \\ 64 & -8 \end{bmatrix}^T = \begin{bmatrix} b & 7 \\ 11 & 3c \end{bmatrix}$$

Solution:
In order to reduce any careless errors, we will work step-by-step, using the order of operations. (Take note that all matrices are 2×2, and, thus, all matrix operations are possible.)

The first operation we will perform is the transpose.

$$\begin{bmatrix} a & 15 \\ b+6 & -2 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 24 & 64 \\ -32 & -8 \end{bmatrix} = \begin{bmatrix} b & 7 \\ 11 & 3c \end{bmatrix}$$

Next, we will multiply the second matrix by the scalar, $\frac{1}{8}$.

$$\begin{bmatrix} a & 15 \\ b+6 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 8 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} b & 7 \\ 11 & 3c \end{bmatrix}$$

Now, we will subtract the two matrices on the left.

$$\begin{bmatrix} a-3 & 15-8 \\ (b+6)-(-4) & -2-(-1) \end{bmatrix} = \begin{bmatrix} b & 7 \\ 11 & 3c \end{bmatrix}$$

Simplification gives us

$$\begin{bmatrix} a-3 & 7 \\ b+10 & -1 \end{bmatrix} = \begin{bmatrix} b & 7 \\ 11 & 3c \end{bmatrix}$$

Setting corresponding entries equal and solving where possible results in

$$\begin{aligned} a-3 &= b \\ a &= b+3 \end{aligned} \qquad 7 = 7 \checkmark \qquad \begin{aligned} b+10 &= 11 \\ b &= 11-10 \\ b &= 1 \end{aligned} \qquad \begin{aligned} -1 &= 3c \\ -\frac{1}{3} &= c \end{aligned}$$

Since $b = 1$, then $a = b + 3 = 1 + 3 = 4$.

Thus, for the matrix equation to be true, $a = 4$, $b = 1$, and $c = -\frac{1}{3}$.

The reader should verify these values indeed satisfy the original equation.

■

Try It # 6:
Determine the values of a, b, and c, if $\begin{bmatrix} a & 5 & c \end{bmatrix} + \begin{bmatrix} -1 & b & c \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$.

■ **Example 8** Solve the following matrix equation for matrix X.

$$\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

Solution:
We solve the equation $a + x = b$ for x by subtracting a from both sides of the equals sign to obtain $x = b - a$ (as was shown in the solution to the previous example). This also works for matrices.

To solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, simply subtract the matrix $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ from both sides to get

$$\begin{aligned} X &= \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-3 & 0-2 \\ -1-(-1) & 2-1 \end{bmatrix} \\ X &= \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

■

The solution in the previous example solves the single matrix equation $A + X = B$ for X directly via matrix subtractions: $X = B - A$. This ability to work with matrices as a single unit, rather than individual entries, lies at the heart of matrix algebra.

It is important to note that the sizes of matrices involved in some calculations are often determined by the context. For example, if

$$A + C = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

then A and C must be the same size (so that $A + C$ makes sense), and that size must be 2×3 (so that the sum is 2×3). For simplicity we shall often omit reference to such facts when they are clear from the context.

Try It # 7:
Solve the following matrix equation for matrix X. Assume matrices B, C, D, and X are all the same size.

$$3X - (B + 5X) = C + \frac{1}{5}D$$

Using Matrices to Solve Real-World Scenarios

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs, due to increased enrollment. They estimate that 15% more supplies are needed in both labs. The school's current inventory is displayed in **Table 1.2**.

	Lab A	Lab B
Computers	15	27
Computer Tables	16	34
Chairs	16	34

Table 1.2: School's Current Inventory

Converting the current inventory data to a matrix, we have

$$C = \begin{array}{l} \\ \text{computers} \\ \text{tables} \\ \text{chairs} \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} \end{array}$$

To compute the *additional* 15% needed, we multiply all of the entries in matrix C by 0.15.

$$\begin{aligned} E &= (0.15)C \\ &= \begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix} \\ &= \begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix} \end{aligned}$$

Due to the fact that we cannot have a partial piece of a computer, table, or chair, we must round each decimal entry up to the next integer. Thus, the amount of extra supplies needed in each lab is

$$E = \begin{array}{l} \\ \text{computers} \\ \text{tables} \\ \text{chairs} \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} \end{array}$$

Adding the additional supplies needed to the current inventory, as shown below, gives us the new inventory amounts for each lab.

$$\begin{aligned} N &= C + E \\ &= \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix} \end{aligned}$$

This means

$$N = \begin{array}{l} \\ \text{computers} \\ \text{tables} \\ \text{chairs} \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix} \end{array}$$

Thus, to accommodate the increased enrollment, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

■ **Example 9** Using the hot dog/corn dog scenario from the beginning of the section, construct two matrices, J and F, to represent the sales (in hundreds) for each month at each location.

a. Compute $J + F$, and interpret your results.
b. Compute $J - F$, and interpret your results.

Solution:
Converting the given data into matrices, based upon monthly sales (in hundreds), we have

$$J = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 10 & 8 \\ 8 & 6 \\ 6 & 4 \end{bmatrix} \end{array} \quad \text{and} \quad F = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 8 & 7 \\ 6 & 7 \\ 6 & 5 \end{bmatrix} \end{array}$$

a. $$J + F = \begin{bmatrix} 10 & 8 \\ 8 & 6 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} 8 & 7 \\ 6 & 7 \\ 6 & 5 \end{bmatrix} = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 18 & 15 \\ 14 & 13 \\ 12 & 9 \end{bmatrix} \end{array}$$

We have added the sales of hot dogs and corn dogs at each location in January to the corresponding sales at each location in February. The resulting matrix means that over the months of January and February,

1800 hot dogs and 1500 corn dogs were sold at Location I.

1400 hot dogs and 1300 corn dogs were sold at Location II.

1200 hot dogs and 900 corn dogs were sold at Location III.

b. $J - F = \begin{bmatrix} 10 & 8 \\ 8 & 6 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 7 \\ 6 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 0 & -1 \end{bmatrix}$

We have subtracted the sales of hot dogs and corn dogs at each location in February from the corresponding sales at each location in January. The resulting matrix gives the change in sales over the two months, with positive values indicating how many more items were sold in January, negative values indicating how many more items were sold in February, and values of zero indicating no change in sales over the two months. In other words,

At Location I, 200 more hot dogs and 100 more corn dogs were sold in January than in February.

At Location II, 200 more hot dogs were sold in January than in February, but 100 more corn dogs were sold in February than in January.

At Location III, the same amount of hot dogs were sold in January and February, but 100 more corn dogs were sold in February than in January.

Applications of matrices can be used in a more realistic sense with larger sets of data or once matrix multiplication is defined. We will see real-world situations involving matrix multiplication in the next section.

▪

Try It Answers

1. $c_{21} = 2, c_{52} = -20, c_{13}$ does not exist

2. **a.** A is a 3×2 matrix; B is a 3×2 matrix

   b. $A + B = \begin{bmatrix} 2+f & d-2 \\ 2 & c \\ e-4 & 0 \end{bmatrix}$

   $B + A = \begin{bmatrix} f+2 & -2+d \\ 2 & c \\ -4+e & 0 \end{bmatrix}$

   $A + B = B + A$

   c. $A - B = \begin{bmatrix} 2-f & d+2 \\ 0 & -c \\ e+4 & -6 \end{bmatrix}$

   $B - A = \begin{bmatrix} f-2 & -2-d \\ 0 & c \\ -4-e & 6 \end{bmatrix}$

   $A - B \neq B - A$ (Matrix subtraction is NOT commutative.)

d. $(A+B)+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3+f & d \\ 5 & c+4 \\ e+1 & 6 \end{bmatrix}$

$A+\left(B+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\right) = \begin{bmatrix} 3+f & d \\ 5 & c+4 \\ e+1 & 6 \end{bmatrix}$

Let $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $(A+B)+C = A+(B+C)$.

e. $(A-B)+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3-f & d+4 \\ 3 & -c+4 \\ e+9 & 0 \end{bmatrix}$

$A-\left(B+\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\right) = \begin{bmatrix} 1-f & d \\ -3 & -c-4 \\ e-1 & -12 \end{bmatrix}$

Let $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $(A-B)+C \neq A-(B+C)$.

f. $A+(-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(-A)+A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A+(-A) = (-A)+A = 0_{3\times 2}$

3. $-4A+9B = \begin{bmatrix} -13 & 26 & 9 \\ 0 & -23 & 10 \\ -16 & -3 & -12 \end{bmatrix}$

$\frac{1}{2}A-6B = \begin{bmatrix} \frac{13}{2} & -13 & -6 \\ 0 & \frac{35}{2} & -11 \\ 2 & -\frac{9}{2} & 21 \end{bmatrix}$

4. $C^T = \begin{bmatrix} 2 & 6 & -5 \\ 1 & 0 & 8 \\ 7 & f & \frac{1}{2} \\ d & \frac{2}{3} & -11 \end{bmatrix}$; C^T is a 4×3 matrix.

5. $x=6, y=3, z=-4$

6. $a=4, b=-3, c=-\frac{1}{2}$

7. $X = -\frac{1}{2}\left(C+\frac{1}{5}D+B\right)$

EXERCISES

BASIC SKILLS PRACTICE (Answers)

For Exercises 1 - 7, use the given matrices below.

$$A = \begin{bmatrix} 2 & 4 \\ -7 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -8 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 0.5 & -3 & 0.15 \\ 10 & -1 & -0.25 \end{bmatrix} \qquad D = \begin{bmatrix} -6 & 0 \\ -4 & 9 \\ 0.3 & -2 \\ 1 & 7 \end{bmatrix}$$

1. State the dimensions of each matrix.
2. If it exists, state the value of entry a_{11}.
3. If it exists, state the value of entry b_{13}.
4. If it exists, state the value of entry c_{21}.
5. If it exists, state the value of entry d_{24}.
6. If it exists, state the value of entry b_{31}.
7. If it exists, state the value of entry d_{42}.

For Exercises 8 - 15, use the given information about the sizes of matrices A, B, C, D, and E to determine the size of the resulting matrix, if the computation is possible.

$$A \text{ is a } 2\times 3, \quad B \text{ is a } 2\times 2, \quad C \text{ is a } 1\times 2, \quad D \text{ is a } 3\times 2, \quad E \text{ is a } 2\times 2$$

8. $B+E$
9. $A-D$
10. $4C$
11. C^T
12. $E-B$
13. $3C+D$
14. A^T-D
15. $5D^T+A$

For Exercises 16 - 39, use the given matrices below to compute the indicated operation, if possible.

$$A = \begin{bmatrix} 2 & 4 \\ -7 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -8 \\ 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 0.5 & -3 \\ 0.15 & 10 \\ -1 & -0.25 \end{bmatrix} \qquad D = \begin{bmatrix} -6 & 0 \\ -4 & 9 \\ 0.3 & -2 \end{bmatrix}$$

16. $A+B$
17. $A-B$
18. $B+A$
19. $B-A$
20. $3A$
21. $\frac{1}{2}B$
22. $-2A+4B$
23. $5A-B$
24. A^T
25. $4B^T$
26. $(A+B)^T$
27. A^T-B
28. $C+D$
29. $C-D$
30. $D+C$
31. $D-C$
32. $100C$
33. $-6D$
34. $10C-10D$
35. $D+5C$
36. C^T
37. $0.1D^T$
38. $(D-C)^T$
39. $10C^T+D$

For Exercises 40 - 43, determine the value of *a*, *b*, *c*, and *d*.

40. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

41. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -2\begin{bmatrix} 1 & 2 \\ -3 & 9 \end{bmatrix}$

42. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^T$

43. $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 3\begin{bmatrix} 4 & 0 \\ -8 & 1 \end{bmatrix}$

For Exercises 44 - 45, write a matrix that represents the data given.

44. A nutrition expert publishes an article about the breakdown of a breakfast in each of four fad diets: the Carb Buster, the Fat Meltway, the Veggie Maximum, and the More Meat. A Carb Buster diet breakfast requires 5 grams of fat, 0 grams of carbohydrates, and 7 grams of protein. A Fat Meltway breakfast requires 0 grams of fat, 5 grams of carbohydrates, and 15 grams of protein. The Veggie Maximum breakfast needs 10 grams of fat, 9 grams of carbohydrates, and 0 grams of protein. The More Meat breakfast has 6 grams of fat, 0 grams of carbohydrates, and 12 grams of protein. Write a 4×3 matrix representing the information, as well as a 3×4 matrix that also represents the information.

45. The Bryan-College Station area has 3 HEB stores near the university campus. The Jones Crossing location stocks 100 lbs of bananas, 20 lbs of guava fruit, 90 lbs of apples, and 150 lbs of oranges each week. The Texas Avenue location stocks 70 lbs, 10 lbs, 120 lbs, and 75 lbs of bananas, guava fruit, apples, and oranges respectively. The Villa Maria location stocks 110 lbs, 35 lbs, 180 lbs, and 100 lbs of bananas, guava fruit, apples, and oranges respectively. Write a 3×4 matrix to represent the weekly stock in all three HEBs, as well as a 4×3 matrix.

Intermediate Skills Practice (Answers)

For Exercises 46 - 52, use the given matrices below.

$$A = \begin{bmatrix} 2 \\ 4s \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} w & 1 & z \\ t & y & -5 \\ -9 & 0 & x \end{bmatrix} \qquad C = \begin{bmatrix} e & \frac{1}{2} \\ \frac{1}{4} & f \\ 6 & -11 \end{bmatrix} \qquad D = \begin{bmatrix} 100g \end{bmatrix}$$

46. State the dimensions of each matrix.

47. If it exists, state the value of entry c_{22}.

48. If it exists, state the value of entry d_{12}.

49. If it exists, state the value of entry a_{31}.

50. If it exists, state the value of entry b_{23}.

51. If it exists, state the value of entry c_{13}.

52. If it exists, state the value of entry d_{11}.

For Exercises 53 - 67, use the given matrices below to compute the indicated operation, if possible.

$$A = \begin{bmatrix} 2 & x \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 3y & 1 & 2 \\ z & 1 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 \\ 2 & k \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ g & 4h \end{bmatrix} \qquad E = \begin{bmatrix} 1 & p & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

53. $5C$

54. $B+D$

55. $A-C$

56. $3E^T$

57. $3A+2C$

58. $2B-3E$

59. $4A+2B$

60. $A-D$

61. $4A^T-3C$

62. $(A+C)^T$

63. $(B-2E)^T$

64. $0.25D$

65. $B+E+D^T$

66. $5E^T-2D$

67. A^T+C^T

For Exercises 68 - 70, determine the value of a, b, c, and d.

68. $\begin{bmatrix} 0 & a & 1 \\ c & -5 & 0 \end{bmatrix} = 5\begin{bmatrix} 0 & 3 & b \\ 1 & -1 & d \end{bmatrix}$

69. $3\begin{bmatrix} a & 0 \\ -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 6 & -2 \\ b & c \end{bmatrix} = 7\begin{bmatrix} 8 & d \\ 0 & 1 \end{bmatrix}$

70. $\begin{bmatrix} 3 & b & 2 & 0 \end{bmatrix} + 4\begin{bmatrix} a & 3 & 1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 11 & c & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 3 & d \end{bmatrix}$

For Exercises 71 - 72, solve the matrix equation for matrix A.

71. $5A - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3A - \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$

72. $A - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5A - 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

For Exercises 73 - 74, use the following scenario.

> In the small village of Pedimaxus in the country of Sasquatchia, all 150 residents get one of the two local newspapers. Market research has shown that in any given week, 90% of those who subscribe to the Pedimaxus Tribune want to keep receiving it, but 10% want to switch to the Sasquatchia Picayune. Of those who receive the Picayune, 80% want to continue with it and 20% want to switch to the Tribune.

73. In the context of the scenario, what would the columns of matrix P represent?

$$P = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix}$$

74. If 80 people subscribe to the Tribune and 70 people subscribe to the Picayune, write a 2×1 matrix to represent the number of people who subscribe to each paper.

Mastery Practice (Answers)

For Exercises 75 - 84, use the given matrices below.

$A = \begin{bmatrix} x & 2 \\ 3y & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 2 \\ -3 & -5w \end{bmatrix}$ $\quad C = \begin{bmatrix} 10 & e & -1 \\ f & 5 & 9 \end{bmatrix}$ $\quad D = \begin{bmatrix} g & 8 \\ 10 & 5 \\ -h & k \end{bmatrix}$

75. State the value of $a_{21} - b_{22} + 3d_{32}$.

76. Compute, if possible, $4B - 3A$, when $w = 2$.

77. Compute, if possible, $\left(C^T\right)^T$.

78. If $E = A^T + B$, state e_{21}.

79. State the value of d_{24}.

80. State the value of b_{22}.

81. State the value of d_{31}.

82. If $A = B$, determine the value of w, x, and y.

83. If $C^T = D$, determine the value of e, f, g, h, and k.

84. If $\begin{bmatrix} c & -1 \\ 2 & 1 \end{bmatrix}^T - 2\begin{bmatrix} 1 & -a \\ d & 3 \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ -5 & b \end{bmatrix}$, determine the value of a, b, c, and d.

For Exercises 85 - 87, solve the matrix equation for matrix X, assuming matrices X, A, and B are all the same size.

85. $X + A = 3X + 2A$

86. $2X - B = 5(X + 2B)$

87. $\frac{1}{2}A + 6X = B + 2X$

88. A study on teaching methods in Math 140 was conducted over two years in five sections. The first year traditional teaching methods were used and resulted in class averages in the fall of 78.2, 71.5, 74.3, 73.8, and 76.9, while the spring semester had averages of 72.2, 70.5, 69.8, 71.8, and 73.4, respectively. The second year active learning methods were used by the same teachers with resulting averages of 79.4, 73.8, 71.9, 75.1, and 76.9 in the fall and 71.6, 72.7, 73.1, 72.8, and 74.9 in the spring, respectively.

    (a) Write a 2×5 matrix for the averages during the first year and a second 2×5 matrix for the averages during the second year.

    (b) Use matrix subtraction to find the change in the averages based on the two teaching techniques, from the first to the second year.

    (c) If the study was hoping for a 2% increase in averages after the change in teaching methods, what should the class averages have been the second year to achieve their goal?

COMMUNICATION PRACTICE (Answers)

89. Explain why your answers to Exercises 62 and 67 are equal.

90. Determine the values of a, b, c, and d if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & c \\ d & a \end{bmatrix}$, and explain your answer.

1.2 Matrix Multiplication

© Photo provided by MarkyB, 1995

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

Table 1.3: Soccer Teams' Equipment Needs

The Wildcats and Mud Cats need new equipment. If a goal costs \$300, a ball costs \$10, and a jersey costs \$30, how can we find the total cost for the equipment needed for each team? We will discover a method in which the data in the soccer equipment table can be displayed, as a matrix, and used for calculating other information. Then, we will be able to calculate the cost of the equipment.

Learning Objectives:

In this section, you will learn matrix multiplication and its proper usage. Upon completion you will be able to:

- State whether or not the product of two matrices is defined.
- Demonstrate matrix multiplication.
- Apply multiplication of matrices to real-world applications.

Multiplying Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices, AB, is only possible when the *inner dimensions are the same*, meaning that the number of columns of the first matrix, A, is equal to the number of rows of the second matrix, B. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product matrix AB is defined and is an $m \times n$ matrix.

$$\begin{array}{ccccc} A & \cdot & B & = & AB \\ (m \times r) & & (r \times n) & & (m \times n) \end{array}$$

$$= \checkmark$$

Example 1 If A is a 2×4 matrix, B is a 2×3 matrix, and C is a 4×3 matrix, determine whether or not the following matrix products are defined. If so, give the size of the resulting matrix, and if not show why not.

a. AC
b. AB
c. CA
d. CB^T

Solution:

a.

$$\begin{array}{ccc} A & \cdot & C \\ (2\times 4) & & (4\times 3) \\ & = \checkmark & \end{array} \quad \Rightarrow AC \text{ is a } 2\times 3 \text{ matrix}$$

b.

$$\begin{array}{ccc} A & \cdot & B \\ (2\times 4) & & (2\times 3) \\ & \neq & \end{array} \quad \Rightarrow AB \text{ is not defined (the inner dimensions do not match)}$$

c.

$$\begin{array}{ccc} C & \cdot & A \\ (4\times 3) & & (2\times 4) \\ & \neq & \end{array} \quad \Rightarrow CA \text{ is not defined (the inner dimensions do not match)}$$

d.

$$\begin{array}{ccc} C & \cdot & B^T \\ (4\times 3) & & (3\times 2) \\ & = \checkmark & \end{array} \quad \Rightarrow CB^T \text{ is a } 4\times 2 \text{ matrix}$$

Recall that transposing a matrix interchanges rows and columns. Because B is a 2×3 matrix, its transpose, B^T, is a 3×2 matrix.

■

In the example above, the matrix product AC was defined, but the matrix product CA was not defined. The order you multiply matrices is important. Matrix multiplication is NOT commutative!

To actually find the matrix product AB, we multiply entries of A with entries of B according to a specific pattern, as outlined below.

If $D = AB$, then to compute the entry d_{ij} proceed as follows:

Going *across* row i of A, the left matrix, and *down* column j of B, the right matrix, multiply corresponding entries, and add the results.

$$\begin{array}{ccccc} & A & B & & D = AB \\ & & c_j & & \\ AB = r_i & \begin{bmatrix} \rightarrow \end{bmatrix} & \begin{bmatrix} \downarrow \end{bmatrix} & = & \begin{bmatrix} \; d_{ij} \; \end{bmatrix} \\ & \text{row } i & \text{column } j & & \end{array}$$

The process of matrix multiplication requires that the rows of A must be the same length as the columns of B, which is why the inner dimensions must be the same for the matrix product to be defined.

Matrix Multiplication Process

Given matrices A and B, where the dimensions of A are 2×3 and the dimensions of B are 3×3, the product of A and B will be a 2×3 matrix.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\underset{(2\times 3)}{A} \cdot \underset{(3\times 3)}{B} \Longrightarrow \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

$$= \checkmark$$

To obtain each entry in row i of AB, we multiply the entries in row i of A by the corresponding entries in column j of B, and add the products.

1. To obtain the entry in row 1, column 1 of AB, multiply the entries in the first row of A by the corresponding entries in the first column of B, and add, as shown below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+a_{13}\cdot b_{31}$$

2. To obtain the entry in row 1, column 2 of AB, multiply the entries in the first row of A by the corresponding entries in the second column of B, and add, as shown below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}\cdot b_{12}+a_{12}\cdot b_{22}+a_{13}\cdot b_{32}$$

3. To obtain the entry in row 1, column 3 of AB, multiply the entries in the first row of A by the corresponding entries in the third column of B, and add, as shown below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = a_{11}\cdot b_{13}+a_{12}\cdot b_{23}+a_{13}\cdot b_{33}$$

We proceed the same way to obtain the second row of AB. In other words, row 2 of A times column 1 of B; row 2 of A times column 2 of B; row 2 of A times column 3 of B. When complete, the product matrix will be

$$AB=\begin{bmatrix} \underline{a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+a_{13}\cdot b_{31}} & \underline{a_{11}\cdot b_{12}+a_{12}\cdot b_{22}+a_{13}\cdot b_{32}} & \underline{a_{11}\cdot b_{13}+a_{12}\cdot b_{23}+a_{13}\cdot b_{33}} \\ \underline{a_{21}\cdot b_{11}+a_{22}\cdot b_{21}+a_{23}\cdot b_{31}} & \underline{a_{21}\cdot b_{12}+a_{22}\cdot b_{22}+a_{23}\cdot b_{32}} & \underline{a_{21}\cdot b_{13}+a_{22}\cdot b_{23}+a_{23}\cdot b_{33}} \end{bmatrix}$$

The notation above can be overwhelming to the reader. We will now provide several numerical examples to help the reader understand this process.

■ **Example 2** Given matrices A and B, determine AB, if possible.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution:

First, we check the dimensions of the matrices.

$$\begin{array}{ccc} A & \cdot & B \\ (2\times 2) & & (2\times 2) \end{array}$$

$$= \checkmark$$

The inner dimensions are the same so we can perform the multiplication. The product will be a 2×2 matrix.

We perform the operations outlined previously.

$$\begin{aligned}
AB &= \begin{array}{c} r_1 \\ {\color{blue}r_2} \end{array} \begin{bmatrix} 1 & 2 \\ {\color{blue}3} & {\color{blue}4} \end{bmatrix} \begin{array}{c} \begin{array}{cc} {\color{red}c_1} & {\color{green}c_2} \end{array} \\ \begin{bmatrix} {\color{red}5} & {\color{green}6} \\ {\color{red}7} & {\color{green}8} \end{bmatrix} \end{array} \\
&= \begin{bmatrix} \underset{r_1c_1}{\underline{\qquad\qquad}} & \underset{r_1c_2}{\underline{\qquad\qquad}} \\ \underset{r_2c_1}{\underline{\qquad\qquad}} & \underset{r_2c_2}{\underline{\qquad\qquad}} \end{bmatrix} \\
&= \begin{bmatrix} 1{\color{red}(5)}+2{\color{red}(7)} & 1{\color{green}(6)}+2{\color{green}(8)} \\ {\color{blue}3}{\color{red}(5)}+{\color{blue}4}{\color{red}(7)} & {\color{blue}3}{\color{green}(6)}+{\color{blue}4}{\color{green}(8)} \end{bmatrix} \\
&= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}
\end{aligned}$$

■

■ **Example 3** Given A and B, below

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$

calculate the following, if possible.

a. AB

b. BA

Solution:

a. First, we check the dimensions of the matrices.

$$\begin{array}{ccc} A & \cdot & B \\ (2\times 3) & & (3\times 2) \end{array}$$

$$= \checkmark$$

The inner dimensions are the same so we can perform the multiplication; the result is a 2×2 matrix.

We perform the operations outlined previously.

$$
\begin{aligned}
AB &= \begin{array}{c} r_1 \\ r_2 \end{array}\begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \end{array} \\
&= \begin{bmatrix} \overline{\qquad r_1c_1 \qquad} & \overline{\qquad r_1c_2 \qquad} \\ \overline{\qquad r_2c_1 \qquad} & \overline{\qquad r_2c_2 \qquad} \end{bmatrix} \\
&= \begin{bmatrix} (-1)(5)+2(-4)+3(2) & (-1)(-1)+2(0)+3(3) \\ 4(5)+0(-4)+5(2) & 4(-1)+0(0)+5(3) \end{bmatrix} \\
&= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix}
\end{aligned}
$$

b. First, we check the dimensions of the matrices.

$$
\begin{array}{ccc} B & \cdot & A \\ (3\times \underbrace{2) & & (2}_{=\checkmark} \times 3) \end{array}
$$

The inner dimensions are the same so we can perform the multiplication. The product will have dimensions 3×3.

We perform the operations outlined previously.

$$
\begin{aligned}
BA &= \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \end{array} \\
&= \begin{bmatrix} \overline{\quad r_1c_1 \quad} & \overline{\quad r_1c_2 \quad} & \overline{\quad r_1c_3 \quad} \\ \overline{\quad r_2c_1 \quad} & \overline{\quad r_2c_2 \quad} & \overline{\quad r_2c_3 \quad} \\ \overline{\quad r_3c_1 \quad} & \overline{\quad r_3c_2 \quad} & \overline{\quad r_3c_3 \quad} \end{bmatrix} \\
&= \begin{bmatrix} 5(-1)+(-1)(4) & 5(2)+(-1)(0) & 5(3)+(-1)(5) \\ (-4)(-1)+0(4) & (-4)(2)+0(0) & (-4)(3)+0(5) \\ 2(-1)+3(4) & 2(2)+3(0) & 2(3)+3(5) \end{bmatrix} \\
&= \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix}
\end{aligned}
$$

In this example the products AB and BA are not equal.

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA$$

This, again, illustrates the fact that matrix multiplication is not commutative.

■

When performing matrix multiplication, each entry in the product matrix will be found by adding multiplied terms. The number of terms added together for each entry is equal to the matched inner dimensions. For instance, when finding AB above, the inner matched dimension was 3, and three terms were added together to find each entry. However, with BA, the inner dimension was 2 which is equal to the number of terms added to find each entry of the product.

We can verify our results of matrix multiplication by using the TI-84 calculator, as long as all entries are numerical.

Using the matrices from the previous example, enter matrices A and B into the calculator. (See **Figures 1.2.2** and **1.2.3**)

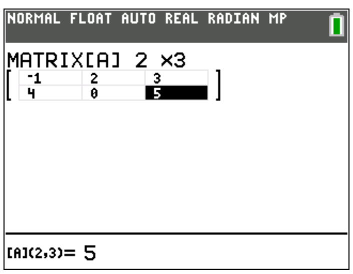


Figure 1.2.2: Calculator screenshot of the input of matrix A.

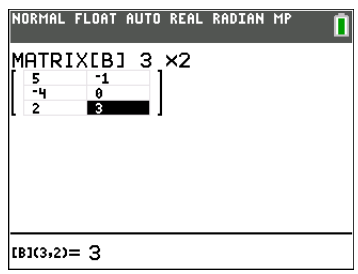


Figure 1.2.3: Calculator screenshot of the input of matrix B.

Return to the Home Screen, and call up and multiply the matrices, as seen in **Figures 1.2.4** and **1.2.5**.

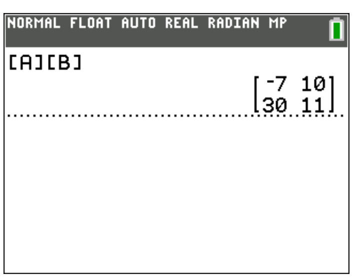


Figure 1.2.4: Calculator screenshot of the matrix product AB.

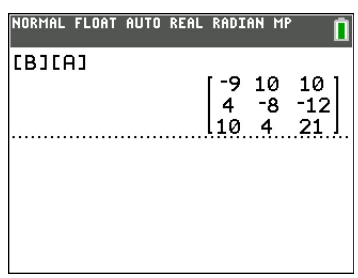


Figure 1.2.5: Calculator screenshot of the matrix product BA.

■ **Example 4** Determine BA, using technology, if

$$A = \begin{bmatrix} 1 & 5 & -2 \\ 9 & 3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -9 & 0 \\ -4 & 7 & 2 \\ 10 & 1 & 8 \end{bmatrix}.$$

Solution:

We can begin by entering the matrices A and B in the calculator. The product BA is given below in **Figure 1.2.6**.

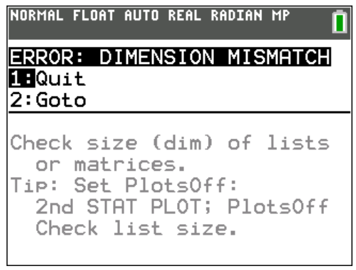


Figure 1.2.6: Calculator screenshot, including a dimension mismatch error, when the product BA is entered.

"Dimension mismatch" means the inner dimensions are not equal. Therefore, the product, BA, is not defined.

■

To save the time used when entering matrices into the calculator, the authors would encourage the readers to begin by checking the sizes of the matrices, by hand, before trying matrix multiplication on the calculator.

■ **Example 5** Determine the product AB given

$$A = \begin{bmatrix} 1 & a & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 0 \\ -11 & b \\ 10 & -4 \end{bmatrix}.$$

Solution:

Checking the dimensions first, we notice the product is defined and the result will be a 1×2 matrix.

$$\begin{array}{ccc} A & \cdot & B \\ (1 \times \underbrace{3) & & (3}_{= \checkmark} \times 2) \end{array} \implies [\underline{\qquad\qquad} \quad \underline{\qquad\qquad}]$$

Because these matrices do not both contain all numerical values, we must find AB by hand. So, using the rows of A and columns of B, we obtain

$$\begin{aligned}
AB &= \begin{array}{cc} & \\ r_1 & \begin{bmatrix} 1 & a & 4 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} 7 & 0 \\ -11 & b \\ 10 & -4 \end{bmatrix} \end{array} \\
&= \begin{bmatrix} \underline{\hspace{2cm}}_{r_1c_1} & \underline{\hspace{2cm}}_{r_1c_2} \end{bmatrix} \\
&= \begin{bmatrix} (1)(7)+a(-11)+4(10) & (1)(0)+a(b)+4(-4) \end{bmatrix} \\
&= \begin{bmatrix} 7-11a+40 & 0+ab-16 \end{bmatrix} \\
&= \begin{bmatrix} 47-11a & ab-16 \end{bmatrix}
\end{aligned}$$

■

Try It # 1:
Determine the matrix products below, if possible, given

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix}.$$

a. AB
b. BA

Similar to real number multiplication, matrix multiplication, when defined, has the following properties.

Theorem 1.3 Properties of Matrix Multiplication

For matrices A, B, and C, the following properties hold.

- Matrix multiplication is associative: $(AB)C = A(BC)$
- Matrix multiplication is distributive: $C(A \pm B) = CA \pm CB$

and

$$(A \pm B)C = AC \pm BC$$

Remember, *matrix* multiplication is not commutative.

■ **Example 6** Use the stated matrices and corresponding dimensions to determine the dimensions of each operation, if possible.

$$A \text{ is } 2\times 2, \quad B \text{ is } 3\times 4, \quad C \text{ is } 2\times 3, \quad D \text{ is } 4\times 2$$

a. $CB + D^T$
b. $4A - DC$
c. $(3A - 6A)C$
d. CBA

Solution:
The following properties must be remembered when determining whether or not the given operations are defined:

- In order to multiply matrices, inner dimensions must be the same.
  The resulting matrix product has size equal to the outer dimensions.
- In order to add or subtract matrices, matrices must be the same size.
  The resulting sum or difference is also the same size.
- Transposing a matrix interchanges the number of rows and the number of columns.
- Multiplying a matrix by a scalar changes all entries in a matrix, but does not change the size of a matrix.

a. Start by determining if the matrix product, CB, exists.

$$\begin{array}{ccccc} C & \cdot & B & \Longrightarrow & CB \\ (2\times 3) & & (3\times 4) & & (2\times 4) \end{array}$$

$$= \checkmark$$

Because the matrix product exists, we continue and determine the size of the transposition, D^T.

$$\begin{array}{ccc} D & \Longrightarrow & D^T \\ (4\times 2) & & (2\times 4) \end{array}$$

Now, we determine if it is possible to add the two resulting matrices.

$$\begin{aligned} CB + D^T &\Longrightarrow (2\times 4) + (2\times 4) \\ &\Longrightarrow 2\times 4 \end{aligned}$$

b. Start by determining the size of the matrix resulting from the scalar multiplication and if the matrix product, DC, exists.

$$\begin{array}{ccc} 4A & \Longrightarrow & 4A \\ 4(2\times 2) & & (2\times 2) \end{array} \qquad \begin{array}{ccccc} D & \cdot & C & \Longrightarrow & DC \\ (4\times 2) & & (2\times 3) & & (4\times 3) \end{array}$$

$$= \checkmark$$

Next, we determine if it is possible to subtract the two matrices.

$$\begin{aligned} 4A - DC &\Longrightarrow (2\times 2) - (4\times 3) \\ &\Longrightarrow \text{The operation is not defined, because you can only subtract matrices of the same size.} \end{aligned}$$

c. Start by subtracting $3A$ and $6A$.

$$\begin{aligned}(3A-6A)C &\implies (-3A)C \\ &\implies -3AC, \text{ since scalar multiplication is associative.}\end{aligned}$$

Due to the fact that scalar multiplication is possible, no matter the size of the matrix, we will check if the matrix product, AC, exists.

$$\begin{array}{ccc} A & \cdot & C \\ (2\times 2) & & (2\times 3) \end{array} \implies \begin{array}{c} AC \\ (2\times 3) \end{array}$$

$$= \checkmark$$

$$\begin{aligned}\text{Thus, we have, } -3AC &\implies -3(2\times 3) \\ &\implies 2\times 3\end{aligned}$$

d. Start by organizing the product.

$$CBA \implies (CB)A$$

Next, we determine if the first product, CB, exists. If so, we then determine if the remaining product exists.

$$\begin{array}{ccc} C & \cdot & B \\ (2\times 3) & & (3\times 4) \end{array} \implies \begin{array}{c} CB \\ (2\times 4) \end{array}$$

$$= \checkmark$$

$$CBA \implies \begin{array}{ccc} CB & \cdot & A \\ (2\times 4) & & (2\times 2) \end{array}$$

$$\neq$$

$\implies$ The operation is not defined, because the inner dimensions of CB and A are not the same.

■

Applying Matrix Multiplication to Real-World Scenarios

Let's return to the problem presented at the opening of this section. We have **Table 1.4**, representing the equipment needs of two soccer teams.

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

Table 1.4: Soccer Teams' Equipment Needs

We were also given the costs of the equipment: a goal costs \$300; a ball costs \$10; a jersey costs \$30.

Our goal is to determine the total costs of the equipment needed for each team.

For convenience we will convert the given data to matrices. The matrix representing the equipment needed could be written as a 3×2 matrix, with the rows representing the different equipment and the columns representing the teams.

$$E = \begin{array}{c} \\ G \\ B \\ J \end{array} \begin{array}{c} \begin{array}{cc} W & MC \end{array} \\ \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \end{array}$$

To calculate the total costs of each team, we need to multiply the quantities needed by their corresponding costs. We will need the costs listed in a matrix in such a way that it will make sense to multiply with matrix E to give the total costs for each team. We will place the costs in a 1×3 matrix, where each column represents the specific equipment costs. The itemized cost matrix will be written as

$$C = \begin{array}{c} \begin{array}{ccc} G & B & J \end{array} \\ \begin{bmatrix} 300 & 10 & 30 \end{bmatrix} \end{array}$$

We can then perform the only logical matrix multiplication from our defined matrices, CE, to obtain the total costs for the equipment of each team. When performing the matrix multiplication the quantity needed is multiplied by its corresponding cost, and each entry of the product is the sum of all costs for a particular team.

$$\begin{aligned} CE &= \begin{bmatrix} 300 & 10 & 30 \end{bmatrix} \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= \begin{bmatrix} \underline{300(6)+10(30)+30(14)} & \underline{300(10)+10(24)+30(20)} \end{bmatrix} \\ &= \$ \begin{array}{c} \begin{array}{cc} W & MC \end{array} \\ \begin{bmatrix} 2520 & 3840 \end{bmatrix} \end{array} \end{aligned}$$

The total cost of new equipment for the Wildcats is \$2520, and the total cost of new equipment for the Mud Cats is \$3840.

■ **Example 7** Recall the scenario from the opening in **Section 1.1**.

A vendor sells hot dogs and corn dogs at three different locations. His total sales (in hundreds) for January and February from the three locations are given in **Table 1.5** below.

	January		February	
	Hot Dogs	Corn Dogs	Hot Dogs	Corn Dogs
Location I	10	8	8	7
Location II	8	6	6	7
Location III	6	4	6	5

Table 1.5: Hot Dog and Corn Dog Sales

If at each location hot dogs sell for \$3 and corn dogs for \$2, find the total revenue from the sale of hot dogs and corn dogs over both months at all three locations.

Solution:

The table gives the quantities sold (in hundreds) of hot dogs and corn dogs in three locations for two months. In the previous section we converted the data from the table into two matrices, J and F, as shown below.

$$J = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 10 & 8 \\ 8 & 6 \\ 6 & 4 \end{bmatrix} \end{array} \quad \text{and} \quad F = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 8 & 7 \\ 6 & 7 \\ 6 & 5 \end{bmatrix} \end{array}$$

We then added the two matrices to find the total number of hot dogs and corn dogs sold (in hundreds) at each location during the two months.

$$Q = J + F = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 10 & 8 \\ 8 & 6 \\ 6 & 4 \end{bmatrix} \end{array} + \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 8 & 7 \\ 6 & 7 \\ 6 & 5 \end{bmatrix} \end{array} = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 18 & 15 \\ 14 & 13 \\ 12 & 9 \end{bmatrix} \end{array}$$

Revenue is found by multiplying price times quantity. Thus, we need to input the price information into a matrix in such a way that it will make sense to multiply with the total sales matrix, Q, to give the total revenue.

As Q is already given as a 3×2 matrix, we will list the prices in a 2×1 column matrix, P, as shown below.

$$P = \begin{array}{c} \\ HD \\ CD \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{array}$$

Performing the only matrix multiplication which is possible, QP, we obtain

$$QP = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{cc} HD & CD \end{array} \\ \begin{bmatrix} 18 & 15 \\ 14 & 13 \\ 12 & 9 \end{bmatrix} \end{array} \begin{array}{c} \\ HD \\ CD \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 18(3)+15(2) \\ 14(3)+13(2) \\ 12(3)+9(2) \end{bmatrix}$$

$$= \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 84 \\ 68 \\ 54 \end{bmatrix} \end{array}$$

Because the quantity matrix Q was given in hundreds, we multiply matrix QP by the scalar 100. Thus, the total revenue at Location I is \$8400, at Location II is \$6800, and at Location III is \$5400.

■

N *When multiplying matrices that represent real-world data, there are many ways to set up the product. It is important to ensure the labels on the columns of the first matrix match the labels on the rows of the second matrix and that the entries that result from the product are logical.*

Try It # 2:
Refer to the table given at the start of the last example.

In March sales are projected to increase from February by 10%, 15%, and 20% at Location I, Location II, and Location III, respectively. Use matrix multiplication to find the expected total number of hot dogs and corn dogs to be sold in March.

Try It Answers

1. **a**. $AB = \begin{bmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{bmatrix}$

b. BA is not defined.

2. 2290 hot dogs and 2175 corn dogs.

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 8, use the given information about matrices A, B, C, D, and E to determine the size of the resulting matrix product, if the product is possible.

$$A \text{ is } 2\times 3, \quad B \text{ is } 2\times 2, \quad C \text{ is } 1\times 2, \quad D \text{ is } 3\times 2, \quad E \text{ is } 2\times 2$$

1. AB
2. BA
3. AC
4. CB
5. EB
6. CD^T
7. $A^T D$
8. EC

For Exercises 9 - 13, compute the following matrix products, by hand. Check your answers using technology.

9. $\begin{bmatrix} 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}$

10. $\begin{bmatrix} 4 \\ 2 \\ -7 \end{bmatrix}\begin{bmatrix} 0 & 5 & 9 \end{bmatrix}$

11. $\begin{bmatrix} 5 & 0 & -7 \\ 1 & 5 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$

Intermediate Skills Practice (Answers)

For Exercises 14 - 21, use the given information about matrices A, B, C, D, and E to determine the size of the resulting matrix operation, if the operation is possible.

$$A \text{ is } 2\times 3, \quad B \text{ is } 2\times 2, \quad C \text{ is } 1\times 2, \quad D \text{ is } 3\times 2, \quad E \text{ is } 2\times 2$$

14. BAD
15. ACE
16. $A^T E$
17. BEA^T
18. $(E-B)A$
19. $(D+A^T)E$
20. $AD+3B$
21. $\frac{1}{2}E-CA$

For Exercises 22 - 29, use the given matrices below to compute each matrix product, if possible.

$$A = \begin{bmatrix} 2 \\ 4s \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} w & 1 & z \\ t & y & -5 \\ -9 & 0 & x \end{bmatrix} \qquad C = \begin{bmatrix} e & \frac{1}{2} \\ \frac{1}{4} & f \\ 6 & -11 \end{bmatrix} \qquad D = \begin{bmatrix} 100g \end{bmatrix}$$

22. BA
23. AD
24. AC
25. BC
26. $D^T A$
27. $A^T B$
28. BAD
29. ADA^T

For Exercises 30 - 37, use the given matrices below to compute the indicated operation, if possible.

$$A = \begin{bmatrix} 2 & x \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 3y & 1 & 2 \\ z & 1 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 \\ 2 & k \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ g & 4h \end{bmatrix} \qquad E = \begin{bmatrix} 1 & p & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

30. $5AB$
31. $\frac{1}{2}DE$
32. $AC - 3A$
33. $CB - \frac{1}{5}E$
34. $CD^T + 4B$
35. $DA - 3B^T$
36. $BD - AC$
37. $ED + AB$

Mastery Practice (Answers)

38. If matrix A is size $5 \times c$ and matrix B is size $d \times 4$, then what are the values of c and d such that the product AB is defined?

39. Given $A = \begin{bmatrix} -1 & 0 & 2d \\ 8 & -4 & g \\ -7 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 17 & 11 & -9 & 0 & h \\ -6 & k & -3 & 1 & m \\ 4 & 6 & 10 & 8p & -4 \end{bmatrix}$

    a. If $E = AB$, determine e_{25}.

    b. Find the value of each variable (d, g, h, k, m, p) given $C = \begin{bmatrix} -17 & -11 & 9 & 0 & -1 \\ 156 & 70 & -70 & -20 & -24 \\ -111 & -65 & 83 & 32 & -15 \end{bmatrix}$ and $AB = C$.

40. Use the following scenario to answer parts **a** and **b**.

    In the small village of Pedimaxus in the country of Sasquatchia, all 150 residents get one of the two local newspapers. Market research has show that in any given week, 90% of those who subscribe to the Pedimaxus Tribune want to keep getting it, but 10% want to switch to the Sasquatchia Picayune. Of those who receive the Picayune, 80% want to continue with it and 20% want to switch to the Tribune.

$$P = \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix}$$

    a. Let's assume that when Pedimaxus was founded, all 150 residents got the Tribune. Write a 2×1 matrix, S, to represent the number of people who subscribe to each paper at the founding.

    b. Compute PS and describe the meaning of each entry in the resulting product matrix.

Communication Practice (Answers)

41. Explain, in your own words, why the inner dimensions of two matrices which are multiplied must be equal.
42. Explain why C^2 only exists if matrix C is a square matrix.

Chapter Review

Directions: Read each reflection question. If you are able to answer the question, then move on to the next question. If not, use the mathematical questions included to help you review. (Answers)

1. Given a matrix A, can you list its dimensions and any element, a_{ij}?

   a. State the dimensions of A and the value of the element a_{32}.

$$A = \begin{bmatrix} 7 & 11 & -8 & 2 \\ 3 & 0 & 19 & -1 \\ -1 & 17 & 2 & 0 \end{bmatrix}$$

   b. State the dimensions of B and the value of the element b_{23}.

$$B = \begin{bmatrix} 4 & 3 & -1 \\ 10 & -6 & 0 \end{bmatrix}$$

   c. Write the 3×3 matrix with a 2 in entry d_{33} and 0's in all other entries.

2. How would you use a matrix to organize information?

   a. **Set up the following information as a** 3×2 **matrix**: A campus store stocks two brands of soup, Earth Rite and Campus Discount. According to the store database there are 24 cans of Earth Rite chicken noodle, 30 cans of Earth Rite vegan vegetable, and 10 cans of Earth Rite butternut squash, while there are 2, 11, and 18 cans of Campus Discount butternut squash, Campus Discount vegan vegetable, and Campus Discount chicken noodle, respectively.

   b. **Set up the following information as a** 1×4 **matrix**: A local food truck sells waffles, chicken tenders, fries, and bottles of soda for \$5, \$6, \$2.50, and \$1.75, respectively.

3. Can you perform appropriate matrix algebra operations on two or more matrices, without the use of technology?

   Given $A = \begin{bmatrix} 4 & 2 \\ a & (b-c) \end{bmatrix}$, $B = \begin{bmatrix} b & 3 \\ 2 & -5 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 4 & 0 \\ 5 & -8 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 10 & -2 \\ 0 & -3 \\ 12 & 9 \end{bmatrix}$, compute the following, if possible. If not possible, explain why not.

   a. $3A - B$

   b. $DB - AC$

   c. $D^T + C$

   d. $CD - \frac{1}{2}B$

   e. DAC^T

   f. $(AB)^T$

   g. Determine the values of a, b, and c, such that $B - A^T = 4\begin{bmatrix} 1 & -8 \\ 0 & -9 \end{bmatrix}$.

4. How would you construct matrices and apply matrix algebra to solve a real-world application?

   An auto parts manufacturer makes fenders, doors, and hoods. Each part requires assembly and packaging which can be carried out at three different plants: Plant 1, Plant 2, and Plant 3. Matrix A, below, gives the total number of hours for assembly and packaging for each part, and matrix B gives the cost per hour for assembly and packaging at each of the three plants.

$$A = \begin{array}{r} \\ Fenders \\ Doors \\ Hoods \end{array} \begin{array}{c} \begin{array}{cc} Assembly & Packaging \end{array} \\ \begin{bmatrix} 12 & 2 \\ 21 & 3 \\ 10 & 2 \end{bmatrix} \end{array} \qquad B = \begin{array}{r} \\ Assembly \\ Packaging \end{array} \begin{array}{c} \begin{array}{ccc} Plant\ 1 & Plant\ 2 & Plant\ 3 \end{array} \\ \begin{bmatrix} 21 & 18 & 20 \\ 14 & 10 & 13 \end{bmatrix} \end{array}$$

   a. Compute $2B$ and explain the meaning of $2B$ in the context of the application.

   b. Would it be logical to compute $A+B$? Why or why not?

5. Can you describe what steps are taken to multiply two matrices by hand or explain why the multiplication is not possible?

   a. Suppose A is a 2×2 matrix, B is a 3×2 matrix, C is a 2×3 matrix, and D is a 3×4 matrix.

      i. Is the product CD possible? If so, what are the dimensions of the product matrix?

      ii. Explain why the product AC is possible, but the product CA is not possible.

      iii. Is the product $\left(D^T\right)^T B$ possible? If so, what are the dimensions of the product matrix?

   b. Given $A = \begin{bmatrix} -4 & 2 & x \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -2 \\ y & 4 \\ 0 & 1 \end{bmatrix}$, for the product $C = AB$, what is the value of the entry c_{12}?

   c. Given $E = \begin{bmatrix} g & 3 \\ -5 & 7 \end{bmatrix}$ and $F = \begin{bmatrix} -1 & h \\ 6 & -2 \\ k & 3 \\ 0 & m \end{bmatrix}$, determine the product FE, if possible. If the product is not possible, explain why not.

   d. If $A = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & -5 \\ 0 & 6 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 17 \\ 20 \\ -12 \end{bmatrix}$, compute the matrix product AX, and write the system (list) of equations resulting from $AX = B$.

6. How would you determine the meaning of a matrix multiplication, if possible, in a real-world application?

   a. An auto parts manufacturer makes fenders, doors, and hoods. Each part requires assembly and packaging which can be carried out at three different plants: Plant 1, Plant 2, and Plant 3. Matrix A, below, gives the total number of hours for assembly and packaging for each part, and matrix B gives the cost per hour for assembly and packaging at each of the three plants.

$$A = \begin{array}{l} \\ Fenders \\ Doors \\ Hoods \end{array} \begin{array}{c} \begin{array}{cc} Assembly & Packaging \end{array} \\ \left[\begin{array}{cc} 12 & 2 \\ 21 & 3 \\ 10 & 2 \end{array}\right] \end{array} \qquad B = \begin{array}{l} \\ Assembly \\ Packaging \end{array} \begin{array}{c} \begin{array}{ccc} Plant\ 1 & Plant\ 2 & Plant\ 3 \end{array} \\ \left[\begin{array}{ccc} 21 & 18 & 20 \\ 14 & 10 & 13 \end{array}\right] \end{array}$$

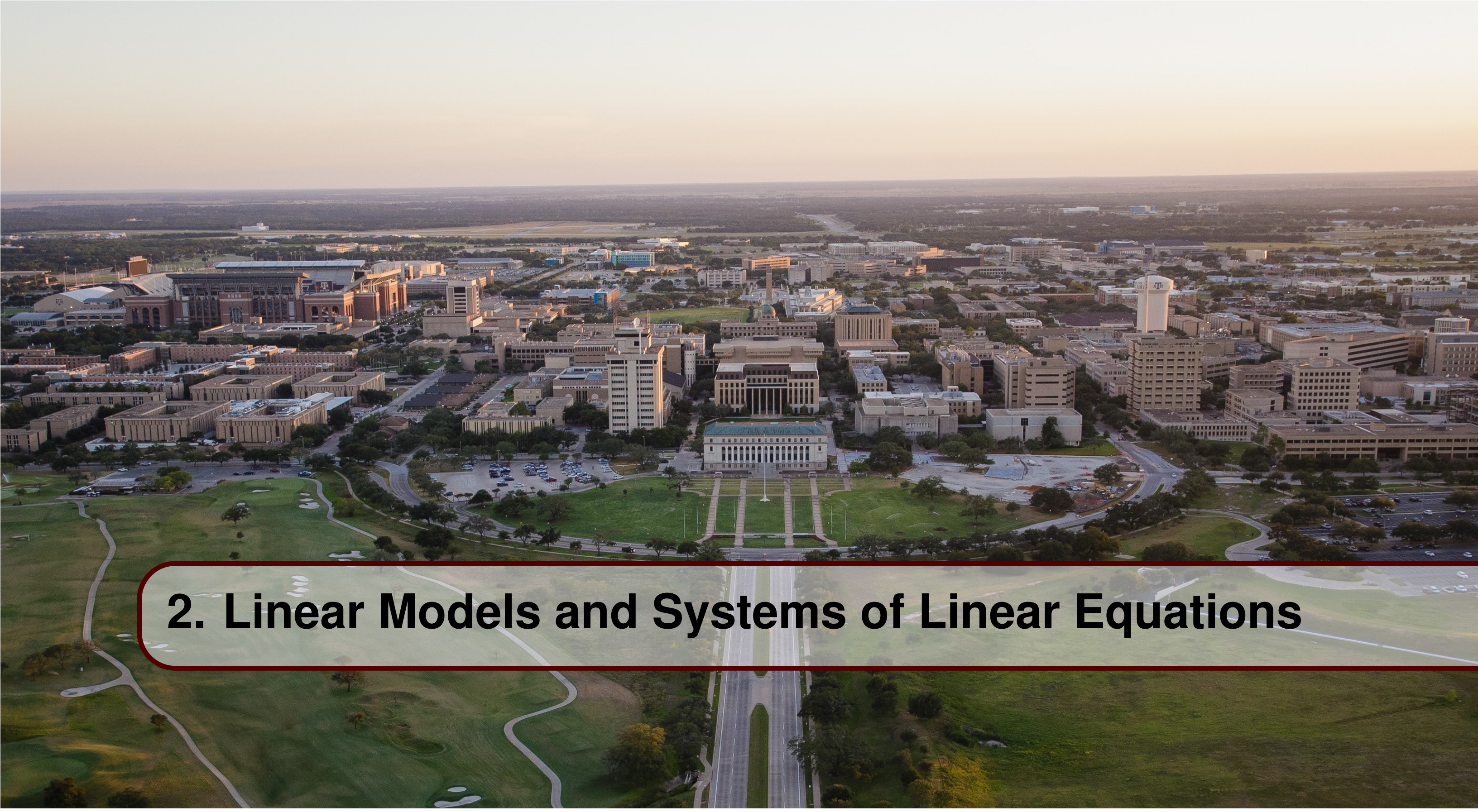
   i. Compute the product, $P = AB$. Make sure to include row and column labels in the resulting product matrix.

   ii. Explain the meaning of the entry p_{32} in the matrix P.

   iii. Which plant is the most economical to operate? Explain your reasoning.

   b. In the fall semester, 3400 students enroll in Math 140, while 2900 students enroll in Math 151. If 40% of Math 140 students are first generation students and 25% of Math 151 students are first generation students, use matrix multiplication to compute the total number of first generation students enrolled in these mathematics classes in the fall semester.

Chapter 2



2. Linear Models and Systems of Linear Equations

In this chapter we are going to discuss lines, linear models, and systems of linear equations in two variables.

The authors recommend all students are comfortable with the topics below prior to beginning this chapter. If you would like additional support on any of these topics, please refer to the Appendix.

A.1 - Integers

A.1 - Fractions

A.2 - Simplifying Expressions

A.1 - Decimals

A.1 - Properties of Real Numbers

A.1 - Systems of Time Measurement

A.2 - Using Variables and Algebraic Symbols

A.2 - Simplifying Expressions

A.2 - Evaluating an Expression

A.2 - Translating an English Phrase to an Algebraic Expression or Equation

A.2 - Solving Linear Equations with One Variable)

A.2 - Using Problem-Solving Strategies

2.1 Review of Lines

Figure 2.1.1: Google Map of a portion of the Texas A&M Campus in College Station

Haley is rushing across campus from her lab in Heldenfels Hall to her lecture in the Blocker Building, both shown in **Figure 2.1.1** above. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines, and we could use the grid to track her movements.

Learning Objectives:
In this section, you will review concepts of lines, including writing the equation of a line and graphing in the coordinate plane. Upon completion you will be able to:

- Derive an equation of a line given two points or a point and a slope.
- Graph a line, by hand, using any appropriate technique.
- Describe the change in the coordinates of points on a line by applying the definition of slope.

Plotting Ordered Pairs in the Cartesian Coordinate System

An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebraic geometry while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the **Cartesian coordinate system**, a grid system having perpendicular axes. Descartes named the horizontal axis the ***x*-axis** and the vertical axis the ***y*-axis**.

The Cartesian coordinate system, also called the *rectangular coordinate system*, is based on a two-dimensional plane consisting of the x-axis and the y-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a **quadrant**; the quadrants are numbered counterclockwise. The center of the plane is the point at which the two axes cross and is known as the **origin**. (See **Figure 2.1.2**).

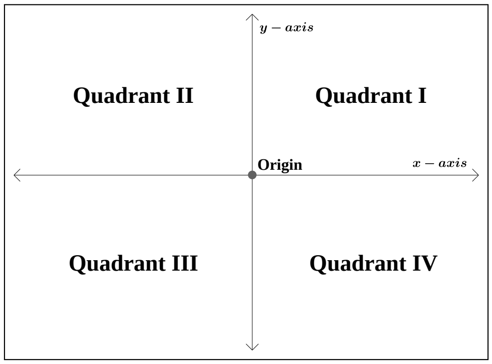


Figure 2.1.2: The coordinate plane with axes, quadrants, and origin labeled.

The two axes are similar to number lines, with positive increasing numbers to the right and above the origin and negative decreasing numbers to the left and below the origin. The axes extend to positive and negative infinity as shown by the arrowheads in **Figure 2.1.3** below. Each point in the plane is identified first by its ***x*-coordinate**, or horizontal displacement from the origin, and second by its ***y*-coordinate**, or vertical displacement from the origin. Together, we write them as an **ordered pair** indicating the combined distance from the origin, $(0,0)$, in the form (x,y).

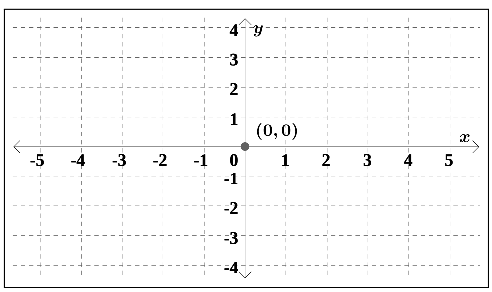


Figure 2.1.3: The coordinate plane with axes labeled as unit number lines and the origin given with coordinates.

The point $(3,-2)$ is represented in the plane by moving three units to the right of the origin in the horizontal direction, and then two units down in the vertical direction. (See **Figure 2.1.4**.)

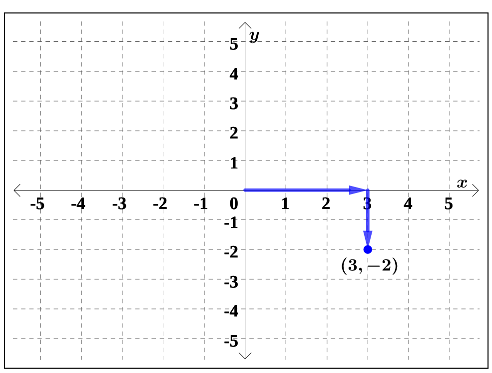


Figure 2.1.4: The coordinate plane with axes labeled and the ordered pair $(3,-2)$ labeled.

When dividing the axes into equally spaced increments the x-axis may be considered separately from the y-axis. In other words, while the x-axis *may* be divided and labeled according to consecutive integers, the y-axis *may* be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent specific and different units, such as years against the balance in a savings account, or quantity against cost, and so on. Therefore, it is imperative to label both axes with proper units to convey the meaning of the graph. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Example 1 Plot the points $(-2,4)$, $(3,3)$, and $(0,-3)$ in the coordinate plane.

Solution:
To plot the point $(-2,4)$, begin at the origin. The x-coordinate is -2, so move two units to the left. The y-coordinate is 4, so then move four units up in the positive y direction.

To plot the point $(3,3)$, begin again at the origin. The x-coordinate is 3, so move three units to the right. The y-coordinate is also 3, so then move three units up in the positive y direction.

To plot the point $(0,-3)$, begin again at the origin. The x-coordinate is 0. This tells us not to move in either direction along the x-axis. The y-coordinate is -3, so then move three units down in the negative y direction.

See the points plotted in **Figure 2.1.5**, below.

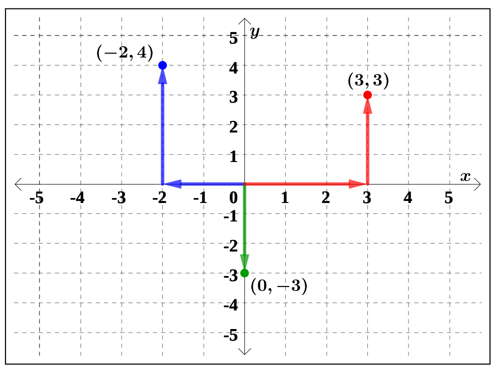


Figure 2.1.5: The coordinate plane with ordered pairs $(-2,4)$, $(3,3)$, and $(0,-3)$ labeled.

When either coordinate of a point is zero, the point must be on an axis. If the x-coordinate is zero, the point is on the y-axis. If the y-coordinate is zero, the point is on the x-axis.

Graphing Linear Equations by Plotting Points

We can plot a set of points to represent an equation. When such an equation contains both an x and a y variable, it is called an equation in two variables; its graph is called a graph in two variables. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for x into the equation and determining the resulting value of y. Each pair of x- and y-values is an ordered pair that can be plotted. **Table 2.1** lists integer values of x from -3 to 3 and the resulting values for y, when $y = 2x - 1$.

x	$y = 2x - 1$	(x, y)
-3	$y = 2(-3) - 1 = -7$	$(-3, -7)$
-2	$y = 2(-2) - 1 = -5$	$(-2, -5)$
-1	$y = 2(-1) - 1 = -3$	$(-1, -3)$
0	$y = 2(0) - 1 = -1$	$(0, -1)$
1	$y = 2(1) - 1 = 1$	$(1, 1)$
2	$y = 2(2) - 1 = 3$	$(2, 3)$
3	$y = 2(3) - 1 = 5$	$(3, 5)$

Table 2.1: A table of x-values and corresponding y-values for $y = 2x - 1$.

We can plot the points from the table on the coordinate plane. The selected points for this particular equation appear to form a line, as shown in **Figure 2.1.6**. While only a few ordered pairs from the equation, $y = 2x - 1$, were plotted, by connecting these points the line represents *all* ordered pairs that would satisfy this equation. The arrows on each end of the line indicate there are points outside of the graphed area. Not all equations form a line when the corresponding points are plotted, but we will shortly discuss how to recognize, before plotting points, when the equation will represent a line.

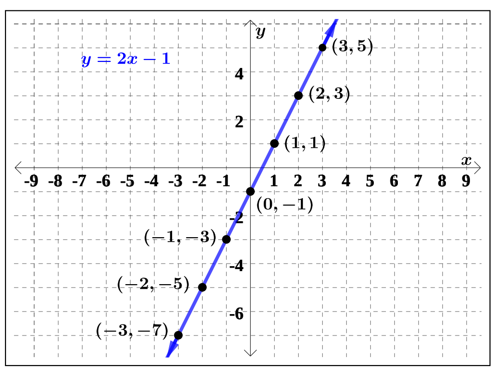


Figure 2.1.6: The coordinate plane with points on the line $y = 2x - 1$ labeled.

It is important to know that, when plotting points to graph an equation, the x-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of x to be plotted in order to see a particular characteristic. Otherwise, it is logical to choose values that can be easily calculated with, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two points to graph a line (a concept that will become clear shortly). Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

■ **Example 2** Graph the equation $y = -x + 2$, by plotting points.

Solution:
First, choose x-values at random, calculate the corresponding y-values, and construct a table similar to **Table 2.1**.

x	$y = -x + 2$	(x, y)
-5	$y = -(-5) + 2 = 7$	$(-5, 7)$
-3	$y = -(-3) + 2 = 5$	$(-3, 5)$
-1	$y = -(-1) + 2 = 3$	$(-1, 3)$
0	$y = -(0) + 2 = 2$	$(0, 2)$
1	$y = -(1) + 2 = 1$	$(1, 1)$
3	$y = -(3) + 2 = -1$	$(3, -1)$
5	$y = -(5) + 2 = -3$	$(5, -3)$

Table 2.2: A table of x-values and corresponding y-values for $y = -x + 2$.

Now, we plot the points and connect them to see they form a line, as shown in **Figure 2.1.7**.

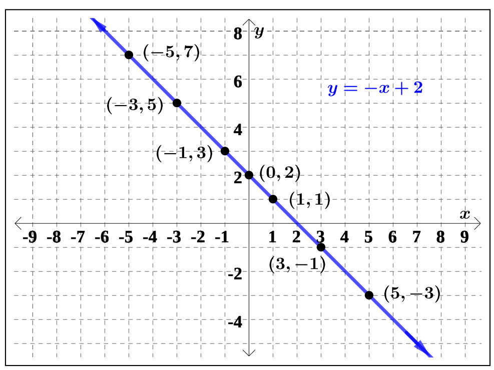


Figure 2.1.7: The coordinate plane with points on the line $y = -x + 2$ labeled.

■

Try It # 1:
Construct a table and graph the equation $y = \frac{1}{2}x + 2$, by plotting points.

Graphing Linear Equations using Technology

Most graphing calculators allow you to graph an equation without plotting points, but may instead require the equation to be manipulated so it is written in the style $y =$ __________. The TI-84, and many other calculator makes and models have a window function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

For example, the equation $y = 2x - 20$ has been entered in the TI-84, as shown in **Figure 2.1.8**. The standard viewing window on the TI-84 shows $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. (See **Figure 2.1.9**) In **Figure 2.1.10**, the resulting graph is shown in the standard window. Notice that we cannot see on the screen where the graph crosses the axes.

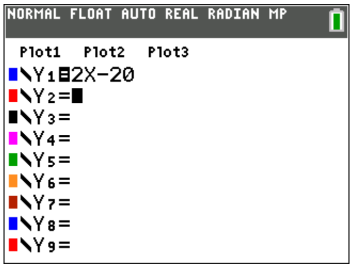


Figure 2.1.8: Calculator screenshot showing the equation $y = 2x - 20$ entered as Y_1.

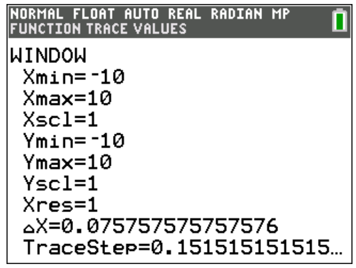


Figure 2.1.9: Calculator screenshot showing the standard window settings.

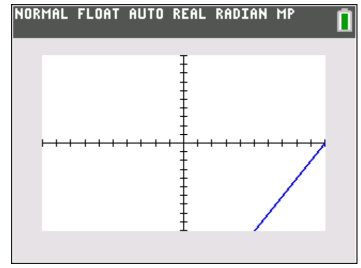


Figure 2.1.10: Calculator screenshot showing the graph of Y_1 in the standard window.

By changing the window to show more of the positive x-axis and more of the negative y-axis, we have a much better view of the graph and where the graph crosses the axes. (See **Figure 2.1.11** and **Figure 2.1.12**.)

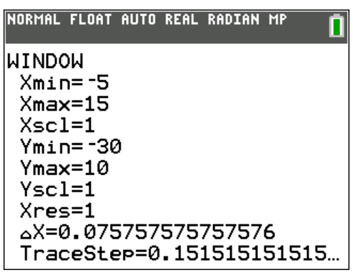


Figure 2.1.11: Calculator screenshot showing the new adjusted window settings, with a larger range of x- and y-values.

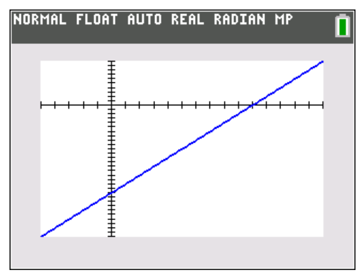


Figure 2.1.12: Calculator screenshot showing the graph in the new window, including where the graph crosses each axis.

Thus far, we have discussed techniques for graphing a line given its equation. Now we will turn our attention to representing a line algebraically by writing the equation which describes the relationship between the x- and corresponding y-values.

Writing a Linear Equation

When writing the equation of a line, we describe both its 'steepness' and the set of points that define the line.

Slope

To get a sense of the 'steepness' of a line, we compute the **slope** of the line by simply using *any* two ordered pairs on the line and the formula below.

Definition

The **slope**, $\boldsymbol{m}$, of the line containing the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

provided $x_1 \neq x_2$.

When $x_1 = x_2$, the slope of the line between the two points P and Q is undefined.

When $y_1 = y_2$, the slope of the line between the two points P and Q is 0. ■

N *A couple of comments about the above definition are in order. First, don't ask why we use the letter 'm' to represent slope; there are many explanations out there, but apparently no one really knows for sure. Secondly, the stipulation $x_1 \neq x_2$ ensures that we aren't trying to divide by zero. The reader is invited to pause and think about what is happening geometrically.*

■ **Example 3** Compute the slope of the line that passes through the points $(2, -1)$ and $(-5, 3)$.

Solution:
Let $(x_1, y_1) = (2, -1)$ and $(x_2, y_2) = (-5, 3)$. Then, we can substitute the y-values and the x-values into the slope formula, as follows.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m &= \frac{3 - (-1)}{-5 - 2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

The slope of the line through the given points is $-\frac{4}{7}$.

■

N *It does not matter which point is called (x_1, y_1) or (x_2, y_2). As long as we are consistent with the order of the y-terms and the order of the x-terms in the numerator and denominator, the calculation will yield the same result.*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Try It # 2:
Compute the slope of the line that passes through the points $(2,3)$ and $(2.1,4)$.

Whether we are given the coordinates of two points on a line, as ordered pairs, or only given a visual representation of the line, we can still determine the slope of the line using the slope formula.

Example 4 Using **Figure 2.1.13**, determine the slope of each line: A, B, C, and D.

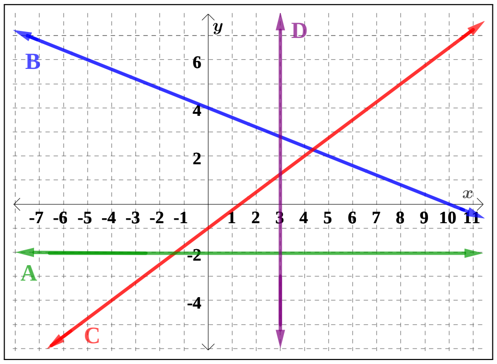


Figure 2.1.13: The graph of four different lines.

Solution:
We can compute the slope of each line by selecting any two points on the line. Considering we can choose *any* two points, we will choose points with "nice" integer coordinates.

A. For line A, we can identify two points: $(0,-2)$ and $(6,-2)$. The slope can then be computed using the formula.

$$\begin{aligned} m &= \frac{-2-(-2)}{6-0} \\ &= \frac{-2+2}{6} \\ &= \frac{0}{6} \\ &= 0 \end{aligned}$$

B. For line B, we can identify two points: $(0,4)$ and $(10,0)$. The slope can then be computed using the formula.

$$\begin{aligned} m &= \frac{0-4}{10-0} \\ &= \frac{-4}{10} \\ &= -\frac{2}{5} \end{aligned}$$

C. For line C, we can identify two points: $(-4,-4)$ and $(0,-1)$. The slope can then be computed using the formula.

$$\begin{aligned} m &= \frac{-1-(-4)}{0-(-4)} \\ &= \frac{-1+4}{0+4} \\ &= \frac{3}{4} \end{aligned}$$

D. For line D, we can identify two points: $(3,-2)$ and $(3,0)$. Notice $x_1 = x_2$, which would also be true for *any* two points on line D. Thus, by the definition, the slope of line D is undefined.

If we would have substituted the coordinates of the two points, $(3,-2)$ and $(3,0)$, into the slope formula, the result would have included a denominator of zero. Because we cannot divide by zero, we would have again concluded the slope of line D is undefined.

N *If the slope of a line is positive (as with line C), the line slants to the right. This means positive (negative) changes in x result in positive (negative) changes in y. If the slope of a line is negative (as with line B), the line slants to the left. This means positive (negative) changes in x result in negative (positive) changes in y. If the slope of a line is zero (as with line A), the line has no slant and is horizontal. If the slope of a line is undefined (as with line D), the line again has no slant, but is vertical.*

■

Horizontal and Vertical Lines

Notice in the previous example that every point on the horizontal line (line A) has the same y-coordinate. In general, the y-coordinate of every point on a horizontal line is the same, b. Therefore, the common equation of a horizontal line is $y = b$.

Then notice again in the previous example that every point on the vertical line (line D) has the same x-coordinate. In general, the x-coordinate of every point on a vertical line is the same, a. Therefore, the common equation of a vertical line is $x = a$.

Definition

- The equation of a **horizontal line** through the point (a,b) is $\boldsymbol{y = b}$.
- The equation of a **vertical line** through the point (a,b) is $\boldsymbol{x = a}$.

■

N *The x-axis is the horizontal line $y = 0$, and the y-axis is the vertical line $x = 0$.*

Example 5 Write the equation of line A and line D, from the previous example.

Solution:
Line A is horizontal, and every point on the line has a y-coordinate of -2. Thus, the equation of line A is $y = -2$.

Line D is vertical, and every point on the line has an x-coordinate of 3. Thus, the equation of line D is $x = 3$.

Point-Slope Form

We have discussed the equations of vertical and horizontal lines. Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_1, y_1). Let (x, y) be an arbitrary point on the line, as indicated in **Figure 2.1.14**.

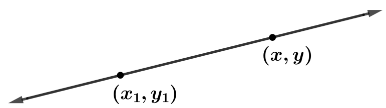


Figure 2.1.14: A line drawn with points (x_1, y_1) and (x, y) labeled on the line.

The slope between these two points yields

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1} \\ m(x - x_1) &= y - y_1 \\ y - y_1 &= m(x - x_1) \end{aligned}$$

After manipulating the slope formula, we have just derived one form of the equation of a line, known as the **point-slope form** of a line.

Definition

The **point-slope form** of the line, with slope m and containing the point (x_1, y_1), is the equation

$$\boldsymbol{y - y_1 = m(x - x_1)}$$

N *When using the point-slope form of the line, any point on the line can be used to determine the equation. If done correctly, equivalent equations will be obtained.*

■ **Example 6** Write the equation of the line with slope $m = -3$ and passing through the point $(4,8)$, using the point-slope form of a line.

Solution:
Using the point-slope form, substitute -3 for m and the point $(4,8)$ for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -3(x - 4) \end{aligned}$$

■

■ **Example 7** Write the equation of the line containing the points $(-1,3)$ and $(2,1)$, using the point-slope form of a line.

Solution:
First, we need to determine the slope of the line in question. Using the given points and slope formula, we have

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{1-3}{2-(-1)} \\ &= -\frac{2}{3} \end{aligned}$$

We can use either point in the point-slope form, but we'll use $(-1,3)$ as (x_1, y_1) and leave it to the reader to check that using $(2,1)$ results in an equivalent equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -\frac{2}{3}(x - (-1)) \\ y - 3 &= -\frac{2}{3}(x + 1) \end{aligned}$$

■

Recall that slope can be described as the ratio $\frac{\text{rise}}{\text{run}}$. We found the slope in the previous example to be $-\frac{2}{3} = \frac{-2}{3}$. We can interpret this as a 'rise' of 2 units downward for every 3 unit 'run' to the right, as we travel along the line to the right. This interpretation is shown in the graph of the line, $y - 3 = -\frac{2}{3}(x+1)$, in **Figure 2.1.15** below. (Notice both of the given points are on the line.)

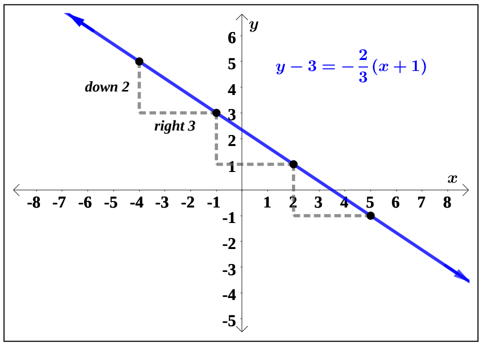


Figure 2.1.15: A line drawn on the coordinate plane, with the 'rise' and 'run' of the slope labeled.

N *Mathematically the slope* $-\frac{2}{3} = \frac{-2}{3}$ *is also equal to* $\frac{2}{-3}$, *the slope can be interpreted as a 'rise' of 2 units upward for every 3 unit 'run' to the left, as you travel along the line to the left.*

Slope-Intercept Form

We can further simplify the equation of the line in the previous example, to produce another form of a line called the **slope-intercept form**. This is the familiar $y = mx + b$ form you have probably seen in algebra. The "intercept" in the slope-intercept form comes from the fact that if we set $x = 0$, we get $y = b$. This point, $(0, b)$, is the ***y*-intercept** of the line $y = mx + b$. Intercepts of a line will be discussed in more detail later in this section.

Definition

The **slope-intercept form** of the line, with slope m and y-intercept $(0, b)$, is the equation

$$\boldsymbol{y = mx + b}$$

■

N *When using the slope-intercept form of a line, if we have slope* $m = 0$, *we get the equation* $y = b$ *which matches our formula for a horizontal line. The slope-intercept form of a line can be used to describe all lines except vertical lines.*

Further simplification of the resulting line in the previous example gives us

$$\begin{aligned} y - 3 &= -\frac{2}{3}(x + 1) \\ y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\ y &= -\frac{2}{3}x + \frac{7}{3} \end{aligned}$$

In this format, $y = mx + b$, we can verify the slope of the line is $m = -\frac{2}{3}$, and we now know the y-intercept of the line is $\left(0, \frac{7}{3}\right)$.

■ **Example 8** Write the equation of the line passing through the points $(3, 4)$ and $(0, -3)$, using slope-intercept form.

Solution:
First, we calculate the slope using the slope formula and the given two points.

$$\begin{aligned} m &= \frac{-3 - 4}{0 - 3} \\ &= \frac{-7}{-3} \\ &= \frac{7}{3} \end{aligned}$$

Next, we use the point-slope form with the slope of $\frac{7}{3}$, and either point. Let's pick the point $(3,4)$ for (x_1,y_1).

$$y-4=\frac{7}{3}(x-3)$$

To write the equation in slope-intercept form, we start by distributing the $\frac{7}{3}$, and then simplify.

$$\begin{aligned} y-4 &= \frac{7}{3}x-7 \\ y &= \frac{7}{3}x-3 \end{aligned}$$

In slope-intercept form, the equation of the line is written as $y=\frac{7}{3}x-3$.

N *To prove that either point can be used, when using the point-slope form, let us use the second point,* $(0,-3)$*, and show that we get the same the equation in slope-intercept form.*

$$\begin{aligned} y-(-3) &= \frac{7}{3}(x-0) \\ y+3 &= \frac{7}{3}x \\ y &= \frac{7}{3}x-3 \end{aligned}$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Also, notice we were given the y-intercept, $(0,-3)$*, as our second point. We could have skipped substitution into the point-slope form and substituted directly into the slope-intercept form of a line.*

■

Try It # 3:
Write the equation of the line passing through the points $(2,3)$ and $(2.1,4)$, in both point-slope form and slope-intercept form.

Standard Form

Another way that we can represent the equation of a line is in **standard form**.

Definition

The **standard form** of a line is given as $\boldsymbol{Ax+By=C}$, where A, B, and C are integers. ■

N *When a line is written in standard form, the x- and y-terms are on one side of the equals sign and the constant term is on the other side.*

Example 9 Write the equation of the line with a slope of -6 which passes through the point $\left(\frac{1}{4}, -2\right)$ in standard form.

Solution:
We begin by using the point-slope form of a line.

$$y-(-2) = -6\left(x-\frac{1}{4}\right)$$

We next simplify to the slope-intercept form of a line.

$$\begin{aligned} y+2 &= -6x+\frac{3}{2} \\ y &= -6x-\frac{1}{2} \end{aligned}$$

We can then rearrange terms and multiply both sides of the equation by 2, the least common denominator, to obtain integers coefficients and an integer constant term.

$$\begin{aligned} 6x+y &= -\frac{1}{2} \\ 12x+2y &= -1 \end{aligned}$$

This equation is now written in standard form, $12x+2y=-1$.

N *There is a flow to the forms of equations of a line. We often begin with substituting into the point-slope form, then transition into slope-intercept form, and conclude with standard form. Throughout this course you will see each form and its usefulness.*

Finding x-intercepts and y-intercepts

Previously, the y-intercept of a line was mentioned in our discussion of the slope-intercept form of a line. In general, the **intercepts** of any graph, including that of a line, are the points at which the graph crosses/touches the axes.

Definition

- A **y-intercept** is a *point* at which the graph crosses/touches the y-axis, $(0, y)$.
- An **x-intercept** is a *point* at which the graph crosses/touches the x-axis, $(x, 0)$.

To determine a y-intercept, we set x equal to zero and solve for y. Similarly, to determine an x-intercept, we set y equal to zero and solve for x. In the case of lines, there will be at most one y-intercept.

For example, let's determine the intercepts of the line $12x+2y=-1$ from the previous example.

To solve for the y-intercept, set $x=0$.

$$\begin{aligned} 12x+2y &= -1 \\ 12(0)+2y &= -1 \\ 2y &= -1 \\ y &= -\frac{1}{2} \\ \left(0,-\frac{1}{2}\right) &\text{ is the } y-\text{intercept} \end{aligned}$$

To solve for the x-intercept, set $y=0$.

$$\begin{aligned} 12x+2y &= -1 \\ 12x+2(0) &= -1 \\ 12x &= -1 \\ x &= -\frac{1}{12} \\ \left(-\frac{1}{12},0\right) &\text{ is the } x-\text{intercept} \end{aligned}$$

N *The intercepts are easily identifiable points that can be used to graph a line.*

We can confirm that our results make sense by observing a graph of $12x+2y=-1$, as shown in **Figure 2.1.16**. Notice the graph crosses the axes where we predicted it would.

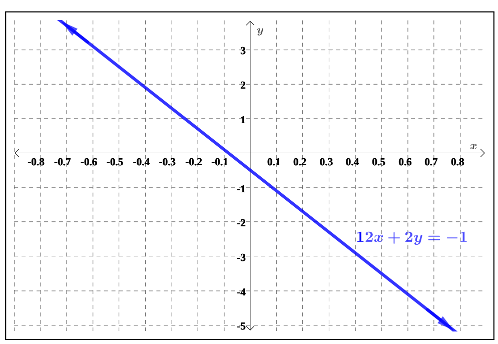


Figure 2.1.16: The coordinate plane with the line $12x+2y=-1$.

■ **Example 10** State the intercepts of the line $y=-3x-4$. Then, sketch a graph of the line, using only the intercepts.

Solution:
Set $x=0$ to solve for the y-intercept.

$$\begin{aligned} y &= -3x-4 \\ y &= -3(0)-4 \\ y &= -4 \\ (0,-4) &\text{ is the } y-\text{intercept} \end{aligned}$$

Set $y = 0$ to solve for the x-intercept.

$$\begin{aligned} y &= -3x - 4 \\ 0 &= -3x - 4 \\ 4 &= -3x \\ -\frac{4}{3} &= x \\ \left(-\frac{4}{3}, 0\right) &\text{ is the } x-\text{intercept} \end{aligned}$$

To sketch a graph of the line, plot both intercepts on the coordinate plane, and draw a line passing through them, as in **Figure 2.1.17**.

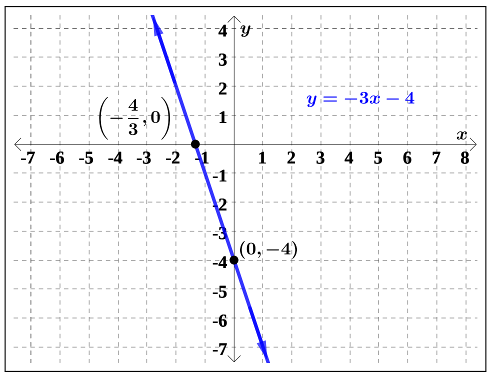


Figure 2.1.17: The coordinate plane with the line through the points $\left(-\frac{4}{3}, 0\right)$ and $(0, -4)$.

■

Another method for graphing a line, using a known point on the line and the slope of the line, is to plot the point and use the understanding of slope in terms of 'rise' and 'run.' Starting at the given point, move the appropriate number of units up/down and then right/left, according to the slope, to plot another point on the line. You can now draw your line through the two points.

Interpreting Slope

Given the line $y = -\frac{11}{8}x + 17$, if we know x increases by 3 units as you move along the line, how can we find the corresponding change in y?

As the slope of the line is negative, we know the graph of the line will slant to the left and that a positive change in x (increase) should produce a negative change in y (decrease). From its definition, we know slope can be written as

$$m = \frac{\Delta y}{\Delta x}$$

which shows the constant relationship between changes in the x-coordinates of points on a line and changes in the y-coordinates of points on a line.

From the given equation we have $m = -\frac{11}{8} = \frac{\Delta y}{\Delta x}$. If x increases by 3 units, then $\Delta x = 3$. Using substitution we have

$$-\frac{11}{8} = \frac{\Delta y}{3}$$

Through multiplication and simplification we can find the corresponding change in y, Δy.

$$\begin{aligned} -11(3) &= 8(\Delta y) \\ -33 &= 8(\Delta y) \\ -\frac{33}{8} &= \Delta y \end{aligned}$$

Thus, y decreases by $\frac{33}{8}$ units when x increases by 3 units.

N *When the **change** in a variable is **negative**, this corresponds to a **decrease** in the variable value, while a **positive change** corresponds to an **increase** in the variable value.*

■ **Example 11** You have a line which passes through the points $(2,-1)$ and $(-5,3)$.

a. If x decreases by 9 units, what is the corresponding change in y?
b. If y increases by 5 units, what is the corresponding change in x?

Solution:
First, we calculate the slope using the slope formula with the given two points.

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{3-(-1)}{-5-2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

Because the slope is negative, we should recognize that an increase in one variable will produce a decrease in the other and vice versa.

a. If x decreases by 9 units, then $\Delta x = -9$. So,

$$m = -\frac{4}{7} = \frac{\Delta y}{-9}$$

Through multiplication and simplification we can find the corresponding change in y.

$$\begin{aligned} -4(-9) &= 7(\Delta y) \\ 36 &= 7(\Delta y) \\ \frac{36}{7} &= \Delta y \end{aligned}$$

Thus, y increases by $\frac{36}{7}$ units when x decreases by 9 units.

b. If y increases by 5 units, then $\Delta y = 5$. So,

$$m = -\frac{4}{7} = \frac{5}{\Delta x}$$

Through multiplication and simplification we can find the corresponding change in x, Δx.

$$\begin{aligned} -4(\Delta x) &= 7(5) \\ -4(\Delta x) &= 35 \\ \Delta x &= \frac{35}{-4} \\ \Delta x &= -\frac{35}{4} \end{aligned}$$

Thus, x decreases by $\dfrac{35}{4}$ units when y increases by 5 units.

∎

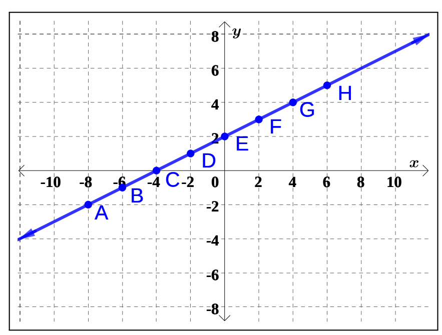
Watch your signs when simplifying fractions. For example, $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ *but* $-\dfrac{a}{b} \neq \dfrac{-a}{-b}$.

Try It # 4:
Given the line $-7x+3y=-9$, if y decreases by 2 units, what is the corresponding change in x?

Try It Answers

1. A table of points and corresponding graph are given below.

x	$y=\frac{1}{2}x+2$	(x,y)
-8	-2	$A=(-8,-2)$
-6	-1	$B=(-6,-1)$
-4	0	$C=(-4,0)$
-2	1	$D=(-2,1)$
0	2	$E=(0,2)$
2	3	$F=(2,3)$
4	4	$G=(4,4)$
6	5	$H=(6,5)$



2. $m = 10$

3. Point-Slope Form: $y-3=10(x-2)$ or $y-4=10(x-2.1)$

Slope-Intercept Form: $y=10x-17$

4. x decreases by $\dfrac{6}{7}$ units

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 4, determine the slope of the line passing through the given points.

1. $(0,0)$ and $(3,5)$
2. $(6,1)$ and $(9,11)$
3. $(5,0)$ and $(0,-8)$
4. $(3,-5)$ and $(-7,-4)$

For Exercises 5 - 8, determine the x- and y-intercept, without graphing. Write each intercept as an ordered pair.

5. $y=-3x+6$
6. $y=5x-14$
7. $3x-2y=6$
8. $4x+8y=9$

For Exercises 9 - 12, write the equation of the line with the given slope which passes through the given point, in both point-slope form and slope-intercept form, if possible.

9. $m=3$ and $(3,-1)$
10. $m=\frac{2}{3}$ and $(-2,1)$
11. $m=-2$ and $(-5,8)$
12. $m=-\frac{1}{5}$ and $(10,4)$

For Exercises, 13 - 16, write the equation of the line which passes through the given points.

13. $(-3,10)$ and $(5,-6)$.
14. $(1,3)$ and $(5,5)$.
15. $(-6,-15)$ and $(-2,-7)$.
16. $(4,-9)$ and $(-14,20)$.

For Exercises 17 - 21, graph the line, without the use of technology, using the given information.

17. The line has a slope of $m=\frac{4}{7}$ and passes through the origin.
18. $y=-3x+2$
19. $y-5=\frac{2}{3}(x-8)$
20. $y=7$
21. $x=-2$

For Exercise 22, describe the changes in x or y, using the given information.

22. Given a line has a slope of $m = \frac{3}{4}$,
    a. If x increases by 4 units, what is the corresponding change in y?
    b. If y increases by 3 units, what is the corresponding change in x?
    c. If x decreases by 4 units, what is the corresponding change in y?
    d. If y decreases by 3 units, what is the corresponding change in x?

Intermediate Skills Practice (Answers)

For Exercises 23 - 26, determine the slope of the line passing through the given points.

23. $\left(\frac{1}{2}, \frac{3}{4}\right)$ and $\left(\frac{5}{2}, -\frac{7}{4}\right)$
24. $(-1,-2)$ and $(3,-2)$
25. $\left(\frac{11}{5}, 7\right)$ and $\left(-\frac{11}{5}, -4\right)$
26. $(8,1)$ and $(8,2)$

For Exercises 27 - 30, determine the x- and y-intercept without graphing.

27. $4y = -2x - 1$
28. $4x - 3 = 2y$
29. $21 - 6y = 7x$
30. $2x = 5y - 8$

For Exercises 31 - 34, write the equation of the line with the given slope which passes through the given point, in both point-slope form and slope-intercept form, if possible.

31. $m = 0$ and $(3,117)$
32. m is undefined and $(10,6)$
33. $m = 1.75$ and $(-4,8)$
34. $m = -0.1$ and $(0.2,0.9)$

For Exercises 35 - 38, write the equation of the line which passes through the given points.

35. $(1,7)$ and $(3,7)$
36. $(-6,10)$ and $(-6,16)$
37. $(-2,6)$ and an x-intercept of $(1,0)$
38. $(4,-1)$ and a y-intercept of $(0,2)$

For Exercises 39 - 43, graph the line, without the use of technology, using the given information.

39. $y = 2(x+1) - 4$

40. $y = \frac{1}{3}(x-3) + 1$

41. $y + 7 = \frac{4}{5}(x+2)$

42. The line has a slope of zero and passes through the point $(-5, 3)$.

43. The line has an undefined slope and passes through the point $(-3, -4)$.

For Exercise 44, describe the changes in x or y, using the given information.

44. Given the line $y = -\frac{9}{7}x - 1$,
    a. If x increases by 1 unit, what is the corresponding change in y?
    b. If y increases by 3 units, what is the corresponding change in x?
    c. If x decreases by 10 units, what is the corresponding change in y?
    d. If y decreases by 1 unit, what is the corresponding change in x?

Mastery Practice (Answers)

45. Determine the slope of the line passing through the points $(a, -5)$ and $(4, 2a)$, in terms of a. For what value(s) of a is the slope of the line undefined?

46. Determine the slope, y-intercept and x-intercept, if any exist, for $y = \frac{1-x}{2}$.

47. Write the equation of the line with a slope of $\frac{6}{5}$ which passes through the point $(c-5, -4)$, in point-slope and slope-intercept form, where c is any real number.

48. Write the equation of the line which intersects the x-axis when $x = -10$ and intersects the y-axis when $y = \frac{7}{11}$.

49. Graph the line $5x - 7y = 70$, by finding and using the x- and y-intercepts.

50. Given the line $6x + 11y = 19$,
    a. If x increases by 1 unit, what is the corresponding change in y?
    b. If y increases by 6 units, what is the corresponding change in x?
    c. If x decreases by 7 units, what is the corresponding change in y?
    d. If y decreases by 11 units, what is the corresponding change in x?

51. If when x decreases by 9 units, y decreases by 2 units, what is the slope of the line containing any point (x, y)?

Communication Practice (Answers)

52. Describe, in your own words, what the y-intercept of a line is.

53. Describe the process for finding the x-intercept and y-intercept of a line, algebraically.

54. If a point is located on the y-axis, what is the x-coordinate and why does it have this value?

55. If a point is located on the x-axis, what is the y-coordinate and why does it have this value?

2.2 Modeling with Linear Functions

© Photo by Texas A&M University, 2020

Josh is hoping to get an A in his college math class. He has scores of 75, 82, 95, 91, and 94 on his first five exams. Only the final exam remains, and the maximum number of points that can be earned is 100. Is it possible for Josh to end the course with an A, if all exams are weighted equally? A simple linear equation will give Josh his answer.

Many real-world applications we come across every day can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce x widgets per month plus monthly operating costs; a car rental company charges a daily fee plus an amount per mile driven.

Learning Objectives:
In this section, you will learn about concepts related to linear depreciation, cost, revenue, profit, supply, and demand. Upon completion you will be able to:

- Formulate the equation for a linear depreciation, cost, revenue, profit, supply, and demand function.
- Solve problems involving linear depreciation, including the rate of depreciation, initial value, and scrap value.
- Describe the relationships between the cost, revenue, and profit functions, graphically and verbally.
- Describe the differences between a supply and demand function.

Defining a Linear Function

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or *independent variable*, and a corresponding output quantity, or *dependent variable*. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and their age, it is easy enough to determine their height.

A **function** is a rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value *uniquely* determines one output value. We say "the output is a function of the input."

To simplify writing out expressions and equations involving functions, an algebraic notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age," we could use the descriptive variables h to represent height, a to represent age, and if we name the function f we could then write:

"h is f of a" or $h = f(a)$

We could instead name the function h and more simply write:

$$h(a)$$

For example, consider a 20-year old who is 5 feet 7 inches tall. Then according to our notation, we say

$$h = f(20) = 5 \text{ feet } 7 \text{ inches}$$
or
$$h(20) = 5 \text{ feet } 7 \text{ inches}$$

The notation $h(a)$ is read "h of a". Remember we can use any variable to name the function; the notation $h(a)$ shows us that h depends on a. The value 'a' must be put into the function 'h' to get a result. Be careful when reading function notation.

! *Do not confuse the parentheses in function notation with multiplication! In the last scenario, the parentheses indicate that age is input into the function.*

For any function,
- The set of all values which can be put into the function is known as the **domain of the function**.
- The resulting outputs from the function are known as the **range of the function**.

Definition

A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$$\boldsymbol{f(x) = mx + b} \quad \text{or} \quad f(x) = b + mx \quad \text{(they're equivalent)}$$

where

b is the **initial** or **starting value** of the function (when input, x, is zero),

and

m is the **constant rate of change** of the function. ▪

Modeling with Linear Functions

A **linear model** is a linear function that helps explain real-world scenarios involving quantities changing linearly. Because we are modeling real-world applications, we use descriptive variables appropriate for the scenario, rather than the standard x and y discussed in the previous section. Due to this change, the independent variable values will *replace* the x-values on the horizontal axis, while the dependent variable values will *replace* the y-values on the vertical axis. Also, when looking at linear models, domain constraints will be dependent on the problem.

For the purposes of this text, we will focus on economic and business scenarios such as depreciation, cost, revenue, profit, supply, and demand.

Linear Depreciation

Definition

Linear depreciation models the loss in value of an asset over time. It can be represented by

$$\boldsymbol{V(t) = mt + b},$$

where

V represents the **value** of the asset at **time** t,

b represents the **initial value** of the asset,

and

m represents the constant **rate of change** in value of the asset with respect to time, $\dfrac{\triangle V}{\triangle t}$. ■

A general graph of a linear depreciation function is given below in **Figure 2.2.2**.

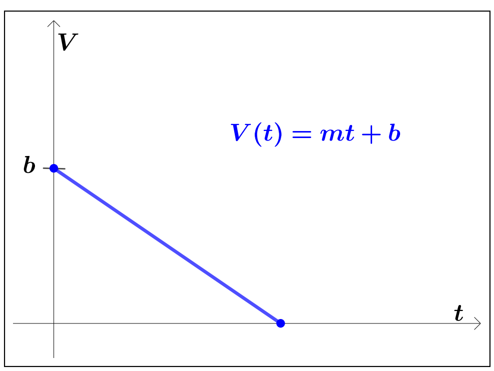


Figure 2.2.2: The first quadrant of the coordinate plane with a line segment sloping in the downward direction. The line segment starts on the V-axis at $(0, b)$ and ends when it reaches the t-axis.

N *Here, we will restrict our discussion to situations where the value of the asset decreases (depreciates) linearly over time. Thus, the slope of a linear depreciation model should always be negative.*

■ **Example 1** A company purchased \$120,000 of new office equipment and expects the value of the equipment to depreciate by \$16,000 per year.

a. Construct a linear model for the value of the equipment as a function of time.
b. What is the rate of depreciation of the equipment?
c. Compute and interpret the horizontal intercept.
d. Determine a reasonable domain and range for this function.

Solution:

a. In the problem, there are two changing quantities: time and value. The remaining value of the equipment depends on how long the company has owned it. We can start by defining our variables, including units.

Let the outputs, V, and inputs, t, be defined as follows:

$V :=$ the value of the equipment, in dollars

$t :=$ the time, in years, since purchase

Reading the problem, we identify two important values. The first, \$120,000, is the initial value for V (the value of b in a linear depreciation model). The other value, \$16,000 per year, appears to be a rate of change – the units of dollars per year match the units of our output variable divided by our input variable. The value of the equipment is depreciating, so we know that the value remaining is *decreasing* each year and the rate of change will produce a negative slope (the value of m in a linear depreciation model).

Using the information provided in the problem, we can write the following linear depreciation model:

$$V(t) = -16000t + 120000$$

b. The slope represents the rate of change in value with respect to time, and is equal to -16000 in this problem, which represents a loss of \$16,000 per year. By definition the slope of a linear depreciation model will always be negative, so the rate of depreciation is the rate at which the value decreases and is defined to be $|m|$, with units included. Thus, the rate of depreciation for the equipment is \$16,000/year.

c. Here the horizontal intercept is the t-intercept, as t is the independent variable. To solve for the t-intercept, we set the dependent variable, V, to zero, and solve for t:

$$\begin{aligned} 0 &= -16000t + 120000 \\ t &= \frac{120000}{16000} \\ t &= 7.5 \end{aligned}$$

The horizontal intercept is (7.5,0); this represents the input value where the output will be zero. Interpreting this, we could say: "The equipment will be worth \$0, or will have no remaining value, after 7.5 years."

d. When modeling any real-world scenario with functions, there is typically a limited domain over which the model will be valid – almost no trend continues indefinitely. In this problem, it certainly doesn't make sense to talk about input values less than zero. This model is also not valid after the horizontal intercept, as it does not make sense to talk about the equipment having a negative value.

Domain represents the set of input values. The reasonable domain for this function is $0 \le t \le 7.5$. Range represents the set of output values; the value starts at \$120,000 and ends with \$0 after 7.5 years. So, the corresponding range is $0 \le V(t) \le 120,000$.

Most importantly remember that domain and range are tied together, and whatever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

■

In the previous example, we discussed the rate of depreciation and the restricted domain. Related formal definitions can be found below.

Definition

Rate of depreciation is the amount of value an asset loses per time unit. It can be represented by $|m|$, with specified units. ■

Definition

Scrap value is the lowest value an asset obtains. Once an asset reaches scrap value, it remains at that value indefinitely. ■

The scrap value is assumed to be zero, unless given other information regarding the actual scrap value or the time when scrap value is reached.

In the previous example, the horizontal intercept is (7.5, 0), which means the equipment has no value after 7.5 years. This signifies the scrap value is \$0, and it occurs after 7.5 years.

Try It # 1:
A refrigerator has a value (in dollars) given by $V(t) = -400t + 5000$, where t represents the number of years since the refrigerator was purchased. Determine and interpret

- **a.** The rate of depreciation.
- **b.** $V(0)$ and $V(3)$
- **c.** The amount of time before the refrigerator reaches scrap value.

In the previous example, we were given the initial value and rate of depreciation. Oftentimes, this is not the case, but instead you may be given different, yet sufficient, information to determine the linear model. In the next example, we will demonstrate how to use skills learned about lines in the previous section to construct a linear model.

■ **Example 2** A new phone is purchased for \$839 and is worth \$543 two years later. Assuming the value of the phone decreases at a constant rate, answer the following.

- **a.** Determine the value (in dollars) of the phone, V, as a function of the number of years since its purchase, t.
- **b.** Determine the value of the phone after 18 months.
- **c.** If the scrap value of the phone is \$10, what is the phone worth after 15 years?

Solution:

a. Because the value of the phone decreases at a *constant rate*, we are trying to find a linear function of the form

$$V(t) = mt + b$$

From the information given, we know that $(0,839)$ and $(2,543)$ are two points defined by the function. Thus,

$$\begin{aligned} m &= \frac{\triangle V}{\triangle t} \\ &= \frac{543-839}{2-0} \\ &= \frac{-296}{2} \\ m &= -148 \end{aligned}$$

The purchase value of \$839 not only gives us the point (0, 839), but also tells us that $b = 839$. Therefore,

$$V(t) = -148t + 839$$

b. In our function, t represents the amount of *years* since the phone was purchased. Due to the fact that there are 12 months in 1 year, then

$$\begin{aligned} 18 \text{ months} &= \frac{18 \text{ months}}{12\frac{\text{months}}{\text{year}}} \\ &= 1.5 \text{ years} \end{aligned}$$

Thus, the value after 18 months can be found by evaluating our function when $t = 1.5$.

$$\begin{aligned} V(1.5) &= -148(1.5) + 839 \\ &= 617 \end{aligned}$$

So, after 18 months, the phone is worth \$617.

c. First, we need to determine how long it takes the phone to reach a scrap value of \$10. To do this we set the value, V, equal to 10 and solve for t:

$$\begin{aligned} 10 &= -148t + 839 \\ 148t &= 829 \\ t &= \frac{829}{148} \\ t &\approx 5.6 \text{ years} \end{aligned}$$

As the scrap value is the lowest value an item obtains, at any time after approximately 5.6 years, the phone will have a value of \$10. Thus, after 15 years the phone is still worth \$10.

The actual value at time t may differ from the calculated function value at time t, $V(t)$, for any time after an item reaches scrap value. Here, if we compute $V(15) = -148(15) + 839 = -1381$, we see the actual value, \$ 10, and the function value, $-\$1381$, differ greatly. Notice that, in the context of our problem, it is not possible for the phone to have a value of negative \$1381. This would mean that instead of someone paying \$10 for the phone after 15 years, you would have to pay them \$1381 to take the phone from you.

■

Cost, Revenue, and Profit

Businesses must keep track of and manage their expenditures, their sales, and the profits they realize. In this text, we will first examine what happens if these transactions are linear in nature.

Definition

When a company produces x items, the total cost function, $C(x)$, models the total cost of producing those items. The total cost includes both **fixed costs**, which are startup costs (like equipment and buildings), and **variable costs**, which are costs that depend on the number of items produced (like materials and labor).

In the most simple case,

$$\begin{aligned}\textbf{Total Cost} &= \textbf{(Variable Costs) + (Fixed Costs)}\\ &= \textbf{(production cost per item)(quantity) + (fixed costs)}\\ \boldsymbol{C(x)} &= \boldsymbol{mx+F},\end{aligned}$$

where x is the number of items (quantity) produced. ▪

A general graph of a linear total cost function is given below in **Figure 2.2.3**. Notice that even when 0 items are produced, the company still has fixed costs of $\$F$.

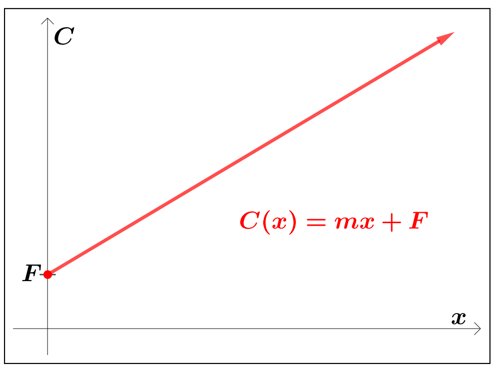


Figure 2.2.3: The first quadrant of the coordinate plane with a line sloping in the upward direction. The line starts on the C-axis at $(0,F)$.

▪ **Example 3** It costs a company a total of \$1900 to manufacture 60 items, and the company has fixed costs of \$700. If x represents the number of items manufactured, write the company's linear total cost function.

Solution:

We are looking for a function of the form $C(x) = mx+F$. From the information given, we know that $(60,1900)$ and $(0,700)$ are two points defined by the function. Thus,

$$\begin{aligned} m &= \frac{\triangle C}{\triangle x}\\ &= \frac{700-1900}{0-60}\\ &= \frac{-1200}{-60}\\ m &= 20\end{aligned}$$

With the fixed costs being \$700, the linear total cost function then becomes

$$C(x) = 20x + 700$$

■

It is important to remember that total costs include both variable and fixed costs. Thus, it is not correct to divide total costs by quantity to obtain the slope. In the previous example, $m \neq 1900/60$.

Definition

Revenue, $R(x)$, is the amount of money a company brings in from sales. In the most simple case,

$$\textbf{Revenue} = \textbf{(price per item)(quantity)}$$
$$\boldsymbol{R(x) = px},$$

where x is the number of items (quantity) sold. ■

A general graph of a linear revenue function is given below in **Figure 2.2.4**. Notice that when 0 items are sold, the company brings in \$0 of revenue.

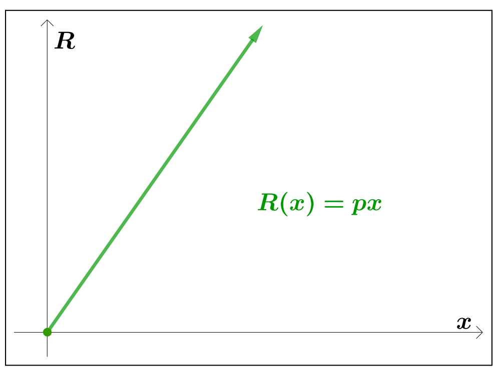


Figure 2.2.4: The first quadrant of the coordinate plane with a line sloping in the upward direction, starting from the origin.

■ **Example 4** What is the revenue function from selling the items in the previous example, if each item sells for \$35?

Solution:
By definition revenue is found by multiplying price per item times quantity sold, so we have

$$R(x) = p \cdot x = 35x$$

where x represents the number of items sold.

■

Definition

Profit, $P(x)$, is the amount of money a company brings in, after expenses.

$$\textbf{Profit} = \textbf{Revenue} - \textbf{Total Cost}$$
$$\boldsymbol{P(x) = R(x) - C(x)},$$

where x is the number of items (quantity) made and sold. ■

A general graph of a linear profit function is given below in **Figure 2.2.5**. Notice that when 0 items are produced and sold, the company will take a loss, due to the fixed costs.

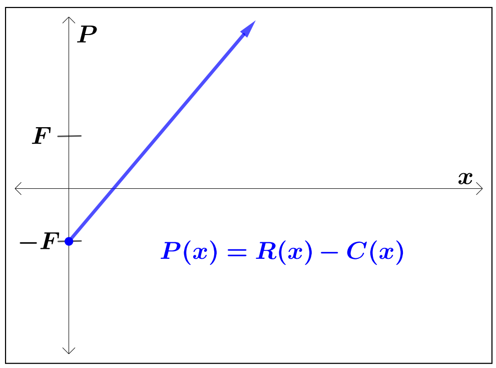


Figure 2.2.5: The first quadrant of the coordinate plane with a line sloping in the upward direction. The line starts on the P-axis at $(0, -F)$.

■ **Example 5** What is the profit function for the manufacturing and selling of the items produced in the previous examples?

Solution:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 35x - (20x + 700) \\ &= 35x - 20x - 700 \\ P(x) &= 15x - 700 \end{aligned}$$

■

Always subtract ALL costs; remember to distribute the negative. In the previous example, $P(x) \neq 35x - 20x + 700$.

When modeling a cost, revenue, or profit function of a company, we must use the appropriate information. At times this requires us to differentiate between types of costs to the company and money received by the company.

▪ **Example 6** To manufacture 100 items, it costs a company a total of \$32,000, and to manufacture 200 of these items, it costs the company a total of \$40,000. If each of these items sells for \$150, determine the profit function for the company making and selling these items. (Assume linear cost and revenue functions.)

Solution:
To determine the company's profit function, we first need to find the cost and revenue functions.

For total cost, we have the points $(100, 32000)$ and $(200, 40000)$ defined by the function. We can now calculate the slope:

$$\begin{aligned} m &= \frac{\triangle C}{\triangle x} \\ &= \frac{40000 - 32000}{200 - 100} \\ &= \frac{8000}{100} \\ m &= 80 \end{aligned}$$

Then, using the point $(100, 32000)$ and the point-slope form of a line, $C - C_1 = m(x - x_1)$, we get

$$\begin{aligned} C - 32000 &= 80(x - 100) \\ C - 32000 &= 80x - 8000 \\ C(x) &= 80x + 24000 \end{aligned}$$

Because each item sells for \$150, the revenue function is given by $R(x) = 150x$.

Therefore, the profit function will be

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 150x - (80x + 24000) \\ &= 150x - 80x - 24000 \\ P(x) &= 70x - 24000 \end{aligned}$$

▪

Previously we mentioned that domain constraints of linear models are dependent on the scenario being modeled. As shown in the general graphs of cost, revenue, and profit, the domain of each of these functions is restricted to non-negative values, as x always represents a quantity of items. One could logically say "it is not possible to produce/sell a negative number of items."

Try It # 2:
A donut shop estimates their fixed daily expenses are \$600. If each donut costs about \$0.05 to make and sells for \$0.60, determine the shop's daily total cost, revenue, and profit functions. Then, sketch all three functions on the same graph.

Supply and Demand

In economics, there are models for how prices are determined in a free market which state that **supply** and **demand** for a product are related to price. For the purposes of this text, we will assume supply and demand are linear.

Definition

Demand, $D(x)$, shows the relationship between the quantity of a certain product consumers desire and the price, p, they are willing to pay per item. Typically, as the number of items a consumer needs to purchase increases, the price the consumer is willing to pay per item decreases.

$$\boldsymbol{D(x) = \text{price}}$$
$$\boldsymbol{D(x) = p}$$
$$\text{or}$$
$$\boldsymbol{p(x) = mx + b},$$

where x is the number of items purchased. ■

A general graph of a linear demand function is given below in **Figure 2.2.6**. In the graph, the x-intercept represents the maximum number of items taken by consumers, if the item is free. Moreover, the p-intercept represents the price above which the item will not be purchased by consumers.

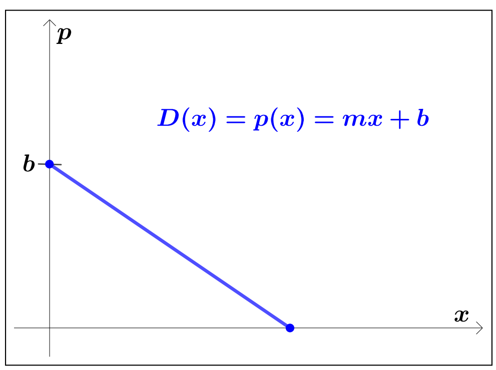


Figure 2.2.6: The first quadrant of the coordinate plane with a line segment sloping in the downward direction. The line segment starts on the p-axis at $(0, b)$ and ends when it reaches the x-axis.

N *Due to consumers' purchasing habits, all linear demand functions will have a negative slope.*

■ **Example 7** It has been determined that at a price of \$2 a store can sell 2400 of a particular type of toy doll, and for a price of \$8 the store can sell 600 such dolls. Determine the price of a doll, p, as a linear function of the number of dolls demanded, x.

Solution:

We are trying to find a demand function, $D(x)$, of the form $p(x) = mx + b$.

With the information given, we have the points $(2400, 2)$ and $(600, 8)$ defined by the function. Calculating the slope:

$$\begin{aligned} m &= \frac{\triangle p}{\triangle x} \\ &= \frac{8-2}{600-2400} \\ &= \frac{6}{-1800} \\ m &= -\frac{1}{300} \end{aligned}$$

Then, using the point $(2400, 2)$ and the point-slope form of a line: $p - p_1 = m(x - x_1)$, we have demand of

$$\begin{aligned} p - 2 &= -\frac{1}{300}(x - 2400) \\ p - 2 &= -\frac{1}{300}x + 8 \\ p(x) &= -\frac{1}{300}x + 10 \end{aligned}$$

■

N *We can double check our demand function by substituting the other given point,* $(x, p) = (600, 8)$.

$$\begin{aligned} 8 &\stackrel{?}{=} -\frac{1}{300}(600) + 10 \\ 8 &\stackrel{?}{=} -2 + 10 \\ 8 &= 8 \checkmark \end{aligned}$$

Definition

Supply, $S(x)$, shows the relationship between the quantity of a certain product producers are willing to supply based on the price per item, p, that is being offered. Typically, the more items a producer has to sell, the higher the price the producer desires to sell each item.

$$\begin{gathered} \boldsymbol{S(x) = \text{price}} \\ \boldsymbol{S(x) = p} \\ \text{or} \\ \boldsymbol{p(x) = mx + b}, \end{gathered}$$

where x is the number of items being sold. ■

A general graph of a linear supply function is given below in **Figure 2.2.7**. In the graph, the p-intercept represents the price below which no items will be supplied to the market.

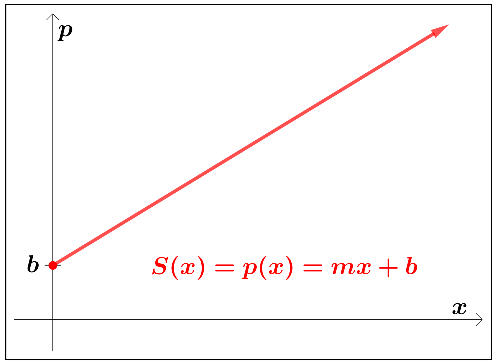


Figure 2.2.7: The first quadrant of the coordinate plane with a line sloping in the upward direction, starting on the p-axis at $(0,b)$.

N *Due to producers wanting the most money possible for the sale of their product, all linear supply functions will have a positive slope.*

Example 8 A manufacturer of toy dolls can supply 3000 dolls if the dolls are sold for \$8 each, but can supply only 1000 dolls if the dolls are sold for \$4 each. Determine the price per doll, p, as a linear function of the number of dolls supplied, x.

Solution:
We are trying to find a function of the form $p(x) = mx + b$.

With the information given, we have the points $(3000,8)$ and $(1000,4)$ defined by the function. Calculating the slope:

$$\begin{aligned} m &= \frac{\triangle p}{\triangle x} \\ &= \frac{4-8}{1000-3000} \\ &= \frac{-4}{-2000} \\ m &= \frac{1}{500} \end{aligned}$$

Then, using the point $(3000,8)$ and the point-slope form of a line, we have supply of

$$\begin{aligned} p-8 &= \frac{1}{500}(x-3000) \\ p-8 &= \frac{1}{500}x-6 \\ p(x) &= \frac{1}{500}x+2 \end{aligned}$$

■

Often we want to study both supply and demand simultaneously to compare the desires of both the consumers and producers.

■ **Example 9** At a price of \$2.50 per gallon, there is a demand in a certain town for 42.5 thousand gallons of gas and a supply of 20 thousand gallons. At a price of \$3.50, there is demand for 25.5 thousand gallons and a supply of 28 thousand gallons. Determine the linear supply and demand functions.

Solution:
We will use price, p (in dollars), as the output and quantity, x (in thousands of gallons of gas), as the input for both functions.

For supply, we have the points $(20, 2.50)$ and $(28, 3.50)$ defined by the function.
Calculating the slope:

$$\begin{aligned} m &= \frac{\triangle p}{\triangle x} \\ &= \frac{3.50 - 2.50}{28 - 20} \\ &= \frac{1}{8} \\ m &= 0.125 \end{aligned}$$

Then, using the point $(20, 2.50)$ and the point-slope form of a line,

$$\begin{aligned} p - 2.50 &= 0.125\,(x - 20) \\ p - 2.5 &= 0.125x - 2.5 \\ p(x) &= 0.125x \qquad \text{(Supply)} \end{aligned}$$

For demand, we have the points $(42.5, 2.50)$ and $(25.5, 3.50)$ defined by the function.
Calculating the slope:

$$\begin{aligned} m &= \frac{\triangle p}{\triangle x} \\ &= \frac{3.50 - 2.50}{25.5 - 42.5} \\ &= \frac{1}{-17} \\ m &= -\frac{1}{17} \end{aligned}$$

Then, using the point $(42.5, 2.50)$ and the point-slope form of a line,

$$\begin{aligned} p - 2.50 &= -\frac{1}{17}(x - 42.5) \\ p - 2.5 &= -\frac{1}{17}x + 2.5 \\ p(x) &= -\frac{1}{17}x + 5 \qquad \text{(Demand)} \end{aligned}$$

■

Try It # 3:
At a price of \$3, consumers will purchase 600 mechanical pencils. For every 50 cent decrease in price, consumers will purchase an additional 75 pencils. Determine the linear demand function, $p(x)$.

Try It # 4:
The producers of mechanical pencils will not supply pencils to the market if the price is below \$1, but for every 25 cent increase above \$1 the producers will supply an additional 25 pencils. Determine the linear supply function, $p(x)$.

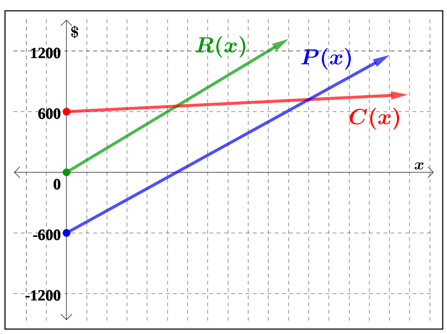
Try It Answers

1. **a.** The rate of depreciation equals \$400/year $\Rightarrow$ the refrigerator's value decreases by \$400 each year.

   b. $V(0) = 5000 \Rightarrow$ The refrigerator is initially worth \$5000.
   $V(3) = 3800 \Rightarrow$ After 3 years, the refrigerator is worth \$3800.

   c. 12.5 years $\Rightarrow$ After 12.5 years, the refrigerator has no value. Moreover, at any point in time after 12.5 years, the refrigerator will continue to have zero value.

2. Cost: $C(x) = 0.05x + 600$, $x :=$ the number of donuts produced
   Revenue: $R(x) = 0.60x$, $x :=$ the number of donuts sold
   Profit: $P(x) = 0.55x - 600$, $x :=$ the number of donuts made and sold.



3. $p(x) = -\dfrac{1}{150}x + 7$, where $x :=$ the number of pencils demanded, and $p :=$ the price of a pencil (in dollars).

4. $p(x) = \dfrac{1}{100}x + 1$, where $x :=$ the number of pencils supplied, and $p :=$ the price of a pencil (in dollars).

EXERCISES

Basic Skills Practice (Answers)

Writing a linear depreciation function.

1. An item is worth \$500 when it is purchased. After 3 years, it is worth \$320. Assuming the item is depreciating linearly with time, write the value of the item (in dollars) as a function of time (in years since purchase).

2. An item is bought and 7 years later is worth \$1000. The same item is worth \$200 fifteen years after purchase. Assuming the item is depreciating linearly with time, write the value of the item (in dollars) as a function of time (in years since purchase).

3. An item purchased for \$850 reaches scrap value after 8 years. Assuming the item is depreciating linearly with time, write the value of the item (in dollars) as a function of time (in years since purchase).

For Exercises 4 - 8, the value of an item is given to be $V(t) = -25t + 1250$, where t is the number of years since the item was purchased and V is given in dollars.

4. What was the purchase price of the item?

5. After how many years will the item be worth only \$1000?

6. After how many years will the item achieve scrap value?

7. What was the item worth after 30 years?

8. What is rate of depreciation?

Writing a linear total cost, revenue, or profit function.

9. A company has monthly fixed costs of \$10,000 and a production cost of \$25 for each item it produces. What is the company's monthly linear total cost function?

10. A company sells an item it produces to the public for \$150 each. What is the company's linear revenue function?

11. A company has a total cost function of $C(x) = 5x + 800$ and revenue function of $R(x) = 22x$. What is the company's profit function?

Linear supply and demand functions.

12. Given a demand function of $p(x) = -\frac{1}{2}x + 40$, where x represents the number of items demanded and $p(x)$ represents the price/item (in dollars),

    a. At what price will 50 items be demanded?

    b. Above what price will consumers not buy this item?

    c. How many items will consumers demand if the items are priced at \$20?

    d. How many items will consumers demand if the items are free?

13. Given a supply function of $p(x) = 3x + 15$, where x represents the number of items supplied and $p(x)$ represents the price/item (in dollars),

    a. At what price will producers supply 100 items to the market?

    b. Producers will only supply the items if the price is above what value?

    c. How many items will producers provide to the market at a price of \$45?

Writing a linear supply or demand function.

14. At a price of \$20 per item, 100 items are demanded by consumers, but at a price of \$50 per item, only 20 items are demanded. Construct the linear demand function, $p(x)$, for this item.

15. At a price of \$40/item, producers will provide 125 items to the market. At a price of \$80/item, producers will provide 325 items. Construct the linear supply function, $p(x)$, for this item.

Intermediate Skills Practice (Answers)

Writing a linear depreciation function.

16. An item is worth \$10 when it is purchased. After 10 years, it is worth 50 cents. Assuming the item is depreciating linearly with time, write the value of the item (in dollars) as a function of time (in years since purchase).

17. An item is purchased in 1996 for \$400,000 and in 2010 it is worth \$120,000. Assuming the item is depreciating linearly with time, write the value of the item (in dollars) as a function of time (in years since 1996).

18. An item purchased for \$3500 has a scrap value of \$200 after 11 years. Assuming the item is depreciating linearly with time, write the value of the item (in dollars) as a function of time (in years since purchase).

For Exercises 19 - 23, the value of an item is given to be $V(t) = -85000t + 1000000$, where t is the number of years since the item was purchased and V is given in dollars.

19. What was the purchase price of the item?

20. How many years will it take the item to reach a scrap value of \$150,000?

21. What was the item worth after 5 years?

22. What is the value of the item after 12 years?

23. What is rate of depreciation?

Writing a linear total cost, revenue, or profit function.

24. A company has fixed costs of $15,000 each month and production costs of $1/item. If each of these items is sold to the public for $5, determine the company's monthly linear profit function.

25. A company can produce 20 items for a total cost of $795. If the company has fixed costs of $495, determine the company's linear total cost function.

26. A company has production costs of $20 per item. If the company can produce 100 items for a total cost of $2850, determine the company's linear cost function.

27. A company brings in a total of $1275 in revenue from selling 85 items. What is the company's linear revenue function?

Writing a linear supply or demand function.

28. Consumers will demand 500 items at a price of $2/item. For every $2 increase in price per item, 4 fewer items will be demanded. Construct the linear demand function, $p(x)$, for this item.

29. Consumers will demand 1000 items at a price of $3.50 per item. When the price per item increases to $4, the number of items demanded will decrease by 50. Construct the linear demand function, $p(x)$, for this item.

30. Producers will supply 50 items at a price of $10 per item. For every $5 increase in price per item, 500 more items will be provided. Construct the linear supply function, $p(x)$, for this item.

31. A producer will not supply any items when the price is $475 or lower, but when the price per item is $500, the producer is willing to supply 800 items. Construct the linear supply function, $p(x)$, for this item.

32. Consumers will buy 8000 items at a price of $20/item. If the price goes up to $25/item, they they will only buy 6000 items. Manufacturers will not market this item below $10, but for every $5 increase in price per item, 2000 more items will be provided to the market.

    a. Construct the linear demand function, $p(x)$.

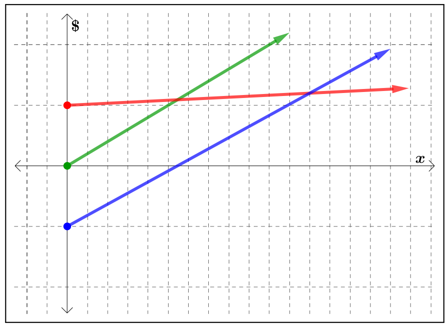
    b. Construct the linear supply function, $p(x)$.

Mastery Practice (Answers)

33. An item is bought and after 2 years has a value $5000. Ten years after its purchase, the item is worth $4500. Assuming the item is depreciating linearly, what was the purchase price of the item?

34. An item purchased for $500 has a scrap value of $150 after 14 years. Compute the value of the item after 36 months, assuming the item depreciates linearly.

35. Given the graph below, identify the total cost function, $C(x)$, the revenue function, $R(x)$, and the profit function, $P(x)$.



36. A company finds the total cost of producing 30 items is \$9150, while they can produce 55 items for a total cost of \$10,775. Each of these items is sold to the public for \$80.

    a. Compute the production cost per item.

    b. Write the company's linear profit function.

37. A manufacturer has fixed costs of \$180 and a production cost of \$30 per item manufactured. What is the selling price of the item, if the company has a profit of \$4320 when selling 100 items? (Assume total cost and revenue are linear.)

38. Given x represents the number of items supplied or demanded each month, p represents the unit price of the items (in dollars), Equation A is $-5x+2p=60$ and Equation B is $3x+2p=300$, answer the following.

    a. Which equation is the demand equation? Why?

    b. How many items will consumers purchase if the items are free?

    c. Above what price will consumers not buy the item?

    d. Producers will only provide the items if the price is above what value?

Communication Practice (Answers)

39. Explain, in your own words, the meaning of the rate of depreciation.

40. If a linear depreciation function, $V(t)$, is given for $0 \leq t \leq 20$, explain how you determine the value of the item for values of $t > 20$.

41. Illustrate the relationships between linear total cost, revenue, and profit functions, graphically.

42. Explain, in your own words, why the linear supply function $p(x)$ has a positive slope.

43. Explain, in your own words, why the linear demand function $p(x)$ has a negative slope.

2.3 Systems of Two Equations in Two Unknowns

© Photo by Texas A&M University, 2020

A water sports equipment manufacturer introduces a new line of kayaks. The manufacturer tracks its costs, which is the amount it spends to produce the kayak, and its revenue, which is the amount it earns through sales of its kayaks. How can the company determine if it is making a profit with its new line? How many kayaks must be produced and sold before a profit is possible?

Learning Objectives:
In this section, you will learn techniques for solving systems of two linear equations in two unknowns and their applications. Upon completion you will be able to:

- Specify the number of solutions to a system of two linear equations in two unknowns.
- Use algebraic techniques to determine the solution, if one exists, of a system of two linear equations in two unknowns.
- Use technology to graph and determine the solution, if one exists, of a system of two linear equations in two unknowns
- Compute break-even points for business related applications.
- Compute equilibrium points for applications relating to supply and demand scenarios.

Identifying Solutions to Linear Equations

In order to investigate situations such as that of the water sports manufacturer, we need to recognize that we are dealing with more than one variable and, likely, more than one equation. A **system of linear equations** consists of two or more linear equations made up of two or more variables, such that all equations in the system are considered simultaneously.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$2x + y = 15$$
$$3x - y = 5$$

To determine the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time.

The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(x, y) = (4, 7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations.

$$2(4) + (7) = 15 \checkmark$$
$$3(4) - (7) = 5 \checkmark$$

Shortly, we will investigate methods of finding such a solution, if it exists.

■ **Example 1** Determine whether the ordered pair $(5, 1)$ is a solution to the given system of linear equations.

$$x + 3y = 8$$
$$2x - 9 = y$$

Solution:
Substitute $x = 5$ and $y = 1$ into both equations.

$$(5) + 3(1) \stackrel{?}{=} 8$$
$$8 = 8 \checkmark$$

$$2(5) - 9 \stackrel{?}{=} 1$$
$$1 = 1 \checkmark$$

The ordered pair $(x, y) = (5, 1)$ satisfies both equations, so it is a solution to the system.

N *By plotting the graph of each equation, we can see the solution to a system of two linear equations. Because the solution is an ordered pair that satisfies both equations, it is a point that lies on both of the lines, and, thus, the point of intersection of the two lines. (**See Figure 2.3.2**)*

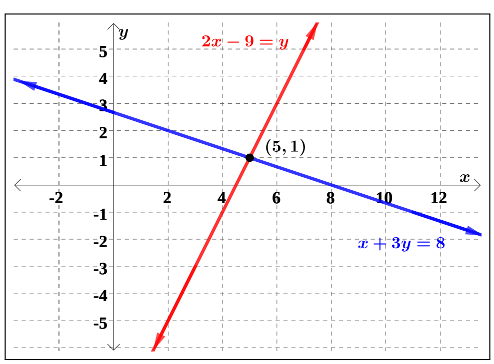


Figure 2.3.2: The coordinate plane with the lines $2x - 9 = y$ and $x + 3y = 8$ and the intersection point, $(5, 1)$, graphed.

■

Try It # 1:
Determine whether the ordered pair $(8,5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Types of Solutions

While some linear systems have one solution, some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A **consistent system** is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes ($m_1 \neq m_2$) and intersect at one point in the plane. A **consistent system** is considered to be a **dependent system** if the equations have the same slope and the same y-coordinate of the y-intercept ($m_1 = m_2$ and $b_1 = b_2$). In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**. Here the equations represent two parallel lines, where the lines have the same slope and different y-coordinates of the y-intercepts ($m_1 = m_2$ and $b_1 \neq b_2$). There are no points common to both lines; hence, there is no solution to the system.

Definition

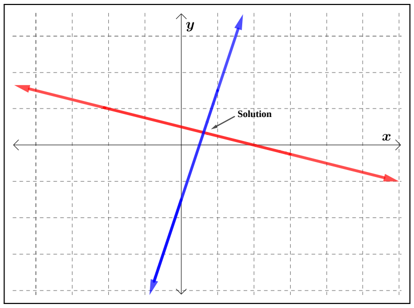
There are three types of systems of two linear equations in two variables, each with a different type of solutions.

- An **independent system** has exactly one solution, the coordinate pair (x, y).
  The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution.
  The two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions.
  The lines are coincident; they are the same line.
  Every coordinate pair on the line is a solution to both equations.

Given the system of linear equations

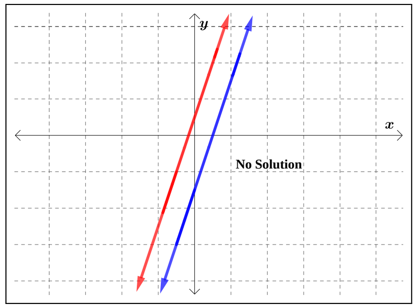
$$\begin{aligned} y_1 &= m_1 x + b_1 \\ y_2 &= m_2 x + b_2, \end{aligned}$$

Figures 2.3.3, **2.3.4**, and **2.3.5** below show graphical representations of each type of system.



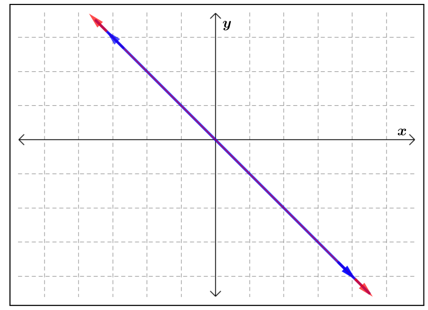
Independent System
$m_1 \neq m_2$

Figure 2.3.3: The coordinate plane with two lines drawn, and the one intersection point labeled.



Inconsistent System
$m_1 = m_2$ and $b_1 \neq b_2$

Figure 2.3.4: The coordinate plane with two lines drawn. The lines never intersect.



Dependent System
$m_1 = m_2$ and $b_1 = b_2$

Figure 2.3.5: The coordinate plane with two lines drawn. The graph appears to only have one line, as the two lines are layered on top of each other.

We can use the above conditions on slopes and y-intercepts to determine the type of linear system we are working with.

▪ **Example 2** How many solutions do the following systems of linear equations have? Justify your answer without graphing or actually calculating a solution.

a. $\begin{cases} x = 9 - 2y \\ x + 2y = 13 \end{cases}$

b. $\begin{cases} x + 3y = 2 \\ 3x + 9y = 6 \end{cases}$

c. $\begin{cases} 2x + 3y = -16 \\ 5x - 10y = 30 \end{cases}$

Solution:
To determine the number of solutions to a system of two linear equations in two unknowns, we can write each equation in slope-intercept form and compare slopes and y-intercepts.

a.

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \\ \left(m_1 = -\frac{1}{2}\right.&, \left. b_1 = \frac{9}{2}\right) \end{aligned}$$

$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\frac{1}{2}x + \frac{13}{2} \\ \left(m_2 = -\frac{1}{2}\right.&, \left. b_2 = \frac{13}{2}\right) \end{aligned}$$

Notice that $m_1 = m_2$, but $b_1 \neq b_2$, which means the lines are parallel and the system is inconsistent. Thus, the system has no solution.

b.

$$\begin{aligned} x+3y &= 2 \\ 3y &= -x+2 \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned} \qquad \begin{aligned} 3x+9y &= 6 \\ 9y &= -3x+6 \\ y &= -\frac{3}{9}x+\frac{6}{9} \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

$$\left(m_1 = -\frac{1}{3},\ b_1 = \frac{2}{3}\right) \qquad \left(m_2 = -\frac{1}{3},\ b_2 = \frac{2}{3}\right)$$

Notice that $m_1 = m_2$ and $b_1 = b_2$, which means the lines coincide and are the same line. Thus, the system is dependent and the system has infinitely many solutions.

c.

$$\begin{aligned} 2x+3y &= -16 \\ 3y &= -2x-16 \\ y &= -\frac{2}{3}x-\frac{16}{3} \end{aligned} \qquad \begin{aligned} 5x-10y &= 30 \\ -10y &= -5x+30 \\ y &= \frac{5}{10}x-\frac{30}{10} \\ y &= \frac{1}{2}x-3 \end{aligned}$$

$$\left(m_1 = -\frac{2}{3},\ b_1 = -\frac{16}{3}\right) \qquad \left(m_2 = \frac{1}{2},\ b_2 = -3\right)$$

Notice that $m_1 \neq m_2$, which means that the system is independent and has exactly one solution.

▪

Try It # 2:

How many solutions do the following systems of linear equations have? Justify your answer without graphing or actually calculating a solution.

a. $\begin{cases} 2y-2x &= 2 \\ 2y-2x &= 6 \end{cases}$

b. $\begin{cases} y-2x &= 5 \\ -3y+6x &= -15 \end{cases}$

Finding an Unknown Coefficient

Knowledge of the number of solutions a system has can help identify any unknown coefficients.

Example 3 State the value(s) of k such that the following system of linear equations has no solution.

$$\begin{aligned} 2x + ky &= 15 \\ -3x + 8y &= 24 \end{aligned}$$

Solution:
If a system of two linear equations in two unknowns has no solution, then the lines must be parallel, meaning

$$m_1 = m_2 \qquad \text{AND} \qquad b_1 \neq b_2.$$

To find and compare the values of m_1 to m_2 and b_1 to b_2, we must first write each equation is slope-intercept form.

$$\begin{aligned} 2x + ky &= 15 \\ ky &= -2x + 15 \\ y &= -\frac{2}{k}x + \frac{15}{k} \end{aligned}$$

$$\begin{aligned} -3x + 8y &= 24 \\ 8y &= 3x + 24 \\ y &= \frac{3}{8}x + \frac{24}{8} \\ y &= \frac{3}{8}x + 3 \end{aligned}$$

$$\left(m_1 = -\frac{2}{k},\ b_1 = \frac{15}{k}\right) \qquad \left(m_2 = \frac{3}{8},\ b_2 = 3\right)$$

For the slopes of the lines to be equal, then

$$\begin{aligned} m_1 &= m_2 \\ -\frac{2}{k} &= \frac{3}{8} \\ -2(8) &= 3k \\ -16 &= 3k \\ -\frac{16}{3} &= k \end{aligned}$$

This means that if $k = -\frac{16}{3}$, the slopes will be equal.

We must now check that this value of k gives different y-intercepts. For $k = -\frac{16}{3}$,

$$b_1 = \frac{15}{k} = \frac{15}{\left(-\frac{16}{3}\right)} = 15 \cdot \left(-\frac{3}{16}\right) = -\frac{45}{16}.$$

Considering $b_2 = 3$, we know $b_1 \neq b_2$, when $k = -\frac{16}{3}$.

Thus, if $k = -\dfrac{16}{3}$, the system will have no solution.

■

Example 4 State the value(s) of k such that the following system of linear equations has infinitely many solutions.

$$\begin{aligned} kx + 8y &= 36 \\ x + 2y &= 9 \end{aligned}$$

Solution:
If a system of two linear equations in two unknowns has infinitely many solutions, then the lines must be the same, meaning

$$m_1 = m_2 \qquad \text{AND} \qquad b_1 = b_2.$$

To find and compare the value of m_1 to m_2 and b_1 to b_2, we must first write each equation is slope-intercept form.

$$\begin{aligned} kx + 8y &= 36 \\ 8y &= -kx + 36 \\ y &= -\frac{k}{8}x + \frac{36}{8} \\ y &= -\frac{k}{8}x + \frac{9}{2} \end{aligned}$$

$$\left(m_1 = -\frac{k}{8},\ b_1 = \frac{9}{2}\right)$$

$$\begin{aligned} x + 2y &= 9 \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned}$$

$$\left(m_2 = -\frac{1}{2},\ b_2 = \frac{9}{2}\right)$$

Upon inspection, it is clear $b_1 = b_2$.

For the slopes of the lines to be equal,

$$\begin{aligned} m_1 &= m_2 \\ -\frac{k}{8} &= -\frac{1}{2} \\ -k(2) &= -8(1) \\ -k &= -4 \\ k &= 4 \end{aligned}$$

Thus, if $k = 4$, then the slopes are the same and the y-intercepts are the same, so the system will have infinitely many solutions.

■

Try It # 3:
State the value(s) of k such that the following system of linear equations has exactly one solution.

$$\begin{aligned} \frac{1}{4}x + ky &= -\frac{1}{4} \\ -7x + 7y &= 21 \end{aligned}$$

Solving Systems of Two Linear Equations in Two Unknowns

Graphical Methods

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution(s), if they exist, by graphing the system of linear equations on the same set of axes.

Example 5 Solve the following system of linear equations, by graphing. Then, identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution:
To graph each equation, by hand, we need two points that lie on each line. As the lines were both given in standard form, the two easiest points to compute are the x- and y-intercepts.

For the first equation, we have intercepts of $(-4, 0)$ and $(0, -8)$, as calculated below.

$$\begin{aligned} 2x + y &= -8 \\ 2x + (0) &= -8 \\ 2x &= -8 \\ x &= -4 \end{aligned} \qquad \begin{aligned} 2x + y &= -8 \\ 2(0) + y &= -8 \\ y &= -8 \end{aligned}$$

For the second equation, we have intercepts of $(-1, 0)$ and $(0, 1)$, as calculated below.

$$\begin{aligned} x - y &= -1 \\ x - (0) &= -1 \\ x &= -1 \end{aligned} \qquad \begin{aligned} x - y &= -1 \\ (0) - y &= -1 \\ -y &= -1 \\ y &= 1 \end{aligned}$$

Using the intercepts of each line, we can graph both equations on the same set of axes, as shown in **Figure 2.3.6**.

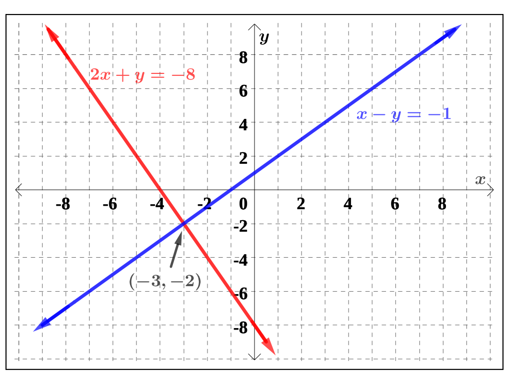


Figure 2.3.6: The coordinate plane with the lines $2x + y = -8$ and $x - y = -1$ graphed.

Upon inspection of the graph above, the lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is indeed the solution to the system, by substituting the ordered pair into both original equations.

$$2(-3)+(-2) \stackrel{?}{=} -8 \qquad (-3)-(-2) \stackrel{?}{=} -1$$
$$-8 = -8 \checkmark \qquad -1 = -1 \checkmark$$

Thus, the solution to the system is the ordered pair $(x,y) = (-3,-2)$, so the system is independent.

■

Not all systems have integer solutions and graphing equations by hand may not be precise enough to identify the exact solution. An easier way to determine a graphical solution to a system of two linear equations in two unknowns is through the use of a graphing calculator.

Previously in this chapter, we graphed equations in the TI-84 if they were written in the style $y =$ ______. For a system of two linear equations you can graph both equations in the same window and identify the point(s) of intersection, if any exist. We will demonstrate this process in the following example.

■ **Example 6** Solve the following system of linear equations, by graphing with the use of technology. Also, identify the type of system.

$$\begin{aligned} 2x+5y &= -4 \\ 7x+2y &= 9 \end{aligned}$$

Solution:
First, we will write each linear equation in slope-intercept form.

$$\begin{aligned} 2x+5y &= -4 \\ 5y &= -2x-4 \\ y &= -\frac{2}{5}x-\frac{4}{5} \end{aligned} \qquad \begin{aligned} 7x+2y &= 9 \\ 2y &= -7x+9 \\ y &= -\frac{7}{2}x+\frac{9}{2} \end{aligned}$$

Looking at the slopes, we see that they are different ($m_1 \neq m_2$). Thus, we know we will have exactly one solution to the system and that the system is independent.

To find this solution, we will enter each equation into the calculator, as seen in **Figure 2.3.7**. Next, we will graph the lines on the same set of axes, as shown in **Figure 2.3.8**.

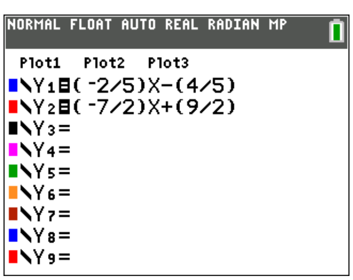


Figure 2.3.7: Calculator screenshot showing the equations as Y_1 and Y_2.

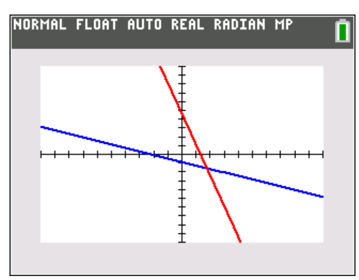


Figure 2.3.8: Calculator screenshot of the graphs of the lines in the standard window.

The graph in **Figure 2.3.8** is graphed in the standard window. In order to proceed and find the solution, the point of intersection must be visible in the window. If you cannot see the point of intersection on your calculator screen, remember to adjust your window.

Once you have graphed both equations and the point of intersection is shown in your window, we use the intersect operation (press $\boxed{\text{2ND}}$, $\boxed{\text{TRACE}}$, and scroll down to 5:intersect), as in **Figure 2.3.9**.

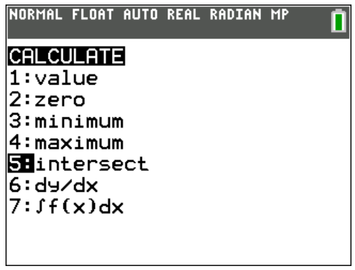


Figure 2.3.9: Calculator screenshot showing the CALC menu, with the intersect operation.

This function will prompt you to choose the lines you would like to intersect, as shown in **Figures 2.3.10** and **2.3.11**. When the correct first line is listed at the top of the screen, press $\boxed{\text{ENTER}}$ to be able to choose the second line.

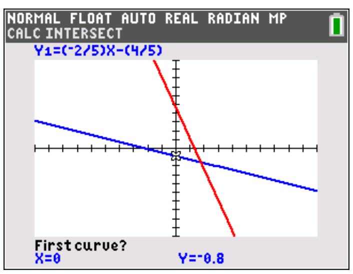


Figure 2.3.10: Calculator screenshot focused on the first curve.

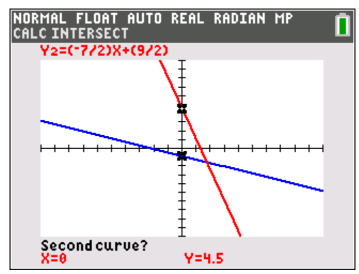


Figure 2.3.11: Calculator screenshot focused on the second curve.

Finally, you will be asked to "Guess" the solution, as shown in **Figure 2.3.12**. You can move your cursor to the point of intersection using the arrow buttons and press $\boxed{\text{ENTER}}$. Afterwards, the coordinates of the intersection point will be displayed at the bottom of the screen, as shown in **Figure 2.3.13**.

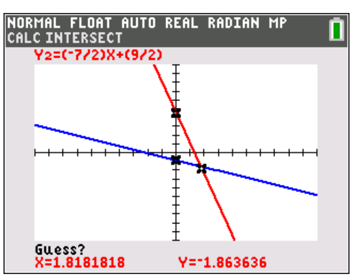


Figure 2.3.12: Calculator screenshot showing the guess.

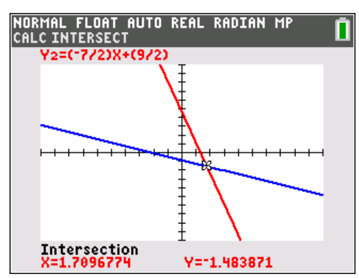


Figure 2.3.13: Calculator screenshot showing the approximate values of the coordinates of the intersection.

The coordinates of the intersection point shown in **Figure 2.3.13** are approximately $(x, y) = (1.710, -1.484)$, but are not exact. To find the *exact* coordinates, return to the home screen. Then recall the x-coordinate of the intersection point (press $\boxed{X,T,\theta,n}$ and then $\boxed{\text{ENTER}}$), as in **Figure 2.3.14**. Convert to a exact fraction (press $\boxed{\text{MATH}}$, choose 1:Frac, and press $\boxed{\text{ENTER}}$), as in **Figure 2.3.15**.

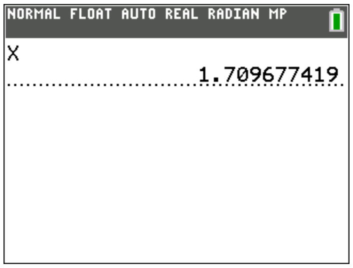


Figure 2.3.14: Calculator screenshot showing the x-value saved in the calculator.

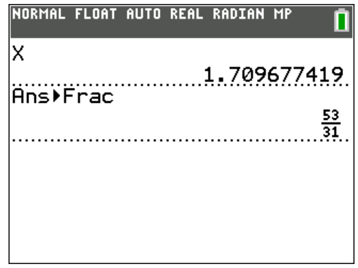


Figure 2.3.15: Calculator screenshot with X converted to a fraction.

To find the exact y-value, you can manually calculate its value using the exact x-coordinate or use function notation on your calculator (press $\boxed{\text{VARS}}$, cursor right to Y-Vars, choose 1:Function, select Y_1, and then press $\boxed{(}$, $\boxed{X,T,\theta,n}$, and $\boxed{)}$) and convert to an exact value, as shown in **Figure 2.3.16**.

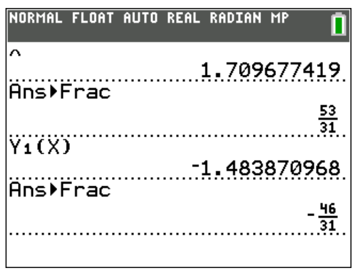


Figure 2.3.16: Calculator screenshot with the y-value converted to a fraction.

Thus, the solution to the system is the ordered pair $(x, y) = \left(\frac{53}{31}, -\frac{46}{31}\right)$, and the system is independent.

■

N *In inconsistent or dependent cases we can still graph the system to determine the existence of a solution and type of system. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinitely many solutions and is a dependent system. The solution to a dependent system will be explained shortly.*

Try It # 4:
Solve the following system of linear equations, by graphing.

$$\begin{aligned} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{aligned}$$

Algebraic Methods

Solving a linear system in two variables through graphing by hand works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method, as illustrated in the previous example. Instead of using technology to graph the system, we will consider two algebraic methods of solving a system of linear equations.

One such method is solving a system of equations by the **Substitution Method**, in which we solve one of the equations for one variable and then *substitute* the result into the second equation to solve for the second variable.

Suppose we are looking for the solution to the system

$$\begin{aligned} -x+y &= -5 \\ 2x-5y &= 1 \end{aligned}$$

First, we will solve the first equation for y, because its coefficient is 1.

$$\begin{aligned} -x+y &= -5 \\ y &= x-5 \end{aligned}$$

Next, we can substitute the expression $x-5$ for y in the second equation, and solve for x.

$$\begin{aligned} 2x-5y &= 1 \\ 2x-5(x-5) &= 1 \\ 2x-5x+25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

$$\begin{aligned} -(8)+y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(x,y)=(8,3)$.

Check the solution by substituting $(8,3)$ into both of the *original* equations.

$$\begin{aligned} -x+y &= -5 \\ -(8)+(3) &= -5 \ \checkmark \end{aligned} \qquad \begin{aligned} 2x-5y &= 1 \\ 2(8)-5(3) &= 1 \ \checkmark \end{aligned}$$

The Substitution Method for solving a system of linear equations in two variables is generalized, as follows.

Substitution Method

1. Solve one equation for one of the variables.
2. Substitute the result into the second equation to solve for the remaining variable, if possible.
3. Then substitute the result of Step 2 back into the equation found in Step 1, and solve for the variable value not known.

When no variable has a coefficient of 1, it is often easier to solve both equations for y. The Substitution Method then results in 'setting the two y-values equal to each other,' solving for x, and continuing with the process discussed above to determine the solution.

Example 7 Solve the following system of linear equations, by substitution.

$$\begin{aligned} 3x + 6y &= 27 \\ 2x + 4y &= 26 \end{aligned}$$

Solution:
As none of the variables have a coefficient of 1, first manipulate the equations so that they are both in slope-intercept form.

$$\begin{aligned} 3x + 6y &= 27 \\ 6y &= -3x + 27 \\ y &= -\frac{3}{6}x + \frac{27}{6} \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned} \qquad \begin{aligned} 2x + 4y &= 26 \\ 4y &= -2x + 26 \\ y &= -\frac{2}{4}x + \frac{26}{4} \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

Now, we can substitute the expression $-\frac{1}{2}x + \frac{13}{2}$ from the second equation for y into the first equation, setting the y-values equal to each other:

$$\begin{aligned} y &= -\frac{1}{2}x + \frac{9}{2} \\ -\frac{1}{2}x + \frac{13}{2} &= -\frac{1}{2}x + \frac{9}{2} \\ \frac{13}{2} - \frac{9}{2} &= -\frac{1}{2}x + \frac{1}{2}x \\ 2 &= 0 \text{ ✗} \end{aligned}$$

Clearly this statement is a contradiction, because $2 \neq 0$. Therefore, the system has no solution and is inconsistent.

N *In general when searching for an algebraic solution to an inconsistent system, the result will be a contradiction.*

Comparing the slope-intercept equations, we see that they have the same slope, but different y-intercepts. Therefore, the lines are parallel and do not intersect, which confirms the result found algebraically through substitution, and the existence of an inconsistent system. The graphs of the linear equations in this example are shown in **Figure 2.3.17**.

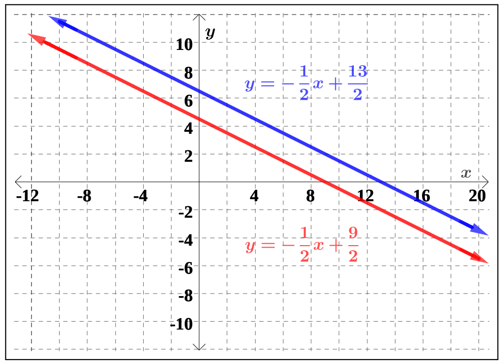


Figure 2.3.17: The coordinate plane with lines $y = -\frac{1}{2}x + \frac{9}{2}$ and $y = -\frac{1}{2}x + \frac{13}{2}$.

■

■ **Example 8** Solve the following system of linear equations, by substitution.

$$\begin{aligned} 2x - 4y &= 6 \\ 3x - 6y &= 9 \end{aligned}$$

Solution:

Again we have no coefficients of 1, so we first manipulate the equations so that they are both in slope-intercept form.

$$\begin{aligned} 2x - 4y &= 6 \\ -4y &= -2x + 6 \\ y &= \frac{-2}{-4}x + \frac{6}{-4} \\ y &= \frac{1}{2}x - \frac{3}{2} \end{aligned} \qquad \begin{aligned} 3x - 6y &= 9 \\ -6y &= -3x + 9 \\ y &= \frac{-3}{-6}x + \frac{9}{-6} \\ y &= \frac{1}{2}x - \frac{3}{2} \end{aligned}$$

Now, we can substitute the expression $\frac{1}{2}x - \frac{3}{2}$ from the second equation for y into the first equation, setting the y-values equal to each other:

$$\begin{aligned} y &= \frac{1}{2}x - \frac{3}{2} \\ \frac{1}{2}x - \frac{3}{2} &= \frac{1}{2}x - \frac{3}{2} \\ \frac{1}{2}x - \frac{1}{2}x &= -\frac{3}{2} + \frac{3}{2} \\ 0 &= 0 \checkmark \end{aligned}$$

Clearly this statement is always true, no matter the values of x and y. Therefore, the system has infinitely many solutions and is dependent.

N *In general when searching for an algebraic solution to a dependent system, the result will be an identity, often* $0 = 0$.

We can also notice both equations are equivalent to $y = \frac{1}{2}x - \frac{3}{2}$, once we have written both equations in slope-intercept form and simplified. Thus, graphing both on the same axes will produce overlapping lines showing every point on 'the line' will be a solution to the system. (See **Figure 2.3.18**)

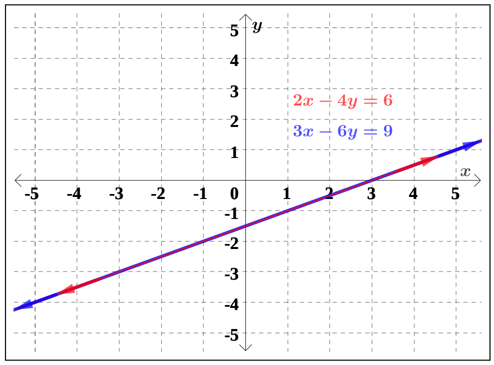


Figure 2.3.18: The coordinate plane with lines $2x-4y=6$ and $3x-6y=9$.

In other words, if an ordered pair (x, y) satisfies the equation $2x-4y=6$, it automatically satisfies the equation $3x-6y=9$. One way to describe the solution set to this dependent system is using what is called a **parametric solution to a system**.

For each value of x, the formula $y = \frac{1}{2}x - \frac{3}{2}$ determines the corresponding y-value of a solution. Because we have no restrictions on x, it is called a **free variable**. We define the free variable, x, with a **parameter**, often t, so that $x = t$ and then $y = \frac{1}{2}t - \frac{3}{2}$.

Our set of solutions can then be described as $(x, y) = \left(t, \frac{1}{2}t - \frac{3}{2}\right)$, where t is any real number.

For specific values of t, we can generate **specific (particular) solutions**. For example, $t = 0$ gives us the solution $\left(0, -\frac{3}{2}\right)$, while $t = 117$ gives us $(117, 57)$, and we could readily check each of these particular solutions satisfy both equations.

▪

Try It # 5:
Solve the following system of linear equations, by substitution. If the system is dependent, write the parametric solution.

$$\begin{aligned} y - 2x &= 5 \\ -3y + 6x &= -15 \end{aligned}$$

Another algebraic method of solving systems of linear equations is the **Addition Method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero and a variable is eliminated. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations using multiplication, so that one variable will be eliminated by addition.

Suppose we are looking for the solution to the system:

$$\begin{aligned} x + 2y &= -1 \\ -x + y &= 3 \end{aligned}$$

It is important to note all variables are on the same side of the equals sign and both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x, without needing to multiply either equation by a constant.

$$\begin{array}{r} x + 2y = -1 \\ \mathbf{+}\ (-x + y = 3) \\ \hline 3y = 2 \end{array}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$\begin{aligned} 3y &= 2 \\ y &= \frac{2}{3} \end{aligned}$$

Then, we can substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x + y &= 3 \\ -x + \frac{2}{3} &= 3 \\ -x &= 3 - \frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $(x, y) = \left(-\frac{7}{3}, \frac{2}{3}\right)$.

We can verify the solution by substituting the solution in the first equation.

$$\begin{aligned} x + 2y &= -1 \\ \left(-\frac{7}{3}\right) + 2\left(\frac{2}{3}\right) &\stackrel{?}{=} -1 \\ -\frac{7}{3} + \frac{4}{3} &\stackrel{?}{=} -1 \\ -\frac{3}{3} &\stackrel{?}{=} -1 \\ -1 &= -1 \checkmark \end{aligned}$$

■ **Example 9** Solve the given system of linear equations, using the Addition Method.

$$\begin{aligned} 2x + 3y &= -16 \\ 5x - 10y &= 30 \end{aligned}$$

Solution:

Because none of the variables have coefficients which are additive inverses, we will need to multiply at least one equation by a constant in order to eliminate one variable. Let's choose to eliminate x. The least common multiple between $2x$ and $5x$ is $10x$, so we will multiply the first equation by -5 and the second equation by 2. (The final result would be the same if we chose to multiply the first equation by 5 and the second by -2. We leave it to the reader to verify.)

$$\begin{aligned} -5(2x+3y) &= -5(-16) \\ -10x-15y &= 80 \end{aligned} \qquad \begin{aligned} 2(5x-10y) &= 2(30) \\ 10x-20y &= 60 \end{aligned}$$

Then, we will add the two resulting equations, and solve for y.

$$\begin{aligned} -10x-15y &= 80 \\ + \ (10x-20y &= 60) \\ \hline -35y &= 140 \\ y &= -4 \end{aligned}$$

Substituting $y=-4$ into the original first equation, we can solve for x.

$$\begin{aligned} 2x+3(-4) &= -16 \\ 2x-12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(x,y)=(-2,-4)$.

We can verify by substituting the solution into the other original equation.

$$\begin{aligned} 5x-10y &= 30 \\ 5(-2)-10(-4) &\stackrel{?}{=} 30 \\ -10+40 &\stackrel{?}{=} 30 \\ 30 &= 30 \checkmark \end{aligned}$$

The solution to the system, $(-2,-4)$, is also seen graphically in **Figure 2.3.19**.

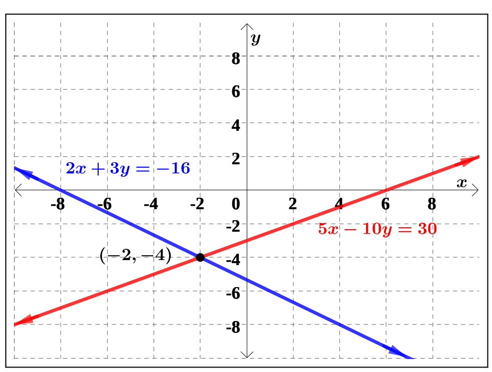


Figure 2.3.19: The coordinate plane with the solution to the given system identified.

■

N *In the previous example, we chose to eliminate the variable x by multiplying each equation by an appropriate constant. We could have easily decided to eliminate the variable y instead. Different constants would have been used, but the final solution to the system would have been the same.*

The Addition Method for solving a system of linear equations in two variables is generalized, step-by-step, as follows.

Addition Method for Solving a System of Linear Equations

1. Write both equations so the variable terms are on the left-hand side and the constant term is on the right-hand side of the equals sign.

2. Multiply one or both equations by the necessary constants so that coefficients of one of the variables are the same, but have opposite signs (additive inverses).

3. Add the two equations; the resulting equation should have at most one variable.

4. Solve the resulting equation from Step 3 for the remaining variable, if one exists.

5. Substitute the result of Step 4 back into one of the *original* equations and solve for the variable value not known.

■ **Example 10** Solve the following system of linear equations, using the Addition Method.

$$\begin{aligned} 6x+3y &= 9 \\ 4x+2y &= 12 \end{aligned}$$

Solution:
While there is no need to reformat either equation, we need to multiply both equations by a constant in order to eliminate one variable. Let's choose to eliminate y. The least common multiple between $3y$ and $2y$ is $6y$, so we will multiply the first equation by -2 and the second equation by 3.

$$\begin{aligned} -2(6x+3y) &= -2(9) \\ -12x-6y &= -18 \end{aligned} \qquad\qquad \begin{aligned} 3(4x+2y) &= 3(12) \\ 12x+6y &= 36 \end{aligned}$$

Then, we will add the two resulting equations.

$$\begin{array}{rr} & -12x-6y = -18 \\ + & (12x+6y = 36) \\ \hline & 0 = 18 \text{ ✗} \end{array}$$

Clearly this statement is a contradiction, because $0 \neq 18$. Therefore, the system is inconsistent and has no solution. ■

■ **Example 11** Solve the following system of linear equations, using the Addition Method.

$$\begin{aligned} x+3y &= 2 \\ 3x+9y &= 6 \end{aligned}$$

Solution:
Again, no reformatting of the equations is necessary. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable. Multiplying the first equation by -3, we have

$$\begin{aligned} x+3y &= 2 \\ (-3)(x+3y) &= (-3)(2) \\ -3x-9y &= -6 \end{aligned}$$

Now, we can add the result to the second equation.

$$\begin{array}{rr} & -3x-9y=-6 \\ + & (3x+9y=6) \\ \hline & 0=0 \end{array}$$

From the resulting identity we can see that we have a dependent system, and there will be an infinite number of solutions that satisfy both equations. If we rewrite both equations in slope-intercept form, we will know what the solution looks like.

$$\begin{aligned} x+3y &= 2 \\ 3y &= -x+2 \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned} \qquad \begin{aligned} 3x+9y &= 6 \\ 9y &= -3x+6 \\ y &= -\frac{3}{9}x+\frac{6}{9} \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

For each value of x, the formula $y=-\frac{1}{3}x+\frac{2}{3}$ determines the corresponding y-value of a solution. We let $x=t$ and get $y=-\frac{1}{3}t+\frac{2}{3}$.

Our set of solutions can be then described as $(x,y)=\left(t,-\frac{1}{3}t+\frac{2}{3}\right)$, where t is any real number.

■

Try It # 6:
Solve the following system of linear equations, using the Addition Method.

$$\begin{aligned} 2x+3y &= 8 \\ 3x+5y &= 10 \end{aligned}$$

Relating Solutions of Systems to Business Applications

In the last section we discussed cost, revenue, and profit of a company and supply and demand for an item in the market place. The relationships between these functions makes it necessary to solve a system of linear equations at times. The water sports equipment manufacturer discussion at the beginning of this section is one such instance.

Break-Even Points

When talking about a company's profit, it is common to hear the terminology "a company is in the black/red." When we say a company is "in the red," we mean the company's costs for producing an item are greater than the revenue generated from the sales of the item. On the other hand, when we say a company is "in the black," we mean the company's costs for producing an item are less than the revenue generated from the sales of the item. When the company's costs are equal to its revenue (its profit is zero), then we say the company 'breaks even.'

Definition

The **break-even point** is the ordered pair (**break-even quantity**, **break-even revenue**), where a company's revenue equals its costs.

A general graph of a break-even point is shown below in **Figure 2.3.20**. Notice, when the sales are equal to the break-even quantity the profit of the company is zero. Any sales below the break-even quantity result in a loss for the company. Any sales above the break-even quantity result in a profit gain for the company. Thus, the break-even quantity is typically the minimum level of sales necessary for the company to make a profit.

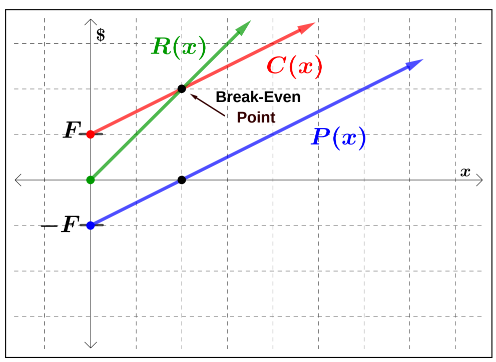


Figure 2.3.20: The coordinate plane with profit, $P(x)$, total cost, $C(x)$, and revenue, $R(x)$, functions drawn. The break-even point is labeled.

N *Depending upon the product being sold, the break-even point may or may not be realistically attainable. Suppose the break-even quantity is 500.25. If the product being sold is lamps, then it is impossible to sell a fraction of a lamp. Rounding down to 500 lamps would result in a small loss for the company, and rounding up to 501 lamps would result in a small profit gain for the company. This means the company would never truly have a profit of exactly zero and would never truly break-even. However, if the product being sold was ounces of granola, then it would be possible to sell 500.25 oz of granola and the company would be able to have exactly zero profit and truly break-even.*

■ **Example 12** Given the cost function $C(x) = 0.85x + 35000$ and the revenue function $R(x) = 1.55x$ for a company (both in dollars), calculate the company's break-even point.

Solution:
Write the system of equations using y to replace the function notation, $C(x)$ and $R(x)$.

$$y = 0.85x + 35000$$
$$y = 1.55x$$

Substitute the expression $0.85x + 35000$ from the first equation into the second equation for y, and solve for x.

$$0.85x + 35000 = 1.55x$$
$$35000 = 0.7x$$
$$50000 = x$$

Then, we substitute $x = 50000$ into either the total cost function or the revenue function to find the corresponding 'y'-value.

$$0.85(50000) + 35000 = 77500 \quad \text{or} \quad 1.55(50000) = 77500$$

Therefore, the break-even point is $(50000, 77500)$.

This means the cost to produce $50,000$ units is $\$77,500$, and the revenue from the sale of the $50,000$ units is also $\$77,500$. Thus, the revenue completely covers the total costs, and the company has zero profit. To make a profit, the business must produce and sell more than $50,000$ units. (See **Figure 2.3.21**.)

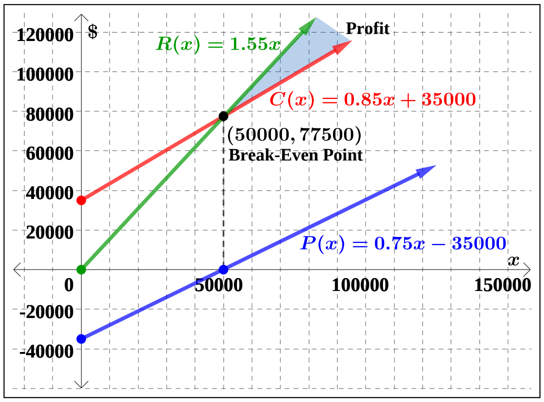


Figure 2.3.21: A first quadrant graph of the cost, revenue, and profit functions with the break-even point labeled.

■

Try It # 7:
A firm producing computer diskettes has fixed costs of \$10,725, and a variable cost of 20 cents a diskette. Determine the break-even point, if the diskettes sell for \$1.50 each. Explain the meaning of the values in the context of the problem.

Market Equilibrium Points

Economic theory says that supply and demand will interact, and the market will adjust until both consumer and producer are happy and the market is in 'equilibrium.'

Definition

The **equilibrium point** is the ordered pair (x_0, p_0), where supply and demand intersect. The quantity produced at this point, x_0, is called the **equilibrium quantity**, and the corresponding price, p_0, is called the **equilibrium price**. ▪

A general graph of an equilibrium point is shown below in **Figure 2.3.22**.

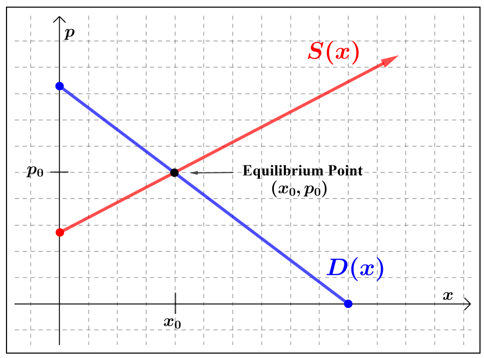


Figure 2.3.22: The coordinate plane with the supply function, $S(x)$, the demand function, $D(x)$, and the equilibrium point labeled.

▪ **Example 13** Compute the equilibrium point for the sale of toy dolls described in examples from the previous section in this chapter.

Recall: Demand for the toy dolls is given by $p(x) = -\frac{1}{300}x + 10$ and supply of the toy dolls is given by $p(x) = \frac{1}{500}x + 2$, both in dollars.

Solution:
Substitute the expression $\frac{1}{500}x + 2$ from the supply equation into the demand equation for $p(x)$, and solve for x.

$$\frac{1}{500}x + 2 = -\frac{1}{300}x + 10$$

$$\frac{1}{500}x + \frac{1}{300}x = 10 - 2$$

$$\frac{2}{375}x = 8$$

$$x = 8\left(\frac{375}{2}\right)$$

$$x = 1500$$

Then, we substitute $x = 1500$ into either the demand or supply function, to solve for p.

$$\begin{aligned} p(1500) &= \left(-\frac{1}{300}\right)(1500) + 10 \\ &= 5 \end{aligned} \qquad \begin{aligned} p(1500) &= \left(\frac{1}{500}\right)(1500) + 2 \\ &= 5 \end{aligned}$$

Thus, the equilibrium point is $(1500, 5)$. At a price of \$5.00, 1500 toy dolls will be supplied to the market and at the same price, all 1500 toy dolls will be bought by consumers.

■

Try It # 8:
A company estimates that at a price of \$140 there will be demand for 4000 computer monitors, and for each \$5 increase in price the demand will drop by 200 monitors. The supply for the monitors is given by $p(x) = \frac{1}{20}x$. Assuming linear demand, calculate the equilibrium point for the computer monitor market, and explain the meaning of the values in the context of the problem.

Try It Answers

1. $(8, 5)$ is NOT a solution to the system.
2. **a.** No solution $\Rightarrow m_1 = m_2$ $(1 = 1)$ AND $b_1 \neq b_2$ $(1 \neq 3)$

   b. Infinitely many solutions $\Rightarrow m_1 = m_2$ $(2 = 2)$ AND $b_1 = b_2$ $(5 = 5)$
3. k can be any value except $-\dfrac{1}{4}$
4. $(x, y) = (-5, 3)$
5. $(x, y) = (t, 2t + 5)$, where t = any real number
6. $(x, y) = (10, -4)$
7. Break-Even Point: $(8250, 12375)$

   Meaning: The cost to produce 8250 diskettes is \$12,375 and the revenue from the sale of 8250 diskettes is also \$12,375. Thus, revenue completely covers the costs when the firm makes and sells 8250 diskettes, and the firm has zero profit.
8. Equilibrium Point: $(3200, 160)$

   Meaning: At a price of \$160, 3200 monitors will be supplied to the market, and at the same price, all 3200 monitors will be bought by consumers.

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 3, determine if the given ordered pair is a solution to the given system of equations.

1. $\begin{cases} 5x - y = 5 \\ x + 6y = 2 \end{cases}$ and $(1,0)$

2. $\begin{cases} 3x + 7y = 1 \\ 2x + 4y = 0 \end{cases}$ and $(2,3)$

3. $\begin{cases} -2x + 5y = 7 \\ 2x + 9y = 7 \end{cases}$ and $(-1,1)$

For Exercises 4 - 9, state the type of linear system given (independent, inconsistent, or dependent), without graphing or actually computing the solution. Then, state the number of solutions (1, 0, or infinitely many).

4. $\begin{cases} y = 2x + 4 \\ y = 2x - 3 \end{cases}$

5. $\begin{cases} y = -5x - 8 \\ y = 3x - 10 \end{cases}$

6. $\begin{cases} y = -\frac{3}{4}x + 2 \\ y = -0.75x + 2 \end{cases}$

7. $\begin{cases} 2y = -x + 9 \\ y = -0.5x + 4.5 \end{cases}$

8. $\begin{cases} 3y = 12x - 14 \\ y = 4x - 14 \end{cases}$

9. $\begin{cases} y = \frac{6}{7}x + 8 \\ 5y = 11x - 6 \end{cases}$

For Exercises 10 - 13, use the Graphical Method to solve each system of two linear equations. Write any solutions as ordered pairs with exact values. For parametric solutions use t as your parameter.

10. $\begin{cases} y = -\frac{1}{2}x + 10 \\ y = \frac{7}{6}x \end{cases}$

11. $\begin{cases} y = \frac{3}{5}x - \frac{7}{5} \\ y = 0.5x - 1.5 \end{cases}$

12. $\begin{cases} 3x - 2y = 18 \\ 5x + 10y = -10 \end{cases}$

13. $\begin{cases} x + 2y = 7 \\ 2x + 6y = 12 \end{cases}$

For Exercises 14 - 17, use the Substitution Method to solve each system of two linear equations. Write any solutions as ordered pairs with exact values. For parametric solutions use t as your parameter.

14. $\begin{cases} y = -\frac{1}{3}x \\ y = -2x - 5 \end{cases}$

15. $\begin{cases} x = 1 + 0.2y \\ -10x + 2y = 5 \end{cases}$

16. $\begin{cases} y = 3x + 2 \\ 12x - 4y = -8 \end{cases}$

17. $\begin{cases} y = 3x - 0.6 \\ x = 2y + 1.3 \end{cases}$

For Exercises 18 - 21, use the Addition Method to solve each system of two linear equations. Write any solutions as ordered pairs with exact values. For parametric solutions use t as your parameter.

18. $\begin{cases} 3x + 2y = 2 \\ -12x - 2y = 1 \end{cases}$

19. $\begin{cases} 7x - 4y = \frac{7}{6} \\ 2x + 4y = \frac{1}{3} \end{cases}$

20. $\begin{cases} -x + 2y = -1 \\ 5x - 10y = 6 \end{cases}$

21. $\begin{cases} 7x + 6y = 2 \\ -28x - 24y = -8 \end{cases}$

Determining Break-Even Points

22. A stuffed animal business has a total production cost of $C(x) = 5x + 30$ and a revenue function $R(x) = 20x$, where x represents the number of stuffed animals made or sold. Compute the break-even point for this business.

23. A cell phone factory has a total production cost of $C(x) = 150x + 10000$ and a revenue function $R(x) = 200x$, where x represents the number of cell phones made or sold. Determine the break-even point for this factory.

24. A hair bow company has a total production cost of $C(x) = 5x + 120$ and a revenue function $R(x) = 11x$, where x represents the number of hair bows made or sold. Calculate the break-even point for this company.

Determining Equilibrium Points

25. A company is willing to supply x items for a price, in dollars, of $p(x) = 3x + 29$, while consumers are willing to buy x items at a price, in dollars, of $p(x) = -5x + 125$. Compute the market equilibrium point.

26. A company is willing to supply x items for a price, in dollars, of $p(x) = 2x + 10$, while consumers are willing to buy x items at a price, in dollars, of $p(x) = -4x + 94$. Determine the market equilibrium point.

27. A company is willing to supply x items for a price, in dollars, of $p(x) = 5x + 140$, while consumers are willing to buy x items at a price, in dollars, of $p(x) = -7x + 500$. Calculate the market equilibrium point.

Intermediate Skills Practice (Answers)

For Exercises 28 - 30, determine if the given ordered pair is a solution to the given system of equations.

28. $\begin{cases} \frac{1}{2}x - \frac{3}{4}y = 8 \\ -\frac{7}{8}x + \frac{1}{16}y = -3 \end{cases}$ and $(4,8)$

29. $\begin{cases} 6x - 9y = 29 \\ x - y = 4 \end{cases}$ and $\left(\frac{7}{3}, -\frac{5}{3}\right)$

30. $\begin{cases} 4x + 2y = -\frac{9}{2} \\ 8x + y = 2 \end{cases}$ and $(-0.25, -1.75)$

For Exercises 31 - 36, state the type of linear system given, without graphing or actually computing the solution, and then state the number of solutions.

31. $\begin{cases} x + 3y = 5 \\ 2x + 3y = 4 \end{cases}$

32. $\begin{cases} 5x - y = 1 \\ 10x - 2y = -5 \end{cases}$

33. $\begin{cases} -2x + 4y = 6 \\ 7x - 14y = -21 \end{cases}$

34. $\begin{cases} -0.1x + 0.2y = 0.6 \\ 5x - 10y = 1 \end{cases}$

35. $\begin{cases} \frac{5}{6}x + \frac{1}{4}y = 0 \\ \frac{1}{8}x - \frac{1}{2}y = -\frac{43}{120} \end{cases}$

36. $\begin{cases} -0.5x - y = -3 \\ 0.3x + 0.6y = 1.8 \end{cases}$

For Exercises 37 - 40, use the Graphical Method to solve each system of two linear equations. Write any solutions as ordered pairs with exact values. For parametric solutions use t as your parameter.

37. $\begin{cases} 2x + 4y = -3.8 \\ 9x - 5y = 1.3 \end{cases}$

38. $\begin{cases} 3x + 5y = 9 \\ 30x + 50y = -90 \end{cases}$

39. $\begin{cases} 6x - 8y = -1 \\ 3x + 2y = 0.7 \end{cases}$

40. $\begin{cases} -x + 2y = 4 \\ 2x - 4y = 1 \end{cases}$

For Exercises 41 - 44, use the Substitution Method to solve each system of two linear equations. Write any solutions as ordered pairs with exact values. For parametric solutions use t as your parameter.

41. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 16 \\ \frac{1}{6}x + \frac{1}{4}y = 9 \end{cases}$

42. $\begin{cases} -2x + 3y = 1.2 \\ -3x - 6y = 1.8 \end{cases}$

43. $\begin{cases} x - \frac{5}{12}y = -\frac{55}{12} \\ -6x + \frac{5}{2}y = \frac{55}{2} \end{cases}$

44. $\begin{cases} 3x + 6y = 11 \\ 2x + 4y = 9 \end{cases}$

For Exercises 45 - 48, use the Addition Method to solve each system of two linear equations. Write any solutions as ordered pairs with exact values. For parametric solutions use t as your parameter.

45. $\begin{cases} -2x + 5y = -42 \\ 7x + 2y = 30 \end{cases}$

46. $\begin{cases} 2x + 6y = 5 \\ -3x - 9y = 4 \end{cases}$

47. $\begin{cases} -4x + 8y = -12 \\ 3x - 6y = 9 \end{cases}$

48. $\begin{cases} 6x - 5y = -34 \\ 2x + 6y = 4 \end{cases}$

Break-Even Points

49. A guitar factory has a cost of production given by $C(x) = 75x + 50000$, where x represents the number of guitars produced. If the company needs to break even after 400 units are sold, at what price should they sell each guitar? Write the revenue function.

50. The cost for a venue to hold a concert is given by $C(x) = 64x + 20,000$, where x is the total number of attendees at the concert. The venue charges \$80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

51. The start-up cost for a restaurant is \$120,000, and each meal costs \$10 for the restaurant to make. If each meal is then sold for \$15, after how many meals does the restaurant break even?

Equilibrium Points

52. At a price of \$20 per item, 100 items are demanded by consumers, but at a price of \$50 per item, only 20 items are demanded. If the supplier is willing to supply x items for a price, in dollars, of $p(x) = 0.125x + 5.5$, what is the market equilibrium point?

53. At a price of \$40 per item, producers will provide 125 items to the market. At a price of \$80 per item, producers will provide 325 items. If the consumers demand x items at a price, in dollars, of $p(x) = -0.4x + 90$, what is the market equilibrium point?

54. Consumers will demand 500 items at a price of \$2 per item. For every \$2 increase in price per item, 4 fewer items will be demanded. If the supplier is willing to supply x item for a price, in dollars, of $p(x) = 1.5x + 40$, what is the market equilibrium point?

Mastery Practice (Answers)

55. State the value(s) of k such that the following system of linear equations has exactly one solution, if possible.

$$\begin{aligned} 5x + ky &= 16 \\ x + 2y &= 4 \end{aligned}$$

56. State the value(s) of k such that the following system of linear equations has infinitely many solutions, if possible.

$$\begin{aligned} 3x - 2y &= 5 \\ kx + 6y &= -15 \end{aligned}$$

57. Given $3x - 4y = 8$, write an equation such that the system consisting of this equation and your equation would be inconsistent.

58. Use the Graphical Method to solve the following system of two linear equations. Write the solution, if one exists, as an ordered pair with exact values. For parametric solutions use t as your parameter.

$$\frac{7}{3}x - \frac{1}{6}y = 2$$
$$-\frac{7}{2}x + \frac{1}{4}y = -3$$

59. A company selling DVDs has fixed costs of \$10,500 and sells the DVDs for \$5 each. If the company breaks even when making and selling 3500 DVDs, what is the company's break-even point?

60. The profit function for a company selling x mini projectors is $P(x) = 150x - 12000$. Each mini projector sells for \$200. Determine the company's break-even point.

61. A company finds the daily total cost of producing 200 items is \$28,000. It also finds if no items are produced, the company still has daily total costs of \$24,000. Each item is sold to the public for \$50. How many items must be sold for the company to turn a profit?

62. At a price of \$27.50, producers will provide 10 items, while at a price of \$38.75, they will provide 25 items. Consumers will purchase 50 of these items if the price is \$97.50, but will purchase 125 items if the price decreases to \$93.75. Calculate the market equilibrium point.

Communication Practice (Answers)

63. You are asked to solve a system of two linear equations with two unknowns. Give a reason when each of the methods discussed (graphical, substitution, and addition) is best used.

64. If when using the Substitution Method to solve a system of two linear equations with two unknowns, all variables cancel leaving you with the equation $0 = 0$, how many solutions does the system you are solving have? Explain your answer.

65. If when using the Addition Method to solve a system of two linear equations with two unknowns, all variables cancel leaving you with the equation $0 = 5$, how many solutions does the system you are solving have? Explain your answer.

66. A company specializes in making gluten-free flour, sold by the pound. The company sells this flour for \$8 per pound and has a break-even point of $(360.5, 2884)$. Interpret the meaning of the break-even point in the context of the situation and include in your interpretation whether the company can truly break even.

67. If a company's break-even point is $(315, 45675)$, what point do you know will be on the graph of the company's profit function?

68. The equilibrium point for a leisure boat market is $\left(\frac{13}{5}, 4.89\right)$, where x represents the quantity of boats (in thousands) and $p(x)$ represents the price per boat (in hundreds of dollars). Write a sentence expressing the meaning of this point in the context of the situation.

2.4 Setting Up and Solving Systems of Linear Equations

© Photo by Vanessa Coffelt, 2020

John received an inheritance of \$12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested three times as much in mutual funds as in municipal bonds. He earned \$670 in total interest the first year. How much did John invest in each type of account?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving two or more linear equations and two or more variables in this section. Doing so uses similar techniques as those we have previously discussed. However, finding solutions to systems of three or more linear equations requires a bit more organization and a touch of visual gymnastics.

Learning Objectives:

In this section, you will learn techniques for solving a system of linear equations using matrices and the Gauss-Jordan Elimination Method and how these techniques can be used for solving real-world applications. Upon completion you will be able to:

- Formulate a system of linear equations from a given scenario.
- Convert a system of linear equations to the corresponding augmented matrix.
- State the different row operations available when solving systems of linear equations with matrices.
- Identify whether or not an augmented matrix is in reduced row-echelon form (RREF).
- Show the row operations that would be used to transform a matrix which is not in reduced row-echelon form into one in reduced row-echelon form.
- Solve a system of linear equations that has one real solution, using the Gauss-Jordan Elimination Method.
- Use technology to reduce a matrix representing a system of linear equations.
- Write a solution from a reduced row-echelon form matrix representing a system of linear equations.
- Solve a system of linear equations that has one real solution.
- Solve a system of linear equations that has no real solutions.
- Solve a system of linear equations that has infinitely many real solutions.

- Apply augmented matrices to solve real-world problems.
- Justify the value(s) of the parameter where there appears to be more than one real solution, but all solutions may not be consistent with the real-world scenario.

Setting up Systems of Linear Equations

Solutions to problems similar to John's inheritance problem above require a conversion from words to algebraic notation in the form of a system of linear equations. Recall that a system of linear equations consists of two or more linear equations made up of two or more variables, such that all equations in the system are considered simultaneously.

Let's first focus on the conversion of such problems into a system of linear equations, and then later in the section, we will learn techniques to solve the system.

Consider the following:

> Suppose a company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. The company has staffing for 65 hours of assembly and 55 hours of painting each week. If the company wants to fully utilize all staffed hours, how many of each item should they produce?

Let's convert this problem into a system of linear equations. (The actual solution will be found at a later point in this section.)

Notice first from the question posed in the problem, we are looking for "how many of each item" should be produced. The "how many" indicates we are looking for a quantity, while the "of each item" directs you to the first sentence which states "a basic and premium version." Therefore, we know we are looking for two unknowns.

Our next step is to clearly define our two unknown variables. Often it is best to define your variables using letters related to the unknown quantities. Thus, we will let

$b :=$ the *number of* basic products produced

and

$p :=$ the *number of* premium products produced.

Notice that, because we are looking for quantities, our definition includes "the number of."

Now, we can create equations involving our variables based on the assembly and painting time constraints. We begin with the amount of time available for assembling these products. Each basic product requires 20 minutes of assembly, so producing b items will require $20b$ minutes. Each premium product requires 30 minutes of assembly, so producing p items will require $30p$ minutes. Thus, $20b+30p$ represents the number of minutes required to assemble all products. We are told we have 65 hours available for assembly. In order to write an equation, all the units must be the same throughout. Therefore, we must convert all time units to either minutes or hours, so that the time units are consistent. For ease, we will convert 65 hours to an equivalent number of minutes. Due to the fact that there are 60 minutes in one hour, we have $65 \cdot 60 = 3900$ minutes. Utilizing all assembly time gives us the equation:

$$20b+30p = 3900$$

For our second equation we turn our focus to the amount of time available for painting the products. Each basic product requires 15 minutes of painting, so producing b items will require $15b$ minutes. Each premium product requires 30 minutes of painting, so producing p items will require $30p$ minutes. Again, we will convert the available 55 hours to $55 \cdot 60 = 3300$ minutes. Utilizing all painting time gives us the equation:

$$15b + 30p = 3300$$

Together, these form a system of linear equations, which represent the given scenario.

$$\begin{aligned} 20b + 30p &= 3900 \\ 15b + 30p &= 3300 \end{aligned}$$

Sometimes product names start with letters such as O or l which can be visually confused with numbers 0 and 1, so it is a good idea not to use these letters. Also, product names might all begin with the same letter and the same letter cannot be used for multiple unknowns. Choose your variable names wisely!

Example 1 Set up, but do not solve, the following problem as a system of linear equations.

> Julia has just retired and has \$600,000 in her retirement account that she needs to reallocate to produce income. She is looking at two investments: a very safe guaranteed annuity that will provide 3% annual interest and a somewhat riskier bond fund that averages 7% annual interest. She would like to invest as little as possible in the riskier bond fund, but she needs to produce \$40,000 a year, in interest, to live on. How much should she invest in each account?

Solution:
Notice there are two unknowns in this problem: the amount of money she should invest in the annuity and the amount of money she should invest in the bond fund. We can start by clearly defining variables for the unknowns.

Let

$a :=$ the amount of money (in dollars) she invests into the annuity

and

$b :=$ the amount of money (in dollars) she invests into the bond fund.

One equation comes from noting that together she is going to invest \$600,000:

$$a + b = 600000$$

Another equation will come from the annual interest. We must convert all percentages to decimals when writing an algebraic expression. She earns 3% annually on the annuity, so the interest earned in a year on the amount invested would be $0.03a$. Likewise, the interest earned on the bond fund in a year would be $0.07b$. Together, these need to total $\$40,000$, giving the equation:

$$0.03a + 0.07b = 40000$$

Together these form a system of linear equations:

$$\begin{aligned} a + b &= 600000 \\ 0.03a + 0.07b &= 40000 \end{aligned}$$

■

■ **Example 2** Set up, but do not solve, the following problem.

> An animal shelter has a total of 340 animals comprised of cats, dogs, and rabbits. If there are 10 more cats than dogs, and if there are twice as many cats as rabbits, how many of each animal are at the shelter?

Solution:
Notice there are three unknowns in this problem: the number of cats, the number of dogs, and the number of rabbits. We can start by clearly defining variables for each of the unknowns.

Let

$c :=$ the number of cats at the shelter,

$d :=$ the number of dogs at the shelter,

and

$r :=$ the number of rabbits at the shelter.

One equation comes from noting there are a total of 340 animals in the shelter:

$$c + d + r = 340$$

Another equation comes from the statement "there are 10 more cats than dogs:"

$$c = d + 10$$

We can reformat the equation so that the variables are on the same side of the equals sign.

$$c - d = 10$$

Our last equation comes from the statement "there are twice as many cats as rabbits." Statements of this form lead to equations often referred to as 'ratio equations.' While statements of this form appear straight-forward, they can oftentimes be converted into an equation incorrectly. We will reword phrases, including "there are twice as many", "three times as much", and the like, to avoid some of the common errors when converting to a 'ratio equation.' Here, "there are twice as many cats as rabbits" can be reworded as

"The number of cats you have is twice the number of rabbits you have."

which becomes the ratio equation,

$$c = 2r$$

Again, we can reformat to get

$$c - 2r = 0$$

Together these form a system of linear equations:

$$\begin{aligned} c + d + r &= 340 \\ c - d &= 10 \\ c - 2r &= 0 \end{aligned}$$

■

N *Each equation is developed based on a constraint, not on a variable. Every equation does not need to contain all of the variables. The number of constraints, not the number of variables, indicates the number of equations in the system.*

Try It # 1:
Set up, but do not solve, a system of linear equations which could be used to solve John's inheritance problem at the beginning of this section.

In the previous examples, we produced systems of linear equations. We can use techniques discussed in in **Section 2.3** to determine a solution to a system of two linear equations in two unknowns. However, if you have more than two linear equations in two unknowns the techniques may be cumbersome or not applicable. Thus, we will now introduce a simpler method for solving systems of linear equations, using matrices.

Converting between Systems and Augmented Matrices

A matrix may be used to represent a system of linear equations. Matrices often make solving systems of equations easier, because they are not encumbered with variables.

Definition

To express a system of linear equations in matrix form, we extract the coefficients of the variables and the constants from the equations, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equals signs. When a linear system is written in this form, we call it an **augmented matrix**. ▪

For example, consider the following system of linear equations.

$$\begin{aligned} 3x+4y&=7\\ 4x-2y&=5 \end{aligned}$$

We can write this system as the following corresponding augmented matrix.

$$\left[\begin{array}{rr|r} 3 & 4 & 7\\ 4 & -2 & 5 \end{array}\right]$$

Another system of linear equations such as

$$\begin{aligned} 3x-y-z&=0\\ x+y&=5\\ 2x-3z&=2 \end{aligned}$$

is represented by the corresponding augmented matrix

$$\left[\begin{array}{rrr|r} 3 & -1 & -1 & 0\\ 1 & 1 & 0 & 5\\ 2 & 0 & -3 & 2 \end{array}\right]$$

Any augmented matrix is written so that the variables line up in their own columns. For instance, coefficients of x-terms go in the first column, coefficients of y-terms in the second column, coefficients of z-terms in the third column, etc., with constants all on the right-hand side of the vertical line. Thus, it is very important that each equation is written in the form $ax+by+cz+\cdots=d$, so that the variables line up before placing a system into an augmented matrix. When there is a missing variable term in an equation, the coefficient is 0.

■ **Example 3** Write the augmented matrix corresponding to the given system of linear equations.

$$\begin{aligned} x+2y-z&=3 \\ 2z-6&=y-2x \\ x+3z&=4+3y \end{aligned}$$

Solution:
Recall when setting up a system of linear equations, we can rewrite all equations so the variables are on the left-hand side of the equals sign and constants are on the right. To put the system in an augmented matrix we must, additionally, have the variables in the same order in each equation.

Our reformatted equations can be written as

$$\begin{aligned} x+2y-z&=3 \\ 2x-y+2z&=6 \\ x-3y+3z&=4 \end{aligned}$$

The corresponding augmented matrix displays the coefficients of the variables, and an additional column for the constants, as shown below.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right]$$

■

N *The number of equations in the system tells you the number of rows your augmented matrix will have, and the number of columns of your augmented matrix will be equal to the number of variables plus 1 (for the constants).*

We can use augmented matrices to help us solve systems of linear equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the corresponding system of linear equations in standard form.

■ **Example 4** Write the system of linear equations which corresponds to the given augmented matrix. (Assume variables are x, y, and z, respectively.)

$$\left[\begin{array}{rrr|r} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right]$$

Solution:
When the columns represent the variables x, y, and z, respectively, we have

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right] \implies \begin{array}{rcrcrcr} x & - & 3y & - & 5z & = & -2 \\ 2x & - & 5y & - & 4z & = & 5 \\ -3x & + & 5y & + & 4z & = & 6 \end{array}$$

∎

Try It # 2:
Write the augmented matrix corresponding to the given system of linear equations.

$$4x - 3y = 11$$
$$3x + 2y = 4$$

Try It # 3:
Write the system of linear equations which corresponds to the given augmented matrix. (Assume variables are x, y, and z, respectively.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{array}\right]$$

Solving Systems using Gauss-Jordan Elimination

Two systems are said to be **equivalent** if they have the same set of solutions. A system can be solved by writing a series of systems, one after the other, each equivalent to the previous system. Each of these systems has the same set of solutions as the original one; the aim is to end up with a system that is easy to solve. Each system in the series is obtained from the preceding system by a simple algebraic manipulation chosen so that it does not change the set of solutions.

As an illustration, we will solve the system given below, using the Addition Method discussed in the previous section. At each stage, the corresponding augmented matrix is displayed. The original system is

$$\begin{array}{l} x + 2y = -2 \\ 2x + y = 7 \end{array} \quad \text{or} \quad \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 2 & 1 & 7 \end{array}\right]$$

First, multiply the first equation by -2 and add the result to the second. The resulting system is

$$-2(x + 2y = -2) \rightarrow \begin{array}{r} -2x - 4y = 4 \\ +\ (2x + y = 7) \\ \hline 0x - 3y = 11 \end{array} \implies \begin{array}{r} x + 2y = -2 \\ -3y = 11 \end{array} \quad \text{or} \quad \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -3 & 11 \end{array}\right]$$

which is equivalent to the original. At this stage we obtain $y = -\frac{11}{3}$ by multiplying the second equation by $-\frac{1}{3}$. The result is the equivalent system.

$$\begin{aligned} x + 2y &= -2 \\ y &= -\frac{11}{3} \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -\frac{11}{3} \end{array}\right]$$

Finally, we multiply the second equation by -2 and add the result to the first equation to get another equivalent system.

$$-2\left(y = -\frac{11}{3}\right) \rightarrow \quad \begin{array}{r} x + 2y = -2 \\ + \quad -2y = \dfrac{22}{3} \\ \hline x + 0y = \dfrac{16}{3} \end{array} \quad \Longrightarrow \quad \begin{aligned} x &= \frac{16}{3} \\ y &= -\frac{11}{3} \end{aligned} \qquad \text{or} \qquad \left[\begin{array}{cc|c} 1 & 0 & \frac{16}{3} \\ 0 & 1 & -\frac{11}{3} \end{array}\right]$$

Because this final system is equivalent to the original system, it provides the solution to that system;

$$(x, y) = \left(\frac{16}{3}, -\frac{11}{3}\right)$$

Observe that at each step, a certain operation is performed on the system (and, thus, on the augmented matrix) to produce an equivalent system.

Gauss-Jordan (Row) Operations

There are only a few algebraic manipulations, called **elementary operations**, that do not change the set of solutions to the system of linear equations in which we are trying to solve.

Definition

The following operations, called **elementary operations**, can routinely be performed on systems of n linear equations, $E_1 - E_n$, to produce equivalent systems.

I. Interchange (reorder) two equations. ($E_i \leftrightarrow E_j$)
II. Multiply one equation by a nonzero constant, k. ($kE_i \rightarrow E_i$)
III. Add a multiple of one equation to a different equation. ($kE_i + E_j \rightarrow E_j$)

Elementary operations performed on a system of linear equations produce manipulations of the rows in the corresponding augmented matrix. Thus, interchanging two equations means interchanging the entries of two rows of a matrix. Multiplying one equation by a nonzero constant, k, means multiplying every entry of a row in the matrix by k. Adding a multiple of one equation to another means adding a multiple of each entry of one row to the corresponding entry of another row.

In hand calculations (and in computer programs) it is easier to manipulate the rows of the augmented matrix rather than the equations. For this reason we restate these elementary operations for matrices, and use them for the purposes of this book. Again, these operations produce a series of augmented matrices corresponding to equivalent systems of linear equations, all with the same set of solutions.

Definition

The following, called **elementary row operations**, can be performed on a matrix with rows $R_1 - R_n$.

I. Interchange (reorder) two rows. $(R_i \leftrightarrow R_j)$
II. Multiply one row by a nonzero constant, k. $(kR_i \to R_i)$
III. Add a multiple of one row to a different row. $(kR_i + R_j \to R_j)$

N *Operations are performed on rows not columns, as rows represent equations.*

Example 5 Perform each specified row operation, and give the resulting matrix.

a. $\left[\begin{array}{cc|c} 9 & -4 & -11 \\ 6 & 1 & 12 \end{array}\right] \quad R_1 \leftrightarrow R_2 \implies ?$

b. $\left[\begin{array}{cc|c} 3 & -7 & 5 \\ 1 & 4 & -6 \end{array}\right] \quad \frac{1}{3}R_1 \to R_1 \implies ?$

c. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ -5 & 6 & 7 & 8 \\ 9 & -10 & 11 & 12 \end{array}\right] \quad 5R_1 + R_2 \to R_2 \implies ?$

Solution:

a. $R_1 \leftrightarrow R_2$ notation means we interchange the entries of Row 1 with the corresponding entries of Row 2.

$$\left[\begin{array}{cc|c} 9 & -4 & -11 \\ 6 & 1 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 6 & 1 & 12 \\ 9 & -4 & -11 \end{array}\right]$$

b. $\frac{1}{3}R_1 \to R_1$ notation means every entry of Row 1 is multiplied by $\frac{1}{3}$; the new entries are placed in Row 1 of the resulting matrix, while the second row remains unchanged.

$$\frac{1}{3}R_1 = \frac{1}{3}\left[\begin{array}{cc|c} 3 & -7 & 5 \end{array}\right] = \left[\begin{array}{cc|c} \left(\frac{1}{3}\right)(3) & \left(\frac{1}{3}\right)(-7) & \left(\frac{1}{3}\right)(5) \end{array}\right] = \left[\begin{array}{cc|c} 1 & -\frac{7}{3} & \frac{5}{3} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -7 & 5 \\ 1 & 4 & -6 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -\frac{7}{3} & \frac{5}{3} \\ 1 & 4 & -6 \end{array}\right]$$

c. $5R_1 + R_2 \to R_2$ notation means we multiplying every entry of Row 1 by 5, and then add the results to the corresponding entries in Row 2. The sum is then placed in Row 2 of the resulting matrix. Rows 1 and 3 remain unchanged in the resulting matrix.

$$5R_1 = 5\left[\begin{array}{ccc|c} 1 & 2 & -3 & -4 \end{array}\right] = \left[\begin{array}{ccc|c} 5 & 10 & -15 & -20 \end{array}\right]$$

$$\begin{array}{rrcrrrr} & 5R_1 & : & 5 & 10 & -15 & -20 \\ + & R_2 & : & -5 & 6 & 7 & 8 \\ \hline & R_2 & : & 0 & 16 & -8 & -12 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & -4 \\ -5 & 6 & 7 & 8 \\ 9 & -10 & 11 & 12 \end{array}\right] \xrightarrow{5R_1+R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & 2 & -3 & -4 \\ 0 & 16 & -8 & -12 \\ 9 & -10 & 11 & 12 \end{array}\right]$$

■

N *When using the row operation $kR_i + R_j \to R_j$, the only row that is changed in the resulting matrix is R_j. Row i is just "temporarily" multiplied by k so that the result can be added to Row j.*

Reduced Row-Echelon Form

In the example before row operations were formally introduced, a series of such operations led to a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$$

where the asterisks represent arbitrary numbers. In the case of three linear equations in three variables, the goal of performing row operations is to produce a matrix of the following form, if possible.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

This will not always happen, as we will see with later systems.

Given a system of linear equations, we will use a sequence of elementary row operations to convert the equivalent augmented matrix to a "nice" matrix (meaning that the corresponding equations are easy to solve). The matrices shown above are examples of "nice" matrices. The following definition states the conditions necessary for identification of a "nice" matrix.

Definition

A matrix is said to be in **reduced row-echelon form** if it satisfies each of the following four conditions:

1. All **zero rows** (rows consisting entirely of zeros) are below all nonzero rows, if they exist.
2. The first nonzero entry from the left in each nonzero row must be a 1, called the **leading 1** for that row.
3. Each leading 1 is to the right of all leading 1's in the rows above it.
4. Each leading 1 is the only nonzero entry in its *column*.

■

The following are examples of augmented matrices that are in reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 \end{array}\right]$$

■ **Example 6** Which of the following matrices are in reduced row-echelon form? If the matrix is not in reduced row-echelon form, state which of the four conditions is first violated, as stated in the definition.

a. $\left[\begin{array}{cc|c} 1 & 2 & -5 \\ 0 & 1 & 7 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$ **c.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -11 \end{array}\right]$ **d.** $\left[\begin{array}{cc|c} -1 & 0 & -3 \\ 0 & 1 & 6 \end{array}\right]$

Solution:
For each matrix we will check the four necessary conditions to determine whether or not the matrix is in reduced row-echelon form. We will check the conditions in the order presented in the definition and stop checking once the matrix does not satisfy a condition. We will only be concerned with checking the portion of the matrix to the left of the vertical bar. While specific constants are given for each matrix to the right of the vertical bar, each of these could be changed to a different constant and that would not change our answer.

a. $\left[\begin{array}{cc|c} 1 & 2 & -5 \\ 0 & 1 & 7 \end{array}\right]$

Not in reduced row-echelon form.

✓#1 : No row consists of all zeros, so this condition is met.

✓#2 : The first nonzero entry in each row (reading left to right) is a 1. These entries, leading 1's, are shown circled in the matrix.

✓#3 : The leading 1's tend to the right as you move down the matrix.

✗#4 : Column 1 meets the criteria, but Column 2 does not. The entry (2) would need to be a zero for this condition to be met.

b. $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{array}\right]$

In reduced row-echelon form.

✓#1 : The row of all zeros is below the two nonzero rows, so this condition is met.

✓#2 : The first nonzero entry in each row is a 1. These entries, leading 1's, are shown circled in the matrix.

✓#3 : The leading 1's tend to the right as you move down the matrix.

✓#4 : Column 1 and Column 2 are the only columns containing leading 1's, and in each of these columns all other entries are zero.

c. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -11 \end{array}\right]$

In reduced row-echelon form.

✓#1 : No row consists of all zeros, so this condition is met.

✓#2 : The first nonzero entry in each row is a 1. These entries, leading 1's, are shown circled in the matrix.

✓#3 : The leading 1's tend to the right as you move down the matrix.

✓#4 : Column 1 and Column 3 are the only columns containing leading 1's, and in each of these columns all other entries are zero.

d. $\left[\begin{array}{cc|c} -1 & 0 & -3 \\ 0 & 1 & 6 \end{array}\right]$

Not in reduced row-echelon form.

✓#1 : No row consists of all zeros, so this condition is met.

✗#2 : The first nonzero entry in each row is circled. The entry in Row 1 should be a 1 (not -1), which means this condition is not met.

▪

N *For a matrix to be in reduced row-echelon form, Condition #3 only requires that each leading 1 is to the right of all leading 1's in the rows above it. In other words, as you move down the matrix, the leading 1's must 'tend' to the right, but do not have to be on the main diagonal from upper left to lower right.*

Try It # 4:
Which of the following matrices are in reduced row-echelon form? If the matrix is not in reduced row-echelon form, state which of the four conditions is first violated, as stated in the definition.

a. $\left[\begin{array}{cc|c} 0 & 1 & 4 \\ 1 & 0 & -1 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ **c.** $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ **d.** $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solving Systems using Gauss-Jordan Elimination by Hand

Every matrix can be brought to reduced row-echelon form by a sequence of elementary row operations. In fact, we can give a step-by-step procedure for actually converting a matrix to a reduced row-echelon matrix. Observe that while there are many sequences of row operations that will bring a matrix to reduced row-echelon form, the one we use is systematic, easy to perform, and is called the **Gauss-Jordan Elimination Method**.

Gauss-Jordan Elimination Method

1. Write the corresponding augmented matrix for the given system of linear equations.
2. Interchange rows, if necessary, to obtain a nonzero number in the first row, first column.
3. Use a row operation to make the entry in the first row, first column a '1'.
4. Use row operation(s) to make all other entries in the first column '0'.
5. Interchange rows, if necessary, to obtain a nonzero number in the second row, second column. Use a row operation to make this entry 1. Use row operations to make all other entries in the second column zero.
6. Repeat Step 5 for Row 3, Column 3 and the third column. Continue moving along the main diagonal from upper left to lower right until you reach the last row, or until the number is zero, which cannot be removed, and the solution can easily be read.

At the completion of this process, the final matrix will be in reduced row-echelon form.

Example 7 Solve the following system of linear equations, by using the Gauss-Jordan Elimination Method.

$$\begin{aligned} 2x+y+2z &= 10 \\ x+2y+z &= 8 \\ 3x+y-z &= 2 \end{aligned}$$

Solution:
Because all of the equations are written in the correct aligned form, we can write the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We want a '1' in Row 1, Column 1. This can be obtained by multiplying the first row by $\frac{1}{2}$, or interchanging the second row with the first. Interchanging the rows is a better choice, because that way we avoid fractions. Performing this operation we get,

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

We need to make all other entries in the first column '0', using our new leading 1. To make the entry (2) in Row 2, Column 1 a '0', we multiply the first row by -2 and add the result to the second row. Performing this operation we get,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad \begin{array}{c} \xrightarrow{-2R_1+R_2 \to R_2} \\ \begin{array}{rrcrrrr} & -2R_1 & : & -2 & -4 & -2 & -16 \\ + & R_2 & : & 2 & 1 & 2 & 10 \\ \hline & R_2 & : & 0 & -3 & 0 & -6 \end{array} \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

To make the entry (3) in Row 3, Column 1 a '0', we multiply the first row by -3 and add the result to the third row, giving

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 3 & 1 & -1 & 2 \end{array}\right] \quad \xrightarrow{-3R_1+R_3 \to R_3} \quad \left[\begin{array}{rrr|r} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right]$$

$$\begin{array}{rrcrrrr} & -3R_1 & : & -3 & -6 & -3 & -24 \\ + & R_3 & : & 3 & 1 & -1 & 2 \\ \hline & R_3 & : & 0 & -5 & -4 & -22 \end{array}$$

So far we have made a '1' in the left corner and all other entries in the first column '0'. Now we move to the next diagonal entry, Row 2, Column 2. We need to make this entry (-3) a '1' and then make all other entries in the second column '0'. To make the Row 2, Column 2 entry a '1', we multiply the entire second row by $-\frac{1}{3}$.

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad \xrightarrow{-\frac{1}{3}R_2 \to R_2} \quad \left[\begin{array}{rrr|r} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right]$$

$$-\tfrac{1}{3}R_2 \; : \; 0 \quad 1 \quad 0 \quad 2$$

Next, we make all other entries in the second column '0', as shown below.

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad \xrightarrow{-2R_2+R_1 \to R_1} \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right]$$

$$\begin{array}{rrcrrrr} & -2R_2 & : & 0 & -2 & 0 & -4 \\ + & R_1 & : & 1 & 2 & 1 & 8 \\ \hline & R_1 & : & 1 & 0 & 1 & 4 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right] \quad \xrightarrow{5R_2+R_3 \to R_3} \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

$$\begin{array}{rrcrrrr} & -5R_2 & : & 0 & 5 & 0 & 10 \\ + & R_3 & : & 0 & -5 & -4 & -22 \\ \hline & R_3 & : & 0 & 0 & -4 & -12 \end{array}$$

We now make the last diagonal entry a '1', by multiplying the third row by $-\frac{1}{4}$.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad \xrightarrow{-\frac{1}{4}R_3 \to R_3} \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$-\tfrac{1}{4}R_3 \; : \; 0 \quad 0 \quad 1 \quad 3$$

Finally, we make all other entries in the third column '0', as shown below.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \xrightarrow{-R_3+R_1 \to R_1} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rrcrrrr} & -R_3 & : & 0 & 0 & -1 & -3 \\ + & R_1 & : & 1 & 0 & 1 & 4 \\ \hline & R_1 & : & 1 & 0 & 0 & 1 \end{array}$$

The resulting matrix is in reduced row-echelon form, and the Gauss-Jordan Elimination Method is complete. We can now write the resulting corresponding system of linear equations.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \implies \begin{array}{c} x+0y+0z=1 \\ 0x+y+0z=2 \\ 0x+0y+z=3 \end{array} \implies \begin{array}{c} x=1 \\ y=2 \\ z=3 \end{array}$$

The solution to our given system is the solution to this system of equations, so, clearly, the solution to the given system is $(x, y, z) = (1, 2, 3)$.

We can check our solution by substituting 1 for x, 2 for y and 3 for z into all three equations of the *original* system.

$$\begin{array}{rcc}
2x+y+2z=10 & x+2y+z=8 & 3x+y-z=2 \\
2(1)+(2)+2(3)\stackrel{?}{=}10 & (1)+2(2)+(3)\stackrel{?}{=}8 & 3(1)+(2)-(3)\stackrel{?}{=}2 \\
2+2+6\stackrel{?}{=}10 & 1+4+3\stackrel{?}{=}8 & 3+2-3\stackrel{?}{=}2 \\
10=10\ \checkmark & 8=8\ \checkmark & 2=2\ \checkmark
\end{array}$$

■

N *In general, to obtain a '1' we multiply a row by a constant, k, and to obtain a '0' in a row we multiply the row with the leading '1' by a constant and add the result to the row to be changed.*

■ **Example 8** Use the Gauss-Jordan Elimination Method to solve the following system of linear equations.

$$\begin{aligned} 4x-8y&=7 \\ 3x-6y&=9 \end{aligned}$$

Solution:

All variables and constants are aligned so we can write the corresponding augmented matrix, as follows.

$$\left[\begin{array}{cc|c} 4 & -8 & 7 \\ 3 & -6 & 9 \end{array}\right]$$

We want a '1' in Row 1, Column 1. This can be obtained by multiplying the first row by $\frac{1}{4}$.

$$\left[\begin{array}{cc|c} 4 & -8 & 7 \\ 3 & -6 & 9 \end{array}\right] \quad \xrightarrow{\frac{1}{4}R_1\to R_1} \quad \left[\begin{array}{cc|c} 1 & -2 & \frac{7}{4} \\ 3 & -6 & 9 \end{array}\right]$$

$$\tfrac{1}{4}R_1 \ : \ 1 \quad -2 \quad \tfrac{7}{4}$$

Next, we need to make the remaining entry in the first column, (3) in Row 2, Column 1, a '0'. We will multiply the first row by -3, and add the result to the second row.

$$\left[\begin{array}{cc|c} 1 & -2 & \frac{7}{4} \\ 3 & -6 & 9 \end{array}\right] \quad \xrightarrow{-3R_1+R_2\to R_2} \quad \left[\begin{array}{cc|c} 1 & -2 & \frac{7}{4} \\ 0 & 0 & \frac{15}{4} \end{array}\right]$$

$$\begin{array}{rrcrrr} & -3R_1 & : & -3 & 6 & -\frac{21}{4} \\ + & R_2 & : & 3 & -6 & 9 \\ \hline & R_2 & : & 0 & 0 & \frac{15}{4} \end{array}$$

The resulting matrix is in reduced row-echelon form, and the Gauss-Jordan Elimination Method is complete. We can now write the resulting corresponding system of linear equations.

$$\left[\begin{array}{cc|c} 1 & -2 & \frac{7}{4} \\ 0 & 0 & \frac{15}{4} \end{array}\right] \Rightarrow \begin{aligned} x-2y &= \frac{7}{4} \\ 0x+0y &= \frac{15}{4} \end{aligned}$$

The solution to our given system is the solution to the system.

$$x - 2y = \frac{7}{4}$$
$$0 = \frac{15}{4} \text{ ✗}$$

Clearly this statement is a contradiction, because $0 \neq \frac{15}{4}$. Therefore, the given system is inconsistent and has no solution.

■

N *It is important to know that ANY system of linear equations of any size can be solved by the Gauss-Jordan Elimination Method.*

Try It # 5:
Use the Gauss-Jordan Elimination Method to solve the following system of linear equations. (Remember to show all row operations with intermediate matrices.)

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Before we move on, we mention something that will be needed in the next chapter. The process of obtaining a '1' in a specific location and then making all other entries '0' in that column is called **pivoting**.

With this terminology, the Gauss-Jordan Elimination Method can be remembered as pivoting on Row 1, Column 1, then pivoting on Row 2, Column 2, and continuing to pivot on the entries along the main diagonal from upper left to lower right until you reach the last row or until an entry along the diagonal is zero, which cannot be removed.

Solving Systems using Gauss-Jordan Elimination with Technology

With systems of two linear equations in two unknowns, the Gauss-Jordan Elimination Method may require up to four row operations. Larger systems may require even more row operations, as seen in a previous example. The calculator can be used to complete the Gauss-Jordan Elimination Method quickly, but without showing all intermediate row operations. The intermediate row operations and resulting matrices are taking place behind the screen.

N *Gauss-Jordan Elimination on the TI-84 calculator only works for matrices where the number of rows is less than or equal to the number of columns, using the process which follows.*

Suppose you are asked to determine the solution to the given the system of linear equations:

$$\begin{aligned} 5x+3y+9z &= -1 \\ -2x+3y-z &= -2 \\ -x-4y+5z &= 1 \end{aligned}$$

Begin by recognizing the system is already in the correct aligned format, so we can write the corresponding augmented matrix.

$$\begin{aligned} 5x+3y+9z &= -1 \\ -2x+3y-z &= -2 \\ -x-4y+5z &= 1 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{array}\right]$$

To reduce this matrix using the calculator, we must first enter the augmented matrix into the calculator using the matrix operation. Choosing to use matrix [A] in the calculator, the process and resulting matrix are shown in **Figures 2.4.2**, **2.4.3**, and **2.4.4**.

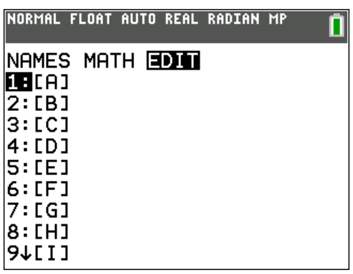


Figure 2.4.2: Calculator screenshot showing the matrix menus, including EDIT.

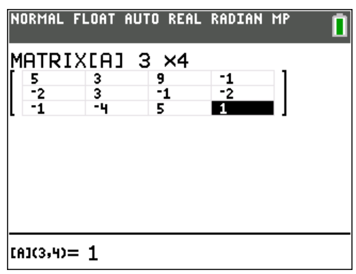


Figure 2.4.3: Calculator screenshot showing how to enter our matrix.

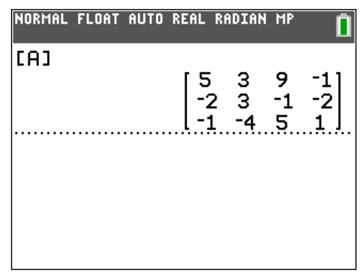


Figure 2.4.4: Calculator screenshot showing both the letter name of the matrix and the numerical values in the matrix.

When entering augmented matrices into the calculator, no vertical bar will separate the coefficients from the constants. When writing the result on paper, you will need to add the vertical bar.

Using the **rref** operation in the calculator on matrix [A] will result in the equivalent reduced row-echelon matrix. (See **Figures 2.4.5**, **2.4.6**, and **2.4.7**.)

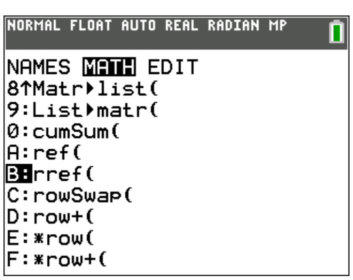


Figure 2.4.5: Calculator screenshot showing the matrix MATH menu, where rref is listed.

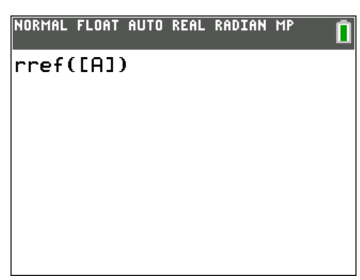


Figure 2.4.6: Calculator screenshot showing the call of the rref operation, applied to matrix A.

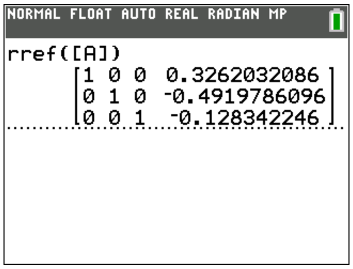


Figure 2.4.7: Calculator screenshot showing rref([A]) and the reduced matrix.

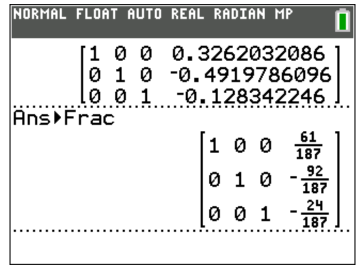


Figure 2.4.8: Calculator screenshot showing the exact values of the reduced matrix, using the fraction operation.

Notice **Figure 2.4.7** contains the non-exact values, so we must convert them to exact values using the fraction operation, as shown in **Figure 2.4.8**. On paper, this calculator process can be written as follows:

$$\left[\begin{array}{rrr|r} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & 4 & 5 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{61}{187} \\ 0 & 1 & 0 & -\frac{92}{187} \\ 0 & 0 & 1 & -\frac{24}{187} \end{array}\right]$$

To determine the solution for the given system, we must now write the final augmented matrix as its corresponding system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{61}{187} \\ 0 & 1 & 0 & -\frac{92}{187} \\ 0 & 0 & 1 & -\frac{24}{187} \end{array}\right] \Longrightarrow \begin{aligned} x+0y+0z &= \frac{61}{187} \\ 0x+y+0z &= -\frac{92}{187} \\ 0x+0y+z &= -\frac{24}{187} \end{aligned} \Longrightarrow \begin{aligned} x &= \frac{61}{187} \\ y &= -\frac{92}{187} \\ z &= -\frac{24}{187} \end{aligned}$$

So, clearly, the solution to the given system is $(x,y,z) = \left(\frac{61}{187}, -\frac{92}{187}, -\frac{24}{187}\right)$.

N *The authors assume, from this point forward, that where a zero coefficient exists in an augmented matrix the variable will not exist in the corresponding equation.*

■ **Example 9** Solve the following system of linear equations. If the system is dependent, write the parametric solution.

$$\begin{aligned} 2x+3y-4z &= 7 \\ 3x+4y-2z &= 9 \\ 5x+7y-6z &= 20 \end{aligned}$$

Solution:
We enter the corresponding augmented matrix in the calculator and use the rref calculator operation to obtain the equivalent matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 7 \\ 3 & 4 & -2 & 9 \\ 5 & 7 & -6 & 20 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Writing the final augmented matrix as its corresponding system of linear equations gives us

$$\begin{aligned} x + 10z &= 0 \\ y - 8z &= 0 \\ 0 &= 1 \text{✗} \end{aligned}$$

Clearly this statement is a contradiction, because $0 \neq 1$. Therefore, the given system is inconsistent and has no solution.

■

■ **Example 10** Solve the following system of linear equations. If the system is dependent, write the parametric solution.

$$\begin{aligned} x + y + z &= 2 \\ 2x + y - z &= 3 \\ 3x + 2y &= 5 \end{aligned}$$

Solution:
We enter the corresponding augmented matrix in the calculator and use the rref calculator operation to obtain the equivalent matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & 0 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Writing the final augmented matrix as its corresponding system of linear equations gives us

$$\begin{aligned} x - 2z &= 1 \\ y + 3z &= 1 \\ 0 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} x - 2z &= 1 \\ y + 3z &= 1 \end{aligned}$$

Notice we have no contradiction present, and we do not have each variable equal to a specific value. This means we have a dependent system (with infinitely many solutions), and, thus, a parametric solution to the system. An easy way to determine how many free variables we need is to see how many, and which, columns of the final augmented matrix do not contain a leading 1. Here, leading 1's are present in the 'x' and 'y' columns, but not in the 'z' column. Therefore, z will be our free variable and we will let $z = t$.

Using the substitution $z = t$, and solving for x and y gives us

$$\begin{aligned} x - 2z &= 1 \\ x - 2t &= 1 \\ x &= 1 + 2t \end{aligned} \qquad \text{and} \qquad \begin{aligned} y + 3z &= 1 \\ y + 3t &= 1 \\ y &= 1 - 3t \end{aligned}$$

So, the solution to our system is given by $(x, y, z) = (1 + 2t, 1 - 3t, t)$, where t represents any real number.

■

Try It # 6:
Solve each system of linear equations, using technology. If the system is dependent, write the parametric solution.

a. $\begin{aligned} -x - 2y + z &= -1 \\ 2x + 3y &= 2 \\ y - 2z &= 0 \end{aligned}$ **b.** $\begin{aligned} x + 2y + 3z &= 9 \\ 3x + 4y + z &= 5 \\ 2x - y + 2z &= 11 \end{aligned}$ **c.** $\begin{aligned} 3x - 9y + 6z &= -12 \\ x - 3y + 2z &= -4 \\ 8 - 2x &= -6y + 4z \end{aligned}$

Solving Applications of Systems

Our discussion now returns to the beginning of the section where we posed the following situation:

> Suppose a company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. The company has staffing for 65 hours of assembly and 55 hours of painting each week. If the company wants to fully utilize all staffed hours, how many of each item should they produce?

Recall we defined the variables b and p as

$b :=$ the number of basic products produced

and

$p :=$ the number of premium products produced

and found the system of linear equations as

$$\begin{aligned} 20b + 30p &= 3900 \\ 15b + 30p &= 3300 \end{aligned}$$

Now, using an augmented matrix and the calculator, we can easily obtain the solution.

$$\left[\begin{array}{cc|c} 20 & 30 & 3900 \\ 15 & 30 & 3300 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 120 \\ 0 & 1 & 50 \end{array}\right] \Rightarrow \begin{aligned} b &= 120 \\ p &= 50 \end{aligned}$$

Clearly, the solution to the system is $(b, p) = (120, 50)$. Thus, the company should produce 120 basic versions and 50 premium versions of its product to fully utilize all staffed hours.

If you are solving a system related to a word problem, your solution should address the question posed in the word problem.

Example 11 Let's solve another problem we set up at the beginning of the section.

Julia has just retired and has \$600,000 in her retirement account that she needs to reallocate to produce income. She is looking at two investments: a very safe guaranteed annuity that will provide 3% annual interest and a somewhat riskier bond fund that averages 7% annual interest. She would like to invest as little as possible in the riskier bond fund, but she needs to produce \$40,000 a year, in interest, to live on. How much should she invest in each account?

Solution:
Recall we defined the variables a and b as

$a :=$ the amount (in dollars) she invests into the annuity

and

$b :=$ the amount (in dollars) she invests into the bond fund

and found the system of linear equations as

$$\begin{aligned} a + b &= 600000 \\ 0.03a + 0.07b &= 40000 \end{aligned}$$

Using an augmented matrix and the calculator we obtain the solution.

$$\left[\begin{array}{cc|c} 1 & 1 & 600000 \\ 0.03 & 0.07 & 40000 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 50000 \\ 0 & 1 & 550000 \end{array}\right] \Rightarrow \begin{array}{l} a = 50000 \\ b = 550000 \end{array}$$

Clearly, the solution to the system is $(a,b) = (50000, 550000)$. Thus, Julia should invest \$50,000 into the guaranteed annuity and \$550,000 into the riskier bond.

■

Example 12 Elise has a collection of 12 coins consisting of nickels, dimes, and quarters. If the total worth of the coins is \$1.80, how many are there of each? State all possible solutions.

Solution:
Notice there are three unknowns in this problem: the number of nickels, the number of dimes, and the number of quarters. We can start by clearly defining variables for these unknowns.

Let

$n :=$ the number of nickels in Elise's collection,

$d :=$ the number of dimes in Elise's collection,

and

$q :=$ the number of quarters in Elise's collection.

For her total number of coins, we have

$$n+d+q=12$$

As nickels are worth 5 cents each, dimes are worth 10 cents each, and quarters are worth 25 cents each, for the total worth of her coins (in dollars), we then have

$$0.05n+0.10d+0.25q=1.80$$

Therefore, to determine a solution to the problem, we just need to solve the following system of linear equations:

$$\begin{aligned} n+d+q&=12\\ 0.05n+0.10d+0.25q&=1.80 \end{aligned}$$

Using an augmented matrix and the calculator we can obtain the solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12\\ 0.05 & 0.10 & 0.25 & 1.80 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -12\\ 0 & 1 & 4 & 24 \end{array}\right] \Rightarrow \begin{aligned} n-3q&=-12\\ d+4q&=24 \end{aligned}$$

Because we did not contain a contradiction or a specific numerical value for each variable, we know we will have a parametric solution. The free variable will be q (as q is present in both equations of the corresponding system and its column does not contain a leading 1), so we let $q=t$ and obtain the following:

$$\begin{aligned} n-3q&=-12 \qquad & d+4q&=24\\ n-3t&=-12 \qquad & d+4t&=24\\ n&=-12+3t \qquad & d&=24-4t \end{aligned}$$

This gives us $(n,d,q)=(-12+3t,24-4t,t)$, where t is a parameter, but does have some restrictions.

Considering n, d, and q all represent an amount of coins, they must all have non-negative integer values.

To ensure all variables are non-negative, we set each variable greater than or equal to zero, and solve for t. The overlap of the resulting inequalities of t will produce restrictions on t which guarantee non-negative values for the variables, n, d, and q.

$\underline{q=t}$	$\underline{d=24-4t}$	$\underline{n=-12+3t}$
$t\geq 0$	$24-4t\geq 0$	$-12+3t\geq 0$
	$-4t\geq -24$	$3t\geq 12$
	$t\leq 6$	$t\geq 4$

The overlapping interval of $t\geq 0$, $t\geq 4$, and $t\leq 6$ is $4\leq t\leq 6$.

For n, d, and q to be integers, t must be an integer which falls in the above interval, so $t=4$, 5, or 6.

Thus, $(n,d,q)=(-12+3t,24-4t,t)$, where $t=4$, 5, or 6.

So there are three possible solutions for Elise's coin collection and they are as follows:

When $t = 4$,

$$\begin{aligned} n &= -12 + 3(4) = -12 + 12 = 0 \\ d &= 24 - 4(4) = 24 - 16 = 8 \\ q &= 4 \end{aligned}$$

⇒ Elise has 0 nickels, 8 dimes, and 4 quarters.

When $t = 5$,

$$\begin{aligned} n &= -12 + 3(5) = -12 + 15 = 3 \\ d &= 24 - 4(5) = 24 - 20 = 4 \\ q &= 5 \end{aligned}$$

⇒ Elise has 3 nickels, 4 dimes, and 5 quarters.

When $t = 6$,

$$\begin{aligned} n &= -12 + 3(6) = -12 + 18 = 6 \\ d &= 24 - 4(6) = 24 - 24 = 0 \\ q &= 6 \end{aligned}$$

⇒ Elise has 6 nickels, 0 dimes, and 6 quarters.

▪

Try It # 7:
The latest reports indicate that there are altogether 20,000 American, French, and Russian troops in Bosnia. The sum of the number of Russian troops and twice the number of American troops equals 10,000. Furthermore, the Americans have 5,000 more troops than the French. Are these reports consistent?

Try It Answers

1. Let
   m := the amount of money (in dollars) invested in the money-market fund
   b := the amount of money (in dollars) invested in municipal bonds
   f := the amount of money (in dollars) invested in mutual funds

$$\begin{aligned} m + b + f &= 12000 \\ f - 3b &= 0 \\ 0.03m + 0.04b + 0.07f &= 670 \end{aligned}$$

2. $\left[\begin{array}{rr|r} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array}\right]$

3.

$$\begin{aligned} x-y+z &= 5 \\ 2x-y+3z &= 1 \\ y+z &= -9 \end{aligned}$$

4. **a.** NOT in reduced row-echelon form; Condition #3
b. NOT in reduced row-echelon form; Condition #1
c. IN Reduced row-echelon form
d. IN Reduced row-echelon form

5.

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right] \xrightarrow{-2R_1+R_3 \to R_3}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow{2R_2+R_1 \to R_1} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-3R_3+R_2 \to R_2}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-9R_3+R_1 \to R_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(x,\ y,\ z) = (1,\ -1,\ 2)$

6. **a.** $\left[\begin{array}{rrr|r} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution: $(x,y,z) = (1-3t, 2t, t)$, where t = any real number.

b. $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 3 & 4 & 1 & 5 \\ 2 & -1 & 2 & 11 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Solution: $(x,y,z) = (2,-1,3)$

c. $\left[\begin{array}{rrr|r} 3 & -9 & 6 & -12 \\ 1 & -3 & 2 & -4 \\ -2 & 6 & -4 & -8 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

No Solution

7. The reports are not consistent.

EXERCISES

BASIC SKILLS PRACTICE (Answers)

For Exercises 1 - 5, set up, but do not solve, a system of linear equations which could be used to solve the problem.

1. A local buffet charges \$7.50 per person for the basic buffet and \$9.25 for the deluxe buffet (which includes crab legs). If 27 diners went out to eat and the total bill was \$227.00 before taxes, how many chose the basic buffet and how many chose the deluxe buffet?

2. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

3. Skippy has a total of \$10,000 to split between two investments. One account offers 3% simple interest, and the other account offers 8% simple interest. For tax reasons, he can only earn \$500, in interest, the entire year. How much money should Skippy invest in each account to earn \$500 in interest for the year?

4. Three co-workers (the warehouse manager, the office manager, and a truck driver) all have different annual salaries. The sum of the annual salaries of the warehouse manager and office manager is \$82,000. The office manager makes \$4,000 more than the truck driver annually, and the annual salaries of the warehouse manager and the truck driver total \$78,000. What is the annual salary of each of the co-workers?

5. In a bag, a child has 325 coins worth \$19.50. There are three types of coins: pennies, nickels, and dimes. If the bag contains the same number of nickels as dimes, how many of each type of coin is in the bag?

For Exercises 6 - 9, write the corresponding augmented matrix for each of the following systems of linear equations.

6. $\begin{cases} 7x - 3y = 5 \\ 2x + 8y = 1 \end{cases}$

7. $\begin{cases} 5x + 3y - 9z = 15 \\ -2x - 4y + 3z = 10 \\ 6x - 11y + 2z = 20 \end{cases}$

8. $\begin{cases} x + 2y = 0 \\ y = 1 \end{cases}$

9. $\begin{cases} x - y + z = 2 \\ x - z = 1 \\ y + 2z = 0 \end{cases}$

For Exercises 10 - 13, write a system of linear equations which corresponds to the augmented matrix. Assume the variables are x and y or x, y, and z.

10. $\left[\begin{array}{cc|c} 2 & 5 & 32 \\ 3 & 6 & 67 \end{array}\right]$

11. $\left[\begin{array}{ccc|c} 3 & -4 & 10 & 50 \\ 1 & 6 & -8 & -20 \\ -7 & 1 & 3 & 66 \end{array}\right]$

12. $\left[\begin{array}{cc|c} 1 & -4 & 9 \\ 0 & 8 & 24 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 0 & 3 & -5 & 25 \\ 1 & -4 & 0 & 41 \\ 2 & 0 & -6 & 37 \end{array}\right]$

For Exercises 14 - 18, perform the indicated row operation, and write the resulting matrix.

14. $\left[\begin{array}{rr|r} 2 & 5 & -6 \\ -3 & 7 & 8 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1}$

15. $\left[\begin{array}{rr|r} 1 & -2 & -6 \\ 5 & 4 & -3 \end{array}\right] \xrightarrow{-5R_1 + R_2 \to R_2}$

16. $\left[\begin{array}{rrr|r} 2 & 5 & -3 & 8 \\ 3 & -1 & 6 & 10 \\ 1 & 4 & -1 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3}$

17. $\left[\begin{array}{rrr|r} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 9 & 10 \end{array}\right] \xrightarrow{\frac{1}{9}R_3 \to R_3}$

18. $\left[\begin{array}{rrr|r} 1 & -7 & 2 & -2 \\ 0 & 1 & -4 & 5 \\ 0 & 10 & -8 & 9 \end{array}\right] \xrightarrow{7R_2 + R_1 \to R_1}$

For Exercises 19 - 24, state if the matrix is in reduced row-echelon form. If the matrix is not in reduced row-echelon form, state which of the four conditions is first violated, as stated in the definition.

19. $\left[\begin{array}{rr|r} -1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

20. $\left[\begin{array}{rrr|r} 1 & 0 & 4 & 3 \\ 0 & 1 & 7 & 9 \\ 0 & 0 & 0 & 1 \end{array}\right]$

21. $\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$

22. $\left[\begin{array}{rr|r} 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$

23. $\left[\begin{array}{rrr|r} 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 20 \\ 1 & 0 & 0 & 19 \end{array}\right]$

24. $\left[\begin{array}{rrr|r} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right]$

For Exercises 25 - 30, the matrices are in reduced row-echelon form. Determine the solution of the corresponding system of linear equations. Write all answers as ordered pairs or ordered triples, as appropriate.

25. $\left[\begin{array}{rr|r} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

26. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{array}\right]$

27. $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

28. $\left[\begin{array}{rr|r} 1 & 2 & -5 \\ 0 & 0 & 1 \end{array}\right]$

29. $\left[\begin{array}{rrr|r} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

30. $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$

Intermediate Skills Practice (Answers)

For Exercises 31 - 35, set up, but do not solve, a system of linear equations which could be used to solve the problem.

31. There were 130 faculty at a conference. If there were 18 more non-tenure track faculty than tenure track faculty attending, how many of each category of faculty attended the conference?

32. CDs cost \$5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost, if 5 CDs and 2 DVDs cost \$127.73?

33. A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested \$80,500 into three accounts: the first paid 4% simple interest, the second paid 3.125% simple interest, and the third paid 2.5% simple interest. He earned \$2,670 in interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

34. Your roommate, Sarah, offered to buy groceries for you and your other roommate, Tara. The total spent on all groceries was \$82. Sarah forgot to save the individual receipts but remembered that your groceries were 5 cents cheaper than half of her groceries, and that Tara's groceries were \$2.10 more than your groceries. How much money was each person's share of the grocery bill?

35. A local band sells out for their concert. They sell all 1175 tickets for a total revenue of \$28,112.50. The tickets were priced at \$20 for student tickets, \$22.50 for children tickets, and \$29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type of ticket was sold?

For Exercise 36 - 39, write the corresponding augmented matrix for each of the following systems of linear equations.

36. $\left\{\begin{array}{rcrcr} x & = & -20 & - & 2y \\ 2 & = & 3x & - & y \end{array}\right.$

37. $\left\{\begin{array}{rcrcr} y & = & 1 & - & x \\ z & = & -y & & \\ z & - & x & = & 2 \end{array}\right.$

38. $\left\{\begin{array}{rcrcr} 4y & = & -x & + & 6 \\ \frac{1}{12}x & - & \frac{1}{2} & = & -\frac{1}{3}y \end{array}\right.$

39. $\left\{\begin{array}{rcrcrcr} 3x & - & \frac{1}{2}y & - & z & = & -\frac{1}{2} \\ & & 4x & + & z & = & 3 \\ & & -x & + & \frac{3}{2}y & = & \frac{5}{2} \end{array}\right.$

For Exercise 40 - 43, write a system of linear equations which corresponds to the augmented matrix.

40. $\left[\begin{array}{cc|r} 1 & 0 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{array}\right]$

41. $\left[\begin{array}{rrr|r} 1 & -2 & 0 & 12 \\ 0 & 5 & 9 & 14 \end{array}\right]$

42. $\left[\begin{array}{cc|c} 1 & 0 & 9 \\ 2 & 4 & 24 \\ 0 & 0 & 0 \end{array}\right]$

43. $\left[\begin{array}{ccr|r} 1 & 0 & 8 & 11 \\ 0 & 1 & -6 & 22 \\ 0 & 0 & 0 & 1 \end{array}\right]$

For Exercises 44 - 47, determine the next row operation used in the Gauss-Jordan Elimination Method, and write the resulting matrix.

44. $\left[\begin{array}{cc|c} 1 & 4 & 5 \\ 3 & -2 & -1 \end{array}\right] \xrightarrow{?}$

45. $\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 4 & -12 \end{array}\right] \xrightarrow{?}$

46. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -10 \\ 0 & 5 & 10 & 30 \\ 5 & -8 & 2 & 20 \end{array}\right] \xrightarrow{?}$

47. $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 8 \\ 0 & -3 & 12 & 15 \\ 0 & -1 & 4 & -9 \end{array}\right] \xrightarrow{?}$

For Exercises 48 - 57, solve each system of linear equations, using the techniques discussed in this section.

48. $\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$

49. $\begin{cases} x + y + z = 3 \\ 2x - y + z = 0 \\ -3x + 5y + 7z = 7 \end{cases}$

50. $\begin{cases} x - y + z = -4 \\ -3x + 2y + 4z = -5 \\ x - 5y + 2z = -18 \end{cases}$

51. $\begin{cases} 2x - y + z = -1 \\ 4x + 3y + 5z = 1 \\ 5y + 3z = 4 \end{cases}$

52. $\begin{cases} 4x - y + z = 5 \\ 2y + 6z = 30 \\ x + z = 5 \end{cases}$

53. $\begin{cases} 4x - 6y = -18 \\ -2x + 3y = 9 \end{cases}$

54. $\begin{cases} 2x - 3y + z = -1 \\ 4x - 4y + 4z = -13 \\ 6x - 5y + 7z = -25 \end{cases}$

55. $\begin{cases} 2x - y + z = 1 \\ 2x + 2y - z = 1 \\ 3x + 6y + 4z = 9 \end{cases}$

56. $\begin{cases} x - 3y - 4z = 3 \\ 3x + 4y - z = 13 \\ 2x - 19y - 19z = 2 \end{cases}$

57. $\begin{cases} x + y + z = 4 \\ 2x - 4y - z = -1 \\ x - y = 2 \end{cases}$

For Exercises 58 - 62, solve using the system of linear equations created in Exercises 31 - 35, respectively.

58. There were 130 faculty at a conference. If there were 18 more non-tenure track faculty than tenure track faculty attending, how many of each category of faculty attended the conference?

59. CDs cost \$5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost, if 5 CDs and 2 DVDs cost \$127.73?

60. A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested \$80,500 into three accounts: the first paid 4% simple interest, the second paid 3.125% simple interest, and the third paid 2.5% simple interest. He earned \$2,670 in interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

61. Your roommate, Sarah, offered to buy groceries for you and your other roommate, Tara. The total spent on all groceries was \$82. Sarah forgot to save the individual receipts but remembered that your groceries were 5 cents cheaper than half of her groceries, and that Tara's groceries were \$2.10 more than your groceries. How much money was each person's share of the grocery bill?

62. A local band sells out for their concert. They sell all 1175 tickets for a total revenue of \$28,112.50. The tickets were priced at \$20 for student tickets, \$22.50 for children tickets, and \$29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type of ticket was sold?

Mastery Practice (Answers)

63. Give the row operation used between each augmented matrix:

$$\left[\begin{array}{rr|r} 3 & -18 & 9 \\ -4 & 2 & -1 \end{array}\right] \xrightarrow{\quad ? \quad} \left[\begin{array}{rr|r} 1 & -6 & 3 \\ -4 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{\quad ? \quad} \left[\begin{array}{rr|r} 1 & -6 & 3 \\ 0 & -22 & 11 \end{array}\right]$$

$$\xrightarrow{\quad ? \quad} \left[\begin{array}{rr|r} 1 & -6 & 3 \\ 0 & 1 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{\quad ? \quad} \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \end{array}\right]$$

64. Give the row operation used between each augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 0 & 6 & 0 \\ 1 & 2 & 0 & 4 \\ 2 & 0 & -4 & 10 \end{array}\right] \xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 0 & 6 & 0 \\ 2 & 0 & -4 & 10 \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -6 & 6 & -12 \\ 2 & 0 & -4 & 10 \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -6 & 6 & -12 \\ 0 & -4 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -4 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -4 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -8 & 10 \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -\frac{5}{4} \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & -\frac{5}{4} \end{array}\right]$$

$$\xrightarrow{?} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & -\frac{5}{4} \end{array}\right]$$

65. An investor who dabbles in real estate invested \$1.1 million in two different land developments. The first development, Swan Peak, returned 110% on her investment. The second investment, Riverside Community, returned 50% on her investment. If she earned \$1 million in returns, how much did she invest in each of the land developments?

66. At The Crispy Critter's Head Shop and Patchouli Emporium, along with their dried up weeds, sunflower seeds, and astrological postcards, they also sell herbal tea blends. By weight, Rosy Tea is 30% peppermint, 40% rose hips and 30% chamomile, Minty Tea is 40%, 20%, and 40%, respectively, and Sleepy Tea is 35%, 30% and 35%, respectively. How much of each type of tea is needed to make 2 pounds of a new blend of tea that is equal parts peppermint, rose hips, and chamomile?

Communication Practice (Answers)

67. How would you approach Exercise 66 if they needed to use up a pound of Rosy Tea to make room on the shelf for a new canister?

68. Explain why the following matrix in reduced row-echelon form represents a system of linear equations that requires two parameters, when writing the solution to the system in parametric form.

$$\left[\begin{array}{rrrr|r} 1 & 0 & -8 & 1 & 7 \\ 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Chapter Review

Directions: Read each reflection question. If you are able to answer the question, then move on to the next question. If not, use the mathematical questions included to help you review. (Answers)

1. Given information about a line, how would you determine if the information provided was sufficient to formulate the corresponding equation?

   a. Write the equation of the line passing through the points $\left(7, \frac{4}{5}\right)$ and $(2a, 0)$ in point-slope form, slope-intercept form, and standard form. Note a is any real number greater than 5.

   b. Write the equation of the line which has undefined slope and passes through the point $(4, -9)$.

   c. Write the equation of the line with an x-intercept of $(8, 0)$ and an y-intercept of $(0, -11)$ in point-slope form, slope-intercept form, and standard form.

   d. Write the equation of a horizontal line passing through the point $\left(-\frac{2}{3}, -7\right)$.

2. Can you graph a linear equation, without the use of technology?

   a. Graph the line $y = -\frac{3}{7}x - 2$, by hand.

   b. Graph a line which intersects the horizontal axis at 1 and the vertical axis at 2, by hand.

   c. Graph a vertical line passing through the point $(-8, 22)$, by hand.

   d. Graph a line with a slope of zero which passes through the point $(145, 35)$, by hand.

3. What is the relationship between the change in x and the change in y for the coordinates of points on a line?

   a. If the slope of a line is $m = \frac{3}{5}$ and y increases by 7 units, what is the corresponding change in x?

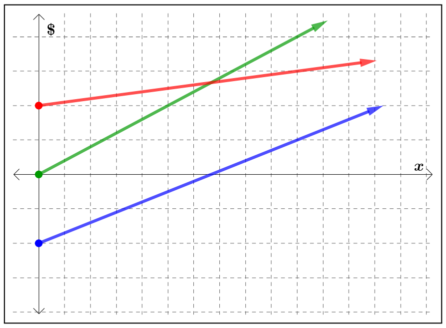
   b. A line passes through the points $(-3, 8)$ and $(4, -6)$. When x decreases by 4 units, what is the corresponding change in y?

   c. On a given line, when x increases by 6, y decreases by 11. What is the slope of this line?

4. Can you explain the mathematical relationship of cost, revenue, and profit in terms a person outside of a mathematics class would understand, and can you illustrate the important features when all three functions are graphed on the same coordinate plane?

   a. The cost and revenue functions (in dollars) for a certain product are $C(x) = 3x + 27$ and $R(x) = 7x$, respectively, where x is the number of items produced and sold. For what values of x is the company not making a profit (in the red)?

b. Label the cost, revenue, and profit functions on the graph below. Explain your reasoning.



5. Can you compare and contrast the features of the supply and demand functions for a certain product in the market?

   a. True or False: Demand functions have a positive slope.

   b. True or False: The vertical intercept of a supply function represents the highest price any consumer is willing to pay for the item.

   c. True or False: The horizontal intercept of a demand function represents the number of items consumers will take if the item is free.

6. What approach would you use to find the equation for a linear depreciation, cost, revenue, profit, supply, or demand function?

   a. An industrial refrigerator is purchased for $12,000. The manufacturer of the refrigerator says the product will depreciate linearly by $450 per year. State a linear model for the value of the refrigerator as a function of time. Include a reasonable domain and range for the function.

   b. A new television manufacturer has fixed costs of $225,000. If they make and sell 1500 televisions for a total cost of $375,000, while bringing in $600,000 in revenue, what is the manufacturer's linear cost, revenue, and profit functions for producing and selling x televisions?

   c. For a price of $100 each, consumers will purchase 1000 computers. If the price per computer increases by $20, then consumers will purchase 50 less computers. If the demand is assumed to be linear, write the demand equation, and use it to determine how many computers will be purchased at a unit price $110. Is this more or less than when the price was $100?

   d. A local farmer is willing to supply 3000 watermelons at a price of $4.50 each. If the price increases to $5.25 each, the farmer is willing to supply 1000 more watermelons. If the supply is assumed to be linear, write the supply equation, and use it to determine the price per watermelon when the farmer supplies 8000 watermelons.

7. How do you identify the rate of depreciation and the value of an item (including initial and scrap) from a linear depreciation model?

   a. A new moped is purchased for $600 in 2019. The dealer tells the buyer the moped will depreciate linearly over a 5-year period with a scrap value of $100. What is the value of the moped in 2021 and in 2030?

b. An item is purchased for \$9800 and 5 years later is valued at \$2800. If the item depreciates linearly over 10 years, what is the rate of depreciation for the item?

c. A car is purchased for \$34,000 and 7 years later is valued at \$20,000. The car depreciates linearly over 10 years, after which point it reaches scrap value. Compute and interpret the scrap value.

8. How would you determine the number of solutions to a system of two linear equations in two unknowns without computing the solution(s)?

a. Determine the type of system and number of solutions for the following system of linear equations.

$$\begin{aligned} 2x + 5y &= 24 \\ 3x - 4y &= 24 \end{aligned}$$

b. For what value of k does the following system of linear equations below have no solution?

$$\begin{aligned} 6x - ky &= 24 \\ -2x + 8y &= 24 \end{aligned}$$

c. For what value of k does the following system of linear equations have infinitely many solutions?

$$\begin{aligned} -2x + 5y &= 25 \\ kx - 4y &= -20 \end{aligned}$$

9. Can you solve a system of two linear equations in two unknowns **without the use of technology**?

a. State the solution(s) as an ordered pair(s), if one exists, for the following system of linear equations.

$$\begin{aligned} 2x + y &= 7 \\ 3x - y &= 3 \end{aligned}$$

b. State the solution(s) as an ordered pair(s), if one exists, for the following system of linear equations.

$$\begin{aligned} 6x - 9y &= 24 \\ -10x + 15y &= -40 \end{aligned}$$

c. State the solution(s) as an ordered pair(s), if one exists, for the following system of linear equations.

$$\begin{aligned} x &= 17 - 5y \\ -x - 5y &= 31 \end{aligned}$$

10. Can you solve a system of two linear equations in two unknowns **with the use of technology**?

    a. Solve the following system of linear equations, if possible. Write your answer as an ordered pair, if one exists.

    $$\begin{aligned} 2x+y&=7 \\ 3x-y&=3 \end{aligned}$$

    b. Solve the following system of linear equations, if possible. Write your answer as an ordered pair, if one exists.

    $$\begin{aligned} 6x-9y&=24 \\ -10x+15y&=-40 \end{aligned}$$

    c. Solve the following system of linear equations, if possible. Write your answer as an ordered pair, if one exists.

    $$\begin{aligned} x&=17-5y \\ -x-5y&=31 \end{aligned}$$

11. Can you find and explain the meaning of the break-even or equilibrium point in terms relating to the appropriate business application?

    a. State how you can use a system of equations to find the break-even point.

    b. The cost and revenue functions (in dollars) for a certain product are $C(x) = 95x+4400$ and $R(x) = 645x$, respectively, where x is the number of items produced and sold.

    i. What is the profit function?

    ii. What is the break-even point? Explain each coordinate of the break-even point in terms of the context of the application.

    c. The producers of hummingbird feeders will supply 50 feeders for \$6, but for every \$1 increase in price, they are willing to produce 10 additional feeders. The demand equation for the hummingbird feeders is given by $p(x) = -\frac{1}{10}x+40$.

    i. What is the equilibrium point?

    ii. Write a sentence explaining the meaning of the equilibrium point in the context of this scenario.

12. Given an application in paragraph form, would you be able to convert the problem to an equivalent system of linear equations?

    a. You invested \$100,000 in three different stocks. The first stock has a 12% return on the investment, while the second and third stocks have a 10% and 8% return, respectively. The total returns on all your stocks is \$10,400. The sum of the investments in the second and third stocks is the same as the amount invested in the first stock. How much did you invest in each stock? **Convert the application to an equivalent system of linear equations, but do not solve.**

b. A local university is putting together some home soccer ticket packages for students to purchase:

	Number of Tickets	Price for Package
Package A	1	\$8.00
Package B	2	\$14.00
Package C	3	\$20.00

A total of 240 home game tickets are available, and the total revenue from all packages sold is \$1692. If the revenue generated from the sales of Package C is \$32 less than twice the revenue generated from the sales of Package A, how many of each package were sold? **Convert the application to an equivalent system of linear equations, but do not solve.**

13. How would you input a system of linear equations into a graphing calculator?

a. Write the following system of linear equations as an augmented matrix.

$$\begin{aligned} x &= 17 - 5y \\ -x - 5y &= 31 \end{aligned}$$

b. Write the following system of linear equations as an augmented matrix.

$$\begin{aligned} 2x + 7y &= 9 - z \\ x - 8 &= 5y + 9z \end{aligned}$$

c. Write the following system of linear equations as an augmented matrix.

$$\begin{aligned} 4x_1 + 8x_2 - 9x_3 &= 0 \\ x_1 + x_3 &= 17 \\ 18x_1 - 11x_2 &= 2 \end{aligned}$$

14. Can you list the four conditions required for a matrix to be in reduced row-echelon form (RREF)?

a. Determine if the matrix is in reduced row-echelon form. If not, state the first condition which fails.

$$\left[\begin{array}{rrr|r} 1 & -9 & 0 & 19 \\ 0 & 1 & 3 & -23 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

b. Determine if the matrix is in reduced row-echelon form. If not, state the first condition which fails.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 19 \\ 0 & 1 & 3 & -23 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

c. Determine if the matrix is in reduced row-echelon form. If not, state the first condition which fails.

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 19 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

15. Can you list and perform the three row operations allowed in the Gauss-Jordan elimination method?

 a. Perform the row operation $\frac{1}{4}R_1 \to R_1$ on the given matrix.

$$\left[\begin{array}{rr|r} 4 & 3 & 16 \\ 10 & -6 & 18 \end{array}\right]$$

 b. Perform the row operation $-3R_2 + R_1 \to R_1$ on the given matrix.

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & 16 \\ 0 & 1 & 2 & 18 \\ 0 & 0 & 4 & 2 \end{array}\right]$$

 c. State and perform the row operations necessary to pivot on Row 1, Column 1.

$$\left[\begin{array}{rr|r} 3 & 9 & -18 \\ 10 & -6 & 24 \end{array}\right]$$

16. Can you use the Gauss-Jordan Elimination Method to solve a system of linear equations, by hand?

 a. Solve the following system of linear equations using the Gauss-Jordan Elimination Method, by hand. If the system is dependent, write the parametric solution.

$$\begin{aligned} -x - 6y &= 30 \\ 4x - 8y &= 28 \end{aligned}$$

 b. Solve the following system of linear equations using the Gauss-Jordan Elimination Method, by hand. If the system is dependent, write the parametric solution.

$$\begin{aligned} 4x - 12y &= 16 \\ -3x + 2y &= 22 \end{aligned}$$

17. How do you use technology to perform all necessary Gauss-Jordan Elimination row operations in one step?

 a. Solve the following system of linear equations. If the system is dependent, write the parametric solution.

$$\begin{aligned} 3x - 6y + 4z &= 1 \\ 2x - z &= 8 \\ 9x - 18y + 12z &= 3 \end{aligned}$$

 b. Solve the following system of linear equations. If the system is dependent, write the parametric solution.

$$\begin{aligned} 4x &= -8y + 9z \\ x + y &= 17 \\ -11y - 2 &= -18x \end{aligned}$$

c. Solve the following system of linear equations. If the system is dependent, write the parametric solution.

$$\begin{aligned} x+2y+4z &= 26 \\ 3x-6y+z &= 4 \\ 4x+5z &= 4y \end{aligned}$$

18. How would you read and determine the solution(s), if any exists, to a system of linear equations solved with the use of matrices?

a. Use the augmented matrix in reduced row-echelon form to write the solution to the corresponding system of linear equations, as an ordered pair, if one exists.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{4}{5} \\ 0 & 0 & 0 \end{array}\right]$$

b. Use the augmented matrix in reduced row-echelon form to write the solution to the corresponding system of linear equations, as an ordered pair, if one exists.

$$\left[\begin{array}{cc|c} 1 & 3 & \frac{4}{5} \\ 0 & 0 & 1 \end{array}\right]$$

c. Use the augmented matrix in reduced row-echelon form to write the solution to the corresponding system of linear equations, as an ordered pair, if one exists.

$$\left[\begin{array}{cc|r} 1 & 0 & 6 \\ 0 & 1 & -14 \\ 0 & 0 & 0 \end{array}\right]$$

19. How would you use an augmented matrix to solve a real-world application, including those with reasonable parameters?

a. You invested \$100,000 in three different stocks. The first stock has a 12% return on the investment, while the second and third stocks have a 10% and 8% return, respectively. The total returns on all your stocks is \$10,400. The sum of the investments in the second and third stocks is the same as the amount invested in the first stock. How much did you invest in each stock?

b. A local university is putting together some home soccer ticket packages for students to purchase:

	Number of Tickets	Price for Package
Package A	1	\$8.00
Package B	2	\$14.00
Package C	3	\$20.00

A total of 240 home game tickets are available, and the total revenue from all packages sold is \$1692. If the revenue generated from the sales of Package C is \$32 less than twice the revenue generated from the sales of Package A, how many of each package were sold?

c. Assume your solution to a real-world application problem was $(x, y, z) = (32 - 4t, 9 + 2t, t)$. If x, y, and z represent the number of whole items produced,

i. How many solutions does the problem actually have?

ii. Is $t = 2$ a value for the parameter? If so, state the solution when $t = 2$.

Chapter 3

3. Linear Programming

In this chapter we are going to discuss linear programming problems.

The authors recommend all students are comfortable with the topics below prior to beginning this chapter. If you would like additional support on any of these topics, please refer to the Appendix.

A.1 - Properties of Real Numbers

A.1 - Systems of Time Measurement

A.2 - Evaluating an Expression

A.2 - Translating an English Phrase to an Algebraic Expression or Equation

A.2 - Solving Linear Equations with One Variable

A.2 - Using Problem-Solving Strategies

A.2 - Graphing Inequalities on the Number Line and Interval Notation

A.2 - Solving Inequalities using the Addition and Subtraction Properties of Inequality

A.2 - Solving Inequalities using the Multiplication and Division Properties of Inequality

3.1 Setting Up Linear Programming Problems

© Photo by Susan Giefer, 2020

A department store sells two sizes of televisions: 21 inch and 40 inch. A 21 inch television requires 6 cubic feet of storage space, and a 40 inch television requires 18 cubic feet of space. A maximum of 1080 cubic feet of storage space is available. The 21 inch and the 40 inch televisions take up, respectively, 2 and 3 sales hours of labor, and the store has a maximum of 198 hours of labor available. If the profit earned from each of these sizes of televisions is \$60 and \$80, respectively, how many of each size of television should be sold to maximize the store's profit, and what is the maximum profit?

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. These conditions or constraints often take the form of inequalities. In this section, we will look at setting up such problems, which are called **linear programming problems**.

Learning Objectives:
In this section, you will learn to set up linear programming problems based on real-world scenarios. Upon completion you will be able to:

- Define the variables needed to solve a linear programming problem.
- Recognize and formulate the objective function of a linear programming problem.
- Recognize and formulate the constraints of a linear programming problem.

Linear Programming Problems

Definition

A **linear programming problem** consists of finding an extreme value (maximum or minimum) of a linear function, subject to certain constraints. Thus, we will classify linear programming problems as **maximization** or **minimization** problems, both known as **optimization** problems.

- The function being optimized is called the **objective function**.
- The conditions that must be satisfied are called the **constraints**.

The steps of converting a linear programming application into a usable mathematical form are given below.

Setting Up a Linear Programming Problem

1. Identify and *clearly* define all variables.
2. Indicate and write the objective function.
3. Indicate and write the constraints.

We will demonstrate this process through the question posed at the start of the section.

1. Identify and clearly define all variables.

Notice first from the question posed in the problem, we are looking for "how many of each size of television should be sold to maximize the store's profit." As we did when setting up a system of linear equations, the "how many" indicates we are looking for a quantity, while the "of each size of television" directs you to the first sentence which states "two sizes of televisions: 21 inch and 40 inch." We will start by defining our variables as follows:

t := the number of 21 inch TVs sold

and

f := the number of 40 inch TVs sold.

Unlike when setting up a system of linear equations, we also need a variable for the name of the function we are trying to optimize.

In the question posed, we are asked "to maximize the store's profit." Thus, we will define profit as

P := the profit earned on the sale of TVs, in dollars.

2. Indicate and write the objective function.

When looking for the objective, we often search out the word *maximize* or *minimize*.

The function being *maximized* in our scenario is the profit from the sale of two sizes of TVs, and "the profit earned on each of these sizes of televisions is \$60 and \$80, respectively." Because the store profits \$60 on each 21 inch TV, if t of these televisions are sold, the total profit earned on these TVs is $60t$ dollars. Similarly, the store profits \$80 on each 40 inch TV, so if f of these televisions are sold, the total profit earned on these TVs is $80f$ dollars. Thus, $60t+80f$ dollars represents the total profit earned on the sale of all 21 inch and 40 inch televisions. This gives us the following objective:

Objective: **Maximize** $P = 60t + 80f$

N *Your objective function should include all of your variables, and it will never be set equal to a numerical value, as it is a value that varies while you try to obtain its largest or smallest value.*

3. Indicate and write the constraints.

The constraints are developed through the remaining sentences of the given scenario and are often written as inequalities. They represent the conditions all variables are 'subject to.'

Each 21 inch TV requires 6 cubic feet of storage space, so t of these TVs take up $6t$ cubic feet of storage space. A 40 inch TV requires 18 cubic feet of storage space, so f of these TVs take up $18f$ cubic feet of storage space. Thus, $6t + 18f$ cubic feet of storage space is used by the department store for these televisions. It is stated that the department store has a maximum of 1080 cubic feet of storage space available. Realistically, the amount of storage space used *must* be less than or equal to the amount available, so we have the following constraint:

$$6t + 18f \leq 1080$$

Similarly, a total of $2t + 3f$ hours of labor is used to sell these TVs, and the store has a maximum of 198 hours of labor available for the sale of the televisions. Again, the number of hours of labor required *must* be less than or equal to the number of hours of labor available. Thus, we have the following constraint:

$$2t + 3f \leq 198$$

Now, we have found all the **stated constraints**:

$$\begin{aligned} 6t + 18f &\leq 1080 \\ 2t + 3f &\leq 198 \end{aligned}$$

Besides the stated constraints, we must also consider any unwritten constraints which arise from real-world factors; these constraints are normally implied. Logically, in the given scenario, we cannot sell a negative number of televisions; thus, it *must* be true that $t \geq 0$ and $f \geq 0$. These constraints are called **non-negativity constraints**. When setting up a linear programming problem for a given real-world scenario, it is almost always necessary to include non-negativity constraints.

So, all of our constraints can be written as:

Subject to:
$$\begin{aligned} &6t + 18f \leq 1080 \\ &2t + 3f \leq 198 \\ &t \geq 0, \ f \geq 0 \end{aligned}$$

For the final setup of the linear programming problem we combine the three steps and have

$t :=$ the number of 21 inch TVs sold

$f :=$ the number of 40 inch TVs sold

$P :=$ the profit earned on the sale of TVs, in dollars

Objective: **Maximize** $P = 60t + 80f$

Subject to:
$$\begin{aligned} &6t + 18f \leq 1080 && \text{(Cubic Feet of Storage Space)} \\ &2t + 3f \leq 198 && \text{(Hours of Labor)} \\ &t \geq 0, \ f \geq 0 \end{aligned}$$

We will return to solve this problem in a later section.

■ **Example 1** Set up, but do not solve, the following linear programming problem.

> A catering company is to make lunch for a business meeting. It will serve double ham sandwiches, ham sandwiches, and vegetarian sandwiches. A double ham sandwich has 1 serving of vegetables, 4 slices of ham, 1 slice of cheese, and 2 slices of bread. A ham sandwich has 2 servings of vegetables, 2 slices of ham, 1 slice of cheese and 2 slices of bread. A vegetarian sandwich has 3 servings of vegetables, 2 slices of cheese, and 2 slices of bread. A total of 10 bags of ham are available (each of which has 40 slices), 18 loaves of bread are available (each with 14 slices), 200 servings of vegetables are available, and 15 bags of cheese (each with 60 slices) are available. Given the resources, how many of each sandwich can be produced, if the goal is to maximize the number of sandwiches made by the company?

Solution:

1. We wish to maximize the number of sandwiches made, so let:

d := the number of double ham sandwiches made

h := the number of ham sandwiches made

v := the number of vegetarian sandwiches made

S := the total number of sandwiches made

2. The total number of sandwiches is given by $S = d + h + v$, which we are maximizing. So, the objective is given by

Objective: **Maximize** $S = d + h + v$

3. The constraints will be given by considering the total amount of each ingredient available. That is, the company has a limited amount of ham (10 bags), bread (18 loaves), vegetables (200 servings), and cheese (15 bags).

In total, the company has 10(40) = 400 slices of ham, 18(14) = 252 slices of bread, 200 servings of vegetables, and 15(60) = 900 slices of cheese available. At most, the company can use these amounts.

There are two sandwiches that use ham: each double ham sandwich requires 4 slices of ham, while each ham sandwich requires only 2. Because the total number of slices of ham cannot exceed 400, then

$$4d + 2h \leq 400$$

Each sandwich requires 2 slices of bread, and there are 252 slices available, so

$$2d + 2h + 2v \leq 252$$

Each double ham sandwich has 1 serving of vegetables, each ham sandwich has 2 servings of vegetables, and each vegetarian sandwich has 3 servings of vegetables. With 200 servings of vegetables available, we have

$$1d + 2h + 3v \leq 200$$

Both types of ham sandwiches require 1 slice of cheese, while each vegetarian sandwich requires 2 slices of cheese. With 900 slices of cheese available, it must be true that

$$1d + 1h + 2v \leq 900$$

Realistically, it is not possible to make a negative number of sandwiches, so we additionally have non-negativity constraints:

$$d \geq 0, h \geq 0, \text{and } v \geq 0$$

Our final setup is then:

d := the number of double ham sandwiches made

h := the number of ham sandwiches made

v := the number of vegetarian sandwiches made

S := the total number of sandwiches made

Objective: **Maximize** $S = d + h + v$

Subject to:

$$
\begin{aligned}
4d + 2h &\leq 400 && \text{(Slices of Ham)} \\
2d + 2h + 2v &\leq 252 && \text{(Slices of Bread)} \\
d + 2h + 3v &\leq 200 && \text{(Servings of Vegetables)} \\
d + h + 2v &\leq 900 && \text{(Slices of Cheese)} \\
d \geq 0, h \geq 0, v &\geq 0
\end{aligned}
$$

■

■ **Example 2** Set up, but do not solve, the following linear programming problem.

A company is creating a meal replacement bar. They plan to incorporate peanut butter, oats, and bananas as the primary ingredients. The nutritional content of one serving, 10 grams, of each ingredient is listed below, along with the cost of each, in cents. Determine the amount of each ingredient the company should use to minimize the cost of producing a bar containing a minimum of 15 grams of each ingredient, at least 10 grams of protein, and at most 14 grams of fat.

	Peanut Butter, 10g	Oats, 10g	Bananas, 10g
Protein (grams)	2.5	1.7	0.11
Fat (grams)	5	0.7	0.03
Cost (cents)	6	1	2

Table 3.1: The nutritional and cost information for each ingredient in a meal replacement bar.

Solution:

1. We start by defining the necessary variables.

p := the number of 10 gram servings of peanut butter

a := the number of 10 gram servings of oats

b := the number of 10 gram servings of bananas

C := the cost of producing a meal replacement bar, in cents

2. The total cost of producing a bar, in cents, will be $C = 6p + 1a + 2b$, which we are minimizing. So the objective is given by

Objective: **Minimize** $C = 6p + a + 2b$

3. Our first constraints come from the requirement for each bar to have *at least* 15 g of each ingredient, which is 1.5 servings (1.5 servings at 10 g per serving = 15 g). Constructing those constraints we have:

$$p \geq 1.5, a \geq 1.5, b \geq 1.5$$

Next, we look at the nutritional components. For protein, p servings of peanut butter will contain $2.5p$ grams of protein. Likewise, a servings of oats will have $1.7a$ grams of protein, and b servings of bananas will have $0.11b$ grams of protein. One bar needs to have *at least* 10 grams of protein, giving the constraint

$$2.5p + 1.7a + 0.11b \geq 10$$

We can construct a similar constraint for fat, in this case noting we want the fat contained in one bar to be *at most* 14 grams:

$$5p + 0.7a + 0.03b \leq 14$$

We can now state the complete setup of the problem:

p := the number of 10 gram servings of peanut butter

a := the number of 10 gram servings of oats

b := the number of 10 gram servings of bananas

C := the cost of producing a meal replacement bar, in cents

Objective: **Minimize** $C = 6p + a + 2b$

Subject to: $2.5p + 1.7a + 0.11b \geq 10$ (Grams of Protein)

$5p + 0.7a + 0.03b \leq 14$ (Grams of Fat)

$p \geq 1.5,\ a \geq 1.5,\ b \geq 1.5$

N *In this example, while our variables cannot have negative values, it is not necessary to include non-negativity constraints, as it is already required that each variable be at least 1.5 (making them non-negative).*

■

Try It # 1:
Set up, but do not solve, the following linear programming problem.

For her classes, Dr. Reid gives three types of quizzes: knowledge, application and evaluation. To keep her students on their toes, she has decided to give at least 20 quizzes next semester. The three types of quizzes, knowledge, application and evaluation, require a student to spend, respectively, 10 minutes, 30 minutes, and 60 minutes for preparation, and Dr. Reid would like them to spend at least 12 hours preparing for these quizzes. An average number of points on an evaluation quiz is 5, on an application quiz is 6, and on a knowledge quiz is 7. Dr. Reid would like students to score at least 130 points on all quizzes. It takes Dr. Reid one minute to grade a knowledge quiz, 2 minutes to grade an application quiz, and 3 minutes to grade an evaluation quiz. How many of each type of quiz should she give in order to minimize her grading time per student?

Try It Answers

1. Let

k := the number of knowledge quizzes given next semester

a := the number of application quizzes given next semester

e := the number of evaluation quizzes given next semester

G := the amount of Dr. Reid's grading time per student over the next semester, in minutes

Objective: **Minimize** $G = k + 2a + 3e$

Subject to: $k + a + e \geq 20$

$10k + 30a + 60e \geq 720$

$7k + 6a + 5e \geq 130$

$k \geq 0,\ a \geq 0,\ e \geq 0$

EXERCISES

Basic Skills Practice (Answers)

1. A student holds two part time jobs, tutoring for MATH 140 and grading for HIST 105, and they never want to work more than a total of 12 hours per week. The student has determined for every hour they tutor, they need two hours of preparation time, and for every hour they grade, the student needs one hour of preparation time. The student can not spend more than 16 hours each week in preparing for the two jobs. If the student makes $30 an hour tutoring and $10 an hour grading, how many hours should they work per week at each job to maximize their income?

   a. Identify and clearly define the variables needed to solve the linear programming problem.

   b. State the objective function, including whether the function is being minimized or maximized.

   c. Write the constraints, including any non-negativity constraints.

2. A factory manufactures and sells two types of gadgets, regular and premium. Each gadget requires the use of two operations, assembly and finishing, and there are at most 12 hours available each day for each operation. A regular gadget requires 1 hour of assembly and 2 hours of finishing, while a premium gadget needs 2 hours of assembly and 1 hour of finishing. Due to other restrictions, the company can make at most 7 gadgets a day. If a profit of $20 is realized for each regular gadget and $30 for each premium gadget, how many of each type of gadget should be manufactured each day to maximize the factory's profit?

   a. Identify and clearly define the variables needed to solve the linear programming problem.

   b. State the objective function, including whether the function is being minimized or maximized.

   c. Write the constraints, including any non-negativity constraints.

3. The law office of Lifelong Decisions wishes to employ two temporary paralegals, Kit and Kat, to draft legal documents for pending lawsuits. Kit can draft 20 documents per day and earns $150 per day, while Kat can draft 30 documents per day and earns $300 per day. Each paralegal must be employed at least 1 day per week to justify their employment. If the law office has at least 110 legal documents each week, how many days per week should the office employ each paralegal to minimize its costs?

   a. Identify and clearly define the variables needed to solve the linear programming problem.

   b. State the objective function, including whether the function is being minimized or maximized.

   c. Write the constraints, including any non-negativity constraints.

4. Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, pasta or tofu. The table below lists the amount of protein, carbohydrates, and Vitamin C each meal provides, along with the amount of cholesterol each meal contains – which he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of Vitamin C from these lunches each month. If Professor Hamer eats lunch in the cafeteria no more than 25 days each month, how many days should he select the pasta meal, and how many days should he choose the tofu meal so that he gets the adequate amount of protein, carbohydrates, and Vitamin C, while at the same time minimizing his cholesterol intake?

	Pasta	Tofu
Protein	8 g	17 g
Carbohydrates	60 g	40 g
Vitamin C	2 g	2 g
Cholesterol	60 mg	50 mg

a. Identify and clearly define the variables needed to solve the linear programming problem.

b. State the objective function, including whether the function is being minimized or maximized.

c. Write the constraints, including any non-negativity constraints.

Intermediate Skills Practice (Answers)

For Exercises 5 - 8, set up, but do not solve the given linear programming problem.

5. You have at most $24,000 to invest in bonds and stocks. You have decided that the amount of money invested in bonds must be at least twice as much as that in stocks, but the money invested in bonds must not be greater than $18,000. If you receive 6% profit on bonds and 8% profit on stocks, how much money should you place in each type of investment to maximize your profit?

6. A professor gives two types of quizzes, objective and recall. The professor is planning to give at least 15 quizzes this semester. The student preparation time for an objective quiz is 15 minutes and for a recall quiz is 30 minutes. The professor would like a student to spend at least 5 hours preparing for these quizzes, above and beyond the normal study time. The average score on an objective quiz is a 7 and on a recall quiz is a 5, and the professor would like the students to score at least 85 points on all quizzes. It takes the professor one minute to grade an objective quiz, and 1.5 minutes to grade a recall type quiz. How many of each type of quiz should the professor give in order to minimize the amount of time spent grading?

7. A factory manufactures two different washing machines, Model 650 and Model 800. During the manufacturing of each washing machine the machine requires time in three bays: electrical, mechanical, and assembly. The time requirements (in hours) and the total hours available for each bay is listed below.

	Electrical	Mechanical	Assembly
Model 650	1	2	4
Model 800	2	2	2
Total hours	70	90	160

If each Model 650 generates a profit of $600 and each Model 800 generates a profit of $500, how many of each model should be manufactured to maximize profit? What is the maximum profit?

8. A computer store sells two types of computers, desktops and laptops. The supplier demands that at least 150 computers be sold each month. In order to keep profits up, the number of desktops sold must be at least twice the number of laptops sold. The store spends $75 a week to market each desktop and $50 a week to market each laptop. How many of each type of computer must be sold to minimize weekly marketing costs? What is the minimum weekly marketing cost?

Mastery Practice (Answers)

9. A small candy company makes chocolate pumpkins and chocolate ghosts. Each pumpkin requires 3 minutes to manufacture and 1 minute to package. Each ghost requires 4 minutes to manufacture and 2 minutes to package. There are a total of 1.5 hours available for manufacturing and 0.5 hours available for packaging. The company stipulates that they must produce at least three times as many chocolate pumpkins as ghosts. Determine the production amounts of each candy to maximize profit, if the company's profit on the sale of each pumpkin is 50 cents and the profit on the sale of each ghost is 60 cents. **Set up, but do not solve.**

10. A company produces three types of shoes (formal, casual, and athletic) at its South College Station and North Bryan factories. Daily production of each factory for each type of shoe is listed below.

	Formal	Casual	Athletic
South College Station	100	100	300
North Bryan	100	200	100

To fulfill a particular order, the company must produce at least 6000 pairs of formal shoes, 8000 pairs of casual shoes, and 9000 pairs of athletic shoes. The cost of operating the South College Station factory is $1500 per day, and the cost of operating the North Bryan factory is $2000 per day. The South College Station factory must operate at least twice as many days as the North Bryan factory, and the North Bryan factory must operate at least 5 days during this production time. How many days should each factory operate to complete the order at a minimum cost, and what is the minimum cost? **Set up, but do not solve.**

Communication Practice (Answers)

11. Explain, in your own words, why non-negativity constraints are necessary.

12. Explain why it is beneficial to start by reading the last sentence or two of an application problem.

3.2 Graphing Systems of Linear Inequalities in Two Variables

© Photo by Science in HD on Unsplash

In the chapter on systems of linear equations, we looked at a company that produces a basic and premium version of its product, and we determined how many of each version they should produce to fully utilize all staffed hours. In some cases, though, it might not make the most sense for the company to utilize all the staffed hours; if the premium product has high demand and a high price, it might make more sense for the company to make as many of those, even if some staff hours go unused. In this case, the company might be most interested in what combinations of basic and premium product production are possible. For that, we need linear inequalities.

Learning Objectives:
In this section, you will learn to graph a system of linear inequalities in two variables in order to find the solution set. Upon completion you will be able to:

- Recognize when to graph linear inequalities with a solid or dashed line.
- Graph the solution set for a given linear inequality.
- Graph the solution set for a system of linear inequalities.
- State whether a solution set to a system of linear inequalities is a bounded or unbounded region.
- Identify and calculate the coordinates of the corner points of a solution set for a system of linear inequalities.

A linear *equation* in two variables is an equation like $2x+y=1$. A **linear *inequality*** in two variables is similar, but involves an inequality:

$$2x+y<1 \quad \text{or} \quad 2x+y>1 \quad \text{or} \quad 2x+y\leq 1 \quad \text{or} \quad 2x+y\geq 1$$

The $<$ or $>$ indicates a **strict** inequality. For example, $2x+y<1$ means $2x+y$ must be *less than* 1, while $2x+y>1$ means $2x+y$ must be *greater than* 1. The $\leq$ or $\geq$ inequality adds that equality is allowed and is known as a **non-strict** inequality. For example, $2x+y\leq 1$ means $2x+y$ must be *less than or equal* to 1, while $2x+y\geq 1$ means $2x+y$ must be *greater than or equal to* 1.

Because of the strong relationship between linear equations and linear inequalities, we could also write a linear inequality in slope-intercept or point-slope form. For the ease of graphing, we recommend rewriting, if necessary, all inequalities in the standard form, $ax+by \boxed{}\ c$, where any inequality may appear in the box.

Graphing Linear Inequalities

Recall the solution set to a linear equation is the set of all points, (x, y), that satisfy the equation, and the graph of the solution set forms a line. The solution set to a linear inequality is similar to the solution set of a linear equation, but the graph of the solution set will be a region, rather than a line.

Definition

The **solution set to a linear inequality** is the set of all points, (x, y), that satisfy the inequality. The graph of the solution set for a linear inequality will be half of the coordinate plane, with the corresponding linear equation's graph as the **boundary line**. ▪

The general steps for graphing the solution set for a single linear inequality, by hand, are as follows:

1. Graph the boundary line, which is the corresponding equation, $ax + by = c$.
   a. Determine the $x-$ and $y-$intercepts.
   b. Determine whether the boundary line is included in the solution set of the linear inequality.
      i. For a *strict* inequality ($<$ or $>$), draw a *dashed* line to show that the points on the boundary line are *not* part of the solution.
      ii. For a *non-strict* inequality, one that includes the equal sign ($\leq$ or $\geq$), draw a *solid* line to show that the points on the boundary line *are* part of the solution.
   c. Label the boundary line with the corresponding linear equation, $ax + by = c$.
2. Determine which side of the boundary line contains the remaining points that form the **solution set**, **S**.
   a. Choose a test point **not** on the boundary line.
      The authors encourage the reader to use $(0, 0)$, when it is not on the boundary line.
   b. Substitute the test point into the *inequality* and simplify.
   c. If the inequality is true, write an **S** on the half-plane including the test point, indicating the solution set.
   d. If the inequality is false, write an **S** on the half-plane *not* including the test point, indicating the solution set.

This process is easier seen with an example. For instance, consider the linear inequality $5x - 3y \geq 15$.

Using the above process we can determine a graphical representation of its solution set. First, we graph the boundary line. To do so, we start by rewriting the inequality as an equality.

$$5x - 3y = 15$$

The x-intercept of this line is $(3, 0)$.

$$\begin{aligned} 5x - 3(0) &= 15 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

The y-intercept is $(0,-5)$.

$$\begin{aligned} 5(0)-3y &= 15 \\ -3y &= 15 \\ y &= -5 \end{aligned}$$

The original inequality, $5x-3y \geq 15$, is non-strict, so we will use a solid line to represent the boundary line, as it is included in the solution set. The boundary line is shown, and labeled, below in **Figure 3.2.2**.

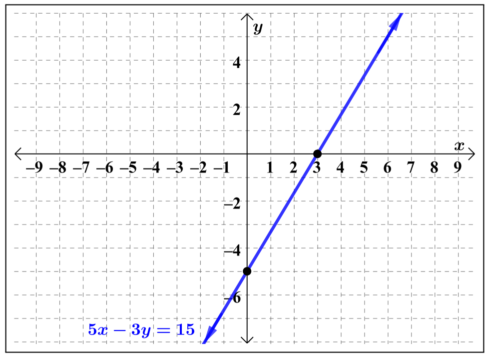


Figure 3.2.2: The coordinate plane with the line $5x-3y=15$ labeled.

Now, we determine which side of the line contains the remaining points of the solution set. Due to the fact that $(0,0)$ is not on the boundary line, again refer to **Figure 3.2.2**, we will use $(x,y)=(0,0)$ as our test point.

Substituting (0,0) into the original inequality we have

$$\begin{aligned} 5x-3y &\geq 15 \\ 5(0)-3(0) &\overset{?}{\geq} 15 \\ 0 &\geq 15 \text{ ✗} \end{aligned}$$

Clearly, the resulting inequality is false. Thus, our test point, $(0,0)$, is not in the solution set. We will identify the half plane not including $(0,0)$ as our solution set, using an **S**. (See **Figure 3.2.3**.)

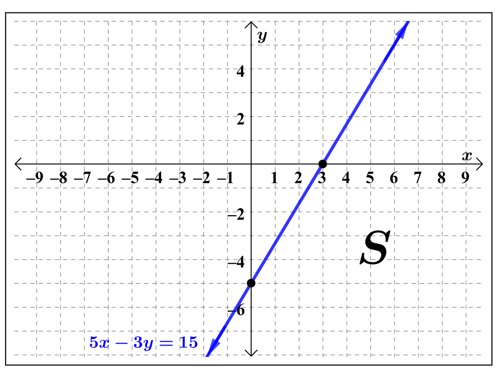


Figure 3.2.3: The coordinate plane with the solution set of $5x-3y \geq 15$ labeled.

The standard way to highlight the solution set is through shading. At this point the reader has two options for how to proceed in highlighting the solution set of the graphed inequality. In **Figure 3.2.4**, the shaded region is the solution set; this is called **true shading**. In **Figure 3.2.5**, the shaded region is not the solution set, but instead the solution set is the unshaded region; this is called **reverse shading**.

True Shading

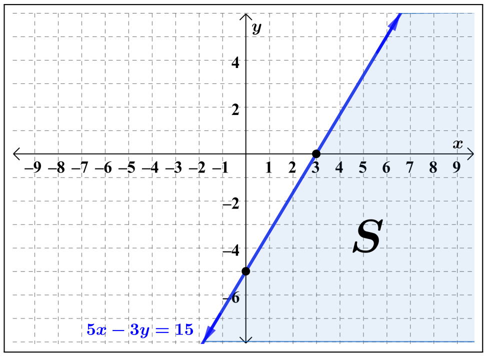


Figure 3.2.4: The graph of $5x-3y \geq 15$. The highlighted region, below the line, is the solution set, **S**.

Reverse Shading

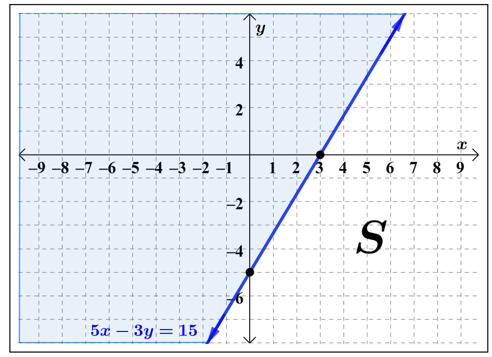


Figure 3.2.5: The graph of $5x-3y \geq 15$. The highlighted region, above the line, is not the solution set, but makes the solution set, **S**, stand out.

N *When graphing one inequality the highlighting may seem unnecessary to the reader. However, when graphing a system of linear inequalities, the highlighting can be visually helpful in determining* **S**. *The authors will leave it to the reader to decide which shading method is preferred.*

■ **Example 1** Graph the solution set for $x-2y<0$.

Solution:

For the boundary line, we rewrite the inequality as an equality.

$$x-2y=0$$

To graph this line, we identify its intercepts. The x-intercept is $(0,0)$, which is also the y-intercept. Because we need two *different* points to graph a line, we must locate another point on the line. To do so, you may select any value for x or y and solve for the remaining variable, in the equality. To avoid fractions we will substitute 1 for y, and solve for x.

$$\begin{aligned} x-2y &= 0 \\ x-2(1) &= 0 \\ x-2 &= 0 \\ x &= 2 \end{aligned}$$

So, the point $(2,1)$ is also on the boundary line.

The original inequality, $x-2y<0$, is strict, so we will use a dashed line to represent the boundary line, as shown in **Figure 3.2.6**.

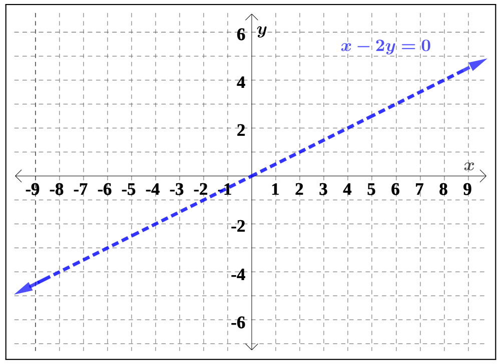


Figure 3.2.6: The coordinate plane with the dashed line $x-2y=0$.

Due to the fact that $(0,0)$ is on the boundary line, we need choose a test point which is off the boundary line. We select a point on the x-axis, but clearly not on the boundary line, such as $(-5,0)$. Substituting $(-5,0)$ into the inequality we get

$$\begin{aligned} x-2y &< 0 \\ (-5)-2(0) &\stackrel{?}{<} 0 \\ -5-0 &\stackrel{?}{<} 0 \\ -5 &< 0 \checkmark \end{aligned}$$

Clearly the inequality is true. Thus, our test point is in the solution set. In **Figure 3.2.7**, we identify the half plane including $(-5,0)$ as our solution set, using an **S**.

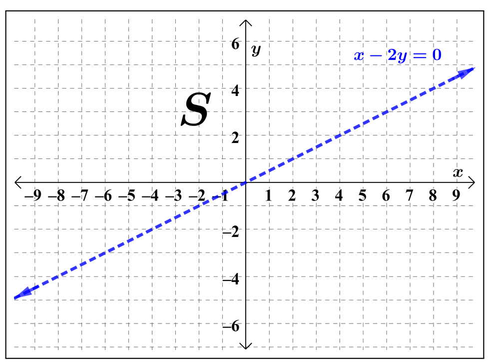


Figure 3.2.7: The coordinate plane with the solution set of $x-2y<0$ labeled.

For highlighting the solution set, our options are

True Shading

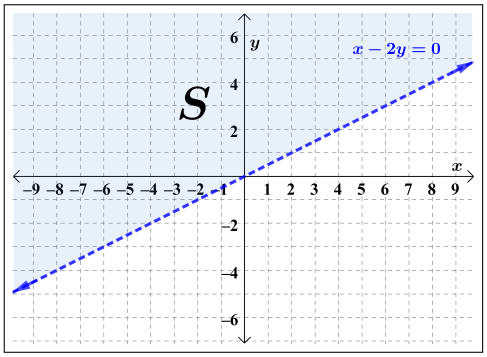


Figure 3.2.8: The graph of $x - 2y < 0$. The highlighted region, above the line, is the solution set, **S**.

Reverse Shading

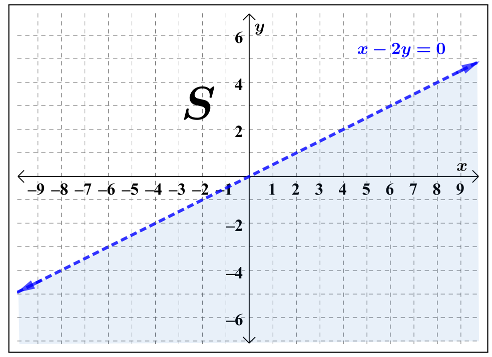


Figure 3.2.9: The graph of $x - 2y < 0$. The highlighted region, below the line, is not the solution set. The solution set is labeled **S**.

■

N *When selecting a test point, it is advised to choose a point with "nice" coordinates, so the computations are easy. This is why we use* $(0,0)$ *when possible, or a point on an axis.*

Try It # 1:
Graph the solution set for $-7x + 6y \leq 35$.

Try It # 2:
Graph the solution set for $4x + 9y > 72$.

Graphing Systems of Linear Inequalities

In the previous chapter, we looked for solutions to a system of linear equations (a point that would simultaneously satisfy all the equations in the system). Likewise, we can consider a system of linear inequalities. The solution to a system of linear inequalities is the set of points that simultaneously satisfy all the inequalities in the system.

As with a single linear inequality, we can show the solution set to a system of linear inequalities graphically. We identify the solution set by looking for where the solution regions indicated by the individual linear inequalities overlap.

■ **Example 2** Graph the solution set for the following system of linear inequalities.

$$\begin{aligned} -x + y &\leq 2 \\ x + y &\geq 1 \end{aligned}$$

Solution:

If we graph the solution set to each inequality individually,

	$-x+y \leq 2$	$x+y \geq 1$
Boundary Line:	$-x+y=2$ (solid)	$x+y=1$ (solid)
***x*-intercept**:	$(-2,0)$	$(1,0)$
***y*-intercept**:	$(0,2)$	$(0,1)$
Test Point:	$(0,0)$	$(0,0)$
	$-1(0)+0 \overset{?}{\leq} 2$	$0+0 \overset{?}{\geq} 1$
	$0 \leq 2$ ✓	$0 > 1$ ✗

we obtain the two solution sets, $\mathbf{S_1}$ and $\mathbf{S_2}$, shown in **Figures 3.2.10** and **3.2.11**, below.

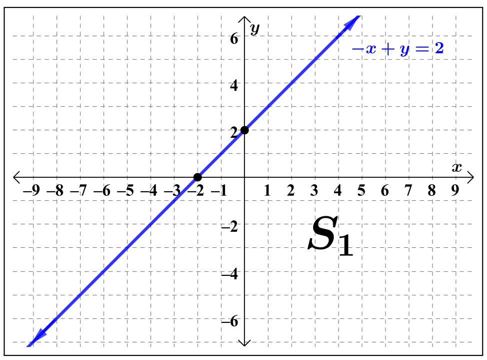


Figure 3.2.10: The graph of $-x+y \leq 2$, with a solid boundary line and the solution set labeled $\mathbf{S_1}$.

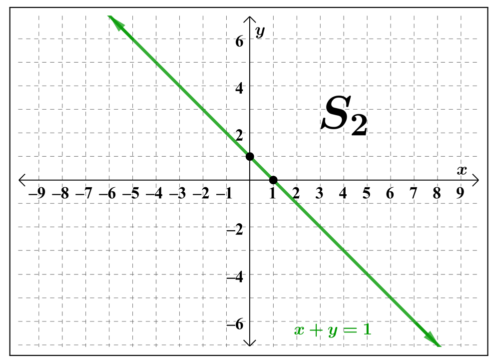


Figure 3.2.11: The graph of $x+y \geq 1$, with a solid boundary line and the solution set labeled $\mathbf{S_2}$.

Graphing these solution sets on the same coordinate plane reveals the solution set, **S**, to the system of linear inequalities as the region where the two overlap. To clearly see the overlap, we will utilize our highlighting techniques. Again, we will demonstrate both true shading (**Figure 3.2.12**) and reverse shading (**Figure 3.2.13**). Remember when using true shading the solution set is the overlap of all shadings, and when using reverse shading the solution set remains completely unshaded.

True Shading

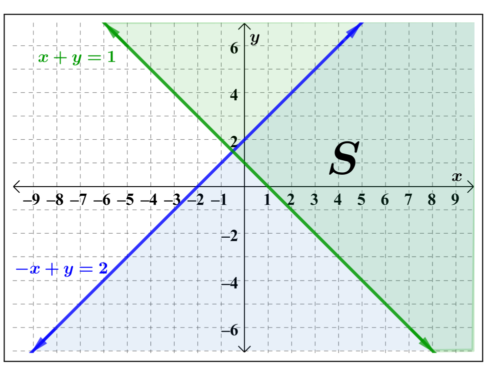


Figure 3.2.12: The solution set to the system $-x+y \leq 2$ and $x+y \geq 1$, illustrated using true shading.

Reverse Shading

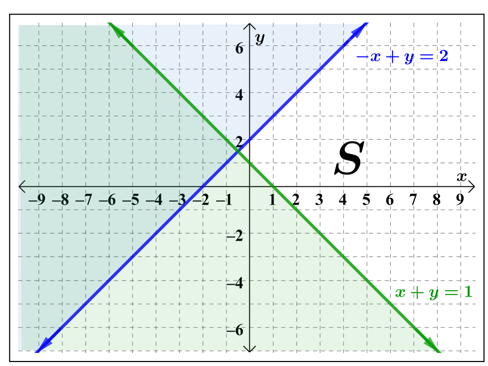


Figure 3.2.13: The solution set to the system $-x+y \leq 2$ and $x+y \geq 1$, illustrated using reverse shading.

■

■ **Example 3** Graph the solution set for the following system of linear inequalities.

$$x+y \le 12$$
$$2x+y \le 16$$
$$x \ge 0,\ y \ge 0$$

Solution:
If we graph the solution set to each inequality individually,

	$x+y \le 12$	$2x+y \le 16$	$x \ge 0$	$y \ge 0$
Boundary Line:	$x+y=12$ (solid)	$2x+y=16$ (solid)	$x=0$ (solid) y-axis	$y=0$ (solid) x-axis
***x*-intercept:**	$(12,0)$	$(8,0)$	$(0,0)$	$(0,0)$
***y*-intercept:**	$(0,12)$	$(0,16)$	$(0,0)$	$(0,0)$
Test Point:	$(0,0)$	$(0,0)$	$(5,0)$	$(0,6)$
	$0+0 \overset{?}{\le} 12$	$2(0)+0 \overset{?}{\le} 16$	$5 \overset{?}{\ge} 0$	$6 \overset{?}{\ge} 0$
	$0 \le 12$ ✓	$0 \le 16$ ✓	$5 \ge 0$ ✓	$6 \ge 0$ ✓

we obtain $\mathbf{S_1}$, $\mathbf{S_2}$, $\mathbf{S_3}$, and $\mathbf{S_4}$ in **Figures 3.2.14, 3.2.15, 3.2.16**, and **3.2.17**, respectively.

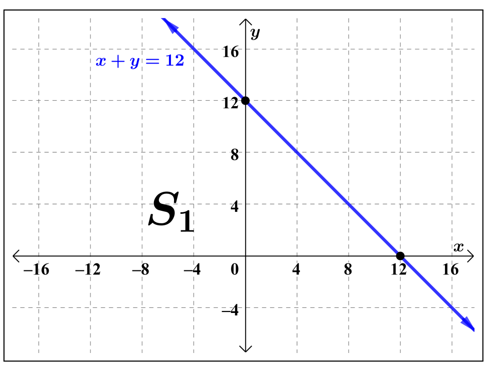


Figure 3.2.14: The graph of $x+y \le 12$, with a solid boundary line and the solution set labeled $\mathbf{S_1}$.

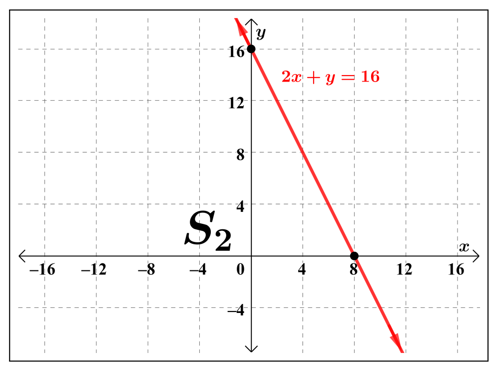


Figure 3.2.15: The graph of $2x+y \le 16$, with a solid boundary line and the solution set labeled $\mathbf{S_2}$.

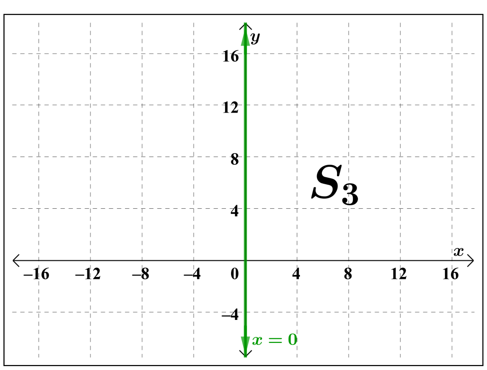


Figure 3.2.16: The graph of $x \ge 0$, with a solid boundary line and the solution set labeled $\mathbf{S_3}$.

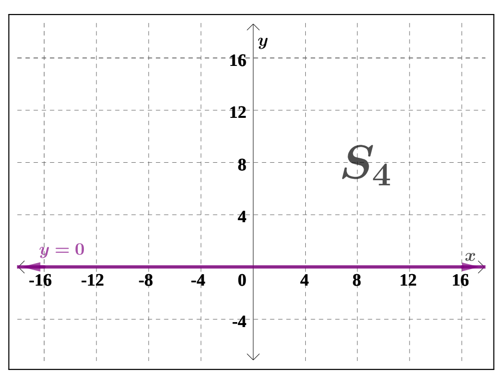


Figure 3.2.17: The graph of $y \ge 0$, with a solid boundary line and the solution set labeled $\mathbf{S_4}$.

Graphing these solution sets on the same coordinate plane reveals the solution set to the system of linear inequalities as the region where the individual solution sets overlap. See **Figure 3.2.18** for the solution set illustrated using true shading and **Figure 3.2.19** for the solution set illustrated using reverse shading.

True Shading

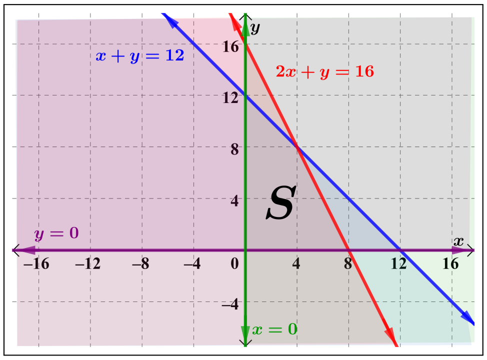


Figure 3.2.18: The solution set, **S**, of the system, illustrated using true shading.

Reverse Shading

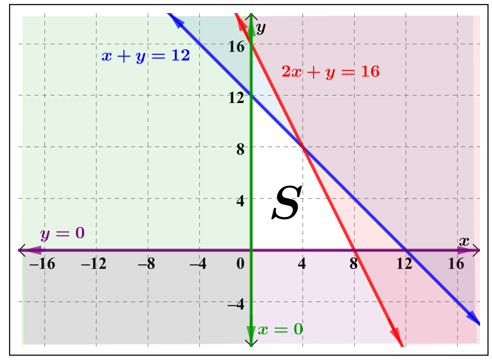


Figure 3.2.19: The solution set, **S**, of the system, illustrated using reverse shading.

▪

When discussing the graphical solution set to a system of linear inequalities, we have been graphing each individual inequality, and then overlapping them on the same coordinate plane to identify the solution set, **S**, of the system. In actuality, when graphing a system of linear inequalities, you will graph each inequality, one-by-one (including shading), on the same coordinate plane and only indicate the final solution set of the system, **S**, after all inequalities have been graphed.

Try It # 3:
Graph the solution set to the following system of linear inequalities.

$$\begin{aligned} 2x+y &\leq 3 \\ x-2y &\leq -2 \\ x &\geq -5 \end{aligned}$$

Determining the Types of Solutions Sets and Corner Points

Definition

A solution set to a system of linear inequalities is **bounded** if the region can be completely enclosed by a circle.

A solution set to a system of linear inequalities is **unbounded** if the region cannot be completely enclosed by a circle.

▪

Recall the reverse shaded solution sets from the two previous examples. In **Figure 3.2.20** the solution set is unbounded, because the region spreads without bound to the right and cannot be completely enclosed by a circle. In **Figure 3.2.21** the solution set is bounded, because it is an enclosed figure.

Unbounded

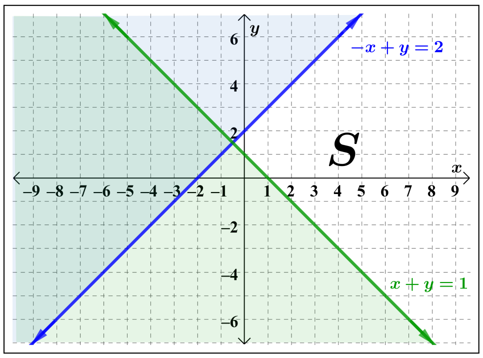


Figure 3.2.20: An unbounded solution set, **S**.

Bounded

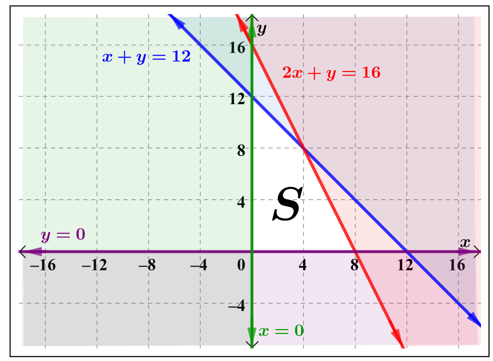


Figure 3.2.21: A bounded solution set, **S**.

Whether the solution set is bounded or unbounded, the solution set has edges created by the boundary lines of the inequalities in the system. (See **Figures 3.2.22** and **3.2.23**.)

Unbounded

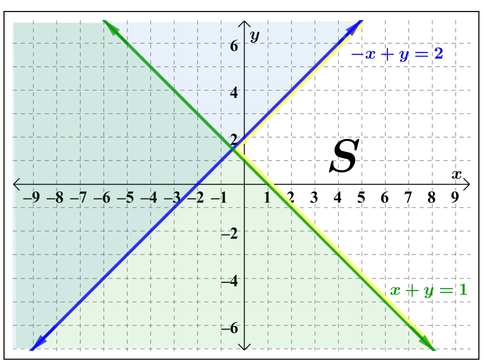


Figure 3.2.22: An unbounded solution set, **S**, with the edges highlighted.

Bounded

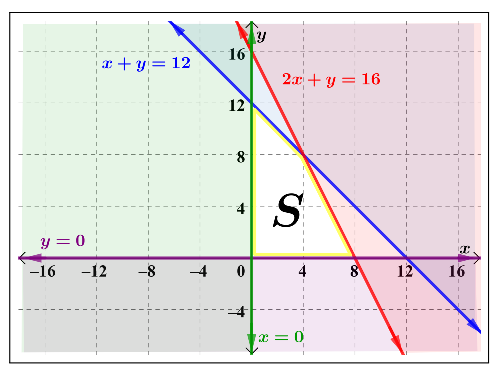


Figure 3.2.23: A bounded solution set, **S**, with the edges highlighted.

Definition

The point where the edge of the solution set changes from one boundary line to the next is called a **corner point**. ▪

■ **Example 4** Indicate the corner point(s) for each of the previous solution sets to a system of linear inequalities in this section, and then determine the exact coordinates of each.

Solution:

a. $-x+y \leq 2$
$x+y \geq 1$

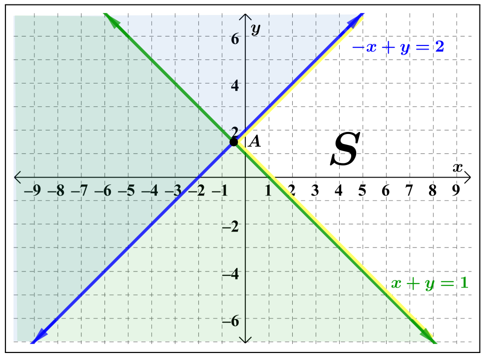


Figure 3.2.24: The coordinate plane with the solution set, **S**, to the system $-x+y \leq 2$ and $x+y \geq 1$ shown. The intersection of the lines $-x+y=2$ and $x+y=1$ is labeled with an A.

Notice in reverse shaded **Figure 3.2.24** there is only one corner point, and it is indicated by A. To determine the exact coordinates of A, we need to identify where the two boundary lines intersect, which amounts to solving the following system of linear equations.

$$\begin{aligned} -x+y &= 2 \\ x+y &= 1 \end{aligned}$$

While you can use any of the previously discussed methods to solve a system of linear equations, the authors choose to use technology and the **rref** operation, as discussed in **Section 2.4**. For this system, we have

$$\left[\begin{array}{rr|r} -1 & 1 & 2 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rr|r} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right] \implies \begin{array}{l} x = -\dfrac{1}{2} \\ y = \dfrac{3}{2} \end{array}$$

Thus, corner point $A = \left(-\dfrac{1}{2}, \dfrac{3}{2}\right)$ or $(-0.5, 1.5)$.

b. $x+y \leq 12$
$2x+y \leq 16$
$x \geq 0,\ y \geq 0$

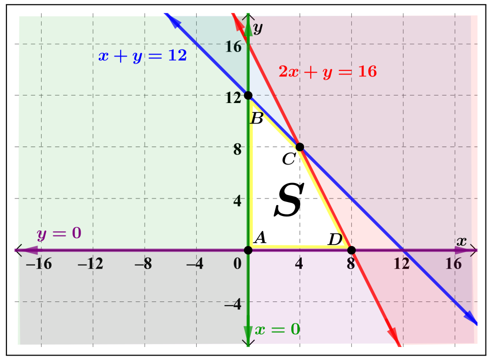


Figure 3.2.25: The coordinate plane with the solution set, **S**, to the system $x+y \leq 12$, $2x+y \leq 16$, $x \geq 0$, and $y \geq 0$ shown, with corner points A-D labeled.

In **Figure 3.2.25**, the four corner points, A, B, C, and D, are indicated. The coordinates of each corner point can be found by solving the system formed by the corresponding intersecting lines.

$\boldsymbol{A}$:
$$\begin{aligned} x &= 0 \\ y &= 0 \quad \Longrightarrow \text{ Clearly, the point } A = (0,0). \end{aligned}$$

$\boldsymbol{B}$:
$$\begin{aligned} x &= 0 \\ x+y &= 12 \end{aligned}$$

Because $x=0$, substitute 0 for x into $x+y=12$.

$$\begin{aligned} x+y &= 12 \\ 0+y &= 12 \\ y &= 12 \quad \Longrightarrow \text{ So, corner point } B = (0,12). \end{aligned}$$

$\boldsymbol{C}$:
$$\begin{aligned} x+y &= 12 \\ 2x+y &= 16 \end{aligned}$$

Using technology, we find

$$\left[\begin{array}{cc|c} 1 & 1 & 12 \\ 2 & 1 & 16 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 8 \end{array}\right] \Longrightarrow \begin{aligned} x &= 4 \\ y &= 8 \end{aligned} \Longrightarrow \text{ So, corner point } C = (4,8).$$

$\boldsymbol{D}$:
$$\begin{aligned} 2x+y &= 16 \\ y &= 0 \end{aligned}$$

Because $y=0$, substitute 0 for y into $2x+y=16$.

$$\begin{aligned} 2x+0 &= 16 \\ 2x &= 16 \\ x &= 8 \quad \Longrightarrow \text{ So, corner point } D = (8,0). \end{aligned}$$

■

After calculating the coordinates of a corner point, it is always a good idea to verify it matches the position on the corresponding graph. If it does not, then either the solution set to at least one of the linear inequalities was graphed incorrectly or an error occurred in the calculation of the coordinates.

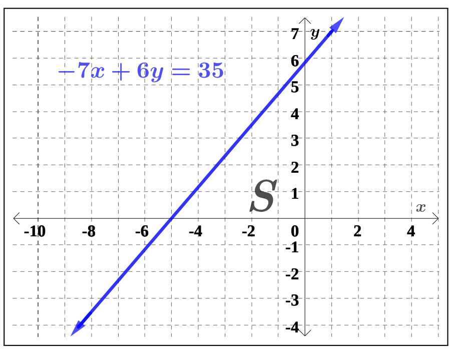
Try It # 4:
Consider the solution set for the given system. (This system was used in Try It # 3.)

$$\begin{aligned} 2x + y &\leq 3 \\ x - 2y &\leq -2 \\ x &\geq -5 \end{aligned}$$

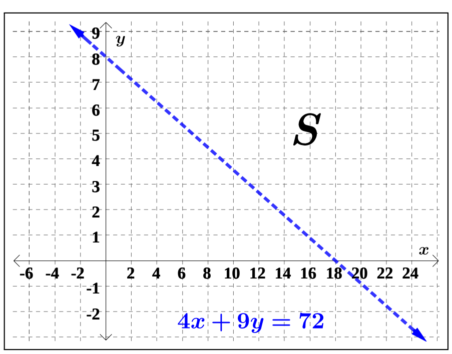
Determine whether the solution set is bounded or unbounded, and state the exact coordinates of all corner points.

Try It Answers

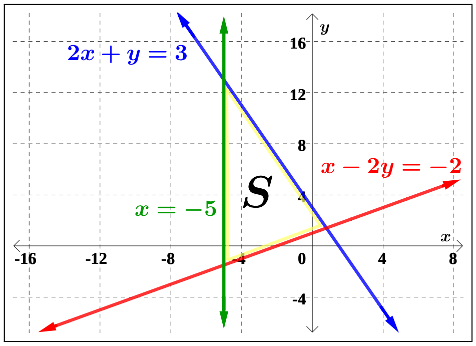
1. Solution Set, **S**



2. Solution Set, **S**



3. Solution Set, **S**



4. **S** is bounded with corner points $\left(-5, -\frac{3}{2}\right)$, $(-5, 13)$, and $\left(\frac{4}{5}, \frac{7}{5}\right)$.

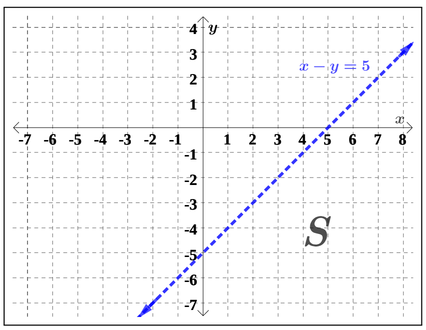
EXERCISES

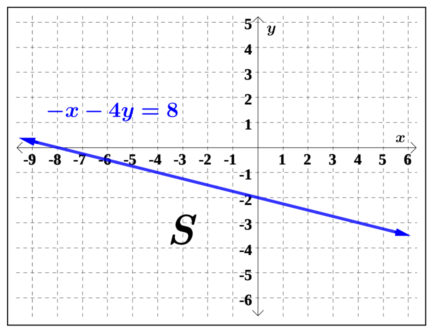
Basic Skills Practice (Answers)

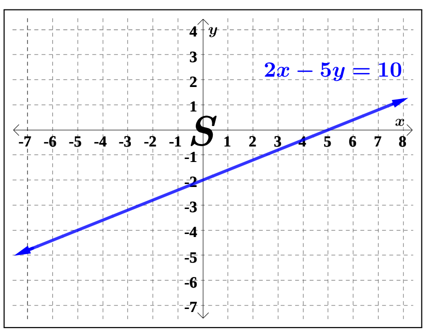
For Exercises 1 - 6, graph the inequality, labeling the solution set with **S**.

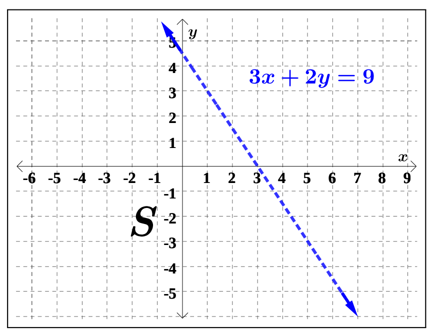
1. $x \geq 0$
2. $y < 0$
3. $y < -x+2$
4. $y > \frac{2}{3}x - 3$
5. $x + y \leq -4$
6. $x - 2y \geq 0$

For Exercises 7 - 10, write the inequality shown by the graph with the given boundary line.

7. 

9. 

8. 

10. 

For Exercises 11 - 14, graph the system of linear inequalities, labeling the solution set with **S**.

11. $\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$

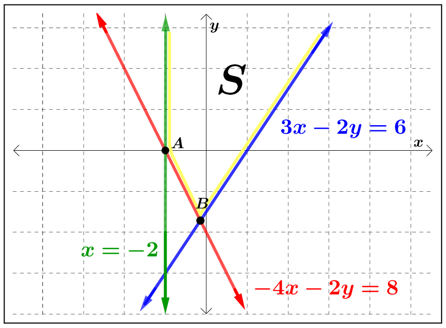
12. $\begin{cases} 3x+y > 5 \\ 2x-y \leq 10 \end{cases}$

13. $\begin{cases} 4x-y < 12 \\ -2x+2y > -8 \end{cases}$

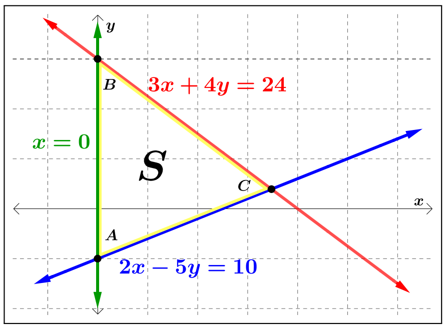
14. $\begin{cases} 7x+2y > 14 \\ 5x-y \leq 8 \end{cases}$

For Exercises 15 - 18, use the graph of the system of linear inequalities to determine if the solution set is bounded or unbounded. Then, determine the exact coordinates of all labeled corner points.

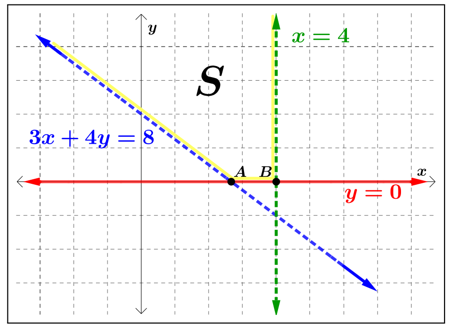
15.



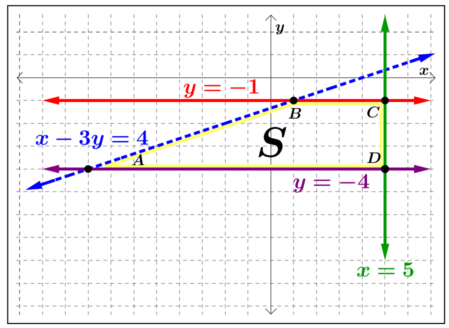
16.



17.



18.



INTERMEDIATE SKILLS PRACTICE (Answers)

For Exercises 19 - 22, graph the system of linear inequalities, labeling the solution set with **S**. Then, state whether the solution set is bounded or unbounded.

19. $\begin{cases} 2x+y > 2 \\ x-y \leq 1 \\ x \geq 0 \\ y \geq 0 \end{cases}$

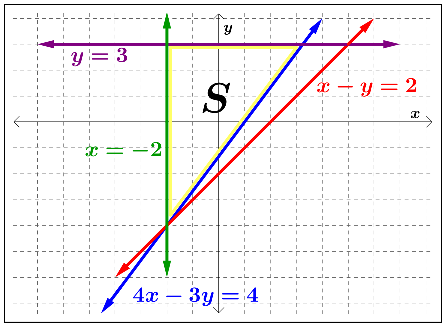
20. $\begin{cases} x+2y \leq 4 \\ x-y \leq 2 \\ x \geq 0 \\ y \geq 0 \end{cases}$

21. $\begin{cases} 2x+2y < -4 \\ -x+3y \leq 9 \\ x \leq -2 \\ y \geq -1 \end{cases}$

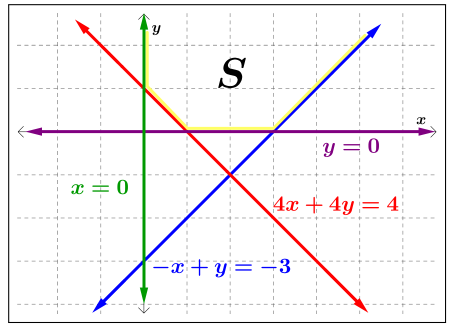
22. $\begin{cases} 3x+y \leq 2 \\ -3x+y \leq 5 \\ x \leq 0 \\ y \leq 0 \end{cases}$

For Exercises 23 - 26, using the graph, write the corresponding system of linear inequalities. Then, determine the corner points of the solution set.

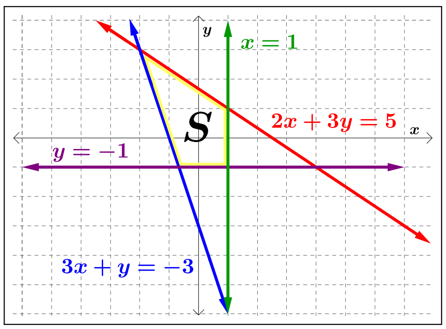
23.



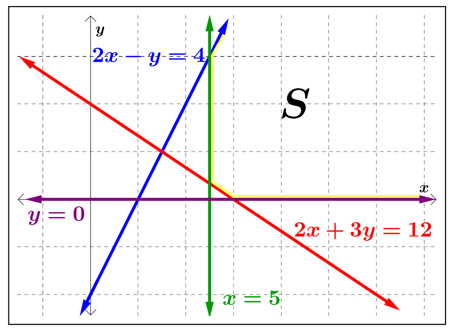
25.



24.



26.



Mastery Practice (Answers)

27. For $\begin{cases} 8x-5y & \leq & 40 \\ 3x+2y & \geq & 12 \\ x & \geq & 0 \\ 0 \leq & y & \leq 8 \end{cases}$

a. Graph the system of linear inequalities. Identify the solution set with an **S**.
b. Determine if the solution set is bounded or unbounded.
c. State the exact corner points of the solution set.

28. For $\begin{cases} x+5y & \geq & 50 \\ x+y & \geq & 30 \\ 5x+2y & \geq & 100 \\ x & \geq & 0 \\ y & \geq & 0 \end{cases}$

a. Graph the system of linear inequalities. Identify the solution set with an **S**.
b. Determine if the solution set is bounded or unbounded.
c. State the exact corner points of the solution set.

Communication Practice (Answers)

29. How is solving a system of linear equations related to finding the corner points of the solution set to a system of linear inequalities?
30. Describe what the graph of a system of linear inequalities looks like when there is no solution set, both in terms of true and false shading.

3.3 Graphical Solution of Linear Programming Problems

© Photo by Vanessa Coffelt, 2020

Wouldn't it be nice if we could simply produce and sell infinitely many units of a product, and thus, make a never-ending amount of money? In business (and in day-to-day living) we know that some things are just unreasonable or impossible, but linear programming allows us to optimize quantities under *realistic* boundaries.

Learning Objectives:
In this section, you will learn the Method of Corners to solve a linear programming problem. Upon completion you will be able to:

- Graph the constraints of a linear programming problem.
- Identify and calculate the coordinates of the corner points for the feasible region of a linear programming problem.
- State whether a linear programming problem has an unbounded or bounded feasible region.
- Recognize whether a solution to a linear programming exists based on the properties of the feasible region.
- Compute the maximum and/or minimum value for a given objective function and feasible region, using the Method of Corners.
- Using the Method of Corners, solve a linear programming problem graphically, including real-world applications.
- Identify any leftover resources from the solution to a real-world application where the Method of Corners is used.

Recall from the beginning of the chapter, the following scenario:

> A department store sells two sizes of televisions: 21 inch and 40 inch. A 21 inch television requires 6 cubic feet of storage space, and a 40 inch television requires 18 cubic feet of space. A maximum of 1080 cubic feet of storage space is available. The 21 inch and the 40 inch televisions take up, respectively, 2 and 3 sales hours of labor, and the store has a maximum of 198 hours of labor available. If the profit earned from each of these sizes of televisions is \$60 and \$80, respectively, how many of each size of television should be sold to maximize the store's profit, and what is the maximum profit?

The algebraic problem we formulated is given below:

$t :=$ the number of 21 inch TVs sold

$f :=$ the number of 40 inch TVs sold

$P :=$ the profit earned on the sale of TVs, in dollars

Objective: **Maximize** $P = 60t + 80f$

Subject to: $6t + 18f \leq 1080$ (Cubic Feet of Storage Space)

$2t + 3f \leq 198$ (Hours of Labor)

$t \geq 0,\ f \geq 0$

In order to determine a **solution** to this linear programming problem, we need to compute all the points, (t, f), which satisfy all of the constraints and which maximizes the objective function, P. To calculate all points which satisfy the constraints, we need to identify the solution set for the system of inequalities given by the constraints. In this section, we will approach all problems graphically. Thus, it makes sense to change our variables while graphing the system of inequalities: $(t, f) \Rightarrow (x, y)$.

Our set-up in (x, y) becomes

$x :=$ the number of 21 inch TVs sold

$y :=$ the number of 40 inch TVs sold

$P :=$ the profit earned on the sale of TVs, in dollars

Objective: **Maximize** $P = 60x + 80y$

Subject to: $6x + 18y \leq 1080$ (Cubic Feet of Storage Space)

$2x + 3y \leq 198$ (Hours of Labor)

$x \geq 0,\ y \geq 0$

Using the techniques discussed in the previous section, we obtain the following solution set, **S**, for the constraints.

True Shading

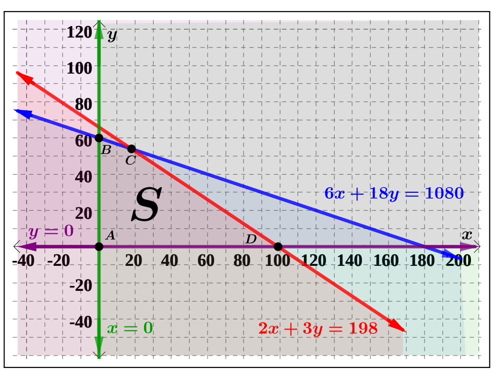


Figure 3.3.2: The solution set to the system $6x + 18y \leq 1080$, $2x + 3y \leq 198$, $x \geq 0$, and $y \geq 0$, illustrated using true shading. The corner points $A - D$ are labeled.

Reverse Shading

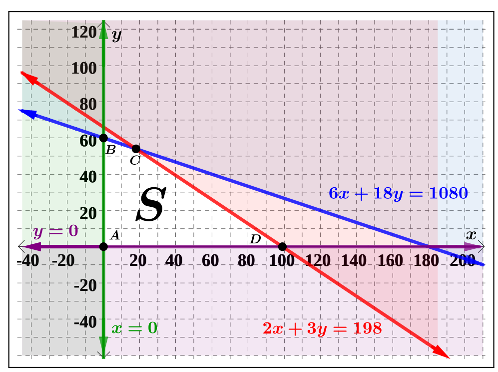


Figure 3.3.3: The solution set to the system $6x + 18y \leq 1080$, $2x + 3y \leq 198$, $x \geq 0$, and $y \geq 0$, illustrated using reverse shading. The corner points $A - D$ are labeled.

The corner points for the solution set, determined by computing the intersections of the appropriate boundary lines, are as follows:

$$A = (0,0),\ B = (0,60),\ C = (18,54),\ \text{and } D = (99,0).$$

Observe that we have determined the solution set where all constraints are satisfied. This region is called the **feasible region**.

Definition

The solution set, **S**, where all constraints of a linear programming problem are satisfied is called the **feasible region**. Each point in the feasible region is a *candidate* for the optimal solution to the linear programming problem. ▪

We are looking for the point(s) in this region which optimize(s) the objective function.

To consider how the objective function connects, suppose we considered all the possible sales combinations, (x,y), that gave a profit of $P = \$3400$, so that $3400 = 60x + 80y$. That set of combinations would form a line in the graph of the feasible region. Repeating this process for a profit of \$5400 and \$6400 would give additional lines. If we graph these lines on top of our feasible region, we obtain the graph shown in **Figure 3.3.4**, below.

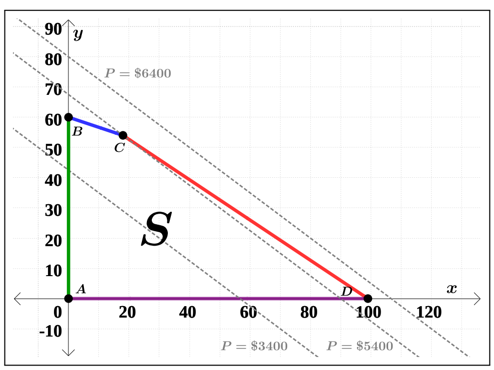


Figure 3.3.4: The feasible region for the linear programming problem, with three isoprofit lines labeled.

Notice that all the constant-profit (**isoprofit**) lines are parallel, and that, in general, the profit increases as we move up and to the right in the first quadrant. Also, for positive x and y, with a profit of \$5400 there are some sales levels inside the feasible region, and some that are outside. This means we could feasibly make \$5400 profit by selling, for example, 18 of the 21 inch TVs and 54 of the 40 inch TVs, but we cannot make \$5400 by selling 6 of the 21 inch and 63 of the 40 inch televisions, because that point falls outside our feasible region. We can also notice that a profit of \$6400 is not possible under the given constraints, as the entire isoprofit line falls outside our feasible region.

The optimal solution will be the largest possible profit that is still feasible. Graphically for this problem, that means the isoprofit line furthest to the upper-right that still touches the feasible region on at least one point. The solution is the one given below in **Figure 3.3.5**.

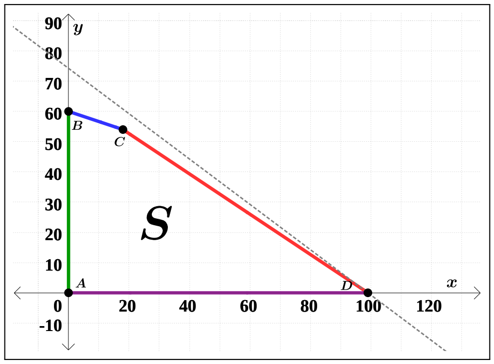


Figure 3.3.5: The feasible region for the linear programming problem, with the optimal profit line shown.

This profit line touches the feasible region, at corner point D, where $x = 99$ and $y = 0$, giving a profit of $P = 60(99) + 80(0) = \$5940$.

In **Figure 3.3.5** above, the line of maximum profit touched the feasible region at a single corner point. This observation inspires the Fundamental Theorem of Linear Programming.

Theorem 3.1 Fundamental Theorem of Linear Programming

1. If a feasible region is bounded, then a maximum and a minimum value for the objective function exists.
2. If a feasible region is unbounded, in Quadrant I, and the objective function has only positive coefficients, then:
   a. A maximum value for the objective function does not exist.
   b. A minimum value for the objective function exists.
3. If there is no feasible region, as it is not possible for all constraints to be met simultaneously, then there is no solution to the linear programming problem.
4. If a solution exists to a linear programming problem, then it will occur at a corner point of the feasible region.
   - If the objective function is optimized at a *single* corner point, then the linear programming problem has an optimal solution at one unique point.
   - If the objective function is optimized at two *adjacent* corner points, then it is optimized at those two points and at every point along the boundary line segment connecting the two points. Thus, the linear programming problem has an optimal solution at infinitely many points (along the boundary line segment).

N *Two corner points are **adjacent** if they are connected by a single boundary line.*

Previously, we solved a linear programming problem somewhat intuitively by "sliding" the profit line up. Typically, we use a more procedural approach, called the **Method of Corners**.

Method of Corners

1. Set up a linear programming problem algebraically.
2. Graph the constraints and determine the feasible region, **S**.
3. Identify the exact coordinates of all corner points of the feasible region, **S**.
4. Decide whether or not the linear programming problem will have a solution, based upon the Fundamental Theorem of Linear Programming.
5. If a solution will exist, evaluate the objective function at each corner point. The 'optimal' point is the point that optimizes the objective function. If there is a "tie" for where the optimal objective function value occurs, then there are infinitely many optimal solution points.

N *If a linear programming problem is being used to solve a real-world application, then if the solution is not unique, it cannot be assumed to have infinitely many solutions. You must consider all real-world constraints. (i.e. Whole units, ...)*

Using the Method of Corners with the scenario, we begin at Step 4.

4. There is a solution to the linear programming problem, as the feasible region is bounded. Thus, an optimal objective function value exists (and will occur at a corner point).

5. To determine the maximum value, we can set up a table of values for the objective function at each corner point.

Corner Points	Maximize $P = 60x + 80y$		Profit ($)	
A: (0,0)	$P = 60(0) + 80(0)$	=	0	
B: (0, 60)	$P = 60(0) + 80(60)$	=	4800	
C: (18, 54)	$P = 60(18) + 80(54)$	=	5400	
D: (99, 0)	$P = 60(99) + 80(0)$	=	5940	Maximum

Table 3.2: Corner points with corresponding profit values.

From **Table 3.2**, we can see the maximum profit is $5940 at the point $(x, y) = (99, 0)$.

Thus, the maximum profit, $5940, is achieved when the department store sells 99 of the 21 inch TVs and 0 of the 40 inch TVs.

Definition

When solving a linear programming problem representing a real-world scenario, sometimes not all resources are utilized, despite reaching an optimal solution. Any unused portion of a resource is called a **leftover**. ▪

Each resource usually corresponds to an inequality, as seen when setting up constraints. So to determine any leftover resources, we first substitute the corner point producing the optimal objective function value into the left-hand side of each inequality to see how much of each resource was used. Then, we subtract the amount of the resource used from the amount available; the difference is the leftover amount for each resource.

You cannot have a negative amount of leftovers. This would mean you used more of a resource than was available.

In our TV problem, let's investigate whether or not there are any leftover resources, when the department store sells 99 of the 21 inch TVs and 0 of the 40 inch TVs for a maximum profit of \$5940.

Resource	**Amount Used at *Optimal Solution***	**Amount Available**	**Leftovers?**
Storage Space	$6(99)+18(0) = 594$ cubic feet	1080 cubic feet	$1080-594 = 486$ cubic feet
Hours of Labor	$2(99)+3(0) = 198$ hours	198 hours	$198-198 = 0$ hours

We can see we have 486 cubic feet of storage space leftover, but no extra time, when selling TVs for maximum profit.

Example 1 Solve the following linear programming problem, graphically.

Objective: **Minimize** $P = 10x + 10y$

Subject to:
$$x + y \geq 1$$
$$x + 2y \leq 6$$
$$2x + y \leq 6$$
$$x \geq 0, \; y \geq 0$$

Solution:
To solve this graphically, we will use the Method of Corners. The problem has been stated algebraically, so we move to Step 2.

2. Graph the feasible region, as shown in **Figure 3.3.6**.

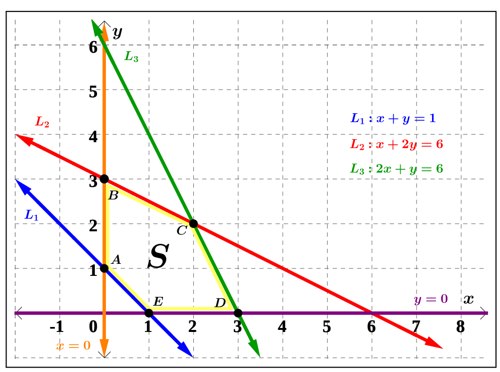


Figure 3.3.6: The feasible region for the system $x+y \geq 1$, $x+2y \leq 6$, $2x+y \leq 6$, $x \geq 0$, and $y \geq 0$. The corner points of the feasible region, **S**, are labeled $A-E$.

3. Using the graph and calculating the appropriate intersection points (see **Figure 3.3.6**), we identify the corner points as

$$A = (0,1), \; B = (0,3), \; C = (2,2), \; D = (3,0), \text{ and } E = (1,0).$$

4. According to the Fundamental Theorem of Linear Programming, there is a solution exists, as the feasible region is bounded. Thus, an optimal objective function value exists (and will occur at at least one corner point).

5. To determine the minimum value, we set up a table of values for the objective function at each corner point.

Corner Points	Minimize $P = 10x + 10y$	
$A:\ (0,1)$	10	Minimum
$B:\ (0,3)$	30	
$C:\ (2,2)$	40	
$D:\ (3,0)$	30	
$E:\ (1,0)$	10	Minimum

Table 3.3: Corner points with corresponding objective function values.

The minimum value is achieved at more than one corner point, at both A and E. The two points are adjacent; thus, the minimum value of P is 10 and it occurs at an infinite number of points on the boundary line segment connecting A and E. The boundary line connecting A and E is $x + y = 1$, so the line segment containing all solution points is $x + y = 1$ for $0 \le x \le 1$.

N *This problem is not connected to a real-world scenario. Therefore we will not discuss leftovers.*

■

■ **Example 2** Given the objective function $L = 7x + 5y$, determine the maximum and minimum values of the objective function over the given feasible region in **Figure 3.3.7**, if they exist, and where they occur.

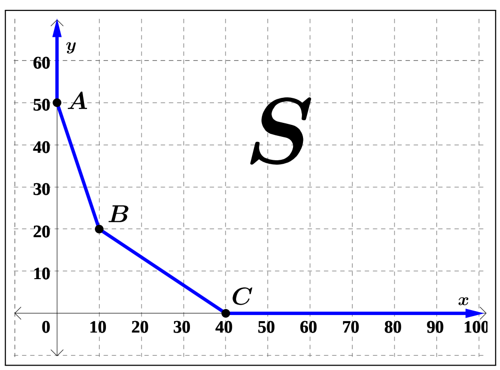


Figure 3.3.7: An unbounded feasible region, opening up in QI, with corner points $A = (0,50)$, $B = (10,20)$, and $C = (40,0)$.

Solution:
Using the Method of Corners, we have all the information needed to start at Step 4.

4. The feasible region is unbounded in Quadrant I and the coefficients of the objective function are positive. Thus, the objective function will have a minimum value, but no maximum value on the given feasible region, according to the Fundamental Theorem of Linear Programming.

5. To determine the minimum value, we set up a table of values for the objective function at each corner point.

Corner Points	Minimize $L = 7x + 5y$	
A : $(0,50)$	250	
B : $(10,20)$	170	Minimum
C : $(40,0)$	280	

Table 3.4: Corner points with corresponding objective function values.

In conclusion, the minimum value of the objective function, L, is 170, and it occurs at corner point B, where $(x,y) = (10,20)$; there is no maximum value.

■

■ **Example 3** Recall the following scenario introduced in Chapter 2:

> Suppose a company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. The company has staffing for 65 hours of assembly and 55 hours of painting each week. If the company wants to fully utilize all staffed hours, how many of each item should they produce?

As stated, the goal was to utilize *all* painting and assembly time. Suppose the company has now turned its focus to maximizing its profits, without concern for the time utilized. If the company sells the basic product for a profit of \$30 each and the premium product for a profit of \$40 each, how many of each version should be produced and sold to maximize the company's profits? Will any resources be leftover with this new goal in mind?

Solution:
Using the Method of Corners, we begin with Step 1, setting up the problem algebraically.

1. Let

   $x :=$ the number of basic versions produced
   $y :=$ the number of premium versions produced
   $P :=$ the profit (in dollars) earned from the sale of the basic and premium products

   Objective: **Maximize** $P = 30x + 40y$

   Subject to: $20x + 30y \leq 3900$ (Minutes of Assembly Time)

   $15x + 30y \leq 3300$ (Minutes of Painting Time)

   $x \geq 0,\ y \geq 0$

2. Graph the constraints. The feasible region, **S**, is labeled in **Figure 3.3.8**.

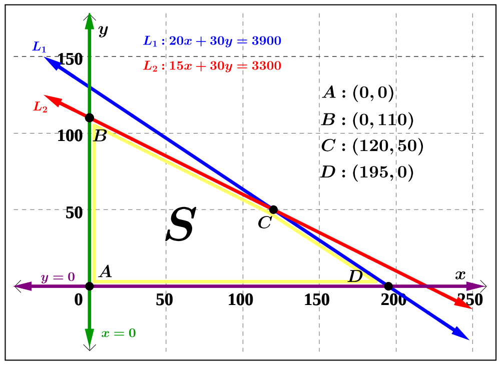


Figure 3.3.8: A bounded feasible region with corner points $A = (0,0)$, $B = (0,110)$, $C = (120,50)$, and $D = (195,0)$.

3. From **Figure 3.3.8** above, the corner points have been calculated as

$$A = (0,0),\ B = (0,110),\ C = (120,50),\ \text{ and } D = (195,0).$$

4. As the feasible region is bounded, there is a maximum value for the objective function (and it will occur at at least one corner point).

5. To determine the maximum value, we set up a table of values for the objective function at each corner point.

Corner Points	Maximize $P = 30x + 40y$	
A : $(0,0)$	0	
B : $(0,110)$	4400	
C : $(120,50)$	5600	
D : $(195,0)$	5850	Maximum

Table 3.5: Corner points with corresponding profit values.

So, the maximum value of $P = 5850$, and it occurs at corner point D where $(x,y) = (195,0)$.

Therefore, the company makes a maximum profit of \$5850, when they produce and sell 195 basic and no premium products.

Next, let's investigate whether or not all resources are being utilized at this production level.

Resource	**Amount Used at Optimal Solution**	**Amount Available**	**Leftovers?**
Assembly Time	$20(195) + 30(0) = 3900$ minutes	3900 minutes	$3900 - 3900 = 0$ minutes
Painting Time	$15(195) + 30(0) = 2925$ minutes	3300 minutes	$3300 - 2925 = 375$ minutes

Thus, not all resources are being utilized at this level. While all assembly time is used, there are 375 minutes of painting time leftover.

■

Try It # 1:
Solve the following linear programming problem, using the Method of Corners.

Objective : Maximize $P = 14x + 9y$

Subject to :
$$\begin{aligned} x + y &\le 9 \\ 3x + y &\le 15 \\ x \ge 0,\ y &\ge 0 \end{aligned}$$

Try It # 2:
A health-food business would like to create a high-potassium blend of dried fruit bars, to be sold by the box. It decides to use dried apricots, which have 407 mg of potassium per serving, and dried dates, which have 271 mg of potassium per serving. The company can purchase its fruit in bulk for a reasonable price. Dried apricots cost \$9.99/lb (3 servings) and dried dates cost \$8.00/lb (4 servings). The company would like each box of bars to have at least the recommended daily potassium intake of about 4700 mg and contain at least 1 serving of each fruit. In order to minimize costs, how many servings of each dried fruit should go into producing a box of fruit bars? Are any resources leftover when producing at minimum cost?

Try It Answers

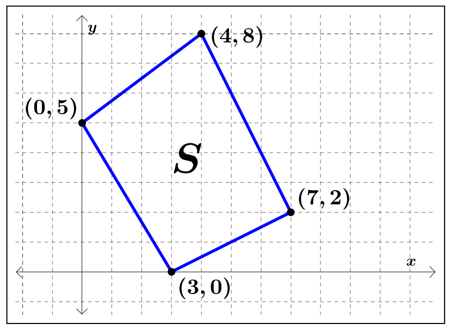
1. The maximum value of $P = 96$, and it occurs at the point $(x, y) = (3, 6)$.

2. One serving of dried apricots and approximately 15.84 servings of dried dates should go into a box of bars, for a minimum cost of \$35.01/box. There are no leftovers when producing at a minimum cost.

EXERCISES

Basic Skills Practice (Answers)

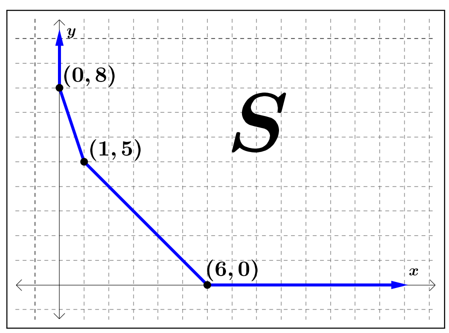
For Exercises 1 - 2, use the given feasible region to determine the maximum and minimum values of the objective functions P and Q over the region, if they exist, and where they occur.



1. $P = 10x + 3y$

2. $Q = -4x + 7y$

For Exercises 3 - 4, use the given feasible region to determine the maximum and minimum values of the objective functions P and Q over the region, if they exist, and where they occur.

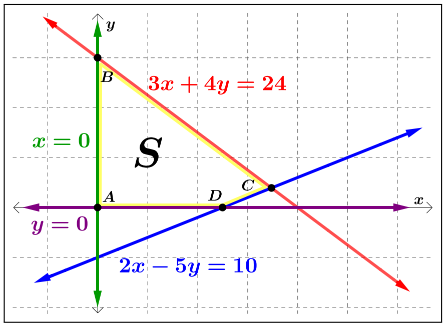


3. $P = x + 2y$

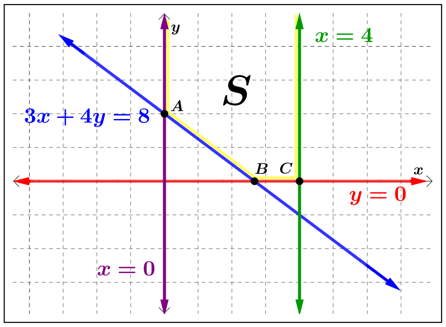
4. $Q = 80x + 25y$

For Exercises 5 - 8, solve each linear programming problem, using the Method of Corners and the provided graphs of the corresponding constraints.

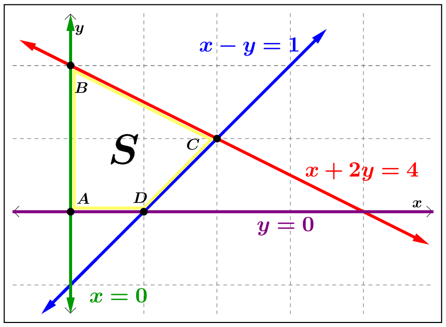
5. Objective: Maximize $P = 8x + 5y$
   Subject to: $2x - 5y \leq 10$
   $3x + 4y \leq 24$
   $x \geq 0,\ y \geq 0$



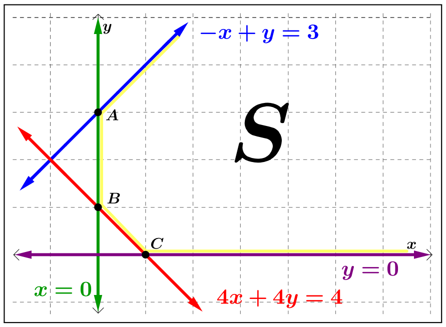
6. Objective: Minimize $P = 2x + 7y$
   Subject to: $x \leq 4$
   $3x + 4y \geq 8$
   $x \geq 0,\ y \geq 0$



7. Objective: Maximize $P = 9x + 4y$
   Subject to: $x - y \leq 1$
   $x + 2y \leq 4$
   $x \geq 0,\ y \geq 0$

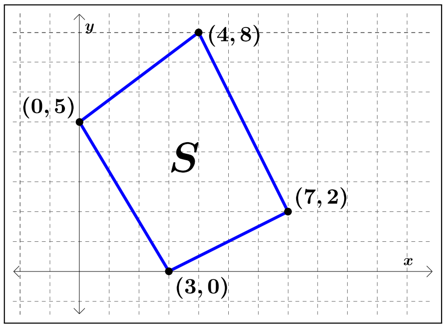


8. Objective: Minimize $P = 7x + 6y$
   Subject to: $-x + y \leq 3$
   $4x + 4y \geq 4$
   $x \geq 0,\ y \geq 0$



INTERMEDIATE SKILLS PRACTICE (Answers)

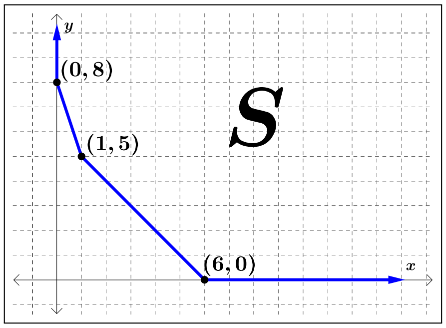
For Exercises 9 - 10, use the given feasible region to determine the maximum and minimum values of the objective functions P and Q over the region, if they exist, and where they occur.



9. $P = 50x + 30y$

10. $Q = 24x + 12y$

For Exercises 11 - 12, use the given feasible region to determine the maximum and minimum values of the objective functions P and Q over the region, if they exist, and where they occur.



11. $P = 120x + 40y$

12. $Q = 15x + 15y$

For Exercises 13 - 16, solve each linear programming problem, using the Method of Corners.

13. Objective: Maximize $P = 17x + 11y$
    Subject to: $x - y \leq 1$
    $x - y \geq -3$
    $x \geq 0,\ 0 \leq y \leq 4$

14. Objective: Minimize $P = -30x + 20y$
    Subject to: $x - y \leq 2$
    $4x - 3y \leq 4$
    $x \geq -2,\ y \leq 3$

15. Objective: Maximize $P = x + 1.5y$
    Subject to: $3x + y \geq -3$
    $2x + 3y \leq 5$
    $x \leq 1,\ y \geq -1$

16. Objective: Minimize $P = 12x + 18y$
    Subject to: $2x - y \geq 4$
    $2x + y \geq 12$
    $x \geq 5,\ y \geq 0$

For Exercises 17 - 18, determine any leftover resources.

17. Pies Galore specializes in chocolate cream and tart cherry pies. Each chocolate cream pie uses 1 pie crust, 1 serving of whipped cream, and 3 servings of sugar. Each tart cherry pie uses 2 pie crusts, 1 serving of whipped cream, and 1 serving of sugar. Pies Galore has not received a food shipment in a while and only has 50 pie crusts, 100 servings of whipped cream, and 120 servings of sugar on hand. In order to maximize their profit using the ingredients on hand, Pies Galore produces 38 chocolate cream pies and 6 tart cherry pies. Does Pies Galore have any pie crusts, whipped cream, or sugar leftover after producing these pies? If so, how much of each ingredient is leftover?

18. The Weather Byall company manufactures raincoats, umbrellas, and hiking backpacks. The company has 600 labor hours, 500 yards of weather resistant fabric, and 400 units of hardware available this week to produce these products. The specifications for each product is given in the table below.

	Raincoat	Backpack	Umbrella
Labor (hours)	1	1.5	2
Fabric (yards)	2	2	1
Hardware (units)	1	1	2

For a maximum profit, Weather Byall produces no raincoats, 200 backpacks, and 100 umbrellas this week. At this production level, are any resources leftover, and if so, how much of each?

Mastery Practice (Answers)

19. A system of linear inequalities has a bounded feasible region with corner points, $(3,1)$, $(2,5)$, $(3,10)$, $(6,8)$, and $(7,2)$. Use the Method of Corners to determine the maximum value of $Z = 27x+18y$ over this region.

20. A system of linear inequalities has an unbounded feasible region in the first quadrant with corner points, $(2,8)$, $(4,6)$, and $(9,3)$. Use the Method of Corners to determine the minimum value of $Z = 4x+5y$ over this region.

For Exercises 21 - 25, solve each linear programming problem, using the Method of Corners.

21. A small candy company makes chocolate pumpkins and chocolate ghosts. Each pumpkin requires 3 minutes to manufacture and 1 minute to package. Each ghost requires 4 minutes to manufacture and 2 minutes to package. There are a total of 1.5 hours available for manufacturing and 0.5 hours available for packaging. The company stipulates that they must produce at least three times as many chocolate pumpkins as ghosts. Determine the production amounts of each candy to maximize profit, if the company's profit on the sale of each pumpkin is 50 cents and the profit on the sale of each ghost is 60 cents.

22. A computer store sells two types of computers, desktops and laptops. The supplier demands that at least 150 computers be sold each month. In order to keep profits up, the number of desktops sold must be at least twice the number of laptops sold. The store spends \$75 a week to market each desktop and \$50 a week to market each laptop. How many of each type of computer must be sold to minimize weekly marketing costs? What is the minimum weekly marketing cost?

23. A factory manufactures two different washing machines, Model 650 and Model 800. During the manufacturing of each washing machine, the machine requires time in three bays: electrical, mechanical, and assembly. The time requirements (in hours) and the total hours available for each bay are listed below.

	Electrical	Mechanical	Assembly
Model 650	1	2	4
Model 800	2	2	2
Total hours	70	90	160

If Model 650 generates a profit of \$600 per washing machine and Model 800 generates a \$500 per washing machine, how many of each model should be manufactured to maximize profit? What is the maximum profit? Do any of the bays have leftover time, when operating at the optimal level?

24. You have at most \$24,000 to invest in bonds and stocks. You have decided that the amount of money invested in bonds must be at least twice as much as that in stocks, but the money invested in bonds must not be greater than \$18,000. If you receive 6% profit on bonds and 8% profit on stocks, how much money should you place in each type of investment to maximize your profit?

25. A professor gives two types of quizzes, objective and recall. The professor is planning to give at least 15 quizzes this semester. The student preparation time for an objective quiz is 15 minutes and for a recall quiz is 30 minutes. The professor would like a student to spend at least 5 hours preparing for these quizzes, above and beyond the normal study time. The average score on an objective quiz is a 7 and on a recall quiz is a 5, and the professor would like the students to score at least 85 points on all quizzes. It takes the professor one minute to grade an objective quiz, and 1.5 minutes to grade a recall type quiz. How many of each type of quiz should the professor give in order to minimize the amount of time spent grading?

Communication Practice (Answers)

26. When using the Method of Corners, describe how you know whether or not an objective function has an optimal value at infinitely many points.

27. Explain why all isoprofit lines are parallel.

3.4 Simplex Method

© Photo provided by Kathryn Bollinger, 1945

In the previous section, we used a graphical method to solve linear programming problems, but the graphical approach will not work for problems that have more than two variables. In real-world situations, linear programming problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, five variables and ten constraints, including the five non-negativity constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if one exists. The trouble with this method is that even for a problem with so few variables, we could have more than 250 corner points, and testing each point would be very tedious. So, we need a method that has a systematic algorithm and could be written as a computer program. The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the **Simplex Method**.

The Simplex Method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm which was supposed to be revolutionary, but, as it turned out, was not any better than the Simplex Method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories, developed Karmarkar's algorithm which has been proven to be four times faster than the Simplex Method for certain problems. However, the Simplex Method still works the best for most problems.

Learning Objectives:

In this section, you will learn the Simplex Method to solve a linear programming problem. Upon completion you will be able to:

- Recognize whether or not a linear programming problem is a standard maximization problem.
- Convert the constraints of a linear programming problem to linear equations with slack variables.
- Construct a simplex tableau for a standard maximization linear programming problem.

- Identify the pivot column for a given simplex tableau.
- Determine the pivot row for a given simplex tableau.
- Identify the pivot element for a given simplex tableau.
- Identify the basic and non-basic variables of a simplex tableau.
- Express the solution represented by a given simplex tableau, and then state whether or not it is the optimal solution.
- Use technology to perform pivots on a simplex tableau to put the tableau in final form.
- Compare each tableau in the Simplex Method to the corresponding corner point in the Method of Corners.
- Solve a standard maximization linear programming problem using the Simplex Method.
- Identify any leftover resources from the solution to a real-world application where the Simplex Method is used.

Performing the Simplex Method on a Standard Maximization Problem

To handle linear programming problems that contain upwards of two variables, we will use the **Simplex Method**.

Definition

The **Simplex Method** is an algorithm that 'toggles' through the corner points of the feasible region of a linear programming problem until it has efficiently located the one that *optimizes* the objective function. ▪

While the Simplex Method can be used to solve many types of linear programming problems, in this text we will use the Simplex Method only to solve a type of linear programming problem known as a **standard maximization problem**.

Identifying a Standard Maximization Problem

Conditions for a Standard Maximization Problem

A **standard maximization problem** will include

- An objective function which is being maximized.
- *Real-world* non-negativity constraints. ($x \geq 0, y \geq 0, z \geq 0, \ldots$)
- All other constraints of the form, $a_1x + a_2y + \ldots + a_nz \leq V$, where V and all a_i are real numbers with $V \geq 0$.

An example of a standard maximization problem is given here:

Objective: Maximize $P = 7x + 12y$

Subject to:
$$2x + 3y \leq 6$$
$$3x + 7y \leq 12$$
$$x \geq 0,\ y \geq 0$$

The constraints may include inequalities that at first glance do not appear to fit the condition of a standard maximization problem. However, if these inequalities can be manipulated mathematically into the form, $a_1x + a_2y + \ldots + a_nz \leq V$, where V and all a_i are real numbers with $V \geq 0$, then the rewritten inequalities satisfy the condition.

■ **Example 1** Are the following linear programming problems considered standard maximization problems? State why or why not.

a. Objective: Maximize $P = 20x + 30y$

Subject to:
$$x + 2y \leq 40$$
$$x + y \geq 2$$
$$x \geq 0,\ y \geq 0$$

b. Objective: Maximize $P = 15x + 40y$

Subject to:
$$x \geq y$$
$$y \leq -3x + 15$$
$$x \geq 0,\ y \geq 0$$

Solution:

a. Objective: Maximize $P = 20x + 30y$

Subject to:
$$x + 2y \leq 40$$
$$x + y \geq 2$$
$$x \geq 0,\ y \geq 0$$

- **Objective**: Maximize $P = 20x + 30y$ ✓ The objective function is being maximized.
- $x \geq 0,\ y \geq 0$ ✓ All variables are non-negative.
- $x + 2y \leq 40$ ✓ The expression involving the variables is $\leq$ a non-negative number, 40.
- $x + y \geq 2$ ✗ The expression involving the variables is $\geq$ a non-negative number, 2, and the inequality cannot be rearranged to satisfy the criteria.

$\Longrightarrow$ This is NOT a standard maximization problem.

b. Objective: Maximize $P = 15x + 40y$

Subject to:
$$x \geq y$$
$$y \leq -3x + 15$$
$$x \geq 0,\ y \geq 0$$

- **Objective**: Maximize $P = 15x + 40y$ ✓ The objective function is being maximized.
- $x \geq 0,\ y \geq 0$ ✓ All variables are non-negative.
- $x \geq y$?
  All variables need to be on the same side of the inequality symbol, so we rewrite this inequality as an equivalent inequality in such a format. $x - y \geq 0$ is the common way to rewrite $x \geq y$, yet this format does not satisfy the criteria. However, we can multiply by -1 to change the direction of the inequality:

$$x - y \geq 0$$
$$(-1)(x - y \geq 0)$$
$$(-1)(x - y) \leq (-1)(0)$$
$$-x + y \leq 0 \checkmark$$

The expression involving the variables is ≤ a non-negative number, 0.

- $y \leq -3x + 15$?

  All variables need to be on the same side of the inequality symbol, so we rewrite as an equivalent inequality in such a format.

  $$3x + y \leq 15 \checkmark$$

  The expression involving the variables is ≤ a non-negative number, 15.

$\Longrightarrow$ This IS a standard maximization problem.

■

Constructing the Initial Tableau

Once we determine that we have a standard maximization problem, in order to use the Simplex Method, we must set up an initial simplex tableau. This tableau is a matrix containing information about the linear programming problem we wish to solve. However, inequalities do not translate properly into matrices. For one, a matrix does not have a simple way of keeping track of the direction of an inequality; this alone discourages the use of inequalities in matrices. How then do we still use matrices, while addressing the constraints of a linear programming problem?

Consider the following linear programming problem. Notice it is a standard maximization problem.

Objective: Maximize $P = 7x + 12y$

Subject to: $2x + 3y \leq 6$

$3x + 7y \leq 12$

$x \geq 0,\ y \geq 0$

Once we recognize we have a standard maximization problem, we ignore the real-world non-negativity constraints ($x \geq 0,\ y \geq 0$) when constructing the initial simplex tableau. Because we know that the left-hand sides of both of the remaining inequalities will be quantities that are no larger than the corresponding values on the right, we can be sure that adding "something" (s_1, s_2) to the left-hand side will balance the inequality, making the two sides exactly equal. That is:

$$2x + 3y + s_1 = 6$$
and
$$3x + 7y + s_2 = 12$$

For instance, suppose that $x = 1$ and $y = 1$. Then for $s_1 = 1$ and $s_2 = 2$, the equalities hold true.

$$2\overset{x}{(1)} + 3\overset{y}{(1)} + \overset{s_1}{1} = 6$$
and
$$3\overset{x}{(1)} + 7\overset{y}{(1)} + \overset{s_2}{2} = 12$$

It is important to note that these two variables, s_1 and s_2, are not necessarily the same. They each simply act on an inequality by picking up the 'slack' that keeps the left-hand side from equalling the right-hand side. Hence, we call them **slack variables**. With the addition of these variables, we no longer have inequalities.

> **N** *Slack variables must always be non-negative, as they represent the unused amount (**leftovers**) of each constraint.*

Because an augmented matrix contains all variables on the left and constants on the right, we will rewrite the objective function to match this format:

$$-7x - 12y + P = 0$$

To accommodate all of the information from the objective function and constraints, we will be required to have a matrix that can handle x, y, s_1, s_2, and P; we will put the columns of the matrix in this order. Finally, the Simplex Method requires that the objective function be listed as the bottom row in the matrix so that we have:

$$\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -7 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

We have now constructed the initial simplex tableau. Note that the vertical line is used to separate constraint coefficients (with included slack variable coefficients) from constants, and the horizontal line separates constraint coefficients from the objective function coefficients.

Try It # 1:
Set up the initial simplex tableau corresponding to the following linear programming problem.

Objective: Maximize $P = 2x + 6y + 9z$

Subject to: $x + y + 3z \leq 85$

$2x + 5z \leq 120$

$4y + z \leq 20$

$x \geq 0,\ y \geq 0,\ z \geq 0$

Try It # 2:
Write the linear programming problem which corresponds to the following initial simplex tableau.

$$\begin{array}{c} \begin{array}{ccccccc} x & y & s_1 & s_2 & s_3 & s_4 & P \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 10 & 1 & 0 & 0 & 0 & 0 & 200 \\ 3 & 5 & 0 & 1 & 0 & 0 & 0 & 155 \\ 4 & 1 & 0 & 0 & 1 & 0 & 0 & 110 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 50 \\ \hline -8 & -7 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Manipulating the Initial Tableau to Solve a Linear Programming Problem

We will now present a step-by-step algorithm for solving the problem we have just set up. Later, we will summarize the entire process.

Identify the pivot column.

We first select a **pivot column**, which will be the column that contains the most negative coefficient in the bottom row. Note that the most negative number belongs to the term that contributes the most to the objective function. This is intentional, as we want to focus on values that make the output, P, as large as possible.

Here, our pivot column, Column 2, is the y column.

$$\begin{array}{c} \begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -7 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \\ \uparrow \\ \text{most negative} \end{array}$$

Identify the pivot row.

We choose a **pivot row** by computing the ratio of each constraint constant to its respective positive coefficient in the pivot column; this is called the **test ratio**. Select the row with the smallest non-negative test ratio.

Calculating the test ratios we have:

$$\begin{array}{cl} \begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \end{array} & \text{Ratios} \\ \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -7 & -12 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} \frac{6}{3} = 2 \\ \frac{12}{7} \approx 1.7 \quad \leftarrow \text{smallest non-negative ratio} \\ \\ \end{array} \end{array}$$

The test ratio is smaller for the second row, so we select it as the pivot row. We select the smallest non-negative ratio to ensure we do not increase the pivot column variable value by more than the resources available.

Determine the pivot element.

The entry in the matrix (tableau) where the pivot column and pivot row intersect is called our **pivot element**. As shown circled in the tableau below, our pivot element will be the 7, in Row 2, Column 2.

$$\begin{array}{c} \begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -7 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \leftarrow \\ \\ \end{array} \\ \uparrow \end{array}$$

Pivot on the pivot element.

Recall, from Chapter 2, that pivoting refers to the process of obtaining a '1' in a specific matrix entry and then zeroing out the rest of the column entries, using elementary row operations.

We will pivot on the pivot element, 7, in Row 2, Column 2.

$$\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -7 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{7}R_2 \to R_2} \left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 6 \\ \frac{3}{7} & 1 & 0 & \frac{1}{7} & 0 & \frac{12}{7} \\ \hline -7 & -12 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[12R_2+R_3 \to R_3]{-3R_2+R_1 \to R_1} \left[\begin{array}{ccccc|c} \frac{5}{7} & 0 & 1 & -\frac{3}{7} & 0 & \frac{6}{7} \\ \frac{3}{7} & 1 & 0 & \frac{1}{7} & 0 & \frac{12}{7} \\ \hline -\frac{13}{7} & 0 & 0 & \frac{12}{7} & 1 & \frac{144}{7} \end{array}\right]$$

At this point, have we optimized the objective function? Because we still have a negative entry in the bottom row after pivoting, this indicates that the objection function is not yet optimized. Continue to pivot, after identifying appropriate pivot elements, until no more negative entries exist in the bottom row.

Continue pivoting, if necessary.

$$\begin{array}{c} \begin{array}{cccccc} x & y & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{5}{7} & 0 & 1 & -\frac{3}{7} & 0 & \frac{6}{7} \\ \frac{3}{7} & 1 & 0 & \frac{1}{7} & 0 & \frac{12}{7} \\ \hline -\frac{13}{7} & 0 & 0 & \frac{12}{7} & 1 & \frac{144}{7} \end{array}\right] \\ \uparrow \\ \text{most negative} \end{array} \quad \begin{array}{l} \text{Ratios} \\ \frac{\left(\frac{6}{7}\right)}{\left(\frac{5}{7}\right)} = \left(\frac{6}{7}\right)\left(\frac{7}{5}\right) = \frac{6}{5} = 1.2 \quad \leftarrow \text{smallest non-negative ratio} \\ \frac{\left(\frac{12}{7}\right)}{\left(\frac{3}{7}\right)} = \left(\frac{12}{7}\right)\left(\frac{7}{3}\right) = \frac{12}{3} = 4 \end{array}$$

We will pivot on the pivot element, $\frac{5}{7}$, in Row 1, Column 1.

$$\left[\begin{array}{ccccc|c} \frac{5}{7} & 0 & 1 & -\frac{3}{7} & 0 & \frac{6}{7} \\ \frac{3}{7} & 1 & 0 & \frac{1}{7} & 0 & \frac{12}{7} \\ \hline -\frac{13}{7} & 0 & 0 & \frac{12}{7} & 1 & \frac{144}{7} \end{array}\right] \xrightarrow{\frac{7}{5}R_1 \to R_1} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{7}{5} & -\frac{3}{5} & 0 & \frac{6}{5} \\ \frac{3}{7} & 1 & 0 & \frac{1}{7} & 0 & \frac{12}{7} \\ \hline -\frac{13}{7} & 0 & 0 & \frac{12}{7} & 1 & \frac{144}{7} \end{array}\right]$$

$$\xrightarrow[\frac{13}{7}R_1+R_3 \to R_3]{-\frac{3}{7}R_1+R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{7}{5} & -\frac{3}{5} & 0 & \frac{6}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{2}{5} & 0 & \frac{6}{5} \\ \hline 0 & 0 & \frac{13}{5} & \frac{3}{5} & 1 & \frac{114}{5} \end{array}\right]$$

With no negatives in the bottom row, we are finished pivoting and are prepared to state the optimal solution.

Identify the optimal solution.

$$\begin{array}{c} \begin{array}{cccccc} x & y & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{7}{5} & -\frac{3}{5} & 0 & \frac{6}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{2}{5} & 0 & \frac{6}{5} \\ \hline 0 & 0 & \frac{13}{5} & \frac{3}{5} & 1 & \frac{114}{5} \end{array}\right] \end{array}$$

Reading columns from left to right, highlight each column which has a single 1 and all other entry values of 0. The corresponding variable in each of these columns is called a **basic variable**. All other variables are called **non-basic variables**.

$$\begin{array}{c} \begin{array}{cccccc} x & y & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{7}{5} & -\frac{3}{5} & 0 & \frac{6}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{2}{5} & 0 & \frac{6}{5} \\ \hline 0 & 0 & \frac{13}{5} & \frac{3}{5} & 1 & \frac{114}{5} \end{array}\right] \end{array}$$

For this example, our basic variables are x, y, and P, while our non-basic variables are s_1 and s_2. As in the previous chapter, we can convert the tableau to a corresponding system of equations.

$$x + \frac{7}{5}s_1 - \frac{3}{5}s_2 = \frac{6}{5}$$

$$y - \frac{3}{5}s_1 + \frac{2}{5}s_2 = \frac{6}{5}$$

$$\frac{13}{5}s_1 + \frac{3}{5}s_2 + P = \frac{114}{5}$$

We will always set all non-basic variables equal to 0, and solve the subsequent equations for the basic variables.

$$x + \frac{7}{5}(0) - \frac{3}{5}(0) = \frac{6}{5} \qquad y - \frac{3}{5}(0) + \frac{2}{5}(0) = \frac{6}{5} \qquad \frac{13}{5}(0) + \frac{3}{5}(0) + P = \frac{114}{5}$$

$$x + 0 - 0 = \frac{6}{5} \qquad y - 0 + 0 = \frac{6}{5} \qquad 0 + 0 + P = \frac{114}{5}$$

$$x = \frac{6}{5} \qquad y = \frac{6}{5} \qquad P = \frac{114}{5}$$

Thus, the maximum value for P is $\frac{114}{5}$, and it occurs when $(x, y) = \left(\frac{6}{5}, \frac{6}{5}\right)$. At this optimal solution, $s_1 = s_2 = 0$.

In the above linear programming problem, s_1 and s_2 (the slack variables) were our non-basic variables, while all original variables (x, y, and P) were the basic variables. The reader must not make the assumption that this will always be the case. When determining the basic and non-basic variables, the reader should always refer to the column entries of the corresponding simplex tableau.

As illustrated in the previous problem, we will now summarize the algorithmic process for solving a standard maximization linear programming problem.

The Simplex Method for a Standard Maximization Problem

1. **Set up the problem, algebraically.**

   That is, write the objective function to be maximized and the constraints in standard maximization form.

2. **Convert the linear constraint inequalities into linear equations**.

   This is done by adding a different slack variable to each inequality, not including non-negativity constraints.

3. **Construct the initial simplex tableau.**

   Align variables from each ‘equality’ and objective function; place the coefficients into a matrix, with the objective function as the bottom row, below a horizontal line.

4. **Identify the pivot column.**

   The most negative entry in the bottom row identifies the **pivot column**.

5. **Identify the pivot row.**

   Calculate the test ratios for all rows except the bottom row. The ratios are computed by dividing the far right (‘constant’) column by the corresponding value in the identified column from Step 4. A ratio that has a zero or negative number in the denominator is ignored. The smallest non-negative ratio computed identifies the **pivot row**.

6. **Identify the pivot element.**

   The entry in the intersection of the pivot column and the pivot row is the **pivot element**.

7. **Pivot on the pivot element.**

   Pivoting can be done using elementary row operations, as previously discussed. However, there are calculator and computer programs which will perform these calculations for you. Often the programs only require the user to input the initial tableau and indicate subsequent pivot elements.

8. **When there are no negative entries in the bottom row after pivoting, we are finished pivoting and can identify the optimal solution; otherwise, we start again from Step 4.**

9. **Identify the optimal solution.**

   Determine basic and non-basic variables. Columns which contain a single 1, with all other entries 0, represent **basic variables**, and all other columns represent **non-basic variables**. Set all non-basic variables equal to zero and solve for the basic variables.

N *The authors of this text will use technology to perform any necessary pivots. While the determination of each pivot element may be discussed, no elementary row operations will be shown.*

Example 2 Use the Simplex Method to solve the following linear programming problem.

Objective: Maximize $P = 5x + 7y + 9z$

Subject to: $x + 4y + 2z \le 8$

$3x + 5y + z \le 6$

$x \ge 0,\ y \ge 0,\ z \ge 0$

Solution:

Upon inspection, it is clear the given problem is a standard maximization problem.

To set up the initial tableau, first add a slack variable to each constraint, not including the non-negativity constraints, and rewrite the objective function. Then, align the variables and convert to a corresponding matrix as follows:

$$\begin{array}{rcrcrcrcrcrcl} x & + & 4y & + & 2z & + & s_1 & & & & & = & 8 \\ 3x & + & 5y & + & z & & & + & s_2 & & & = & 6 \\ -5x & - & 7y & - & 9z & & & & & + & P & = & 0 \end{array} \Longrightarrow \begin{array}{c} \begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 8 \\ 3 & 5 & 1 & 0 & 1 & 0 & 6 \\ \hline -5 & -7 & -9 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

We identify the first pivot element by finding the intersection of the first pivot column and pivot row.

$$\left[\begin{array}{cccccc|c} 1 & 4 & 2 & 1 & 0 & 0 & 8 \\ 3 & 5 & 1 & 0 & 1 & 0 & 6 \\ \hline -5 & -7 & -9 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \text{Ratios} \\ \frac{8}{2} = 4 \quad \leftarrow \text{smallest non-negative ratio} \\ \frac{6}{1} = 6 \\ \end{array}$$

$$\uparrow$$

most negative

After pivoting on the entry (2) in Row 1, Column 3, we have

$$\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \end{array}$$
$$\left[\begin{array}{cccccc|c} \frac{1}{2} & 2 & 1 & \frac{1}{2} & 0 & 0 & 4 \\ \frac{5}{2} & 3 & 0 & -\frac{1}{2} & 1 & 0 & 2 \\ \hline -\frac{1}{2} & 11 & 0 & \frac{9}{2} & 0 & 1 & 36 \end{array}\right]$$

Because we still have a negative in the bottom row, we determine the next pivot element.

$$\left[\begin{array}{cccccc|c} \frac{1}{2} & 2 & 1 & \frac{1}{2} & 0 & 0 & 4 \\ \frac{5}{2} & 3 & 0 & -\frac{1}{2} & 1 & 0 & 2 \\ \hline -\frac{1}{2} & 11 & 0 & \frac{9}{2} & 0 & 1 & 36 \end{array}\right] \begin{array}{l} \text{Ratios} \\ \frac{4}{\left(\frac{1}{2}\right)} = 4 \cdot 2 = 8 \\ \frac{2}{\left(\frac{5}{2}\right)} = 2 \cdot \frac{2}{5} = \frac{4}{5} \quad \leftarrow \text{smallest non-negative ratio} \\ \end{array}$$

$$\uparrow$$

most negative

After pivoting on the entry $\left(\frac{5}{2}\right)$ in Row 2, Column 1, we have

$$\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \end{array}$$
$$\left[\begin{array}{cccccc|c} 0 & \frac{7}{5} & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{18}{5} \\ 1 & \frac{6}{5} & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{4}{5} \\ \hline 0 & \frac{58}{5} & 0 & \frac{22}{5} & \frac{1}{5} & 1 & \frac{182}{5} \end{array}\right]$$

As we no longer have a negative entry in the bottom row, we can identify the optimal solution.

$$\begin{array}{ccccccc} x & y & z & s_1 & s_2 & P & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 0 & \frac{7}{5} & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{18}{5} \\ 1 & \frac{6}{5} & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{4}{5} \\ \hline 0 & \frac{58}{5} & 0 & \frac{22}{5} & \frac{1}{5} & 1 & \frac{182}{5} \end{array}\right]$$

The basic variables are x, z, and P, while the non-basic variables are y, s_1, and s_2.

The corresponding system to our tableau is

$$\frac{7}{5}y + z + \frac{3}{5}s_1 - \frac{1}{5}s_2 = \frac{18}{5}$$

$$x + \frac{6}{5}y - \frac{1}{5}s_1 + \frac{2}{5}s_2 = \frac{4}{5}$$

$$\frac{58}{5}y + \frac{22}{5}s_1 + \frac{1}{5}s_2 + P = \frac{182}{5}$$

We set the non-basic variables (y, s_1, and s_2) equal to 0, and solve for the basic variables (x, z, and P).

$$0 + z + 0 - 0 = \frac{18}{5} \qquad x + 0 - 0 + 0 = \frac{4}{5} \qquad 0 + 0 + 0 + P = \frac{182}{5}$$

$$z = \frac{18}{5} \qquad x = \frac{4}{5} \qquad P = \frac{182}{5}$$

Thus, the maximum value of P is $\frac{182}{5}$ when $(x,y,z) = \left(\frac{4}{5}, 0, \frac{18}{5}\right)$. Here, $s_1 = s_2 = 0$.

■

■ **Example 3** Is the following simplex tableau in final form? If so, read the optimal solution. If not, state the next pivot element.

$$\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 0 & 5 & 1 & 0 & 28 \\ 2 & 1 & 3 & 0 & 0 & 16 \\ \hline 2 & 0 & 8 & 0 & 1 & 48 \end{array}\right] \end{array}$$

Solution:
We can see there are no negative entries in the bottom row; therefore, the simplex tableau IS in final form.

The basic variables are y, s_2, and P, while x and s_1 are non-basic variables. By setting the non-basic variables equal to zero, $y = 16$, $s_2 = 28$, and $P = 48$.

Thus, the maximum value of P is 48 when $(x,y) = (0,16)$. Here, $s_1 = 0$ and $s_2 = 28$.

■

Try It # 3:
Solve the following linear programming problem, using the Simplex Method.

Objective: Maximize $P = x_1 + 2x_2 + 3x_3$

Subject to:
$$x_1 + x_2 + x_3 \le 12$$
$$2x_1 + x_2 + 3x_3 \le 18$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

Comparing the Simplex Method and the Method of Corners

The Simplex Method uses an approach that is very efficient. It does not compute the value of the objective function at every corner point; instead, it begins with a corner point of the feasible region where all the main variables are zero, and then, systematically moves from corner point to adjacent corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

In this subsection, we will focus on the relationship between the simplex tableaus and the corresponding corner points of a feasible region.

Consider the following standard maximization linear programming problem:

Objective: Maximize $P = 4x + 3y$

Subject to:
$$2x + y \le 10$$
$$2x + 3y \le 18$$
$$x \ge 0,\ y \ge 0$$

The bounded feasible region for the problem is shown in **Figure 3.4.2** below, with corner points

$$A : (0,0),\ B : (0,6),\ C : (3,4),\ \text{and } D : (5,0).$$

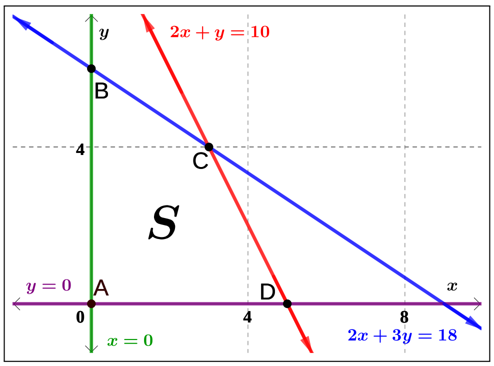


Figure 3.4.2: The graph of the feasible region for $2x+y \le 10$, $2x+3y \le 18$, $x \ge 0$, and $y \ge 0$ with corner points, $A-D$, labeled.

The initial simplex tableau for this problem is given by

$$\begin{array}{rcrcrcrcrcr} 2x & + & y & + & s_1 & & & & & = & 10 \\ 2x & + & 3y & & & + & s_2 & & & = & 18 \\ -4x & - & 3y & & & & & + & P & = & 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 10 \\ 2 & 3 & 0 & 1 & 0 & 18 \\ \hline -4 & -3 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

At this point the tableau has basic variables s_1, s_2, and P, while the non-basic variables are x and y.

$$\left.\begin{array}{c} x = 0 \\ y = 0 \\ \text{and} \\ s_1 = 10 \\ s_2 = 18 \\ P = 0 \end{array}\right\} \quad \Longrightarrow \quad P = 0 \text{ when } (x, y) = (0, 0).$$

This indicates we are starting at corner point A, with an objective function value of 0.

This is a solution, but not an *optimal* solution, as we still have negative entries in the bottom row of our tableau. After pivoting on the appropriate pivot element, the result is

$$\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 0 & 2 & -1 & 1 & 0 & 8 \\ \hline 0 & -1 & 2 & 0 & 1 & 20 \end{array}\right] \end{array}$$

At this point the tableau has basic variables of x, s_2, and P, while the non-basic variables are y and s_1.

$$\left.\begin{array}{c} y = 0 \\ s_1 = 0 \\ \text{and} \\ x = 5 \\ s_2 = 8 \\ P = 20 \end{array}\right\} \quad \Longrightarrow \quad P = 20 \text{ when } (x, y) = (5, 0).$$

This indicates we are now at corner point D, with an objective function value of 20.

Pivoting has moved us from A to the location of the next best solution, D, based on our criteria − contributing the most to the objective function and without using more resources than we have available.

While this is a better solution than the first, is it the optimal solution? We know it is not, because there is still a negative entry in the bottom row of our last tableau.

After pivoting on the appropriate pivot element, we obtain

$$\begin{array}{c} \begin{array}{cccccc} x & y & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{1}{4} & 0 & 3 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 4 \\ \hline 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & 24 \end{array}\right] \end{array}$$

At this point the tableau has basic variables of x, y, and P, while the non-basic variables are s_1 and s_2.

$$\left.\begin{array}{c} s_1 = 0 \\ s_2 = 0 \\ \text{and} \\ x = 3 \\ y = 4 \\ P = 24 \end{array}\right\} \implies \quad P = 24 \text{ when } (x, y) = (3, 4).$$

This indicates we are now at corner point C, with an objective function value of 24.

Because the Simplex Method systematically moves from one corner point to a better **adjacent** corner point, it should make sense that we are now at corner point C.

At this point in the process, there are no negative entries in the bottom row of our tableau, so $P = 24$ when $(x, y) = (3, 4)$ is the optimal solution. Here $s_1 = s_2 = 0$.

Unlike with the Method of Corners where all four corner points were necessary for determining the optimal solution, the Simplex Method only required us to examine three corner points.

! *When performing the Simplex Method, if you seem to end up in an infinite loop and you never attain a tableau with the bottom row having no negative values, that is equivalent to choosing the wrong pivot element and landing at a point outside of the feasible region. If this occurs, double check your initial tableau and all subsequent pivot element calculations.*

Applying the Simplex Method to Real-World Scenarios

Let's revisit the department store selling televisions.

> A department store sells two sizes of televisions: 21 inch and 40 inch. A 21 inch television requires 6 cubic feet of storage space, and a 40 inch television requires 18 cubic feet of space. A maximum of 1080 cubic feet of storage space is available. The 21 inch and the 40 inch televisions take up, respectively, 2 and 3 sales hours of labor, and the store has a maximum of 198 hours of labor available. If the profit earned from each of these sizes of televisions is \$60 and \$80, respectively, how many of each size of television should be sold to maximize the store's profit, and what is the maximum profit?

Recall the formulated algebraic problem:

t := the number of 21 inch TVs sold

f := the number of 40 inch TVs sold

P := the profit earned on the sale of TVs, in dollars

Objective: **Maximize** $P = 60t + 80f$

Subject to: $6t + 18f \leq 1080$ (Cubic Feet of Storage Space)

$2t + 3f \leq 198$ (Hours of Labor)

$t \geq 0,\ f \geq 0$

Previously we solved this problem using the Method of Corners, by substituting variables such that $(t, f) \to (x, y)$. We can now verify our answer, using the Simplex Method, without needing to make such a variable substitution.

The initial tableau will be

$$\begin{array}{c} \begin{array}{cccccc} t & f & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 6 & 18 & 1 & 0 & 0 & 1080 \\ 2 & 3 & 0 & 1 & 0 & 198 \\ \hline -60 & -80 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Once *all* pivots have been performed, we are left with the following *final* tableau. (We leave it to the reader to determine each pivot element, to pivot as needed, and verify our result).

$$\begin{array}{c} \begin{array}{cccccc} t & f & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 9 & 1 & -3 & 0 & 486 \\ 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 99 \\ \hline 0 & 10 & 0 & 30 & 1 & 5940 \end{array}\right] \end{array}$$

The basic variables are t, s_1, and P, while the non-basic variables are f and s_2.

$$\left.\begin{array}{c} f = 0 \\ s_2 = 0 \\ \text{and} \\ s_1 = 486 \\ t = 99 \\ P = 5940 \end{array}\right\} \implies P = 5940 \text{ when } (t, f) = (99, 0) \text{ and } s_1 = 486 \text{ and } s_2 = 0.$$

In the context of our problem, this means a maximum profit of \$5940 is made when selling 99 of the 21 inch TVs and 0 of the 40 inch TVs.

Remember that slack variables represent leftover resources. The first constraint, $6t + 18f \leq 1080$, was an indication of storage space (in cubic feet), so $s_1 = 486$ means the department store will have 486 cubic feet of unused storage space. The second constraint, $2x + 3y \leq 198$, was an indication of available sales hours of labor, so $s_2 = 0$ means the department store used all available labor hours, with none left over.

Notice, this solution does indeed match the solution found when using the Method of Corners.

Try It # 4:
Revisit the following problem, and use the Simplex Method to find the optimal solution.

> Suppose a company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. The company has staffing for 65 hours of assembly and 55 hours of painting each week. If the company sells the basic product for a profit of \$30 each and the premium product for a profit of \$40 each, how many of each version should be produced and sold to maximize the company's profits? Will any resources be leftover?

Compare your optimal solution with the solution found in **Section 3.3**, when the Method of Corners was used.

Try It Answers

1.

$$\begin{array}{c} \begin{array}{ccccccc} x & y & z & s_1 & s_2 & s_3 & P \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & 3 & 1 & 0 & 0 & 0 & 85 \\ 2 & 0 & 5 & 0 & 1 & 0 & 0 & 120 \\ 0 & 4 & 1 & 0 & 0 & 1 & 0 & 20 \\ \hline -2 & -6 & -9 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

2. **Objective**: **Maximize** $P = 8x + 7y$

Subject to:

$$\begin{aligned} 2x + 10y &\leq 200 \\ 3x + 5y &\leq 155 \\ 4x + y &\leq 110 \\ x &\leq 50 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

3. The maximum value of P is 27 when $(x_1,\ x_2,\ x_3) = (0, 9, 3)$ and $s_1 = s_2 = 0$.

4. A maximum profit of \$5850 is made when selling 195 basic products and no premium products. At this optimal level, all assembly time is used, but there are 375 minutes of painting time leftover. (This is the same solution as was found using the Method of Corners.)

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 4, determine if the linear programming problem is a standard maximization problem. If not, state why not.

1. **Objective**: Minimize $P = 2x + 3y$
   Subject to: $x - y \leq 2$
   $5x + 8y \leq 25$
   $x \geq 0,\ y \geq 0$

2. **Objective**: Maximize $P = 4x + 7y$
   Subject to: $x + 3y \leq 0$
   $6x + 2y \leq 12$
   $x \leq 0,\ y \leq 0$

3. **Objective**: Maximize $P = 20x + 45y$
   Subject to: $y \geq x - 8$
   $6y \geq 4x$
   $x \geq 0,\ y \geq 0$

4. **Objective**: Maximize $P = 9x + 24y$
   Subject to: $5x + 2 \leq 6y$
   $y - x \leq 11$
   $x \geq 0,\ y \geq 0$

For Exercises 5 - 8, set up the initial simplex tableau corresponding to the given linear programming problem.

5. **Objective**: Maximize $P = 2.5x + 3.75y$
   Subject to: $2x + y \leq 35$
   $x + 4y \leq 100$
   $x \geq 0,\ y \geq 0$

6. **Objective**: Maximize $P = 20x + 40y + 90z$
   Subject to: $3x + 0.5y + z \leq 40$
   $0.65x + 0.35y + 0.85z \leq 250$
   $30x + 20y + 45z \leq 3200$
   $x \geq 0,\ y \geq 0,\ z \geq 0$

7. **Objective**: Maximize $P = 3x + 4y + 5z$
   Subject to: $6x + 7y \leq 8$
   $5y + z \leq 10$
   $x \geq 0,\ y \geq 0,\ z \geq 0$

8. **Objective**: Maximize $P = 10x + 15y$
   Subject to: $7x + 11y \leq 24$
   $2x + 4y \leq 6$
   $x \leq 8$
   $x \geq 0,\ y \geq 0$

For Exercises 9 - 11, state the following for the given simplex tableau.

a. Pivot column
b. Pivot row
c. Pivot element

9.
$$\begin{array}{c} \begin{array}{cccccc} x & y & s_1 & s_2 & P & \text{constant} \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 2 & 1 & 0 & 0 & 60 \\ 4 & 10 & 0 & 1 & 0 & 100 \\ \hline -19 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

10. $$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 14 & 21 & 30 & 1 & 0 & 0 & 630 \\ 0.25 & 0.125 & 0.5 & 0 & 1 & 0 & 250 \\ \hline -1 & -9 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

11. $$\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} 6 & 0 & 2 & 1 & 0 & 0 & 0 & 120 \\ 0 & 1 & 4 & 0 & 1 & 0 & 0 & 84 \\ 7 & 8 & 0 & 0 & 0 & 1 & 0 & 100 \\ \hline -6 & -10 & -13 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

For Exercises 12 - 14, state the value of each variable and whether the variable is basic or non-basic, using the provided final simplex tableau.

12. $$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 20 \\ 0 & \frac{22}{3} & -\frac{4}{3} & 1 & 0 & 20 \\ \hline 0 & \frac{2}{3} & \frac{19}{3} & 0 & 1 & 380 \end{array}\right]$$

13. $$\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} \frac{2}{3} & 1 & \frac{10}{7} & \frac{1}{21} & 0 & 0 & 30 \\ \frac{1}{6} & 0 & \frac{9}{28} & -\frac{1}{168} & 1 & 0 & \frac{985}{4} \\ \hline 5 & 0 & \frac{90}{7} & \frac{3}{7} & 0 & 1 & 270 \end{array}\right]$$

14. $$\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{16}{103} & -\frac{8}{103} & \frac{1}{103} & 0 & \frac{1348}{103} \\ 0 & 0 & 1 & \frac{7}{206} & \frac{24}{103} & -\frac{3}{103} & 0 & \frac{2136}{103} \\ 0 & 1 & 0 & -\frac{14}{103} & \frac{7}{103} & \frac{12}{103} & 0 & \frac{108}{103} \\ \hline 0 & 0 & 0 & \frac{3}{206} & \frac{334}{103} & \frac{87}{103} & 1 & \frac{36936}{103} \end{array}\right]$$

Intermediate Skills Practice (Answers)

For Exercises 15 - 18, set up the initial simplex tableau corresponding to the given linear programming problem.

15. **Objective**: Maximize $P = 4x + 8y$
    Subject to: $11y \leq 8x + 5$
    $4y \geq x$
    $x \geq 0,\ y \geq 0$

16. **Objective**: Maximize $P = x + 1.75y + 1.25z$
    Subject to: $\frac{3}{4}x + \frac{1}{3}y + \frac{2}{5}z \leq 60$
    $x + \frac{1}{2}y \leq 16 - z$
    $4x + 5z \leq 10$
    $x \geq 0,\ y \geq 0,\ z \geq 0$

17. **Objective**: Maximize $P = 75x + 52y$
    Subject to: $8x + 14y \leq 91$
    $25x + 10y \leq 522$
    $0 \leq x \leq 10,\ y \geq 0$

18. **Objective**: Maximize $P = 17x + 15y$
    Subject to: $\frac{1}{7}x + \frac{3}{8}y \leq 42$
    $9 \geq -3x + 6y$
    $x - y \leq 283$
    $x \geq 0,\ y \geq 0$

For Exercises 19 - 21, state the following for the given simplex tableau.

a. Pivot column
b. Pivot row
c. Pivot element

19. $$\begin{array}{cccccc|c} x & y & s_1 & s_2 & P & & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 0 & 1 & -\frac{1}{4} & 0 & \frac{255}{2} \\ \frac{1}{2} & 1 & 0 & \frac{1}{8} & 0 & \frac{375}{4} \\ \hline -13 & 0 & 0 & \frac{3}{2} & 1 & 1125 \end{array}\right]$$

20. $$\begin{array}{cccccccc} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} \frac{2}{7} & \frac{1}{5} & \frac{3}{4} & 1 & 0 & 0 & 0 & 630 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{2} & 0 & 1 & 0 & 0 & 250 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 & 480 \\ \hline -1 & -9 & -4 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

21. $$\begin{array}{ccccccccc} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccccc|c} 6 & -0.5 & 0 & 1 & -0.5 & 0 & 0 & 0 & 78 \\ 0 & 0.25 & 1 & 0 & 0.25 & 0 & 0 & 0 & 21 \\ 7 & 8 & 0 & 0 & 0 & 1 & 0 & 0 & 100 \\ 9 & -0.75 & 0 & 0 & 0.25 & 0 & 1 & 0 & 21 \\ \hline -6 & -6.75 & 0 & 0 & 3.25 & 0 & 0 & 1 & 273 \end{array}\right]$$

For Exercises 22 - 24,

a. State the value of each variable, and whether the variable is basic or non-basic.
b. Determine if the given simplex tableau is a final tableau.

22.
$$\begin{array}{cccccc} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 0 & 1 & -\frac{1}{4} & 0 & \frac{255}{2} \\ \frac{1}{2} & 1 & 0 & \frac{1}{8} & 0 & \frac{375}{4} \\ \hline -13 & 0 & 0 & \frac{3}{2} & 1 & 1125 \end{array}\right]$$

23.
$$\begin{array}{cccccccc} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} 0 & 0 & -\frac{1}{20} & 1 & -\frac{8}{5} & -\frac{4}{35} & 0 & 230 \\ 0 & 1 & 4 & 0 & 8 & 2 & 0 & 2000 \\ 1 & 0 & 6 & 0 & 8 & 3 & 0 & 2480 \\ \hline 0 & 0 & 22 & 0 & 72 & 17 & 1 & 18000 \end{array}\right]$$

24.
$$\begin{array}{ccccccccc} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccccc|c} \frac{103}{16} & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{16} & 0 & 0 & \frac{337}{4} \\ -\frac{7}{32} & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{32} & 0 & 0 & \frac{143}{8} \\ \frac{7}{8} & 1 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{25}{2} \\ \frac{309}{32} & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{32} & 1 & 0 & \frac{243}{8} \\ \hline -\frac{3}{32} & 0 & 0 & 0 & \frac{13}{4} & \frac{27}{32} & 0 & 1 & \frac{2859}{8} \end{array}\right]$$

For Exercises 25 - 27, use the given constraints and optimal solution to state the value of each slack variable.

25. **Subject to**: $6x+5y \le 500$
$20x+30y \le 900$
$x \ge 0,\ y \ge 0$

Optimal Solution: $(x,y) = (0,30)$

26. **Subject to**: $0.5x+0.75y+z \le 40$
$2x+10y+4z \le 200$
$x+2y+3z \le 600$
$x \ge 0,\ y \ge 0,\ z \ge 0$

Optimal Solution: $(x,y,z) = \left(\frac{500}{7}, \frac{40}{7}, 0\right)$

27. **Subject to**: $7x+3y \le 21$
$x+0.5y \le 300$
$2x+5y \le 100$
$x \ge 0,\ y \ge 0$

Optimal Solution: $(x,y) = (0,7)$

For Exercises 28 - 31, solve each linear programming problem, using the Simplex Method, if possible. If the Simplex Method cannot be used to solve the linear programming problem, state why not.

28. **Objective**: Maximize $P = 17x+11y$
Subject to: $x-y \le 1$
$x+2y \le 3$
$x \ge 0,\ y \ge 0$

29. **Objective**: Maximize $P = 30x+20y$
Subject to: $x+y \le 18$
$4x-3y \le 4$
$x \ge 0,\ y \ge 0$

30. **Objective**: Maximize $P = 1.5x+y$
Subject to: $3x+y \le 300$
$2x+3y \le 45$
$x \ge 0,\ y \ge 0$

31. **Objective**: Maximize $P = 12x+18y$
Subject to: $2x-y \ge 4$
$2x+y \ge 12$
$x \ge 5,\ y \ge 0$

Mastery Practice (Answers)

32. Write the linear programming problem which corresponds to the following initial simplex tableau.

$$\begin{array}{c}
\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array} \\
\left[\begin{array}{cccccc|c}
1 & 2 & 1 & 0 & 0 & 0 & 100 \\
4 & 3 & 0 & 1 & 0 & 0 & 75 \\
0 & 1 & 0 & 0 & 1 & 0 & 23 \\
\hline
-8 & -11 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

33. If the given simplex tableau is the final tableau, state the optimal solution. If the given simplex tableau is not the final tableau, explain why not.

$$\begin{array}{c}
\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \text{constant} \end{array} \\
\left[\begin{array}{cccccc|c}
0 & 0 & 1 & 0.5 & 1 & 0 & 850 \\
1 & 0.25 & 0 & 4 & 3 & 0 & 125 \\
\hline
0 & 15 & 0 & 2 & 6 & 1 & 38000
\end{array}\right]
\end{array}$$

34. Write a final simplex tableau that satisfies the following conditions:
    - The linear programming problem has two main variables, x and y, and an objective function variable, P.
    - The linear programming problem has 3 constraints, not including the non-negativity constraints.
    - The basic variables are y, s_1, s_2, and P, with values of 78, 136, 20, and 15000, respectively. All other variables are non-basic.

35. A linear programming problem was solved using the Simplex Method. All tableaus are given, in order, below. If the Method of Corners had been used, state the corner point corresponding to each tableau.

$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 2 & 6 & 1 & 0 & 0 & 0 & 12 \\ 1 & 0.25 & 0 & 1 & 0 & 0 & 25 \\ 3 & 0.5 & 0 & 0 & 1 & 0 & 60 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} \frac{1}{3} & 1 & \frac{1}{6} & 0 & 0 & 0 & 2 \\ \frac{11}{12} & 0 & -\frac{1}{24} & 1 & 0 & 0 & \frac{49}{2} \\ \frac{17}{6} & 0 & -\frac{1}{12} & 0 & 1 & 0 & 59 \\ \hline -1 & 0 & \frac{1}{2} & 0 & 0 & 1 & 6 \end{array}\right]$$

$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 3 & \frac{1}{2} & 0 & 0 & 0 & 6 \\ 0 & -\frac{11}{4} & -\frac{1}{2} & 1 & 0 & 0 & 19 \\ 0 & -\frac{17}{2} & -\frac{3}{2} & 0 & 1 & 0 & 42 \\ \hline 0 & 3 & 1 & 0 & 0 & 1 & 12 \end{array}\right]$$

For Exercises 36 - 39, use the Simplex Method to solve the linear programming problem.

36. A factory manufactures two different washing machines, Model 650 and Model 800. During the manufacturing of each washing machine the machine requires time in three bays: electrical, mechanical, and assembly. The time requirements (in hours) and the total hours available for each bay are listed below.

	Electrical	Mechanical	Assembly
Model 650	1	2	4
Model 800	2	2	2
Total hours	70	90	160

If Model 650 generates a profit of \$600 per washing machine and Model 800 generates a \$500 per washing machine, how many of each model should be manufactured to maximize profit? What is the maximum profit? Do any of the bays have leftover time, when operating at the optimal level?

37. Pies Galore specializes in chocolate cream, lemon meringue, and tart cherry pies. Each chocolate cream pie uses 1 pie crust, 1 serving of whipped cream, and 3 servings of sugar. Each lemon meringue pie uses 1 pie crust, 2 servings of whipped cream, and 3 servings of sugar. Each tart cherry pie uses 2 pie crusts, 1 serving of whipped cream, and 1 serving of sugar. Pies Galore has not received a food shipment in a while and only has 50 pie crusts, 100 servings of whipped cream, and 120 servings of sugar on hand. They sell each chocolate cream pie for \$15, each lemon meringue pie for \$14, and each tart cherry pie for \$20. How many of each type of pie should Pies Galore make and sell in order to maximize their revenue, given their current inventory? Does Pies Galore have any ingredients leftover, when maximizing their revenue?

38. You have at most \$24,000 to invest in bonds and stocks. You have decided that the amount of money invested in bonds must be at least twice as much as that in stocks, but the money invested in bonds must not be greater than \$18,000. If you receive 6% profit on bonds and 8% profit on stocks, how much money should you place in each type of investment to maximize your profit?

39. A small candy company makes chocolate pumpkins, chocolate cats, and chocolate ghosts. Each pumpkin requires 3 minutes to manufacture and 1 minute to package. Each cat requires 5 minutes to manufacture and 1.5 minutes to package. Each ghost requires 4 minutes to manufacture and 2 minutes to package. There are a total of 1.5 hours available for manufacturing and 0.5 hours available for packaging. The company stipulates that they must produce at least three times as many chocolate pumpkins as ghosts. Determine the production amount of each candy to maximize profit, if the profit on the sale of each pumpkin, cat, and ghost is 50 cents, 75 cents, and 60 cents, respectively. What is the company's maximum profit, and at this production level, is there any manufacturing or packaging time left over?

Communication Practice (Answers)

40. Explain why the pivot column is chosen from the most negative entry in the bottom row when using the Simplex Method to solve a standard maximization linear programming problem.

41. Explain why the pivot row is chosen from the smallest non-negative ratio when using the Simplex Method to solve a standard maximization linear programming problem.

42. Explain why slack variables must be non-negative.

43. What determines the number of slack variables needed when using the Simplex Method to solve a standard maximization linear programming problem?

44. Explain what a negative entry in the last *column* of a simplex tableau means.

Chapter Review

Directions: Read each reflection question. If you are able to answer the question, then move on to the next question. If not, use the mathematical questions included to help you review. (Answers)

1. Can you set up a linear programming problem to be used to solve a real-world optimization problem under a set of constraints? **Set up each, but do not solve**.
   a. A natural shampoo maker produces two types of shampoo for a locally owned specialty shop. One bottle of gentle shampoo will have 14 oz of water and 4 oz of surfactant, while one bottle of regular shampoo will have 12 oz of water and 6 oz of surfactant. The maker estimates they will generate a profit of \$3 on each bottle of gentle shampoo and \$4 on each bottle of regular shampoo. How many bottles of each shampoo should be made in order to maximize the shampoo maker's profit, if only 840 oz of water and 348 oz of surfactant are available?
   b. Over the last three years, market prices have indicated that Kansas farmers make a profit of \$9000 per acre of wheat planted and \$8000 per acre of sorghum planted. One Kansas farmer manages 500 acres of land, in which he can plant wheat or sorghum. Due to market demands, the farmer needs to harvest at least twice as many acres of wheat as he does acres of sorghum. It takes the farmer 10 hours to harvest an acre of wheat and 15 hours to harvest an acre of sorghum. The window to harvest both the wheat and sorghum is at most 1500 hours. How many acres of wheat and sorghum should be planted to maximize the farmer's profits?
   c. A university football team has made it to a bowl game. The team needs to drive at least 75 people and carry at least 3890 lbs of gear from College Station, Texas to New Orleans, Louisiana. The university has 9-passenger vans which can each carry 550 lbs of gear, while they have 12-passenger vans which can each carry 10 additional pounds of gear. The gas and maintenance costs on a 9-passenger van are \$250 for the trip, and on a 12-passenger van they are \$270. How many of each type of van should the team use to minimize the gas and maintenance costs for the trip?

2. How would you graph the solution set for a linear inequality or a system of linear inequalities, without the use of technology?
   a. Graph the system of linear inequalities, by hand. Identify the solution set.

$$\begin{aligned} &x-2y<-7\\ &2x+y>5\\ &x\geq 0,\ y\geq 0 \end{aligned}$$

   b. Graph the system of linear inequalities, by hand. Identify the solution set.

$$\begin{aligned} y&\leq \frac{3}{2}x\\ -6x+4y&\geq -12\\ x<0,\ y&>0 \end{aligned}$$

   c. Graph the system of linear inequalities, by hand. Identify the solution set.

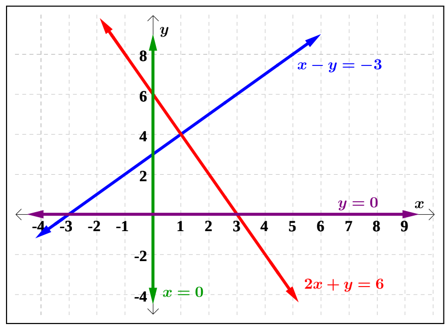
$$\begin{aligned} 8x+5y&\leq 40\\ 2x+3y&\leq 18\\ 1\leq y&\leq 5 \end{aligned}$$

3. Can you describe the process for finding the solution set for a system of linear inequalities?
   a. Write the steps for finding the solution set for a system of linear inequalities.
   b. Is $(4,6)$ a solution to the following system of linear inequalities?

$$\begin{aligned} 8x+7y &\geq 56 \\ -x+\frac{3}{4}y &\geq -3 \end{aligned}$$

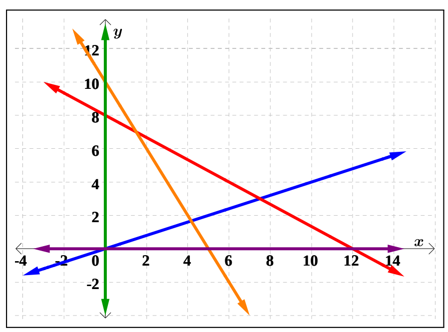
4. What is a corner point and how do you find the coordinates of one?
   a. Explain what a corner point is, in the context of a solution set to a system of linear inequalities.
   b. Determine the corner points of the solution set to the given system of linear inequalities. (The graph of the boundary lines is provided below.)

$$\begin{aligned} x-y &\geq -3 \\ 2x+y &\leq 6 \\ x\geq 0,&\ y\geq 0 \end{aligned}$$



   c. Determine the corner points of the solution set to the given system of linear inequalities. (The graph of the boundary lines is provided below.)

$$\begin{aligned} 2x-5y &\geq 0 \\ 4x+6y &\geq 48 \\ 14x+7y &\geq 70 \\ x\geq 0,&\ y\geq 0 \end{aligned}$$



5. How would you explain the differences between a bounded and unbounded solution set for a system of linear inequalities?
   a. Determine if the solution set for the following system of linear inequalities is bounded or unbounded.

$$\begin{aligned} 3x+4y &\leq 12 \\ -x+y &\leq 1 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

   b. Determine if the solution set for the following system of linear inequalities is bounded or unbounded.

$$\begin{aligned} -x+y &\leq 0 \\ x+y &\leq 3 \end{aligned}$$

   c. Determine if the solution set for the following system of linear inequalities is bounded or unbounded.

$$\begin{aligned} 4x+4y &\geq 12 \\ y &\leq 5 \\ y &\geq x+1 \end{aligned}$$

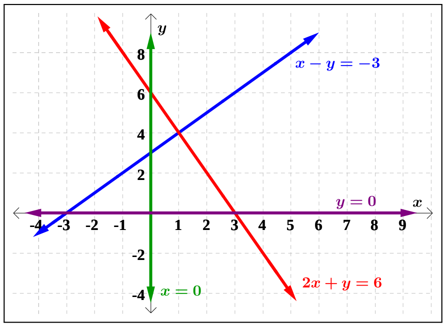
6. How is the solution set to a system of linear inequalities related to a linear programming problem?
   a. Use the graph provided to solve the linear programming problem.

Objective: **Maximize** $R = 9x + 3y$

Subject to: $x - y \geq -3$

$2x + y \leq 6$

$x \geq 0,\ y \geq 0$



   b. **True/False**: In order for a linear programming problem to have a solution, the feasible region must be bounded.
   c. **True/False**: A linear programming problem may have infinitely many solutions represented by the ordered pairs on the line segment connecting two corner points.

7. How would you recognize whether a maximum or a minimum value exists for a given objective function and a particular feasible region? If it exists, could you find the value?
   a. Determine the maximum value, if possible, of the objective function $P = 8x + 5y$, over the bounded feasible region with corner points of $(0,0)$, $(0,8.5)$, $(7,0)$, and $(5,3)$.
   b. Determine the maximum value, if possible, of the objective function $P = 8x + 5y$, over the unbounded feasible region with corner points of $(0,0)$, $(0,8.5)$, and $(5,3)$.
   c. Determine the minimum value of the objective function $P = 4x + 3y$, over the bounded feasible region with corner points of $(1,1)$, $(0,10)$, $(7,0)$, and $(8,6)$.
   d. Determine the minimum value of the objective function $P = 4x + 3y$, over the unbounded feasible region with corner points of $(5,0)$, $(0,8.5)$, and $(2,4)$.

8. Can you solve a linear programming problem, graphically?
   Use the Method of Corners to solve each of the following.
   a. A natural shampoo maker produces two types of shampoo for a locally owned specialty shop. One bottle of gentle shampoo will have 14 oz of water and 4 oz of surfactant, while one bottle of regular shampoo will have 12 oz of water and 6 oz of surfactant. The maker estimates they will generate a profit of \$3 on each bottle of gentle shampoo and \$4 on each bottle of regular shampoo. How many bottles of each shampoo should be made in order to maximize the shampoo maker's profit, if only 840 oz of water and 348 oz of surfactant are available?
   b. Over the last three years, market prices have indicated that Kansas farmers make a profit of \$9000 per acre of wheat planted and \$8000 per acre of sorghum planted. One Kansas farmer manages 500 acres of land in which he can plant wheat or sorghum. Due to market demands, the farmer needs to harvest at least twice as many acres of wheat as he does acres of sorghum. It takes the farmer 10 hours to harvest an acre of wheat, and 15 hours to harvest an acre of sorghum. The window to harvest both the wheat and sorghum is at most 1500 hours. How many acres of wheat and sorghum should be planted to maximize the farmer's profits?
   c. A university football team has made it to a bowl game. The team needs to drive at least 75 people and carry at least 3890 lbs of gear from College Station, Texas to New Orleans, Louisiana. The university has 9-passenger vans which can each carry 550 lbs of gear, while they have 12-passenger vans which can each carry 10 additional pounds of gear. The gas and maintenance costs on a 9-passenger van are \$250 for the trip, and on a 12-passenger van they are \$270. How many of each type of van should the team use to minimize the gas and maintenance costs for the trip?

9. After finding the optimal solution of a linear programming problem, using the Method of Corners, how would you determine if any leftover resources exist?
   a. A salsa company produces and sells two different salsas, mild and hot. Each jar of salsa is sold in a 50 ounce jar. A jar of mild salsa contains 20 ounces of onion and 30 ounces of tomatoes. A jar of hot salsa contains 10 ounces of onion, 20 ounces of tomatoes, and the rest is a pepper mix. The profit on each jar of mild salsa is \$1.20, and the profit on each jar of hot salsa is \$0.90. How many jars of each salsa should the salsa company make to maximize its profits, if 2750 ounces of onion and 4500 ounces of tomatoes are currently available? If it has already been determined the optimal solution is to make 225 jars of hot salsa and no jars of mild salsa, are there any onions or tomatoes leftover?

10. Can you construct the initial simplex tableau for a standard maximization linear programming problem or explain why the problem is not a standard maximization linear programming problem?

a. Write the initial simplex tableau, if possible, for the linear programming problem. If not possible, explain why not.

$$\begin{array}{ll} \text{Objective: } \textbf{Maximize} & P = 15x + 22y + 8z \\ \text{Subject to:} & 5x + 2y + z \le 24 \\ & 4x + 3y + 2z \le 29 \\ & 6x + y + 5z \le 26 \\ & x \ge 0,\ y \ge 0,\ z \ge 0 \end{array}$$

b. Write the initial simplex tableau, if possible, for the linear programming problem. If not possible, explain why not.

$$\begin{array}{ll} \text{Objective:} \textbf{Maximize} & P = 3.6x + 4.8y \\ \text{Subject to:} & 3x \le -24 + 2y \\ & 6x \le y \\ & x \ge 0,\ y \ge 0 \end{array}$$

c. Write the initial simplex tableau, if possible, for the linear programming problem. If not possible, explain why not.

$$\begin{array}{ll} \text{Objective:} \textbf{Maximize} & P = 15x + 12y \\ \text{Subject to:} & 5 \ge 2x + y \\ & 3x + 6y \le 25 \\ & 3x \ge 4y \\ & x \ge 0,\ y \ge 0 \end{array}$$

11. Can you read off the solution of any simplex tableau, and explain how to determine whether or not it represents the optimal solution to the corresponding linear programming problem?

a. Read the solution from the simplex tableau, and determine if the solution is the optimal solution.

$$\begin{array}{c} \begin{array}{cccccc} x & y & s_1 & s_2 & P & \text{constant} \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{1}{2} & -\frac{2}{3} & 0 & 18 \\ 1 & 0 & \frac{2}{3} & -\frac{1}{2} & 0 & 22 \\ \hline 0 & 0 & 2 & 14 & 1 & 9 \end{array}\right] \end{array}$$

b. Read the solution from the simplex tableau, and determine if the solution is the optimal solution.

$$\begin{array}{c} \begin{array}{cccccc} x & y & s_1 & s_2 & P & \text{constant} \end{array} \\ \left[\begin{array}{ccccc|c} -1 & 1 & 0 & 1 & 0 & 2 \\ 4 & 0 & 1 & -2 & 0 & 3 \\ \hline -1 & 0 & 0 & 2 & 1 & 4 \end{array}\right] \end{array}$$

c. Read the solution from the simplex tableau, and determine if the solution is the optimal solution.

$$\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 1 & 0 & 10 & 0 & 120 \\ 1 & 0 & 15 & 0 & 5 & 20 & 0 & 210 \\ 0 & 1 & 3 & 0 & -1 & -10 & 0 & 102 \\ \hline 0 & 0 & -60 & 0 & 40 & 40 & 1 & 380 \end{array}\right]$$

12. Can you describe the process for finding each pivot element in the Simplex Method?

a. State the pivot column, row, and element for the initial tableau below.

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 4 & 1 & 0 & 0 & 20 \\ 3 & 5 & 0 & 1 & 0 & 35 \\ \hline -10 & -12 & 0 & 0 & 1 & 0 \end{array}\right]$$

b. State the pivot column, row, and element for the initial tableau below.

$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & \frac{1}{2} & 1 & 0 & 0 & 0 & 18 \\ 4 & 2 & 0 & 1 & 0 & 0 & 24 \\ 3 & 0 & 0 & 0 & 1 & 0 & 60 \\ \hline -6 & -9 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

c. State the pivot column, row, and element for the initial tableau below.

$$\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} 2 & 4 & \frac{2}{3} & 1 & 0 & 0 & 0 & 40 \\ 4 & 3 & -10 & 0 & 1 & 0 & 0 & 60 \\ 0 & 1 & 9 & 0 & 0 & 1 & 0 & 90 \\ \hline -5 & -3 & -6 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

13. Can you pivot through the Simplex Method, using provided technology?

a. Solve the following linear programming problem, using the Simplex Method. Include the initial and final tableau as part of your work.

Objective: **Maximize** $P = 15x + 42y$

Subject to:
$$28x + 34y \leq 210$$
$$14x + 22y \leq 110$$
$$x \geq 0,\ y \geq 0$$

b. Solve the following linear programming problem, using the Simplex Method. Include the initial and final tableau as part of your work.

Objective: **Maximize** $P = 0.8x + y + 0.7z$

Subject to:
$$x + 3y + 2z \leq 10$$
$$x \leq 8 - 5y - z$$
$$x \geq 0,\ y \geq 0,\ z \geq 0$$

c. Solve the following linear programming problem, using the Simplex Method. Include the initial and final tableau as part of your work.

$$\begin{array}{ll} \text{Objective: } \textbf{Maximize} & P = 3x + y + 8z \\ \textbf{Subject to:} & 6x + 8y + z \le 118 \\ & 220 \ge 5x + 10y + z \\ & x \ge 0,\ y \ge 0,\ z \ge 0 \end{array}$$

14. Can you explain each tableau of the Simplex Method in terms of its corresponding corner point in the Method of Corners?

a. Compute the corner points of the system below. Then, use the Simplex Method to show that at each pivot one of the corner points is represented.

$$\begin{array}{ll} \text{Objective: } \textbf{Maximize} & P = 6x + 4y \\ \text{Subject to:} & 5x + y \le 25 \\ & -3x + 4y \le 12 \\ & x \ge 0,\ y \ge 0 \end{array}$$

15. Can you solve a linear programming problem, algebraically?
Use the Simplex Method to solve each of the following.
   a. Over the last three years, market prices have indicated that Kansas farmers make a profit of \$9000 per acre of wheat planted and \$8000 per acre of sorghum planted. One Kansas farmer manages 500 acres of land in which he can plant wheat or sorghum. Due to market demands, the farmer needs to harvest at least twice as many acres of wheat as he does acres of sorghum. It takes the farmer 10 hours to harvest an acre of wheat, and 15 hours to harvest an acre of sorghum. The window to harvest both the wheat and sorghum is at most 1500 hours. How many acres of wheat and sorghum should be planted to maximize the farmer's profits?
   b. A meal prep service creates pre-made meals for single people on the go. The "College Student" meal includes 2 protein sources and 1 vegetable. The "Retire" meal includes 2 protein sources, 1 vegetable, and 1 carbohydrate. The "Single on the Go" meal includes 3 protein sources, 1 vegetable, and 1 carbohydrate. The service profits \$3 on the "College Student" meal, \$4 on the "Retiree," and \$6 on the "Single on the Go." The service has 800 protein sources, 400 vegetables, and 200 carbohydrates in stock. How many of each type of meal should be made and sold, in order to maximize the service's profit?

16. After finding the optimal solution of a linear programming problem using the Simplex Method, how would you determine if there are any leftover resources?

a. State the value of any nonzero slack variables in the following problem and describe what they tell you about the leftover resources.

A local construction company builds three different models of homes. The luxury model takes 1000 hours to build, 4 hours to paint the interior, 10 hours to brick the outside, and 2 hours to run the electrical wiring. The standard model takes 600 hours to build, 2 hours to paint the interior, 5 hours to brick the outside, and 1 hour to run the electrical wiring. The basic model takes 300 hours to build, 2 hours to paint the interior, 5 hours to brick the outside, and 1 hour to run the electrical wiring. The construction company estimates they have available at most 2,800,000 hours to build, 10,000 hours to paint the interior, 25,000 to brick, and 6,000 hours for wiring. They will earn a profit of \$38,000 on each luxury model, \$22,000 on each standard model, and \$12,000 on each basic model. How many homes of each model should be built and sold to maximize the construction company's profit?

IV

Chapter 4

4. Basic Probability and Applications

In this chapter we are going to discuss experiments and probability.

The authors recommend all students are comfortable with the topics below prior to beginning this chapter. If you would like additional support on any of these topics, please refer to the Appendix.

A.1 - Integers

A.1 - Fractions

A.1 - Simplifying Fractions

A.1 - Decimals

A.1 - Properties of Real Numbers

A.2 - Using Variables and Algebraic Symbols

A.2 - Simplifying Expressions

A.2 - Translating an English Phrase to an Algebraic Expression or Equation

A.2 - Solving Linear Equations with One Variable

A.2 - Using Problem-Solving Strategies

A.2 - Graphing Inequalities on the Number Line and Interval Notation

4.1 Mathematical Experiments

© Photo by Vanessa Coffelt, 2020

Toss a thumb tack one time. Do you think the tack will land with the point up or the point down?

We cannot predict which way the tack will land before we toss it. Sometimes it will land with the point up, and other times it will land with the point down. Tossing a tack is a **random experiment**, as we cannot predict what the outcome will be. However, we do know that there are only two possible outcomes for each trial of the experiment: lands "point up" or lands "point down." If we repeat the experiment of tossing the tack many times, we might be able to guess how likely it is that the tack will land a particular way.

Learning Objectives:
In this section, you will learn about concepts related to mathematical experiments, including sample spaces and events. Upon completion you will be able to:

- State the sample space and events of an experiment.
- Classify an event as simple, certain, or impossible.
- State the number of outcomes, simple events, and total events of an experiment.
- Use a tree diagram to determine the outcomes of an experiment.
- Shade a Venn diagram to represent the union, intersection, and/or complement of events.
- Justify whether two events are mutually exclusive.
- Construct a symbolic notation for a verbal description of an operation of events using unions, intersections and/or complements.
- Construct a verbal description of an operation of events given in symbolic notation.

Defining a Sample Space and Events

If we roll a die and note the number rolled, pick a card from a deck of playing cards and note the suit, or randomly select a person and observe their hair color, we are conducting a **mathematical experiment**. We will begin with some terminology relating to mathematical experiments.

Definition

- A **random experiment** is an activity or an observation whose results cannot be predicted ahead of time.
- **Outcomes** are the results of a random experiment.
- The **sample space**, **S**, is the set of all possible outcomes for a random experiment.

As previously stated, the experiment in the introduction to this section, tossing a tack and noting how it lands, is a random experiment. The possible outcomes for the experiment are the tack lands "point up" or the tack lands "point down," so the sample space is $\mathbf{S} = \{\text{point up}, \text{point down}\}$.

Many mathematical experiments involve selecting a card from a standard deck of cards or rolling a standard die.

To help the reader, a **standard deck of cards** is described below. In each deck

- There are 52 cards.
- There are 2 colors - Black (♠ and ♣) and Red (♡ and ♢).
- There are 4 suits - Hearts, Clubs, Spades, and Diamonds (♡,♣,♠,♢).
- Each suit has 13 ranks - Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen (Q), and King (K).
- There are 12 face cards - Jack (J), Queen (Q), and King (K). (All the cards with faces).

Figure 4.1.2: An image of all 52 cards in a standard deck of cards.

A **standard die** is named by its number of sides, and the sides are numbered as follows:

- Four-sided die: $\mathbf{S} = \{1,2,3,4\}$
- Five-sided die: $\mathbf{S} = \{1,2,3,4,5\}$
- Six-sided die: $\mathbf{S} = \{1,2,3,4,5,6\}$

  $\vdots$
- 20-sided die: $\mathbf{S} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$

N *An 'n'-sided die is numbered 1 to n.*

Often when we roll two distinguishable dice, we create a table with all outcomes of the first die listed in the first column of the table and all outcomes of the second die listed as the first row of the table. Each intersection of these rows and columns represent a possible outcome of the two dice in a single roll of the experiment.

Suppose we roll two standard six-sided dice, the first die is green and the second die is blue. The outcomes are shown, as follows, in **Table 4.1**.

	1	2	3	4	5	6
1	(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
2	(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
3	(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
4	(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
5	(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
6	(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

Table 4.1: Two Standard Six-Sided Dice Chart

N *Each outcome shown in the table above is unique. For example, the outcomes* $(1,2)$ *and* $(2,1)$ *are different outcomes.*

Example 1 State the sample space for each given experiment.

a. Drawing a card from a standard 52-card deck and noting the suit.
b. Tossing a 2-cent coin and noting the side facing up.
c. Rolling two distinguishable standard six-sided dice and noting the sum.

Solution:

a. Drawing a card from a standard 52-card deck and noting the *suit*.

Because a standard 52-card deck has four suits, the sample space would be

$$\mathbf{S} = \{\text{clubs, diamonds, hearts, spades}\} = \{\clubsuit,\ \diamondsuit,\ \heartsuit,\ \spadesuit\}.$$

b. Tossing a 2-cent coin and noting the *side facing up*.

A 2-cent coin has two sides, normally noted as “heads” and “tails.” Thus, the sample space would be

$$\mathbf{S} = \{\text{heads}, \text{tails}\}.$$

c. Rolling two distinguishable standard six-sided dice and noting the *sum*.

The chart for rolling two distinguishable standard six-sided dice is shown in **Table 4.1**. Notice all outcomes in each diagonal, as you move from lower left to upper right, have the same sum. Because we are noting the sum of the rolls and not the numbers actually rolled, the sample space would be

$$\mathbf{S} = \{2,3,4,5,6,7,8,9,10,11,12\}.$$

■

When writing a sample space associated with an experiment, it is important to pay attention to what is 'noted' when an action is taken in the experiment. For instance, when a card is drawn from a standard 52-card deck, the experiment may be noting the color, the suit, the rank, or one of many other characteristics. The sample space is dependent upon the observation being made.

Try It # 1:
State the sample space for rolling a standard four-sided die and a standard five-sided die, noting the values on each die.

All outcomes of the experiment are not always the focus of a scenario. Sometimes only a portion of the outcomes that occur becomes the center of the discussion.

Definition

- An **event**, **E**, is a subset of the sample space (a collection of outcomes from an experiment).
- A **simple event** is an event with *exactly one* outcome.
- The **certain event** is the event containing *all* outcomes of an experiment (the entire sample space). The certain event is often denoted by **S**.
- An **impossible event** is an event containing *no* outcomes. An impossible event is often denoted as { } or $\varnothing$.

$\{\varnothing\}$ *does not represent an impossible event.*

When notating events mathematically, we can either list the outcomes satisfying the conditions of the event in a comma-separated list surrounded by curly brackets or we can describe the event in words.

Returning to our section introduction of the tack experiment, the sample space was **S** = {point up, point down}. Then, the two simple events are

$$E_1 = \{\text{point up}\} \text{ and } E_2 = \{\text{point down}\}.$$

The certain event would be

$$\{\text{point up, point down}\} = \mathbf{S}.$$

An impossible event means there are no outcomes in the sample space that satisfy the conditions of the event. In our sample space, one impossible event would be the tack lands both point up and point down *at the same time.* Because this is not possible to occur, we could write

$$\{\ \} \text{ or } \varnothing.$$

If we list all *possible* events, we consider all subsets of the sample space:

$$\{\}, \{\text{point up}\}, \{\text{point down}\}, \{\text{point up, point down}\}$$

■ **Example 2** An experiment consists of drawing a letter from a bag containing the first three letters of the English alphabet. The sample space would be $\mathbf{S} = \{a, b, c\}$. Use this information to determine the following.

a. The certain event
b. All simple events
c. An example of an impossible event
d. All possible events

Solution:

a. The certain event is the set containing all outcomes of an experiment, which is equal to the sample space;

$$\mathbf{S} = \{a, b, c\}$$

b. Simple events are subsets of the sample space containing exactly one outcome, so we have

$$E_1 = \{a\},\ E_2 = \{b\},\ E_3 = \{c\}$$

c. An impossible event contains no outcomes, $\{\}$. For this example, we could say

The event "the letter 'd' is drawn."

d. An event is a subset of the sample space. So far we have the following events:

$$\{a, b, c\},\ \{a\},\ \{b\},\ \{c\},\ \{\}$$

Are there any other events? In other words, are there other subsets of $\{a, b, c\}$?

The answer is yes. We could have two outcomes in a single event, such as the event "the letter 'a' or the letter 'b' is drawn." The two-outcome events would be

$$\{a, b\} \text{ or } \{b, c\} \text{ or } \{a, c\}$$

Thus, the list of *all possible events* of the sample space would be

$$\{\},\ \{a\},\ \{b\},\ \{c\},\ \{a, b\},\ \{b, c\},\ \{a, c\},\ \{a, b, c\}$$

■

In general, for a sample space with n outcomes,

- There are n simple events.
- The total number of possible events is 2^n.

In the previous example, the sample space contained 3 outcomes. Thus, there were $2^3 = 8$ *total possible events.*

■ **Example 3** Consider the experiment of rolling a standard five-sided die, noting the number rolled, and then tossing a 2-cent coin, noting the side facing up.

a. Write the sample space for the experiment.
b. State the total number of possible events.
c. List the outcomes in the event, F, "an even number is rolled."

Solution:

This experiment involves a multiple step process with two observations (one on the die *and* one on the coin). First, we observe the number rolled on the standard five-sided die, which has five possible outcomes, $1-5$. Then, we observe the side of the 2-cent coin landing up, which has two possible outcomes, heads and tails. (For ease in notation, let $H :=$ tossing a heads on the coin and $T :=$ tossing a tails on the coin.) So, all together, there are

$$\underset{\text{die}}{\underline{5}} \cdot \underset{\text{coin}}{\underline{2}} = 10 \text{ possible outcomes.}$$

As with rolling two distinguishable dice, the number of outcomes of this experiment is dependent on two observations. A visual representation is helpful to determine all of the possible outcomes. We could use a table, but in doing so we are limited to two observations in an experiment. Another visual representation that can be used with two or more observations is a **tree diagram**.

In a tree diagram, you draw a branch starting from a single point to each possible outcome of the first observation. Here, these are the numbers $1-5$. Then, from the end of each branch, you draw a different branch leading to each possible outcome of the second observation. Here, these are H and T. If there are more than two observations, you continue in a similar manner, until all observations have been considered. Then, each outcome of the experiment can be found by following each branch path from the single starting point of the tree.

The tree diagram in **Figure 4.1.3**, below, gives us all 10 possible outcomes for the experiment.

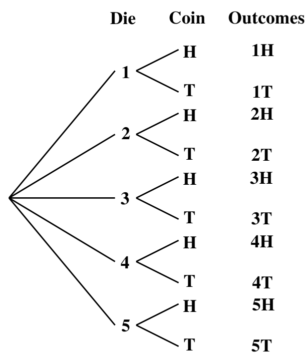


Figure 4.1.3: A tree diagram for the experiment in Example 3.

a. The sample space is

$$\mathbf{S} = \{\text{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T}\},$$

which can also be written as

$$\mathbf{S} = \{\text{(1,H), (1,T), (2,H), (2,T), (3,H), (3,T), (4,H), (4,T), (5,H), (5,T)}\}.$$

b. The sample space has $n = 10$ outcomes. Thus, the total number of events possible is equal to $2^{10} = 1024$. (This is a lot of events to list ... you would not want to try to list and count them all to arrive at this answer.)

c. The event "an even number is rolled" contains all outcomes from **S**, which include an even number. Therefore,

$$F = \{\text{2H, 2T, 4H, 4T}\} = \{\text{(2,H), (2,T), (4,H), (4,T)}\}$$

N *F is not equal to $\{2,4\}$, as the experiment was a two-step process and each outcome includes both a number rolled and a face visible.*

■

Try It # 2:
Consider the experiment of drawing a card from a standard 52-card deck, noting the color, rolling a standard eight-sided die, noting whether an even or odd number is rolled, and picking a letter from the word MATH, noting the letter chosen.

a. Write the sample space for the experiment.
b. State the total number of events possible.
c. List the outcomes in the event, F, "the letter 'M' is drawn from MATH."

Operating on Events

Early in our mathematical education we are taught how to perform operations on real numbers, which result in another real number. In Chapter 1, we saw how to use operations to transform and combine matrices, in order to form other matrices. Next we will describe how operations (**complements**, **intersections**, and **unions**) can be performed on events within a sample space, which result in another event.

Using Venn Diagrams to Visualize Operations

In the late 1800s, an English logician named John Venn developed a method to represent relationships between sets. He represented these relationships using diagrams, which are now known as **Venn diagrams**.

Definition

A **Venn diagram** represents the outcomes of an event as the interior of a circle. Often two or more circles are enclosed in a rectangle, where the rectangle represents the sample space. ▪

We will use Venn diagrams to illustrate the results of operations on events.

Definition

The **complement** of event A is the *event* that contains all outcomes in the sample space which are **NOT** in event A. The complement of event A is denoted by $\boldsymbol{A^C}$. ▪

Depending on the number of events defined, some Venn diagrams corresponding to A^C are shaded in **Figures 4.1.4**, **4.1.5**, and **4.1.6.** In each figure the region(s) in **S**, but outside of A is/are shaded.

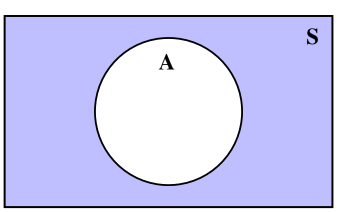


Figure 4.1.4: A circle representing one event, A, inside a rectangle representing the sample space, **S**.

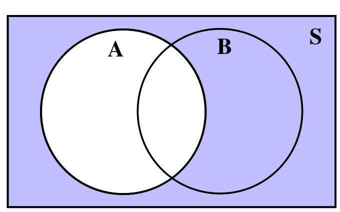


Figure 4.1.5: Two circles representing two events, A and B, inside a rectangle representing the sample space, **S**.

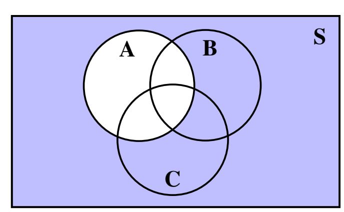


Figure 4.1.6: Three circles representing three events A, B, and C, inside a rectangle representing the sample space, **S**.

Definition

The **intersection** of events A and B is the *event* that contains all outcomes in the sample space which **BOTH** A **AND** B have in common. The intersection of events A and B is denoted by $\boldsymbol{A \cap B}$.

Depending on the number of events defined, some Venn diagrams corresponding to $A \cap B$ are shaded in **Figures 4.1.7** and **4.1.8**. The overlapping region(s) of A and B is/are shaded.

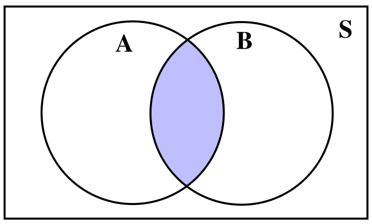


Figure 4.1.7: Two circles representing two events, A and B, inside a rectangle representing the sample space, **S**.

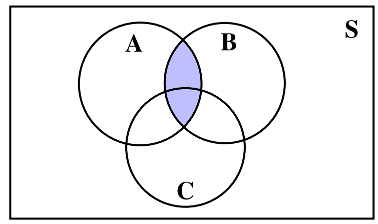


Figure 4.1.8: Three circles representing three events A, B, and C, inside a rectangle representing the sample space, **S**.

N *Intersection of events is commutative:* $A \cap B = B \cap A$.

Definition

If two events, A and B, have no outcomes in common, then their intersection is impossible, $\boldsymbol{A \cap B = \varnothing}$; the events A and B are called **mutually exclusive** events.

Venn Diagrams corresponding to mutually exclusive events A and B are shown in **Figures 4.1.9** and **4.1.10**. Because $A \cap B = \varnothing$, no region is shaded in either figure.

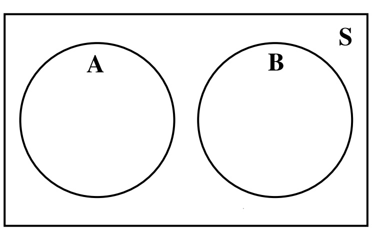


Figure 4.1.9: Two circles representing two events, A and B, inside a rectangle representing the sample space, **S**.

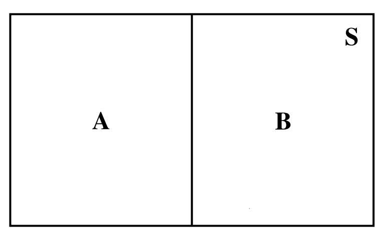


Figure 4.1.10: A rectangle representing the sample space, **S**, is divided down the middle into events A and B.

Definition

The **union** of events A and B, is the *event* that contains all outcomes in the sample space which are in A **OR** are in B **OR** are in both A and B. The union of events A and B is denoted by $\boldsymbol{A \cup B}$. ■

In the English language, the word "or" has two interpretations: inclusive and exclusive. Inclusive means that at least one of the options is true, whereas exclusive means exactly one of the options is true. In mathematics, we interpret "or" as inclusive.

Depending on the number of events defined, some Venn diagrams corresponding to $A \cup B$ are shaded in **Figures 4.1.11**, **4.1.12**, and **4.1.13**. All regions of A and all regions of B are shaded.

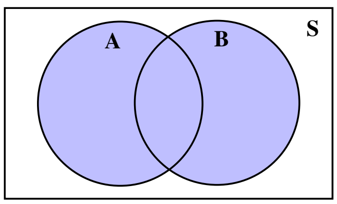


Figure 4.1.11: Two circles representing two events, A and B, inside a rectangle representing the sample space, **S**.

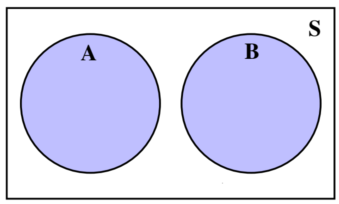


Figure 4.1.12: Two circles representing two mutually exclusive events, A and B, inside a rectangle representing the sample space, **S**.

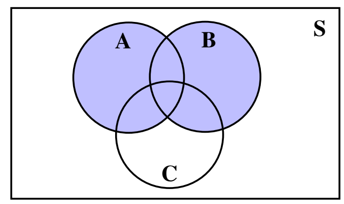


Figure 4.1.13: Three circles representing three events A, B, and C, inside a rectangle representing the sample space, **S**.

N *Union of events is commutative:* $A \cup B = B \cup A$.

As with real numbers and matrices, we can combine events, using operations, to produce another event. Again, Venn diagrams can be used to visually highlight the operations and resulting event.

For the remainder of the section, we will focus on the relationships between *two* events. In a Venn diagram depicting a sample space, **S**, with two events, there are four mutually exclusive regions, a, b, c, and d. (See **Figure 4.1.14**.)

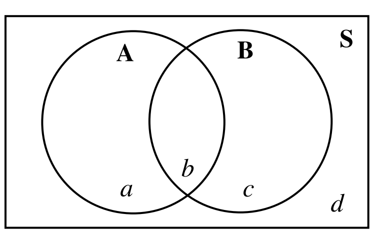


Figure 4.1.14: Two circles representing events A and B intersecting inside a rectangle for the sample space, **S**. Region a is in circle A only, region b is the overlap of A and B, region c is in circle B only, and region d is outside circles A and B, but still within the rectangle.

$$A = \{a, b\}$$
$$B = \{b, c\}$$
$$\mathbf{S} = \{a, b, c, d\}$$

▪ **Example 4** Let A and B be two events of the sample space, **S**, as shown in **Figure 4.1.15**. Shade the region(s) of the two-circle Venn diagram corresponding to the event resulting from the given operation(s).

a. B^C
b. $A \cup B^C$
c. $A^C \cap B$
d. $(A \cup B) \cap B^C$

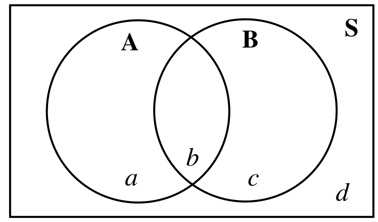


Figure 4.1.15: Unshaded Two-Circle Venn Diagram.

Solution:

a. B^C is the event containing all outcomes in the sample space which are not in B. Thus, $B^C = \{a, d\}$, which is illustrated by the shading in **Figure 4.1.16**.

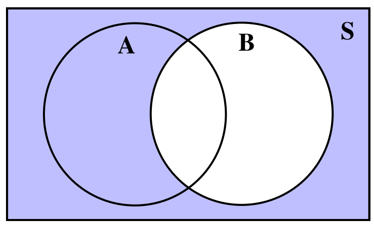


Figure 4.1.16: A visual representation of B^C.

b. $A \cup B^C$ is the event containing all outcomes in the sample space which are in A OR in B^C OR are in both A and B^C.

$$\begin{array}{ccc} A = \{a, b\} & & \\ & \Longrightarrow & A \cup B^C = \{a, b, d\} \\ B^C = \{a, d\} & & \uparrow \\ & & \text{combine together} \end{array}$$

Region 'a' is in both A and B^C but is only listed once when writing the union. Repeating the same outcome (region) is unnecessary and may cause difficulty in later sections.

$A \cup B^C$ is illustrated in **Figure 4.1.17**.

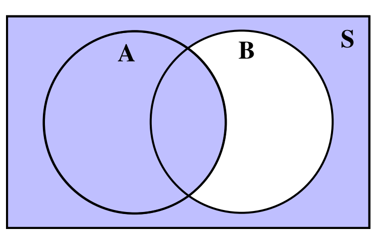


Figure 4.1.17: A visual representation of $A \cup B^C$.

c. $A^C \cap B$ is the event containing all outcomes in the sample space which BOTH A^C AND B have in common.

$$\begin{array}{ccc} A^C = \{c, d\} & & \\ & \Longrightarrow & A^C \cap B = \{c\} \\ B = \{b, c\} & & \uparrow \\ & & \text{in both} \end{array}$$

$A^C \cap B$ is illustrated in **Figure 4.1.18**.

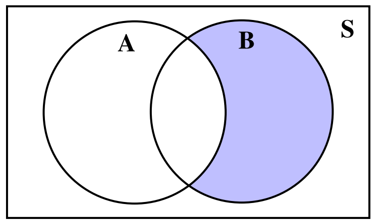


Figure 4.1.18: A visual representation of $A^C \cap B$.

d. For $(A \cup B) \cap B^C$, we use order of operations (as with real numbers) and start within the parentheses first, $A \cup B$.

$$\begin{array}{ccc} A = \{a, b\} & & \\ & \Longrightarrow & A \cup B = \{a, b, c\} \\ B = \{b, c\} & & \uparrow \\ & & \text{combine together} \end{array}$$

Then, we use the above result and move to the next operation, $(A \cup B) \cap B^C$.

$$\begin{array}{ccc} A \cup B = \{a, b, c\} & & \\ & \Longrightarrow & (A \cup B) \cap B^C = \{a\} \\ B^C = \{a, d\} & & \uparrow \\ & & \text{in both} \end{array}$$

$(A \cup B) \cap B^C$ is illustrated in **Figure 4.1.19**.

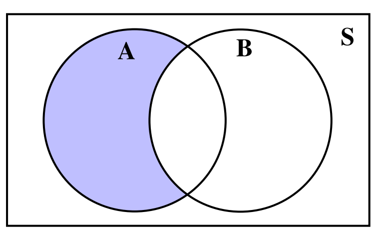


Figure 4.1.19: A visual representation of $(A \cup B) \cap B^C$.

■

■ **Example 5** Let A and B be two events of the sample space, **S**, as shown in **Figure 4.1.20**. Shade the region(s) of the two-circle Venn diagram corresponding to the event resulting from the given operation(s).

a. $A^C \cup B^C$
b. $A^C \cap B^C$
c. $(A \cup B)^C$
d. $(A \cap B)^C$

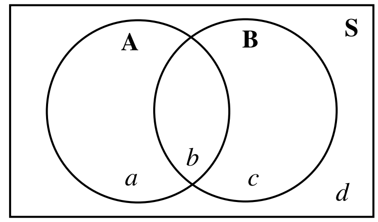


Figure 4.1.20: Unshaded Two-Circle Venn Diagram

Solution:

a.

$$A^C = \{c,d\} \qquad B^C = \{a,d\} \quad \Longrightarrow \quad \underset{\substack{\uparrow \\ \text{combine together}}}{A^C \cup B^C} = \{a,c,d\}$$

$A^C \cup B^C$ is illustrated in **Figure 4.1.21**.

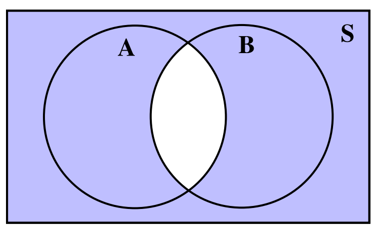


Figure 4.1.21: A visual representation of $A^C \cup B^C$.

b.

$$A^C = \{c,d\} \qquad B^C = \{a,d\} \quad \Longrightarrow \quad \underset{\substack{\uparrow \\ \text{in both}}}{A^C \cap B^C} = \{d\}$$

$A^C \cap B^C$ is illustrated in **Figure 4.1.22**.

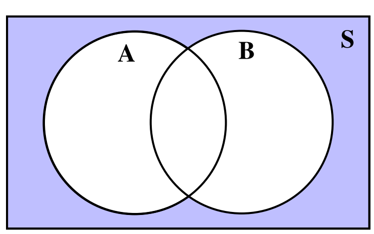


Figure 4.1.22: A visual representation of $A^C \cap B^C$.

c.

$$A = \{a, b\} \qquad\qquad B = \{b, c\}$$

$$\Longrightarrow \quad \underbrace{A \cup B = \{a, b, c\}}_{\uparrow \text{ combine together}}$$

$$\underbrace{(A \cup B)^C = \{d\}}_{\uparrow \text{ what's not in } A \cup B, \text{ but still in } \mathbf{S}}$$

$(A \cup B)^C$ is illustrated in **Figure 4.1.23**.

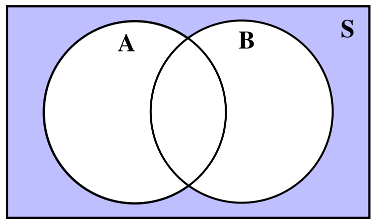


Figure 4.1.23: A visual representation of $(A \cup B)^C$.

d.

$$A = \{a, b\} \qquad\qquad B = \{b, c\}$$

$$\Longrightarrow \quad \underbrace{A \cap B = \{b\}}_{\uparrow \text{ in both}}$$

$$\underbrace{(A \cap B)^C = \{a, c, d\}}_{\uparrow \text{ what's not in } A \cap B, \text{ but still in } \mathbf{S}}$$

$(A \cap B)^C$ is illustrated in **Figure 4.1.24**.

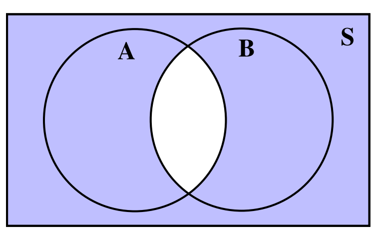


Figure 4.1.24: A visual representation of $(A \cap B)^C$.

■

In the previous example, notice the resulting events in parts **a** and **d** were identical and the resulting events in parts **b** and **c** were identical. These results illustrate **DeMorgan's Laws**.

Theorem 4.1 **DeMorgan's Laws**

If A and B are two events of the sample space, **S**, then

$$(A \cap B)^C = A^C \cup B^C$$
and
$$(A \cup B)^C = A^C \cap B^C$$

Try It # 3:
Let A and B be two events of the sample space, **S**. Shade the region(s) of a two-circle Venn diagram corresponding to the event resulting from the given operation(s).

a. $(A \cup B^C)^C$
b. $A \cup (B^C \cap B)$

Converting between Verbal and Symbolic Notation

When defining the complement, intersection, and union of events, the symbolic notation *and* a verbal descriptor were both given.

	Symbolic	Verbal
Complement:	A^C	**NOT** A
Intersection:	$A \cap B$	A **AND** B
Union:	$A \cup B$	A **OR** B

Suppose we return to Example 3 of this section, where a standard five-sided die was rolled and a 2-cent coin was tossed. The sample space was $\mathbf{S} = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T\}$
$= \{(1,H),(1,T),(2,H),(2,T),(3,H),(3,T),(4,H),(4,T),(5,H),(5,T)\}$.

If the following events are defined:

A := the event "an even number is rolled" $= \{2H, 2T, 4H, 4T\} = \{(2,H),(2,T),(4,H),(4,T)\}$
B := the event "a number larger than 2 is rolled" $= \{3H, 3T, 4H, 4T, 5H, 5T\}$
$= \{(3,H),(3,T),(4,H),(4,T),(5,H),(5,T)\}$
D := the event "heads is visible on the coin" $= \{1H, 2H, 3H, 4H, 5H\} = \{(1,H),(2,H),(3,H),(4,H),(5,H)\}$
F := the event "a 1 is rolled" $= \{1H, 1T\} = \{(1,H),(1,T)\}$,

then we can describe A^C as

the event "an even number is *not* rolled."

The event A^C would be all outcomes in the sample space which do not include an even number being rolled on the die.

$$A^C = \{1H, 1T, 3H, 3T, 5H, 5T\} = \{(1,H), (1,T), (3,H), (3,T), (5,H), (5,T)\}$$

$B \cap D$ can be described as

the event "a number larger than 2 is rolled *and* heads is visible on the coin."

This event includes all outcomes that events B and D have in common.

$$B \cap D = \{3H, 4H, 5H\} = \{(3,H), (4,H), (5,H)\}$$

$B \cap F$ can be described as

the event "a number larger than 2 is rolled *and* a 1 is rolled."

The event $B \cap F$ would be all outcomes that events B and F have in common. Due to the fact that there exists no common outcomes, $B \cap F = \varnothing$. (It is not possible in a single roll to have both a '1' and a number greater than 2.) We can also say events B and F are mutually exclusive.

$F \cup D$ can be described as

the event "a 1 is rolled *or* heads is visible on the coin."

This event includes all outcomes that are in event F or are in event D or are in both events F and D.

$$F \cup D = \{1H, 1T, 2H, 3H, 4H, 5H\} = \{(1,H), (1,T), (2,H), (3,H), (4,H), (5,H)\}$$

■ **Example 6** An experiment consists of rolling two standard six-sided die, the first die is green and the second die is blue.

Let

$A :=$ the event "a 4 is rolled on the green die"
$B :=$ the event "doubles are rolled"
$D :=$ the event "a sum of 6 is rolled"

a. List the outcomes in A, B, and D.
b. Describe and list the outcomes of the event $B \cap D^C$.
c. Describe the event $(A \cup D)^C$.
d. Write the symbolic notation for the event "a 4 is rolled on the green die, but no doubles are rolled."
e. Write an event which is non-empty and mutually exclusive to event A^C.

Solution:

The sample space can be determined from the dice chart in **Table 4.2**.

	1	2	3	4	5	6
1	(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
2	(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
3	(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
4	(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
5	(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
6	(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

Table 4.2: Two Standard Six-Sided Dice Chart

a. $A = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$

$B = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

$D = \{(5,1),(4,2),(3,3),(2,4),(1,5)\}$

b. $B \cap D^C$ is the event "doubles are rolled, but a sum of 6 is not rolled." The "but" is used to verbalize the "and" with a "not," as it sounds more natural when speaking. The outcomes of the event are

$$B \cap D^C = \{(1,1),(2,2),(4,4),(5,5),(6,6)\}$$

c. From DeMorgan's Laws we know $(A \cup D)^C = A^C \cap D^C$. The latter is easier to verbalize, so $(A \cup D)^C$ is the event "a 4 is not rolled on the green die and a sum of 6 is also not rolled."

d. The event "a 4 is rolled on the green die" is event A. The event "no doubles are rolled" is event B^C. Thus, the event "a 4 is rolled on the green die, but no doubles are rolled" would become $A \cap B^C$.

e. It is possible to write many events which are mutually exclusive to event A^C. One example is A. Another example is "a 3 is rolled on the green die."

■

Try It # 4:

Consider the experiment of drawing a card from a standard 52-card deck, noting the suit, and selecting a letter from the word LUNCH, noting the letter drawn.

Let

A := the event "a heart is drawn"

B := the event "a 'U' is selected"

D := the event "a consonant is selected"

a. List the sample space, **S**, for the experiment described.
b. Describe and list the outcomes of the event $A \cap B^C$.
c. Write the symbolic notation for the event "a consonant is selected or a heart is drawn."
d. Are events B and D mutually exclusive? Why or why not?

Try It Answers

1. $\mathbf{S} = \{(1,1),(1,2),(1,3),(1,4),(1,5),$
$(2,1),(2,2),(2,3),(2,4),(2,5),$
$(3,1),(3,2),(3,3),(3,4),(3,5),$
$(4,1),(4,2),(4,3),(4,4),(4,5)\}$

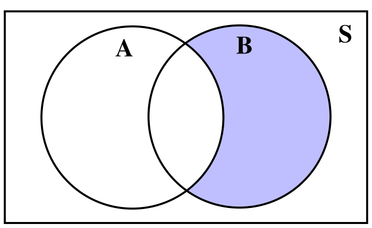
2. $B :=$ a black card is drawn
$R :=$ a red card is drawn
$E :=$ an even number is rolled
$D :=$ an odd number is rolled

a. **S** = { BEM, BEA, BET, BEH, BDM, BDA, BDT, BDH, REM, REA, RET, REH, RDM, RDA, RDT, RDH }
= { (B,E,M), (B,E,A), (B,E,T), (B,E,H), (B,D,M), (B,D,A), (B,D,T), (B,D,H), (R,E,M), (R,E,A), (R,E,T), (R,E,H), (R,D,M), (R,D,A), (R,D,T), (R,D,H) }

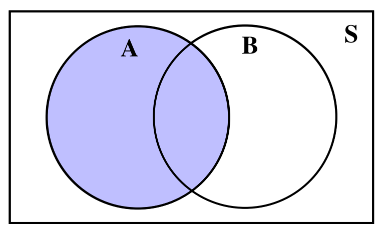
b. $2^{16} = 65,536$

c. F = {BEM, BDM, REM, RDM} = { (B,E,M), (B,D,M), (R,E,M), (R,D,M) }

3. **a.**



b.



4. $h :=$ a heart is drawn
$d :=$ a diamond is drawn
$c :=$ a club is drawn
$s :=$ a spade is drawn

a. **S** = { hL, hU, hN, hC, hH, dL, dU, dN, dC, dH, cL, cU, cN, cC, cH, sL, sU, sN, sC, sH }
= { (h,L), (h,U), (h,N), (h,C), (h,H), (d,L), (d,U), (d,N), (d,C), (d,H), (c,L), (c,U), (c,N), (c,C), (c,H), (s,L), (s,U), (s,N), (s,C), (s,H) }

b. $A \cap B^C$ = {hL, hN, hC, hH} = {(h,L), (h,N), (h,C), (h,H)}
$\Longrightarrow$ the event "a heart is drawn, but 'U' is not selected"

c. $D \cup A$

d. B and D are mutually exclusive, because $B \cap D = \varnothing$. You cannot select one letter that is both a 'U' and a consonant from the word LUNCH.

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 5, state the sample space for the given experiment.

1. A single card is drawn from a standard 52 card deck, noting the color of the selected card.
2. A single card is drawn from a standard 52 card deck, noting the rank.
3. A standard 20-sided die is rolled, noting the number showing.
4. A marble is drawn from a bag containing 20 identical red marbles, 30 identical blue marbles, and 5 identical green marbles, noting the color of the selected marble.
5. The letters in the word "HOWDY" are written on identical pieces of paper and placed in a cup. A single piece of paper is drawn from the cup, and the letter on the paper is noted.

For Exercises 6 - 8, use the given sample space to determine

a. All simple events

b. The total number of possible events.

c. The list of outcomes in the event, E, described.

6. S = $\{a, e, i, o, u\}$ and E is the event "a letter in the word *house* is chosen."
7. S = $\{2, 4, 6, 8\}$ and E is the event "a multiple of 4 is chosen."
8. S = $\{for,\ against,\ undecided\}$ and E is the event "a person is not *for* the amendment."

For Exercises 9 - 11, for the given pair of events, explain whether or not they are mutually exclusive.

9. $E = \{c, h, r, i, s\}$ and $F = \{d, a, v, e\}$
10. $E = \{c, l, o, s, e\}$ and $F = \{o, p, e, n\}$
11. $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $F = \{2, 0, 1, 9\}$

For Exercises 12 - 17, let E and F be two events of the sample space, S. Shade the region(s) of a two-circle Venn diagram corresponding to the event resulting from the given operation(s).

12. E^C
13. F^C
14. $E \cup F$
15. $E \cap F$
16. $E^C \cup F^C$
17. $E^C \cap F^C$

For Exercises 18 - 23, let S = $\{m, n, p, q, r, t, w, x\}$ with events

$A = \{m, p, r\}$ $\quad B = \{t, w, x\}$ $\quad D = \{p, q, r, t\}$ $\quad E = \{r, t\}$

Describe the given event as a subset.

18. A^C
19. E^C
20. $A \cup B$
21. $B \cup D$
22. $D \cap E$
23. $B \cap E$

Intermediate Skills Practice (Answers)

For Exercises 24 - 28, state the sample space for the given experiment.

24. A standard three-sided die and a standard six-sided are rolled, noting the values on each die.
25. A spinner divided into four equal regions (red, blue, green, and yellow) is spun, noting the color, and then a two-sided coin is tossed, noting the side landing up.
26. The numbers 1, 2, 3, and 4 are written on equal sized pieces of paper and placed in a box. Two pieces of paper are drawn at the same time and the sum is noted.
27. A coin is tossed three times, and the side landing up is noted.
28. A letter is selected at random from the word SOCIAL, noting the letter, and then a card is drawn from a standard 52-card deck, noting the suit.

For Exercises 29 - 31, use the given information to determine

a. The total number of simple events.

b. The total number of possible events.

c. The list of outcomes in the event, E, described.

29. S = $\{(1, heads), (1, tails), (2, heads), (2, tails), (3, heads), (3, tails), (4, heads), (4, tails), (5, heads), (5, tails)\}$ and E is the event "a *heads* and a *tails* is chosen."
30. S = $\{(heart, queen), (heart, king), (spade, queen), (spade, king)\}$ and E is the event "a heart or a queen is chosen."
31. S = $\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$ and E is the event "a one and an even sum is chosen."

For Exercises 32 - 36, let E and F be two events of the sample space, S. Shade the region(s) of a two-circle Venn diagram corresponding to the event resulting from the given operation(s).

32. $E^C \cup F$
33. $(E \cup F) \cap F$
34. $E \cap F^C$
35. $(E \cap F)^C$
36. $\left(E^C \cap F^C\right)^C$

For Exercises 37 - 41, let E and F be two **mutually exclusive** events of the sample space, S. Shade the region(s) of a two-circle Venn diagram corresponding to the event resulting from the given operation(s).

37. E^C
38. $E \cup F$
39. $E \cap F^C$
40. $(E \cap F)^C$
41. $\left(E^C \cap F^C\right) \cup F$

For Exercises 42 - 47, let S = $\{m, n, p, q, r, t, w, x\}$ with events

$A = \{m, p, r\}$ $B = \{t, w, x\}$ $D = \{p, q, r, t\}$ $E = \{r, t\}$

Describe the given event as a subset.

42. $A^C \cup D$
43. $E^C \cap B$
44. $(A \cup B)^C$
45. $B \cup D^C$
46. $(D \cap E)^C$
47. $(B \cap E) \cup A$

For Exercises 48 - 50, an experiment consists of spinning a spinner divided into four equal regions (red, blue, green, and yellow), noting the color, and choosing a letter at random from the word MATRIX.

Let

E := the event "the spinner lands on green"

F := the event "a vowel is drawn"

G := the event "a letter in the word *EXIT* is drawn"

H := the event "the spinner lands on red or yellow"

Write the symbolic notation for the given event.

48. The event that "the spinner doesn't land on green."
49. The event that "a vowel in the word *EXIT* is drawn."
50. The event that "the spinner lands on red or yellow or a consonant is drawn."

For Exercises 51 - 53, an experiment consists of spinning a spinner divided into four equal regions (red, blue, green, and yellow), noting the color, and choosing a letter at random from the word MATRIX.

Let

E := the event "the spinner lands on green"

F := the event "a vowel is drawn"

G := the event "a letter in the word *EXIT* is drawn"

H := the event "the spinner lands on red or yellow"

Write a verbal description of the given event.

51. G^C
52. $H^C \cap E$
53. $F \cup E$

Mastery Practice (Answers)

54. An experiment consists of selecting at random a letter from the word MULTIPLY (noting whether or not the letter is a vowel), then tossing a fair coin (noting the side landing up), and, last, rolling a standard ten-sided die (noting if the number rolled is odd or even). Write the sample space associated with this experiment.

55. A sample space for a given experiment is S = $\{1, a, 2, b\}$. List all possible events for the experiment.

56. An experiment consists of rolling a standard five-sided die, noting the number rolled, and spinning a spinner divided into four equal regions, (red, blue, green, and yellow) is spun, noting the color.
   a. List the sample space, S, for the experiment.
   b. List the outcomes of the certain event.
   c. List two non-empty mutually exclusive events.

57. Let E and F be two events of the sample space, S. Use a two-circle Venn diagram to illustrate which region(s) contain the outcomes of the event $\left(E \cup F^C\right) \cap \left(E \cap F^C\right)^C$.

58. An experiment consists of drawing a card from a standard deck of 52 cards, noting the color, and then rolling a standard eight-sided die, noting the number rolled.

    Let

    E := the event "a black card is drawn"

    F := the event " a number less than 6 is rolled"

    G := the event "a multiple of 2 is rolled"

   a. Describe and list the outcomes in the event $E \cap G$.
   b. Describe and list the outcomes in the event $F^C \cup G$.
   c. Write the symbolic notation for the event "a red card is drawn or a number less than 6 is rolled, but not a multiple of 2."

Communication Practice (Answers)

59. Explain why the sample space is considered the certain event.

60. Explain to a person outside of a mathematics class what it means for two events to be mutually exclusive.

4.2 Basics of Probability

© Photo by Kathryn Bollinger, 2020

We see probabilities almost every day in our real lives. Most times when you pick up the newspaper or read the news on the internet, you encounter probability. For instance, "there is a 65% chance of rain today," or "a pre-election poll shows that 52% of voters approve of a ballot measure."

Did you ever wonder why a flush beats a full house in poker? It's because the probability of getting a flush is smaller than the probability of getting a full house.

You can encounter probability in many different areas of life, and probabilities can be used in many ways, including to make business decisions, determine insurance premiums, and to set the price of raffle tickets.

Learning Objectives:
In this section, you will learn about concepts related to basic probability. Upon completion you will be able to:

- Identify whether or not a sample space is uniform.
- State the theoretical or empirical probability of given events.
- Construct and use a probability distribution table.

Defining Probability

In some experiments all the outcomes have the same chance of occurring. If we roll a standard die, the chances of rolling any of the numbers on the die are the same, or if we draw a single card from a standard deck of cards, each card has the same chance of being selected. We call the outcomes, in either experiment, **equally likely**.

Definition

- An experiment has **equally likely outcomes** if every outcome has the same probability of occurring.
- A sample space is **uniform** if all of its outcomes are equally likely.
- In a uniform sample space with n outcomes, the probability of each outcome is $\frac{1}{n}$.

Given an event of an experiment, sometimes we want to describe how likely the event is to occur. To do so, we will use probabilities; we begin with the simplest case, where the sample space of the experiment is uniform.

Definition

For a uniform sample space, **S**, the **probability of event *A***, denoted ***P*(*A*)**, is calculated as:

$$\boldsymbol{P(A) = \frac{\textbf{total number of outcomes in } A}{\textbf{total number of possible outcomes}}}$$

The definition for the probability of A, P(A), is written as a fraction. While fractions represent exact probabilities, and the format the authors will use most frequently, often times in casual discussions the reader will see probabilities written as percentages or rounded decimals.

A probability is a number that is never negative or never greater than 1. In other words,

$$0 \leq P(A) \leq 1.$$

The closer the probability of an event is to 0, the less likely the event is to occur. The closer the probability of an event is to 1, the more likely the event is to occur.

In the course of this chapter, if you compute a probability and arrive at an answer that is negative or greater than 1, you have made a mistake and should double check your work.

In the previous section, we discussed experiments involving a standard die, a standard deck of cards, and a 2-sided coin, as well as the outcomes associated with each. With *standard* dice and cards we have previously mentioned that all outcomes are equally likely. We can extend this concept to other dice, cards, and/or coins.

Definition

When all outcomes involving the side of a die or coin are equally likely, then we have a **fair die** or a **fair coin**.

If each card in a deck of cards is equally likely to be selected, then we have a **well-shuffled deck**.

Suppose you roll a fair standard six-sided die one time, noting the number showing. The sample space is $S = \{1,2,3,4,5,6\}$. Let's explore some probabilities, based on this experiment.

First, let's consider the event "a 4 is rolled" and compute the probability of the event. The mathematical notation for this event would be P(a 4 is rolled).

As it is stated that we have a *fair* die, all outcomes are equally likely and we have

$$P(\text{a 4 is rolled}) = \frac{\text{total number of ways to roll a four}}{\text{total number of possible outcomes when rolling the die}} = \frac{1}{6}$$

Thus, the probability a 4 is rolled is $\frac{1}{6}$.

The probability of rolling any specific number (1, 2, 3, 4, 5, or 6) with a fair standard six-sided die is $\frac{1}{6}$.

Next, let's consider the event "an odd number is rolled" and compute its probability, P(an odd number is rolled).

The event "an odd number is rolled" is $\{1,3,5\}$. So,

$$P(\text{an odd number is rolled}) = \frac{\text{total number of ways to roll an odd number}}{\text{total number of possible outcomes when rolling the die}} = \frac{3}{6}$$

Therefore, the probability an odd number is rolled is $\frac{3}{6}$.

N *While $\frac{3}{6} = \frac{1}{2}$, the authors will not reduce fractions, when discussing probability, in order to emphasize the relationship between the numerator and denominator in the context of the problem.*

What is the probability of rolling a 7, P(roll a 7)?

It is impossible to have a 7 showing if you roll a fair standard six-sided die. So the event "roll a 7" is $\{\,\} = \varnothing$. Thus,

$$P(\text{roll a 7}) = \frac{\text{total number of ways to roll a seven}}{\text{total number of possible outcomes when rolling the die}} = \frac{0}{6}$$

Therefore, the probability of rolling a 7 is 0.

■ **Example 1** Suppose you draw a single card from a well-shuffled standard deck of cards. Compute each of the following probabilities.

a. $P(\text{card is red})$
b. $P(\text{card is a heart})$
c. $P(\text{card is a red 5})$
d. $P(\text{card is a face card})$

Solution:
As the deck is *well-shuffled*, each card has the same chance of being drawn and we have *equally likely* outcomes. Also, the deck is *standard*, so it has 52 cards.

a. $P(\text{card is red}) = \dfrac{\text{total number of red cards}}{\text{total number of cards in the deck}} = \dfrac{26}{52}$

b. $P(\text{card is a heart}) = \dfrac{\text{total number of hearts}}{\text{total number of cards in the deck}} = \dfrac{13}{52}$

c. $P(\text{card is a red 5}) = \dfrac{\text{total number of red fives}}{\text{total number of cards in the deck}} = \dfrac{2}{52}$

d. $P(\text{card is a face card}) = \dfrac{\text{total number of face cards}}{\text{total number of cards in the deck}} = \dfrac{12}{52}$

■

■ **Example 2** A pair of fair standard distinguishable six-sided dice is cast, and the numbers showing on each die are observed. What is the probability that

a. A 6 is rolled?
b. A sum of 8 is rolled?
c. The two dice are not showing the same number?
d. A sum of 2 or greater is rolled?
e. A 3 or a sum of 6 is rolled?

Solution:
We are using *fair* dice in a two-step experiment. Thus, there are $\underset{\text{1st die}}{\underline{6}} \cdot \underset{\text{2nd die}}{\underline{6}} = 36$ outcomes, which are all *equally likely*.

Recall, as shown below in **Table 4.3**, the outcomes when rolling two standard distinguishable six-sided dice.

	1	2	3	4	5	6
1	(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
2	(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
3	(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
4	(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
5	(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
6	(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

Table 4.3: Two Standard Six-Sided Dice Chart

a. The event "a 6 is rolled" = $\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,1),(6,2),(6,3),(6,4),(6,5)\}$, contains 11 of the possible outcomes.

$$P(\text{6 is rolled}) = \frac{11}{36}$$

b. The event "a sum of 8 is rolled" = $\{(6,2),(5,3),(4,4),(3,5),(2,6)\}$, contains 5 of the possible outcomes.

$$P(\text{a sum of 8 is rolled}) = \frac{5}{36}$$

c. The event "the dice ARE showing the same number" = $\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$, contains 6 of the possible outcomes. The event "the dice ARE NOT showing the same number" will then have $36-6 = 30$ outcomes.

$$P(\text{two dice are not the same number}) = \frac{30}{36}$$

Notice, it is easier to count the number of times the dice show the same number, than to count when the numbers are different.

d. The event "a sum of 2 or greater is rolled" is composed of all outcomes in the sample space.

$$P(\text{a sum of 2 or greater is rolled}) = \frac{36}{36} = 1$$

e. We know from the previous section, that when talking about events, the word "or" indicates the union of the events. Thus, the event "a 3 or a sum of 6 is rolled" = $\{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3),(3,1),(3,2),(3,4),(3,5),(3,6),(1,5),(2,4),(4,2),(5,1)\}$, contains 15 of the possible outcomes.

$$P(\text{a 3 or a sum of 6 is rolled}) = \frac{15}{36}$$

N *As shown in part* ***c****, it is sometimes easier to count the total number of outcomes not in an event (the event's complement), rather than count the total number of outcomes in the event. Considering*

$$\textit{Total \# of Outcomes in } \mathbf{S} = \textit{Total \# of Outcomes in } A + \textit{Total \# of Outcomes in } A^C,$$

a simple subtraction will then give the number of outcomes you are looking for.

■

Try It # 1:
A jar contains 3 red, 4 white, and 3 blue marbles (all the same size). If a marble is chosen at random, what is the probability that the marble is a red marble or a blue marble?

There are three ways to determine probabilities. The probability of being dealt a red Jack in a card game or rolling a five on a fair die can be calculated from mathematical formulas, as seen in the previous examples. These are examples of **theoretical probabilities**.

Another way is experimental in nature, where we repeatedly conduct an experiment. Suppose we flip a coin over and over and over again, and it comes up heads about half of the time; we would expect that, in the future, whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," it is based on prior evidence. These are examples of **empirical probabilities**.

A third view is subjective in nature, or, in other words, an "educated" guess. If someone asks you the probability that the Texas A&M Aggies will win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under precisely the same conditions. Because there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an "educated" guess that there is a 75% chance they will win the game; that is, if the same two teams were to play each other repeatedly under identical conditions, the Aggies would win about three out of every four games. However, this would be just a guess, with no way to verify its accuracy. A **subjective probability** may not be worth much, as it depends upon the knowledge of the educated guesser.

Definition

There are three types of probabilities:

- A probability found using a mathematical formula without the necessity of an experiment is called a **theoretical probability**.
- A probability found by repeatedly conducting an experiment or through repeated observations, to determine the number of occurrences, is called an **empirical probability**.
- If a probability is an estimate (or guess), dependent upon an "educated" guesser with experience or intuition, but with no way to verify their accuracy, then it is called a **subjective probability**.

To explore empirical probabilities, consider the following scenario:

A survey of 665 drivers was conducted, and **Table 4.4**, below, shows the number of surveyed drivers who have received and not received a speeding ticket in the last year, along with the color of their car.

	Speeding Ticket	No Speeding Ticket	**Total**
Red Car	15	135	**150**
Not Red Car	45	470	**515**
Total	**60**	**605**	**665**

Table 4.4: Speeding Ticket and Car Color Results.

Let's begin by computing the probability that a randomly chosen person, from the survey, did not get a speeding ticket.

In **Table 4.4** we are given a set of observations. We can see that 605 people did not get a speeding ticket, regardless of their car color, out of 665 total people surveyed. Thus,

$$P(\text{did not get a speeding ticket}) = \frac{605}{665}.$$

Next, how could we determine the probability that a randomly selected person, from the survey, has a red car and got a speeding ticket?

From the previous section, we know the word "and," when talking about events, indicates the intersection of the events. So we look for the number of people who **both** own a red car and got a speeding ticket.

	Speeding Ticket	No Speeding Ticket	**Total**
Red Car	15	135	**150**
Not Red Car	45	470	**515**
Total	**60**	**605**	**665**

Table 4.5: Speeding Ticket and Car Color Results, with the row and column in question highlighted.

Table 4.5 above shows **Table 4.4** with the "Red Car" row and "Speeding Ticket" column highlighted and their intersection of 15 people circled. Thus,

$$P(\text{has a red car } \textit{and} \text{ got a speeding ticket}) = \frac{15}{665}.$$

Now, how could we determine the probability a randomly selected person, from the survey, has a red car *or* got a speeding ticket?

Again, we know that when talking about events, the word "or" indicates the union of the two events. This means we include any person who owns a red car or received a speeding ticket or both owns a red car and received a speeding ticket. So, we look for the number of people who own a red car, but didn't get a speeding ticket (135) **or** who got a speeding ticket, but doesn't own a red car (45) **or** who both own a red car and got a speeding ticket (15).

	Speeding Ticket	No Speeding Ticket	**Total**
Red Car	15	135	**150**
Not Red Car	45	470	**515**
Total	**60**	**605**	**665**

Table 4.6: Speeding Ticket and Car Color Results, with the drivers in question circled.

Table 4.6 above shows **Table 4.4** with all people within the two events, circled. Therefore,

$$P(\text{has a red car } or \text{ got a speeding ticket}) = \frac{135+45+15}{665} = \frac{195}{665}.$$

Notice that when computing the probability of the union, we did not add the total number of people who own a red car (150) and the total number of people who got a speeding ticket (60). If we had we would have arrived at the INCORRECT answer of $\frac{210}{665}$. This is caused by the 15 people who both own a red car and who got a speeding ticket being counted twice, from them being in both the row and the column.

Example 3 **Table 4.7**, below, shows the distribution, by classification, of students at a small community college who take public transportation and the ones who drive to school.

	Freshman (F)	Sophomore (M)	**Total**
Public Transportation (T)	8	13	**21**
Drive (D)	39	40	**79**
Total	**47**	**53**	**100**

Table 4.7: Students and Transportation Results

Determine the probability that a randomly selected student from this small community college

a. Is not a sophomore.
b. Drives to school.
c. Is a sophomore and takes public transportation.
d. Is a sophomore or takes public transportation.

Solution:

a. Given M^C := the event "a student is NOT a sophomore," then

$$P\left(M^C\right) = \frac{47}{100}.$$

b. We know D := the event "a student drives to school. " Thus,

$$P(D) = \frac{79}{100}.$$

c. We know M := the event "a student is a sophomore" and T := the event "a student takes public transportation." Then, $M \cap T$ = the event "a student is a sophomore **and** takes public transportation." Identifying the intersection of the appropriate column and row gives us,

$$P(M \cap T) = \frac{13}{100}.$$

d. $M \cup T$ = the event "a student is a sophomore **or** takes public transportation." Taking into account all individual students in the public transportation row and sophomore column (not the **Total** row or column entries), we have

$$P(M \cup T) = \frac{13+40+8}{100} = \frac{61}{100}.$$

■

Try It # 2:
The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch, on a particular day, is given in **Table 4.8**, below.

	Main (M)	Branch (B)	**Total**
Fiction (F)	300	100	**400**
Non-Fiction (N)	150	50	**200**
Total	**450**	**150**	**600**

Table 4.8: Book Count Breakdown by Location

Determine the probability a randomly selected checked out book
a. Was non-fiction.
b. Was not checked out at the main library.
c. Was checked out at the smaller branch or was fiction.

Constructing Probability Distributions

One way to organize probabilities from an experiment is by using a **probability distribution**.

Definition

A **probability distribution** for an experiment is a table of all the possible outcomes and their corresponding probabilities. ■

If we toss a fair coin and see which side lands up, there are two possible outcomes, heads or tails. With the coin being *fair*, these are equally likely outcomes and have the same probabilities: $P(\text{heads}) = \frac{1}{2}$ and $P(\text{tails}) = \frac{1}{2}$. The corresponding probability distribution is shown in **Table 4.9**, below.

Outcome	Heads	Tails
Probability	$\frac{1}{2}$	$\frac{1}{2}$

Table 4.9: Probability Distribution for Tossing a Fair Coin Once

Notice, the probability of every outcome is equal, which indicates this experiment has a uniform sample space.

Next, if we roll a fair standard eight-sided die and note the number showing, the probability distribution would be given by

Outcome	1	2	3	4	5	6	7	8
Probability	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Table 4.10: Probability Distribution for Rolling a Fair Standard Eight-Sided Die Once

Again, this distribution shows the experiment of rolling an eight-sided die and noting the number showing has a uniform sample space.

Now, if we roll a pair of fair standard distinguishable six-sided dice and note the sum, we can use the two standard six-sided dice chart given in the previous section to count the number of pairs that result in the same sum. The corresponding probability distribution would be given by

Outcome = Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Table 4.11: The probability distribution for the sum, when rolling a pair of fair standard distinguishable six-sided dice.

Because not *all sums* are equally likely (for instance, it is more likely to roll a sum of 7 than any other sum), the sample space consisting of all possible sums is not a uniform sample space.

As stated earlier, when a sample space is uniform, each outcome has a probability of $\frac{1}{n}$, so it follows that the sum of the probabilities of all n outcomes in the corresponding probability distribution is 1. In fact, no matter if a probability distribution represents a uniform sample space or not, the sum of the probabilities of all possible outcomes is always 1.

In order for a probability distribution with n outcomes $(x_1, \cdots, x_n)$ to be valid, it must satisfy the following conditions:

- Each probability must be a number between 0 and 1, inclusively. ($0 \le P(x_i) \le 1$)
- The sum of the probabilities in a probability distribution must add to 1. $(P(x_1) + P(x_2) + \cdots + P(x_n) = 1)$

A quick check of each of the three probability distributions above, **Tables 4.9**, **4.10**, and **4.11**, shows the validity of each.

■ **Example 4** Are the probability distributions in **Tables 4.12**, **4.13**, and **4.14**, valid probability distributions? If valid, does the distribution represent an experiment with a uniform sample space?

a.

Outcome	A	B	C	D	E
Probability	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Table 4.12

b.

Outcome	A	B	C	D	E	F
Probability	0.45	0.80	−0.20	−0.35	0.10	0.20

Table 4.13

c.

Outcome	A	B	C	D
Probability	0.30	0.20	0.40	0.25

Table 4.14

Solution:

a. This is a valid probability distribution. All the probabilities are between 0 and 1, inclusive, and the sum of the probabilities is 1.00. Because the probabilities of all 5 outcomes (A−E) are not the same, the sample space is not uniform.

b. This is not a valid probability distribution. The sum of the probabilities is 1.00, but some of the probabilities are not between 0 and 1, inclusive (outcomes C and D have negative probabilities).

c. This is not a valid probability distribution. All the probabilities are between 0 and 1, inclusive, but the sum of the probabilities is 1.15 and not 1.00.

■

Try It # 3:

Are the probability distributions in **Tables 4.15**, **4.16**, and **4.17**, valid probability distributions? For each one, explain why or why not. If valid, does the distribution represent an experiment with a uniform sample space?

a.

Outcome	A	B	C	D	E
Probability	0.2	0.4	−0.2	0.4	0.2

Table 4.15

b.

Outcome	A	B	C	D	E
Probability	0.2	0.4	0.1	0.4	0.2

Table 4.16

c.

Outcome	A	B	C	D	E
Probability	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Table 4.17

Try It Answers

1. $\frac{6}{10}$

2.
   a. $\frac{200}{600}$

   b. $\frac{150}{600}$

   c. $\frac{450}{600}$

3.
   a. This is not a valid probability distribution. The sum of the probabilities is 1.00, but some of the probabilities are not between 0 and 1, inclusive.

   b. This is not a valid probability distribution. All of the probabilities are between 0 and 1, inclusive, but the sum of the probabilities is 1.30 and not 1.00.

   c. This is a valid probability distribution. All probabilities are between 0 and 1, inclusive, and the sum of the probabilities is 1.00. It represents a uniform sample space, because all outcomes have the same probability, and are, therefore, equally likely.

EXERCISES

Basic Skills Practice (Answers)

1. A spinner divided into 4 equal regions (red, blue, green, and yellow) is spun, noting the color. What is the probability

   a. The spinner lands on red?
   b. The spinner lands on green or yellow?
   c. The spinner does not land on blue?

2. A marble is drawn from a bag containing 20 identical red marbles, 30 identical blue marbles, and 5 identical white marbles (all the same size), noting the color of the selected marble. What is the probability

   a. The marble is blue?
   b. The marble is red or white?
   c. The marble is not white?
   d. The marble is a color used in the American flag?

3. A Halloween bucket contains 12 hard caramel candies, 15 peppermints, 21 hard fruit-flavored candies, and 18 butterscotch candies (all the same size and shape). A single candy is selected at random. What is the probability that the candy is

   a. A peppermint?
   b. A butterscotch or hard fruit-flavored?
   c. Not a hard caramel?

4. A bag contains 100 \$1 bills, 20 \$5 bills, 10 \$10 bills, and one \$100 bill. A single bill is randomly selected from the bag. What is the probability

   a. The bill is a \$1 bill?
   b. The bill is less than \$10?
   c. The bill is not a \$50 bill?

5. An instructor collected the following data from the students in the instructor's fall classes.

	Freshman	Sophomore	**Total**
MAT 121	43	15	**58**
MAT 142	35	28	**63**
MAT 187	27	32	**59**
Total	**105**	**75**	**180**

   If a student is selected at random, what is the probability

   a. The student is a freshman?
   b. The student is taking MAT 142?
   c. The student is a sophomore and is taking MAT 187?
   d. The student is not taking MAT 187?
   e. The student is a freshman or is taking MAT 121?
   f. The student is taking MAT 142 or MAT 187?

6. Is the following a valid probability distribution? Why or why not?

Outcome	A	B	C	D
Probability	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

7. Is the following a valid probability distribution? Why or why not?

Outcome	A	B	C	D
Probability	$-\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

Intermediate Skills Practice (Answers)

8. A standard five-sided die is rolled, and the number showing is noted. Compute each of the following probabilities.

   a. P(a 1 is rolled)
   b. P(an odd number is rolled)
   c. P(a number no greater than 4 is rolled)
   d. P(a number other than 3 is rolled)
   e. P(an even number greater than 4 is rolled)

9. A card is selected from a well-shuffled standard 52-card deck. Compute each of the following probabilities.

   a. P(a club is drawn)
   b. P(a red card is drawn)
   c. P(a black Jack is drawn)
   d. P(a non-face card is drawn)
   e. P(a 7 is drawn)

10. A fair coin is tossed three times, noting the side landing up on each toss. What is the probability that

    a. The first toss shows heads?
    b. The coin shows tails exactly once?
    c. The coin shows no tails?
    d. The coin shows at least two heads?

11. A letter is randomly selected from the word GRADUATE. Compute each of the following probabilities.

    a. P(a 'D' is selected)
    b. P(an 'A' is selected)
    c. P(a vowel is selected)
    d. P(a letter in the word AGGIE is selected)
    e. P(a letter in the word FOIL is selected)

12. Two fair standard six-sided dice are cast (one green and one blue) and the numbers showing on each die are observed. What is the probability that

   a. At least one 2 is rolled?
   b. A 7 is showing on the green die?
   c. A sum of 6 or a sum of 9 is rolled?
   d. The sum rolled is no more than 12?
   e. A sum of 7 is rolled?
   f. A 1 is rolled on the green die and a sum of 4 is rolled?
   g. An even sum is rolled or the blue die shows a 6?

13. A real estate agent has kept records of the number of bedrooms in the houses they sold during the years of 2009 to 2013. The data is listed in the following table.

	One or Two	Three	Four	Five or more	**Total**
2009	5	12	25	3	**45**
2010	7	15	22	3	**47**
2011	6	18	28	6	**58**
2012	6	16	30	2	**54**
2013	5	17	29	5	**56**
Total	**29**	**78**	**134**	**19**	**260**

If a home sold by the agent is selected at random, what is the probability

   a. The home had exactly four bedrooms?
   b. The home was sold in 2010?
   c. The home has less than four bedrooms?
   d. The home was sold after 2011?
   e. The home was not sold in 2013 or was not sold in 2009?
   f. The home was sold in 2012 or had exactly one or two bedrooms?
   g. The home was not sold in 2011 and had five or more bedrooms?
   h. The home was sold before 2012 and had at least three bedrooms?

14. Is the following probability distribution uniform? Why or why not?

Outcome	-4	0	1	3
Probability	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{4}{10}$

15. Is the following probability distribution uniform? Why or why not?

Outcome	0	1	2	3	4
Probability	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{4}{20}$	$\frac{5}{20}$

Mastery Practice (Answers)

16. A fair standard four-sided green die and a fair standard five-sided blue die are rolled and the numbers showing on each die are observed. What is the probability that

   a. At least one 2 is rolled?
   b. A 3 is showing on the green die?
   c. A sum of 1 is rolled?
   d. The green die shows a number less than 3 and the blue die shows a number greater than 3?
   e. The blue die shows a 4 or the sum of the dice rolled is 6?
   f. A sum of 7 is rolled or the green die shows a 1?
   g. A sum less than 10 is rolled?

17. A spinner divided into 4 equal regions (red, blue, green, and yellow) is spun, noting the color, and then a fair two-sided coin is tossed, noting the side landing up. What is the probability that

   a. The coin shows heads?
   b. The spinner lands on yellow?
   c. The spinner lands on a color other than green or the coin shows tails?
   d. The spinner lands on red and the coin shows heads?
   e. The spinner lands on blue or green or the coin shows tails?

18. The following table shows a distribution of drink preferences by age.

	Water (W)	Coffee (F)	Soda (S)	Juice (J)	**Total**
18 - 24 (Y)	22	50	61	10	**143**
25 - 40 (M)	36	78	31	20	**165**
41 and over (D)	45	57	12	38	**152**
Total	**103**	**185**	**104**	**68**	**460**

   If a surveyed person is selected at random, compute each of the following.

   a. $P(F)$
   b. $P(M)$
   c. $P(D \cap W)$
   d. $P(Y \cup S)$
   e. $P\left(Y^C\right)$
   f. $P\left(D^C \cap S^C\right)$
   g. $P(W \cup J)$
   h. $P((Y \cup M) \cap (F \cup S))$
   i. $P((D \cap J) \cup (Y \cap F))$

19. Is the following probability distribution valid? If valid, does the distribution represent an experiment with a uniform sample space?

Outcome	-100	450
Probability	$\frac{2}{10}$	$\frac{8}{10}$

20. Is the following probability distribution valid? If valid, does the distribution represent an experiment with a uniform sample space?

Outcome	0	1	2	3
Probability	$\frac{3}{12}$	$\frac{2}{8}$	$\frac{1}{4}$	$\frac{4}{16}$

Communication Practice (Answers)

21. Explain to a student not taking a math class what it means for a probability distribution to be uniform.

22. Explain why $0 \leq P(A) \leq 1$ for any event A in the sample space.

23. Give an example of a theoretical probability and an example of an empirical probability.

4.3 Rules of Probability

© Photo by Robert Tyler, 2019

A plane arrives on time at Dallas/Fort Worth International Airport (DFW) 79% of the time, according to https://www.transtats.bts.gov/. Does this mean an arriving plane is late to DFW 21% of the time? The answer is "No." The complement of "on time" is not "late," because there is a third possibility; the plane could be "early." Based on the given information, we cannot find the likelihood a plane arrives late at DFW, but we can say the likelihood a plane arrives late *or* early is 21%.

Learning Objectives:
In this section, you will learn techniques and formulas for finding the probabilities for the union, intersection, and/or complement of multiple events. Upon completion you will be able to:

- Compute the probability of an event, given a probability distribution table.
- State the rules of basic probability, including the union and complement rules.
- Apply the rules of basic probability to given events of an experiment.
- Construct a Venn diagram using information about events, and then use the Venn diagram to find probabilities.

Computing a Probability Using a Probability Distribution

Whether or not a probability distribution is uniform, to compute the probability of a given event, we add the probabilities of all the individual outcomes that make up the event.

▪ **Example 1** The 2010 U.S. Census gathered information on household sizes. The data, with the exception of households of size 5, is given in **Table 4.18**. ("Households by age," 2013.)

Outcome	1	2	3	4	5	6	7 or more
Probability	$\frac{267}{1000}$	$\frac{336}{1000}$	$\frac{158}{1000}$	$\frac{137}{1000}$		$\frac{24}{1000}$	$\frac{15}{1000}$

Table 4.18: Partial Probability Distribution for Household Sizes per 2010 U.S. Census

Let A := the event "a household has less than four people," and
B := the event "a household has between 2 and 6 people, exclusively."

a. Fill in the missing probability in the distribution table.

Determine the following probabilities.

b. $P(A)$

c. $P(B)$

d. $P\left(B^C\right)$

e. $P(A \cap B)$

f. $P(A \cup B)$

g. $P\left(A^C \cap B^C\right)$

h. $P\left(A^C \cup B\right)$

Solution:

a. We notice that while the probability distribution is not uniform, the probabilities must still add to 1. So, $1 - \left(\frac{267}{1000} + \frac{336}{1000} + \frac{158}{1000} + \frac{137}{1000} + \frac{24}{1000} + \frac{15}{1000}\right) = \frac{63}{1000}$ is the missing probability.

Outcome	1	2	3	4	5	6	7 or more
Probability	$\frac{267}{1000}$	$\frac{336}{1000}$	$\frac{158}{1000}$	$\frac{137}{1000}$	$\boldsymbol{\frac{63}{1000}}$	$\frac{24}{1000}$	$\frac{15}{1000}$

Table 4.19: Complete Probability Distribution for Household Sizes per 2010 U.S. Census

b. Because $A = \{1,2,3\}$, to find $P(A)$, we add the probabilities of outcome 1, 2, or 3 occurring. Thus,

$$\begin{aligned} P(A) &= P(1) + P(2) + P(3) \\ &= \frac{267}{1000} + \frac{336}{1000} + \frac{158}{1000} \\ &= \frac{267 + 336 + 158}{1000} \\ &= \frac{761}{1000} \end{aligned}$$

c. $B = \{3,4,5\}$, as exclusive means to not include 2 and 6. So,

$$\begin{aligned} P(B) &= P(3) + P(4) + P(5) \\ &= \frac{158 + 137 + 63}{1000} \\ &= \frac{358}{1000} \end{aligned}$$

d. $B^C = \{1,2,6,7 \text{ or more}\}$, and therefore,

$$P\left(B^C\right) = \frac{267+336+24+15}{1000}$$
$$= \frac{642}{1000}$$

e. $A = \{1,2,3\}$, $B = \{3,4,5\}$ $\implies$ $A \cap B = \{3\}$

Thus, $P(A \cap B) = \dfrac{158}{1000}$

f. $A = \{1,2,3\}$, $B = \{3,4,5\}$ $\implies$ $A \cup B = \{1,2,3,4,5\}$

Then we have

$$P(A \cup B) = \frac{267+336+158+137+63}{1000}$$
$$= \frac{961}{1000}$$

g. $A^C = \{4,5,6,7 \text{ or more}\}$, $B^C = \{1,2,6,7 \text{ or more}\}$ $\implies$ $A^C \cap B^C = \{6,7 \text{ or more}\}$

This gives us,

$$P\left(A^C \cap B^C\right) = \frac{24+15}{1000}$$
$$= \frac{39}{1000}$$

h. $A^C = \{4,5,6,7 \text{ or more}\}$, $B = \{3,4,5\}$ $\implies$ $A^C \cup B = \{3,4,5,6,7 \text{ or more}\}$

So,

$$P\left(A^C \cup B\right) = \frac{158+137+63+24+15}{1000}$$
$$= \frac{397}{1000}$$

■

Try It # 1:
Let $\mathbf{S} = \{s_1, s_2, s_3, s_4\}$ be the sample space for an experiment with the distribution given in **Table 4.20**.

Outcome	s_1	s_2	s_3	s_4
Probability	$\frac{1}{50}$	$\frac{1}{25}$	$\frac{19}{50}$	

Table 4.20: The probability distribution for $\mathbf{S}$, with the probability for s_4 missing.

Let $A = \{s_1, s_3\}$ and $B = \{s_1, s_4\}$.

a. Fill in the missing probability in the distribution table.

Determine the following probabilities.

b. $P(A)$

c. $P(B)$

d. $P(A \cup B)$

e. $P(A \cap B)$

f. $P\left(A^C\right)$

g. $P\left(A \cup B^C\right)$

Applying the Rules of Probability

The following are more general rules for computing probabilities that can be helpful when you do not know the specific outcomes in an experiment or when there may be too many to list. The union and complement rules follow from examples discussed in the previous section.

Rules of Probability

Let $\mathbf{S}$ be the sample space of an experiment and suppose A and B are events of the experiment. Then,

- $0 \leq P(A) \leq 1$, for any A
- $P(\varnothing) = 0$
- $P(\mathbf{S}) = 1$

Union Rule:

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Complement Rule:

- $P\left(A^C\right) = 1 - P(A)$ or $P(A) = 1 - P\left(A^C\right)$

N **S** *includes all outcomes of an experiment, and because the probabilities of all outcomes of an experiment must add to 1, it should follow that $P(S) = 1$. Moreover, the probability of an event equaling 1 implies the event is certain. However, while $P(\varnothing) = 0$, $P(A) = 0$ does not necessarily mean $A = \varnothing$, as we commonly assign a probability of 0 to events that are extremely unlikely.*

The union rule has four probabilities. If you know the values of any three probabilities, you can solve for the fourth.

If events A and B are mutually exclusive, then $A \cap B = \emptyset$, and the probability of the union of A and B is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - 0 \\ &= P(A) + P(B) \end{aligned}$$

Returning to our previous example from the 2010 U.S. Census, we could have computed some of the probabilities in question by using these new rules.

Previously, we completed the given distribution by computing $P(5) = \frac{63}{1000}$.

Outcome	1	2	3	4	5	6	7 or more
Probability	$\frac{267}{1000}$	$\frac{336}{1000}$	$\frac{158}{1000}$	$\frac{137}{1000}$	$\frac{63}{1000}$	$\frac{24}{1000}$	$\frac{15}{1000}$

Table 4.21: Completed Probability Distribution for Household Sizes per 2010 U.S. Census

and we found

$$A = \{1, 2, 3\} \quad \Longrightarrow \quad P(A) = \frac{761}{1000}$$

$$B = \{3, 4, 5\} \quad \Longrightarrow \quad P(B) = \frac{358}{1000}$$

For part **d**, we were asked to compute $P\left(B^C\right)$. As there were a limited number of outcomes in B^C, we were able to list them and then add their corresponding probabilities. Instead, we could have used the complement rule without needing to identify the outcomes in B^C, because we had already calculated $P(B)$.

$$\begin{aligned} P\left(B^C\right) &= 1 - P(B) \\ &= 1 - \frac{358}{1000} \\ &= \frac{642}{1000} \end{aligned}$$

For part **f**, we were asked to compute $P(A \cup B)$. Originally, we determined the outcomes in $A \cup B$ and added their corresponding probabilities. However, because we were asked to compute $P(A), P(B)$, and $P(A \cap B)$ in parts **b**, **c**, and **e**, respectively, we could have used the union rule to calculate the probability, instead.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{761}{1000} + \frac{358}{1000} - \frac{158}{1000} \\ &= \frac{961}{1000} \end{aligned}$$

By using the union rule, we avoid having to list all outcomes and possibly double counting the households of size 3.

In part **g**, we were asked to compute $P\left(A^C \cap B^C\right)$. While this does not appear to fit any of our probability rules, let's investigate the event in question more closely.

$A^C \cap B^C = (A \cup B)^C$ by DeMorgan's Laws. So $P\left(A^C \cap B^C\right) = P\left((A \cup B)^C\right)$, which is the probability of the complement of the event $A \cup B$.

Applying the complement rule, we have:

$$\begin{aligned} P\left(A^C \cap B^C\right) &= P\left((A \cup B)^C\right) \\ &= 1 - P(A \cup B) \\ &= 1 - \frac{961}{1000} \\ &= \frac{39}{1000} \end{aligned}$$

In the previous section we found probabilities for events in an experiment of drawing a single card from a well-shuffled standard 52-card deck. Now let's turn our attention to applying the rules of probability to such an experiment, where possible.

▪ **Example 2** An experiment consists of selecting one card from a well-shuffled standard 52-card deck. Calculate the probability that the card drawn is

a. The King of hearts.
b. A heart or a King.
c. A Queen or a King.
d. Not a heart.

Solution:
Considering the deck is well-shuffled, each card selected is equally likely to occur, out of the 52 possible cards.

a. There are four Kings in the deck, but only one of the Kings is also a heart, so

$$\begin{aligned} P(\text{King of hearts}) &= P(\text{King} \cap \text{heart}) \\ &= \frac{1}{52} \end{aligned}$$

b. There are 13 hearts in the deck, so $P(\text{heart}) = \frac{13}{52}$.

There are four kings in the deck, so $P(\text{King}) = \frac{4}{52}$.

There is one king that is also a heart, so $P(\text{King} \cap \text{heart}) = \frac{1}{52}$.

We can then use the union rule to calculate the probability of drawing a heart or a King.

$$\begin{aligned} P(\text{heart} \cup \text{King}) &= P(\text{heart}) + P(\text{King}) - P(\text{King} \cap \text{heart}) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{16}{52} \end{aligned}$$

c. There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus, the probability of drawing a Queen or a King is

$$P(\text{King} \cup \text{Queen}) = \frac{8}{52}$$

Note that, in this case, there are no cards that are *both* a Queen and a King, so $P(\text{King} \cap \text{Queen}) = 0$. Using the union rule, we could have said

$$\begin{aligned} P(\text{King} \cup \text{Queen}) &= P(\text{King}) + P(\text{Queen}) - P(\text{King} \cap \text{Queen}) \\ &= \frac{4}{52} + \frac{4}{52} - 0 \\ &= \frac{8}{52} \end{aligned}$$

d. There are 13 hearts in the deck, so $P(\text{heart}) = \frac{13}{52}$.

Not drawing a heart is the complement of drawing a heart, thus the probability of not drawing a heart is

$$\begin{aligned} P(\text{not heart}) &= P\left(\text{heart}^C\right) \\ &= 1 - P(\text{heart}) \\ &= 1 - \frac{13}{52} \\ &= \frac{39}{52} \end{aligned}$$

■

Try It # 2:
An experiment consists of selecting one card from a well-shuffled standard deck of cards. Compute the probability that the card drawn is

a. A black card.
b. A black card or a 10.
c. Not a face card or a not a number card.
d. A red Jack.

In the examples thus far, we knew the outcomes of the experiment in question. Now we will turn our attention to an example where the specific outcomes of the experiment are unknown.

Example 3 Let B and R be two events of an experiment. Suppose $P(B) = 0.6, P(R) = 0.7$, and $P(B \cap R) = 0.5$. Calculate the following probabilities.

a. $P(B \cup R)$

b. $P\left(R^C\right)$

c. $P\left((B \cup R)^C\right)$

d. $P\left(R^C \cap B\right)$

e. $P\left(B^C \cup R\right)$

Solution:

Because the specific outcomes of the experiment are unknown, we will use the probability rules.

a. Using the given probabilities ($P(B)$, $P(R)$, and $P(B \cap R)$) and the union rule,

$$\begin{aligned} P(B \cup R) &= P(B) + P(R) - P(B \cap R) \\ &= 0.6 + 0.7 - 0.5 \\ &= 0.8 \end{aligned}$$

b. Using the given probability of R and the complement rule,

$$\begin{aligned} P\left(R^C\right) &= 1 - P(R) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

c. Using the probability found in part **a** and the complement rule,

$$\begin{aligned} P\left((B \cup R)^C\right) &= 1 - P(B \cup R) \\ &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

Without the actual outcomes, it is more difficult to compute the probabilities in parts **d** and **e** with only the stated probability rules. To help visualize the relationships between the events, we will return to a Venn diagram.

With two events we have a two-circle Venn diagram. Instead of labeling the mutually exclusive regions a, b, c, and d, as was done previously, we will use w, x, y, and z to avoid confusion with the event name B.

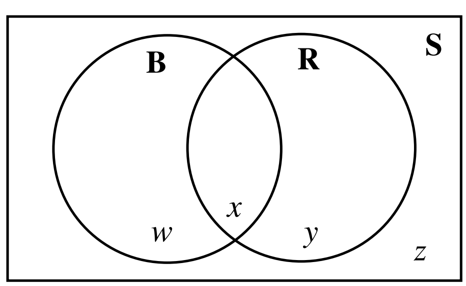


Figure 4.3.2: Unshaded Two-Circle Venn Diagram

Due to the fact that $P(\mathbf{S}) = 1$, then it is true that $P(w) + P(x) + P(y) + P(z) = 1$.

Using the given information, we can rewrite the probabilities, $P(B)$, $P(R)$, and $P(B \cap R)$, in terms of the probabilities of the regions in the Venn diagram.

$$\begin{aligned} P(B) &= P(w) + P(x) = 0.6 \\ P(R) &= P(x) + P(y) = 0.7 \\ P(B \cap R) &= P(x) = 0.5 \end{aligned}$$

Now we have a system of equations which we can solve. With $P(x) = 0.5$, then

$$\begin{aligned} P(w) + P(x) &= 0.6 \\ P(w) + 0.5 &= 0.6 \\ P(w) &= 0.1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} P(x) + P(y) &= 0.7 \\ 0.5 + P(y) &= 0.7 \\ P(y) &= 0.2 \end{aligned}$$

Thus,

$$\begin{aligned} P(w) + P(x) + P(y) + P(z) &= 1 \\ 0.1 + 0.5 + 0.2 + P(z) &= 1 \\ 0.8 + P(z) &= 1 \\ P(z) &= 0.2 \end{aligned}$$

Placing this information into the appropriate regions of the Venn diagram produces **Figure 4.3.3**.

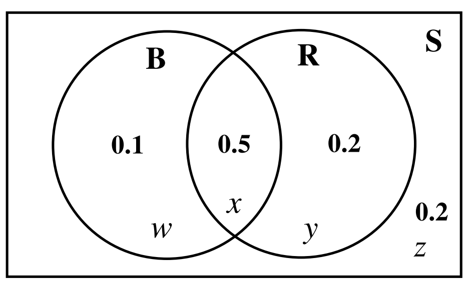


Figure 4.3.3: The two-circle Venn diagram with the probabilities for regions w, x, y, and z inserted.

d. Using the Venn diagram,

$$\begin{aligned} R^C &= \{w, z\} \\ B &= \{w, x\} \end{aligned} \qquad \Longrightarrow \qquad R^C \cap B = \{w\}$$

So,

$$P\left(R^C \cap B\right) = P(w) = 0.1$$

e. Using the Venn diagram,

$$\begin{aligned} B^C &= \{y,z\} \\ R &= \{x,y\} \end{aligned} \quad \Longrightarrow \quad B^C \cup R = \{x,y,z\}$$

So,

$$\begin{aligned} P\left(B^C \cup R\right) &= P(x) + P(y) + P(z) \\ &= 0.5 + 0.2 + 0.2 \\ &= 0.9 \end{aligned}$$

N *We could have computed parts **a-c** using the Venn diagram, instead of by using the probability rules. We leave it to the reader to verify.*

■

Try It # 3:
Let A and B be two events of an experiment. Suppose $P(A) = 0.70, P(B) = 0.75$, and $P(A \cup B) = 0.90$. Calculate the following probabilities.

a. $P\left(A^C\right)$
b. $P(A \cap B)$
c. $P\left(A^C \cup B^C\right)$
d. $P\left((A \cup B)^C\right)$

In our next example, we will focus on a survey in which the results are given as a list, rather than in a table.

■ **Example 4** Two hundred fifty people who recently purchased a car (new or used) were surveyed, and the following information was compiled.

- 120 people purchased a new car
- 175 people were happy with their purchase
- 47 people did not buy a new car and were not happy with their purchase

Compute the probability that a surveyed person bought a new car or was unhappy with their purchase.

Solution:
Let
$N :=$ the event "a person purchased a new car," and
$H :=$ the event "a person was happy with their recent purchase."

Then, we know from the information given:

$$P(N) = \frac{120}{250}, \quad P(H) = \frac{175}{250}, \quad \text{and} \quad P\left(N^C \cap H^C\right) = \frac{47}{250}$$

The probability we are looking for is $P\left(N \cup H^C\right)$.

Using the union rule, we get $P\left(N \cup H^C\right) = P(N) + P\left(H^C\right) - P\left(N \cap H^C\right)$. While $P(N)$ and $P\left(H^C\right)$ can easily be found with the known probability rules, $P\left(N \cap H^C\right)$ cannot. Thus, we will again draw a Venn diagram to help determine $P\left(N \cup H^C\right)$.

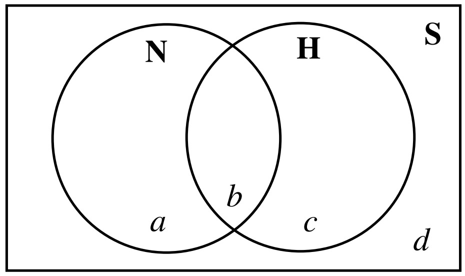


Figure 4.3.4: Unshaded Two-Circle Venn Diagram

Again, $P(\mathbf{S}) = 1$, so $P(a) + P(b) + P(c) + P(d) = 1$.

From the given information:

$$P(N) = P(a) + P(b) = \frac{120}{250}$$

$$P(H) = P(b) + P(c) = \frac{175}{250}$$

$$\begin{array}{l} N^C = \{c, d\} \\ H^C = \{a, d\} \end{array} \implies N^C \cap H^C = \{d\} \implies P\left(N^C \cap H^C\right) = P(d) = \frac{47}{250}$$

Now we have a system of equations which we can solve, using simple substitutions.

$$\begin{aligned} P(a) + P(b) + P(c) + P(d) &= 1 \\ \frac{120}{250} + P(c) + \frac{47}{250} &= 1 \\ P(c) + \frac{167}{250} &= \frac{250}{250} \\ P(c) &= \frac{83}{250} \end{aligned}$$

Using $P(c) = \frac{83}{250}$,

$$\begin{aligned} P(b) + P(c) &= \frac{175}{250} \\ P(b) + \frac{83}{250} &= \frac{175}{250} \\ P(b) &= \frac{92}{250} \end{aligned}$$

then

using $P(b) = \frac{92}{250}$,

$$\begin{aligned} P(a) + P(b) &= \frac{120}{250} \\ P(a) + \frac{92}{250} &= \frac{120}{250} \\ P(a) &= \frac{28}{250} \end{aligned}$$

Placing these values into the appropriate regions of the Venn diagram produces **Figure 4.3.5**.

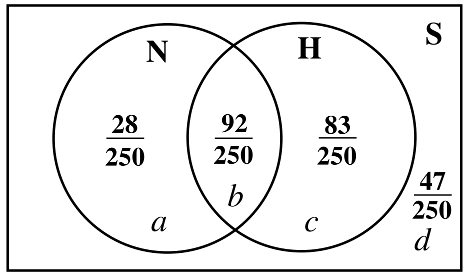


Figure 4.3.5: The two-circle Venn diagram with the probabilities for regions *a*, *b*, *c*, and *d* inserted.

Now we can use the Venn diagram, to calculate the probability in question, $P\left(N \cup H^C\right)$.

$$\begin{aligned} N &= \{a,b\} \\ H^C &= \{a,d\} \end{aligned} \quad \Longrightarrow \quad N \cup H^C = \{a,b,d\}$$

Therefore,

$$\begin{aligned} P\left(N \cup H^C\right) &= P(a) + P(b) + P(d) \\ &= \frac{28}{250} + \frac{92}{250} + \frac{47}{250} \\ &= \frac{167}{250} \end{aligned}$$

We can then say the probability that a surveyed person bought a new car or was unhappy with their purchase is $\frac{167}{250}$.

■

■ **Example 5** A survey of 350 students at a university about their classification and place of residence revealed the data shown in **Table 4.22**.

	Freshman (F)	Sophomore (M)	Junior (J)	Senior (R)	**Total**
Dormitory (D)	89	34	46	15	**184**
Apartment (A)	32	17	22	48	**119**
With Parents (W)	13	31	3	0	**47**
Total	**134**	**82**	**71**	**63**	**350**

Table 4.22: Student Living Arrangements

Compute the probability that a randomly selected student from the survey,

a. Is not a junior.
b. Is a senior living with their parents.
c. Is a freshman or lives in the dorms.

Solution:

a. Using the complement rule and the information given,

$$\begin{aligned} P\left(J^C\right) &= 1 - P(J) \\ &= 1 - \frac{71}{350} \\ &= \frac{279}{350} \end{aligned}$$

The probability a randomly selected student is not a junior is $\frac{279}{350}$.

b. A senior living with their parents is both a senior and a student who lives with their parents, so

$$P(R \cap W) = \frac{0}{350} = 0$$

The probability is 0, which may not be surprising.

c. We are looking for $P(F \cup D)$. Using the union rule and given information,

$$\begin{aligned} P(F \cup D) &= P(F) + P(D) - P(F \cap D) \\ &= \frac{134}{350} + \frac{184}{350} - \frac{89}{350} \\ &= \frac{229}{350} \end{aligned}$$

The probability a randomly selected student is a freshman or lives in the dorms is $\frac{229}{350}$.

■

N *When calculating probabilities based on information given in a table, no matter the level of difficulty, you can always add the individual cells instead of using the* ***Totals*** *with the probability rules. Always make sure to avoid double counting any intersections, no matter the method you choose.*

Try It Answers

1. **a.** $\frac{28}{50}$
   b. $\frac{20}{50}$
   c. $\frac{29}{50}$
   d. $\frac{48}{50}$
   e. $\frac{1}{50}$
   f. $\frac{30}{50}$
   g. $\frac{22}{50}$

2. **a.** $\frac{26}{52}$
   b. $\frac{28}{52}$
   c. 1
   d. $\frac{2}{52}$

3. **a.** 0.3
   b. 0.55
   c. 0.45
   d. 0.10

EXERCISES

Basic Skills Practice (Answers)

1. Suppose you roll a fair standard six-sided die, noting the number showing. What is the probability of rolling a 4 or a 5?

2. Suppose you roll a fair standard six-sided die, noting the number showing. What is the probability of rolling a 2 or an even number?

3. Let S = $\{s_1, s_2, s_3, s_4, s_5\}$ be a sample space with events $A = \{s_2, s_3, s_5\}$, $B = \{s_1, s_5\}$, and $D = \{s_2, s_4\}$. Use the probability distribution of the outcomes below to compute each of the following probabilities.

Outcome	s_1	s_2	s_3	s_4	s_5
Probability	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{7}{20}$

a. $P(A)$
b. $P\left(D^C\right)$
c. $P(A \cap B)$
d. $P(B \cup D)$
e. $P(B \cap D)$
f. $P\left(A^C \cup D\right)$
g. $P(B \cap D)^C$
h. $P\left(A^C \cup B^C\right)$

4. A marble is drawn from a bag containing 20 identical plain red marbles, 30 identical plain blue marbles, 5 identical white marbles with red stars, and 10 identical white marbles with blue stars (all the same size). What is the probability

a. The marble includes blue coloring?
b. The marble includes stars?
c. The marble includes white coloring, but not red coloring?
d. The marble includes blue coloring or white coloring?
e. The marble does not include stars?

5. Given F and G are two events of an experiment with $P(F) = 0.44$, $P(G) = 0.64$, and $P(F \cap G) = 0.38$, calculate the following probabilities.

a. $P\left(F^C\right)$
b. $P\left(G^C\right)$
c. $P(F \cup G)$

6. Given F and G are two *mutually exclusive* events of an experiment with $P(F) = 0.26$ and $P(G) = 0.47$, calculate the following probabilities.

a. $P\left(F^C\right)$
b. $P\left(G^C\right)$
c. $P(F \cap G)$
d. $P(F \cup G)$

7. A teacher asks a class of 40 students about the classes they are taking.
   - 17 of the students are taking math
   - 31 are taking English
   - 15 are taking both math and English

   What is the probability that a randomly selected student is taking

   a. Math or English?
   b. Only math?
   c. Only English?
   d. Neither course?

8. An instructor collected the following data from the students in the instructor's fall classes.

	Freshman (F)	Sophomore (M)	**Total**
MAT 121 (T)	33	15	**48**
MAT 187 (E)	40	38	**78**
MAT 194 (N)	22	52	**74**
Total	**95**	**105**	**200**

   Use the table above to answer the following.

   a. If a student is selected at random, what is the probability the student is a freshman or a sophomore?
   b. If a student is selected at random, what is the probability the student is not taking MAT 121 or is not taking MAT 194?
   c. Write the symbolic probability notation for "the probability a randomly selected student is a freshman or is taking MAT 187."
   d. Write the symbolic probability notation for "the probability a randomly selected student is a sophomore and is not taking MAT 121."
   e. Compute $P(T \cup F)$.
   f. Compute $P\left(E \cap M^C\right)$.

INTERMEDIATE SKILLS PRACTICE (Answers)

9. Suppose you draw a card from a well-shuffled standard 52-card deck. What is the probability that the card is an Ace or a diamond?

10. Suppose you draw a card from a well-shuffled standard 52-card deck. What is the probability that the card is an Ace or the King of diamonds?

11. Let S = $\{s_1, s_2, s_3, s_4, s_5\}$ be a sample space with events $A = \{s_1, s_3, s_4, s_5\}$, $B = \{s_1, s_5\}$, and $D = \{s_2, s_4, s_5\}$. Use the partial probability distribution of outcomes below to compute each of the following probabilities.

Outcome	s_1	s_2	s_3	s_4	s_5
Probability	$\frac{27}{100}$	$\frac{9}{50}$		$\frac{1}{10}$	$\frac{3}{20}$

a. $P(s_3)$
b. $P\left(A^C\right)$
c. $P(A \cap D)$
d. $P\left(B^C \cup D\right)$
e. $P(A \cap B \cap D)$
f. $P\left(A^C \cap D\right) \cup B$
g. $P(B \cup D)^C \cap B$
h. $P\left(A^C \cap D\right)^C \cup D$

12. Two fair standard ten-sided dice are cast (one green and one blue), and the numbers showing on each die are observed. What is the probability that

a. The two dice show different numbers?
b. The green die shows a 4 and the blue die shows an even number?
c. The green die shows a 4 or the blue die shows an even number?
d. The green die shows an odd number or the blue die shows an even number?

13. Given W and Y are two events of an experiment with $P(W) = 0.60$, $P(Y) = 0.45$, and $P(W \cup Y) = 0.84$, calculate the following probabilities.

a. $P\left(W^C\right)$
b. $P\left(Y^C\right)$
c. $P(W \cap Y)$
d. $P\left(W^C \cap Y^C\right)$
e. $P\left(W^C \cup Y^C\right)$
f. $P\left(W^C \cap Y\right)$

14. Given R and T are two mutually exclusive events of an experiment with $P(R) = 0.43$ and $P(T) = 0.15$, calculate the following probabilities.

a. $P\left(R^C\right)$
b. $P\left(T^C\right)$
c. $P(R \cap T)$
d. $P(R \cup T)$
e. $P(R \cap T)^C$
f. $P\left(R^C \cup T^C\right)$

15. A teacher observes their class and notices a few trends. Out of the 60 students in the class, 23 have brown hair, seven are wearing glasses, and 27 have brown hair or are wearing glasses. What is the probability that a randomly selected student from the class

a. Has brown hair and is wearing glasses?
b. Has brown hair, but is not wearing glasses?
c. Is wearing glasses, but does not have brown hair?
d. Does not have brown hair and is not wearing glasses?

16. A real estate agent has kept records of the number of bedrooms in the houses they sold during the years of 2009 to 2013. The data is listed in the following table.

	One or Two (1)	Three (3)	Four (4)	Five or more (5)	**Total**
2009 (N)	5	12	25	3	**45**
2010 (T)	7	15	22	3	**47**
2011 (E)	6	18	28	6	**58**
2012 (W)	6	16	30	2	**54**
2013 (X)	5	17	29	5	**56**
Total	**29**	**78**	**134**	**19**	**260**

Use the table above to answer the following.

a. If a home is selected at random, what is the probability the home was sold in 2011 with four or more bedrooms?
b. If a home is selected at random, what is the probability the home was sold before 2012 or had exactly three bedrooms?
c. Write the symbolic probability notation for "the probability a randomly selected home was sold in 2013, but did not have more than two bedrooms."
d. Write the symbolic probability notation for "the probability a randomly selected home was not sold in 2010 or has less than five bedrooms."
e. Compute $P((W \cup X) \cap 3)$.
f. Compute $P\left(N^C \cap 4\right)$.

Mastery Practice (Answers)

17. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ be a sample space with events $A = \{s_1, s_3, s_4, s_5\}$ and $B = \{s_1, s_2, s_5\}$. If $P(A \cap B) = 0.45$ and the probability distribution of the outcomes is given below, compute the missing probabilities.

Outcome	s_1	s_2	s_3	s_4	s_5
Probability	0.25	0.15		0.07	

18. A fair standard four-sided die is rolled, noting the number shown, and a spinner divided into 4 equal regions (red, blue, green, and yellow) is spun, noting the color. What is the probability that

a. The die shows a 2 or the spinner lands on blue?
b. The die shows an even number or the spinner lands on a color other than green?
c. The spinner lands on red or green and the die does not show a 1?

19. Given D and E are two events of an experiment with $P\left(D^C\right) = 0.65$, $P\left(E^C\right) = 0.30$, and $P(D \cup E) = 0.8$, calculate the following probabilities.

a. $P(D)$
b. $P(E)$
c. $P(D \cap E)$
d. $P\left(D^C \cap E\right)$
e. $P\left(E^C \cup D^C\right)$
f. $P\left(D \cup E^C\right)^C$

20. A student observes 50 vehicles in the parking lot. They notice that 12 of the vehicles are red and that 19 of the vehicles are SUVs. If the probability that a vehicle in the parking lot is red or is an SUV is 0.609, what is the probability that a randomly selected vehicle

a. Is a red SUV?
b. Is red, but is not an SUV?
c. Is a non-red SUV?
d. Is neither red nor an SUV?

21. The following table shows a distribution of drink preferences by age.

	Water (W)	Coffee (F)	Soda (S)	Juice (J)	**Total**
18 - 24 (Y)	22	50	61	10	**143**
25 - 40 (M)	36	78	31	20	**165**
41 and over (D)	45	57	12	38	**152**
Total	**103**	**185**	**104**	**68**	**460**

Use the table above to answer the following.

a. If a person is selected at random, what is the probability the person is not older than 40 or prefers drinking water?
b. If a person is selected at random, what is the probability the person does not prefer juice or does not prefer coffee?
c. Write the symbolic probability notation for "the probability a randomly selected person is aged 25 - 40 and does not prefer drinking soda."
d. Write the symbolic probability notation for "the probability a randomly selected person is older than 24 or prefers to drink coffee."
e. Compute $P(M \cup J)^C$.
f. Compute $P\left(\left(D \cap W^C\right)^C \cup Y\right)$.

Communication Practice (Answers)

22. Explain to someone outside a math class why the probability of the sample space is always 1.

23. Explain to someone outside a math class why the probability of the impossible event is always 0.

24. Use a Venn diagram to illustrate the complement rule.

4.4 Probability Distributions and Expected Value

© Photo by Kathryn Bollinger, 1996

Would you buy a lottery ticket with the numbers 1, 2, 3, 4, and 5? Do you think that a winning ticket with five consecutive numbers is less likely than a winning ticket with the numbers 2, 14, 18, 23, and 32? If you are playing a slot machine in Las Vegas and you have lost the last 10 times, do you keep playing the same machine because you are "due for a win?" Have you ever wondered how a casino can afford to offer meals and rooms at such cheap rates? Should you play a game of chance at a carnival? How much should an organization charge for raffle tickets for their next fund raiser? All of these questions, and more, can be answered using probabilities, specifically expected value.

Learning Objectives:
In this section, you will learn concepts related to expected value. Upon completion you will be able to:

- Express the outcomes of an experiment as values of a random variable.
- Construct probability distributions of random variables.
- Construct histograms to represent probability distributions.
- Use histograms to calculate the probabilities of events.
- Compute the expected value of a random variable.
- Solve real-world applications, using expected values.
- Explain whether or not a mathematical 'game' is fair.

Defining a Random Variable

In probability, a **variable** is represented by a letter and it represents a quantitative (numerical) value that is measured or observed in an experiment. If you have a variable, and can determine a probability associated with that variable, it is called a **random variable**.

In many cases a random variable is what you are measuring or counting in an experiment. For example, in observing how many fleas are on a prairie dog in a colony, the random variable is the number of fleas on a prairie dog in the colony.

When observing non-numerical values in an experiment, it is necessary to assign a numerical value to describe the observation, in order to perform calculations discussed later in this section. For example, if we toss a fair coin three times and observe the side landing up, then we must decide whether to *count* the number of times the coin lands on heads or the number of times the coin lands on tails; our choice would define the random variable.

Definition

- A **random variable**, $\boldsymbol{X}$, is a rule which assigns a numerical value to each outcome in an experiment.
- A **probability distribution** is used to organize the values of a random variable and their corresponding probabilities.

Example 1 For the experiment of tossing a fair coin three times and observing the side landing up on each toss, let X be the random variable counting the number of times the coin lands tails up.

a. List the sample space for the experiment.
b. List all values for the random variable, X.
c. Construct the probability distribution for X.

Solution:

a. We are tossing a coin three times, and each toss has two possible outcomes. Therefore, we should have $2 \cdot 2 \cdot 2 = 8$ total outcomes in the sample space. We can use a tree diagram, as shown in **Figure 4.4.2**, to help us visualize all of these outcomes.

Let H := the event "heads lands up" and T := the event "tails lands up."

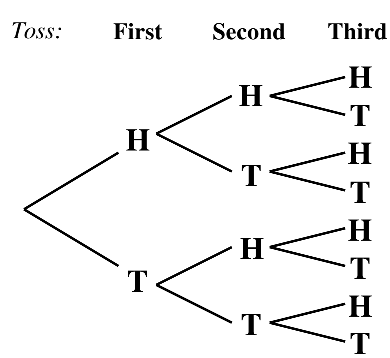


Figure 4.4.2: A visual representation of the outcomes from tossing a coin three times.

The sample space for the experiment is

$$\mathbf{S} = \{HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}$$
$$= \{(H,H,H),\ (H,H,T),\ (H,T,H),\ (H,T,T),\ (T,H,H),\ (T,H,T),\ (T,T,H),\ (T,T,T)\}.$$

b. From our sample space we can count the number of tails in each outcome. We organize this information in **Table 4.23** for the ease of the reader.

Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	0	1	1	2	1	2	2	3

Table 4.23: The random variable value associated with each outcome of the sample space, **S**.

Thus, the values of the random variable $X = 0, 1, 2,$ and 3.

c. A fair coin is being used, making each outcome on each toss equally likely, and so **S** is a uniform sample space. Therefore, when constructing the probability distribution (**Table 4.24**), we will count the number of times each value of the random variable occurs and divide by the total number of outcomes in the sample space, to arrive at the probability for each value of the random variable, X.

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Table 4.24: Probability Distribution for X

■

A probability distribution is most often given in table form (as in **Table 4.24**), but can also be represented graphically.

Definition

The graph of a probability distribution is called a **histogram**. ■

Constructing a Histogram

A histogram is drawn in the first and second quadrants of the coordinate plane.

- All axes must be labeled.
- The x-axis is a number line representing the values of the random variable, X.
- The y-axis represents probability, and should be labeled 0 to 1 for the discussions in this book.
  Note: *If the probabilities were given as percentages, the y-axis would be labeled 0% to 100%.*
- A rectangle is constructed at each random variable value, with the height corresponding to the probability of the random variable value occurring, and the width being one unit. The value of the random variable should be placed at the center of the width.

The heights of all rectangles in a histogram must add to 1 (or 100%).

The area of each rectangle in a histogram is equal to the probability of the random variable value occurring, because the width of each rectangle is 1.

■ **Example 2** Recall, from the previous section, the 2010 U.S. Census gathered information on household sizes. The data given, and completed in the previous section, is in **Table 4.25** below. Draw a histogram of the probability distribution.

Outcome	1	2	3	4	5	6	7 or more
Probability	$\frac{267}{1000}$	$\frac{336}{1000}$	$\frac{158}{1000}$	$\frac{137}{1000}$	$\frac{63}{1000}$	$\frac{24}{1000}$	$\frac{15}{1000}$

Table 4.25: Probability Distribution for Household Sizes per 2010 U.S. Census

Solution:

Begin by stating the random variable; $X :=$ the number of people in a household.

To draw the histogram, label the values of the random variable along the x-axis (for the "7 or more" category, just call it 7). The probabilities are on the y-axis.

Then, construct a rectangle at each random variable value, as previously described, noting the probabilities found in **Table 4.25**. The completed histogram is shown in **Figure 4.4.3**.

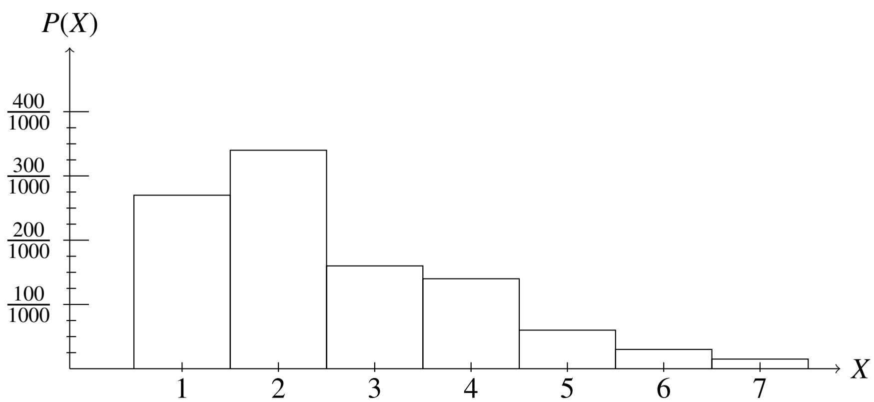


Figure 4.4.3: The corresponding histogram for Table 4.25.

■

Not only is a histogram visually helpful, but because it contains the same information as a probability distribution, we can compute probabilities of events directly from a histogram.

■ **Example 3** Using the 2010 U.S. Census histogram (**Figure 4.4.3**), determine the following probabilities.

a. $P(X \leq 4)$
b. $P(2 < X < 6)$

Solution:

a. $P(X \leq 4)$ means we are looking for the probability that the number of people in a household is 4 or less and is represented by the shaded region in **Figure 4.4.4**, below.

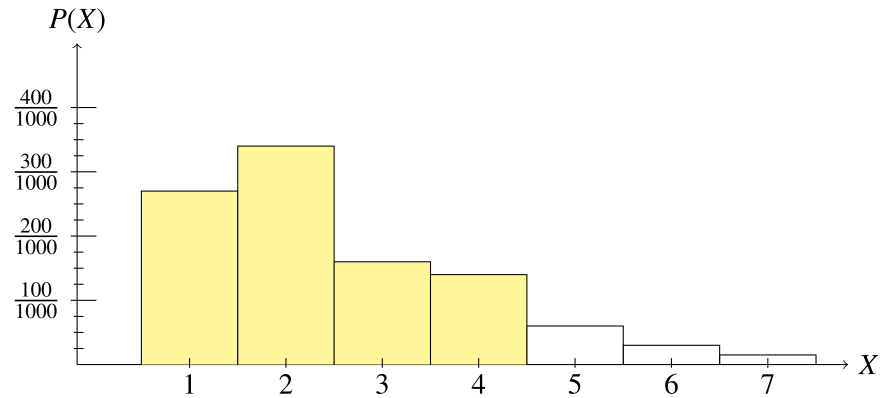


Figure 4.4.4: A graphical representation of $P(X \leq 4)$.

We add the areas of the shaded rectangles to find the probability that the value of the random variable is less than or equal to 4.

$$\begin{aligned} P(X \leq 4) &= \frac{267}{1000} + \frac{336}{1000} + \frac{158}{1000} + \frac{137}{1000} \\ &= \frac{898}{1000} \end{aligned}$$

b. $P(2 < X < 6)$ is the probability a household has between 2 and 6 people, exclusively, and is represented by the shaded region in **Figure 4.4.5**, below.

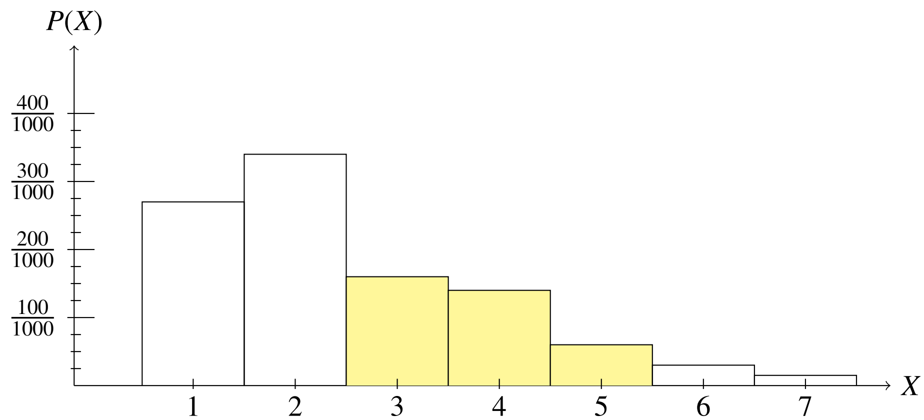


Figure 4.4.5: A graphical representation of $P(2 < X < 6)$.

Again, we add the areas of the shaded rectangles, to find the probability in question.

$$\begin{aligned} P(2 < X < 6) &= \frac{158}{1000} + \frac{137}{1000} + \frac{63}{1000} \\ &= \frac{358}{1000} \end{aligned}$$

■

N *In the 2010 U.S. Census histogram, the exact probabilities are hard to read, but are known from the probability distribution. In cases where the only probability information given is presented in a histogram, the exact probabilities should be easy to read.*

Try It # 1:
Recall the probability distribution (**Table 4.26**) for tossing a coin three times and counting the number of tails.

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Table 4.26

a. Draw the corresponding histogram.
b. Compute the probability that more than one tail is tossed.
c. Compute the probability that four tails are tossed.

Computing Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Definition

Suppose the random variable, X, can take on the n values $x_1, x_2, \ldots, x_n$, with the probability that each of these values occurs being $p_1, p_2, \ldots, p_n$, respectively.

X	x_1	x_2	x_3	$\cdots$	x_n
$P(X)$	p_1	p_2	p_3	$\cdots$	p_n

Then, the **expected value** of the random variable, X, is given by

$$\boldsymbol{E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n}$$

■

N *Expected value is the average gain or loss of an event if the experiment (procedure) is repeated many times.*

Consider the following scenario:

> In a town, 10% of the families have three children, 60% of the families have two children, 20% of the families have one child, and 10% of the families have no children. What is the expected number of children in a family in this town?

We begin by defining a random variable, X. We can let $X :=$ the number of children in a family in this town.

We are asked to determine "the *expected number* of children in a family in this town," so we look for the value of $E(X)$.

Converting the given percentages into their corresponding decimals, we can list the given information in a probability distribution table, as shown in **Table 4.27** below.

X	0	1	2	3
$P(X)$	0.10	0.20	0.60	0.10

Table 4.27: Probability Distribution for X

Using the probability distribution and formula for expected value,

$$E(X) = 0(0.10) + 1(0.20) + 2(0.60) + 3(0.10)$$
$$E(X) = 1.7$$

This means that, on average, there are expected to be 1.7 children in a family in this town. It is important to note this is a mathematical average; the authors are not saying a family has 0.7 of a child. Realistically, we would expect to see 1 or 2 children, on average, per family in this town.

■ **Example 4** A real estate investor buys a parcel of land for \$150,000. He estimates the probability that he can sell the land for \$200,000 to be 0.40, the probability that he can sell it for \$160,000 to be 0.45, and the probability that he can sell it for \$125,000 to be 0.15. What is the expected profit from the sale of this land?

Solution:
If we let $X :=$ the profit from the sale of the parcel of land, in dollars, we are looking to compute $E(X)$.

We can calculate the profit for each selling price first.

$$\$200,000 - \$150,000 = \$50,000 \text{ profit}$$
$$\$160,000 - \$150,000 = \$10,000 \text{ profit}$$
$$\$125,000 - \$150,000 = -\$25,000 \text{ profit (loss)}$$

Using the profit for each selling price, with the corresponding probability, the probability distribution for X can be constructed, as shown in **Table 4.28**.

X	$-\$25,000$	\$10,000	\$50,000
$P(X)$	0.15	0.45	0.40

Table 4.28: Probability Distribution for X

The expected value of X is then calculated to be

$$\begin{aligned} E(X) &= (-25000)(0.15) + (10000)(0.45) + (50000)(0.40) \\ &= 20750 \end{aligned}$$

Thus, the expected profit from the sale of the parcel of land is \$20,750.

■

Try It # 2:
In a particular town, the attendance at a football game depends on the weather. On a sunny day the attendance is 60,000, on a snowy day the attendance is 40,000, and on a rainy day the attendance is 30,000. If for the next football season, the weatherman has predicted that 30% of the game days will be sunny, 50% of the game days will be snowy, and 20% of game days will be rainy, what is the expected attendance for a single game during the next football season?

Expected value is very common in making insurance decisions. Our next two examples illustrate some insurance scenarios.

■ **Example 5** An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake is estimated by the insurance company to be \$60,000. If the premium for earthquake insurance is \$100, what is the company's expected profit from the sale of the policy?

Solution:
Let $X :=$ the company's profit from the sale of the policy, and determine $E(X)$.

There are two situations to consider here,

1. The company sells the policy, but no earthquake occurs.
2. The company sells the policy and an earthquake does occur.

The **premium** is what a person pays to obtain an insurance policy (the insurance company's revenue). In Situation 1, the company's profit is \$100, as no payout was made by the company. However in Situation 2, an earthquake does occur and a payout is made, so the company's profit is $\$100 - \$60,000 = -\$59,900$.

With this information, and the stated probability, the probability distribution for X is given in **Table 4.29**, below.

X	100	-59900
$P(X)$	1 - 0.0013 = 0.9987	0.0013

Table 4.29: Probability Distribution for X

The expected value of X is then

$$\begin{aligned} E(X) &= (100)(0.9987) + (-59900)(0.0013) \\ &= 22 \end{aligned}$$

Thus, the expected profit for the company is \$22. Realistically the company is not profiting \$22 on a particular policy. Instead, if the company sold the same policy under the same earthquake conditions to many people, the company would profit an average of \$22 per policy sold.

■

Not surprisingly, the expected value in the previous example is positive; the insurance company can only afford to offer policies if they, on average, expect to make money on each policy. They can afford to pay out the occasional benefit, because they offer enough policies that those benefit payouts are balanced by the rest of the insured people who do not receive a payout.

For people buying insurance, there is then a corresponding negative expected value. However, people still purchase insurance policies, as there is a security that comes from insurance that is worth the cost.

It should be noted that the lowest premium a policy will be sold for is the one in which the company's expected profit is \$0.

■ **Example 6** What is the minimum the company will sell the earthquake policy for in the previous example?

Solution:
We will ignore the given \$100 premium in the example, as we are looking for a minimum the company is willing to charge for the premium.

If we let $X :=$ the company's profit from the sale of the policy and $m :=$ the minimum premium for which the earthquake policy is sold, the probability distribution for X can now be written as shown in **Table 4.30**, below.

X	m	$m - 60000$
$P(X)$	0.9987	0.0013

Table 4.30: Probability Distribution for X

The expected value of X is now

$$E(X) = (m)(0.9987) + (m - 60000)(0.0013)$$

For a minimum premium, the company's expected profit is \$0; $E(X) = 0$. So,

$$\begin{aligned} 0 &= 0.9987m + 0.0013(m - 60000) \\ 0 &= 0.9987m + 0.0013m - 78 \\ 0 &= 1m - 78 \\ 78 &= m \end{aligned}$$

Therefore, the minimum amount the company will sell the policy for is \$78.

■

Determining if a Mathematical Game is Fair

Expected value also has applications outside of business. Many applications involve mathematical 'games,' where money is paid to play the game.

■ **Example 7** Valley View Elementary is trying to raise money to buy tablets for its classrooms. The PTA sells 2000 raffle tickets at $3 each. First prize is a flat-screen TV worth $500. Second prize is an android tablet worth $375. Third prize is an e-reader worth $200. Five $25 gift certificates will also be awarded. What are the expected net winnings for a person who buys one ticket?

Solution:
A person's NET winnings are equal to their profit. If we let $X :=$ the NET winnings (profit) for a person buying one ticket, then we need to determine $E(X)$.

Out of the 2000 raffle tickets sold, a total of eight tickets are winners, so the other 1992 tickets are losers.

Knowing each ticket is sold for $3, **Table 4.31** gives the probability distribution for X.

X	$500-3=497$	$375-3=372$	$200-3=197$	$25-3=22$	$0-3=-3$
$P(X)$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{5}{2000}$	$\frac{1992}{2000}$

Table 4.31: Probability Distribution for X

Using the formula for expected value, we have

$$\begin{aligned} E(X) &= 497\left(\frac{1}{2000}\right)+372\left(\frac{1}{2000}\right)+197\left(\frac{1}{2000}\right)+22\left(\frac{5}{2000}\right)+(-3)\left(\frac{1992}{2000}\right) \\ &= -2.4 \end{aligned}$$

So, the expected **net** winnings for a person who buys one ticket are $-\$2.40$. Therefore, a person would expect to lose, on average, $-\$2.40$ every time they purchase a ticket.

■

In general, if the expected profit of a game is negative, it is not a good idea to play the game, because, on average, you will lose money. It would be better to play a game with a positive expected profit (good luck trying to find one!). Not surprisingly, the expected profit for casino games is negative for the player, which is correspondingly positive for the casino (house); the expected profit must be positive or the casinos would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected profit is negative. That is fine so long as you enjoy playing the game and think it is worth the cost, but it would be unwise to expect to come out ahead. Even if the *average* net winnings are positive, it could be the case that most people lose money and only one very fortunate individual wins a great deal of money.

If the expected profit for a game is 0, we call it a **fair game**, as neither side has an advantage.

Definition

A mathematical 'game' (money is paid to play) is **fair**, if the expected profit for both sides is 0. ■

*Expected profit is also known as expected **net** winnings. In a fair game, for these to be 0 for both sides, the player should pay an amount equal to their expected winnings.*

■ **Example 8** Suppose you and your friend play a game that consists of rolling a fair standard six-sided die. Your friend offers you the following deal: If the die shows any number from 1 to 5, he will pay you the face value of the die in dollars, that is, if the die shows a 4, he will pay you \$4. But, if the die shows a 6, you will have to pay him \$12. Is this a fair game?

Solution:
Start by letting $X :=$ your profit.

First, we will organize the information given, as shown in **Table 4.32**.

Outcome of the die	1	2	3	4	5	6
\$ amount you win or lose	win \$1	win \$2	win \$3	win \$4	win \$5	lose \$12
X = your profit	$1-0=1$	$2-0=2$	$3-0=3$	$4-0=4$	$5-0=5$	$0-12=-12$

Table 4.32: The random variable values associated with each outcome of the experiment.

When working with a fair six-sided die, each outcome has the same probability of occurring, $\frac{1}{6}$.

Thus, the probability distribution for X can be written, as shown in **Table 4.33**.

X	1	2	3	4	5	-12
$P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Table 4.33: Probability Distribution for X

The expected value of X is then

$$\begin{aligned} E(X) &= 1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+(-12)\left(\frac{1}{6}\right) \\ &= 0.5 \end{aligned}$$

Therefore, you expect to profit \$0.50, on average, when playing this game; your friend should expect to lose \$0.50 on average. Because your expected profit is not \$0, the game is not fair; someone (you) has an advantage.

■

■ **Example 9** A roulette wheel consists of 38 slots numbered 0, 00, and 1 through 36, evenly spaced around a wheel. The wheel is spun in one direction, and a ball is rolled around the wheel in the opposite direction. Eventually the ball will drop into one of the numbered slots. A player bets \$1 on a single number. If the ball lands in the slot for that number, the player wins \$36; otherwise, the player wins nothing. Is this game fair?

Solution:
There are 38 slots on the roulette wheel. Only one slot wins, and the other 37 slots lose. Considering the slots are evenly spaced, each slot and corresponding number is equally likely, so

$$P(\text{win}) = \frac{1}{38} \quad \text{and} \quad P(\text{lose}) = \frac{37}{38}$$

Betting \$1, if $X :=$ the player's profit, then the probability distribution of X is given in **Table 4.34**.

X	$36-1=35$	$0-1=-1$
$P(X)$	$\frac{1}{38}$	$\frac{37}{38}$

Table 4.34: Probability Distribution for X

Thus, the expected value is

$$\begin{aligned} E(X) &= 35\left(\frac{1}{38}\right) + (-1)\left(\frac{37}{38}\right) \\ &\approx -0.0526 \end{aligned}$$

The expected value of the game is approximately $-\$0.05$. This means that the player would expect to lose an average of 5 cents for each game played, and the house would expect to profit an average of 5 cents for each game played. As someone (the house) has an advantage, this game is not fair.

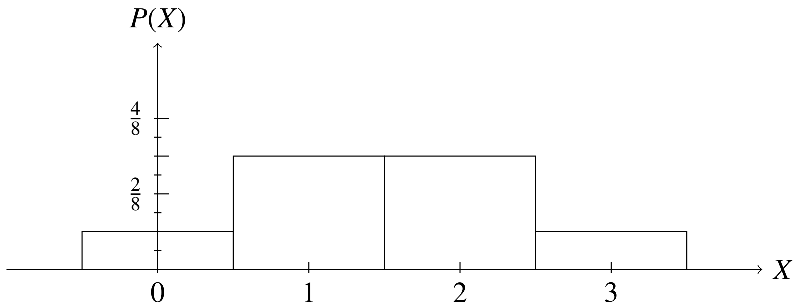
■

Try It # 3:
You and a friend are playing some of the games at the county fair. A particular game consists of drawing a single card from a well-shuffled standard 52-card deck. You win \$5 if you draw an Ace, \$2 if you draw a face card, and \$0.25 if any other card is drawn, besides a 10. If a 10 is drawn, you win nothing.

a. How much should you expect to pay to play this game, in order for it to be fair?
b. If the county fair charges \$2, is it a good idea to play the game?

Try It Answers

1. **a.**



b. $\frac{4}{8}$

c. 0

2. 44,000 people

3. **a.** \$1
   b. No, it's not a good idea.

EXERCISES

Basic Skills Practice (Answers)

1. Suppose you roll four fair standard six-sided dice, noting the number showing on each die. Let X be the random variable denoting the number of 5's showing. Write all values of X.

2. A Houstonian is looking to purchase flood insurance for their \$20,000 car. The insurance premium for a 1-year policy is \$508. Houston has a 12% chance of a flood over the next year. Let X be the insurance company's net gain or loss on the policy described. Write all possible values of X.

3. You are going on a European vacation and decide to purchase travel insurance on your brand new luggage worth \$2000 (excluding the value of the contents). The premium for the policy is \$48. In the event your luggage is damaged to the point of needing duct tape, then you will receive 40% of the value of the luggage. In the event your luggage is lost or stolen, then you will receive 100% of the value of the luggage. According to airline data, the probability of your luggage being damaged and needing duct tape is 9%, while the probability your luggage is lost or stolen is 3.5%. Let X be your net gain or loss on the policy described. Write all possible values of X.

4. You pay \$3 to play a game where you draw a numbered ball from a bucket containing twelve identical balls numbered 1 - 12. If you draw ball numbered less than 3, you win a dollar amount equal to the number on the ball. If you draw a ball numbered with a multiple of 3, you win \$8. If you draw a ball with any other number, you win nothing. Let X be your winnings and Y be your net winnings. Write all possible values of both X and Y.

5. Given the probability distribution below, draw the corresponding histogram.

X	1	2	3	4	5
$P(X)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{7}{20}$

6. Given the *partial* probability distribution below, draw the corresponding *complete* histogram.

X	-1	0	1	2	3
$P(X)$	$\frac{27}{100}$	$\frac{9}{50}$		$\frac{1}{10}$	$\frac{3}{20}$

7. Given the probability distribution below, compute the expected value of X, $E(X)$.

X	1	2	3	4	5
$P(X)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{7}{20}$

8. Given the probability distribution below, compute the expected value of X, $E(X)$.

X	-5	0	2	7	25
$P(X)$	0.38	0.29	0.16	0.03	0.14

Intermediate Skills Practice (Answers)

9. Suppose you roll two fair standard distinguishable six-sided dice, noting the number showing on each die. Let X be the random variable denoting the number of 3's showing. Create the probability distribution for X.

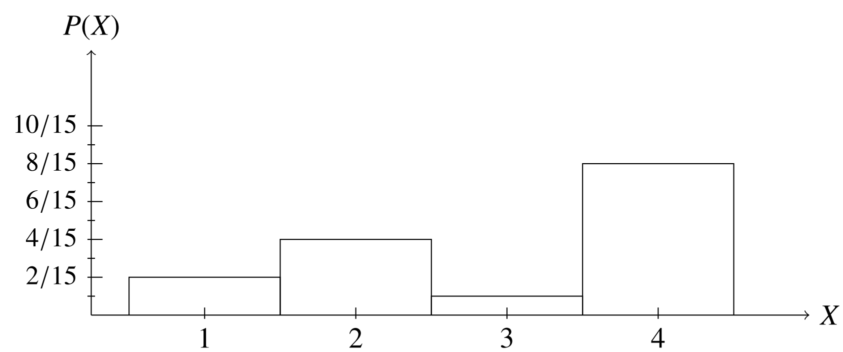
10. Suppose you roll a fair standard 30-sided die, noting whether or not the number rolled is a multiple of seven. Let X be the random variable denoting a multiple of seven as 1 and not a multiple of seven as 0. Create the probability distribution for X.

11. A Houstonian is looking to purchase flood insurance for their \$20,000 car. The insurance premium for a 1-year policy is \$508. Houston has a 12% chance of a flood over the next year. Let X be the insurance company's net gain or loss on the policy described. Create the probability distribution for X.

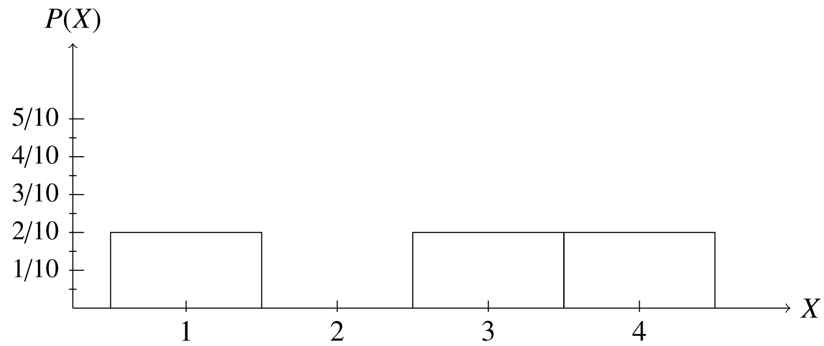
12. You are going on a European vacation and decide to purchase travel insurance on your brand new luggage worth \$2000 (excluding the value of the contents). The premium for the policy is \$48. In the event your luggage is damaged to the point of needing duct tape, then you will receive 40% of the value of the luggage. In the event your luggage is lost or stolen, then you will receive 100% of the value of the luggage. According to airline data, the probability of your luggage being damaged and needing duct tape is 9%, while the probability your luggage is lost or stolen is 3.5%. Let X be your net gain or loss on the policy described. Create the probability distribution for X.

13. You pay \$3 to play a game where you draw a numbered ball from a bucket containing twelve identical balls numbered 1 - 12. If you draw ball numbered less than 3, you win a dollar amount equal to the number on the ball. If you draw a ball numbered with a multiple of 3, you win \$8. If you draw a ball with any other number, you win nothing. Let X be your winnings and Y be your net winnings. Create the probability distribution for X.

14. Given the histogram below, create the corresponding probability distribution.



15. Use the given *partial* histogram below, to answer the questions which follow.



a. Compute the value of $P(X = 2)$.

b. Compute the value of $P(X \leq 3)$.

c. Compute the value of $P(1 < X < 4)$.

d. Compute the value of $P(X > 4)$.

e. Calculate $E(X)$.

16. A department tracks the number of days it takes a instructor to respond to a student email. If D represents the number of days that a student will wait for a response from an instructor, the data collected is summarized in the probability distribution below. On average, how many days can a student expect to wait for a response from their instructor in this department?

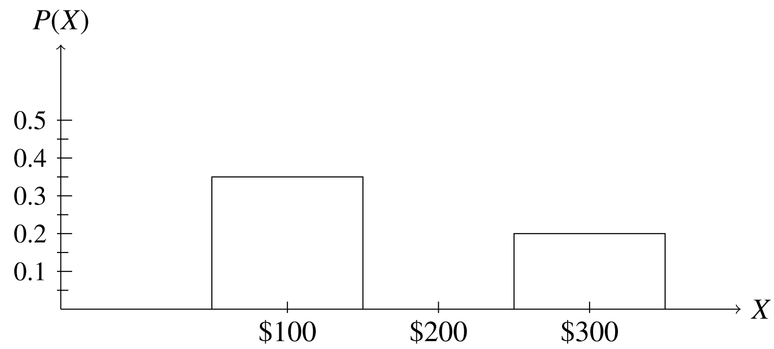
D	0	1	2	3	4
$P(D)$	$\frac{17}{100}$	$\frac{28}{100}$	$\frac{32}{100}$	$\frac{15}{100}$	$\frac{8}{100}$

Mastery Practice (Answers)

17. Given the probability distribution below, draw the corresponding histogram.

X	-1	2	4	7	9
$P(X)$	0.1	0.15	0.4	0.05	0.3

18. Create the complete probability distribution corresponding to the given partial histogram.



19. A quality control department tracks the number of visible chocolate chips on a cookie in random packages from the company's assembly line. If X represents the number of visible chocolate chips, the data collected is summarized in the frequency distribution below. On average, how many chocolate chips are visible on a cookie from this company?

X	0	2	4	5	7
Frequency	14	52	298	173	22

20. You purchase a $250,000 long-term disability policy. The insurance premium for this 1-year policy is $4500. Based on your work environment, the probability of a long-term disability is 0.007. Let X be the insurance company's net gain or loss on the policy described.

    a. Create a probability distribution for X.

    b. Compute the company's expected net gain on this policy.

21. You purchase a brand new yacht for $1,375,000 and insure it. The policy pays 20% of the yacht's value if it is involved in a minor accident (as defined by the insurance company) or 85% of the yacht's value if it sinks. The probability of a minor yachting accident is 0.13, while the probability your yacht sinks is 0.002. Let X be your net gain from this policy.

    a. Create a probability distribution for X.

    b. Compute the minimum amount the insurance company will charge for this policy.

22. The local fair has a game where you pay $2 to perform a two-stage game: flip a fair coin, noting the side showing, and roll a fair standard eight-sided die, noting whether the number showing is odd or even. If the coin shows heads and the die shows an even number, then you win $20. If the coin shows tails and the die shows an odd number, you win $10. Otherwise, you lose. Let X be your net winnings.

    a. Create a probability distribution for X.

    b. Compute your expected net winnings for the game.

    c. Is this game fair?

23. You play a game where a fair standard four-sided die is rolled, noting the number shown, and a spinner divided into four equal regions (red, blue, green, and yellow) is spun, noting the color. If you roll a 1, you win \$6. If you roll a 3 and the spinner lands on a color other than green, you win \$8. Otherwise, you win nothing. Let X be your winnings.

    a. Create a probability distribution for X.

    b. Compute your expected winnings for the game.

    c. How much should be charged, in order to make the game fair?

COMMUNICATION PRACTICE (Answers)

24. If you expect to win \$5 on a game, how much should you expect to pay to play the game, in order to make the game fair? Explain your reasoning.

25. Explain why an insurance company charges more than the minimum premium.

26. Explain why the height of every rectangle in a probability histogram is between 0 and 1, inclusive.

Chapter Review

Directions: Read each reflection question. If you are able to answer the question, then move on to the next question. If not, use the mathematical questions included to help you review. (Answers)

1. Can you define the terminology used in the description of a mathematical experiment?
   a. Define the sample space of an experiment.
   b. Define an event of an experiment.
   c. Differentiate between a simple, a certain, and an impossible event of an experiment.

2. Can you differentiate between the different terms and formulas of a mathematical experiment?
   a. State the sample space of rolling a standard four-sided die and a standard five-sided die, noting the number rolled on each die.
   b. State the sample space of selecting a letter from the word TEXAS.
   c. List all simple events for the experiment of drawing a card from a standard deck of cards and noting the suit.
   d. List an impossible event for the experiment of spinning a spinner with six equal regions: red, blue, green, yellow, orange, and white, noting the region's color the spinner lands in. (Re-spin if the spinner lands on a boundary line.)
   e. State the total number of possible events for the experiment of rolling a standard five-sided die, noting the number rolled.
   f. An experiment consists of rolling two standard distinguishable six-sided dice, noting the number rolled on each die. Write the outcomes of the event, E, "the sum of the dice is eight, with at least one die showing an odd number."

3. Can you express the operations of events both symbolically and verbally?
   Given an experiment consisting of drawing a card from a standard deck of cards, let
   $A :=$ the event "a face card is drawn,"
   $B :=$ the event "a red card is drawn,"
   $C :=$ the event "a spade is drawn," and
   $D :=$ the event "a diamond is drawn."
   a. Write the symbolic notation for the event "a red face card is drawn."
   b. Write the symbolic notation for the event "a spade is not drawn."
   c. Describe the event $C \cup D$.
   d. Describe the event $A^C \cap B$.

4. Can you explain what it means for two events to be mutually exclusive?
   Given an experiment consisting of rolling a standard ten-sided die, let
   $A :=$ the event "an even number is rolled,"
   $B :=$ the event "a multiple of 3 is rolled," and
   $C :=$ the event "a 5 is rolled."
   a. Determine if the events A and B are mutually exclusive. Explain your reasoning.
   b. Determine if the events B and C are mutually exclusive. Explain your reasoning.
   c. Write an event, not given above, which is mutually exclusive to event B.

5. How do you use a tree diagram to describe an experiment?
   a. Use a tree diagram to write the sample space for the experiment of rolling a standard five-sided die, noting the number rolled, and then flipping a two-sided coin, noting the side landing up.
   b. Use a tree diagram to write the sample space for the experiment of drawing a card from a standard deck of cards, noting the suit, and drawing a card from a second standard deck of cards, noting the color.

6. How do you use a Venn diagram to represent the result of an operation of events?
   For the result of each of the following, shade a two-circle Venn diagram, with events A and B.
   a. $B^C \cup B$
   b. $\left(A \cap B^C\right) \cup B$
   c. $\left((A \cup B)^C\right)^C$
   d. $\left(A^C \cup B^C\right) \cap A$

7. Can you describe a uniform sample space, and then compute the probability of events from a uniform sample space?
   a. Does the experiment of writing each letter of the word AGGIES on a scrap of paper and then drawing a scrap of paper, noting the letter drawn, have a uniform sample space? Why or why not?
   b. Does the experiment of selecting a random student from a class of 100, based on their UIN, have a uniform sample space? Why or why not?
   c. Suppose a fair standard 20-sided die is rolled, and the number rolled is recorded. What is the probability
      i. A six is rolled?
      ii. An even number is rolled?
      iii. A zero is rolled?
      iv. A number less than seven is rolled?
   d. Given a card is drawn from a well-shuffled standard deck of cards, compute
      i. P(the card is the six of hearts)
      ii. P(the card is a King)
      iii. P(the card is a club)
      iv. P(the card is a face card)

8. Can you give an example of an event with a theoretical probability or an empirical probability?
   a. Determine if the event has a theoretical or empirical probability?
      i. Drawing a card from a well-shuffled standard deck of cards, noting the rank. Let E be the event "a 4 is drawn."
      ii. Spinning a game spinner with three equal regions (blue, red, and green), noting the region's color the spinner stops on. (Re-spin if the spinner lands on a boundary line.) Let F be the event "the spinner lands on red."
      iii. Conducting a survey concerning the new building going up in the middle of campus, and using the results to predict the probability of the event, G, that "students will use the learning pods designed inside the building, daily."

9. How would you construct a probability distribution table?
   a. Create a probability distribution table for rolling two fair standard distinguishable six-sided dice, noting if doubles are rolled. Does the experiment have an uniform sample space?
   b. Create a probability distribution table for drawing a card from a well-shuffled standard deck of cards, noting the suit. Does the experiment have an uniform sample space?

10. How would you read a probability distribution table to find the probability of a given event?
    a. Use the following partial probability distribution on the height, in inches, for a sample of 100 male semiprofessional soccer players, to compute the probability of each event.

Height (in)	59.9 - 61.9	62 - 63.9	64 - 65.9	66 - 67.9	68 - 69.9	70 - 71.9	72 - 73.9	74 - 75.9
Probability	$\frac{5}{100}$	$\frac{3}{100}$		$\frac{40}{100}$	$\frac{17}{100}$	$\frac{12}{100}$	$\frac{7}{100}$	$\frac{1}{100}$

i. P(a player is 64 to 65.9 inches tall)
ii. P(a player is no more than 69.9 inches tall)
iii. P(a player is between 62 and 73.9 inches tall, inclusively)
iv. P(a player is not taller than 71.9 inches)
v. P(a player is less than 59.9 inches tall)

11. Can you state the rules of basic probability, and do you know when to apply these rules?
    a. State the union rule for the probability events H or K occur.
    b. State the complement rule for the probability event M does not occur.
    c. Suppose a fair standard 20-sided die is rolled, the number rolled is recorded, and then a fair two-sided coin is flipped, noting the side landing up. What is the probability that
    i. A 6 is rolled or the coin lands on heads?
    ii. An even is rolled and the coin lands on tails?
    iii. A 5 is not rolled?
    iv. A number no greater than 8 is rolled, but the coin lands on tails?
    d. Suppose two fair standard six-sided dice are rolled, the first green and the second blue. What is the probability
    i. The green die does not show a 4?
    ii. The green die shows an even number and the blue die shows any number but 3?
    iii. The green die shows a 1 and doubles are showing?
    e. Given a card is drawn from a well-shuffled standard deck of cards, compute
    i. P(the card is a two or a heart)
    ii. P(the card is a not a red card)
    iii. P(the card is a red face card)
    iv. P(the card is a Queen, but not a heart)
    v. P(the card is an 8 or a 10)
    vi. P(the card is not red, but is also not a club)
    f. The results of an 80-person survey on the idea of reviving a state university football rivalry are shown in the table below.

	In Favor	Opposed	No Opinion	**Total**
Incoming Freshman	1	4	18	**23**
Current Students	10	5	2	**17**
Alumni	39	0	1	**40**
Total	**50**	**9**	**21**	**80**

What is the probability a randomly selected person is
i. In favor of reviving the rivalry?
ii. A current student who opposes reviving the rivalry?
iii. Is an incoming freshman or has no opinion about reviving the rivalry?
iv. Does not favor reviving the rivalry?
v. Is not a current student and has no opinion about reviving the rivalry?
vi. Is not an alumni or is in favor of reviving the rivalry?

12. Can you identify when a Venn diagram is useful for finding probabilities?
Suppose A and B are two events of an experiment, where $P(A) = 0.5$, $P(B) = 0.6$, and $P(A \cap B) = 0.4$. Use a Venn diagram to calculate the following probabilities.
    a. $P(A \cup B)$
    b. $P(A^C \cap B)$
    c. $P\left(A^C \cup B^C\right)$

13. Can you quantify the outcomes of an experiment and display their probabilities using a table?
    a. A fair coin is tossed four times, and the number of times the coin lands tails up, X, is recorded. Construct a probability distribution for X.
    b. Two fair standard distinguishable six-sided dice are rolled, and the sum of the dice, X, is recorded. Construct a probability distribution for X.

14. Can you draw a graphical representation of a probability distribution table?
    a. What is the graphical representation of a probability distribution table called?
    b. A hospital researcher is interested in the number of times the average post-op patient will ring the nurse during a 12-hr shift. For a random sample of 50 patients, the following information was obtained. Let X be the number of times a patient rings the nurse during a 12-hour shift.

X (rings)	0	1	2	3	4	5
$P(X)$	$\frac{4}{50}$	$\frac{8}{50}$	$\frac{16}{50}$	$\frac{14}{50}$		$\frac{2}{50}$

    Construct the corresponding complete histogram for the distribution.

    c. Construct a histogram for the probability distribution found in **Question 13b**.

15. How would you find the expected value of a random variable?
    a. Explain what expected value means in words someone outside of a mathematics class will understand.
    b. A hospital researcher is interested in the number of times the average post-op patient will ring the nurse during a 12-hr shift. For a random sample of 50 patients, the following information was obtained. Let X be the number of times a patient rings the nurse during a 12-hour shift.

X (rings)	0	1	2	3	4	5
$P(X)$	$\frac{4}{50}$	$\frac{8}{50}$	$\frac{16}{50}$	$\frac{14}{50}$		$\frac{2}{50}$

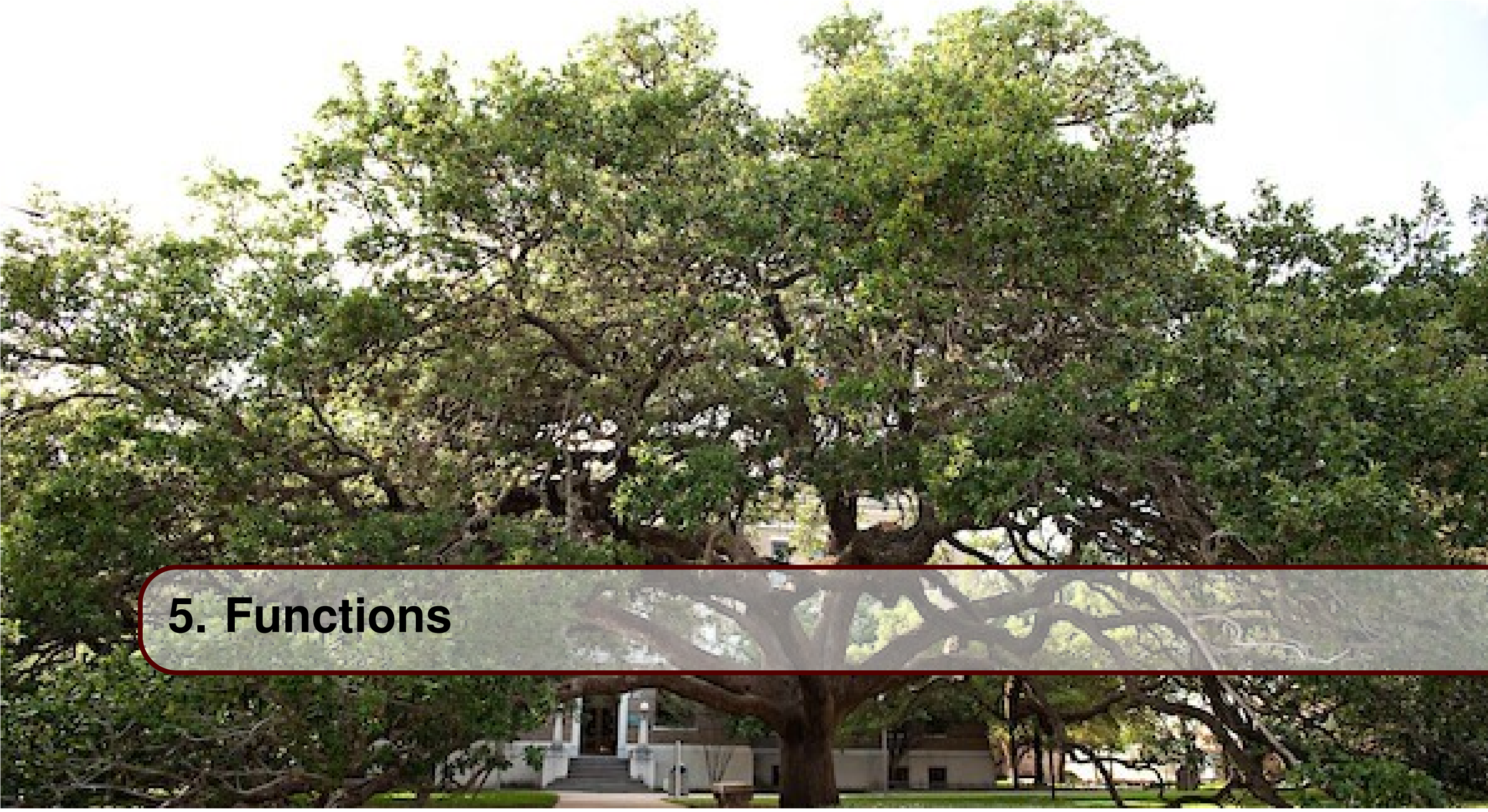
    Use the probability distribution to compute the expected value of the random variable, X. Explain your answer in the context of the application.

    c. Use the probability distribution found in **Question 13b** to compute the expected value of the random variable, X. Explain your answer in the context of the application.

16. How would you find the expected value of a quantity in real-world applications, including mathematical games?
    a. Explain what it means for a mathematical game to be fair.
    b. At age 25 you purchase a 35-year, \$250,000 injury insurance policy for \$10,500. If you suffer an injury that causes you to be unable to work prior to the end of the 35 years, the insurance company will pay you the \$250,000. If you do not suffer an injury before the end of the 35 years, you receive nothing. What is the expected profit on the injury policy, if the likelihood you are injured in the next 35 years is 2%? Is the cost of the policy reasonable for you as the consumer?
    c. A local group is sponsoring a game at a nearby festival. The game costs \$1 to play. The player will toss a fair two-sided coin and call out whether the coin is going to land heads or tails up. If the player guesses correctly, they are given \$2. If they are incorrect, they receive nothing. What are the player's expected net winnings of the game? Is the game fair?

V

Chapter 5



5. Functions

In this chapter we are going to discuss functions.

The authors recommend all students are comfortable with the topics below prior to beginning this chapter. If you would like additional support on any of these topics, please refer to the Appendix.

A.1 - Number Sense

A.2 - Introduction to Algebra

A.3 - Introduction to Algebraic Expressions

A.4 - Factoring

A.5 - Solving Quadratic Equations

5.1 Relations and Functions

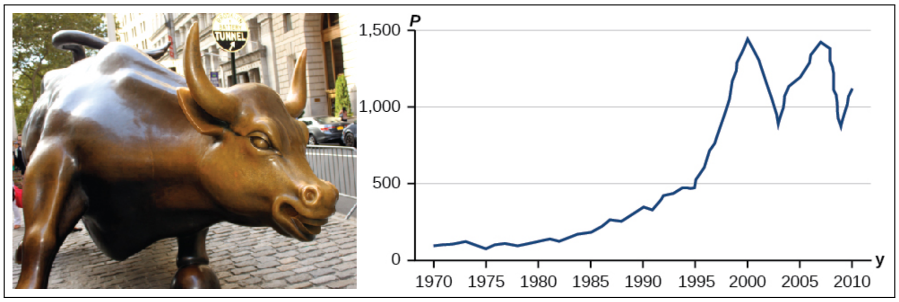


Figure 5.1.1: Credit "bull": Modification of work by Prayitno Hadinata; Credit "graph": Modification of work by MeasuringWorth

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. **Figure 5.1.1** tracks the value of an initial investment of just under \$100 over the 40 years (1970 - 2010). It shows that an investment that was worth less than \$500 until about 1995, skyrocketed up to about \$1500 by the beginning of 2000. That five-year period became known as the "dot-com bubble," because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph, beginning at the end of 2000.

Notice, as we consider this example, that there is a definite relationship between the year and the stock market average. For any year we choose, we can determine the corresponding value of the stock market average.

Learning Objectives:
In this section, you will learn about the relationship between relations and functions. Upon completion you will be able to:

- Translate between a set of real numbers and interval notation.
- State whether or not a relation is a function.
- State the domain and range of a given graphical representation of a function, using interval notation.
- Use and apply function notation to given scenarios.

Writing Interval Notation

We will focus on the use of the real numbers, $\mathbb{R}$. Recall that we may visualize $\mathbb{R}$ as a line. Segments of this line are called **intervals** of numbers, which can be discussed symbolically or graphically. Symbolically, these segments can be written using set-builder or interval notation.

Definition

- **Set-builder notation** is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x \mid \text{statement about } x\}$ which is read as, " the set of all x such that the statement about x is true." For example,
$$\{x \mid 4 < x \leq 12\}$$
would be read as "the set of all x such that four is less than x is less than or equal to 12."

- **Interval notation** is a way of describing sets that include all real numbers between a lower limit, that may or may not be included, and an upper limit, that may or may not be included. The endpoint values are listed, separated by a comma, between brackets or parentheses. A square bracket, "[" or "]", indicates inclusion in the set, and a parenthesis, "(" or ")", indicates exclusion from the set. The example given in set-builder notation above would be written, using interval notation, as
$$(4, 12].$$

Graphically speaking, for intervals with finite endpoints, we use a filled-in or 'closed' dot to indicate an endpoint is included in the interval. On the other hand, an 'open' circle is used to signal an endpoint is not included in the interval.

If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and ∞ to indicate that the interval extends indefinitely to the right. As infinity is a concept, and not a number, when using these symbols we always use parentheses in interval notation, and when graphing we use an appropriate arrow to indicate that the interval extends indefinitely in one (or both) direction(s).

Below is a summary of the different ways to express given sets of numbers.

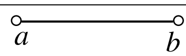
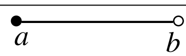
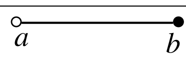
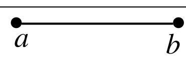
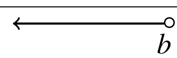
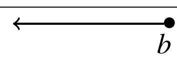
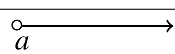
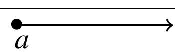
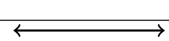
Set-Builder Notation	Segment of the Real Number Line	Interval Notation	Verbal Description
$\{x \mid a < x < b\}$	 a b	(a,b)	x is strictly between a and b
$\{x \mid a \leq x < b\}$	 a b	$[a,b)$	x is between a and b and includes a
$\{x \mid a < x \leq b\}$	 a b	$(a,b]$	x is between a and b and includes b
$\{x \mid a \leq x \leq b\}$	 a b	$[a,b]$	x is between a and b, including a and b
$\{x \mid x < b\}$	 b	$(-\infty,b)$	x is less than b
$\{x \mid x \leq b\}$	 b	$(-\infty,b]$	x is less than or equal to b
$\{x \mid x > a\}$	 a	(a,∞)	x is greater than a
$\{x \mid x \geq a\}$	 a	$[a,\infty)$	x is greater than or equal to a
$\{x \mid -\infty < x < \infty\}$		$(-\infty,\infty)$	x is any real number

Table 5.1: Equivalent Notations for the Description of an Interval

▪ **Example 1** Consider the sets of real numbers described in **Table 5.2** below. Fill in the missing cells of the table with the different equivalencies.

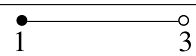
	Set-Builder Notation	Segment of Real Line	Interval Notation	Verbal Description
a.		 1 3		
b.				x is between -1 and 4, and includes both -1 and 4
c.	$\{x \mid x \le 5\}$			
d.			$(-2, \infty)$	

Table 5.2: A table of intervals with missing equivalencies.

Solution:

a. The given segment of the real number line represents all numbers between 1 and 3; the segment includes 1, but does not include 3.

So the set-builder notation is $\{x \mid 1 \le x < 3\}$.

Because 1 is included, but 3 is not included, the interval notation is $[1,3)$.

The verbal description is "x is between 1 and 3 and includes 1."

b. The given verbal description of the interval is "x is between -1 and 4, and includes both -1 and 4."

If we break down the statement and start with "x is between -1 and 4," we get $-1 < x < 4$. Additionally, the interval "includes both -1 and 4," so the set-builder notation is $\{x \mid -1 \le x \le 4\}$.

The corresponding segment of the real number line, shown in **Figure 5.1.2** , starts with a 'closed' dot at -1 and ends with a 'closed' dot at 4.

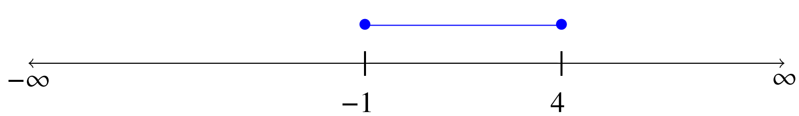


Figure 5.1.2: A graphical representation of "x is between -1 and 4, and includes both -1 and 4."

Because -1 and 4 are both included, the interval notation is $[-1,4]$.

c. The given set-builder notation shows that x must satisfy the inequality $x \le 5$.

So, the corresponding segment of the real number line, shown in **Figure 5.1.3**, extends to $-\infty$ indicated by an arrow to the left and ends with a 'closed' dot at 5.

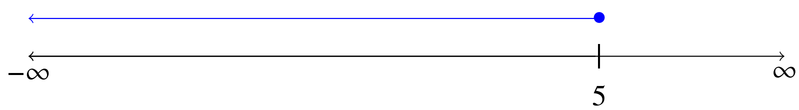


Figure 5.1.3: A graphical representation of $\{x \mid x \le 5\}$.

Due to the fact that the real number line extends indefinitely to the left and has a 'closed' dot at 5, the corresponding interval notation is $(-\infty, 5]$.

The verbal description is "x is less than or equal to 5."

d. The given interval notation is $(-2, \infty)$, which includes all real numbers greater than -2.

The equivalent set-builder notation is $\{x \mid x > -2\}$.

The corresponding segment of the real number line will begin with an 'open' circle and extend indefinitely with an arrow to the right, as shown in **Figure 5.1.4**.

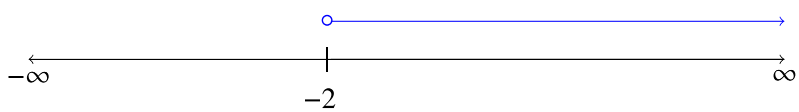


Figure 5.1.4: A graphical representation of $(-2, \infty)$.

The verbal description is "x is greater than -2."

■

We will often have occasion to combine intervals. There are two basic ways to combine intervals: **intersection** and **union**.

Definition

Suppose A and B are two intervals,

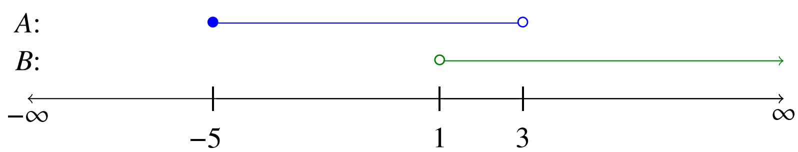
- The **intersection** of A and B, $A \cap B$, is the portion of the real number line that the two intervals have in common.
- The **union** of A and B, $A \cup B$, is the portion of the real number line which includes all points of either interval.

■

Said differently, the intersection of two intervals consists of all real numbers both in the overlap of the two intervals. The union of two intervals consists of all real numbers in one interval or the other interval or in both.

Recall that with events, if $A = \{1,2,3\}$ and $B = \{2,4,6\}$, then $A \cap B = \{2\}$ and $A \cup B = \{1,2,3,4,6\}$.

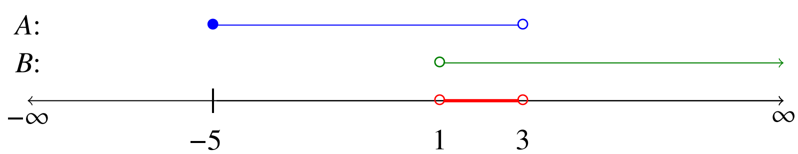
Now, if A is the interval $[-5,3)$ and B is the interval $(1,\infty)$, both shown in **Figure 5.1.5**,



$A : [-5,3)$, $B : (1,\infty)$

Figure 5.1.5: The equivalent line segments for intervals A and B.

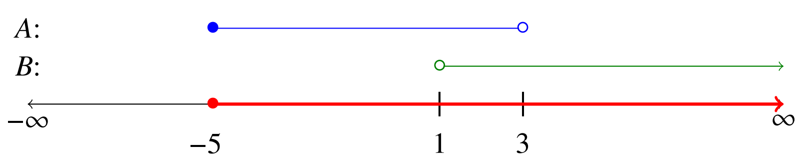
then we can represent $A \cap B$ graphically. To determine $A \cap B$, we shade the overlapping portions of the two line segments and obtain the interval $A \cap B = (1,3)$, shown in **Figure 5.1.6**. Notice the endpoints of the line segment are 'open' circles, as the endpoints are not included in *both* intervals.



$A \cap B : (1,3)$

Figure 5.1.6: The equivalent line segment for $A \cap B$.

We can also represent $A \cup B$, graphically by shading all points included in either A or B or both. The resulting shaded region, $A \cup B = [-5,\infty)$, is shown in **Figure 5.1.7**. Notice the non-included endpoints of A and B are both included in the union, because they were in at least one of the intervals.



$A \cup B : [-5,\infty)$

Figure 5.1.7: The equivalent line segment for $A \cup B$.

▪ **Example 2** Express the following sets of numbers, using interval notation.

a. $\{x \mid x \leq -2 \text{ or } x \geq 2\}$

b. $\{x \mid x \neq 3\}$

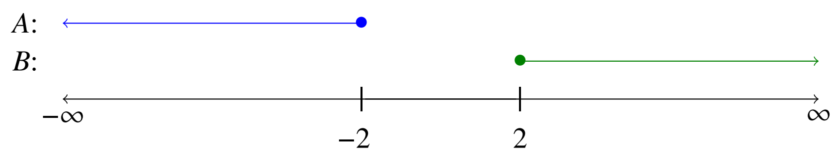
c. $\{x \mid x \neq \pm 3\}$

d. $\{x \mid x \leq 4 \text{ and } x > -2\}$

Solution:

a. First, let the inequality $x \leq -2$ correspond to the interval $A:\ (-\infty,-2]$, and the inequality $x \geq 2$ correspond to the interval $B:\ [2,\infty)$.

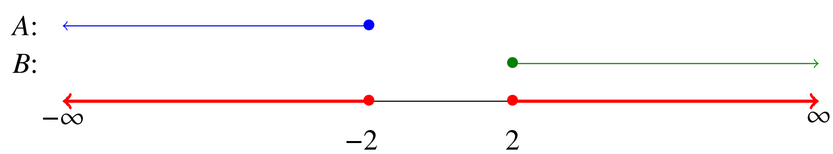
Then, the best way to proceed is to graph each interval as an equivalent line segment, as shown in **Figure 5.1.8**.



$A:(-\infty,-2],\ B:[2,\infty)$

Figure 5.1.8: The equivalent line segments for A and B.

Because of the "or" in the set-builder notation, we are looking to describe the real numbers, x, in A or B, or in both A and B. Considering the intervals have no overlap, we have $\{x \mid x \leq -2 \text{ or } x \geq 2\}$ is equivalent to $(-\infty,-2] \cup [2,\infty)$, in interval notation, as shown on the real number line in **Figure 5.1.9**.



$A \cup B:\ (-\infty,-2] \cup [2,\infty)$

Figure 5.1.9: The equivalent line segment for $\{x \mid x \leq -2 \text{ or } x \geq 2\}$.

b. For $\{x \mid x \neq 3\}$, we shade the entire real number line except at $x = 3$, where we leave an 'open' circle. (See **Figure 5.1.10**.)

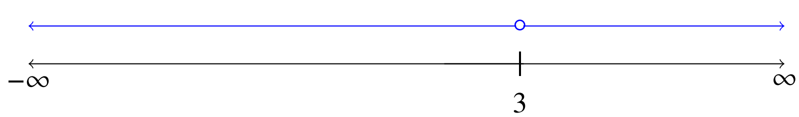
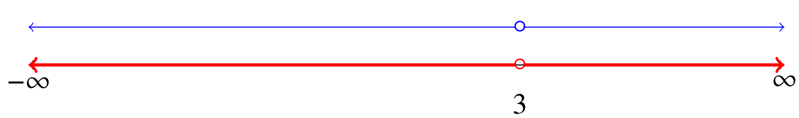


Figure 5.1.10: A number line representing the real numbers with a hole at 3.

Notice this divides the real number line into two non-overlapping intervals. One being to the left of $x = 3$, $(-\infty,3)$, and the other being to the right, $(3,\infty)$. As the values of x could only be in one of these intervals, but not both, we have that $\{x \mid x \neq 3\}$ is equivalent to $(-\infty,3) \cup (3,\infty)$, in interval notation, as shown in **Figure 5.1.11**.



$(-\infty,3) \cup (3,\infty)$

Figure 5.1.11: The equivalent line segment for $\{x \mid x \neq 3\}$.

c. For $\{x \mid x \neq \pm 3\}$, we proceed as in part **b** and exclude both $x = 3$ *and* $x = -3$ from the real number line, as shown in **Figure 5.1.12**.

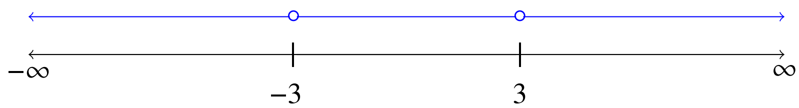
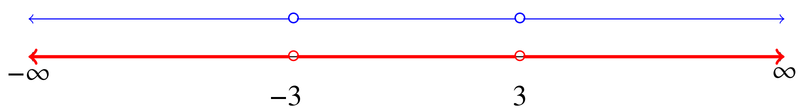


Figure 5.1.12: The number line representing the real numbers with holes at -3 and 3.

This breaks the real number line into three non-overlapping intervals: $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$. The set describes real numbers which come from the first, second, *or* third interval, so we have $\{x \mid x \neq \pm 3\}$ is equivalent to $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$, in interval notation, as shown in **Figure 5.1.13**.

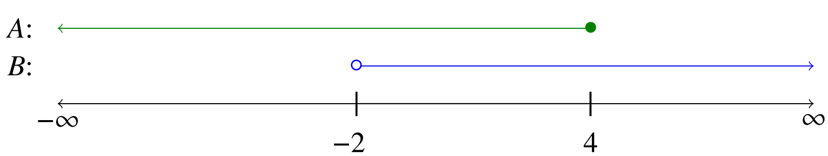


$$(-\infty,-3)\cup(-3,3)\cup(3,\infty)$$

Figure 5.1.13: The equivalent line segment for $\{x \mid x \neq \pm 3\}$.

*While we say x is not equal to -3 **and** x is not equal to 3, it is important to note the graphical representation gives three unique intervals which make our statement true. All possible values of x in these unique intervals must be included in our corresponding interval notation. Because the intervals do not intersect, if we used an intersection symbol, $\cap$, in the interval notation (as the word 'and' usually indicates), the subsequent interval would be empty. Therefore, we use the union symbol, $\cup$, to join intervals when writing interval notation.*

d. Let the inequality $x \leq 4$ correspond to the interval $A : (-\infty,4]$ and the inequality $x > -2$ correspond to the interval $B : (-2,\infty)$. Now proceed as in part **a**, by graphing these intervals as equivalent line segments. (See **Figure 5.1.14**.)



$$A : (-\infty,4],\ B : (-2,\infty)$$

Figure 5.1.14: The equivalent line segments for A and B.

Because of the "and" in the set-builder notation, we are looking to describe the real numbers, x, in the overlapping portions of interval A and interval B. In doing so, we have $\{x \mid x \leq 4 \text{ and } x > -2\}$ is equivalent to $\{x \mid -2 < x \leq 4\} = (-2,4]$, in interval notation. (See **Figure 5.1.15**.)

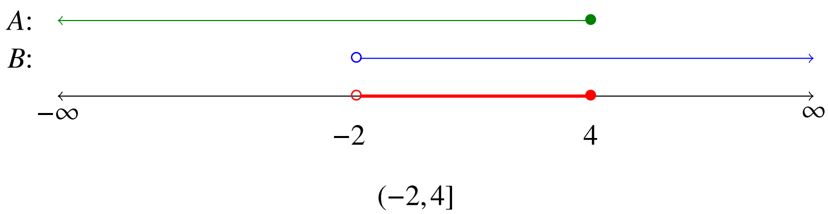


Figure 5.1.15: The equivalent line segment for $\{x \mid x \leq 4 \text{ and } x > -2\}$.

Try It # 1:
Given the shaded real number line in **Figure 5.1.16**,

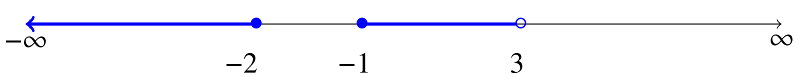


Figure 5.1.16: A real number line with one line segment extending from a solid dot at -2 to the left and another line segment between the solid dot at -1 and an open circle at 3.

specify the graphed set using an equivalent:

a. Verbal description
b. Set-builder notation
c. Interval notation

Try It # 2:
Write the given sets, using interval notation.

a. $\{x \mid x \neq 0 \text{ and } x \neq \pm 6\}$

b. $\{x \mid x < 3 \text{ or } x \geq 2\}$

c. $\left\{x \mid -5 \leq x < 7 \text{ and } x \neq \dfrac{6}{5}\right\}$

Differentiating Between a Relation and a Function

Recall in **Section 2.2**, we said "the natural world is full of relationships between quantities that change." We briefly explored the idea of a linear function to demonstrate relationships between some business quantities. We will now further investigate a broader collection of relationships, called **relations**, which include functions.

Definition

A **relation** is a set of ordered pairs in the coordinate plane.

The set consisting of the first components of each ordered pair is called the **inputs**, and the set consisting of the second components of each ordered pair is called the **outputs**.

We can describe a relation verbally, using a list, or using set-builder notation. By definition the elements in a relation are points in the plane, and, therefore, we often try to describe a relation graphically or algebraically, as well. Depending on the situation, one method may be easier or more convenient to use than another.

As an example, consider the relation $R = \{(-2,1),\ (4,3),\ (0,-3)\}$. As written, R is described as a list. Because R consists of points in the plane, we can plot the points. Doing so produces the graph of R in **Figure 5.1.17**.

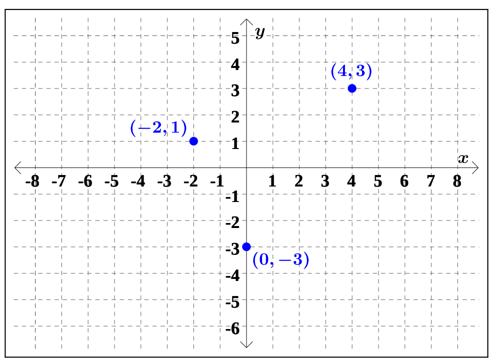


Figure 5.1.17: The coordinate plane with the points of R labeled.

The first number in each ordered pair of the relation makes up the set of inputs, while the second number in each ordered pair makes up the set of outputs. For R, we have

$$\text{Inputs: } \{-2,\ 0,\ 4\}$$

$$\text{Outputs: } \{-3,\ 1,\ 3\}$$

For sets with a finite number of elements like these, the elements do not have to be listed in particular order. Commonly, they are listed in ascending numerical order, as shown here for R.

One of the core concepts in this text, and in calculus, is the **function**. There are many ways to describe a function, and we begin by defining a function as a special kind of relation.

Definition

A relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x. ■

In other words a function, f, is a relation that assigns a single element in the set of outputs to each element in the set of inputs, meaning no x-values are repeated. **Figures 5.1.18** and **5.1.19**, below, show examples of functions, while **Figure 5.1.20** shows a relation which is not a function.

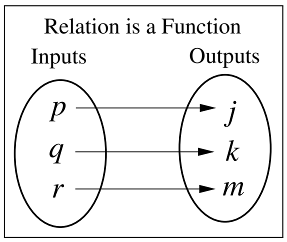


Figure 5.1.18: This relation is a function, because each input is associated with a single output.

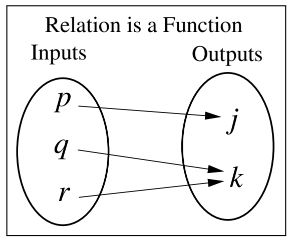


Figure 5.1.19: This relation is a function, because each input is associated with a single output. Note that both input q and r give output k.

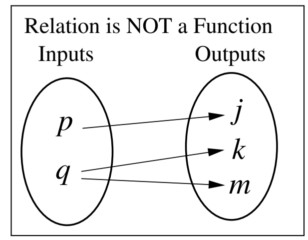


Figure 5.1.20: This relation is not a function, because input q is associated with two different outputs.

Two different inputs can be associated with one output, but one input cannot be associated with two different outputs.

All functions are relations, but not all relations are functions.

■ **Example 3** The coffee shop menu shown in **Figure 5.1.21** consists of items and their prices.

a. Is the price a function of the item?
b. Is the item a function of the price?

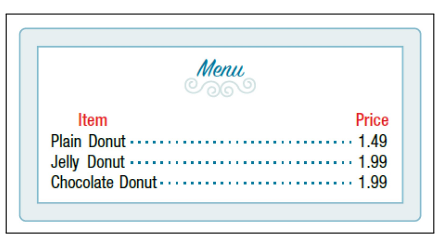


Figure 5.1.21: Coffee shop menu

Solution:

a. To determine whether or not the price is a function of the item, consider the input values as the items on the menu, and the output values as the prices. This input/output relation is shown with arrows from the items to the prices in **Figure 5.1.22**.

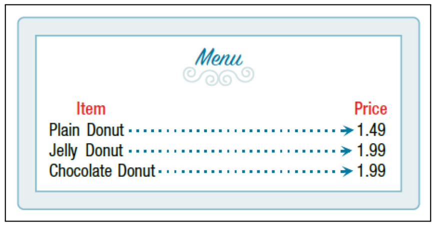


Figure 5.1.22: The coffee shop menu with arrows from items to prices.

As each item on the menu has only one price, the price **IS** a function of the item.

b. To determine whether or not the item is a function of the price, consider the input values as the prices on the menu, and the output values as the items. This input/output relation is shown with arrows from the prices to the items in **Figure 5.1.23**.

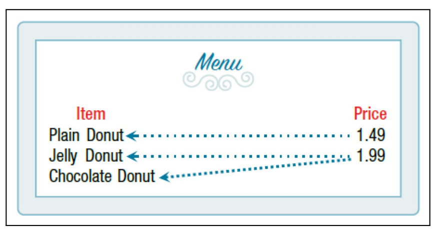


Figure 5.1.23: The coffee shop menu with arrows from prices to items.

Notice, two items on the menu have the same price. So the same input value has more than one output associated with it. Therefore, the item is **NOT** a function of price.

■

In the discussion of functions, we often use the terms **domain** and **range** in place of the terms inputs and outputs.

Definition

For any function,
- Each value in the **domain of the function** is also known as an **input** value, or **independent variable** value, and is often labeled with the lowercase letter x.
- Each value in the **range of the function** is also known as an **output** value, or **dependent variable** value, and is often labeled with the lowercase letter y.

The x and y correspond to their position in an ordered pair. ■

■ **Example 4** Determine whether or not the following relations describe y as a function of x. If the relation describes a function, give the function's domain and range.

a. $R_1 = \{(-2,1), (1,3), (1,4), (3,-1)\}$
b. $R_2 = \{(-2,1), (1,3), (2,3), (3,-1)\}$

Solution:

a. A quick scan of the points in R_1 reveals that the x-coordinate 1 is matched with two different y-coordinates: 3 and 4. Hence, in R_1, y is not a function of x. To verify, we can draw a picture of the relation. (See **Figure 5.1.24**.)

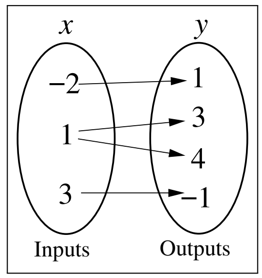


Figure 5.1.24: A visual representation of R_1.

b. Every x-coordinate in R_2 occurs only once which means each x-coordinate has only one corresponding y-coordinate. So, R_2 does represent y as a function of x. To verify, we can draw a picture of the relation. (See **Figure 5.1.25**.)

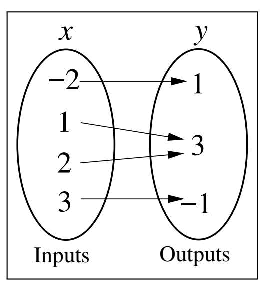


Figure 5.1.25: A visual representation of R_2.

The domain of the function is $\{-2,1,2,3\}$.

The range of the function is $\{1,3,-1\}$. (Note, as we have a finite number of values, the order in which they are listed does not matter.)

■

Try It # 3:
Determine whether or not the relations graphed in **Figures 5.1.26** and **5.1.27** represent y as a function of x. If the relation is a function, state the function's domain and range.

a.

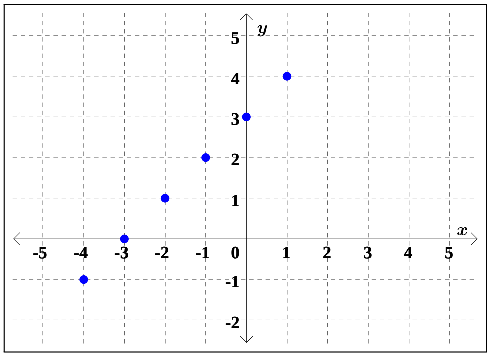


Figure 5.1.26: A coordinate plane with points at $(-4,-1)$, $(-3,0)$, $(-2,1)$, $(-1,2)$, $(0,3)$, and $(1,4)$.

b.

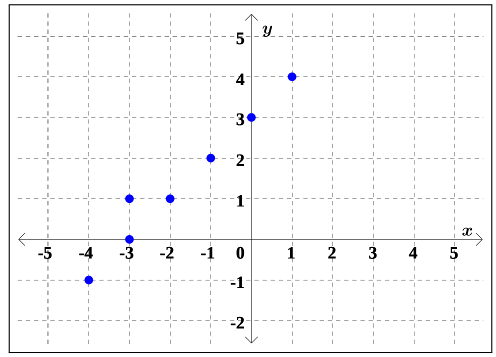


Figure 5.1.27: A coordinate plane with points at $(-4,-1)$, $(-3,0)$, $(-3,1)$, $(-2,1)$, $(-1,2)$, $(0,3)$, and $(1,4)$.

To see what the function concept means geometrically, we graph R_1 and R_2 from the previous example in the coordinate plane in **Figures 5.1.28** and **5.1.29**, respectively.

$$R_1 = \{(-2,1),\ (1,3),\ (1,4),\ (3,-1)\}$$

$$R_2 = \{(-2,1),\ (1,3),\ (2,3),\ (3,-1)\}$$

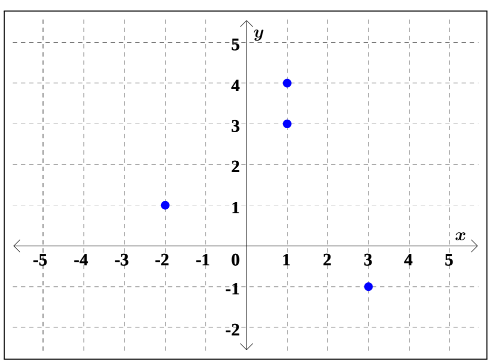


Figure 5.1.28: A graphical representation of R_1.

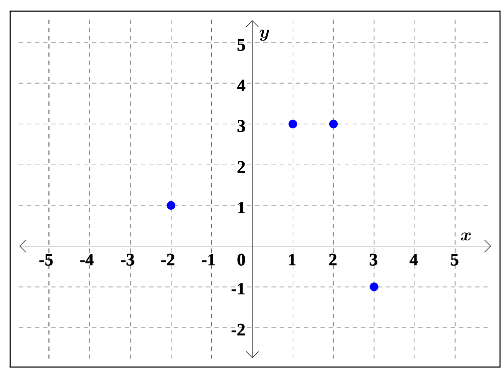


Figure 5.1.29: A graphical representation of R_2.

From the definition of a function, we have already determined that R_1 is not a function, by the fact that the x-coordinate 1 is matched with two different y-coordinates in R_1. This presents itself graphically as the points $(1,3)$ and $(1,4)$ lying on the same vertical line, $x = 1$.

We have already determined that R_2 is a function. In the graph of R_2, we see that no two points of the relation lie on the same vertical line.

We can generalize these ideas in the following test.

Theorem 5.1 The Vertical Line Test

A set of points in the plane represents y as a function of x if and only if no two points lie on the same vertical line.

Consider the following graphs in **Figures 5.1.30**, **5.1.31**, and **5.1.32**.

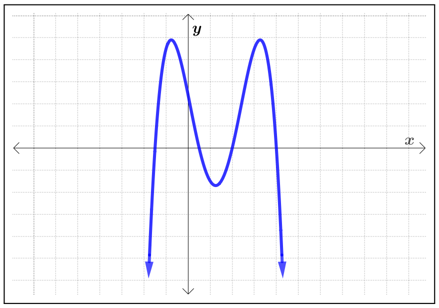


Figure 5.1.30: The coordinate plane with a curve which opens down on the left and right and is shaped like an M.

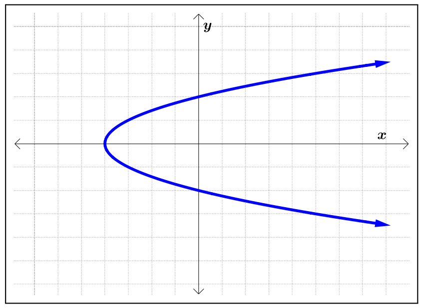


Figure 5.1.31: The coordinate plane with a parabola which opens to the right.

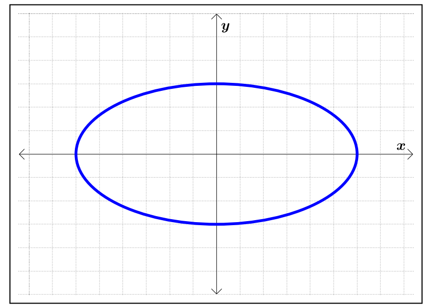


Figure 5.1.32: The coordinate plane with an ellipse.

Practically speaking, the Vertical Line Test says that if we can draw **any** vertical line that intersects a graph more than once, then the graph does not define a function and it *fails* the Vertical Line Test. Using the Vertical Line Test, we can see in **Figures 5.1.33**, **5.1.34**, and **5.1.35** that only the first of the three graphs represents a function, as the last two graphs fail the Vertical Line Test.

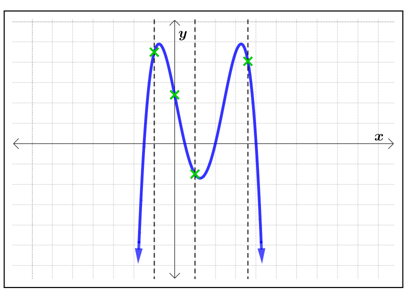


Figure 5.1.33: The graph of Figure 5.1.30 with multiple vertical lines drawn on the graph.

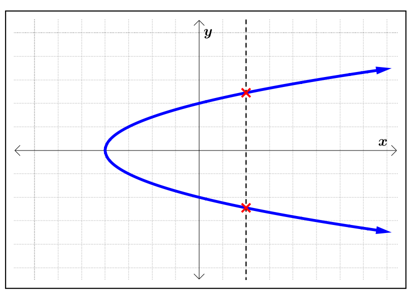


Figure 5.1.34: The graph of Figure 5.1.31 with one vertical line drawn on the graph.

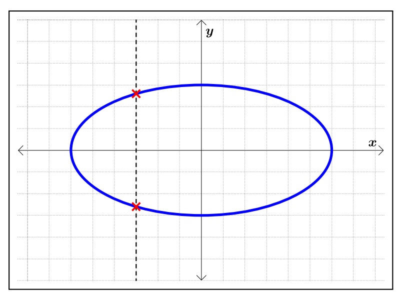


Figure 5.1.35: The graph of Figure 5.1.32 with one vertical line drawn on the graph.

N *In the graphs of **Figures 5.1.34** and **5.1.35** above, we showed only one vertical line passing through the graph of the relation. It is important to mention there are infinitely many vertical lines which could have been drawn, but if a single vertical line intersects the graph more than once, the relation fails the Vertical Line Test, and, thus, the relation is not a function. In general, if all vertical lines intersect the graph in at most one point, the relation passes the Vertical Line Test and is a function, as shown with the multiple vertical lines drawn in **Figure 5.1.33**.*

Example 5 Use the Vertical Line Test to determine if any of the following relations, R, S, T, and U, in **Figures 5.1.36**, **5.1.37**, **5.1.38**, and **5.1.39**, respectively, describe y as a function of x.

a.

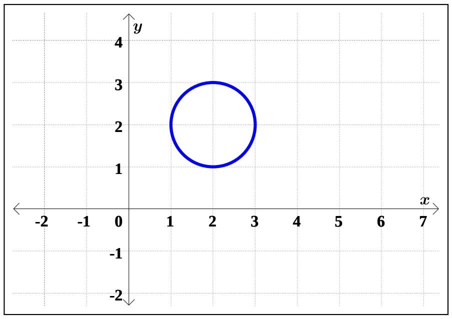


Figure 5.1.36: The graph of R. The coordinate plane with a circle of radius 1 centered at $(2,2)$.

b.

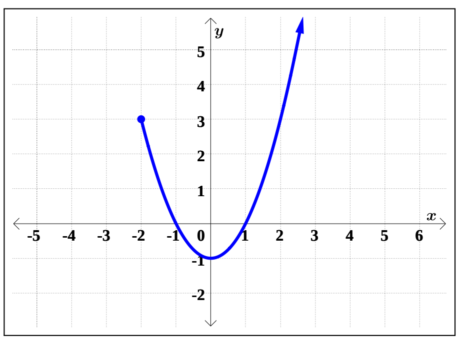


Figure 5.1.37: The graph of S. The coordinate plane with a curve that starts with a 'closed' dot at $(-2,3)$. The curve decreases to $(0,-1)$ and then increases infinitely to the right.

c.

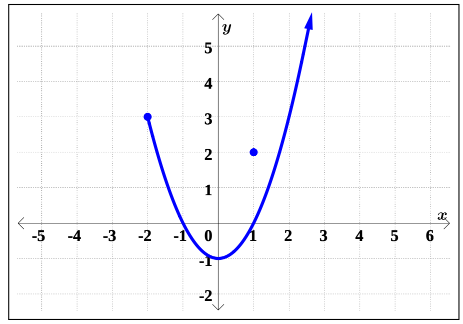


Figure 5.1.38: The graph of T. The coordinate plane with the graph of S and an additional 'closed' dot at $(1,2)$.

d.

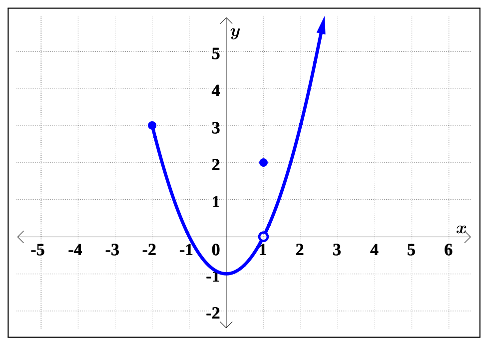


Figure 5.1.39: The graph of U. The coordinate plane with the graph of T, but with an 'open' circle at $(1,0)$.

Solution:

a. Looking at the graph of R, we can easily imagine vertical lines intersecting the graph more than once. An example of one such line is graphed in **Figure 5.1.40**. Hence, R does not represent y as a function of x.

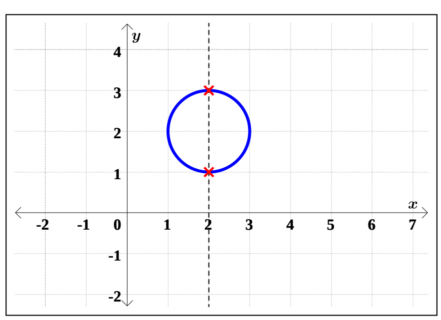


Figure 5.1.40: The graph of R with one vertical line drawn on the graph.

b. In the graph of S, any vertical line will intersect the graph at most once. Use **Figure 5.1.41** to help visualize this. So, S does represent y as a function of x.

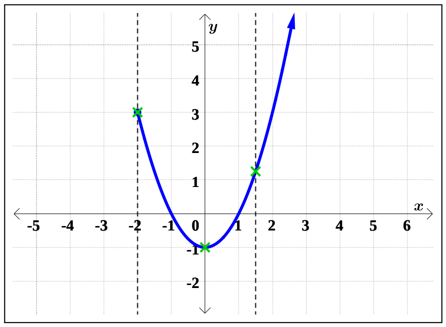


Figure 5.1.41: The graph of S with multiple lines drawn on the graph.

Relations T and U are slight modifications of the relation S, whose graph we determined passed the Vertical Line Test. In both T and U, it is the addition of the point $(1,2)$ which threatens to cause trouble.

c. In the graph of T, there is a point on the curve with x-coordinate 1 just below $(1,2)$, which means that both $(1,2)$ and this point on the curve, $(1,0)$, lie on the vertical line $x = 1$. (See **Figure 5.1.42**.) Hence, the graph of T fails the Vertical Line Test, so y is not a function of x.

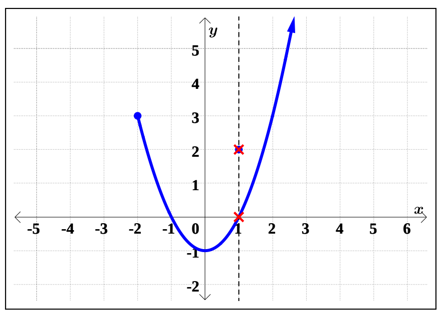


Figure 5.1.42: The graph of T with a vertical line drawn at $x = 1$.

d. In the graph of U, notice that the point with x-coordinate 1 on the curve has been omitted, leaving an 'open' circle there. Hence, the vertical line $x = 1$ intersects the graph of U only at the point $(1,2)$. Indeed, any vertical line will intersect the graph at most once, so we have that the graph of U passes the Vertical Line Test. (See **Figure 5.1.43**.) Thus, U represents y as a function of x.

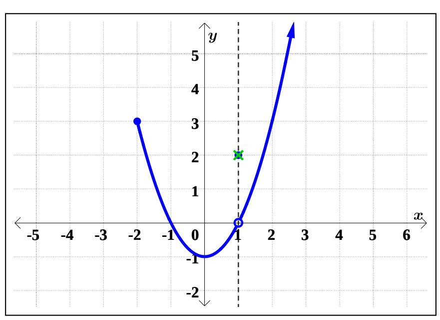


Figure 5.1.43: The graph of U with a vertical line drawn at $x = 1$.

■

Try It # 4:
Determine whether or not the given relation graphed in **Figure 5.1.44**, below, represents y as a function of x.

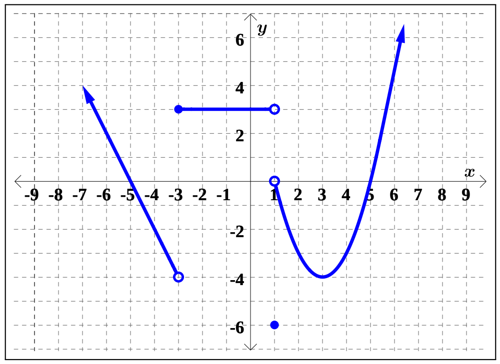


Figure 5.1.44: A graphed relation.

We now will demonstrate how to determine the domain and range of functions given to us graphically.

Suppose we are asked to state the domain and range of the function G, whose graph is given in **Figure 5.1.45**, below.

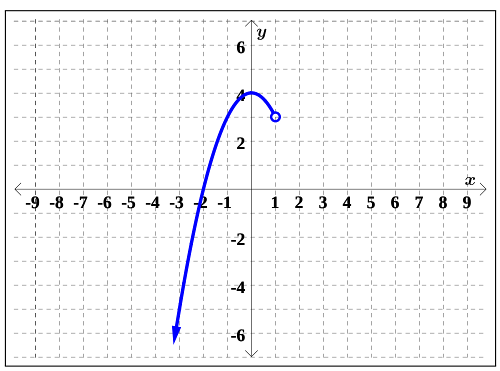


Figure 5.1.45: The graph of function G. The coordinate plane with a portion of a parabola ending at the 'open' circle $(1,3)$.

To state the domain and range of G, we need to determine which x- and y-values occur as coordinates of points on the given graph.

To illustrate the domain, it may be helpful to imagine collapsing the curve onto the x-axis and determining the portion of the x-axis that gets covered. This is called *projecting the curve* to the x-axis (See **Figure 5.1.46**). Before we start projecting, we need to pay attention to two subtle notations on the graph. The arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forevermore, and the open circle at $(1,3)$ indicates that the point $(1,3)$ is not on the graph, but all points on the curve leading up to that point are on the graph.

We see from **Figure 5.1.47** that if we project the graph of G to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of G as $(-\infty,1)$.

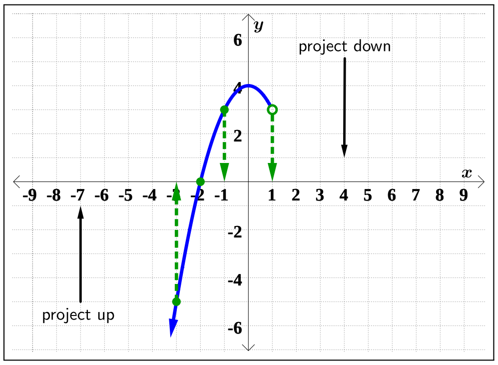


Figure 5.1.46: Modeling the projection process of the graph of G to the x-axis.

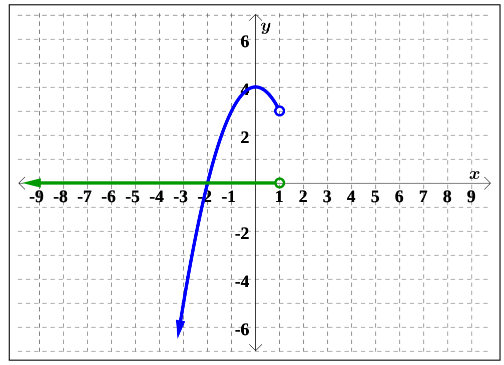


Figure 5.1.47: The graph of G with the domain, $(-\infty, 1)$, drawn on the x-axis.

To determine the range of G, we similarly project the curve to the y-axis, as seen in **Figure 5.1.48**.

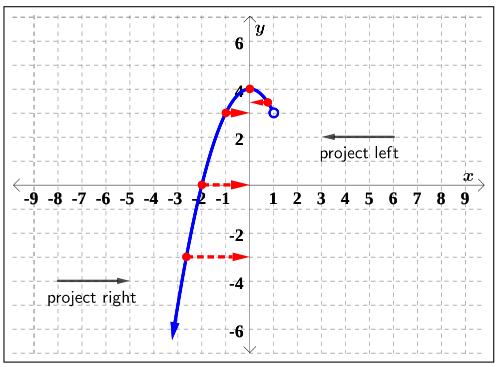


Figure 5.1.48: Modeling the projection process of the graph of G to the y-axis.

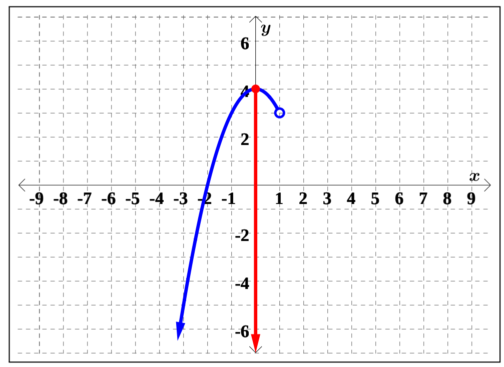


Figure 5.1.49: The graph of G with the range, $(-\infty, 4]$, drawn on the y-axis.

Notice that even through there is an 'open' circle at $(1,3)$, we still included the y-value of 3 in our range, as the point $(-1,3)$ is on the graph of G. We see that the range of G is all real numbers less than or equal to 4, or, in interval notation, $(-\infty, 4]$.

▪ **Example 6** State the domain and range of the function given in **Figure 5.1.50**, using interval notation.

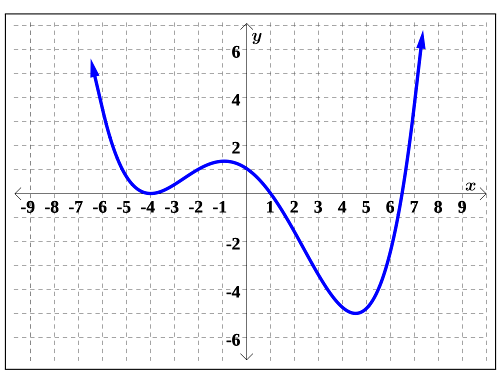


Figure 5.1.50: A function graphed in the coordinate plane.

Solution:
Before we project the function onto the axes to determine the domain and range, notice the graph extends indefinitely upward to the left and right, as indicated by the arrow heads on the ends of the graph.

To illustrate the domain, project the graph up and down to the x-axis, as shown in **Figure 5.1.51**.

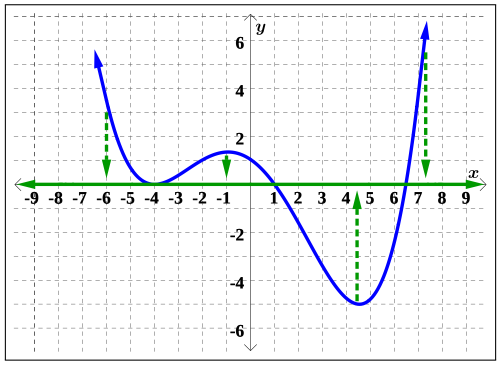


Figure 5.1.51: A model of the projection of the graph of the function to the x-axis.

We can see the entire x-axis is included in **Figure 5.1.51**, if we project the graph to the x-axis. Therefore, the domain includes all real numbers. Using interval notation, we write the domain as $(-\infty, \infty)$.

To illustrate the range, project the graph left and right to the y-axis, as shown in **Figure 5.1.52**.

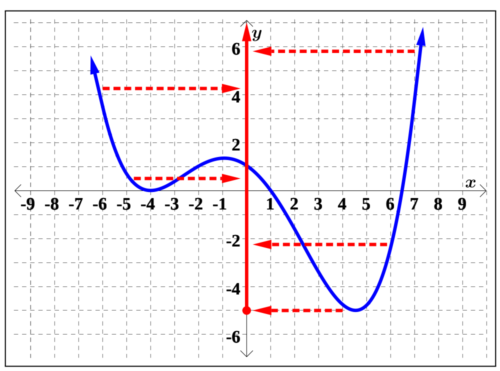


Figure 5.1.52: A model of the projection of the graph of the function to the y-axis.

We can see all real numbers greater than or equal to -5 are included in **Figure 5.1.52**, if we project the graph to the y-axis. Thus, the range includes all real numbers greater than or equal to -5. Using interval notation, we write the range as $[-5, \infty)$.

▪

Try It # 5:
State the domain and range of the function in **Figure 5.1.53**, below, using interval notation.

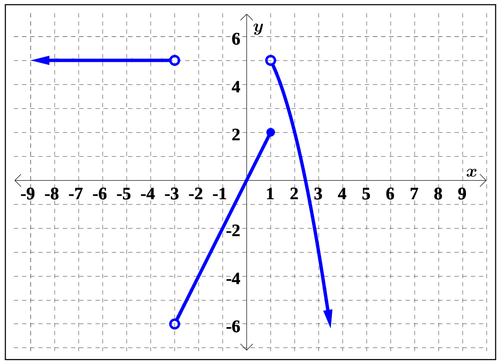


Figure 5.1.53: A function graphed in the coordinate plane.

Using Function Notation

Once we determine that a relation is a function, we need to define and display the functional relationship so that we can understand and use it. Sometimes we also need the function in a form so that we can program the function into graphing calculators and computers. There are various ways of representing functions.

Recall from Chapter 2, we used standard function notation when representing the linear business models. For example, the total cost of producing x items was written as $C(x) = mx + F$, and the revenue from selling x items was written as $R(x) = px$.

Remember, we can use any letter to name a function; the notation $f(x)$ shows us that f *depends* on x. The x-value must be input into the function, f, to obtain a resulting output. The parentheses indicate that x is the input of the function; they do not indicate multiplication.

Definition

The notation $y = f(x)$ defines a **function** named f. This is read as "y is a function of x."

The letter x represents the input value, or independent variable value. The set of all input values is the domain of f. The letter y, or $f(x)$, represents the output value, or dependent variable value. The set of all output values is the range of f. ▪

N *A function can also be called a rule, as it gives the directions for determining the output based on a given input.*

This relationship is typically visualized using a diagram similar to the one in **Figure 5.1.54** below.

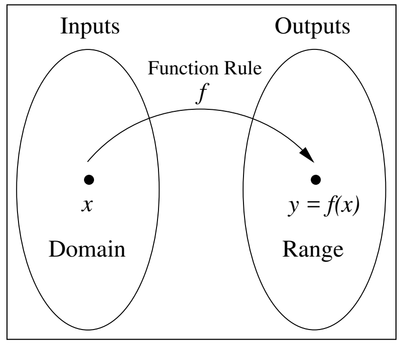


Figure 5.1.54: A visual representation of a function rule.

▪ **Example 7** Use function notation to represent a function whose input is the name of a month and whose output is the number of days in that month.

Solution:
The name of the month is the input to a rule that associates the specific number of days in that month (the output) with each input. If we name the rule, or function, f, we write

$$\text{days} = f(\text{month}) \qquad \text{or} \qquad d = f(m).$$

The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

For example, $f(\text{June}) = 30$, because June has 30 days.

$$\underset{\substack{\uparrow\\ \text{output}}}{30} = \underset{\substack{\uparrow\\ \text{rule}}}{f}\,(\underset{\substack{\uparrow\\ \text{input}}}{\text{June}})$$

▪

The inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

▪ **Example 8** A function $N = f(y)$ gives the number of police officers, N, in a specific town in year y. What does $f(2005) = 300$ represent?

Solution:
We are given $N = f(y)$, so when we read $f(2005) = 300$, we see the input, y, is the year 2005. The value for the output, the number of police officers, N, is 300. Thus, the statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in this specific town.

▪

Try It # 6:
Use function notation to express the weight of a pig, w (in pounds), as a function of its age in days, d.

When a table represents a function, corresponding domain and range values can also be specified using function notation.

Example 9 Use the function notation $g(x) = y$ to represent the function information given in each row of **Table 5.3**.

Domain	Range
2	1
5	3
8	6

Table 5.3: A table of domain values and corresponding range values for the function $g(x)$.

Solution:
The values in the first column are from the domain of the function, and the values from the second column are the corresponding values in the range.

Thus, when $x = 2$, $y = g(x) = 1$. Using function notation the information can be written as $g(2) = 1$.

Therefore, the corresponding function notation for the information in the remaining rows would be

$g(5) = 3$, when $x = 5$ and $y = g(x) = 3$

and

$g(8) = 6$, when $x = 8$ and $y = g(x) = 6$.

■

Try It # 7:
Use the function notation $k(x) = y$ to represent the function information given in each row of **Table 5.4**.

Domain	Range
1	10
2	100
3	1000

Table 5.4: A table of domain values and corresponding range values for the function $k(x)$.

When a function is defined algebraically and we know an input value, we can determine the corresponding output value of a function by *evaluating the function*. Evaluating a function will always produce one result (or function value), because each input value of a function corresponds to exactly one output value.

We can also have an algebraic expression as the input value of a function. For example, $f(a+b)$ means "first add a and b, and the result is the input for the function f." The result when evaluating a function at an algebraic expression is often another algebraic expression in the same variable(s).

Using and evaluating functions are key skills for success in this and many other math courses.

Example 10 Let $f(x) = -x^2 + 3x + 4$.

Evaluate and simplify the following.

a. $f(-1),\ f(0),\ f(2),\ f(a)$

b. $f(2x),\ 2f(x)$

c. $f(a+b),\ f(x+2),\ f(x)+2,\ f(x)+f(2)$

Solution:

a. To evaluate $f(-1)$, we replace every occurrence of x in the function $f(x)$ with -1 . Then, we use order of operations to simplify.

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} f(0) &= -(0)^2 + 3(0) + 4 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(2) &= -(2)^2 + 3(2) + 4 \\ &= -(4) + 6 + 4 \\ &= -4 + 6 + 4 \\ &= 6 \end{aligned}$$

$$\begin{aligned} f(a) &= -(a)^2 + 3(a) + 4 \\ &= -\left(a^2\right) + 3a + 4 \\ &= -a^2 + 3a + 4 \end{aligned}$$

b. To evaluate $f(2x)$, we replace every occurrence of x in the function $f(x)$ with the quantity $2x$, and simplify.

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply the function $f(x)$ by 2 .

$$\begin{aligned} 2f(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8 \end{aligned}$$

c. To evaluate $f(a+b)$, we replace every occurrence of x in the function $f(x)$ with the quantity $a+b$.

$$\begin{aligned} f(a+b) &= -(a+b)^2 + 3(a+b) + 4 \\ &= -[(a+b)(a+b)] + 3(a+b) + 4 \\ &= -(a^2 + 2ab + b^2) + (3a + 3b) + 4 \\ &= -a^2 - 2ab - b^2 + 3a + 3b + 4 \end{aligned}$$

To evaluate $f(x+2)$, we replace every occurrence of x in the function $f(x)$ with the quantity $x+2$.

$$\begin{aligned} f(x+2) &= -(x+2)^2 + 3(x+2) + 4 \\ &= -[(x+2)(x+2)] + 3(x+2) + 4 \\ &= -(x^2 + 4x + 4) + (3x + 6) + 4 \\ &= -x^2 - 4x - 4 + 3x + 6 + 4 \\ &= -x^2 - x + 6 \end{aligned}$$

To compute $f(x)+2$, we add 2 to the function $f(x)$.

$$\begin{aligned} f(x) + 2 &= \left(-x^2 + 3x + 4\right) + 2 \\ &= -x^2 + 3x + 6 \end{aligned}$$

From our work in part **a**, we see $f(2) = 6$, so

$$\begin{aligned} f(x) + f(2) &= \left(-x^2 + 3x + 4\right) + 6 \\ &= -x^2 + 3x + 10 \end{aligned}$$

■

In the previous example, when simplifying $-(2x)^2$, $2x$ is squared and then multiplied by -1, so that $-(2x)^2 = -4x^2$. Also, $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$.

Be careful and watch the order of operations when evaluating functions:

$$\begin{aligned} -(2x)^2 &\neq (-2x)^2 \\ (a+b)^2 &\neq a^2 + b^2 \end{aligned}$$

Try It # 8:
Let $f(x) = -2x^2 - 6x + 5$.

Evaluate and simply the following.

a. $f(1)$
b. $f(-x)$
c. $f(x+h)$

Try It Answers

1. **a.** "all real numbers, x, that are less than or equal to -2 or that are between -1 and 3, including -1"

   b. $\{x \mid x \leq -2 \text{ or } -1 \leq x < 3\}$

   c. $(-\infty, -2] \cup [-1, 3)$

2. **a.** $(-\infty, -6) \cup (-6, 0) \cup (0, 6) \cup (6, \infty)$

   b. $(-\infty, \infty)$

   c. $\left[-5, \frac{6}{5}\right) \cup \left(\frac{6}{5}, 7\right)$

3. **a.** y IS a function of x

   Domain: $\{-4, -3, -2, -1, 0, 1\}$

   Range: $\{-1, 0, 1, 2, 3, 4\}$

   b. y is NOT a function of x

4. y IS a function of x

5. Domain: $(-\infty, -3) \cup (-3, \infty)$ Range: $(-\infty, 5]$

6. $w = f(d)$, where d = age (in days) and w = weight (in pounds).

7. $k(1) = 10$, $k(2) = 100$, $k(3) = 1000$

8. **a.** $f(1) = -3$

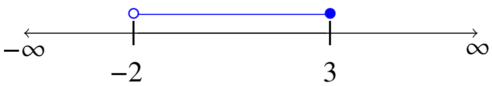
   b. $f(-x) = -2x^2 + 6x + 5$

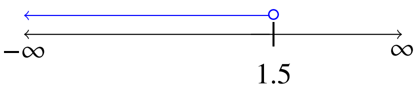
   c. $f(x+h) = -2x^2 - 4xh - 2h^2 - 6x - 6h + 5$

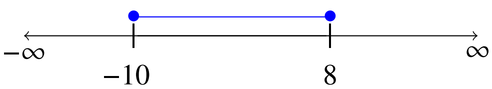
EXERCISES

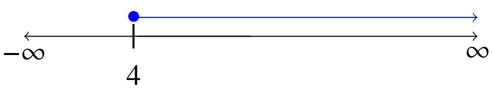
Basic Skills Practice (Answers)

For Exercises 1 - 4, express each of the following using equivalent interval notation.

1. 

2. 

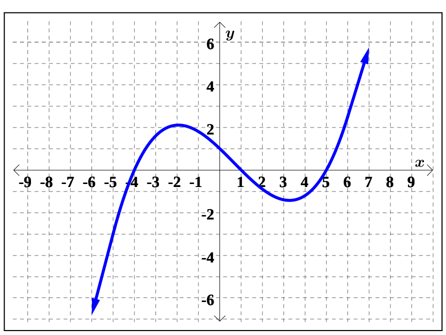
3. 

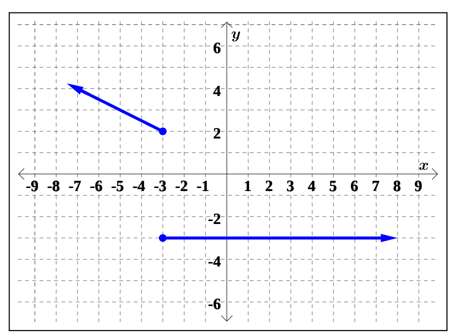
4. 

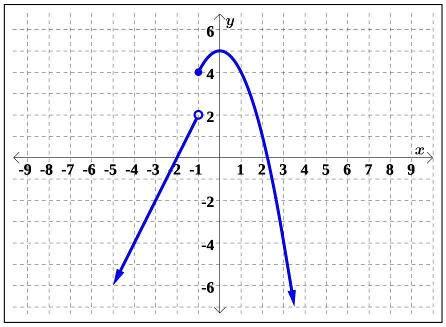
For Exercises 5 - 8, express each of the following using equivalent interval notation.

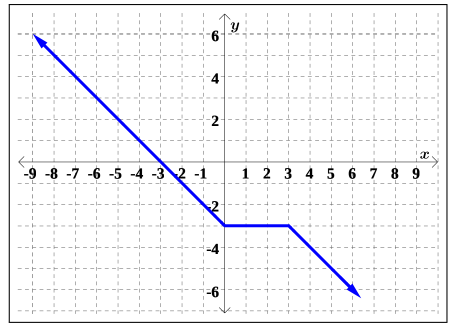
5. $\{x \mid x \leq -3\}$
6. $\{x \mid x > 6\}$
7. $\{x \mid x \geq 20\}$
8. $\left\{x \mid x < -\dfrac{5}{9}\right\}$

For Exercises 9 - 12, determine if the graph of the relation is a function.

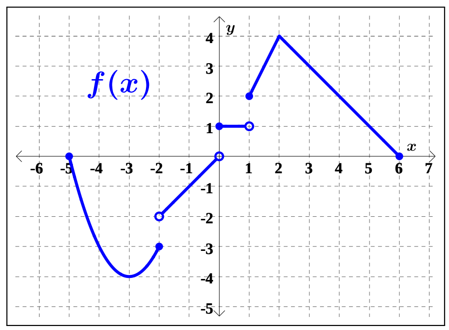
9. 

10. 

11. 

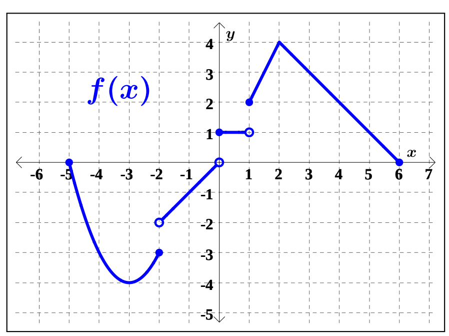
12. 

For Exercises 13 - 16, use the given graph of $f(x)$ to compute each function value.



13. $f(0)$
14. $f(-3)$
15. $f(2)$
16. $f\left(\frac{1}{2}\right)$

For Exercises 17 - 20, use the given graph of $f(x)$ to determine all values of x where the given function value occurs.



17. $f(x) = 0$
18. $f(x) = -3$
19. $f(x) = 2$
20. $f(x) = \frac{1}{2}$

For Exercises 21 - 23, use the function notation $h(x) = y$ to represent the function information given in each row of the tables below.

21.

Domain	Range
0	4
2	1
6	−3

22.

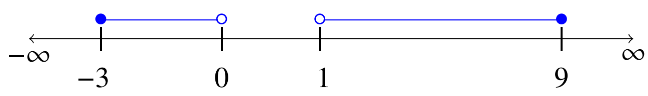
Domain	Range
−5	7
4	1.5
11	2

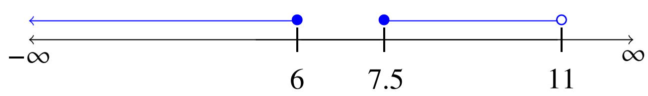
23.

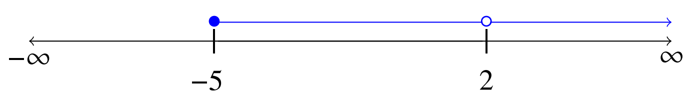
Domain	Range
−1	−2
−2	−1
9	0

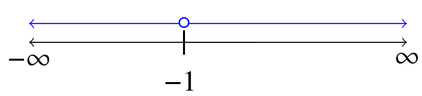
Intermediate Skills Practice (Answers)

For Exercises 24 - 27, express each of the following using equivalent interval notation.

24.


25.


26.


27.


For Exercises 28 - 31, express each of the following using equivalent interval notation.

28. $\{x \mid x \geq -3.4 \text{ and } x < 100\}$

29. $\{x \mid x < -1 \text{ or } x > 7\}$

30. $\{x \mid x > -5 \text{ and } x \leq 9, \text{ but } x \neq 0 \text{ and } x \neq 3\}$

31. $\{x \mid x < 10 \text{ or } x > 20, \text{ but } x \neq 30\}$

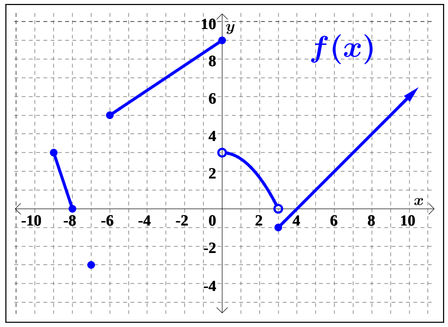
For Exercises 32 - 34, state the inputs and outputs of each relation. Then, determine if the relation is a function.

32. $\{(1,1),\ (2,2),\ (3,4)\}$

33. $\{(-1,-2),\ (0,1),\ (1,-3),\ (2,1)\}$

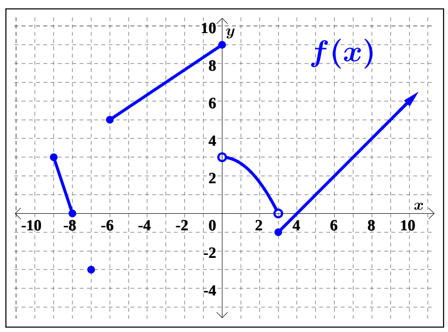
34. $\{(-2,-5),\ (1,3),\ (2,0),\ (1,4)\}$

For Exercises 35 - 40, use the given graph of $f(x)$ to compute each function value.



35. $f(0)$
36. $f(-3)$
37. $f(5)$
38. $f(4)$
39. $f(7)$
40. $f\left(-\frac{3}{2}\right)$

For Exercises 41 - 46, use the given graph of $f(x)$ to determine all values of x where the given function value occurs.



41. $f(x) = 0$
42. $f(x) = -3$
43. $f(x) = 5$
44. $f(x) = 3$
45. $f(x) = -1$
46. $f(x) = 9$

For Exercises 47 - 55, use the function $f(x) = 2x + 1$ to evaluate and simplify each of the following.

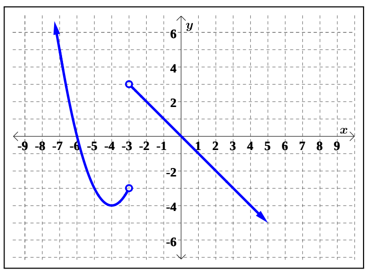
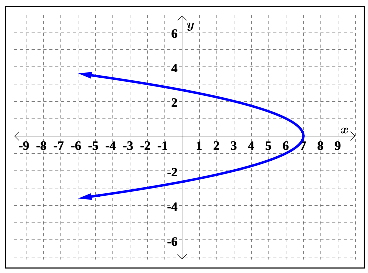
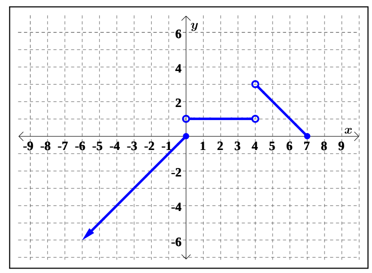
47. $f(3)$
48. $f(-3)$
49. $f\left(\frac{3}{2}\right)$
50. $f(4a)$
51. $4f(a)$
52. $\frac{f(a)}{4}$
53. $f(a-5)$
54. $f(a) - f(5)$
55. $f(a) - 5$

For Exercises 56 - 64, use the function $f(x) = 2x^2 - x - 12$ to evaluate and simplify each of the following.

56. $f(0)$
57. $f(1)$
58. $f\left(\frac{1}{2}\right)$
59. $f(-a)$
60. $-f(a)$
61. $\frac{f(a)}{2}$
62. $f(a+5)$
63. $f(a)+f(5)$
64. $f(a)+5$

Mastery Practice (Answers)

For Exercises 65 - 67, determine if the graph of the relation is a function. If the graph is a function, state the domain and range.

65. 
66. 
67. 

For Exercises 68 - 73, use the function $f(x) = 2 - \frac{1}{6}x$ to evaluate and simplify each of the following.

68. $f(-12)$
69. $f\left(\frac{3}{2}\right)$
70. $f(2a)$
71. $6f(a)$
72. $f(x)+h$
73. $f(x+h)$

For Exercises 74 - 79, use the function $f(x) = -3x^2 + 5x + 1$ to evaluate and simplify each of the following.

74. $f(-4)$
75. $f(1.6)$
76. $f(a+b)$
77. $f(a)+f(b)$
78. $f(x-h)$
79. $f(x+h)-f(x)$

Communication Practice (Answers)

80. Write the verbal description for $(-\infty,-12)\cup(-12,\infty)$.
81. Write the verbal description for $(-\infty,-7]\cup(-1,4)$.
82. Explain why vertical lines are used to determine if the graph of a relation is a function.

5.2 Polynomial Functions

© Photo by Kathryn Bollinger, 2020

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects.

Learning Objectives:

In this section, you will learn about the properties, characteristics, and applications of polynomial functions. Upon completion you will be able to:

- Recognize the degree, leading coefficient, and end behavior of a given polynomial function.
- State the domain of a polynomial function, using interval notation.
- Memorize the graphs of parent polynomial functions (linear, quadratic, and cubic).
- Define what it means to be a root/zero of a function.
- Identify the coefficients of a given quadratic function and the direction the corresponding graph opens.
- Determine the vertex and properties of the graph of a given quadratic function (domain, range, and minimum/maximum value).
- Compute the roots/zeros of a quadratic function by applying the quadratic formula and/or by factoring.
- Sketch the graph of a given quadratic function using its properties.
- Use properties of quadratic functions to solve business and social science problems.

Describing Polynomial Functions

A **polynomial function** consists of either the constant zero or the sum of a finite number of nonzero terms, each of which is the product of a number, called the **coefficient** of the term, and a variable raised to a non-negative integer (a whole number) power.

Definition

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $n \geq 0$ is a whole number. ▪

In a previous chapter we discussed linear functions, $f(x) = mx + b$, and the equation of a horizontal line, $y = b$. Both of these are examples of polynomial functions.

N *The equation of a horizontal line, $y = b$, can be written in function notation as $y = f(x) = b$, or simply as $f(x) = b$.*

Properties of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the powers of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in **general form**.

Definition

Suppose $f(x)$ is a polynomial function, in **general form**,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0, \text{ with } a_n \neq 0.$$

We say

- The whole number n (highest power of the variable which occurs) is called the **degree** of the polynomial.
  - If $f(x) = a_0$, and $a_0 \neq 0$, we say $f(x)$ has degree 0.
  - If $f(x) = 0$, we say $f(x)$ has no degree.
- The term $\boldsymbol{a_n x^n}$ is called the **leading term** of the polynomial $f(x)$.
- The real number $\boldsymbol{a_n}$ is called the **leading coefficient** of the polynomial $f(x)$.
- The real number $\boldsymbol{a_0}$ is called the **constant term** of the polynomial $f(x)$.

$$f(x) = \underbrace{\overset{\text{Leading Coefficient} \searrow \quad \swarrow \text{Degree}}{a_n x^n}}_{\text{Leading Term}} + \ldots + a_2x^2 + a_1x + \underbrace{a_0}_{\text{Constant Term}}$$

▪

For one type of polynomial, linear functions, the general form is $f(x) = mx + b$.

$$f(x) = \underbrace{\overset{\text{Leading Coefficient}}{\overset{\downarrow}{m}}x}_{\substack{\uparrow \\ \text{Leading Term}}} + \underbrace{b}_{\substack{\uparrow \\ \text{Constant Term}}}$$

While the power of x is unwritten, $x = x^1$, so linear functions are polynomials of degree 1. Any polynomial of degree 1 is called a **first degree polynomial**.

A polynomial whose graph is a horizontal line, has a general polynomial form of $f(x) = \underbrace{b}_{\substack{\uparrow \\ \text{Constant Term}}}$.

If $b \neq 0$, the degree is 0 and $f(x)$ is called a **constant polynomial**.

If $b = 0$, there is no degree and $f(x)$ is called the **zero polynomial**.

Example 1 Determine if each function is a polynomial function. If the function is a polynomial, state its degree, leading coefficient, and the constant term.

a. $f(x) = -2x^3 + 7x + 5x^6 - 3x^4 + \pi$

b. $g(q) = 6q - \frac{1}{2}q^3$

c. $h(x) = 4x^3 - 2x + x^{-3} + 8$

d. $k(x) = 5x^9 - \frac{1}{4}x^7 + x^{\frac{9}{2}} - 22$

e. $m(p) = p(p-4)$

Solution:

a. We begin by checking the coefficient and power of each term. For

$$f(x) = -2x^3 + 7x + 5x^6 - 3x^4 + \pi,$$

all coefficients are real numbers (yes, π is a real number), and all powers of the variable x are non-negative integers. Thus, $f(x)$ IS a polynomial.

Rewriting $f(x)$ in general form we have,

$$f(x) = 5x^6 - 3x^4 - 2x^3 + 7x + \pi = 5x^6 + 0x^5 - 3x^4 - 2x^3 + 0x^2 + 7x + \pi$$

- The degree is 6.
- The leading coefficient is 5.
- The constant term is π.

b. We begin by checking the coefficient and power of each term. For

$$g(q) = 6q - \frac{1}{2}q^3,$$

all coefficients are real numbers, and all powers of the variable q are non-negative integers. Thus, $g(q)$ IS a polynomial.

Rewriting $g(q)$ in general form we have,

$$g(q) = -\frac{1}{2}q^3 + 6q = -\frac{1}{2}q^3 + 0q^2 + 6q + 0$$

- The degree is 3.
- The leading coefficient is $-\frac{1}{2}$.
- The constant term is 0.

c. We begin by checking the coefficient and power of each term. For

$$h(x) = 4x^3 - 2x + x^{-3} + 8,$$

while all coefficients are real numbers, the power on the x^{-3} term, -3, is negative. Thus, $h(x)$ IS NOT a polynomial.

d. We begin by checking the coefficient and power of each term. For

$$k(x) = 5x^9 - \frac{1}{4}x^7 + x^{\frac{9}{2}} - 22,$$

while all coefficients are real numbers, the power on the $x^{\frac{9}{2}}$ term, $\frac{9}{2}$, is not an integer. Thus, $k(x)$ IS NOT a polynomial.

e. We begin by checking the coefficient and power of each term. For

$$m(p) = p(p-4) = p^2 - 4p,$$

all coefficients are real numbers, and all powers of the variable p are non-negative integers. Thus, $m(p)$ IS a polynomial.

Rewriting $m(p)$ in general form we have,

$$m(p) = p^2 - 4p = 1p^2 - 4p + 0$$

- The degree is 2.
- The leading coefficient is 1.
- The constant term is 0.

■

Polynomials do not have to contain all powers of the variable from 0 to n. When a term is "missing," the coefficient of that term is 0.

Try It # 1:
Determine if each function is a polynomial function. If so, identify its degree, leading coefficient, and the constant term.

a. $f(x) = 4x^2 - x^6 + 2x - 9$
b. $g(t) = (2t)\left(7t^9 + 8\right)$

Knowing the degree and leading coefficient of a polynomial function is useful in helping us predict its **end behavior**. The end behavior of a function is a way to describe what is happening to the values of the function (the y-values) as the x-values approach the 'ends' of the x-axis. That is, what happens to y as x becomes small without bound (written $x \to -\infty$) and, on the flip side, as x becomes large without bound (written $x \to \infty$).

For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into $f(x)$ for results of $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus, the values of the function are increasing without bound. To describe this behavior, we write: as $x \to -\infty$, $f(x) \to \infty$. If we study the behavior of $f(x)$ as $x \to \infty$, we see that in this case, too, $f(x) \to \infty$.

For $f(x) = x^3$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into $f(x)$ for results of $f(-100) = -1000000$, $f(-1000) = -1000000000$, and so on. Thus, the values of the function are decreasing without bound. To describe this behavior, we write: as $x \to -\infty$, $f(x) \to -\infty$. If we study the behavior of $f(x)$ as $x \to \infty$, we see that in this case, $f(x) \to \infty$.

It turns out for a polynomial, with degree 1 or higher, you only need to look at the leading term of the function to determine its end behavior. Because the power of the leading term is the largest, that term will grow significantly faster than the other terms of the polynomial as $x \to -\infty$ or as $x \to \infty$. So its behavior will dominate the graph. The leading coefficient and the degree found in the leading term both play a role in the end behavior of the polynomial. The general end behavior of polynomials is summarized below.

End Behavior of Polynomial Functions $f(x) = a_n x^n + \ldots + a_0$, when n is **odd**.

Suppose $f(x) = a_n x^n + \ldots + a_0$, where $a_n \neq 0$ is a real number and n is an odd natural number. The end behavior of the graph of $f(x)$ matches one of the following:

- For $a_n > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.
- For $a_n < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$.

Graphically:

$\swarrow \cdots \nearrow$	$\nwarrow \cdots \searrow$
$a_n > 0$	$a_n < 0$

End Behavior of Polynomial Functions $f(x) = a_n x^n + \ldots + a_0$, when n is **even**.

Suppose $f(x) = a_n x^n + \ldots + a_0$, where $a_n \neq 0$ is a real number and n is an even natural number. The end behavior of the graph of $f(x)$ matches one of the following:

- For $a_n > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$.
- For $a_n < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

Graphically:

$$\nwarrow \ldots \nearrow \qquad\qquad \swarrow \overset{\cdots}{} \searrow$$

$$a_n > 0 \qquad\qquad a_n < 0$$

N *While the end behavior is determined solely by the leading term of a polynomial, the interior behavior of the function is dependent on <u>all</u> terms of the polynomial, and, thus, is different for each polynomial. Therefore, the interior behavior is represented by the "…" in each graphical representation of the end behavior.*

Considering the graph of the constant or zero polynomial, $f(x) = b$, is a horizontal line, the end behavior of these polynomials is written as: $f(x) \to b$ as $x \to \pm\infty$.

Example 2 Describe the end behavior of the given polynomial function, both symbolically and graphically.

a. $f(x) = -3x^4 - 9x^3 + 12x^2$

b. $g(x) = 8x^3 - 6x^2 + 3x + 10$

Solution:

a. The polynomial $f(x)$ is written in general form, and its leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (-3). We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

and

$$\text{as } x \to \infty,\ f(x) \to -\infty.$$

Graphically, the end behavior would be seen as

$$\swarrow \overset{\cdots}{} \searrow$$

b. The polynomial $g(x)$ is written in general form, and its leading term is $8x^3$; therefore, the degree of the polynomial is 3. The degree is odd (3) and the leading coefficient is positive (8). We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

and

$$\text{as } x \to \infty,\ f(x) \to \infty.$$

Graphically, the end behavior would be seen as

$$\swarrow \cdots \nearrow$$

■

■ **Example 3** For the polynomial function, $f(x)$, graphed in **Figure 5.2.2**, describe the end behavior symbolically, and determine a possible degree of the polynomial function.

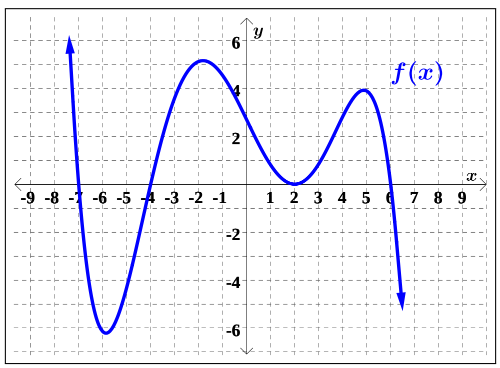


Figure 5.2.2: The coordinate plane with the curve $f(x)$.

Solution:

As we move to the left on the x-axis, the graph of the values of the function, $f(x)$, increase without bound. As we move to the right on the x-axis, the graph of the values of the function, $f(x)$, decrease without bound. We can describe this end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to \infty$$

and

$$\text{as } x \to \infty,\ f(x) \to -\infty.$$

In words, we could say that as the x-values approach negative infinity, the values of the function approach infinity, and as the x-values approach infinity, the values of the function approach negative infinity. This end behavior coincides with the graph of a polynomial of odd degree having a negative leading coefficient.

■

■ **Example 4** Let $h(x) = -ax^{50} + bx^{25} + cx^{10} - d$, where a, b, c, and d are real numbers such that $a < 0$, $b < 0$, $c > 0$, and $d > 0$. Determine the end behavior of $h(x)$, both symbolically and graphically.

Solution:
The leading term of $h(x)$ is $-ax^{50}$; therefore the degree of the polynomial is 50. The degree is even (50) and the leading coefficient is $-a$. As stated, $a < 0$ which means $-a > 0$. Thus, the leading coefficient $(-a)$ is positive. We can describe this end behavior symbolically by writing

$$\text{as } x \to -\infty, \ f(x) \to \infty$$

and

$$\text{as } x \to \infty, \ f(x) \to \infty.$$

Graphically, the end behavior would be seen as

$$\nwarrow \dots \nearrow$$

■

Try It # 2:
Describe the end behavior of the given polynomial function, both symbolically and graphically.

a. $f(x) = -6x^{11} + 9x^2 + 5$

b. $g(x) = 100x^8 - 700x^5 - 10000$

Despite having different end behavior, all polynomial functions are continuous. While this concept is formally defined using calculus, informally, graphs of continuous functions have no 'breaks' or 'holes' in them. Moreover, based on the properties of real numbers, for every real number input into a polynomial function, a single real number will be the output, giving a domain of all real numbers.

> The **domain** of *any* polynomial function is $\boldsymbol{(-\infty, \infty)}$.

■ **Example 5** State the domain of $f(x) = 150x^{200} - 783x^{44}$, using interval notation.

Solution:
$f(x)$ is a polynomial, so its domain is $(-\infty, \infty)$.

■

In this chapter, we will be exploring functions - shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our 'parent functions,' which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable. We will see these parent functions, combinations of parent functions, their graphs, and their transformations, throughout this chapter. It will be very helpful to recognize these parent functions and their features quickly by name, formula, and graph.

The polynomials made up of just the leading term and a leading coefficient of 1 are called **parent polynomial functions**. The graphs, sample table values, and domain are included with each parent polynomial function in **Table 5.5**. As we continue to introduce additional functions in this chapter, we will add to this table of parent functions.

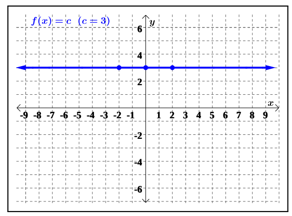
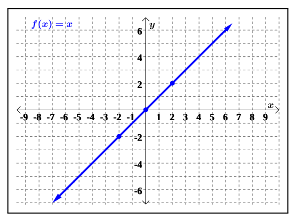
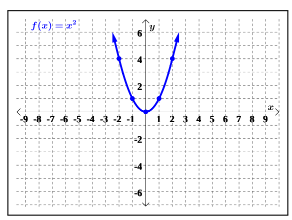
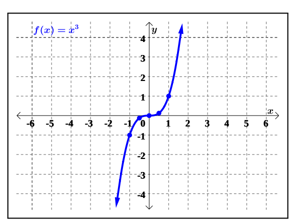
<table>
<tr><th>Name</th><th>Function</th><th>Graph</th><th>Table</th><th>Domain</th></tr>
<tr><td>Constant</td><td>$f(x) = c$
where c is a constant</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>c</td></tr><tr><td>0</td><td>c</td></tr><tr><td>2</td><td>c</td></tr></table></td><td>$(-\infty, \infty)$</td></tr>
<tr><td>Linear
(Identity)
1st degree polynomial</td><td>$f(x) = x$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>-2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td><td>$(-\infty, \infty)$</td></tr>
<tr><td>Quadratic
2nd degree polynomial</td><td>$f(x) = x^2$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>4</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>4</td></tr></table></td><td>$(-\infty, \infty)$</td></tr>
<tr><td>Cubic
3rd degree polynomial</td><td>$f(x) = x^3$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>-8</td></tr><tr><td>-1</td><td>-1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>8</td></tr></table></td><td>$(-\infty, \infty)$</td></tr>
</table>

Table 5.5: Parent Polynomial Functions

Intercepts of a Polynomial Function

Characteristics of the graph such as vertical and horizontal intercepts are part of the *interior* behavior of a polynomial function.

Definition

With any function, $f(x)$, the vertical intercept, or ***y*-intercept**, is the point where the graph crosses/touches the y-axis, and occurs when the input value, x, is zero, $f(0)$.

Seeing as a polynomial is a *function*, there can only be one y-intercept, the point $(0, a_0)$.

The ***x*-intercept(s)** of any function, $f(x)$, is the point(s) where the graph crosses/touches the x-axis, and occur at the x-value(s) that correspond with an output value of zero, $f(x) = 0$.

It is possible for a polynomial, or any function, to have more than one x-intercept, $(x_i, 0)$. ■

Definition

The **real zeros**, or **real roots** of a function, $f(x)$, are

1. the x-value(s) when $f(x) = 0$.
2. the solution(s) to the equation $f(x) = 0$.
3. the x-coordinate(s) of the ***x*-intercept(s)** of $f(x)$.
4. the x-value(s) where the graph of $f(x)$ crosses/touches the x-axis.

■

N *In this text, our discussions only concern real zeros.*

To determine the real zero(s) of any function, we need to identify when the output of the function will be zero. For general polynomials, this can be a challenging prospect. While equations involving a quadratic polynomial set equal to zero can be solved using the quadratic formula (which we will discuss later in this section), the corresponding formulas for equations involving cubic and 4$^{\text{th}}$ degree polynomials are not simple enough to remember, and formulas do not exist for equations involving general higher-degree polynomials. Consequently, we will limit ourselves to three cases when discussing zeros of polynomials.

1. The polynomial can be factored, using known methods - greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to *approximate* the real zeros.

■ **Example 6** Determine the y-intercept and all real zeros of each given polynomial function.

a. $f(x) = 7x + 3$

b. $g(x) = (x-3)^2(x+2)(x^2-1)$

c. $h(x) = x^3 + 4x^2 + x - 6$

Solution:

a. To determine the y-intercept, we substitute 0 for x into $f(x)$ and evaluate.

$$\begin{aligned} f(0) &= 7(0)+3 \\ &= 3 \end{aligned}$$

So, $f(x)$ has a y-intercept of $(0,3)$.

To determine the real zeros, we set $f(x)=0$ and solve for x.

$$\begin{aligned} 7x+3 &= 0 \\ 7x &= -3 \\ x &= -\frac{3}{7} \end{aligned}$$

Therefore, $f(x)$ has a real zero when $x = -\frac{3}{7}$.

b. To determine the y-intercept, evaluate $g(0)$.

$$\begin{aligned} g(0) &= (0-3)^2\ (0+2)\ \left((0)^2-1\right) \\ &= (-3)^2(2)(-1) \\ &= (9)(2)(-1) \\ &= -18 \end{aligned}$$

So, $g(x)$ has a y-intercept of $(0,-18)$.

To determine the real zeros, we set $g(x)=0$ and solve for x.

$$(x-3)^2(x+2)(x^2-1) = 0$$

Using the Zero Product Property, we set each factor equal to 0.

$$\begin{array}{ccccc} (x-3)^2 = 0 & \text{or} & x+2=0 & \text{or} & x^2-1=0 \\ (x-3)(x-3)=0 & & x=-2 & & (x-1)(x+1)=0 \\ x-3=0 \text{ or } x-3=0 & & & & x-1=0 \text{ or } x+1=0 \\ x=3 \text{ or } x=3 & & & & x=1 \text{ or } x=-1 \end{array}$$

Then, $g(x)$ has real zeros when $x = 3, -2, 1, \text{ and } -1$.

c. To determine the y-intercept, we evaluate $h(0)$.

$$\begin{aligned} h(0) &= (0)^3+4(0)^2+(0)-6 \\ &= -6 \end{aligned}$$

So, $h(x)$ has a y-intercept of $(0,-6)$.

Because this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to a graph to help identify the real zeros.

Graphing this function using technology it *appears* there are real zeros when $x = -3, -2,$ and 1. (See **Figure 5.2.3**)

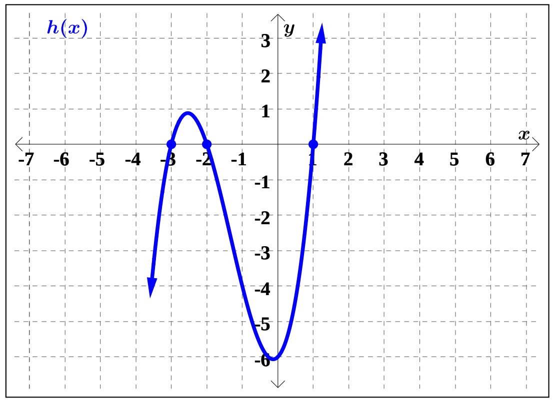


Figure 5.2.3: The coordinate plane with the graph of $h(x) = x^3 + 4x^2 + x - 6$. The graph appears to cross the x-axis when $x = -3,\ x = -2$, and $x = 1$.

We verify these are the correct real zeros by substituting in these values for x and confirming that $h(-3) = h(-2) = h(1) = 0$.

Thus, $h(x)$ has real zeros when $x = -3, -2$, and 1.

■

Try It # 3:
Determine the y-intercept and all real zeros of the function $f(t) = t^3 - 4t^2$.

Describing Quadratic Functions

We will now explore a specific type of polynomial function, **quadratic functions**. Quadratic functions commonly arise from problems involving revenue and profit, providing some interesting applications.

Definition

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers with $a \neq 0$. ■

A quadratic function is a type of polynomial function. As such, the domain of a quadratic function is $(-\infty, \infty)$.

Properties of a Quadratic Function

The graph of a quadratic function, $f(x) = ax^2 + bx + c$, is a U-shaped curve called a **parabola**. One important feature of a parabola is that it has an extreme point, called the **vertex**. If $a > 0$, the parabola opens upward, and the vertex represents the lowest point on the graph, or the **minimum** of the quadratic function. If $a < 0$, the parabola opens downward, and the vertex represents the highest point on the graph, or the **maximum**. In either case, the vertex is a turning point on the graph. The graph is also symmetric about a vertical line drawn through the vertex, called the **axis of symmetry**. The features just described are illustrated in **Figures 5.2.4** and **5.2.5** below.

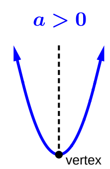


Figure 5.2.4: A parabola opening upward with the axis of symmetry drawn and vertex labeled on the graph.

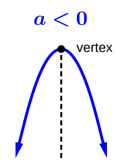


Figure 5.2.5: A parabola opening downward with the axis of symmetry drawn and vertex labeled on the graph.

Properties of Quadratic Functions

- A quadratic function is a polynomial function of degree two.
- The graph of a quadratic function is a **parabola**.
- The **parent quadratic function** is $f(x) = x^2$.
- The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.
- The **vertex** is located at $(h,k) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- The **axis of symmetry** is the vertical line $x = h = \frac{-b}{2a}$.
- The **vertex form of a quadratic function** is $f(x) = a(x-h)^2 + k$, where $a \neq 0$.
- The **domain** is $(-\infty, \infty)$.
- The **range** is dependent on the value of the leading coefficient, a.
  - For $a > 0$, range : $\left[f\left(\frac{-b}{2a}\right), \infty\right)$
  - For $a < 0$, range : $\left(-\infty, f\left(\frac{-b}{2a}\right)\right]$

The vertex of a parabola will inform us of what the maximum ***or*** *minimum value of the output of a quadratic function is, k, and where it occurs, when $x = h$.*

N *In similar texts, authors will develop the vertex form of a quadratic function using a method called "Completing the Square." The authors of this text find the method cumbersome. Thus, we will allow the reader to investigate this method on their own. Information regarding Completing the Square can be found in the Appendix.*

Consider the quadratic function $f(x) = \dfrac{1}{2}x^2 + 2x - 1$, given in general form, where $a = \frac{1}{2}$, $b = 2$, and $c = -1$, and let's investigate its properties.

The leading term of $f(x)$ is $\dfrac{1}{2}x^2$, so the leading coefficient, $a = \dfrac{1}{2}$, is positive. With $a > 0$, the parabola will open upward and the vertex will be a minimum.

The coordinates of the vertex are found using the given formula, as follows:

$$\begin{aligned} x = h &= \frac{-b}{2a} \\ &= \frac{-2}{2\left(\frac{1}{2}\right)} \\ &= \frac{-2}{1} \\ x &= -2 \end{aligned}$$

and

$$\begin{aligned} y = k &= f\left(\frac{-b}{2a}\right) \\ &= f(-2) \\ &= \frac{1}{2}(-2)^2 + 2(-2) - 1 \\ &= \frac{1}{2}(4) - 4 - 1 \\ &= 2 - 4 - 1 \\ y &= -3 \end{aligned}$$

Thus, the vertex is $(-2, -3)$, and we know $f(x)$ has an axis of symmetry of $x = -2$.

Because the vertex is a minimum, the minimum value of $f(x)$ is -3 and it occurs when $x = -2$.

Due to the fact that $a > 0$ and the parabola opens upward, as $x \to \pm\infty$, $f(x) \to \infty$, so there is no maximum value of $f(x)$.

The domain of $f(x)$ is $(-\infty, \infty)$, as it is a 2nd degree *polynomial*.

The range of $f(x)$ is $[-3, \infty)$, as $a > 0$.

■ **Example 7** If $f(x) = -5x^2 + 9x - 1$, determine the vertex, axis of symmetry, the maximum value, the minimum value, domain, and range of the function.

Solution:
$f(x) = -5x^2 + 9x - 1$ is given in general form, where $a = -5$, $b = 9$, and $c = -1$.

The leading term of $f(x)$ is $-5x^2$, so the leading coefficient, $a = -5$, is negative. With $a < 0$, the parabola will open downward and the vertex will be a maximum.

The coordinates of the vertex are found using the given formula, as follows:

$$\begin{aligned} x = h &= \frac{-b}{2a} \\ &= \frac{-(9)}{2(-5)} \\ &= \frac{-9}{-10} \\ x &= \frac{9}{10} \end{aligned}$$

and

$$\begin{aligned} y = k &= f\left(\frac{-b}{2a}\right) \\ &= f\left(\frac{9}{10}\right) \\ &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\ y &= \frac{61}{20} \end{aligned}$$

Thus, the vertex is $\left(\frac{9}{10}, \frac{61}{20}\right)$, and we know $f(x)$ has an axis of symmetry of $x = \frac{9}{10}$.

Because the vertex is a maximum, the maximum value of $f(x)$ is $\frac{61}{20}$ and it occurs when $x = \frac{9}{10}$.

Due to the fact that $a < 0$ and the parabola opens downward, as $x \to \pm\infty$, $f(x) = -\infty$. so there is no minimum value of $f(x)$.

The domain of $f(x)$ is (∞, ∞), as it is a 2^{nd} degree polynomial.

The range of $f(x)$ is $\left(-\infty, \frac{61}{20}\right]$, as $a < 0$.

■

■ **Example 8** If $f(x) = -5(x+4)^2 - 7$, determine the vertex, axis of symmetry, the maximum value, the minimum value, domain, and range of the function.

Solution:

This quadratic function is given in vertex form, $f(x) = a(x-h)^2 + k$, with $a = -5 < 0$. Thus, the parabola will open downward and the vertex, (h,k), will be a minimum.

We know $x - h = x + 4 = x - (-4)$, so we can conclude $h = -4$. We can see $k = -7$; thus, the vertex is $(h,k) = (-4,-7)$.

It follows that the axis of symmetry is $x = -4$.

Because the vertex is a maximum, the maximum value of $f(x)$ is -7 and it occurs when $x = -4$.

As $x \to \pm\infty$, $f(x) \to -\infty$, so there is no minimum value of $f(x)$.

The domain of $f(x)$ is $(-\infty,\infty)$, as it is a 2nd degree polynomial.

The range of $f(x)$ is $(-\infty,-7]$, as $a < 0$.

■

Try It # 4:
Given the quadratic function $g(x) = 13 + x^2 - 6x$, write the function in general form, and determine its vertex, axis of symmetry, minimum value, maximum value, domain, and range.

Try It # 5:
Given the quadratic function $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$, determine its vertex, axis of symmetry, minimum value, maximum value, domain, and range.

Intercepts of a Quadratic Function

As previously discussed, the y-intercept of a function, $f(x)$, is found by evaluating $f(0)$, and the real zeros are where the function is equal to zero. The number of x-intercepts of a quadratic function, which correspond to the real zeros of the function, can vary depending upon the direction it opens and its position in relation to the x-axis. **Figures 5.2.6**, **5.2.7**, and **5.2.8**, below, show graphs of some of the different possibilities.

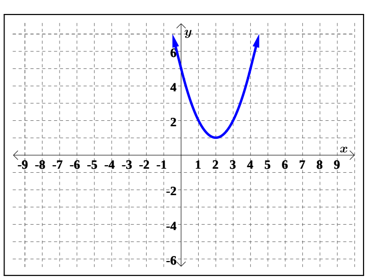


Figure 5.2.6: The coordinate plane with a parabola which does not intersect the x-axis.

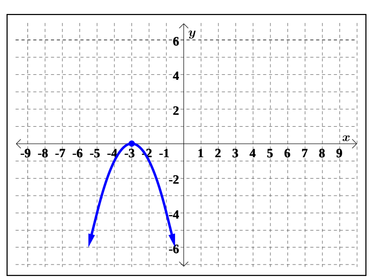


Figure 5.2.7: The coordinate plane with a parabola which intersects the x-axis exactly once.

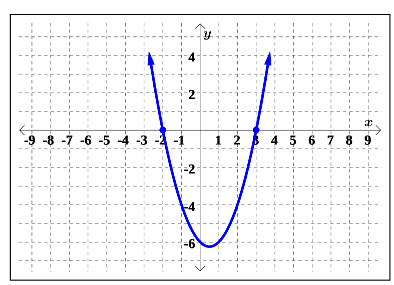


Figure 5.2.8: The coordinate plane with a parabola which intersects the x-axis exactly twice.

Example 9 Compute the y-intercept, real zeros, and x-intercept(s) of the quadratic function $f(x) = 3x^2 + 5x - 2$.

Solution:
We determine the y-intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 3(0)^2 + 5(0) - 2 \\ &= 0 + 0 + (-2) \\ &= -2 \end{aligned}$$

So, the y-intercept of $f(x)$ is $(0, -2)$.

To determine the real zeros, we calculate all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored, providing the simplest method for finding a solution.

$$\begin{gathered} 0 = (3x-1)(x+2) \\ 3x - 1 = 0 \qquad \text{or} \qquad x + 2 = 0 \\ x = \frac{1}{3} \qquad \text{or} \qquad x = -2 \end{gathered}$$

So, the real zeros occur when $x = \frac{1}{3}$ and $x = -2$, meaning the x-intercepts of $f(x)$ are $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. ■

In the last example, the real zeros of the function were found after factoring the quadratic expression, $3x^2 + 5x - 2$. However, there are many quadratic functions that cannot be factored. When a quadratic function cannot be factored, we can determine its real zeros using the **quadratic formula**.

Definition

If a, b, and c are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Or

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

■

We know having negative numbers underneath a square root produces values outside the set of real numbers. Given that $\sqrt{b^2-4ac}$ is part of the quadratic formula, we will need to pay special attention to the radicand, b^2-4ac. It turns out this quantity plays a critical role in determining the nature of the real zeros of a quadratic function and is given a special name.

Definition

If a, b, and c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2+bx+c=0$ is the quantity $\boldsymbol{b^2-4ac}$. ▪

The discriminant 'discriminates' between the possible types of real zeros which result from setting a quadratic function equal to zero and solving. These cases, and their relation to the discriminant, are summarized in **Theorem 5.2** below.

Theorem 5.2 The Discriminant Properties

Let a, b, and c be real numbers, with $a \neq 0$.

- If $b^2-4ac<0$, the equation $ax^2+bx+c=0$ has no real solutions and $f(x)=ax^2+bx+c$ has no real zeros. (See **Figure 5.2.6** for an example.)
- If $b^2-4ac=0$, the equation $ax^2+bx+c=0$ has exactly one real solution and $f(x)=ax^2+bx+c$ has one real zero. (See **Figure 5.2.7** for an example.)
- If $b^2-4ac>0$, the equation $ax^2+bx+c=0$ has exactly two real solutions and $f(x)=ax^2+bx+c$ has two real zeros. (See **Figure 5.2.8** for an example.)

▪ **Example 10** Use the quadratic formula to compute the real zeros of $f(x)=3x^2+5x-2$. Compare your answers to those found by factoring in the solution of Example 9.

Solution:
We identify $a=3$, $b=5$, and $c=-2$.

The discriminant is then

$$\begin{aligned} b^2-4ac &= (5)^2-4(3)(-2) \\ &= 25+24 \\ &= 49 \end{aligned}$$

Given that $49>0$, from **Theorem 5.2** we know $f(x)=3x^2+5x-2$ has exactly two real zeros.

Next, using the discriminant in the quadratic formula, we determine the zeros to be

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(5) \pm \sqrt{49}}{2(3)} \\ &= \frac{-5 \pm 7}{6} \end{aligned}$$

$$x = \frac{-5+7}{6} \quad \text{or} \quad x = \frac{-5-7}{6}$$

$$x = \frac{2}{6} \quad \text{or} \quad x = \frac{-12}{6}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -2$$

The reals zeros found here, when $x = \frac{1}{3}$ and $x = -2$, match the real zeros found by factoring.

■

N *If the reader is not confident in their factoring skills, the quadratic formula can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$.*

! *The quadratic formula is only applicable when solving the equation $ax^2 + bx + c = 0$. It does not apply to any other trinomial equation.*

■ **Example 11** Calculate the real zeros of the function $f(x) = 2x^2 + 4x - 4$.

Solution:
We begin by setting the output equal to zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we will solve the equation using the quadratic formula.

We identify $a = 2$, $b = 4$, and $c = -4$.
Then, the discriminant is

$$\begin{aligned} b^2 - 4ac &= (4)^2 - 4(2)(-4) \\ &= 16 + 32 \\ &= 48 \end{aligned}$$

Given that $48 > 0$, from **Theorem 5.2** we know $f(x) = 2x^2 + 4x - 4$ has exactly two real zeros.

Next, using the quadratic formula, we determine the real zeros to be

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(4) \pm \sqrt{48}}{2(2)} \\ &= \frac{-4 \pm \sqrt{48}}{4} \end{aligned}$$

$$x = \frac{-4 + \sqrt{48}}{4} \quad \text{or} \quad \frac{-4 - \sqrt{48}}{4}$$

As 48 is not a perfect square, the authors have chosen not to simplify the real zeros any further. If we had, the resulting zeros would be written as $x = -1 + \sqrt{3}$ and $x = -1 - \sqrt{3}$.

We can check our work by graphing the given function, using the graphing utility on a calculator, and observing the real zeros. This process is shown on a TI-84 calculator in the screen captures below in **Figures 5.2.9 - 5.2.14**.

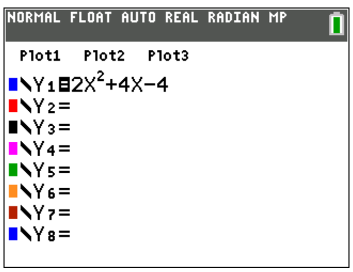


Figure 5.2.9: Calculator screenshot showing the input of $f(x)$ into $\mathrm{Y}_i =$.

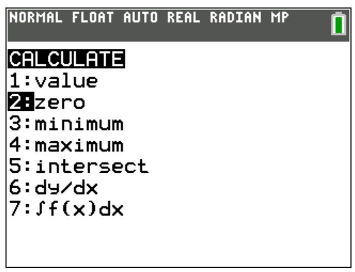


Figure 5.2.10: Calculator screenshot displaying the CALC menu and option 2 highlighted.

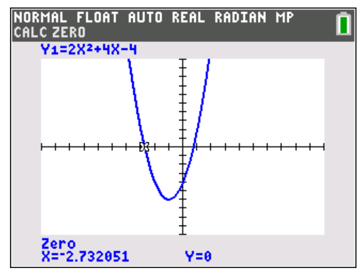


Figure 5.2.11: Calculator screenshot displaying the approximate answer for $x = -1 - \sqrt{3}$ found from the graph.

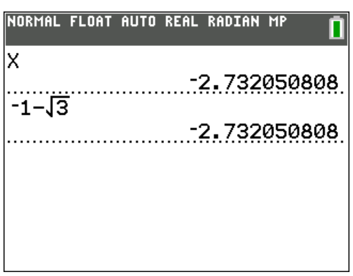


Figure 5.2.12: Calculator screenshot displaying the approximate value for $x = -1 - \sqrt{3}$ to verify the value in the previous screen is the same.

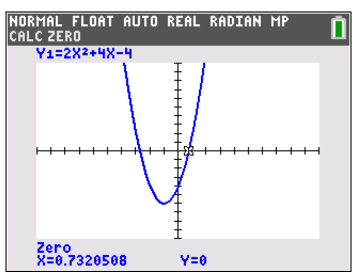


Figure 5.2.13: Calculator screenshot displaying the approximate answer for $x = -1 + \sqrt{3}$ found from the graph.

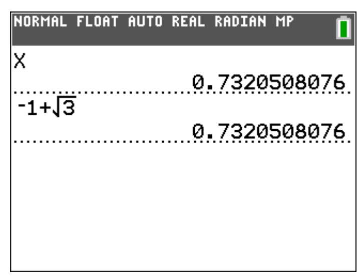


Figure 5.2.14: Calculator screenshot showing the approximate value for $x = -1 + \sqrt{3}$ to verify the value in the previous screen is the same.

When the zeros involve radicals, the calculator is unable to convert the approximation to an exact value. The only way to compute the exact value is by using the quadratic formula.

∎

Try It # 6:
In Try It # 4 we found some properties of the function $g(x) = 13 + x^2 - 6x$. Now, calculate the x- and y-intercept(s), if any exist.

Thus far in this section, we have determined the properties of a quadratic function, given its algebraic representation. Next, we will explore how to determine the graphical representation of a quadratic function, given a list of its properties.

Example 12 Graph the quadratic function with the following properties:

- As $x \to \pm\infty$, $f(x) \to -\infty$.
- There is a maximum value of 4.
- There are real roots when $x = -1$ and $x = 5$.
- The axis of symmetry is $x = 2$.
- The graph intersects the y-axis when $y = 2.5$.

Indicate the vertex and x-intercepts on your graph, and give the coordinates of each.

Solution:

- "As $x \to \pm\infty$, $f(x) \to -\infty$" indicates the end behavior of the quadratic function is

$$\swarrow \cdots \searrow$$

  Thus, the parabola will open downward.

- "There is a maximum value of 4" indicates the vertex has a y-coordinate of 4.

$$\text{Vertex} = (?, 4)$$

- Having "real roots when $x = -1$ and $x = 5$," means there are real zeros when $x = -1$ and $x = 5$; real zeros and x-intercepts are closely related. Thus, the x-intercepts are

$$(-1, 0) \text{ and } (5, 0)$$

- "The axis of symmetry is $x = 2$," indicates the vertex has an x-coordinate of 2. Combining this with what we already know about the vertex, we have

$$\text{Vertex} = (2, 4)$$

- "The graph intersects the y-axis when $y = 2.5$," means the y-intercept is

$$(0, 2.5)$$

The resulting graph is

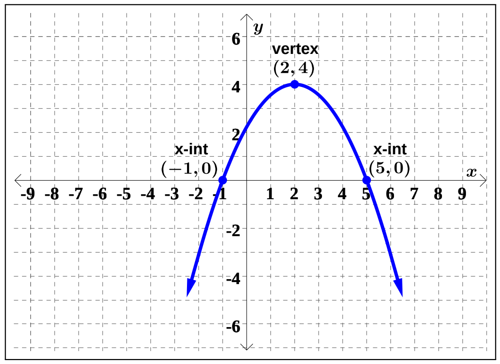


Figure 5.2.15: The coordinate plane with the graph of the function described in the example. The x-intercepts, $(-1,0)$ and $(5,0)$, and vertex, $(2,4)$, are labeled on the graph.

Note the point $(4,2.5)$ is also on the graph, due to the quadratic function's y-intercept, $(0,2.5)$, and its symmetry about $x = 2$.

■

Try It # 7:
Use the properties of quadratic functions to sketch a graph of $g(x) = 13 + x^2 - 6x$, without technology. (The properties of this function were computed in previous Try It # 4 and Try It #6 from this section.)

In Chapter 2, we discussed how to construct a linear revenue function of a company, $R(x)$. If the item being sold had a fixed selling price, p, then $R(x) = px$. However, we also learned that the selling price of an item could be determined by consumers in the form of a price-demand function, $p(x) = mx + b$. So, in general, if you are given a linear price-demand function, $p(x)$, then the revenue function is given by

$$\begin{aligned} R(x) &= p \cdot x \\ &= (mx+b)x \\ R(x) &= mx^2 + bx, \end{aligned}$$

which is a quadratic function. As profit is given by $P(x) = R(x) - C(x)$, if revenue is a quadratic function and total costs are linear, then profit will also be a quadratic function.

■ **Example 13** The total cost, in dollars, to produce x "Math is Awesome" T-shirts is $C(x) = 2x + 26$ for $x \geq 0$. The price-demand function, in dollars per shirt, is given by $p(x) = 30 - 2x$, for $0 \leq x \leq 15$. Compute

a. The profit function, $P(x)$.
b. The number of T-shirts which need to be sold in order to maximize profit.
c. The maximum profit.
d. The price to charge per T-shirt, in order to maximize profit.
e. And interpret the break-even quantity/quantities, if any exist.

Solution:

a. Recall from the chapter on Linear Functions, Profit = Revenue - Cost and Revenue = price · quantity, thus

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= p \cdot x - C(x) \\ &= (30-2x)x - (2x+26) \\ &= 30x - 2x^2 - 2x - 26 \\ P(x) &= -2x^2 + 28x - 26, \text{ for } \ 0 \leq x \leq 15 \end{aligned}$$

Notice profit is a quadratic function with $a = -2 < 0$, thus its graph opens downward and the maximum is at the vertex.

The number of T-shirts which need to be sold in order to maximize the profit is the x-coordinate of the vertex (as x is defined as a quantity of shirts), while the maximum profit is the y-coordinate of the vertex, in dollars.

The profit function, given in general form, is $P(x) = -2x^2 + 28x - 26$, so $a = -2$, $b = 28$, and $c = -26$.

b. The x-coordinate of the vertex is found using

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(28)}{2(-2)} \\ &= \frac{-28}{-4} \\ x &= 7 \end{aligned}$$

So, 7 T-shirts need to be sold, in order to maximize profit.

c. The y-coordinate of the vertex is found using

$$\begin{aligned} y &= f\left(\frac{-b}{2a}\right) \\ &= P(7) \\ &= -2(7)^2 + 28(7) - 26 \\ y &= 72 \end{aligned}$$

So, the maximum profit is \$72, when 7 T-shirts are sold.

d. The price per T-shirt is given by $p(x) = 30 - 2x$. For a maximum profit, 7 T-shirts must be sold. So,

$$\begin{aligned} p(7) &= 30 - 2(7) \\ &= 30 - 14 \\ p(7) &= 16 \end{aligned}$$

Thus, the T-shirts should be sold for \$16, in order to maximize profit.

e. Recall that break-even quantities found where Revenue = Cost or where Profit = 0. So, if $P(x) = -2x^2 + 28x - 26$, then the break-even quantities, x, are found when

$$-2x^2 + 28x - 26 = 0$$
$$-2(x^2 - 14x + 13) = 0$$
$$-2(x-13)(x-1) = 0$$

$$-2 \neq 0 \quad \text{or} \quad x - 13 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 13 \qquad\qquad x = 1$$

So, the break-even quantities are 1 and 13. This means that if 1 or 13 T-shirts are produced and sold, total costs are completely covered and there is no profit loss or gain.

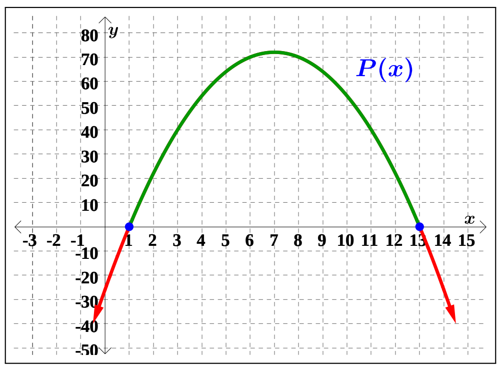


Figure 5.2.16: The graph of the profit function.

Notice from the graph of the profit function in **Figure 5.2.16**, a profit occurs when $1 < x < 13$.

■

■ **Example 14** A soccer stadium holds 62,000 spectators. With a ticket price of \$11, the average attendance has been 26,000. When the price dropped to \$9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price,

a. What is the revenue function for the sale of tickets?
b. How many tickets should be sold to maximize revenue? What is the maximum revenue?
c. At what price should tickets be sold, in order to maximize revenue?

Solution:

a. Revenue = price · quantity, but because attendance is linearly related to ticket price, we must begin by constructing the price-demand function, $p(x) = mx + b$, where x is the number of tickets sold and $p(x)$ is in dollars. With the information given, we have the points $(26000, 11)$ and $(31000, 9)$ on the graph of $p(x)$.

We can find the slope of the price-demand function:

$$\begin{aligned} m &= \frac{\Delta p}{\Delta x} \\ &= \frac{11-9}{26000-31000} \\ &= \frac{2}{-5000} \\ m &= -\frac{1}{2500} \end{aligned}$$

Then, using the point $(26000, 11)$ and the point-slope form of a line, $p - p_1 = m(x - x_1)$, we have a linear price-demand function of

$$\begin{aligned} p - 11 &= -\frac{1}{2500}(x - 26000) \\ p - 11 &= -\frac{1}{2500}x + \frac{52}{5} \\ p(x) &= -\frac{1}{2500}x + \frac{107}{5} \end{aligned}$$

So, revenue is given by

$$\begin{aligned} R(x) &= p \cdot x \\ &= \left(-\frac{1}{2500}x + \frac{107}{5}\right)x \\ R(x) &= -\frac{1}{2500}x^2 + \frac{107}{5}x \end{aligned}$$

b. Notice that revenue, $R(x)$, is a quadratic function with $a = -\frac{1}{2500} < 0$. Thus, its graph opens downward, and the maximum is at the vertex.

The number of tickets which need to be sold in order to maximize revenue is the x-coordinate of the vertex (as x is defined as a quantity of tickets), while the maximum revenue is the y-coordinate of the vertex, in dollars.

The revenue function in general form is $R(x) = -\frac{1}{2500}x^2 + \frac{107}{5}x + 0$, so $a = -\frac{1}{2500}$, $b = \frac{107}{5}$, and $c = 0$.

The x-coordinate of the vertex is found using

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-\frac{107}{5}}{2\left(-\frac{1}{2500}\right)} \\ &= \frac{-\frac{107}{5}}{\left(-\frac{2}{2500}\right)} \\ x &= 26750 \end{aligned}$$

So, 26,750 tickets should be sold to maximize revenue.

The y-coordinate of the vertex is found using

$$\begin{aligned} y &= f\left(\frac{-b}{2a}\right) \\ &= R(26750) \\ &= -\frac{1}{2500}(26750)^2 + \frac{107}{5}(26750) \\ y &= 286255 \end{aligned}$$

Thus, the maximum revenue is \$286,255, when 26750 tickets are sold.

c. The price per ticket is given by $p(x) = -\frac{1}{2500}x + \frac{107}{5}$. For maximum revenue, 26750 tickets must be sold, so

$$\begin{aligned} p(26750) &= -\frac{1}{2500}(26750) + \frac{107}{5} \\ &= -\frac{107}{10} + \frac{107}{5} \\ &= 10.7 \end{aligned}$$

Therefore, the tickets should be sold for \$10.70 each, in order to maximize revenue.

■

Knowing the revenue, price can be found by dividing the revenue by quantity: $R = px \Rightarrow p = \frac{R}{x}$.

Try It # 8:
The total cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is given by $C(x) = 18x + 240$, for $x \geq 0$, and the price-demand function, in cents per cup, is given by $p(x) = 90 - 3x$, $0 \leq x \leq 30$. Compute

a. The profit function, $P(x)$.
b. The number of cups which need to be sold, in order to maximize profit.
c. The maximum profit.
d. The price to charge per cup, in order to maximize profit.
e. The price to charge per cup to maximize revenue.
f. The break-even quantity/quantities, if they exist.

Try It Answers

1. **a.** $f(x)$ IS a polynomial.
- Degree: 6
- Leading coefficient: -1
- Constant term: -9

b. $g(t)$ IS a polynomial.
- Degree: 10
- Leading coefficient: 14
- Constant term: 0

2. **a.** As $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to -\infty$.

$$\nwarrow \cdots \searrow$$

b. As $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.

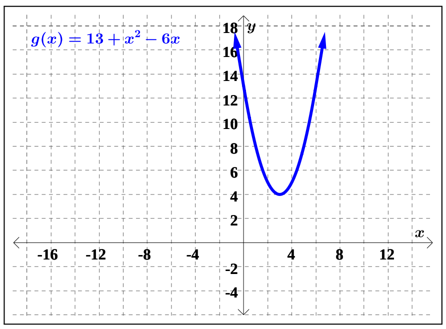
$$\nwarrow \ldots \nearrow$$

3. y-intercept: (0,0); real zeros when $t = 0$ and $t = 4$

4. General Form: $g(x) = x^2 - 6x + 13$
- Vertex: $(3, 4)$
- Axis of Symmetry: $x = 3$
- Minimum Value: 4
- Maximum Value: None
- Domain: $(-\infty, \infty)$
- Range: $[4, \infty)$

5.
- Vertex $\left(\frac{4}{7}, \frac{8}{11}\right)$
- Axis of Symmetry: $x = \frac{4}{7}$
- Minimum Value: $\frac{8}{11}$
- Maximum Value: None
- Domain: $(-\infty, \infty)$
- Range: $\left[\frac{8}{11}, \infty\right)$

6. y-intercept: $(0, 13)$; x-intercepts: None

7. 

8.
a. $P(x) = -3x^2 + 72x - 240$, $0 \leq x \leq 30$
b. 12 cups
c. 192 ¢
d. 54 ¢
e. 45 ¢
f. 4 cups and 20 cups of lemonade

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 6, determine if the function is a polynomial.

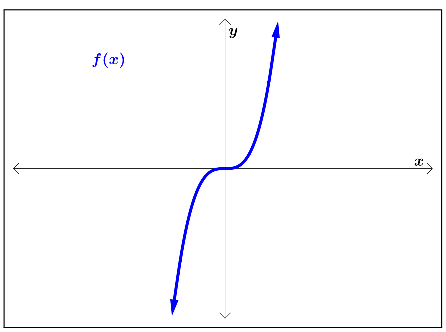
1. $f(x) = 4x^2 + 3x - 5$
2. $g(x) = -8x^7 + 2x^5 - 7x$
3. $h(x) = 6x^4 + 3x^{2/7} - 2$
4. $p(x) = 3$
5. $q(x) = 1 - 16x^{-4}$
6. $r(x) = -2x^2 + x + \pi - 3x^5$

For Exercises 7 - 10, use technology to graph each polynomial function, and then, describe the end behavior of the polynomial symbolically.

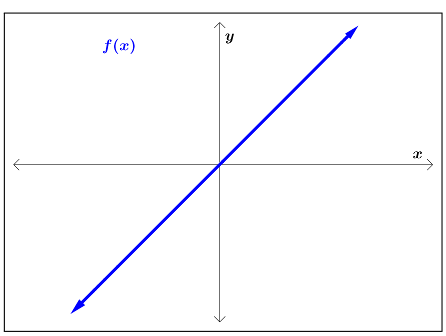
7. $f(x) = -ex + 5$
8. $g(x) = 2x^2 - \sqrt{3}x$
9. $h(x) = -8x^6 - 3x^5 + 2x + 2$
10. $k(x) = 3x^7 + 4x^2 - 1$

For Exercises 11 - 14, write the name of the parent function and corresponding equation for each polynomial function graphed.

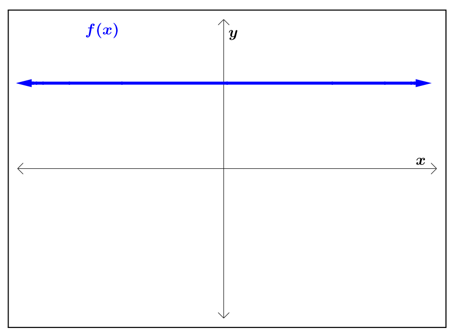
11.



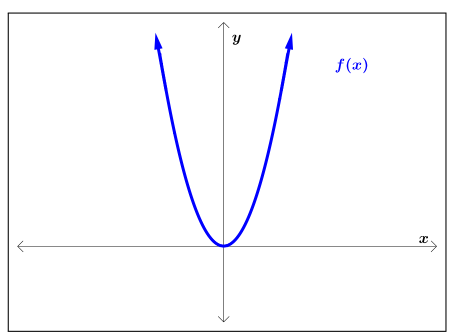
12.



13.

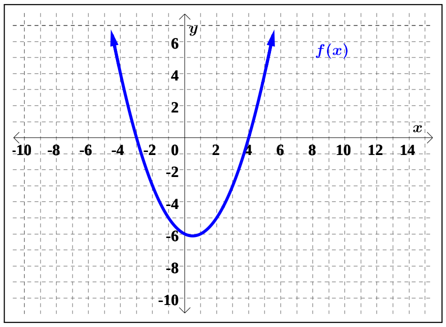


14.

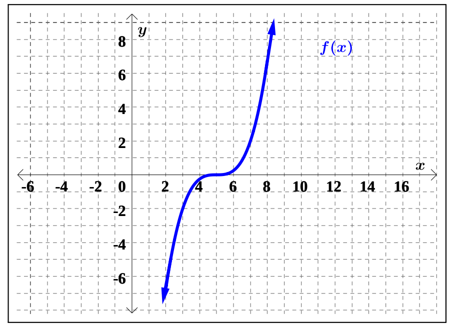


For Exercises 15 - 18, use the given graph of $f(x)$ to compute the real zeros. If none exist, write "Does not exist."

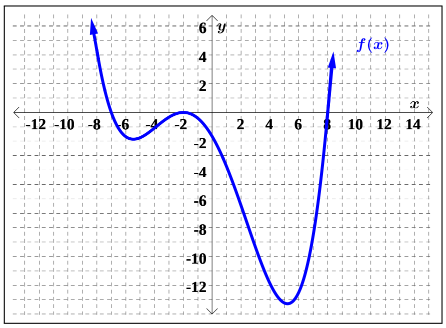
15.



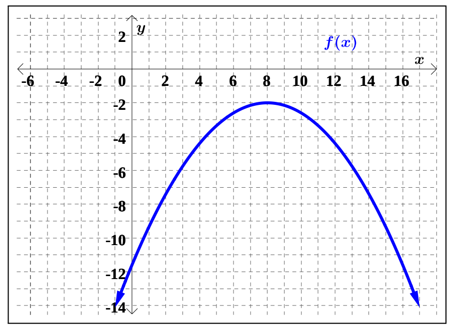
17.



16.



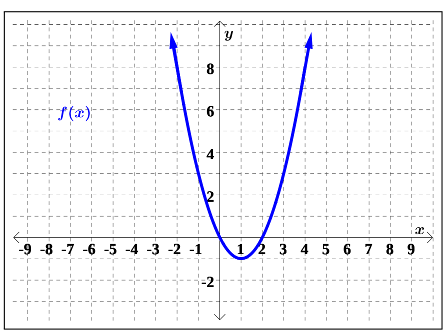
18.



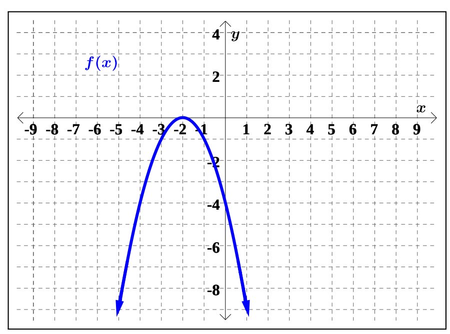
For Exercises 19 - 22, use the given graph to determine each of the following, if they exist.

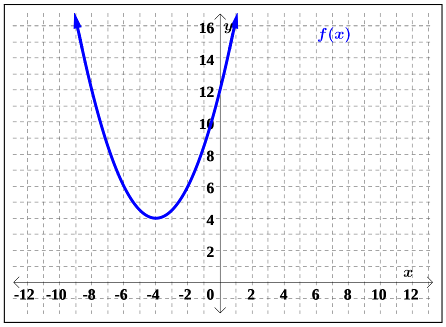
a. Vertex
b. Axis of symmetry
c. Domain
d. Range
e. x-intercept(s)
f. y-intercept
g. Maximum value
h. Minimum value

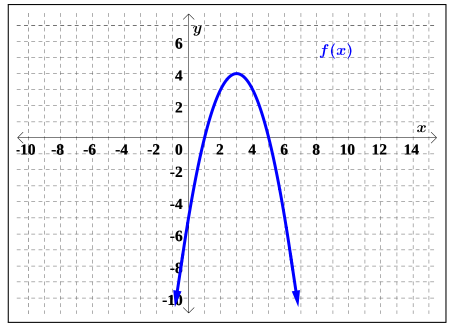
19.



20.



21. 

22. 

For Exercises 23 - 26, use the given price-demand function, $p(x)$, where x is the number of items made and sold and $p(x)$ is the price per item (in dollars), to write the revenue function, $R(x)$, for the sale of x items.

23. $p(x) = -2x + 60$
24. $p(x) = -0.05x + 50$
25. $p(x) = -4x + 24$
26. $p(x) = -\frac{2}{3}x + 200$

For Exercises 27 - 30, use the given revenue function, $R(x)$, and total cost function, $C(x)$, where x is the number of items made and sold, to write the profit function, $P(x)$, from the sale of x items.

27. $R(x) = -5x^2 + 90x$ and $C(x) = 20x + 120$
28. $R(x) = -3x^2 + 120x$ and $C(x) = 30x + 375$
29. $R(x) = -4x^2 + 520x$ and $C(x) = 80x + 4000$
30. $R(x) = -\frac{1}{2}x^2 + 85x$ and $C(x) = 45x + 750$

Intermediate Skills Practice (Answers)

For Exercises 31 - 36, state the degree, leading coefficient, and constant term of the given polynomial function.

31. $f(x) = -x - 3x^2$
32. $g(t) = 22.5t^{10} - \sqrt{3}t^{17} + et^7 - 10^{20}$
33. $h(x) = (x-1)(x-2)$
34. $p(x) = 7x^2 + \sqrt{2}x - 5x^9 + 2.8$
35. $q(n) = \frac{2}{11}n^{12}$
36. $r(x) = -4$

For Exercises 37 - 40, describe the end behavior of each polynomial symbolically.

37. $f(x) = 25x^{100} + x^{55} - 32$
38. $g(x) = -6x^{37} - 5x^{24} + 7x^9$
39. $h(x) = 0.25x^6 - 0.08x^{40} - 3x^8 + 9x^{11}$
40. $k(x) = 32 - 7x^{10} - 2x + x^{99} + x^{50}$

For Exercises 41 - 46, state the real zeros of the given polynomial.

41. $f(x) = (x-3)(7x+4)$
42. $g(x) = 3(x-2)(x+1)$
43. $h(x) = \frac{1}{2}x(4x+5)(x+11)$
44. $k(x) = x^2(x-6)^3(3x+2)^2(9x-5)$
45. $m(x) = (3-x)(x^2+1)$
46. $n(x) = 4\left(\frac{2}{3}x+7\right)\left(6x-\frac{4}{11}\right)$

For Exercises 47 - 54, without the use of technology, determine each of the following, if they exist, for the given polynomial.

a. Vertex
b. Axis of symmetry
c. Domain
d. Range
e. x-intercept(s)
f. y-intercept
g. Maximum value
h. Minimum value

47. $f(x) = -2(x-3)^2+7$
48. $g(x) = x^2-4$
49. $h(x) = -3x^2+5x-7$
50. $k(x) = \frac{1}{2}x^2-x+\frac{3}{4}$
51. $p(x) = 4(x+5)^2-8$
52. $q(x) = 27-3x^2$
53. $r(x) = -3(x+6)(x-1)$
54. $m(x) = -\frac{1}{5}x^2+\frac{8}{5}x-\frac{21}{5}$

For Exercises 55 - 58, use the given revenue function, $R(x)$, and total cost function, $C(x)$, where x is the number of items made and sold, to determine each of the following. Assume both revenue and total cost are given in dollars.

a. The number of items sold when revenue is maximized.

b. The maximum revenue.

c. The number of items sold when profit is maximized.

d. The maximum profit.
e. The break-even quantity/quantities, if they exist.

55. $R(x) = -5x^2+90x$ and $C(x) = 20x+120$
56. $R(x) = -3x^2+120x$ and $C(x) = 30x+375$
57. $R(x) = -4x^2+520x$ and $C(x) = 80x+4000$
58. $R(x) = -\frac{1}{2}x^2+85x$ and $C(x) = 45x+750$

Mastery Practice (Answers)

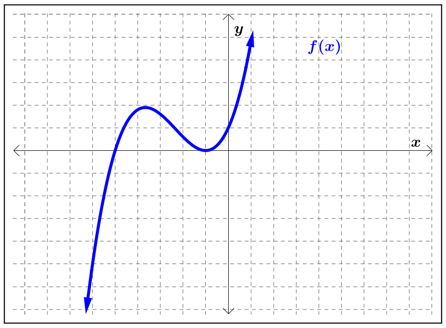
59. Write a 5th degree polynomial function with three terms which has a leading coefficient of 100 and constant term of -245.

60. Write a 10th degree polynomial function with five terms which has a leading coefficient of $\sqrt{73}$ and constant term of $\frac{\pi}{e}$.

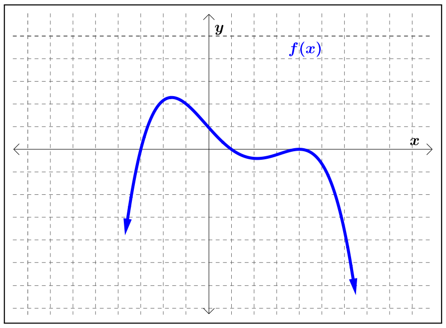
61. Symbolically describe the end behavior of $f(x) = ax^6 + bx^3 + cx - 8$, where a, b, and c are real numbers with $a < 0$, $b > 0$, and $c < 0$.

62. Symbolically describe the end behavior of $f(x) = ax^2 + bx^7 - cx^{10}$, where a, b, and c are real numbers with $a < 0$, $b < 0$, and $c < 0$.

63. Given the graph below is that of a polynomial function, state whether the function has even or odd degree and whether the leading coefficient is positive or negative.



64. Given the graph below is that of a polynomial function, state whether the function has even or odd degree and whether the leading coefficient is positive or negative.



For Exercises 65 - 70, state the real zeros and x-intercept(s) of the given polynomial, if they exist.

65. $f(x) = -4x^2 + x + 9$
66. $g(x) = 4x^5 - 36x^3$
67. $h(x) = -x^4 + 10x$
68. $p(x) = 2x^2 - 3x + 7$
69. $q(x) = 3x^3 + 18x^2 - 48x$
70. $r(x) = 12 - 16x - 3x^2$

71. Graph the quadratic function with the following properties:
    - As $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$
    - There is a minimum value of -3.
    - There are real zeros when $x = -4$ and $x = 2$.
    - The axis of symmetry is $x = -1$.
    - The graph intersects the y-axis when $y = -\dfrac{8}{3}$.

72. Graph the quadratic function with the following properties:
    - As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$
    - There is a maximum value of 9.
    - There are real roots when $x = 0$ and $x = 7$.
    - The axis of symmetry is $x = 3.5$.
    - The graph intersects the y-axis when $y = 0$.

73. A college organization is holding a water balloon tossing contest. You and a friend build a sling to fling your water balloon. A balloon released from the sling follows a path modeled by $b(t) = -2t^2 + 5t + 6$, where $b(t)$ is the height of the balloon t seconds after its release.
    a. What was the maximum height your balloon reached during its flight?

    b. How long after its release did your balloon hit the ground? (Round your answer to the nearest tenth of a second.)

74. Given the price-demand function $p(x) = -6x + 96$, where x is the number of thousands of items made and $p(x)$ is the price per item (in dollars), compute
    a. The revenue function of the sale of x thousands of items.
    b. The number of items sold when revenue is maximized.
    c. The maximum revenue.
    d. The price per item when revenue is maximized.

75. A company making widgets has a price-demand function of $p(x) = -0.1x + 30$ and a total cost function of $C(x) = 5x + 400$, where x represents the number of widgets made/sold and revenue and total cost are given in dollars.
    a. Construct the company's profit function.
    b. How many widgets must be sold, in order to maximize revenue?
    c. How many widgets must be sold, in order to maximize profit?
    d. At what price per widget will the company's maximum profit be achieved?

76. A company sells gadgets. They can sell 600 gadgets when the price is \$25/gadget, and they can sell 500 gadgets when the price is \$30/gadget. Determine
    a. The company's revenue function.
    b. The price per gadget when revenue is maximized.
    c. The maximum profit made from the sale of these gadgets, if the company incurs production costs of \$40 per gadget and has fixed costs of \$600.

Communication Practice (Answers)

77. Explain why the domain of any polynomial function is $(-\infty, \infty)$.
78. Describe the relationship between the real zeros, real roots, and the x-intercepts of a function.

5.3 Rational Functions

© Photo by Vanessa Coffelt, 2015

Suppose we know that the total cost of making a product is dependent on the number of items, x, produced. Suppose we also know this total cost is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the total cost function by the number of items, x.

The average cost function, which yields the average cost per item for x items produced, is then

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many application problems in calculus require finding an average value in a similar way, giving us variables in the denominator. Here we explore **rational functions**, which have variables in the denominator.

Learning Objectives:
In this section, you will learn about the properties and characteristics of rational functions and how to simplify rational expressions. Upon completion you will be able to:

- Identify if a given function is a rational function.
- Determine the domain of a rational function, using interval notation.
- Demonstrate addition, subtraction, multiplication, division, and simplification of rational expressions.
- Compute and fully simplify the difference quotient for given functions.

Describing Rational Functions

Definition

A **rational function** is a function which is the quotient of two *polynomial functions*. Said differently, $r(x)$ is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions. ■

According to this definition, all polynomial functions are also rational functions. (Consider when $q(x) = 1$.)

■ **Example 1** Which of the following are rational functions?

a. $f(x) = \dfrac{3}{x+1}$

b. $g(x) = \dfrac{(2x-3)(x+4)}{(x-6)(x+1)}$

c. $h(x) = \dfrac{x^{\frac{1}{2}}}{4x+5}$

Solution:

A rational function is the quotient of two polynomials, $p(x)$ and $q(x)$: $\dfrac{p(x)}{q(x)}$

a. $f(x) = \dfrac{3}{x+1}$

The numerator is $n(x) = 3$, which is a constant polynomial function.

The denominator is $d(x) = x+1$, which is a 1^{st} degree polynomial function.

Thus, $f(x)$ is the quotient of two polynomials and IS a rational function.

b. $g(x) = \dfrac{(2x-3)(x+4)}{(x-6)(x+1)} = \dfrac{2x^2+5x-12}{x^2-5x-6}$

The numerator is $n(x) = (2x-3)(x+4) = 2x^2+5x-12$, which is a 2^{nd} degree polynomial.

The denominator is $d(x) = (x-6)(x+1) = x^2-5x-6$, which is a 2^{nd} degree polynomial.

Thus, $g(x)$ is the quotient of two polynomials and IS a rational function.

c. $h(x) = \dfrac{x^{\frac{1}{2}}}{4x+5}$

The numerator is $n(x) = x^{\frac{1}{2}}$, which is *not* a polynomial function, because the power, $\frac{1}{2}$, is not a whole number.

Thus, $h(x)$ is NOT a rational function.

■

Notice the product of polynomials results in another polynomial, as shown in part **b** of the previous example.

Properties of a Rational Function

Unlike with polynomials, where any real number input produces a real number output, one must be cautious when selecting inputs for a rational function. For a rational function the ratio must be defined, meaning the denominator must be nonzero. In general, to find the the domain of a rational function, we need to determine and exclude the real number inputs which would cause division by zero.

> The **domain** of a rational function, $r(x) = \dfrac{p(x)}{q(x)}$, includes all real numbers except those that cause the denominator to equal zero.
>
> In other words, **all $\boldsymbol{x}$ such that $\boldsymbol{q(x) \neq 0}$**.

Consider the rational function, $f(x) = \dfrac{2x-1}{x+1}$.

To determine the domain of $f(x)$, we investigate the values of x where the denominator is equal to zero.

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$$

As the denominator is equal to zero at $x = -1$, the domain of $f(x)$ is all real numbers except $x = -1$.

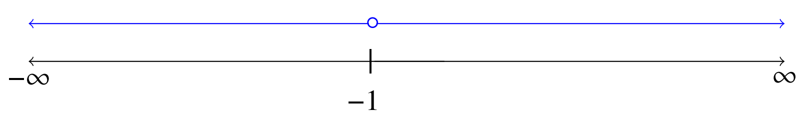


Figure 5.3.2: A graphical representation of $x \neq -1$.

Using interval notation, the domain of $f(x)$ is $(-\infty, -1) \cup (-1, \infty)$.

■ **Example 2** Using interval notation, state the domain of $g(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution:
$g(x)$ is the quotient of two quadratic polynomial functions, and thus, $g(x)$ is a rational function.

To determine the domain of $g(x)$, we investigate the values of x where the denominator is equal to zero.

$$\begin{aligned}
2-x-x^2 = -x^2-x+2 &= 0 \\
-x^2-x+2 &= 0 \\
-1(x^2+x-2) &= 0 \\
-1(x+2)(x-1) &= 0 \\
-1 \neq 0 \quad \text{or} \quad x+2 &= 0 \quad \text{or} \quad x-1=0 \\
x &= -2 \quad \text{or} \quad x=1
\end{aligned}$$

The denominator is equal to zero when $x = -2$ and $x = 1$, so we exclude these from the real numbers when stating the domain.

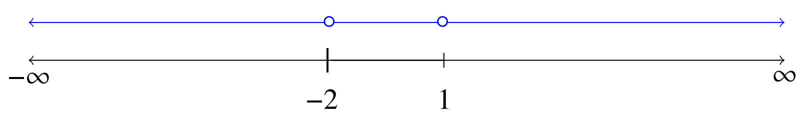


Figure 5.3.3: A graphical representation of $x \neq -2$ and $x \neq 1$.

Using interval notation, the domain of $g(x)$ is $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$.

N *A graph of this function, as shown in* ***Figure 5.3.4****, confirms that the function is not defined when $x = -2$ or $x = 1$.*

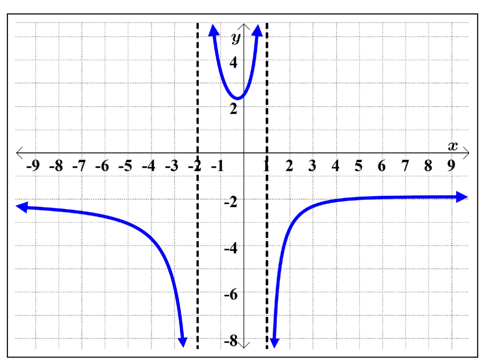


Figure 5.3.4: A graph of $g(x) = \frac{5+2x^2}{2-x-x^2}$.

There are ***vertical asymptotes*** *at $x = -2$ and $x = 1$.*

■

■ **Example 3** Using interval notation, state the domain of $h(x) = \dfrac{x+3}{x^2-9}$.

Solution:
$h(x)$ is the quotient of two polynomial functions (one linear and one quadratic), and thus, $h(x)$ is a rational function.

To determine the domain of $h(x)$, we investigate the values of x where the denominator is equal to zero.

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ x-3 = 0 \quad &\text{or} \quad x+3 = 0 \\ x = 3 \quad &\text{or} \quad \quad x = -3 \end{aligned}$$

Because the denominator is equal to zero when $x = \pm 3$, we exclude these from the real numbers when stating the domain.

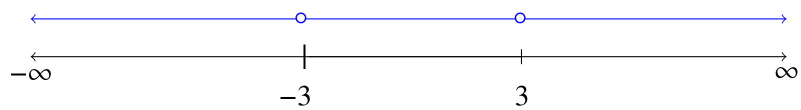


Figure 5.3.5: A graphical representation of $x \neq \pm 3$.

Using interval notation, the domain of $h(x)$ is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

N *A graph of this function, as shown in* ***Figure 5.3.6****, confirms that the function is not defined when* $x = \pm 3$.

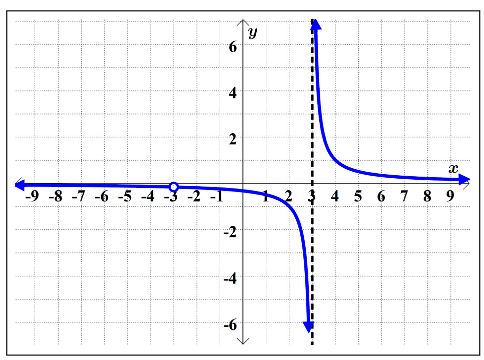


Figure 5.3.6: A graph of $h(x) = \frac{x+3}{x^2-9}$.

There is a vertical asymptote at $x = 3$ *and a* ***hole*** *in the graph when* $x = -3$.

■

In the two previous examples, graphs were provided to demonstrate the behavior of $g(x)$ and $h(x)$ at points not in their respective domains. In both graphs, a vertical asymptote existed. A vertical asymptote, $x = a$, exists when the function is not defined at a and $f(x) \to \pm\infty$ 'near' a. In the graph of $h(x)$, there also existed a hole when $x = a$. A hole is a 'point' on the graph of a function which occurs when the function is not defined at a, but $f(x) \to f(a)$ 'near' a. The concepts of vertical asymptotes and holes will be more formally defined in calculus. For now, we will focus on algebraic techniques for identifying the values of x where a vertical asymptote or a hole exists on a graph of a function.

In the example of $h(x)$, while neither $x = -3$ nor $x = 3$ ise in the domain of $h(x)$, the behavior of the graph of $h(x)$ is drastically different near these x-values. The reason for this difference lies in the simplified form of $h(x) = \frac{(x+3)}{(x-3)(x+3)}$. The reason $x = 3$ is not in the domain of $h(x)$ is because the factor $(x-3)$ appears in the denominator of $h(x)$; similarly, $x = -3$ is not in the domain of $h(x)$ because of the factor $(x+3)$ in the denominator of $h(x)$. The major difference between these two factors is that $(x+3)$ is also a factor in the numerator, whereas $(x-3)$ is not. Loosely speaking, the trouble caused by $(x+3)$ in the denominator is divided out, while the factor $(x-3)$ remains to cause mischief. This is why the graph of $h(x)$ has a vertical asymptote at $x = 3$ but only a hole when $x = -3$. These observations are generalized and summarized in the theorem below, whose proof is found in calculus.

Theorem 5.3 Location of Vertical Asymptotes and Holes

Suppose $r(x)$ is a rational function which can be written as $r(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ have no common real zeros. Let c be a real number which is not in the domain of $r(x)$.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a **hole** at $\left(c, \dfrac{p(c)}{q(c)}\right)$.
- If $q(c) = 0$, then the line $x = c$ is a **vertical asymptote** of the graph of $y = r(x)$.

In English, the previous theorem says that if $x = c$ is not in the domain of $r(x)$, then one of two features will occur on the graph of $y = r(x)$ at $x = c$. If when we simplify $r(x)$ the denominator is no longer 0 when $x = c$, then we have a hole in the graph of $y = r(x)$ when $x = c$. Otherwise, the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

Try It # 1:
State the domain of $f(x) = \dfrac{4x}{5(x+4)(x-8)}$, using interval notation. At any point not in the domain, explain what is happening with the graph of $f(x)$.

Try It # 2:
State the domain of $f(x) = \dfrac{x^2-25}{x(x-1)(x-5)}$, using interval notation. At any point not in the domain, explain what is happening with the graph of $f(x)$.

Recall that a polynomial's end behavior will mirror that of its leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the polynomial functions, that is the ratio of the leading terms. The end behavior of a rational function can approach ∞ or $-\infty$ like polynomials, but, unlike polynomials, the end behavior of a rational function can, instead, approach a finite number called a **horizontal asymptote**. Examples of the end behavior for rational functions can be found in **Figures 5.3.7**, **5.3.8**, and **5.3.9**.

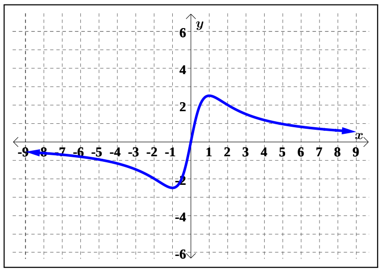


Figure 5.3.7: A rational function with a horizontal asymptote at $y = 0$.

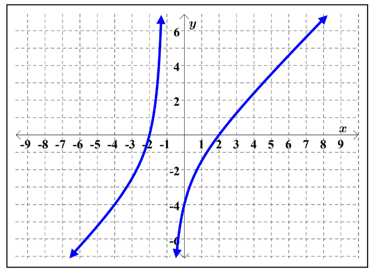


Figure 5.3.8: A rational function with no horizontal asymptote.

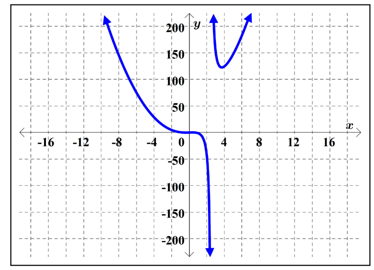


Figure 5.3.9: Another rational function with no horizontal asymptote

The process for computing end behavior of a rational function is left to a calculus course.

Recall from **Section 5.2**, a list of parent functions was started. We can now add two additional parent rational functions to the list, as shown in **Table 5.6** below.

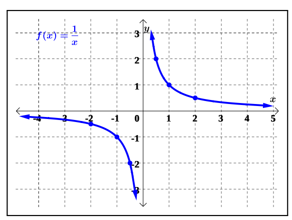
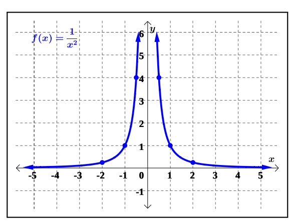
Name	Function	Graph	Table (x)	Table ($f(x)$)	Domain
Reciprocal	$f(x) = \frac{1}{x}$		-2 -1 -0.5 0.5 1 2	-0.5 -1 -2 2 1 0.5	$(-\infty, 0) \cup (0, \infty)$
Reciprocal Squared	$f(x) = \frac{1}{x^2}$		-2 -1 -0.5 0.5 1 2	0.25 1 4 4 1 0.25	$(-\infty, 0) \cup (0, \infty)$

Table 5.6: Parent Rational Functions

Intercepts of a Rational Function

All intercepts of a rational function, $r(x)$, are found *after* finding the domain of $r(x)$ *and after* $r(x)$ has been reduced, such that the numerator and denominator have no common factors (in lowest terms).

A rational function, $r(x)$, will have a y-intercept when the input is zero, $r(0)$, if $r(x)$ is defined at $x = 0$. A rational function will not have a y-intercept if $r(x)$ is not defined at $x = 0$.

As with polynomials, a rational function will have x-intercept(s) where $r(x) = 0$. A fraction is only equal to zero when the numerator is zero; x-intercept(s) can only occur when the numerator of the rational function is equal to zero.

■ **Example 4** Determine the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$, if they exist.

Solution:
First we must identify the domain of the function, as $f(x)$ is a rational function. By setting the denominator equal to 0, we have

$$\begin{array}{ccccc} & & (x-1)(x+2)(x-5) = 0 & & \\ x-1=0 & \text{or} & x+2=0 & \text{or} & x-5=0 \\ x=1 & \text{or} & x=-2 & \text{or} & x=5 \end{array}$$

Thus, the domain of $f(x)$ is $(-\infty,-2)\cup(-2,1)\cup(1,5)\cup(5,\infty)$.

Also notice $f(x)$ is given in lowest terms, as the numerator and denominator have no common factors.

Seeing as $x = 0$ is in the domain of $f(x)$, we can calculate the y-intercept by evaluating the function at zero.

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= \frac{-6}{10} \\ &= -\frac{3}{5} \end{aligned}$$

So, the y-intercept is $\left(0,-\dfrac{3}{5}\right)$.

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Because the expression is zero when the numerator is zero, then

$$\begin{array}{ccc} & 0 = (x-2)(x+3) & \\ x-2=0 & \text{or} & x+3=0 \\ x=2 & \text{or} & x=-3 \end{array}$$

Both $x = 2$ and $x = -3$ are in the domain of $f(x)$; thus, the x-intercepts are $(2,0)$ and $(-3,0)$.

■

▪ **Example 5** Determine the intercepts of $g(x) = \dfrac{(x-4)(2x+1)}{x(2x+1)(x+4)}$, if they exist.

Solution:
First we must identify the domain of the function, as $g(x)$ is a rational function. By setting the denominator equal to 0, we have

$$\begin{array}{ccccc} & & x(2x+1)(x+4)=0 & & \\ x=0 & \text{or} & 2x+1=0 & \text{or} & x+4=0 \\ & & 2x=-1 & & \\ x=0 & \text{or} & x=-\dfrac{1}{2} & \text{or} & x=-4 \end{array}$$

Thus, the domain of $g(x)$ is $(-\infty,-4)\cup\left(-4,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},0\right)\cup(0,\infty)$.

Notice $g(x)$ has a common factor of $2x+1$ that can be divided out from the numerator and denominator, and thus, we know $g(x)$ has a hole when $x=-\frac{1}{2}$. Reducing $g(x)$ to lowest terms results in

$$\begin{aligned} g(x) &= \frac{(x-4)\cancel{(2x+1)}}{x\cancel{(2x+1)}(x+4)} \\ &= \frac{(x-4)}{x(x+4)} \end{aligned}$$

As $x=0$ is not in the domain of $g(x)$, $g(x)$ has no y-intercept.

The x-intercepts will occur when

$$\frac{(x-4)}{x(x+4)} = 0$$

This is only true when the numerator is zero:

$$\begin{aligned} x-4 &= 0 \\ x &= 4 \end{aligned}$$

The x-intercept is $(4,0)$, after verifying $x=4$ is in the domain of $g(x)$.

▪

Try It # 3:
Determine the intercepts of $h(x) = \dfrac{x^2-25}{x(x-1)(x-5)}$, if they exist. (This function was investigated in Try It #2.)

As we have seen in the previous examples, it has been necessary to use simplifications to determine properties of rational functions. The simplifications, so far, have required factoring and dividing out common terms within the function rule. We will now turn our focus to working with the rule without using any function notation.

Combining and Simplifying Rational Expressions

The function rule for a rational function is called a **rational expression**. We can apply the properties of fractions to rational expressions, such as

- Multiplying by a scalar or another rational expression
- Dividing
- Adding/subtracting
- Simplifying
- Reducing to lowest terms

Multiplication Involving Rational Expressions

Multiplication involving rational expressions works the same way as multiplication involving any other fractions. We multiply the numerators to calculate the numerator of the product, and then, multiply the denominators to calculate the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators, just as we did when simplifying rational expressions, so that we are able to simplify the resulting product more easily.

Example 6 Perform the indicated multiplication.

$$-6\left(\frac{4x+1}{x-2}\right)$$

Solution:
The number -6 is equal to $\frac{-6}{1}$.

Therefore, the multiplication becomes:

$$\begin{aligned}-6\left(\frac{4x+1}{x-2}\right) &= \frac{-6}{1}\cdot\left(\frac{4x+1}{x-2}\right)\\ &= \frac{(-6)(4x+1)}{(1)(x-2)}\\ &= \frac{-24x-6}{x-2}\end{aligned}$$

As stated $-6 = \dfrac{-6}{1}$, *but* $-6 \neq \dfrac{-6}{-6}$.

When multiplying a rational expression by an integer, only multiply the numerator of the rational expression by the integer.

■

Example 7 Multiply the rational expressions and simplify.

$$\left(\frac{x^2+4x-5}{3x+18}\right)\cdot\left(\frac{2x-1}{x+5}\right)$$

Solution:
We begin by factoring the numerator and denominator of each expression.

$$\left(\frac{x^2+4x-5}{3x+18}\right)\cdot\left(\frac{2x-1}{x+5}\right)=\frac{(x+5)(x-1)}{3(x+6)}\cdot\frac{(2x-1)}{x+5}$$

Then, we multiply the numerators and denominators of the expressions and divide out common factors to simplify.

$$\begin{aligned}&=\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}\\&=\frac{\cancel{(x+5)}(x-1)(2x-1)}{3(x+6)\cancel{(x+5)}}\\&=\frac{(x-1)(2x-1)}{3(x+6)}\end{aligned}$$

∎

Try It # 4:
Multiply the rational expressions and simplify.

$$\left(\frac{x^2+5x+4}{x+7}\right)\cdot\left(\frac{x-3}{x+1}\right)$$

Division Involving Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second.

Using this approach, we would rewrite $\frac{1}{3}\div\frac{x^2}{3}$ as the product $\frac{1}{3}\cdot\frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as explained previously.

$$\frac{1}{3}\div\frac{x^2}{3}=\frac{1}{3}\cdot\frac{3}{x^2}=\frac{1\cdot 3}{3\cdot x^2}=\frac{1\cdot\cancel{3}}{\cancel{3}\cdot x^2}=\frac{1}{x^2}$$

Similarly, if given $\frac{\frac{2}{x}}{\frac{x}{6}}$, we would rewrite the division as a product, and multiply as follows:

$$\begin{aligned}\frac{\frac{2}{x}}{\frac{x}{6}}&=\frac{\left(\frac{2}{x}\right)}{\left(\frac{x}{6}\right)}\\&=\left(\frac{2}{x}\right)\cdot\left(\frac{6}{x}\right)\\&=\frac{2\cdot 6}{x\cdot x}\\&=\frac{12}{x^2}\end{aligned}$$

Example 8 Divide the rational expressions and simplify.

$$\left(\frac{2x^2+x-6}{x^2-1}\right)\div\left(\frac{x^2-4}{x^2+2x+1}\right)$$

Solution:
We begin by writing the division expression as a multiplication expression.

$$\left(\frac{2x^2+x-6}{x^2-1}\right)\div\left(\frac{x^2-4}{x^2+2x+1}\right)=\left(\frac{2x^2+x-6}{x^2-1}\right)\cdot\left(\frac{x^2+2x+1}{x^2-4}\right)$$

Now, we multiply and simplify.

$$\begin{aligned}
&=\left(\frac{(2x-3)(x+2)}{(x+1)(x-1)}\right)\cdot\left(\frac{(x+1)(x+1)}{(x+2)(x-2)}\right)\\
&=\frac{(2x-3)(x+2)(x+1)(x+1)}{(x+1)(x-1)(x+2)(x-2)}\\
&=\frac{(2x-3)\cancel{(x+2)}\cancel{(x+1)}(x+1)}{\cancel{(x+1)}(x-1)\cancel{(x+2)}(x-2)}\\
&=\frac{(2x-3)(x+1)}{(x-1)(x-2)}
\end{aligned}$$

■

Try It # 5:
Divide the rational expressions and simplify.

$$\frac{3xh+h}{x^2+2x+1}\div h$$

Addition and Subtraction Involving Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to identify a common denominator. Let's look at an example of fraction addition.

$$\begin{aligned}
\frac{1}{10}+\frac{1}{12}&=\frac{1}{2\cdot5}+\frac{1}{2\cdot6}\\
&=\left(\frac{6}{6}\right)\left(\frac{1}{2\cdot5}\right)+\left(\frac{1}{2\cdot6}\right)\left(\frac{5}{5}\right)\\
&=\frac{6}{6\cdot2\cdot5}+\frac{5}{2\cdot6\cdot5}\\
&=\frac{11}{6\cdot2\cdot5}\\
&=\frac{11}{60}
\end{aligned}$$

When adding or subtracting rational expressions, we must also determine a common denominator; the easiest common denominator to use will be the **least common denominator**, or **LCD**. The LCD is the smallest multiple that the denominators have in common. To identify the LCD of two rational expressions, we factor the expressions and multiply all the distinct factors. For instance, if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$, then the LCD would be $(x+3)(x+4)(x+5)$.

Once we have identified the LCD, we need to multiply each expression by the form of '1' that will change the denominator to the LCD. Using the denominators of the two rational expressions just mentioned, we would need to multiply the expression with a denominator of $(x+3)(x+4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x+4)(x+5)$ by $\frac{x+3}{x+3}$.

■ **Example 9** Add the rational expressions and simplify.

$$\frac{5}{x}+\frac{6}{x+1}$$

Solution:
First, we have to identify the LCD of the expressions. In this case the LCD will be $x(x+1)$. We then multiply each expression by the appropriate form of '1' to obtain $x(x+1)$ as the denominator for each fraction.

$$\begin{aligned}
\frac{5}{x}+\frac{6}{x+1} &= \left(\frac{5}{x}\right)\cdot\left(\frac{x+1}{x+1}\right)+\left(\frac{6}{x+1}\right)\cdot\left(\frac{x}{x}\right)\\
&= \frac{5(x+1)}{(x)(x+1)}+\frac{6x}{(x+1)(x)}\\
&= \frac{5x+5}{(x)(x+1)}+\frac{6x}{(x)(x+1)}
\end{aligned}$$

Now that the expressions have the same denominator, we simply add the numerators over the common denominator to compute the sum.

$$\begin{aligned}
&= \frac{5x+5+6x}{(x)(x+1)}\\
&= \frac{11x+5}{(x)(x+1)}
\end{aligned}$$

While we could have expanded the denominator and written the sum as $\frac{11x+5}{x^2+x}$, oftentimes we leave the denominator in factored form to help bring to light any simplification which might be possible. As the numerator here does not factor, no simplification is necessary.

N *Multiplying by* $\left(\frac{x+1}{x+1}\right)$ *or* $\left(\frac{x}{x}\right)$ *does not change the value of the original expression, because any number divided by itself (except 0) is 1, and multiplying an expression by '1' gives an expression equivalent to the original.*

■

■ **Example 10** Subtract the rational expressions and simplify.

$$\frac{6}{x^2+4x+4} - \frac{2}{x^2-4}$$

Solution:
First, we factor the denominator of each expression to determine the LCD.

$$\frac{6}{x^2+4x+4} - \frac{2}{x^2-4} = \frac{6}{(x+2)(x+2)} - \frac{2}{(x+2)(x-2)}$$

Next, we multiply each expression by the appropriate '1' to obtain the LCD of $(x+2)(x+2)(x-2)$.

$$\begin{aligned}
&= \frac{6}{(x+2)(x+2)} \cdot \frac{(x-2)}{(x-2)} - \frac{2}{(x+2)(x-2)} \cdot \frac{(x+2)}{(x+2)} \\
&= \frac{6(x-2)}{(x+2)^2(x-2)} - \frac{2(x+2)}{(x+2)^2(x-2)}
\end{aligned}$$

Then, we expand each numerator and write the difference over the common denominator.

$$= \frac{6x-12-(2x+4)}{(x+2)^2(x-2)}$$

Last, we compute the difference in the numerator and simplify.

$$\begin{aligned}
&= \frac{6x-12-2x-4}{(x+2)^2(x-2)} \\
&= \frac{4x-16}{(x+2)^2(x-2)} \\
&= \frac{4(x-4)}{(x+2)^2(x-2)}
\end{aligned}$$

■

When subtracting rational expressions, after multiplying by the appropriate '1' to achieve a common denominator, remember to distribute the subtraction throughout the second numerator.

Try It # 6:
Subtract the rational expressions and simplify.

$$\frac{3x-2}{x+5} - \frac{4x+1}{x-3}$$

When multiple operations are to be performed with rational expressions, use order of operations as you would with numerical fractions.

■ **Example 11** Perform the indicated operations and simplify.

$$\frac{x-1}{x+1}-2\left(\frac{x+3}{x-1}\right)$$

Solution:
Using order of operations, we begin by multiplying the second expression by the scalar, 2.

$$\begin{aligned}\frac{x-1}{x+1}-2\left(\frac{x+3}{x-1}\right) &= \frac{x-1}{x+1}-\frac{2}{1}\left(\frac{x+3}{x-1}\right)\\ &= \frac{x-1}{x+1}-\left(\frac{2x+6}{x-1}\right)\end{aligned}$$

Now, we subtract the expressions as previously described.

$$\begin{aligned}&= \left(\frac{x-1}{x+1}\right)\cdot\left(\frac{x-1}{x-1}\right)-\left(\frac{2x+6}{x-1}\right)\cdot\left(\frac{x+1}{x+1}\right)\\ &= \frac{x^2-2x+1}{(x+1)(x-1)}-\frac{2x^2+8x+6}{(x-1)(x+1)}\\ &= \frac{x^2-2x+1-2x^2-8x-6}{(x-1)(x+1)}\\ &= \frac{-x^2-10x-5}{(x-1)(x+1)}\\ &= \frac{-(x^2+10x+5)}{(x-1)(x+1)}\end{aligned}$$

■

! *When simplifying, remember to divide out factors, not terms. Factors involve products, but whole terms involve addition and/or subtraction.*

$$\frac{x(x-1)}{(x-1)(x+1)}=\frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}=\frac{x}{x+1} \qquad \textit{but} \qquad \frac{x^2+1}{x^2-4}\neq\frac{1}{-4}$$

Computing the Difference Quotient

In calculus you will spend a lot of time looking at how quickly function outputs change when the input only changes a tiny bit. You will do this by looking at the **difference quotient** for the function.

Definition

Given a function, $f(x)$, the **difference quotient** of $f(x)$ is the expression

$$\frac{f(x+h)-f(x)}{h}$$

■

For reasons which will become clear in calculus, 'simplifying' a difference quotient means rewriting it in a form where the "h" *in the definition* of the difference quotient is no longer in the denominator. Once that happens, we consider our work to be done.

■ **Example 12** Compute and simplify the difference quotient for the function $f(x) = x^2 - x - 2$.

Solution:
First, we determine $f(x+h)$; we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= (x+h)(x+h) - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2 \end{aligned}$$

Substituting the result into the formula and simplifying, the difference quotient is

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x^2+2xh+h^2-x-h-2)-(x^2-x-2)}{h} \\ &= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} \\ &= \frac{2xh+h^2-h}{h} \\ &= \frac{h(2x+h-1)}{h} \\ &= \frac{\cancel{h}(2x+h-1)}{\cancel{h}} \\ &= 2x+h-1 \end{aligned}$$

Seeing as we have divided out the original "h" from the denominator, we have 'simplified' the difference quotient and we are done.

■

Example 13 Compute and simplify the difference quotient for the function $g(x) = \dfrac{3}{2x+1}$.

Solution:

To determine $g(x+h)$, we replace every occurrence of x in the formula $g(x) = \dfrac{3}{2x+1}$ with the quantity $(x+h)$ to get

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x+2h+1} \end{aligned}$$

which yields a simplified difference quotient of

$$\begin{aligned} \frac{g(x+h)-g(x)}{h} &= \frac{\left(\frac{3}{2x+2h+1}\right) - \frac{3}{2x+1}}{h} \\ &= \left(\frac{3}{2x+2h+1} - \frac{3}{2x+1}\right)\cdot\left(\frac{1}{h}\right) \\ &= \left(\frac{1}{h}\right)\cdot\left[\left(\frac{2x+1}{2x+1}\right)\left(\frac{3}{2x+2h+1}\right) - \left(\frac{3}{2x+1}\right)\left(\frac{2x+2h+1}{2x+2h+1}\right)\right] \\ &= \left(\frac{1}{h}\right)\cdot\left[\frac{6x+3-6x-6h-3}{(2x+1)(2x+2h+1)}\right] \\ &= \left(\frac{1}{h}\right)\cdot\left[\frac{-6h}{(2x+1)(2x+2h+1)}\right] \\ &= \frac{-6h}{h(2x+1)(2x+2h+1)} \\ &= \frac{-6\cancel{h}}{\cancel{h}(2x+1)(2x+2h+1)} \\ &= \frac{-6}{(2x+1)(2x+2h+1)} \end{aligned}$$

■

N *When simplifying a difference quotient, if it is not possible to divide out the "h" from the denominator, then an error has occurred in your work.*

Try It # 7:

Compute and simplify the difference quotient for each function.

a. $f(x) = x^2 - 6$

b. $g(x) = \dfrac{x}{x+4}$

Try It Answers

1. Domain $(-\infty,-4)\cup(-4,8)\cup(8,\infty)$
   - $x=-4$: vertical asymptote
   - $x=8$: vertical asymptote

2. Domain $(-\infty,0)\cup(0,1)\cup(1,5)\cup(5,\infty)$
   - $x=0$: vertical asymptote
   - $x=1$: vertical asymptote
   - $x=5$: hole at $\left(5,\dfrac{1}{2}\right)$

3. No y-intercept; x-intercept at $(-5,0)$

4. $\dfrac{(x+4)(x-3)}{x+7}$

5. $\dfrac{3x+1}{(x+1)^2}$

6. $\dfrac{-x^2-32x+1}{(x-3)(x+5)}$ or $\dfrac{-(x^2+32x-1)}{(x-3)(x+5)}$

7.
   a. $2x+h$

   b. $\dfrac{4}{(x+4)(x+h+4)}$

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 6, determine if the function is a rational function.

1. $f(x) = \dfrac{4x^2+3x-5}{-8x^7+2x^5-7x}$
2. $g(x) = \dfrac{9x^{1/3}+26}{5x}$
3. $h(x) = \dfrac{197}{3x+84}$
4. $p(x) = \dfrac{(x-6)(x+9)}{3(x+2)(x-7)}$
5. $r(x) = \dfrac{2x^4+x-3x^{-1}}{x^5-7x^{40}}$
6. $q(x) = \dfrac{x+1}{x-100}$

For Exercises 7 - 14, state the domain of each rational function, using interval notation.

7. $f(x) = \dfrac{x+1}{x+4}$
8. $g(x) = \dfrac{(x-1)(x-3)}{(x+2)(x+5)}$
9. $h(x) = \dfrac{x+7}{x^2-4}$
10. $p(x) = \dfrac{x^2+3x-18}{x^2-2x-8}$
11. $r(x) = \dfrac{2}{5x+2}$
12. $q(x) = \dfrac{x}{x^2-9}$
13. $m(x) = \dfrac{x-6}{x^2+3}$
14. $k(x) = \dfrac{x^2-6}{2x^2+7}$

For Exercises 15 - 18, determine the x- and y-intercepts, if possible, for each function. Write your answers as ordered pairs.

15. $f(x) = \dfrac{x+5}{x^2+4}$
16. $g(x) = \dfrac{x^2+x+6}{x^2-10x+24}$
17. $h(x) = \dfrac{3+7x}{5-2x}$
18. $p(x) = \dfrac{x+7}{(x+3)^2}$

For Exercises 19 - 22, compute the indicated multiplication.

19. $2\left(\dfrac{x}{x+3}\right)$
20. $\dfrac{3}{4}\left(\dfrac{x-6}{8+x}\right)$
21. $-3\left(\dfrac{8}{5-9x}\right)$
22. $-\dfrac{9}{14}\left(\dfrac{3x-1}{2x+7}\right)$

For Exercises 23 - 26, compute the indicated operation.

23. $\dfrac{2}{x+1}+\dfrac{9}{x+1}$

24. $\dfrac{3}{x-4}-\dfrac{8}{x-4}$

25. $\dfrac{4}{2x-5}+\dfrac{7x}{2x-5}$

26. $\dfrac{5x}{11+3x}-\dfrac{1}{3x+11}$

For Exercises 27 - 32, compute and simplify the difference quotient, $\dfrac{f(x+h)-f(x)}{h}$, for each given $f(x)$.

27. $f(x)=x^2$

28. $f(x)=\dfrac{1}{x}$

29. $f(x)=2x^2$

30. $f(x)=\dfrac{2}{x}$

31. $f(x)=-x^2$

32. $f(x)=-\dfrac{1}{x}$

Intermediate Skills Practice (Answers)

For Exercises 33 - 40, state the domain of each rational function, using interval notation.

33. $f(x)=\dfrac{x+4}{(x+4)(x+3)}$

34. $g(x)=\dfrac{(x-6)(x+9)}{(x+2)(x-6)}$

35. $h(x)=\dfrac{(x+1)(x+2)}{x(x+2)(x+3)}$

36. $p(x)=\dfrac{8x^2-8}{3x^2-3x}$

37. $r(x)=\dfrac{2x^2-4x}{x^3-x}$

38. $q(x)=\dfrac{3x^2-3x-60}{2x^2+2x-24}$

39. $m(x)=\dfrac{(x+1)(x^2-x+1)}{x+1}$

40. $k(x)=\dfrac{2x^2+5x-3}{(x+3)(x^2+5x+7)}$

For Exercises 41 - 44, determine the x- and y-intercepts, if possible, for each function.

41. $f(x)=\dfrac{(x+3)^2}{(x-1)^2(x+3)}$

42. $g(x)=\dfrac{2}{x^2-100}$

43. $h(x)=\dfrac{x}{x^2-x}$

44. $p(x)=\dfrac{x^2-x-12}{x^2+x-6}$

For Exercises 45 - 48, compute the indicated multiplication.

45. $\left(\dfrac{x}{5x-4}\right)\left(\dfrac{8x+9}{x}\right)$

46. $\left(\dfrac{x+2}{x^2+4x+3}\right)\left(\dfrac{x+1}{x+2}\right)$

47. $\dfrac{x^2-1}{3x^2-3x}\cdot\dfrac{x^2+5x}{2x^2+19}$

48. $\dfrac{4x}{(x+3)^2}\cdot\dfrac{x-4}{(x+3)(x-2)}$

For Exercises 49 - 52, compute the indicated division.

49. $\left(\dfrac{x-6}{3-x}\right) \div \left(\dfrac{x^2-9}{x-5}\right)$

50. $\left(\dfrac{x-5}{11-x}\right) \div \left(\dfrac{x^2-25}{x-11}\right)$

51. $\left(\dfrac{27x^2}{3x+21}\right) \div \left(\dfrac{3x^2+18}{x^2+13x+42}\right)$

52. $\left(\dfrac{x^2+3x-10}{4x}\right) \div \left(2x^2+20x+50\right)$

For Exercises 53 - 58, compute the indicated operation and simplify the answer.

53. $\dfrac{4}{x-1} + \dfrac{3}{x+6}$

54. $\dfrac{8}{x+7} - \dfrac{9}{x+2}$

55. $\dfrac{3x-1}{x+4} + \dfrac{x-2}{7x+1}$

56. $\dfrac{x}{x+2} - \dfrac{3}{9-x}$

57. $\dfrac{5}{x^2-4x+4} + \dfrac{3x}{x^2-10x+16}$

58. $\dfrac{12}{x+h+3} - \dfrac{12}{x+3}$

For Exercises 59 - 64, compute and simplify the difference quotient, $\dfrac{f(x+h)-f(x)}{h}$, for each given $f(x)$.

59. $f(x) = 3x^2 - 4$

60. $f(x) = \dfrac{1}{x-7}$

61. $f(x) = -0.5x^2 + 5$

62. $f(x) = \dfrac{3}{7x+2}$

63. $f(x) = 10x^2 + x$

64. $f(x) = -\dfrac{2}{x-5}$

Mastery Practice (Answers)

For Exercises 65 - 70, state the domain of each rational function, using interval notation. Then, at any point not in the domain, state what is happening with the graph of the function.

65. $f(x) = \dfrac{4}{2x-3}$

66. $h(x) = \dfrac{(x+2)^2(x-3)}{(x-3)(x+1)(x+4)}$

67. $p(x) = \dfrac{2x^2-3x-20}{x^2-5}$

68. $r(x) = \dfrac{3x^2+13x+4}{3x^2+8x-16}$

69. $g(x) = \dfrac{(x+3)^2}{2x^2-11x+47}$

70. $q(x) = \dfrac{x^2-x-6}{x^3-4x}$

71. Write the equation for a rational function with vertical asymptotes at $x = 5$ and $x = -5$, x-intercepts at $(2,0)$ and $(-1,0)$ and y-intercept at $\left(0, \dfrac{2}{25}\right)$.

72. Write the equation for a rational function with vertical asymptotes at $x = -4$ and $x = -1$, a hole when $x = 2$, and x-intercepts at $(1,0)$ and $(5,0)$.

73. Compute and simplify the product:

$$-8\left(\frac{2x+7}{x^2-9}\right)\left(\frac{18-2x^2}{x+7}\right)$$

74. Compute and simplify the product:

$$\left(\frac{1}{h}\right)\left(\frac{3x^2h-6xh+12h}{(x+h+11)(x+11)}\right)$$

75. Compute and simplify:

$$2\left(\frac{x+5}{x-7}\right)+6\left(\frac{9-x}{2x^2-13x-7}\right)$$

76. Compute and simplify:

$$x\left(\frac{x^2-4}{-2-x}\right)-19\left(\frac{x+8}{x^2-x}\right)$$

For Exercises 77 - 80, compute and simplify the difference quotient for each given function.

77. $f(x)=-4x^2-7x+8$

78. $f(x)=\dfrac{x+1}{x-2}$

79. $f(x)=8x^2+7x-10$

80. $f(x)=\dfrac{3x-4}{2x+9}$

Communication Practice (Answers)

81. Explain, in your own words, how to determine whether a domain restriction is the location of a hole or a vertical asymptote on the graph of a rational function.

82. Explain why it is important to determine the domain of a rational function prior to computing its intercepts.

5.4 Power and Radical Functions

Figure 5.4.1: Mound of Rocks, © Photo by Kathryn Bollinger, 2020

A mound of rocks is in the shape of a cone with the height equal to twice the radius, $h = 2r$, as shown above in **Figure 5.4.1**. The volume of the mound is found using a formula from elementary geometry.

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi r^2 (2r) \\ V(r) &= \frac{2}{3}\pi r^3 \end{aligned}$$

Using the given relationship between the height and radius, we have written the volume, V, in terms of the radius, r. However, in some applications, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of rocks to construct a cone shaped mound with a height twice the radius. What are the radius and height of the new cone? To find the radius, we use the formula

$$r(V) = \sqrt[3]{\frac{3V}{2\pi}} = \left(\frac{3V}{2\pi}\right)^{\frac{1}{3}}$$

Then, to find the height, we use its given relationship to the radius. Here, notice both $V(r)$ and $r(V)$ are functions involving variables raised to a numerical exponent (*power*), known as **power functions**.

Learning Objectives:

In this section, you will learn about the properties and characteristics of power and radical functions. Upon completion you will be able to:

- Identify if a given function is a power or a radical function.
- Convert between radical and power notation.
- Memorize the graphs of the parent even and odd power/radical functions (square root and cube root).
- Determine the domain of a function involving power or radical functions, using interval notation.
- Identify the conjugate of a given expression involving radicals.
- Rationalize the numerator or denominator of a given expression, by multiplying by the conjugate.

Review of Exponent Laws

Before we dive into the properties of power functions, the authors would like to pause and review some basic properties of variables raised to numerical exponents.

Theorem 5.4 Laws of Exponents

Let p and q be any real numbers and x and y be any variables with values greater than zero. Then, the following rules hold true.

$$x^p x^q = x^{p+q} \qquad (xy)^p = x^p y^p \qquad (x^p)^q = x^{pq}$$

$$\frac{1}{x^p} = x^{-p} \qquad x^{\frac{1}{p}} = \sqrt[p]{x} \qquad \frac{1}{x^{-p}} = x^p$$

$$\left(\frac{x}{y}\right)^p = \frac{x^p}{y^p} \qquad \frac{x^p}{x^q} = x^{p-q}$$

Make sure to use the Laws of Exponents, correctly. In particular,

$$(x+y)^p \neq (x^p + y^p)$$

and

$$(x-y)^p \neq (x^p - y^p)$$

While $\sqrt[p]{x} = x^{\frac{1}{p}}$, in mathematics we call $\sqrt[p]{x}$ the *radical* form and $x^{\frac{1}{p}}$ the *exponent* form. Both of these have the same meaning, they just look a bit different. Anytime you see the expression $\sqrt[p]{x}$, you can replace it with the expression $x^{\frac{1}{p}}$ and vice versa.

These rules will all be quite handy in calculus. In both integral and differential calculus, we will have rules that work well when we have a power function, but will not work for other forms of functions. By being able to rewrite functions like $f(x) = \dfrac{1}{x^2}$, as a power function $\left(f(x) = x^{-2}\right)$, other calculations will be simplified.

■ **Example 1** Simplify $\left(\dfrac{x^2y^4}{xy^{\frac{1}{2}}}\right)^2$, assuming $x > 0$ and $y > 0$.

Solution:
Anytime we simplify, we need to remember our order of operations. The order of operations tells us to start with terms that are inside parentheses, so we will work on simplifying the fraction before we worry about the exponent on the outside.

All exponents inside the parentheses are positive, so we will first eliminate the fraction by rewriting terms that are in the denominator as terms in the numerator with negative exponents $\left(\frac{1}{x^p} = x^{-p}\right)$. After rewriting we will combine any like terms $(x^p x^q = x^{p+q})$.

$$\begin{aligned}
\left(\frac{x^2y^4}{xy^{\frac{1}{2}}}\right)^2 &= \left(x^2y^4x^{-1}y^{-\frac{1}{2}}\right)^2 \\
&= \left(x^2x^{-1}y^4y^{-\frac{1}{2}}\right)^2 \\
&= \left(x^{(2-1)}y^{(4-\frac{1}{2})}\right)^2 \\
&= \left(x^1y^{\frac{7}{2}}\right)^2
\end{aligned}$$

Now that everything inside the parentheses is simplified as much as possible, we will use the rule $(xy)^p = x^py^p$ to continue simplifying. We need to make sure we distribute the exponent that is outside of the parentheses to each term inside the parentheses. This gives us

$$\left(x^1y^{\frac{7}{2}}\right)^2 = \left(x^1\right)^2\left(y^{\frac{7}{2}}\right)^2$$

Next, we will use the rule $(x^p)^q = x^{pq}$ to finish simplifying.

$$\begin{aligned}
&= x^2y^{\frac{14}{2}} \\
&= x^2y^7
\end{aligned}$$

So, in the end, we get that $\left(\frac{x^2y^4}{xy^{\frac{1}{2}}}\right)^2 = x^2y^7$.

■

■ **Example 2** Simplify $\left(\sqrt{x}+\sqrt{y}\right)^2$, assuming $x \geq 0$ and $y \geq 0$.

Solution:
We'll start by focusing on the terms inside the parentheses. Due to the fact that $\sqrt{y} = \sqrt[2]{y}$ and $\sqrt[p]{x} = x^{\frac{1}{p}}$, then $\sqrt{y} = y^{\frac{1}{2}}$ and, likewise, $\sqrt{x} = x^{\frac{1}{2}}$. Thus,

$$\left(\sqrt{x}+\sqrt{y}\right)^2 = \left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)^2$$

There is nothing that we can simplify inside the parentheses, so we now need to apply the exponent on the outside of the parentheses. Inside the parentheses we have two terms that are added together, so we cannot apply an exponent rule here. We will need to rewrite the expression, and then, expand and simplify.

$$\begin{aligned}
\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)^2 &= \left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right) \\
&= \left(x^{\frac{1}{2}}\right)^2 + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + \left(y^{\frac{1}{2}}\right)^2 \\
&= x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y
\end{aligned}$$

Considering we do not have any like terms, we cannot simplify the above expression any further.

We have three other ways we could rewrite the final expression, but this is a matter of personal preference. We could use exponent rules to rewrite the middle term since $x^{\frac{1}{2}}y^{\frac{1}{2}} = (xy)^{\frac{1}{2}}$, giving us $x+2(xy)^{\frac{1}{2}}+y$. We could also use radicals and write either $x+2\sqrt{x}\sqrt{y}+y$ or $x+2\sqrt{xy}+y$. All four of these expressions are fully simplified, and equally valid. Probably the most common forms are

$$\left(\sqrt{x}+\sqrt{y}\right)^2 = x+2\sqrt{xy}+y = x+2(xy)^{\frac{1}{2}}+y$$

■

■ **Example 3** Show the following statement is true.

$$2xy^{-1} - x^{-\frac{1}{2}}y = \frac{2x^{\frac{3}{2}} - y^2}{x^{\frac{1}{2}}y}$$

Solution:
To verify this equality, we start with the left-hand side and transform the expression until we obtain the right-hand side. To subtract the expressions we will use both exponent rules and the properties of rational expressions.

$$\begin{aligned}
2xy^{-1} - x^{-\frac{1}{2}}y &= \frac{2x}{y} - \frac{y}{x^{\frac{1}{2}}} \\
&= \left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)\left(\frac{2x}{y}\right) - \left(\frac{y}{x^{\frac{1}{2}}}\right)\left(\frac{y}{y}\right) \\
&= \frac{2x^{\frac{1}{2}}x^1}{x^{\frac{1}{2}}y} - \frac{y^1 \cdot y^1}{x^{\frac{1}{2}}y} \\
&= \frac{2x^{\frac{3}{2}}}{x^{\frac{1}{2}}y} - \frac{y^2}{x^{\frac{1}{2}}y} \\
&= \frac{2x^{\frac{3}{2}} - y^2}{x^{\frac{1}{2}}y} \checkmark
\end{aligned}$$

■

Try It # 1:
Show the following statement is true.

$$-\left(\frac{1}{2}\right)(25-x^2)^{-\frac{1}{2}}(-2x) = \frac{x}{\sqrt{25-x^2}}$$

Defining Power Functions

Definition

A **power function** is a function that can be written in the form

$$f(x) = ax^p,$$

where a is a nonzero real number, p is a real number, and a is known as the coefficient. ▪

In the definition of a power function, the left-hand side, $f(x)$, tells us that x is the variable; a and p are parameters that can be any *fixed* real numbers. Because a and p can be any numerical value, this is a very general function type where the properties of the function can be very different based on those values of a and p.

Categories of Power Functions

Returning to our discussion of power functions, we can categorize a power function, $f(x) = ax^p$, by the value of its exponent (power).

1. If $p = 0$ *and* $x \neq 0$, then $f(x)$ is the constant polynomial function, $f(x) = ax^0 = a$.

2. If p is a positive integer, then $f(x)$ is a p^{th} degree polynomial function, consisting of only the leading term.

   For example: $g(x) = 3x^2$ and $h(x) = -\frac{1}{2}x^7$

3. If p is a negative integer, then $f(x)$ is a rational function.

   For example: $g(x) = 4x^{-1} = \frac{4}{x}$ and $h(x) = \frac{3}{8}x^{-2} = \frac{3}{8x^2}$

   Each of these types of power functions hold the properties discussed in the Polynomial Functions and Rational Functions sections, respectively.

4. If p is an irrational number and $x > 0$, then we have a complicated power function, which is beyond the scope of this text.

   For example: $g(x) = x^\pi$

5. If $p = \frac{1}{n}$, where n is a natural number, such that $n \geq 2$, then $f(x)$ is a **radical (root) function**.

   For example: $g(x) = 6x^{\frac{1}{2}} = 6\sqrt{x}$ and $h(x) = (7x)^{\frac{1}{3}} = \sqrt[3]{7x}$

Let's now turn our focus to the properties of these last types of power functions, radical functions.

Describing Properties of Radical Functions

Definition

A power function of the form $f(x) = x^{\frac{1}{n}}$ is more commonly referred to as a **radical (root) function** and is often written in the form

$$f(x) = \sqrt[n]{x},$$

where n is a natural number, such that $n \geq 2$, called the **index** or **root**.

- If n is an even number, then we say $f(x)$ has an **even root** and $f(x)$ is only defined for $x \geq 0$. (The even root of a negative number is not a real number.)
- If n is an odd number, then we say $f(x)$ has an **odd root** and $f(x)$ is defined for any real number, x.

We can now add the parent functions for radical (root) functions which are $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$. The graphs of these parent radical functions are shown in **Table 5.7**, below.

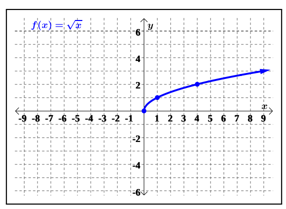
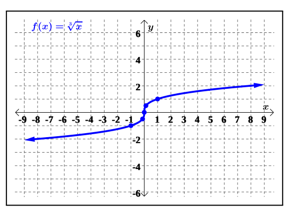
Name	**Function**	**Graph**	**Table**	**Domain**
Square Root	$f(x) = \sqrt{x}$		x | $f(x)$ 0 | 0 1 | 1 4 | 2	$[0, \infty)$
Cube Root	$f(x) = \sqrt[3]{x}$		x | $f(x)$ -1 | -1 -0.125 | -0.5 0 | 0 0.125 | 0.5 1 | 1	$(-\infty, \infty)$

Table 5.7: Parent Root Functions

Many times in calculus we are faced with functions in radical form. It is often necessary to convert these functions from radical *form* to an equivalent exponent (power) *form*.

Example 4 Write the following radicals in their equivalent exponent (power) form.

a. $f(x) = \sqrt[3]{x^2}$

b. $g(x) = \sqrt[4]{3x+2}$

c. $h(x) = \sqrt{(x^2+3x+2)^{-1}}$

Solution:
Using the laws of exponents, we know $\sqrt[p]{x} = x^{\frac{1}{p}}$.

a. For $\sqrt[3]{x^2}$, $p = 3$, so

$$\begin{aligned} f(x) &= \sqrt[3]{x^2} \\ &= \left(x^2\right)^{\frac{1}{3}} = x^{\frac{2}{3}} \end{aligned}$$

Thus, $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$.

b. For $\sqrt[4]{3x+2}$, $p = 4$, so

$$\begin{aligned} g(x) &= \sqrt[4]{3x+2} \\ &= \left[(3x+2)^1\right]^{\frac{1}{4}} = (3x+2)^{\frac{1}{4}} \end{aligned}$$

Thus, $g(x) = \sqrt[4]{3x+2} = (3x+2)^{\frac{1}{4}}$.

c. For $\sqrt{(x^2+3x+2)^{-1}}$, $p = 2$, so

$$\begin{aligned} h(x) &= \sqrt{(x^2+3x+2)^{-1}} \\ &= \left[(x^2+3x+2)^{-1}\right]^{\frac{1}{2}} = \left(x^2+3x+2\right)^{-\frac{1}{2}} \end{aligned}$$

Thus, $h(x) = \sqrt{(x^2+3x+2)^{-1}} = \left(x^2+3x+2\right)^{-\frac{1}{2}}$.

■

Sometimes it is necessary to have a function written with non-negative powers. If the previous example had asked the reader to write their final answer with non-negative powers, then $h(x) = \dfrac{1}{(x^2+3x+2)^{\frac{1}{2}}}$, using the rule $x^{-p} = \dfrac{1}{x^p}$.

In the previous example none of the exponents reduced. If both the root and power are even, then we must proceed with caution. For example $\sqrt{x^2} = |x|$, not x, as the square root of a positive number is always positive. On the other hand, $\left(\sqrt{x}\right)^2 = x$, because in this course, the square root is only defined for $x \geq 0$.

As with all functions, we need to investigate the domain of radical (root) functions. Our investigation will focus on the differences in domain based on whether the root of the function is even or odd.

Recall that if $f(x) = \sqrt[n]{x}$ and

n is even, then $f(x)$ is defined when $x \geq 0$,

or

n is odd, then $f(x)$ is defined when x is any real number.

Now if $f(x) = \sqrt[n]{g(x)}$ and

n is even, then $f(x)$ is defined when $g(x)$ is defined AND $g(x) \geq 0$,

or

n is odd, then $f(x)$ is defined when $g(x)$ is defined.

Similarly, if $f(x) = \dfrac{1}{\sqrt[n]{g(x)}}$ and

n is even, then $f(x)$ is defined when $g(x)$ is defined AND $g(x) > 0$,

or

n is odd, then $f(x)$ is defined when $g(x)$ is defined AND $g(x) \neq 0$.

Example 5 State the domain of each function, using interval notation.

a. $f(x) = x^{\frac{2}{3}}$

b. $h(x) = (3x+2)^{\frac{1}{4}}$

c. $j(x) = \dfrac{1}{\sqrt[5]{x^2+3x+2}}$

d. $k(x) = \dfrac{1}{\sqrt{x-6}}$

Solution:

a. In order to more easily identify the root as even or odd, we begin by converting $f(x)$ into its equivalent radical (root) form.

$$\begin{aligned} f(x) &= x^{\frac{2}{3}} \\ &= \left(x^2\right)^{\frac{1}{3}} \quad \text{or} \quad \left(x^{\frac{1}{3}}\right)^2 \\ &= \sqrt[3]{x^2} \quad \text{or} \quad \left(\sqrt[3]{x}\right)^2 \end{aligned}$$

Here, the root is 3 (odd).

Therefore, $f(x)$ is defined when $g(x)$, which equals x^2 or x, is defined. Considering $g(x)$ is a polynomial, in either form, and is defined for all real numbers, the domain of $f(x)$ is $(-\infty, \infty)$.

b. Again, we will begin by converting $h(x)$ into its equivalent radical (root) form.

$$\begin{aligned} h(x) &= (3x+2)^{\frac{1}{4}} \\ &= \sqrt[4]{3x+2} \end{aligned}$$

Here, the root is 4 (even).

Thus, $h(x)$ is defined when $g(x)$, which equals $3x+2$, is defined AND $g(x) \geq 0$. Because $g(x)$ is a polynomial and defined for all real numbers, we must only determine the values of x such that $g(x) \geq 0$.

$$\begin{aligned} g(x) = 3x+2 &\geq 0 \\ 3x &\geq -2 \\ x &\geq -\frac{2}{3} \end{aligned}$$

Hence, the domain of $h(x)$ is $\left[-\frac{2}{3}, \infty\right)$.

c. The function $j(x)$ is already in radical form, with the radical in the denominator. The root is 5 (odd).

Thus, $j(x)$ is defined when $g(x)$, which equals x^2+3x+2, is defined AND $g(x) \neq 0$.

Again $g(x)$ is a polynomial and defined for all real numbers, so we must only determine the values of x such that $g(x) \neq 0$.

$$\begin{aligned} g(x) = x^2+3x+2 &\neq 0 \\ g(x) = (x+2)(x+1) &\neq 0 \\ x+2 \neq 0 \quad \text{and} \quad x+1 &\neq 0 \\ x \neq -2 \qquad \text{and} \qquad x &\neq -1 \end{aligned}$$

Thus, the domain of $j(x)$ is $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

d. The function is already in radical form, with the radical in the denominator. The root is 2 (even).

Thus, $k(x)$ is defined when $g(x)$, which equals $x-6$, is defined AND $g(x) > 0$.

Given that $g(x)$ is a polynomial and defined for all real numbers, we must only determine the values of x such that $g(x) > 0$.

$$\begin{aligned} g(x) = x-6 &> 0 \\ x &> 6 \end{aligned}$$

Thus, the domain of $k(x)$ is $(6, \infty)$.

■

Try It # 2:
State the domain of each function, using interval notation.

a. $f(x) = (x+11)^{-\frac{1}{3}}$

b. $h(x) = (2x-7)^{\frac{1}{8}}$

c. $j(x) = -2x^{\frac{4}{9}}$

d. $k(x) = 4x^{-\frac{1}{6}}$

We determine the intercepts of power functions, including radicals, using the same techniques discussed in previous sections. As with rational functions, power functions may or may not have both x-intercepts and a y-intercept; it is important to determine the domain of a power function before locating its intercepts. The authors will leave it to the reader to practice finding intercepts in the section exercises.

Computing Domains of Algebraic Functions

By computing the sum, difference, product, or quotient of all types of functions we have discussed thus far, we create a new single function, which the authors call, an **algebraic function**.

The domain of an algebraic function is all real numbers, x, which satisfy all parts of the function. *So far*, our only discussed domain restrictions are

1. Division by zero is undefined.

2. Even roots of negative numbers are undefined.

N *More domain restrictions will be discussed as we investigate other functions in later sections.*

Example 6 State the domain of each algebraic function, using interval notation.

a. $f(x) = \sqrt{6-x} + 12x$

b. $h(x) = \dfrac{x^2+4}{\sqrt{x}-1}$

c. $j(x) = \dfrac{\sqrt{1-x}}{(x-2)(3x+5)}$

d. $k(x) = \dfrac{(x+9)^{\frac{3}{8}}}{\sqrt[5]{x+7}}$

Solution:

a. $f(x)$ is the sum of an even root and a polynomial, so the domain of $f(x)$ must satisfy the domain restrictions for both. As the domain of the polynomial term, $12x$, is all real numbers, $(-\infty, \infty)$, we turn to $\sqrt{6-x}$.

For $\sqrt{6-x}$ (an even root of a polynomial) to be defined,

$$\begin{aligned} 6-x &\geq 0 \\ -x &\geq -6 \\ x &\leq 6. \end{aligned}$$

So, the domain of $f(x)$ is the intersection of $(-\infty,\infty)$ and $(-\infty,6]$.

If we drew $(-\infty,\infty)$ and $(-\infty,6]$ on a number line (using techniques from **Section 5.1**), the overlapping segment of these two intervals would show that the domain of $f(x)$ is

$$(-\infty,6].$$

b. $h(x)$ is the quotient of two functions, so both the numerator and denominator must be defined AND the denominator cannot be zero.

The numerator, x^2+4, is a quadratic polynomial and is defined for all real numbers, $(-\infty,\infty)$.

The denominator, $\sqrt{x}-1$, contains an even root term $\left(\sqrt{x}\right)$, which is defined only when $x \geq 0$ or $[0,\infty)$.

Moreover, the denominator, $\sqrt{x}-1$, cannot be zero, thus

$$\begin{aligned} \sqrt{x}-1 &\neq 0 \\ \sqrt{x} &\neq 1 \\ x &\neq 1 \end{aligned}$$

So, the domain of $h(x)$ is the intersection of

$$(-\infty,\infty) \qquad \text{and} \qquad [0,\infty) \qquad \text{and} \qquad (-\infty,1)\cup(1,\infty)$$

If we drew $(-\infty,\infty)$, $[0,\infty)$, and $(-\infty,1)\cup(1,\infty)$ on a number line, the overlapping segments of these intervals would show that the domain of $h(x)$ is

$$[0,1)\cup(1,\infty).$$

N *Technically for* $\sqrt{x}\neq 1$, $|x|\neq 1$, *however because* $\sqrt{x}$ *indicates* $x \geq 0$, *the use of absolute value bars was unnecessary.*

c. $j(x)$ is again the quotient of two functions.

The numerator, $\sqrt{1-x}$, is an even root and is defined only when the polynomial $1-x \geq 0$.

$$\begin{aligned} 1-x &\geq 0 \\ -x &\geq -1 \\ x &\leq 1 \end{aligned}$$

The denominator, $(x-2)(3x+5)$, is a quadratic polynomial which is defined for all real numbers, $(-\infty,\infty)$. However, the denominator cannot equal zero, so

$$\begin{array}{rcl} & (x-2)(3x+5) \neq 0 & \\ x-2 \neq 0 & \text{and} & 3x+5 \neq 0 \\ & & 3x \neq -5 \\ x \neq 2 & \text{and} & x \neq -\dfrac{5}{3} \end{array}$$

Thus, the domain of $j(x)$ is the intersection of

$$(-\infty,1] \quad \text{and} \quad (-\infty,\infty) \quad \text{and} \quad \left(-\infty,-\frac{5}{3}\right)\cup\left(-\frac{5}{3},2\right)\cup(2,\infty)$$

If we drew $(-\infty,1]$, $(-\infty,\infty)$, and $\left(-\infty,-\frac{5}{3}\right)\cup\left(-\frac{5}{3},2\right)\cup(2,\infty)$ on a number line, the overlapping segments of these intervals would show that the domain of $j(x)$ is

$$\left(-\infty,-\frac{5}{3}\right)\cup\left(-\frac{5}{3},1\right].$$

d. $k(x)$ is again the quotient of two functions.

The numerator, $(x+9)^{\frac{3}{8}} = \sqrt[8]{(x+9)^3} = \left(\sqrt[8]{(x+9)}\right)^3$, is an even root and defined only when the polynomial $x+9 \geq 0$.

$$\begin{aligned} x+9 &\geq 0 \\ x &\geq -9 \end{aligned}$$

The denominator, $\sqrt[5]{x+7}$, is an odd root of a polynomial, and is defined only when the polynomial $x+7$, is defined. The domain of $x+7$ is $(-\infty,\infty)$. However, the denominator cannot equal zero.

$$\begin{aligned} \sqrt[5]{x+7} &\neq 0 \\ x+7 &\neq 0 \\ x &\neq -7 \end{aligned}$$

So, the domain is the intersection of

$$[-9,\infty) \quad \text{and} \quad (-\infty,\infty) \quad \text{and} \quad (-\infty,-7)\cup(-7,\infty)$$

If we drew $[-9,\infty)$, $(-\infty,\infty)$, and $(-\infty,-7)\cup(-7,\infty)$ on a number line, the overlapping segments of these intervals would show that the domain of $k(x)$ is

$$[-9,-7)\cup(-7,\infty).$$

■

Try It # 3:
State the domain of the following functions, using interval notation.

a. $f(x) = \dfrac{\sqrt{x+2}}{\sqrt[4]{7-x}}$

b. $h(x) = \dfrac{\sqrt[3]{2x}-4}{3x+17}$

c. $j(x) = \dfrac{1}{2}(x+8)^{-\frac{1}{3}} + 9x - 15$

d. $k(x) = \dfrac{(x+1)(x-5)}{\left(\sqrt{x-5}\right)(x+1)}$

Rationalizing Numerators and Denominators

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to rewrite expressions that contain radicals in the denominator. To remove square root radicals from the denominators of fractions, we multiply by the form of '1' that will eliminate the radical, using a process call **rationalizing the denominator**.

For a denominator containing the sum or difference of two terms, where at least one term is a square root radical, multiply the numerator and denominator by the **conjugate** of the denominator to eliminate the radical from the denominator. The conjugate is found by changing the sign between the two terms of the denominator. For example, if the denominator is $a+b\sqrt{c}$, then the conjugate is $a-b\sqrt{c}$, but if the denominator is $a-b\sqrt{c}$, then the conjugate is $a+b\sqrt{c}$.

■ **Example 7** Rationalize the denominator and simplify the expression $\dfrac{4}{1+\sqrt{5}}$.

Solution:
We begin by stating the conjugate of the denominator as the denominator with the opposite sign between the two terms; the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$.

Then, we multiply the given expression by $\dfrac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\left(\frac{4}{1+\sqrt{5}}\right)\cdot\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right)=\frac{4\left(1-\sqrt{5}\right)}{\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)}$$
$$=\frac{4\left(1-\sqrt{5}\right)}{1-\sqrt{5}+\sqrt{5}-\left(\sqrt{5}\right)\left(\sqrt{5}\right)}$$
$$=\frac{4\left(1-\sqrt{5}\right)}{1-5}$$
$$=\frac{4\left(1-\sqrt{5}\right)}{-4}$$
$$=\frac{\cancel{-4}\left(-1+\sqrt{5}\right)}{\cancel{-4}}$$
$$=-1+\sqrt{5}\text{ or }\sqrt{5}-1$$

■

■ **Example 8** Rationalize the denominator and simplify the expression

$$\frac{x-16}{\sqrt{x}-4},\text{ for }x\geq 0\text{ and }x\neq 16$$

Solution:

We begin by stating the conjugate of the denominator as the denominator with the opposite sign between the two terms; the conjugate of $\sqrt{x}-4$ is $\sqrt{x}+4$.

Then, we multiply the given expression by $\frac{\sqrt{x}+4}{\sqrt{x}+4}$.

$$\left(\frac{x-16}{\sqrt{x}-4}\right)\cdot\left(\frac{\sqrt{x}+4}{\sqrt{x}+4}\right)=\frac{(x-16)\left(\sqrt{x}+4\right)}{\left(\sqrt{x}-4\right)\left(\sqrt{x}+4\right)}$$
$$=\frac{(x-16)\left(\sqrt{x}+4\right)}{\left(\sqrt{x}\right)\left(\sqrt{x}\right)+4\sqrt{x}-4\sqrt{x}-16}$$
$$=\frac{(x-16)\left(\sqrt{x}+4\right)}{x-16}$$
$$=\frac{\cancel{(x-16)}\left(\sqrt{x}+4\right)}{\cancel{x-16}}$$
$$=\sqrt{x}+4$$

■

Similarly, we can rationalize a numerator containing radicals.

■ **Example 9** Rationalize the numerator and simplify the expression

$$\frac{5\sqrt{x}+7}{x}, \text{ for } x>0.$$

Solution:
We begin by stating the conjugate of the numerator as the numerator with the opposite sign between the two terms; the conjugate of $5\sqrt{x}+7$ is $5\sqrt{x}-7$.

Then, we multiply the given expression by $\frac{5\sqrt{x}-7}{5\sqrt{x}-7}$.

$$\begin{aligned}\left(\frac{5\sqrt{x}+7}{x}\right)\cdot\left(\frac{5\sqrt{x}-7}{5\sqrt{x}-7}\right) &= \frac{\left(5\sqrt{x}+7\right)\left(5\sqrt{x}-7\right)}{x\left(5\sqrt{x}-7\right)}\\ &= \frac{\left(5\sqrt{x}\right)\left(5\sqrt{x}\right)-35\sqrt{x}+35\sqrt{x}-49}{x\left(5\sqrt{x}-7\right)}\\ &= \frac{25x-49}{x\left(5\sqrt{x}-7\right)}\end{aligned}$$

■

For $a \geq > 0$ and $b \geq 0$, $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=a-b$.

Try It # 4:
Rationalize the denominator and simplify the given expression.

$$\frac{7x}{2+\sqrt{3}}$$

Try It # 5:
Rationalize the numerator and simplify the given expression.

$$\frac{\sqrt{3x+1}-8}{x-21}, \text{ for } x \geq -\frac{1}{3} \text{ and } x \neq 21$$

Oftentimes we rationalize the numerator in calculus when simplifying a difference quotient.

Example 10 Compute and simplify the difference quotient for the function $r(x) = \sqrt{x}$.

Solution:
For $r(x) = \sqrt{x}$, we have $r(x+h) = \sqrt{x+h}$, so the difference quotient is

$$\frac{r(x+h)-r(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

In order to divide out the "h" from the denominator and simplify the difference quotient, we rationalize the *numerator* by multiplying by the form of '1' based on its conjugate.

$$\begin{aligned}
\frac{r(x+h)-r(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{\left(\sqrt{x+h}\right)\left(\sqrt{x+h}\right)+\left(\sqrt{x+h}\right)\left(\sqrt{x}\right)-\left(\sqrt{x}\right)\left(\sqrt{x+h}\right)-\left(\sqrt{x}\right)\left(\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

We are done with our simplification, as the original "h" in the denominator is no longer present.

■

Try It # 6:
Compute and simplify the difference quotient for the function $q(x) = 6\sqrt{x-5}$.

Try It Answers

1. $-\left(\frac{1}{2}\right)\left(25-x^2\right)^{-\frac{1}{2}}(-2x)=-\left(\frac{1}{2}\right)(-2x)\left(\frac{1}{(25-x^2)^{\frac{1}{2}}}\right)=\left(\frac{x}{1}\right)\left(\frac{1}{(25-x^2)^{\frac{1}{2}}}\right)=\frac{x}{\sqrt{25-x^2}}$

2. **a.** $(-\infty,-11)\cup(-11,\infty)$

   b. $\left[\frac{7}{2},\infty\right)$

   c. $(-\infty,\infty)$

   d. $(0,\infty)$

3. **a.** $[-2,7)$

   b. $\left(-\infty,-\frac{17}{3}\right)\cup\left(-\frac{17}{3},\infty\right)$

   c. $(-\infty,-8)\cup(-8,\infty)$

   d. $(5,\infty)$

4. $7x\left(2-\sqrt{3}\right)$

5. $\frac{3}{\sqrt{3x+1}+8}$

6. $\frac{6}{\sqrt{x+h-5}+\sqrt{x-5}}$

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 4, simplify each expression, assuming $x, y > 0$. Write all final answers with positive exponents.

1. 4^{-3}
2. $\left(3x^{-5}\right)^{1/2}$
3. $23(x+1)^3(x+1)^6$
4. $\left(-4x^2y^8\right)\left(2xy^{-3}\right)^{-1}$

For Exercises 5 - 10, rewrite each radical in its equivalent exponent (power) form, assuming $x > 0$.

5. $f(x) = \sqrt[5]{x^9}$
6. $g(x) = 7\sqrt{x+36}$
7. $h(x) = \dfrac{8}{\sqrt[3]{(x+11)^4}}$
8. $r(x) = \sqrt[4]{(5x+2)^3}$
9. $q(x) = -2\sqrt[7]{49x^7-63}$
10. $m(x) = \dfrac{\sqrt[8]{15x+125}}{13}$

For Exercises 11 - 13, rewrite each exponent (power) in its equivalent radical form, assuming $x > 0$.

11. $f(x) = 6x^{2/7}$
12. $g(x) = 8(3x+4)^{9/10}$
13. $h(x) = \left(x^2+2x+50\right)^{-1/5}$

For Exercises 14 - 21, state the domain of each function, using interval notation.

14. $f(x) = \sqrt[6]{x}$
15. $g(x) = 3\sqrt{7x+2}$
16. $h(x) = 4\sqrt[7]{5+4x^2}$
17. $j(x) = \sqrt[8]{0.4x}$
18. $k(x) = \sqrt[5]{x}$
19. $m(x) = \sqrt[3]{x-5}$
20. $n(x) = \sqrt{2-3x}$
21. $p(x) = \sqrt[27]{x^2-3x+4}$

For Exercises 22 - 25, rationalize each denominator and simplify the expression.

22. $\dfrac{1}{\sqrt{2}}$
23. $\dfrac{7}{\sqrt{x}+6}$ for $x > 0$
24. $\dfrac{1}{\sqrt{5}}$
25. $-\dfrac{9}{x-2\sqrt{13}}$ for $x \neq 2\sqrt{13}$

For Exercises 26 - 29, rationalize each numerator and simplify the expression.

26. $9+\sqrt{11}$

27. $\sqrt{41}-\sqrt{x}$ for $x>0$

28. $\sqrt{3}-\sqrt{26}$

29. $8+\sqrt{x}$ for $x>0$

For Exercises 30 - 32, compute and simplify the difference quotient, $\dfrac{f(x+h)-f(x)}{h}$, for each given $f(x)$.

30. $f(x)=\sqrt{x}$

31. $f(x)=2\sqrt{x}$

32. $f(x)=\sqrt{3x}$

Intermediate Skills Practice (Answers)

For Exercises 33 - 35, show each of the following statements are true.

33. $\left(x-y^2\right)^2=x^2-2xy^2+y^4$

34. $\left(\sqrt[6]{x+y}\right)\left(\sqrt[5]{x^{-8}y^3z}\right)=\dfrac{(x+y)^{1/6}y^{3/5}z^{1/5}}{x^{8/5}}$

35. $25x^4-6x^2+\dfrac{3}{2}x^{-1/2}=\dfrac{50x^{9/2}-12x^{5/2}+3}{2x^{1/2}}$

For Exercises 36 - 43, state the domain of each function, using interval notation. Then, determine the x- and y-intercepts, if possible, for each function.

36. $f(x)=\dfrac{5}{\sqrt[3]{x}}$

37. $g(x)=\dfrac{1}{\sqrt{x+2}}$

38. $h(x)=\dfrac{8x}{\sqrt[6]{5-7x}}$

39. $j(x)=\dfrac{x^2-5x+6}{4\sqrt[8]{-10x+2}}$

40. $k(x)=\dfrac{1}{\sqrt[4]{x}}$

41. $m(x)=\dfrac{7x}{\sqrt[13]{x+1}}$

42. $n(x)=\dfrac{4x-3}{2\sqrt[3]{6-x}}$

43. $p(x)=\dfrac{x^2+6x-1}{\sqrt[11]{x^2-49}}$

For Exercises 44 - 47, rationalize each numerator and simplify the expression.

44. $\sqrt{2x-5}-11$ for $x\geq 2.5$

45. $\dfrac{7+\sqrt{3}}{7-\sqrt{3}}$

46. $\sqrt{x+h}-\sqrt{x}$ for $x>0$

47. $\dfrac{9\sqrt{x}+4}{3}$ for $x>0$

For Exercises 48 - 51, compute and simplify the difference quotient, $\dfrac{f(x+h)-f(x)}{h}$, for each given $f(x)$.

48. $f(x) = \sqrt{x-4}$
49. $f(x) = 3\sqrt{x+1}$
50. $f(x) = \sqrt{2x+5}$
51. $f(x) = \sqrt{4-3x}$

Mastery Practice (Answers)

For Exercises 52 - 57, state the domain of each function, using interval notation.

52. $f(x) = \dfrac{4\sqrt[3]{x}}{\sqrt[15]{x+9}}$
53. $g(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x}-4}$
54. $h(x) = (8-x)^{1/6} + (x-1)^{1/2}$
55. $p(x) = \dfrac{11\sqrt{x}}{\sqrt{x-4}}$
56. $q(x) = \dfrac{8\sqrt[3]{x}}{9-\sqrt{x}}$
57. $r(x) = (9-x)^{-3/4}$

58. Rationalize the numerator of the following and simplify the expression:

$$\frac{\sqrt{x^2+2xh+h^2-5}-\sqrt{x^2-5}}{h} \text{ for } h \neq 0$$

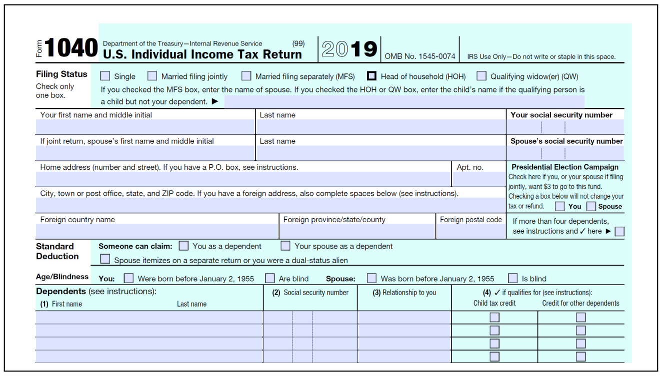
For Exercises 59 - 60, compute and simplify the difference quotient for each given $f(x)$.

59. $f(x) = -5\sqrt{6x+10}$
60. $f(x) = (x-8)^{1/2}$

Communication Practice (Answers)

61. Explain why when determining the domain of $f(x) = \dfrac{3x+1}{x^2-4x-5}$ you only consider the denominator, but when determining the domain of $g(x) = \dfrac{\sqrt{3x+1}}{x^2-4x-5}$ you must consider both the numerator and denominator.
62. Explain why it is always necessary to rationalize the numerator when simplifying the difference quotient for a square root function.

5.5 Piecewise-Defined Functions


Form 1040 Department of the Treasury—Internal Revenue Service (99)
U.S. Individual Income Tax Return 2019 OMB No. 1545-0074 IRS Use Only—Do not write or staple in this space.

Filing Status
Check only one box.
Single | Married filing jointly | Married filing separately (MFS) | Head of household (HOH) | Qualifying widow(er) (QW)
If you checked the MFS box, enter the name of spouse. If you checked the HOH or QW box, enter the child's name if the qualifying person is a child but not your dependent. ▶

Your first name and middle initial | Last name | Your social security number
If joint return, spouse's first name and middle initial | Last name | Spouse's social security number
Home address (number and street). If you have a P.O. box, see instructions. | Apt. no.
City, town or post office, state, and ZIP code. If you have a foreign address, also complete spaces below (see instructions).
Presidential Election Campaign
Check here if you, or your spouse if filing jointly, want $3 to go to this fund. Checking a box below will not change your tax or refund. You Spouse
Foreign country name | Foreign province/state/county | Foreign postal code
If more than four dependents, see instructions and ✓ here ▶

Standard Deduction
Someone can claim: You as a dependent | Your spouse as a dependent
Spouse itemizes on a separate return or you were a dual-status alien

Age/Blindness You: Were born before January 2, 1955 | Are blind Spouse: Was born before January 2, 1955 | Is blind

Dependents (see instructions): (1) First name	Last name	(2) Social security number	(3) Relationship to you	(4) ✓ if qualifies for (see instructions): Child tax credit	Credit for other dependents

Figure 5.5.1: Tax form from www.irs.gov.

Tax Rate	Individual Annual Income for 2019	Married Filing Joint Income for 2019
10%	up to \$9,700	up to \$19,400
12%	over \$9,700	over \$19,400
22%	over \$39,475	over \$78,950
24%	over \$84,200	over \$168,400
32%	over \$160,725	over \$321,450
35%	over \$204,100	over \$408,200
37%	over \$510,300	over \$612,350

Table 5.8: irs.gov/newsroom (Nov 15, 2018)

Every working U.S citizen is required to pay income taxes. The Internal Revenue Service reports the annual federal income tax brackets and deductions. The tax brackets reported for the 2019 year are given in **Table 5.8**.

This and many other real-world applications are modeled by functions in which different rules are used to define the results over different inputs. Such functions are called **piecewise-defined functions**.

Learning Objectives:
In this section, you will learn about the properties and characteristics of piecewise-defined functions. Upon completion you will be able to:

- Compute specific values of a piecewise-defined function.
- Write an absolute value function as an equivalent piecewise-defined function.
- Determine the domain of a piecewise-defined function, using interval notation.
- Memorize the graph of the parent absolute value function.
- Graph piecewise-defined functions.
- Express a real-world scenario as a piecewise-defined function.

Describing Piecewise-Defined Functions

We use piecewise-defined functions to describe situations in which a rule or relationship changes as the input values crosses certain "boundaries." The U.S. tax system described in the introduction is one such example. Other examples one might encounter are situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value.

Definition

A **piecewise-defined function** is a function in which more than one rule is used to define the output. We notate this idea as:

$$f(x) = \begin{cases} \text{Rule A} & \text{if } x \text{ is in Interval A} \\ \text{Rule B} & \text{if } x \text{ is in Interval B} \\ \text{Rule C} & \text{if } x \text{ is in Interval C} \end{cases} = \begin{cases} f(x)_A & \text{if } x \text{ is in Interval A} \\ f(x)_B & \text{if } x \text{ is in Interval B} \\ f(x)_C & \text{if } x \text{ is in Interval C} \end{cases}$$

■

Consider the function:

$$f(x) = \begin{cases} x+5 & \text{if} & x \le -3 \\ 9-x^2 & \text{if} & -3 < x \le 3 \\ -x+5 & \text{if} & x > 3 \end{cases}$$

We read $f(x)$ as follows:

- When the input, x, is a value less than or equal to -3, then the output is defined by the rule $f(x)_A = x+5$.
- When the input, x, is a value greater than -3, but less than or equal to 3, then the output is defined by the rule $f(x)_B = 9-x^2$.
- When the input, x, is a value greater than 3, then the output is defined by the rule $f(x)_C = -x+5$.

So if asked to calculate $f(-6)$, we see the input is $x = -6$. As -6 is in the interval $x \le -3$, the output is defined by the rule $f(x)_A = x+5$.

Thus,

$$\begin{aligned} f(-6) &= -6+5 \\ &= -1 \end{aligned}$$

To calculate $f(3)$, we see the input value is $x = 3$, which is in the interval $-3 < x \le 3$, and the output is defined by the rule $f(x)_B = 9-x^2$.

Thus,

$$\begin{aligned} f(3) &= 9-(3)^2 \\ &= 9-9 \\ &= 0 \end{aligned}$$

■ **Example 1** Let $f(x) = \begin{cases} -8 & \text{if} \quad x < -4 \\ 2x+7 & \text{if} \quad -4 \leq x < 5 \\ \dfrac{1}{x} & \text{if} \quad x > 9 \end{cases}$

Compute the following function values.

a. $f(-5)$

b. $f(-4)$

c. $f\left(-\dfrac{3}{2}\right)$

d. $f(5)$

e. $f(7)$

f. $f(100)$

Solution:

Rule A:	$f(x)_A = 8$	for	Interval A: $(-\infty, -4)$
Rule B:	$f(x)_B = 2x+7$	for	Interval B: $[-4, 5)$
Rule C:	$f(x)_C = \dfrac{1}{x}$	for	Interval C: $(9, \infty)$

a. $f(-5)$ has an input value of $x = -5$, which is in Interval A, so we use Rule A, $f(x)_A = 8$.

$$f(-5) = 8$$

b. $f(-4)$ has an input value of $x = -4$. Upon close inspection we can see that $x = -4$ is not in Interval A, but is in Interval B, so we use Rule B, $f(x)_B = 2x+7$.

$$\begin{aligned} f(-4) &= 2(-4)+7 \\ &= -8+7 \\ &= -1 \end{aligned}$$

c. $f\left(-\frac{3}{2}\right)$ has an input value of $x = -\frac{3}{2} = -1.5$, which is in Interval B, so we use Rule B, $f(x)_B = 2x+7$.

$$\begin{aligned} f\left(-\frac{3}{2}\right) &= 2\left(-\frac{3}{2}\right)+7 \\ &= -3+7 \\ &= 4 \end{aligned}$$

d. $f(5)$ has an input value of $x = 5$, which is NOT in any of the intervals of $f(x)$, and, thus, has no corresponding rule. Therefore, $f(5)$ is undefined, meaning $x = 5$ is not in the domain of $f(x)$. So, $f(x)$ does not exist at $x = 5$.

e. $f(7)$ has an input value of $x = 7$, which is also NOT is any of the intervals of $f(x)$ and, again, has no corresponding rule. Therefore, $f(7)$ is undefined.

f. $f(100)$ has an input value of $x = 100$, which is in Interval C, so we use Rule C, $f(x)_C = \dfrac{1}{x}$.

$$f(100) = \frac{1}{100}$$

■

Try It # 1:

Let $g(x) = \begin{cases} 2x^{\frac{1}{3}} & \text{if} \quad x < 0 \\ x^2 - 10 & \text{if} \quad x = 0 \end{cases}$

Compute the following function values.

a. $g(-27)$

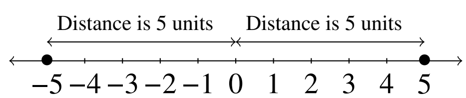
b. $g(2)$

c. $g\left(-\frac{1}{8}\right)$

d. $g(0)$

Absolute Value Functions

There are several ways to describe what is meant by the absolute value, $|x|$, of a real number x. You may have been taught that $|x|$ is the distance from the real number x to 0 on the number line. So, for example, $|5| = 5$ and $|-5| = 5$, since both 5 and -5 are 5 units from 0 on the number line.



Another way to define absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$ and $|5| = \sqrt{(5)^2} = \sqrt{25} = 5$. The long and short of both of these procedures is that $|x|$ takes negative real numbers and assigns them to their positive counterparts, while it leaves non-negative numbers alone.

In terms of functions and function notation, we can explain this procedure in the following manner.

If we input a negative value, the output is the the opposite of the input:

$$f(x) = -x \quad \text{if} \quad x < 0$$

If we input 0 or a positive value, the output is the same as the input:

$$f(x) = x \quad \text{if} \quad x \geq 0$$

Because two different rules, or 'pieces,' are required to describe absolute value, the absolute value function is an example of a piecewise-defined function.

Definition

The **absolute value function** can be defined as

$$f(x) = |x| = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$$

■

■ **Example 2** Write $f(x) = |x+3|$ as an equivalent piecewise-defined function.

Solution:
If the input, $x+3$, is a negative value, the output $|x+3|$, is the opposite of the input.

$$f(x)_B = -(x+3) \quad \text{if} \quad x+3 < 0$$

If the input, $x+3$, is a non-negative value, the output, $|x+3|$, is the same as the input.

$$f(x)_A = x+3 \quad \text{if} \quad x+3 \geq 0$$

Thus,

$$f(x) = |x+3| = \begin{cases} -(x+3) & \text{if} \quad x+3 < 0 \\ x+3 & \text{if} \quad x+3 \geq 0 \end{cases}$$

Simplifying the intervals (inequalities) to intervals (inequalities) of x, instead of $x+3$, we get

$$f(x) = |x+3| = \begin{cases} -(x+3) & \text{if} \quad x < -3 \\ x+3 & \text{if} \quad x \geq -3 \end{cases}$$

N *We could also have distributed the negative in the first rule, and written the rule as* $-x-3$.

■

■ **Example 3** Write $g(x) = |8-5x|$ as an equivalent piecewise-defined function.

Solution:
If the input, $8-5x$, is a negative value, the output, $|8-5x|$, is the opposite of the input.

$$g(x)_B = -(8-5x) \quad \text{if} \quad 8-5x < 0$$

If the input, $8-5x$, is a non-negative value, the output, $|8-5x|$, is the same as the input.

$$g(x)_A = 8-5x \quad \text{if} \quad 8-5x \geq 0$$

Thus,

$$g(x) = |8-5x| = \begin{cases} -(8-5x) & \text{if} \quad 8-5x < 0 \\ 8-5x & \text{if} \quad 8-5x \geq 0 \end{cases}$$

Simplifying the intervals to intervals of x, instead of $8-5x$, we get

$$g(x) = |8-5x| = \begin{cases} -(8-5x) & \text{if} \quad x > \frac{8}{5} \\ 8-5x & \text{if} \quad x \leq \frac{8}{5} \end{cases}$$

While the above notation is mathematically correct, it is standard to write the intervals of a piecewise-defined function, with corresponding rules, in ascending order. For $g(x)$ we have

$$g(x) = |8-5x| = \begin{cases} 8-5x & \text{if} \quad x \leq \frac{8}{5} \\ -(8-5x) & \text{if} \quad x > \frac{8}{5} \end{cases}$$

■

When solving for the intervals of x in the equivalent piecewise-defined function of $f(x) = |g(x)|$, always begin by setting $g(x) < 0$ and $g(x) \geq 0$. Pay close attention to the signs of your numbers and the directions of your inequalities, as you proceed.

Try It # 2:

Write $h(x) = \left|\frac{1}{2}x - 7\right|$ as an equivalent piecewise-defined function.

Domain of a Piecewise-Defined Function

Let's now turn our attention to computing the domain of piecewise-defined functions.

When determining the domain of a piecewise-defined function,

1. Check the first Rule to see whether or not it is defined everywhere on its corresponding Interval.
   - **a.** If there are no restrictions, check the next Rule.
   - **b.** If there are restrictions, rewrite the corresponding Interval to include these restrictions, and check the next Rule.
   - **c.** Proceed until all Rules have been checked.
2. After all Rules have been checked and any necessary Interval changes have been made, find the union of the Intervals.

The union of the intervals found in Step 2 is the domain of the piecewise-defined function.

Consider the previously discussed function

$$f(x) = \begin{cases} x+5 & \text{if} \quad x \leq -3 \\ 9-x^2 & \text{if} \quad -3 < x \leq 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$$

To compute the domain of $f(x)$, we use the process just described.

Step 1:

Check Rule A, $f(x)_A = x+5$, on Interval A: $(-\infty, -3]$. Clearly, $f(x)_A = x+5$ is a linear polynomial function, and it is defined for all real numbers. Thus, $f(x)_A$ is defined for all x-values in Interval A.

Check Rule B, $f(x)_B = 9-x^2$, on Interval B: $(-3, 3]$. Upon close inspection, $f(x)_B = 9-x^2$ is a quadratic polynomial function, and is defined for all real numbers. Thus, $f(x)_B$ is defined for all x-values in Interval B.

Check Rule C, $f(x)_C = -x+5$, on Interval C: $(3, \infty)$. Noting $f(x)_C = -x+5$ is a linear polynomial function, it is defined for all real numbers. Thus, $f(x)_C$ is defined for all x-values in Interval C.

Step 2: Using **Figure 5.5.2**, we can determine the union of the Intervals from Step 1.

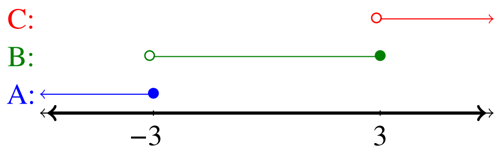


Figure 5.5.2: A graphical representation of the domain of $f(x)$.

The union and, thus, the domain of $f(x)$ is $(-\infty, \infty)$.

N *While $x = -3$ is not included in Interval B, it is included in Interval A. Likewise, $x = 3$ is not included in Interval C, but it is included in Interval B.*

▪ **Example 4** State the domain of $g(x)$, using interval notation.

$$g(x) = \begin{cases} \sqrt[5]{x} & \text{if} \quad x < 0 \\ \dfrac{x}{x+3} & \text{if} \quad 2 < x < 4 \\ \sqrt{3x+6} & \text{if} \quad x \geq 6 \end{cases}$$

Solution:

Step 1:

Check Rule A, $g(x)_A = \sqrt[5]{x}$, on Interval A: $(-\infty, 0)$. Obviously, $g(x)_A = \sqrt[5]{x}$ is a odd root function of a linear polynomial, so it is defined for all real numbers. Thus, $g(x)_A$ is defined for all x-values in Interval A.

Check Rule B, $g(x)_B = \frac{x}{x+3}$, on Interval B: $(2,4)$. As $g(x)_B = \frac{x}{x+3}$ is a rational function, it is defined for all real numbers except where the denominator, $x+3$, equals zero. $x+3 = 0$ when $x = -3$; however, $x = -3$ is not in Interval B. Thus, there is no need to amend Interval B.

Check Rule C, $g(x)_C = \sqrt{3x+6}$, on Interval C: $[6, \infty)$. Seeing as $g(x)_C = \sqrt{3x+6}$ is a even root function of a linear polynomial, it is defined for $3x+6 \geq 0$. $3x+6 \geq 0$ when $x \geq -2$, and all x-values in Interval C are greater than or equal to -2. Thus, there is no need to amend Interval C.

Step 2: We can illustrate the union of the Intervals from Step 1 in **Figure 5.5.3**.

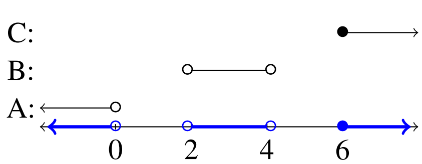


Figure 5.5.3: A graphical representation of the domain of $g(x)$.

The union and, thus, the domain of $g(x)$ is $(-\infty, 0) \cup (2,4) \cup [6, \infty)$.

▪

Example 5 State the domain of $h(x)$, using interval notation.

$$h(x) = \begin{cases} \dfrac{2x-1}{x+4} & \text{if} \quad x \leq 0 \\[2ex] \dfrac{3}{(x-11)(x+8)} & \text{if} \quad x > 0 \end{cases}$$

Solution:

Step 1:

Check Rule A, $h(x)_A = \frac{2x-1}{x+4}$, on Interval A: $(-\infty, 0]$. Here, $h(x)_A = \frac{2x-1}{x+4}$ is a rational function, so it is defined for all real numbers except where the denominator, $x+4$, equals zero. $x+4=0$ when $x=-4$, which is in Interval A, so we must amend Interval A to include the restriction, $x \neq -4$.

$$\text{Amended Interval A: } (-\infty, -4) \cup (-4, 0]$$

Check Rule B, $h(x)_B = \frac{3}{(x-11)(x+8)}$, on Interval B: $(0, \infty)$. Similarly, $h(x)_B = \frac{3}{(x-11)(x+8)}$ is a rational function, and it is defined for all real numbers except where the denominator, $(x-11)(x+8)$, equals 0. $(x-11)(x+8) = 0$ when $x = 11$ and when $x = -8$, so we must amend Interval B to include the restriction, $x \neq 11$ ($x = -8$ is not in Interval B).

$$\text{Amended Interval B: } (0, 11) \cup (11, \infty)$$

Step 2: We can illustrate the union of the amended Intervals from Step 1 in **Figure 5.5.4**.

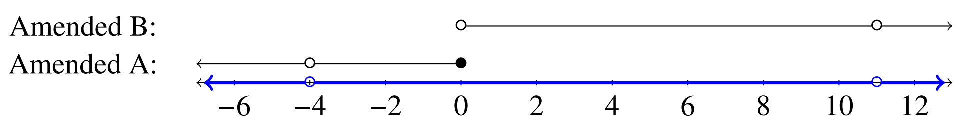


Figure 5.5.4: A graphical representation of the domain of $h(x)$.

The union and, thus, the domain of $h(x)$ is $(-\infty, -4) \cup (-4, 11) \cup (11, \infty)$.

Example 6 Write $f(x) = \dfrac{x-2}{|x-2|}$ as an equivalent piecewise-defined function.

Solution:

If the input of the absolute value, $x-2$, is a non-negative value, the output of the absolute value is the same as the input, and we get

$$f(x)_A = \left\{ \dfrac{x-2}{x-2} \quad \text{if} \quad x-2 \geq 0 \quad \text{or} \quad x \geq 2 \right.$$

However, $f(x)_A$ is a rational function and is defined for all real numbers except where $x-2=0$. Because $x-2=0$ when $x=2$, we must amend the interval $x \geq 2$ to include the restriction, $x \neq 2$.

$$\text{Amended Interval for } f(x)_A : x > 2$$

Thus, $f(x)_A = \dfrac{x-2}{x-2} = 1$ if $x > 2$.

If the input of the absolute value, $x-2$, is a negative value, the output of the absolute value is the opposite of the input, and we get

$$f(x)_B = \begin{cases} \dfrac{x-2}{-(x-2)} & \text{if} \quad x-2<0 \quad \text{or} \quad x<2 \end{cases}$$

As above, $f(x)_B$ is a rational function and is defined for all real numbers except where $-(x-2)=0$. Because $-(x-2)=0$ when $x=2$, we have the restriction $x \neq 2$, but $x=2$ is not in the interval $x<2$. Thus, there is no need to amend the interval for $f(x)_B$.

So, $f(x)_B = \dfrac{x-2}{-(x-2)} = -1$ if $x < 2$.

Therefore, in standard notation,

$$f(x) = \frac{x-2}{|x-2|} = \begin{cases} -1 & \text{if} \quad x<2 \\ 1 & \text{if} \quad x>2 \end{cases}$$

∎

Try It # 3:
State the domain of $f(x)$, using interval notation.

$$f(x) = \begin{cases} \sqrt[8]{x+4} & \text{if} \quad x<5 \\ (x-20)^{\frac{1}{3}} & \text{if} \quad x>10 \end{cases}$$

Graphs of Piecewise-Defined Functions

We can now add the parent absolute value function to our list of parent functions, as shown in **Table 5.9**.

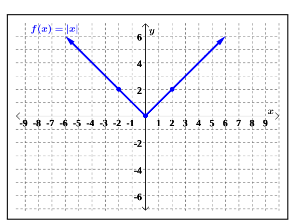
Name	**Function**	**Graph**	**Table**	**Domain**
Absolute Value	$f(x) = \lvert x \rvert = \begin{cases} -x & \text{if} \quad x<0 \\ x & \text{if} \quad x \geq 0 \end{cases}$		$\begin{array}{c\|c} x & f(x) \\ \hline -2 & 2 \\ 0 & 0 \\ 2 & 2 \end{array}$	$(-\infty, \infty)$

Table 5.9: Parent Absolute Value Function

Notice in the graph of the parent absolute value function, to the left of the y-axis ($x < 0$), $y = -x$ is graphed, and for $x \geq 0$, $y = x$ is graphed. In general, we graph piecewise-defined functions by graphing each Rule on its corresponding Interval on a common set of axes.

Let's consider sketching the graph of the function,

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 1 \\ 5 & \text{if} \quad 1 < x \leq 3 \end{cases}$$

Rule A, $f(x)_A = x^2$, is graphed on the interval $(-\infty, 1]$. $f(x)_A = x^2$ is the quadratic parent function, so we know our graph is a portion of the parabola opening upward.

A table of values can be useful when graphing, especially when only graphing a portion of a function. To make an accurate graph, it is important to use any finite endpoint(s) of the interval, whether the endpoint(s) are included or not. Additional values within the interval are necessary to show the correct behavior of the Rule; the number of additional values is dependent upon the type of function rule.

One table of values for helping to graph Rule A is shown in **Table 5.10**.

	x	$f(x)_A = x^2$
numbers less than 1	-3	9
	-2	4
	-1	1
	0	0
included endpoint →	1	1

Table 5.10: A chart of values for $f(x)_A = x^2$ over the interval $[-3, 1]$.

Rule B, $f(x)_B = 5$, will be graphed on the interval $(1, 3]$. $f(x)_B = 5$ is a constant polynomial function, so we know our graph is a horizontal line segment.

One table of values for helping to graph Rule B is shown in **Table 5.11**.

	x	$f(x)_B = 5$
excluded endpoint →	1	5
numbers between 1 and 3	1.75	5
	2.5	5
included endpoint →	3	5

Table 5.11: A chart of values for $f(x)_B = 5$ over the interval $[1, 3]$.

Now, to graph

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 1 \\ 5 & \text{if} \quad 1 < x \leq 3 \end{cases}$$

we use the points found in **Tables 5.10** and **5.11**, along with our knowledge of graphing constant and quadratic functions and of included and excluded endpoints. The resulting graph is given in **Figure 5.5.5**.

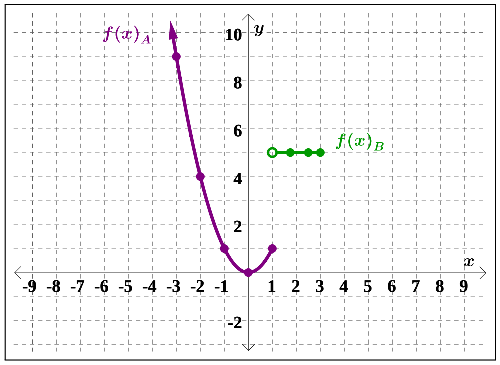


Figure 5.5.5: The graphs of $f(x)_A$ and $f(x)_B$ on the same coordinate plane.

Thus, the graph of $f(x)$ is seen in **Figure 5.5.6**.

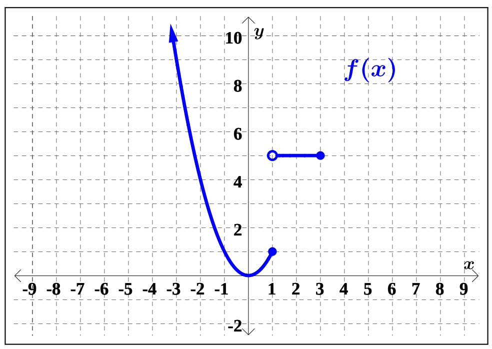


Figure 5.5.6: The graph of $f(x)$.

Whenever graphing functions, especially piecewise-defined functions, it is important to verify the final graph passes the Vertical Line Test.

▪ **Example 7** Sketch a graph of the following function.

$$f(x) = \begin{cases} x-1 & \text{if} \quad x < -2 \\ 1 & \text{if} \quad x = -2 \\ 3 - \dfrac{1}{2}x & \text{if} \quad x > -2 \end{cases}$$

Solution:
We begin by investigating the three Rules of $f(x)$.

Rule A, $f(x)_A = x - 1$, is a linear polynomial function and is graphed on the interval $(-\infty, -2)$.

One corresponding table of values is given in **Table 5.12**.

	x	$f(x)_A = x-1$
numbers less than -2	-5	-6
	-3	-4
excluded endpoint $\rightarrow$	-2	-3

Table 5.12: A chart of values for $f(x)_A = x-1$ over the interval $[-5,-2]$.

Rule B, $f(x)_B = 1$, is a constant polynomial function and is graphed only when $x = -2$. As there is a single x-value corresponding to Rule B, at that x-value there is only one point $(-2, f(-2)) = (-2, 1)$ on then graph of the function, $f(x)$.

Rule C, $f(x)_C = 3 - \dfrac{1}{2}x$, is a linear polynomial function and is graphed on the interval $(-2, \infty)$.
One corresponding table of values is given in **Table 5.13**.

	x	$f(x)_C = 3 - \dfrac{1}{2}x$
excluded endpoint $\rightarrow$	-2	4
numbers greater than -2	0	3
	4	1

Table 5.13: A chart of values for $f(x)_C = 3 - \frac{1}{2}x$ over the interval $[-2, 4]$.

N *In* ***Table 5.13****, we chose x-values which were multiples of 2, so that* $f(x)_C = 3 - \dfrac{1}{2}x$ *would have "nice" integer values.*

Using the information gained from the three Rules, a sketch of $f(x)$ is seen in **Figure 5.5.7**.

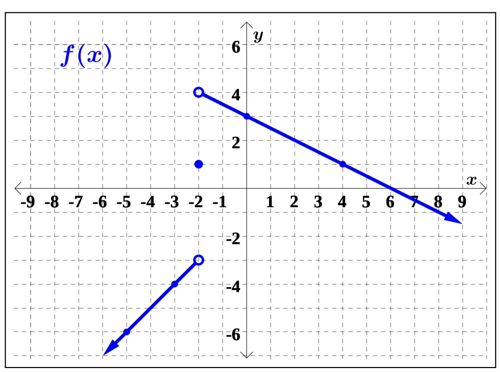


Figure 5.5.7: The graph of $f(x)$.

We can see that $f(x)$ passes the Vertical Line Test.

∎

Try It # 4:
Sketch a graph of the following piecewise-defined function

$$f(x) = \begin{cases} x^3 & \text{if} \quad x < -1 \\ -2 & \text{if} \quad -1 < x < 4 \\ \sqrt{x} & \text{if} \quad x > 4 \end{cases}$$

Real-World Applications

Next, we turn our focus to real-world scenarios where different rules are needed to model the results.

Example 8 The height, h, in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t & \text{if} \quad 0 \le t \le 20 \\ 0 & \text{if} \quad t > 20 \end{cases}$$

a. Compute and interpret $h(10)$ and $h(60)$.
b. Solve $h(t) = 375$, and interpret your answer.

Solution:

a. We first notice that the function is broken up into two rules: one rule for values of t between 0 and 20 inclusive, and another for values of t greater than 20.

Because $t = 10$ satisfies the inequality $0 \le t \le 20$, we use Rule A, $h(t)_A = -5t^2 + 100t$, to calculate $h(10)$. We have $h(10) = -5(10)^2 + 100(10) = 500$. As t represents the number of seconds since lift-off and $h(t)$ is the height above the ground (in feet), the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground.

To compute $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use Rule B, $h(t)_B = 0$. This function rule returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to the ground.

b. Considering the function $h(t)$ is defined in pieces, we need to solve $h(t) = 375$ for each of these pieces.

For $0 \le t \le 20$, $h(t)_A = -5t^2 + 100t$, so for these values of t, we solve

$$\begin{aligned} -5t^2 + 100t &= 375 \\ 0 &= 5t^2 - 100t + 375 \\ 5(t^2 - 20t + 75) &= 0 \\ 5(t-5)(t-15) &= 0 \end{aligned}$$

$$5 \ne 0 \quad \text{or} \quad t - 5 = 0 \quad \text{or} \quad t - 15 = 0$$
$$t = 5 \quad \text{or} \quad t = 15$$

Both $t = 5$ and $t = 15$ lie between 0 and 20, so we keep both solutions.

For $t > 20$, $h(t) = 0$, and in this case, there are no solutions to $0 = 375$.

In terms of the model rocket, solving $h(t) = 375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t = 5$ and $t = 15$ correspond to the rocket reaching this altitude *twice* – once 5 seconds after launch and again 15 seconds after launch. (What goes up must come down!)

■

■ **Example 9** Current U.S. tax laws are complicated and the percentage of your income paid in taxes involves many variables. However, suppose newly elected officials change the laws so that to determine your taxes owed, you simply take your income and pay a percentage of that income, as shown in **Table 5.14**.

Tax Rate	Individual Annual Income for 2019
10%	up to \$9,700
12%	over \$9,700
22%	over \$39,475
24%	over \$84,200
32%	over \$160,725
35%	over \$204,100
37%	over \$510,300

Table 5.14: irs.gov/newsroom (Nov 15, 2018)

Write a function, $T(m)$, representing the amount of taxes paid by an individual U.S. citizen, in dollars, with an individual income of m dollars, under the changed law.

Solution:
Seven different rules $(A-G)$ will be needed for the seven different tax rates given, based on income.

A person making no more than \$9,700 will be taxed at a rate of 10%. Thus, Interval A is $0 \leq m \leq 9700$ (a person's income can't be negative), and Rule A is $T(m)_A = 0.10m$.

A person making more than \$9,700, but not more than \$39,475 will be taxed at a rate of 12%. Thus, Interval B is $9700 < m \leq 39475$, and Rule B is $T(m)_B = 0.12m$.

Similarly for incomes between \$39,475 and \$510,300, Rules $C-F$ can be defined.

For a person making more that \$510,300 they will be taxed at a rate of 37%. Thus, Interval G is $m > 510300$, and Rule G is $T(m)_G = 0.37m$.

We combine the Rules and corresponding Intervals to create the piecewise-defined function $T(m)$, below.

$$T(m) = \begin{cases} 0.10m & \text{if} \quad 0 \leq m \leq 9700 \\ 0.12m & \text{if} \quad 9700 < m \leq 39475 \\ 0.22m & \text{if} \quad 39475 < m \leq 84200 \\ 0.24m & \text{if} \quad 84200 < m \leq 160725 \\ 0.32m & \text{if} \quad 160725 < m \leq 204100 \\ 0.35m & \text{if} \quad 204100 < m \leq 510300 \\ 0.37m & \text{if} \quad m > 510300 \end{cases}$$

■

Example 10 A cell phone company offers a plan where talk and text are unlimited, but data is not. The company charges \$45/month for talk and text and the first 2 gigabytes of data used and an additional \$10/gb for data usage over 2 gigabytes during the month.

Write the function, $C(g)$, representing the monthly cost of the cell phone plan, in dollars, with g gigabytes of data used during the month.

Solution:
Two different rules, A and B, will be needed, as the monthly cost changes with increased use of data.

For data usage up to 2 gigabytes, the monthly cost is \$45. Thus, Interval A is $0 \leq g \leq 2$, and Rule A is $C(g)_A = 45$.

For data usage over 2 gigabytes, the monthly cost is \$45 *plus* \$10 *per gigabyte over 2*. Thus, Interval B is $g > 2$, and Rule B is $C(g)_B = 45 + 10(g - 2)$.

We combine Rules A and B and their corresponding Intervals to create the piecewise-defined function $C(g)$, below.

$$C(g) = \begin{cases} 45 & \text{if} \quad 0 \leq g \leq 2 \\ 45 + 10(g-2) & \text{if} \quad g > 2 \end{cases}$$

Try It # 5:
A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for a group of 10 or more people. Write a function, $C(p)$, representing the cost, in dollars, of a tour with p people.

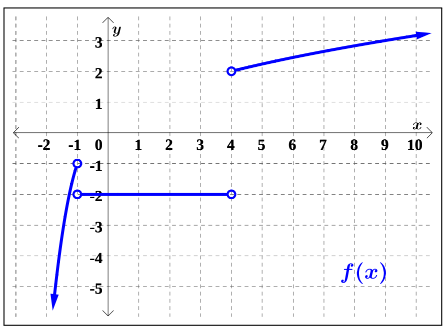
Another reason to create a piecewise-defined function is to redefine a function with holes, so that the function becomes continuous. While this is not a concept for this text, it is often discussed in a calculus text.

Try It Answers

1. **a.** -6
   b. undefined
   c. -1
   d. -10

2. $h(x) = \left|\frac{1}{2}x - 7\right| = \begin{cases} -\left(\frac{1}{2}x - 7\right) & \text{if} \quad x < 14 \\ \frac{1}{2}x - 7 & \text{if} \quad x \geq 14 \end{cases}$

3. $[-4, 5) \cup (10, \infty)$

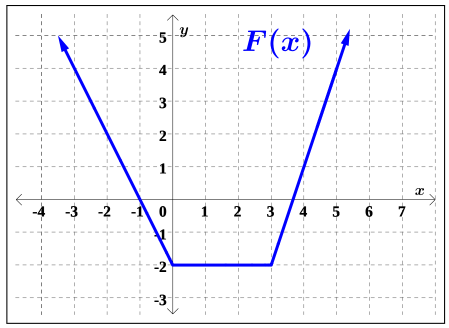
4. 

5. $C(p) = \begin{cases} 5p & \text{if} \quad 1 \leq p \leq 9 \\ 50 & \text{if} \quad p \geq 10 \end{cases}$

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 4, using the graph of $F(x)$ below, compute each function value.



1. $F(0)$
2. $F(-2)$
3. $F(5)$
4. $F\left(\frac{3}{2}\right)$

For Exercises 5 - 10, using the function below, compute each function value.

$$f(x) = \begin{cases} x^2+2x-4 & \text{if } x \leq 3 \\ x+4 & \text{if } x > 3 \end{cases}$$

5. $f(-3)$
6. $f(-1.5)$
7. $f(0)$
8. $f(3)$
9. $f(4)$
10. $f(8.5)$

For Exercises 11 - 19, using the function below, compute each function value.

$$f(x) = \begin{cases} -3x & \text{if } -7 < x \leq 0 \\ 2x^2+6 & \text{if } x > 4 \end{cases}$$

11. $f(-10)$
12. $f(-7)$
13. $f(-5.5)$
14. $f(0)$
15. $f(0.25)$
16. $f(1)$
17. $f(4)$
18. $f(8)$
19. $f(10)$

For Exercises 20 - 21, state the equivalent absolute value function for each piecewise-defined function.

20. $f(x) = \begin{cases} -3x & \text{if } x < 0 \\ 3x & \text{if } x \geq 0 \end{cases}$

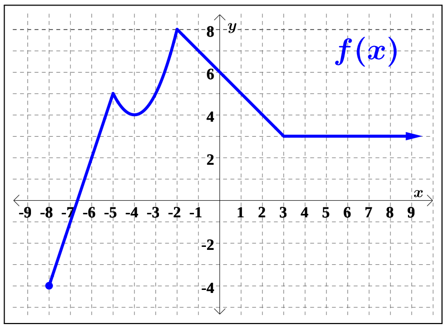
21. $f(x) = \begin{cases} -(x+6) & \text{if } x < -6 \\ x+6 & \text{if } x \geq -6 \end{cases}$

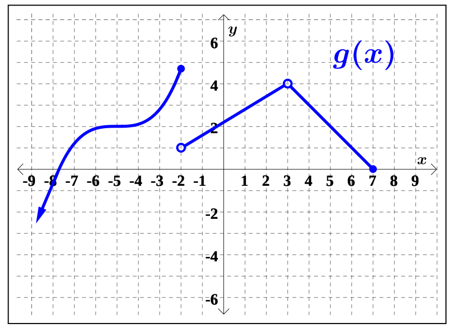
For Exercises 22 - 23, write the equivalent piecewise-defined function for each absolute value function.

22. $f(x) = |\,x - 5\,|$

23. $g(x) = \left|\dfrac{2}{7}x\right|$

For Exercises 24 - 25, state the domain, using interval notation, of each graphed piecewise-defined function.

24. 

25. 

For Exercises 26 - 31, state the domain, using interval notation, of each piecewise-defined function.

26. $f(x) = \begin{cases} x^2 + 2x - 4 & \text{if } x \le 3 \\ x + 4 & \text{if } x > 3 \end{cases}$

27. $f(x) = \begin{cases} 4x + 16 & \text{if } x < 12 \\ 7x^5 - 8x^2 & \text{if } x > 12 \end{cases}$

28. $f(x) = \begin{cases} 3x^2 - 4 & \text{if } x < 8 \\ 7 & \text{if } x = 8 \\ \sqrt[5]{x^2 - 3x - 13} & \text{if } x > 8 \end{cases}$

29. $f(x) = \begin{cases} -3x & \text{if } -7 < x \le 0 \\ 2x^2 + 6 & \text{if } x > 4 \end{cases}$

30. $f(x) = \begin{cases} -1 & \text{if } x < -3 \\ 1 & \text{if } x \ge -3 \end{cases}$

31. $f(x) = \begin{cases} |x - 8| & \text{if } 0 < x < 2 \\ 3x^4 + 2 & \text{if } 2 \le x < 7 \\ 10 & \text{if } x \ge 8 \end{cases}$

For Exercises 32 - 33, graph each function.

32. $f(x) = \begin{cases} x & \text{if } x < 1 \\ x - 5 & \text{if } x \ge 1 \end{cases}$

33. $f(x) = \begin{cases} -3x + 4 & \text{if } x \le 0 \\ 6 & \text{if } x \ge 2 \end{cases}$

For Exercises 34 - 35, express each of the following as an appropriate piecewise-defined function.

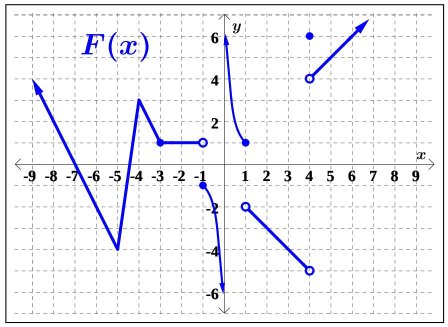
34. A hat store is having a sale this weekend. The first 10 hats you buy cost \$12 each, and each additional hat bought after that costs \$8/hat. Construct the function, $H(x)$, which gives the amount one would pay for x hats.

35. Bob works for ABC Vacuum Cleaners. His weekly salary is \$475, plus a commission based on the number of vacuums he sells. His commission is \$80 per vacuum for the first 9 vacuums he sells each week. For any

vacuum over 9 sold each week, Bob earns a commission of \$200 per vacuum. Construct the function, $I(x)$, that can be used to calculate Bob's weekly income when he sells x vacuums.

INTERMEDIATE SKILLS PRACTICE (Answers)

For Exercises 36 - 41, using the graph of $F(x)$ below, compute each function value.



36. $F(0)$
37. $F(-4)$
38. $F(2)$
39. $F(4)$
40. $F(-6.5)$
41. $F\left(-\frac{5}{2}\right)$

For Exercises 42 - 50, using the function below, compute each function value. Leave all answers in exact form.

$$f(x) = \begin{cases} 5 - 3x & \text{if } -1 \le x < 5 \\ 9 & \text{if } x = 5 \\ \sqrt{35 - x} & \text{if } 5 < x \le 35 \end{cases}$$

42. $f(-1)$
43. $f(10)$
44. $f(5)$
45. $f(-11)$
46. $f(35)$
47. $f(0)$
48. $f(-3)$
49. $f(18)$
50. $f(39)$

For Exercises 51 - 59, using the function below, compute each function value. Leave all answers in exact form.

$$f(x) = \begin{cases} \sqrt[3]{x-8} & \text{if } x < 2 \\ 3x^2 - x + 19 & \text{if } 2 \le x < 7 \\ \dfrac{10}{x-9} & \text{if } x \ge 7 \end{cases}$$

51. $f(-10)$
52. $f(7)$
53. $f(0)$
54. $f(9)$
55. $f(5)$
56. $f(-19)$
57. $f(2)$
58. $f(2.1)$
59. $f\left(\frac{15}{2}\right)$

For Exercises 60 - 61, state the equivalent absolute value function for each piecewise-defined function.

60. $f(x) = \begin{cases} -2x-4 & \text{if } x < -2 \\ 2x+4 & \text{if } x \geq -2 \end{cases}$

61. $f(x) = \begin{cases} 3x-12 & \text{if } x > 4 \\ -3x+12 & \text{if } x \leq 4 \end{cases}$

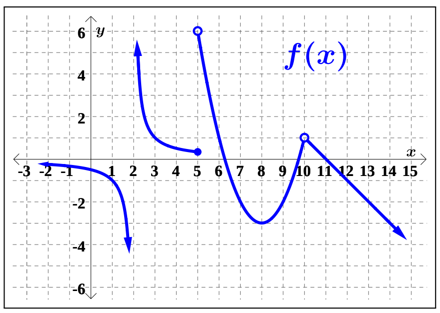
For Exercises 62 - 63, write the equivalent piecewise-defined function for each absolute value function.

62. $f(x) = |9-6x|$

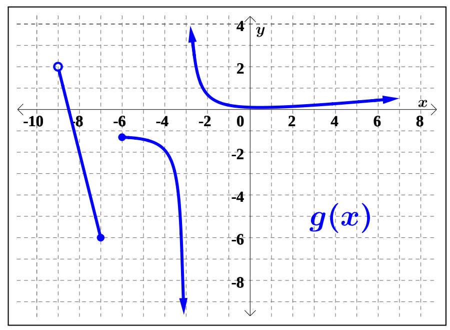
63. $g(x) = |-100x|$

For Exercises 64 - 65, state the domain, using interval notation, of each graphed piecewise-defined function.

64.



65.



For Exercises 66 - 71, state the domain, using interval notation, of each piecewise-defined function.

66. $f(x) = \begin{cases} \dfrac{x+5}{x-3} & \text{if } -1 < x < 4 \\ x^2+2x-10 & \text{if } x \geq 6 \end{cases}$

69. $f(x) = \begin{cases} \dfrac{x+7}{x^2-1} & \text{if } x \leq 0 \\ \dfrac{1}{x} & \text{if } x > 0 \end{cases}$

67. $f(x) = \begin{cases} \dfrac{(x+3)(x-3)}{x+3} & \text{if } -7 < x \leq 2 \\ 4x+2 & \text{if } x > 2 \end{cases}$

70. $f(x) = \begin{cases} \dfrac{(x+8)(x-14)}{x-14} & \text{if } x < 9 \\ \sqrt{x+5} & \text{if } x \geq 9 \end{cases}$

68. $f(x) = \begin{cases} \dfrac{x-2}{x^2-x-2} & \text{if } -5 \leq x \leq 4 \\ \dfrac{3x+9}{x-7} & \text{if } 4 < x \leq 8 \end{cases}$

71. $f(x) = \begin{cases} \dfrac{\sqrt[3]{2x-15}}{x} & \text{if } x \leq 1 \\ \dfrac{9}{\sqrt{x+2}} & \text{if } x > 1 \end{cases}$

For Exercises 72 - 73, graph each function.

72. $f(x) = \begin{cases} -2x-6 & \text{if } x \leq -4 \\ 3 & \text{if } x = -1 \\ x^2 & \text{if } x > 0 \end{cases}$

73. $f(x) = \begin{cases} 3x+2 & \text{if } x < -1 \\ 2 & \text{if } -1 < x \leq 5 \\ 7-x & \text{if } x > 5 \end{cases}$

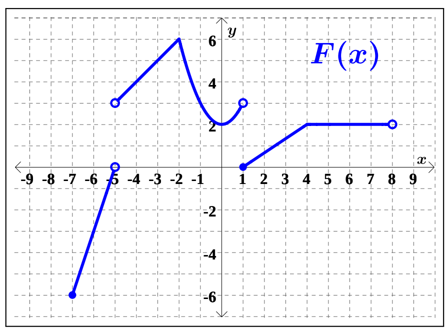
For Exercises 74 - 75, express each of the following as an appropriate piecewise-defined function.

74. 123 Cell Phones offers a monthly plan for cellular phone minutes. The plan consists of a base charge of \$20/month. The first 100 minutes are free, and the next 400 minutes cost \$0.10/minute. Any usage over 500 minutes costs \$0.15/minute. Construct the function, $C(x)$, which gives the amount of a cell phone bill during a month when a customer uses x cell phone minutes.

75. A cellular phone carrier offers a monthly plan for sending text messages. The plan consists of paying a flat rate of \$5.00 each month for up to 100 text messages. For each additional text message over 100 (up to 1200 text messages), the customer pays \$0.20. For each text message over 1200 messages, the customer will pay \$0.40. Construct the function, $B(t)$, which gives the amount of a cell phone bill during a month when a customer sends t text messages.

Mastery Practice (Answers)

For Exercises 76 - 81, using the graph of $F(x)$ below, determine each of the following.



76. $F(0)$
77. $F(-1)$
78. $F(1)$
79. $F(4.5)$
80. The domain of $F(x)$
81. The range of $F(x)$

For Exercises 82 - 93, using the function below, determine each of the following. Leave all answers in exact form.

$$f(x) = \begin{cases} \dfrac{2}{x} & \text{if } x \leq -1 \\ \sqrt{x^2+3} & \text{if } -1 < x < 9 \\ 7 & \text{if } x > 10 \end{cases}$$

82. $f(9)$
83. $f(-4)$
84. $f(10)$
85. $f(-0.99)$
86. $f(-1.01)$
87. $f(12)$
88. $f(2)$
89. $f(-1)$
90. $f(8)$
91. $f(9.5)$
92. The domain of $f(x)$
93. The range of $f(x)$

94. Write the equivalent piecewise-defined function for $f(x) = \dfrac{x-4}{|x-4|}$.

For Exercises 95 - 97, state the domain, using interval notation, of each piecewise-defined function.

95. $f(x) = \begin{cases} \dfrac{x}{x^2+12x+32} & \text{if } x < -6 \\ \dfrac{11x}{x+3} & \text{if } -6 \le x < -2 \\ \sqrt[3]{x-12} & \text{if } x > -2 \end{cases}$

96. $f(x) = \begin{cases} 2x+4 & \text{if } -1 \le x < 5 \\ 9 & \text{if } x = 5 \\ \sqrt{35-x} & \text{if } x > 5 \end{cases}$

97. $f(x) = \begin{cases} \dfrac{1}{\sqrt[3]{x^2+5x+4}} & \text{if } x \le -3 \\ x^2-4 & \text{if } -3 < x < 5 \\ \dfrac{8}{x-6} & \text{if } 5 < x < 7 \end{cases}$

98. Graph $f(x) = \begin{cases} -2x+8 & \text{if } 3 < x < 6 \\ -5 & \text{if } x = 6 \\ 4x-30 & \text{if } 6 < x < 8 \end{cases}$

99. Graph $g(x) = \begin{cases} x^2-4 & \text{if } x < 0 \\ \dfrac{1}{x} & \text{if } x > 0 \end{cases}$

100. Express the following as an appropriate piecewise-defined function.

Suzie's class is holding a fundraiser in which the students are selling boxes of cookies at the school store next Saturday from 8 am - 5 pm. As an incentive, Suzie's teacher is going to give the students "prize" points they can use to "purchase" items from the school store, based on the number of hours they work at the fundraiser that day. For the first 3 hours they work, they will earn 20 points/hour. For the next 2 hours, they will receive 40 points/hours, and any additional hours they work they earn 60 points/hour. Write the function, $P(t)$, for the number of prize points a student will receive for working t hours at the fundraiser.

Communication Practice (Answers)

101. When graphing a piecewise-defined function, when is it appropriate to use an 'open' circle versus a 'closed' dot?

5.6 Exponential Functions

© Photo by Jack Mitchell, 2018

India was the second most populous country in the world, with a population of about 1.25 billion people, in 2013. According to Worldometer (http://www.worldometers.info/world-population/), on February 24, 2014, the population was growing at a rate of about 1.2% each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say the growth is "exponential." To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at **exponential functions**, which model this kind of rapid growth.

Learning Objectives:
In this section, you will learn about the properties and characteristics of exponential functions. Upon completion you will be able to:

- Apply the laws of exponents to simplify exponential expressions.
- Identify an exponential function.
- Classify the graphs of exponential functions as growth or decay models.
- Memorize the graph of the parent exponential growth or decay model.
- Determine the domain of an exponential function, using interval notation.
- Solve equations involving exponential functions with like bases.
- Solve equations involving exponential functions which can be rewritten with like bases.
- Use exponential functions to model real-world situations, and use them to solve application problems.

Reviewing Laws of Exponents

Recall from the Power and Radical Functions section, we reviewed some basic properties of variables raised to numerical exponents. These properties also hold true when numerical values are raised to variable exponents, as follows.

Theorem 5.5 Laws of Exponents

Let a and b be any positive real numbers and x and y be any variables with real number values. The following rules hold true.

$$a^0 = 1 \qquad a^x a^y = a^{x+y}$$

$$\frac{1}{a^x} = a^{-x} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$\frac{1}{a^{-x}} = a^x \qquad (a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Parentheses are stated when needed. Do not assume parentheses where there are none.

$$ab^x \neq a^x b^x$$
$$a \cdot b^x \neq (ab)^x$$

When a real number is raised to a variable exponent, the real number is called the **base**.

Example 1 Using the Laws of Exponents, rewrite 8^{5x-7} as an equivalent expression in base 2.

Solution:
For 8^{5x-7}, 8 is the base. An equivalent form of 8, in base 2, is 2^3. Using this fact and the rule $(a^x)^y = a^{xy}$, we have

$$\begin{aligned} 8^{5x-7} &= (2^3)^{5x-7} \\ &= (2)^{3\cdot(5x-7)} \\ &= 2^{15x-21} \end{aligned}$$

■

Example 2 Using the Laws of Exponents, completely simplify $\frac{27 \cdot 7^{3x+1}}{7^{2x}}$.

Solution:
We begin by eliminating the fraction, using the rule $\frac{a^x}{a^y} = a^{x-y}$.

$$\begin{aligned} \frac{27 \cdot 7^{3x+1}}{7^{2x}} &= 27 \cdot 7^{(3x+1)-(2x)} \\ &= 27 \cdot 7^{x+1} \end{aligned}$$

As 27 does not have a factor of 7, no further simplification is necessary. We cannot combine unlike base factors to make a single base. So,

$$\frac{27 \cdot 7^{3x+1}}{7^{2x}} = 27 \cdot 7^{x+1}$$

■

When combining like base factors, we combine the exponents, but keep the common base. For example,

$$7^{3x+1} \cdot 7^{-2x} = 7^{x+1} \;\; \textit{but} \;\; 7^{3x+1} \cdot 7^{-2x} \neq 49^{x+1}$$

Try It # 1:
Using the Laws of Exponents, determine if the following statements are true or false.

a. $\frac{1}{9} \cdot 3^{7x} = 3^{7x-2}$

b. $2^{x^3-x} = 2^{x^3} - 2^x$

c. $\frac{19 \cdot 4^{2x+1}}{4^{2x}} = 76$

Describing Exponential Functions

Definition

A general **exponential function** has the form $f(x) = ab^x$, where a is a nonzero number, and b is a positive real number not equal to 1.

- If $\boldsymbol{b > 1}$ and $\boldsymbol{a > 0}$ the functions **grows** at a rate proportional to its size.
- If $\boldsymbol{0 < b < 1}$ and $\boldsymbol{a > 0}$ the functions **decays** at a rate proportional to its size.

■

In the definition of a *general* exponential function the base is positive and not equal to 1.

- If the base were a negative number, say $b = -2$, then the function $f(x) = a \cdot (-2)^x$ has trouble. For instance, when $x = \frac{1}{2}$, $a \cdot (-2)^{\frac{1}{2}} = a \cdot \sqrt{-2}$, which is not a real number. In general, if x is any rational number (in simplest terms) with an even denominator, then $(-2)^x$ is not defined. Therefore, we must restrict our attention to bases where $b \geq 0$.

- If the base were equal to 0, the function $f(x) = a \cdot 0^x$ is undefined for $x \leq 0$, because we cannot divide by 0 and 0^x is unclear as x approaches 0 or ∞.

- If the base were equal to 1, the function would be of the form $f(x) = a \cdot 1^x$. So, $f(x)$ can be simplified to $f(x) = a$ for any finite real number x, which has the behavior of a constant function. However, the behavior of $f(x) = a \cdot 1^x$ is unclear as x approaches $\pm\infty$. The authors will leave this special case to higher level mathematics.

Thus, the authors will require the base to be positive and not equal to 1 in discussions concerning exponential functions in this text.

N *Power functions are of the form $f(x) = a \cdot x^p$, and exponential functions are of the form $g(x) = a \cdot b^x$. While the two functions look similar, they are not the same. Power functions have a variable base and a constant exponent, whereas exponential functions have a constant base and a variable exponent.*

▪ **Example 3** Determine whether or not each of the following are exponential functions. If the function is exponential, state whether it represents exponential growth or decay.

a. $f(x) = 4^{3x}$

b. $g(x) = x^3$

c. $h(x) = \left(\frac{1}{3}\right)^x$

d. $j(x) = (-5)^x$

Solution:

a. $f(x) = 4^{3x}$ can also be written as $f(x) = (4^3)^x = 64^x$, using the Laws of Exponents. In both forms, the base is a real number constant which is greater than 0 and not equal to 1, and the exponent contains the variable. Thus, $f(x)$ IS an exponential function. Because the base, in either form, is greater than 1, $f(x)$ is an exponential growth function.

b. $g(x) = x^3$ has a variable base and a real number constant exponent. Thus, $g(x)$ is NOT an exponential function. Instead, $g(x)$ is a power function; in particular, $g(x)$ is a cubic polynomial.

c. $h(x) = \left(\frac{1}{3}\right)^x$ has a base of $b = \frac{1}{3}$ and a variable exponent. With $b > 0$ and $b \neq 1$, $h(x)$ IS an exponential function. Because $0 < b < 1$, $h(x)$ is an exponential decay function.

d. $j(x) = (-5)^x$ has a base of $b = -5$ and a variable exponent. Due to the fact that $b < 0$, $j(x)$ is NOT an exponential function.

▪

Try It # 2:
Determine whether or not each of the following are exponential functions. If the function is exponential, state whether it represents exponential growth or decay.

a. $f(x) = x(x-3)^2$

b. $g(x) = 8(1.04)^x$

c. $h(x) = 6^{-2x}$

Properties of Exponential Functions

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function, $f(x) = 2^x$, is exponential. The second function, $g(x) = 2x$, is linear. We will start with an input of 0, and increase each input by 1. **Table 5.15** shows the consecutive outputs of $f(x)$ are doubled, while the consecutive outputs of $g(x)$ only increase by 2.

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Table 5.15: A chart comparing exponential and linear growth.

From **Table 5.15** we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increasing by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increasing by the *same amount* over equal increments found in the domain.

We can also see from **Table 5.15**, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding two to the output whenever the input was increased by one.

In general, if the constant multiplicative rate of change or constant additive rate of change is known, then the corresponding output for a 1 unit input change can be found from the previous output, without having an actual function rule specified.

Let's now reconsider the function $f(x) = 2^x$ from **Table 5.15** over a different interval, $[-3,3]$. **Table 5.16** shows the corresponding outputs for each input.

x	$f(x) = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

Table 5.16: A chart of values for $f(x) = 2^x$ over the interval $[-3,3]$.

To construct a more accurate graph, we need to consider what would happen as $x \to \pm\infty$. For example, $2^{-100} \approx 7.8886 \times 10^{-31}$ is positive, but very close to 0. On the other hand, $2^{100} \approx 1.2677 \times 10^{30}$, which is also positive, but very large.

Let us examine the graph of $f(x)$ by plotting and connecting the ordered pairs from **Table 5.16** and drawing the end behavior according to our further calculations, as in **Figure 5.6.2** .

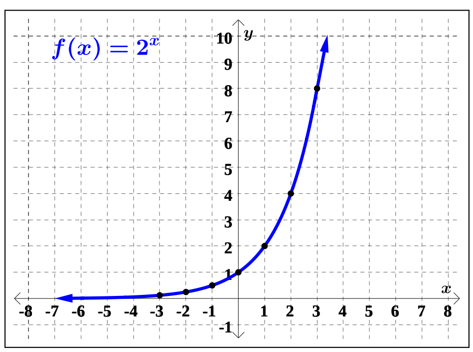


Figure 5.6.2: The graph of $f(x) = 2^x$.

Some observations from **Figure 5.6.2** are

- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- The end behavior shows as $x \to -\infty$, $f(x) \to 0$; there is a horizontal asymptote (HA) at $y = 0$.
- The end behavior shows as $x \to \infty$, $f(x) \to \infty$.
- The graph of $f(x)$ will never touch the x-axis, because base 2 raised to any real number exponent never has the result of zero. Thus, there are no x-intercepts.
- The y-intercept is $(0,1)$.

N *We could have rewritten $f(x) = 2^x$ as $f(x) = \left(\frac{1}{2}\right)^{-x}$, using the Laws of Exponents.*

Now, let's consider the function $g(x) = \left(\frac{1}{2}\right)^x = 0.5^x$. We will again create a table (**Table 5.17**) to determine the corresponding outputs over the interval $[-3,3]$.

x	$g(x) = \left(\frac{1}{2}\right)^x$
-3	$\left(\frac{1}{2}\right)^{-3} = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2$
0	$\left(\frac{1}{2}\right)^{0} = 1$
1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$
3	$\left(\frac{1}{2}\right)^{3} = \frac{1}{8}$

Table 5.17: A chart of values for $g(x) = \left(\frac{1}{2}\right)^x$ over the interval $[-3,3]$.

Again, to construct a more accurate graph, we will consider what would happen as $x \to \pm\infty$. For example, $\left(\frac{1}{2}\right)^{-100} \approx 1.2677 \times 10^{30}$ is positive, but very large. On the other hand, $\left(\frac{1}{2}\right)^{100} \approx 7.8886 \times 10^{-31}$ is also positive, but very close to 0.

We will graph $g(x)$ in **Figure 5.6.3**, using the ordered pairs from **Table 5.17** and drawing the end behavior according to our further calculations.

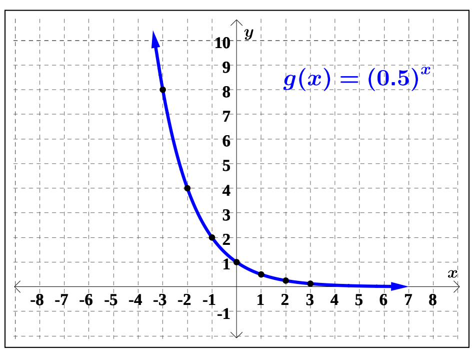


Figure 5.6.3: The graph of $g(x) = \left(\frac{1}{2}\right)^x$.

Some observations from **Figure 5.6.3** are

- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- The end behavior shows as $x \to -\infty$, $g(x) \to \infty$.
- The end behavior shows as $x \to \infty$, $g(x) \to 0$; there is a horizontal asymptote (HA) at $y = 0$.
- The graph of $g(x)$ will never touch the x-axis, because base $\frac{1}{2}$ raised to any real number exponent never has the result of zero. Thus, there are no x-intercepts.
- The y-intercept is $(0,1)$.

N *We could have rewritten $g(x) = \left(\frac{1}{2}\right)^x$ as $g(x) = 2^{-x}$, using the Laws of Exponents.*

We can generalize our observations.

Properties of General Exponential Functions

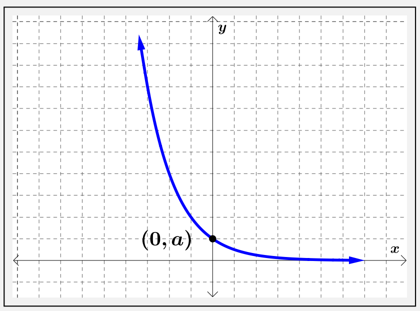
For any real number x, nonzero real number a, and a real number b where $b > 0$ but $b \neq 1$, the general exponential function

$$f(x) = a \cdot b^x$$

has the following properties.

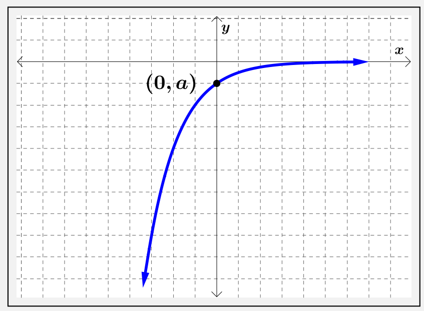
If $\boldsymbol{a > 0}$ and $\boldsymbol{0 < b < 1}$, then

- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- The end behavior is
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0$ (H.A. $y = 0$)
- There are no x-intercepts.
- The y-intercept is $(0, a)$.



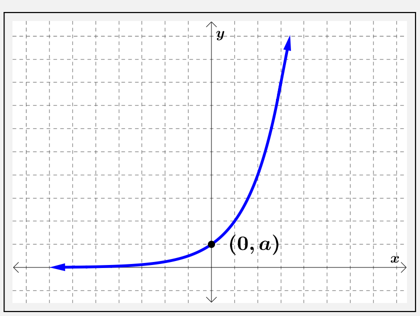
If $\boldsymbol{a < 0}$ and $\boldsymbol{0 < b < 1}$, then

- The domain is $(-\infty, \infty)$.
- The range is $(-\infty, 0)$.
- The end behavior is
  - As $x \to -\infty$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to 0$ (H.A. $y = 0$)
- There are no x-intercepts.
- The y-intercept is $(0, a)$.



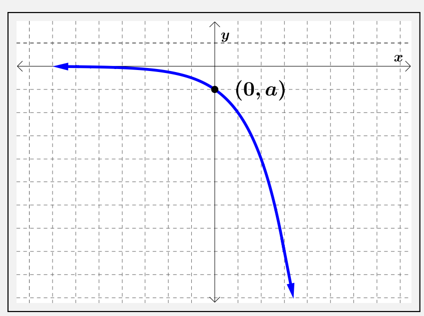
If $\boldsymbol{a > 0}$ and $\boldsymbol{b > 1}$, then

- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- The end behavior is
  - As $x \to -\infty$, $f(x) \to 0$ (H.A. $y = 0$)
  - As $x \to \infty$, $f(x) \to \infty$
- There are no x-intercepts.
- The y-intercept is $(0, a)$.



If $\boldsymbol{a < 0}$ and $\boldsymbol{b > 1}$, then

- The domain is $(-\infty, \infty)$.
- The range is $(-\infty, 0)$.
- The end behavior is
  - As $x \to -\infty$, $f(x) \to 0$ (H.A. $y = 0$)
  - As $x \to \infty$, $f(x) \to -\infty$
- There are no x-intercepts.
- The y-intercept is $(0, a)$.



N *The properties of any general exponential function vary depending on the values of both a and b.*

We will now add the parent functions for exponential growth and decay to our list of parent functions. Notice these parent functions, shown in **Table 5.18**, have $a = 1$.

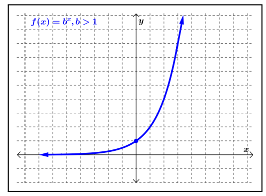
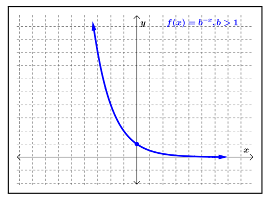
<table>
<tr><th>Name</th><th>Function</th><th>Graph</th><th>Table</th><th>Domain</th></tr>
<tr><td>Exponential Growth</td><td>$f(x) = b^x = \left(\frac{1}{b}\right)^{-x}, b > 1$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>b^{-2}</td></tr><tr><td>0</td><td>1</td></tr><tr><td>2</td><td>b^2</td></tr></table></td><td>$(-\infty, \infty)$</td></tr>
<tr><td>Exponential Decay</td><td>$f(x) = \left(\frac{1}{b}\right)^{x} = b^{-x}, b > 1$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>b^{2}</td></tr><tr><td>0</td><td>1</td></tr><tr><td>2</td><td>b^{-2}</td></tr></table></td><td>$(-\infty, \infty)$</td></tr>
</table>

Table 5.18: Parent Exponential Growth and Decay Functions

■ **Example 4** Given $h(x) = 3 \cdot \left(\frac{1}{5}\right)^x$, state each of the following.

a. Domain
b. Range
c. End behavior
d. x-intercept(s)
e. y-intercept(s)

Solution:
We begin by recognizing $b = \frac{1}{5}$ $(0 < b < 1)$, and $a = 3 > 0$. From this information, we note $h(x)$ is an exponential decay function, and thus, we have the following results.

a. The domain of $h(x)$ is $(-\infty, \infty)$.

b. The range of $h(x)$ is $(0, \infty)$.

c. The end behavior of $h(x)$ is
- As $x \to -\infty$, $h(x) \to \infty$.
- As $x \to \infty$, $h(x) \to 0$.

d. $h(x)$ has no x-intercepts.

e. $h(x)$ has a y-intercept at $(0, h(0)) = (0, a) = (0, 3)$.

■

Try It # 3:
Given $f(x) = \frac{1}{2} \cdot 7^x$, state each of the following.

a. Domain
b. Range
c. End behavior
d. x-intercept(s)
e. y-intercept(s)

Of all the bases for exponential functions, two occur the most often in real-world scenarios. The first, base 10, is often called the **common base**. The second base is the irrational number $e \approx 2.718$, called the **natural base**. So, $f(x) = 10^x$ and $g(x) = e^x$ are both exponential growth functions and $h(x) = 10^{-x}$ and $k(x) = e^{-x}$ are both exponential decay functions.

N *A more formal discussion of the origins of the irrational number e is left to calculus.*

Computing Domain

We have seen for $b > 0$ and $b \neq 1$, $f(x) = b^x$ has a domain of all real numbers. Now suppose $h(x) = b^{g(x)}$, where $b > 0$ and $b \neq 1$, and $g(x)$ is any previously discussed function. Then, $h(x)$ is defined wherever $g(x)$ is defined.

Example 5 State the domain of each function, using interval notation.

a. $f(x) = 3^{-x+2}$

b. $h(x) = 10^{\frac{2}{x-1}}$

c. $j(x) = \dfrac{1}{e^{\sqrt{2x+5}}}$

Solution:

a. $f(x) = 3^{-x+2}$ is defined when $g(x) = -x+2$ is defined. Considering $g(x)$ is a polynomial and defined for all real numbers, the domain of $f(x)$ is $(-\infty, \infty)$.

b. $h(x) = 10^{\frac{2}{x-1}}$ is defined when $g(x) = \dfrac{2}{x-1}$ is defined. As $g(x)$ is a rational function, we must find all values of x where $x - 1 \neq 0$.

$$\begin{aligned} x - 1 &\neq 0 \\ x &\neq 1 \end{aligned}$$

Thus, the domain of $h(x)$ is $(-\infty, 1) \cup (1, \infty)$.

c. $j(x) = \dfrac{1}{e^{\sqrt{2x+5}}}$ can be written as $j(x) = e^{-\sqrt{2x+5}}$, using the Laws of Exponents. So $j(x)$ is defined when $g(x) = -\sqrt{2x+5}$ is defined. Because $g(x)$ is an even root function, it is defined where $2x+5$ is defined and greater than or equal to 0. As $2x+5$ is a polynomial expression defined for all real numbers, we must satisfy

$$\begin{aligned} 2x+5 &\geq 0 \\ 2x &\geq -5 \\ x &\geq \frac{-5}{2} \end{aligned}$$

Thus, the domain of $j(x)$ is $\left[-\frac{5}{2}, \infty\right)$.

■

Try It # 4:
State the domain of each function, using interval notation.

a. $f(x) = 12^{\left(\frac{2x}{\sqrt{x-7}}\right)}$

b. $h(x) = \dfrac{x}{e^{(x-1)}}$

c. $j(x) = 1 + 2e^{-2x}$

Solving Equations Involving Exponential Expressions

In this chapter, thus far, we have solved equations involving polynomials. Now, we will explore equations involving exponential expressions.

Theorem 5.6 Common Base Property of Exponents

For any algebraic expression S and T, and any positive real number $b \neq 1$,

$$b^S = b^T \quad \text{if and only if} \quad S = T.$$

N *This is often called the One-to-One Property of Exponents. We will discuss what one-to-one means in a later section of this chapter.*

In other words, **Theorem 5.6** says that when an equation has an exponential expression with the same base on each side of the equals sign, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many of these types of equations by using the Laws of Exponents to rewrite each side as an exponential expression with the same base. Then, we can set the exponents equal to one another, and solve for the unknown.

Consider how to solve the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$ for x.

We begin by rewriting the right-hand side so that both sides have a single expression with the common base, 3. Then, we apply the Common Base Property of Exponents by setting the exponents equal to each other, and solve for x.

$$\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} \\
3^{4x-7} &= \frac{3^{2x}}{3^1} \\
3^{4x-7} &= 3^{2x-1} \\
4x-7 &= 2x-1 \\
2x &= 6 \\
x &= 3
\end{aligned}$$

■ **Example 6** Solve $2^{5x} = \sqrt{2}$ for x.

Solution:
First, we rewrite $\sqrt{2}$ in its equivalent exponent form.

$$\begin{aligned}
2^{5x} &= \sqrt{2} \\
2^{5x} &= 2^{\left(\frac{1}{2}\right)}
\end{aligned}$$

Now that both sides of the equation have a single exponential expression with a common base of 2, we can apply the Common Base Property, and solve for x.

$$\begin{aligned}
5x &= \frac{1}{2} \\
x &= \frac{1}{10}
\end{aligned}$$

■

■ **Example 7** Solve $9^{x-1} = 9^{2x-4}$ for x.

Solution:
Both sides of the equation already have a single exponential expression with the same base, so we can apply the Common Base Property, and solve for x.

$$\begin{aligned}
9^{x-1} &= 9^{2x-4} \\
x-1 &= 2x-4 \\
-x &= -3 \\
x &= 3
\end{aligned}$$

■

■ **Example 8** Solve $\frac{1}{e^{5x}} = e^{x^2-6}$ for x.

Solution:
We will again start by rewriting the left-hand side to have a single exponential expression with a common base on both sides of the equals sign. To do so, we will use the Laws of Exponents.

$$\frac{1}{e^{5x}} = e^{x^2-6}$$
$$e^{-5x} = e^{x^2-6}$$

Now that there is a single exponential expression with a common base on both sides, we set the exponents equal, and solve for x.

$$-5x = x^2 - 6$$
$$0 = x^2+5x-6$$
$$0 = (x+6)(x-1)$$
$$x+6=0 \quad \text{or} \quad x-1=0$$
$$x=-6 \quad \text{or} \quad x=1$$

■

Try It # 5:
Solve $5^{2x} = 5^{3x+2}$ for x.

Sometimes the common base for a exponential equation is not explicit. In these cases, we simply rewrite the terms in the equation as exponential expressions with a common base raised to an appropriate power, and then solve using Laws of Exponents and the Common Base Property.

Consider the equation $256 = 4^{x-5}$.

While there does not appear to be a common base, we can rewrite both 256 and 4 as powers of 2. Then, we can apply the Laws of Exponents, along with the Common Base Property, to solve for x.

$$256 = 4^{x-5}$$
$$2^8 = \left(2^2\right)^{x-5}$$
$$2^8 = 2^{2x-10}$$
$$8 = 2x-10$$
$$18 = 2x$$
$$9 = x$$
$$x = 9$$

Example 9 Solve $8^{x+2} = 16^{x+1}$ for x.

Solution:
While both sides of the equation have an exponential expression with different bases, we can rewrite each base as a power of 2. Then, we can proceed as before.

$$\begin{aligned} 8^{x+2} &= 16^{x+1} \\ \left(2^3\right)^{x+2} &= \left(2^4\right)^{x+1} \\ 2^{3x+6} &= 2^{4x+4} \\ 3x+6 &= 4x+4 \\ 2 &= x \\ x &= 2 \end{aligned}$$

Try It # 6:
Solve $5^{2x} = 25^{3x+2}$ for x.

Not all equations involving exponential expressions have a solution. Recall that the range of an exponential function is always positive. So, while solving an equation involving exponentials, we may obtain an expression which is undefined.

To demonstrate, consider $3^x = -2$. We have an exponential function (3^x) set equal to a non-positive number, so there is no value of x where $3^x = -2$.

Similarly, $e^{-x} = 0$ has no solution, as 0 is not a positive number.

When an equation has multiple terms on one or both sides of the equal sign, it is important to move all terms to the same side of the equals sign (as with quadratic equations) before solving for x.

Example 10 Solve $x^2e^x - 3xe^x = 10e^x$ for x.

Solution:
First, we move all terms to the same side of the equals sign, leaving 0 on the other side.

$$\begin{aligned} x^2e^x - 3xe^x &= 10e^x \\ x^2e^x - 3xe^x - 10e^x &= 0 \end{aligned}$$

Factoring gives us

$$\begin{aligned} e^x\left(x^2 - 3x - 10\right) &= 0 \\ e^x(x-5)(x+2) &= 0 \end{aligned}$$

$$\begin{array}{ccccc} e^x = 0 & \text{or} & x-5 = 0 & \text{or} & x+2 = 0 \\ e^x \neq 0 & \text{or} & x = 5 & \text{or} & x = -2 \end{array}$$

Thus, $x = -2$ or $x = 5$.

Try It # 7:
Solve $x^2 \cdot 7^x = 8x \cdot 7^x$ for x.

Applying Exponential Functions to Real-World Applications

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. Assuming this situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013 and P is the population (in billions), what will the population be in 2031?

To estimate the population in 2031, we evaluate the model for $t = 18$, because 2031 is 18 years after 2013.

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

So, there will be about 1.549 billion people in India in the year 2031.

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the values time (t), principal (P), APR (r), and number of compounding periods in a year (m).

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

For example, observe **Table 5.19**, which shows the result of investing \$1000 at 10% for one year. Notice how the value of the account increases, as the compounding frequency increases.

Frequency	m	Value after 1 year
Annually	1	\$1100.00
Semiannually	2	\$1102.50
Quarterly	4	\$1103.81
Monthly	12	\$1104.71
Weekly	52	\$1105.06
Daily	365	\$1105.16

Table 5.19: Compounding results for a principal of \$1000.

Definition

Compound Interest can be calculated using the function

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

where

- $A(t)$ is the accumulated value of the account,
- t is the time measured in years,
- P is the starting amount of the account, often called **principal**, or more generally, present value,
- r is the annual percentage rate (APR) *expressed as a decimal*, and
- m is the number of compounding periods in one year.

■

■ **Example 11** If we invest \$3000 in an investment account paying 3% interest per year, compounded quarterly, how much will the account be worth in 10 years?

Solution:
Because we are starting with \$3000, $P = 3000$. Our annual interest rate is 3% so $r = 0.03$. As we are compounding quarterly, we are compounding 4 times per year, so $m = 4$. We are asked the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$. Using this information in the formula, we have

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{m}\right)^{mt} \\ A(10) &= 3000\left(1 + \frac{0.03}{4}\right)^{4\cdot 10} \\ &\approx \$4045.05 \end{aligned}$$

The account will be worth about \$4045.05 in 10 years.

■

N *When dealing with dollar amounts, round your final answer to the nearest cent, when not told otherwise.*

■ **Example 12** A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to \$40,000 over 18 years. She believes the account will earn 6% annual interest, compounded semiannually. To the nearest dollar, how much will Lily need to invest in the account now?

Solution:
The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded semiannually, or twice a year, thus, $m = 2$. The account grows over 18 years, giving us $t = 18$. We want to calculate the initial investment, P, needed so that the value of the account will be worth \$40,000 in 18 years, $A(18) = 40000$. Using the formula gives us

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{m}\right)^{mt} \\ A(18) = 40000 &= P\left(1+\frac{0.06}{2}\right)^{2\cdot 18} \\ 40000 &= P(1.03)^{36} \\ \frac{40000}{(1.03)^{36}} &= P \\ P &\approx 13801 \end{aligned}$$

Lily will need to invest approximately \$13,801 now, in order to have \$40,000 in 18 years.

■

In the applications thus far, we have worked with rational bases for exponential functions. For most real-world phenomena, however, the irrational number e is used as the base for exponential functions. Exponential models that use e as the base are called *continuous growth* or *continuous decay* models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Definition

For all real numbers $t \geq 0$, all positive numbers a, and all nonzero real numbers r, **continuous growth or decay** is represented by the function

$$A(t) = ae^{rt}$$

where
- a is the initial value,
- r is the continuous growth/decay rate per unit time, *expressed as a decimal*, and
- t is the elapsed time.

If $r > 0$, then the function represents continuous growth.

If $r < 0$, then the function represents continuous decay. ■

Definition

For financial applications, the continuous growth function is called the **continuous compounding** function and takes the form

$$A(t) = Pe^{rt}$$

where
- P is the principal or the initial amount invested,
- r is the growth or interest rate per *unit time**, expressed as a decimal, and
- t is the period or term of the investment (*time** of the investment).

*The units of time for r and t must be the same. ∎

N *When using a function involving the natural base, e, always use the e^x option found on your calculator.*

∎ **Example 13** Radon-222 decays at as continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution:
Seeing as the substance is decaying, the rate 17.3% is negative. So, $r = -0.173$. As the initial amount of Radon-222 was 100 mg, $a = 100$. We use the continuous decay function to compute the value after $t = 3$ days, as follows:

$$\begin{aligned} A(t) &= ae^{rt} \\ A(3) &= 100e^{-0.173\cdot 3} \\ &\approx 59.5115 \end{aligned}$$

So, approximately 59.5115 mg of Radon-222 will remain.

∎

∎ **Example 14** A person invested \$15,000 in an account earning 2% interest per year, compounded continuously. How much will be in the account at the end of seven years?

Solution:
Considering the account is growing continuously in value, this is a continuous compounding problem with a growth rate of $r = 0.02$. The initial investment was \$15,000, so $P = 15000$. We use the continuous compounding function to calculate the value after $t = 7$ years, as follows:

$$\begin{aligned} A(t) &= Pe^{rt} \\ A(7) &= 15000e^{0.02\cdot 7} \\ &\approx 17254.11 \end{aligned}$$

The account is worth about \$17,254.11 after seven years.

∎

Try It # 8:
The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. Assuming this situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013 and P is the population (in billions), what will the population of China be in the year 2031?

Try It # 9:
An investment of \$100,000 is held in an account for 30 years and earns 12% interest per year. How much is the account worth at the end of 30 years, if

a. The interest is compounded weekly?
b. The interest is compounded continuously?

Try It # 10:
Calculate the initial deposit of an account that is worth \$20,000 after earning 3.75% interest per year, compounded monthly, for 9 years.

Try It Answers

1.
- **a.** True
- **b.** False
- **c.** True

2.
- **a.** Not exponential
- **b.** Exponential growth
- **c.** Exponential decay

3.
- **a.** $(-\infty, \infty)$
- **b.** $(0, \infty)$
- **c.** As $x \to -\infty$, $f(x) \to 0$

  As $x \to \infty$, $f(x) \to \infty$
- **d.** None
- **e.** $\left(0, \dfrac{1}{2}\right)$

4.
- **a.** $(7, \infty)$
- **b.** $(-\infty, \infty)$
- **c.** $(-\infty, \infty)$

5. $x = -2$

6. $x = -1$

7. $x = 0$ or $x = 8$

8. 1.548 billion people

9. **a.** \$3,644,675.88
b. \$3,659,823.44

10. \$14,278.55

EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 4, rewrite each exponential expression as a single equivalent expression in the stated base.

1. 16^{3x}, base 4
2. 16^{3x}, base 2
3. 125^{x-7}, base 5
4. 81^{x+4}, base 3

For Exercises 5 - 8, determine whether or not each of the following are exponential functions.

5. $f(x) = 7^{11x}$
6. $g(x) = x^{2/3}$
7. $h(x) = (-5)^x$
8. $j(x) = \left(\frac{3}{8}\right)^{-x}$

For Exercises 9 - 12, state the following properties of the exponential function.

a. Domain
b. Range
c. End behavior
d. x - intercept(s)
e. y - intercept(s)

9. $f(x) = 22^x$
10. $g(x) = \left(\frac{1}{15}\right)^x$
11. $h(x) = (0.4)^x$
12. $j(x) = \left(\frac{8}{3}\right)^x$

For Exercises 13 - 16, state the domain of the function, using interval notation.

13. $f(x) = 3^{x-4}$
14. $g(x) = \left(\frac{1}{2}\right)^x$
15. $h(x) = e^{3x-11}$
16. $j(x) = \dfrac{8}{7^{x-9}}$

For Exercises 17 - 19, algebraically solve each equation for x.

17. $15^x = 15^3$
18. $9^{x^2} = 9$
19. $\left(3^{x-1}\right)^4 = 3$

For Exercises 20 - 23, algebraically solve each equation for x.

20. $4^x = 8$
21. $25^{2x} = 125^{9x-4}$
22. $27^{x+5}3^{4x} = 9$
23. $4^{x-1}2^{3x} = 8^4$

For Exercises 24 - 25, the model for continuous (exponential) growth/decay is given by $y = ce^{kt}$, where c is the initial amount, k is the relative growth rate (as a decimal), t is time (in years), and y is the amount after t years.

24. A new piece of equipment worth \$100,000 depreciates continuously at a relative rate of 7% per year. What will it be worth in 8 years, to the nearest dollar?

25. If \$2500 is invested in an account that earns interest at a rate of 3.87% per year, compounded continuously, how much will be in the account (to the nearest cent) after 10 years?

Intermediate Skills Practice (Answers)

For Exercises 26 - 29, rewrite each exponential expression as a single equivalent expression in the stated base.

26. $7 \cdot 49^{2x-3}$, base 7
27. $\left(\frac{1}{64}\right)^{4-6x}$, base 2
28. $121 \cdot 11^{1+4x}$, base 11
29. $\left(\frac{1}{36}\right) \cdot 6^{8x+3}$, base 6

For Exercises 30 - 33, state whether the exponential function represents exponential growth or decay.

30. $f(x) = 0.5^x$
31. $g(x) = \left(\frac{7}{6}\right)^x$
32. $h(x) = 2^{-10x}$
33. $j(x) = \left(\frac{1}{4}\right)^{-3x}$

For Exercises 34 - 37, state the following properties of the exponential function.

a. Domain
b. Range
c. End behavior
d. x - intercept(s)
e. y - intercept(s)

34. $f(x) = 4e^x$
35. $g(x) = -\left(\frac{1}{15}\right) \cdot 35^x$
36. $h(x) = 6 \cdot (0.12)^x$
37. $j(x) = -11 \cdot \left(\frac{2}{9}\right)^x$

For Exercises 38 - 41, state the domain of the function, using interval notation.

38. $f(x) = e^{\frac{x}{x-6}}$

39. $g(x) = \dfrac{e^{x+2}}{\sqrt{7-x}}$

40. $h(x) = 5^{\sqrt[4]{x+3}}$

41. $j(x) = \dfrac{2^{\sqrt[9]{12+x}}}{x^2-1}$

For Exercises 42 - 45, algebraically solve each equation for x.

42. $6^{x^2} = 6^{x+12}$

43. $\dfrac{1}{2^{6x}} = 2^{8x-11}$

44. $e^{x^2} \cdot e^{-4x} = e^{12}$

45. $\dfrac{1}{e^{9x}} = e^{5x+7}$

For Exercises 46 - 49, algebraically solve each equation for x.

46. $8^{5-7x} \cdot 8^{6x} = 1$

47. $\dfrac{2^{x^2+23}}{2^{9x}} = 8$

48. $0.04 = 5^{3x} \cdot 5^{-2x^2}$

49. $36^{x^2} \cdot \dfrac{1}{6^{-4x}} - 1 = 0$

50. How much should you invest now (to the nearest cent) in an account that earns interest at a rate of 7.2% per year, compounded continuously, in order to have \$100,000 in 20 years?

51. If you invest \$1500 in an account that earns interest at a rate of 4.1% per year, compounded quarterly, how much will be in the account (to the nearest cent) after 6 years?

52. How much money should you invest now (to the nearest cent) in an account that earns interest at a rate of 5.75% per year, compounded monthly, in order to have \$85,000 in 14 years?

Mastery Practice (Answers)

53. Using the Laws of Exponents determine if the following statements are true or false.

    a. $\dfrac{5 \cdot 4^{x+7}}{16 \cdot 4^{x-2}} = 81920$

    b. $3 \cdot 25^{6x+1} = 75^{6x+1}$

54. Use the Laws of Exponents, to write $\dfrac{9 \cdot 27^{-4x}}{3 \cdot 81^{x+9}}$ as a single equivalent expression in base 3.

55. Determine if $g(x) = 3 \cdot \left(\dfrac{1}{2}\right)^x$ is an exponential growth or decay function.

56. Determine if $h(x) = c^{-x}$, where c is a real number and $c > 8.5$, is an exponential growth or decay function.

57. For $f(x) = 5^{-x}$, state the following properties of the exponential function.

   a. Domain

   b. Range

   c. End behavior

   d. x - intercept(s)

   e. y - intercept(s)

58. State the domain of $f(x) = \dfrac{e^{\sqrt{x+9}}}{\sqrt[5]{x+1}}$, using interval notation.

59. State the domain of $g(x) = \dfrac{4^{\frac{8x-1}{\sqrt{x+2}}}}{\sqrt[8]{3-x}}$, using interval notation.

60. Algebraically solve $x^2e^{3x} + 2xe^{3x} - 8e^{3x} = 0$ for x.

61. Algebraically solve $x \cdot 10^{-x} = x^2 \cdot 10^{-x}$ for x.

62. Algebraically solve $x \cdot 3^{2x} - x \cdot 27^{3x+1} = 0$ for x.

63. Determine how much money (to the nearest cent) needs to be invested in an account earning 6.7% interest per year, so that the account contains \$90,000 after 15 years, if the interest is compounded

   a. Semiannually.

   b. Weekly.

   c. Daily.

   d. Continuously.

Communication Practice (Answers)

64. Explain why $f(x) = 0.125^{-4x}$ is an exponential growth function.

65. Explain the difference between linear and exponential growth.

5.7 Combining and Transforming Functions

© Photos by Reid Allen, 2020

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival fun house mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world.

Learning Objectives:

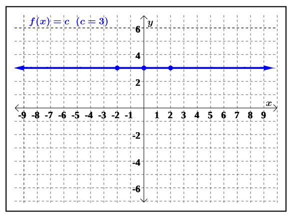
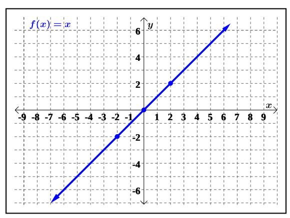
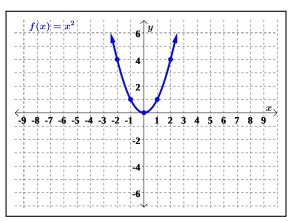
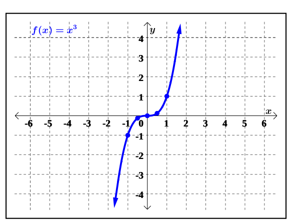
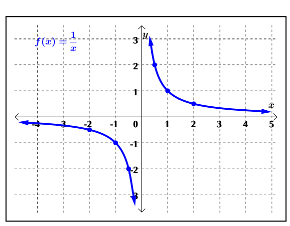
In this section, you will learn about transformations, combinations, and compositions of functions. Upon completion you will be able to:

- State the symbolic forms and draw the graphs of various parent functions.
- Identify vertical and horizontal shifts of parent functions.
- Identify reflections of parent functions about the x-axis.
- Identify vertical expansions and contractions of parent functions.
- Recognize the appropriate parent function and, then, state the series of transformations to be performed on the parent function which would result in the graph of a given function.
- Write the function that results from a given series of transformations to be performed on a stated parent function.
- Graph a function based on the transformations to the corresponding parent or given function.
- Compute the sum, difference, product, and quotient of functions.
- Compute the composition of two or more functions, using function notation, tables, and graphs.
- Recognize the difference in the mathematical notation for combining and composing functions.

Transforming Parent Functions

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, or equations. One method we can employ is to adapt the basic graphs of parent functions to build new models for the given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

In this section, we discuss how the graphs of parent functions change, or **transform**, when certain specialized modifications are made to their rules. We present the list of parent functions amassed thus far in **Table 5.20**, for your recollection.

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th><th>Table</th><th>Domain/Range</th></tr>
<tr><td>Constant</td><td>$f(x)=c$
where c is
a constant</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>c</td></tr><tr><td>0</td><td>c</td></tr><tr><td>2</td><td>c</td></tr></table></td><td>Domain: $(-\infty,\infty)$
Range: $[c,c]$</td></tr>
<tr><td>Linear
(Identity)
1st degree
polynomial</td><td>$f(x)=x$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>-2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td><td>Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$</td></tr>
<tr><td>Quadratic
2nd degree
polynomial</td><td>$f(x)=x^2$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>4</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>4</td></tr></table></td><td>Domain: $(-\infty,\infty)$
Range: $[0,\infty)$</td></tr>
<tr><td>Cubic
3rd degree
polynomial</td><td>$f(x)=x^3$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-1</td><td>-1</td></tr><tr><td>-0.5</td><td>-0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table></td><td>Domain: $(-\infty,\infty)$
Range: $(-\infty,\infty)$</td></tr>
<tr><td>Reciprocal</td><td>$f(x)=\frac{1}{x}$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>-0.5</td></tr><tr><td>-1</td><td>-1</td></tr><tr><td>-0.5</td><td>-2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table></td><td>Domain: $(-\infty,0)\cup(0,\infty)$
Range: $(-\infty,0)\cup(0,\infty)$</td></tr>
</table>

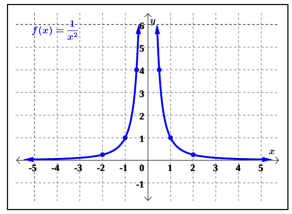
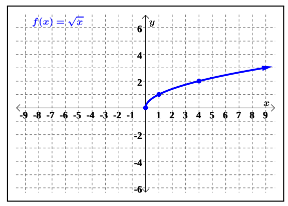
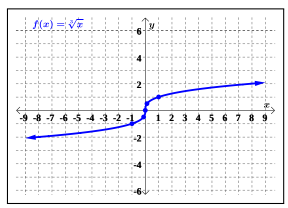
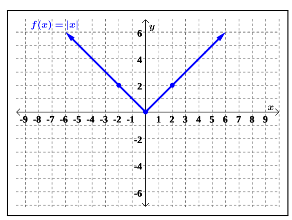
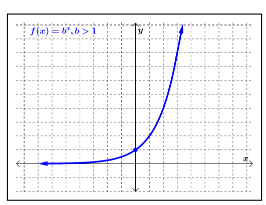
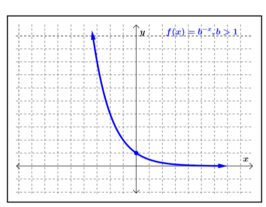
<table>
<tr><th>Name</th><th>Function</th><th>Graph</th><th>Table</th><th>Domain/Range</th></tr>
<tr><td>Reciprocal Squared</td><td>$f(x) = \dfrac{1}{x^2}$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>0.25</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>-0.5</td><td>4</td></tr><tr><td>0.5</td><td>4</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.25</td></tr></table></td><td>Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$</td></tr>
<tr><td>Square Root</td><td>$f(x) = \sqrt{x}$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table></td><td>Domain: $[0, \infty)$
Range: $[0, \infty)$</td></tr>
<tr><td>Cube Root</td><td>$f(x) = \sqrt[3]{x}$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-1</td><td>-1</td></tr><tr><td>-0.125</td><td>-0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table></td><td>Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$</td></tr>
<tr><td>Absolute Value</td><td>$f(x) = \lvert x \rvert$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td><td>Domain: $(-\infty, \infty)$
Range: $[0, \infty)$</td></tr>
<tr><td>Exponential Growth</td><td>$f(x) = b^x = \left(\dfrac{1}{b}\right)^{-x}$,
$b > 1$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>b^{-2}</td></tr><tr><td>0</td><td>1</td></tr><tr><td>2</td><td>b^2</td></tr></table></td><td>Domain: $(-\infty, \infty)$
Range: $(0, \infty)$</td></tr>
<tr><td>Exponential Decay</td><td>$f(x) = \left(\dfrac{1}{b}\right)^{x} = b^{-x}$,
$b > 1$</td><td>
</td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>-2</td><td>b^2</td></tr><tr><td>0</td><td>1</td></tr><tr><td>2</td><td>b^{-2}</td></tr></table></td><td>Domain: $(-\infty, \infty)$
Range: $(0, \infty)$</td></tr>
</table>

Table 5.20: List of Previously Discussed Parent Functions

Additionally, suppose the graph in **Figure 5.7.2**, below, is a complete graph of $f(x)$. For the sake of our discussions, we will consider $f(x)$ to be a 'parent' function.

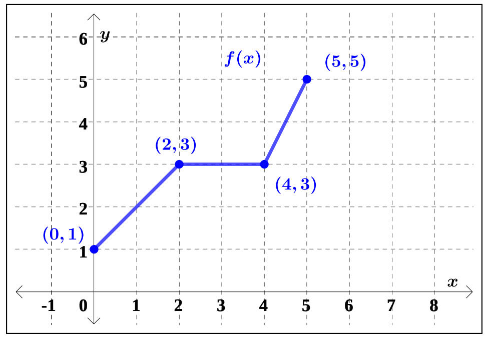


Figure 5.7.2: The function $f(x)$ is drawn on a coordinate plane. The graph includes the points (0,1), (2,3), (4,3), and (5,5), an increasing line between the ordered pairs (0,1) and (2,3), a horizontal line connecting the ordered pairs (2,3) and (4,3), and another increasing line between (4,3) and (5,5).

The transformations of our parent functions will fall into three broad categories: shifts, reflections, and scalings. We will present the transformations in this order.

Vertical and Horizontal Shifts

Our first kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation only involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function, regardless of the input.

Suppose we wanted to graph the function $g(x) = f(x) + 2$, where $f(x)$ is the 'parent' function graphed in **Figure 5.7.2**. Using the points indicated on **Figure 5.7.2**, we can create **Table 5.21** representing the ordered pairs on the graph of the function $g(x)$.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0,1)$	1	3	$(0,3)$
2	$(2,3)$	3	5	$(2,5)$
4	$(4,3)$	3	5	$(4,5)$
5	$(5,5)$	5	7	$(5,7)$

Table 5.21: A chart of ordered pairs on the graphs of $f(x)$ and $g(x) = f(x) + 2$.

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$, so $g(a) = f(a) + 2 = b + 2$. Hence, $(a, b+2)$ is on the graph of $g(x)$. In other words, to obtain the graph of $g(x)$, we add 2 to the y-coordinate of each point on the graph of $f(x)$. Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location, and is "shifting the graph up 2 units." Notice, by comparing the graph of $f(x)$ in **Figure 5.7.3** and the graph of $g(x)$ in **Figure 5.7.4**, that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four points we moved in the same manner in which they were connected before.

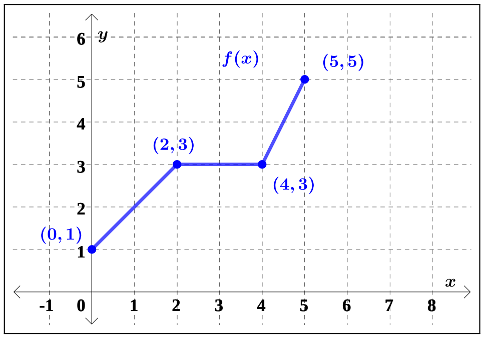


Figure 5.7.3: The graph of $f(x)$.

shift up 2 units $\xrightarrow{}$ add 2 to each y−coordinate

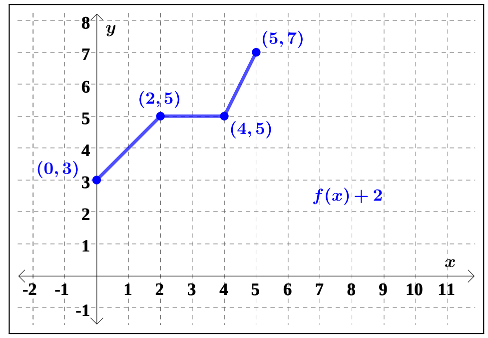


Figure 5.7.4: The graph of $g(x)$.

You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of $f(x)$, we would be subtracting 2. Geometrically, we would be moving the graph down 2 units.

What we have just discussed is generalized in the following theorem.

Theorem 5.7 Vertical Shifts

Suppose f is a function and k is a positive number.

- To graph $y_2 = f(x)+k$, shift the graph of $y_1 = f(x)$ up k units by adding k to the y-coordinates of the points on the graph of $f(x)$.
- To graph $y_2 = f(x)-k$, shift the graph of $y_1 = f(x)$ down k units by subtracting k from the y-coordinates of the points on the graph of $f(x)$.

In the language of inputs and outputs, **Theorem 5.7** can be paraphrased as "adding to, or subtracting from, the *output* of a function causes the graph to shift up or down, respectively."

So what happens if we add to or subtract from the *input* of the function? A change to the input results in a movement of the graph left or right in what is known as a **horizontal shift**.

Keeping with the graph of $f(x)$ in **Figure 5.7.2**, suppose we wanted to graph $h(x) = f(x+2)$. In words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of $h(x)$, based on those we know for $f(x)$. We quickly find that we run into some difficulties. (See **Table 5.22**.)

x	$(x, f(x))$	$f(x)$	$h(x) = f(x+2)$	$(x, h(x))$
0	$(0,1)$	1	$f(0+2) = f(2) = 3$	$(0,3)$
2	$(2,3)$	3	$f(2+2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$f(4+2) = f(6) = ?$	
5	$(5,5)$	5	$f(5+2) = f(7) = ?$	

Table 5.22: A chart of ordered pairs of $f(x)$ and the work to determine the ordered pars of $h(x)$.

When we substitute $x = 4$ into the formula $h(x) = f(x+2)$, we are asked to find $f(4+2) = f(6)$, which does not exist because the domain of $f(x)$ is only $[0,5]$. The same thing happens when we attempt to find $h(5)$. What we need here is a new strategy.

We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of $h(x)$, we need to figure out what value of x we must substitute into $h(x) = f(x+2)$ so that the quantity $x+2$ works out to be 0. Solving $x+2 = 0$ gives $x = -2$, and $h(-2) = f(-2+2) = f(0) = 1$, so $(-2,1)$ is on the graph of $h(x)$. Similarly, we know $f(2) = 3$, and if we set $x+2 = 2$, we get $x = 0$. Substituting, gives $h(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we construct **Table 5.23**.

x	$x+2$	$h(x) = f(x+2)$	$(x,h(x))$
-2	0	$h(-2) = f(0) = 1$	$(-2,1)$
0	2	$h(0) = f(2) = 3$	$(0,3)$
2	4	$h(2) = f(4) = 3$	$(2,3)$
3	5	$h(3) = f(5) = 5$	$(3,5)$

Table 5.23: A chart of ordered pairs on the graph of $h(x) = f(x+2)$.

In summary, the points $(0,1)$, $(2,3)$, $(4,3)$, and $(5,5)$ on the graph of $f(x)$ give rise to the points $(-2,1)$, $(0,3)$, $(2,3)$, and $(3,5)$ on the graph of $h(x)$, respectively. In general, if (a,b) is on the graph of $f(x)$, then $f(a) = b$. Solving $x+2 = a$ gives $x = a-2$ so that $h(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a-2,b)$ is on the graph of $h(x)$. The point $(a-2,b)$ is exactly 2 units to the *left* of the point (a,b), so the graph of $h(x)$ is obtained by shifting the graph $f(x)$ to the left 2 units, as shown in **Figure 5.7.6**.

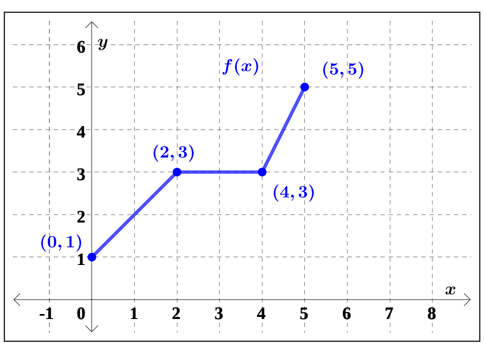


Figure 5.7.5: The graph of $f(x)$.

$\xrightarrow[\text{subtract 2 from each } x\text{−coordinate}]{\text{shift left 2 units}}$

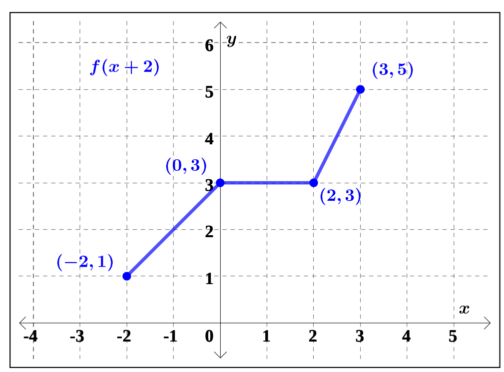


Figure 5.7.6: The graph of $h(x)$.

If we set out to graph $p(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x-values of the points on the graph of $f(x)$ to produce a shift to the *right* 2 units. Generalizing these notions produces the following theorem.

Theorem 5.8 Horizontal Shifts

Suppose f is a function and h is a positive number.

- To graph $y_2 = f(x+h)$, shift the graph of $y_1 = f(x)$ left h units by subtracting h from the x-coordinates of the points on the graph of $f(x)$.
- To graph $y_2 = f(x-h)$, shift the graph of $y_1 = f(x)$ right h units by adding h to the x-coordinates of the points on the graph of $f(x)$.

In other words, **Theorem 5.8** says "adding to, or subtracting from, the *input* to a function amounts to shifting the graph left or right, respectively."

Theorems 5.7 and **5.8** present a theme which will run common throughout this section: changes to the outputs from a function affect the y-coordinates of the graph, resulting in some kind of vertical change; changes to the inputs to a function affect the x-coordinates of the graph, resulting in some kind of horizontal change.

Example 1

a. Graph $f(x) = \sqrt{x}$. Plot at least three points.
b. Use your graph of $f(x)$ to graph $g(x) = \sqrt{x} - 1$.
c. Use your graph of $f(x)$ to graph $j(x) = \sqrt{x-1}$.

Solution:

a. We know the domain of $f(x)$ is $[0, \infty)$, as it is the parent even root function. We choose non-negative numbers which are perfect squares to build **Table 5.24**, in order to graph $f(x)$ in **Figure 5.7.7**, below.

x	$f(x) = \sqrt{x}$	$(x, f(x))$
0	0	(0,0)
1	1	(1,1)
4	2	(4,2)

Table 5.24: A chart of ordered pairs on the graph of $f(x) = \sqrt{x}$.

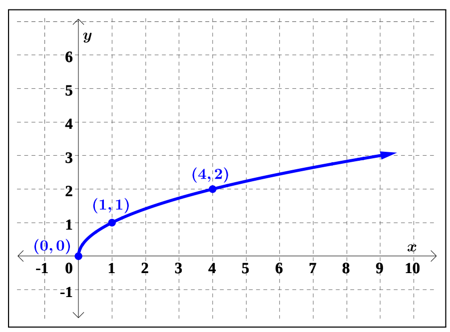


Figure 5.7.7: The graph of $f(x) = \sqrt{x}$.

b. $g(x) = \sqrt{x} - 1 = f(x) - 1$ indicates a vertical shift of -1, moving all points on the graph of $f(x)$ down 1 unit.

The point $(0,0)$ is transformed by subtracting 1 from the y-coordinate.

$$(0,0) \to (0,-1)$$

Similarly,

$$(1,1) \to (1,0)$$
$$(4,2) \to (4,1)$$

Plotting the new points and connecting them in the same manner as the graph of $f(x)$ results in **Figure 5.7.8**.

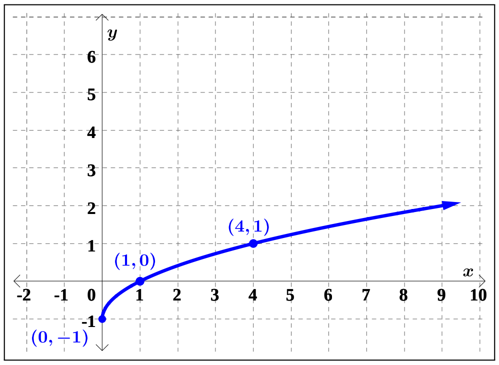


Figure 5.7.8: The graph of $g(x) = \sqrt{x} - 1$.

c. $j(x) = \sqrt{x-1} = f(x-1)$ indicates a horizontal shift of 1, such that all points on the graph of $f(x)$ move right 1 unit.

The point $(0,0)$ is transformed by adding 1 to the x-coordinate.

$$(0,0) \rightarrow (1,0)$$

Similarly,

$$(1,1) \rightarrow (2,1)$$
$$(4,2) \rightarrow (5,2)$$

Plotting the new points and connecting them in the same manner as the graph of $f(x)$ results in **Figure 5.7.9**.

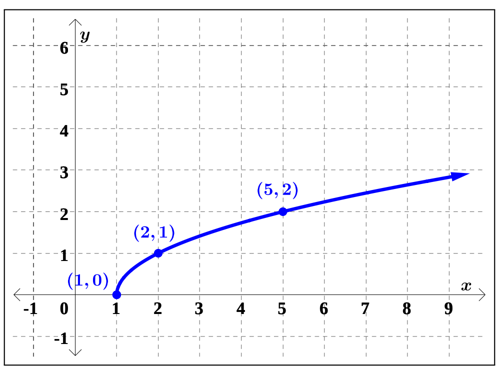


Figure 5.7.9: The graph of $j(x) = \sqrt{x-1}$.

N *For $g(x)$ the domain remains $[0,\infty)$, but the range changes to $[-1,\infty)$, as changes were made only to the outputs. For $j(x)$ the domain changes to $[1,\infty)$, but the range remains $[0,\infty)$, as changes were made only to the inputs.*

∎

Try It # 1:

a. Graph $f(x) = x^2$. Plot at least three points.
b. Use the graph of $f(x)$ to graph $g(x) = x^2 + 3$.
c. Use the graph of $f(x)$ to graph $j(x) = (x+3)^2$.

Reflections

We now turn our attention to reflections. Imagine plotting the point $(2,3)$ on a coordinate plane and folding the paper along the axes.

If the fold is along the x-axis, then where would the point $(2,3)$ be transferred to below the x-axis?

If the fold is along the y-axis, then where would the point $(2,3)$ be transferred to on the left of the y-axis?

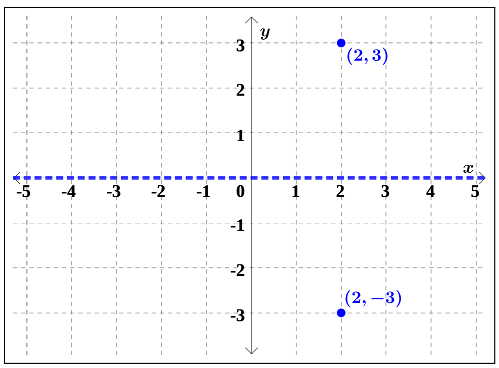


Figure 5.7.10: The coordinate plane with the points (2,3) and (2,−3) labeled on the graph.

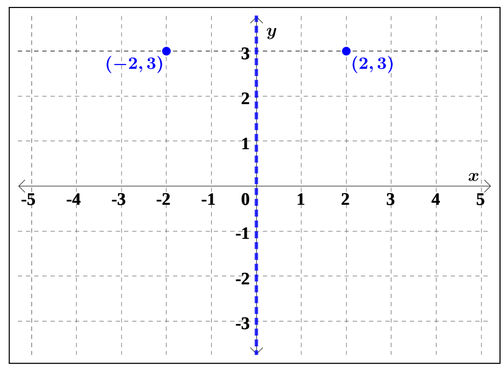


Figure 5.7.11: The coordinate plane with the points (2,3) and (−2,3) labeled on the graph.

Using **Figure 5.7.10**, we can see $(2,3)$ is transferred across the x-axis to the point $(2,-3)$, a mirror image across the x-axis.

Using **Figure 5.7.11**, we can see $(2,3)$ is transferred across the y-axis to the point $(-2,3)$, a mirror image across the y-axis.

To reflect a point (x,y) across the x-axis, we replace y with $-y$. If (x,y) is on the graph of $f(x)$, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of $f(x)$ reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of $f(x)$ reflected across the y-axis.

Returning to the language of inputs and outputs, multiplying the output from a function by -1 reflects its graph across the x-axis, while multiplying the input to a function by -1 reflects the graph across the y-axis.

Theorem 5.9 Reflections

Suppose f is a function.

- To graph $y_2 = -f(x)$, reflect the graph of $y_1 = f(x)$ across the x-axis by multiplying the y-coordinates of the points on the graph of $f(x)$ by -1.
- To graph $y_2 = f(-x)$, reflect the graph of $y_1 = f(x)$ across the y-axis by multiplying the x-coordinates of the points on the graph of $f(x)$ by -1.

Applying **Theorem 5.9** to the graph of the 'parent' function $f(x)$ given at the beginning of the section, we can graph $j(x) = -f(x)$ by reflecting the graph of $f(x)$ about the x-axis. (See **Figure 5.7.13**.)

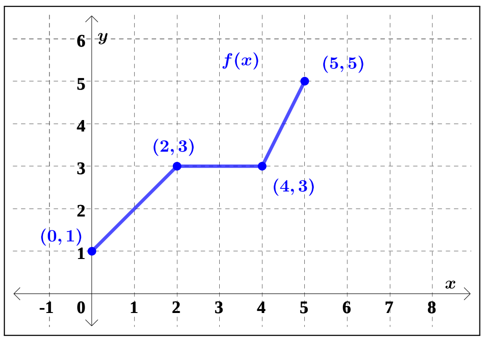


Figure 5.7.12: The graph of $f(x)$.

multiply each y–coordinate by -1 $\longrightarrow$ reflect across x–axis

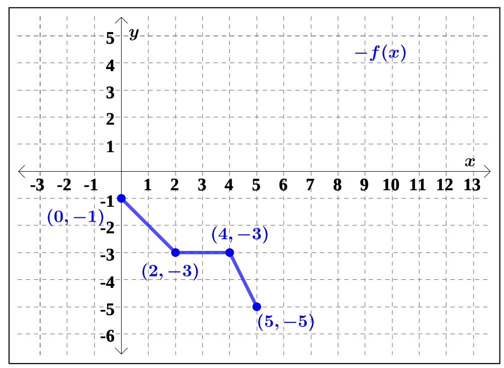


Figure 5.7.13: The graph of $j(x)$.

By reflecting the graph of $f(x)$ across the y-axis, we obtain the the graph of $k(x) = f(-x)$, shown in **Figure 5.7.15**.

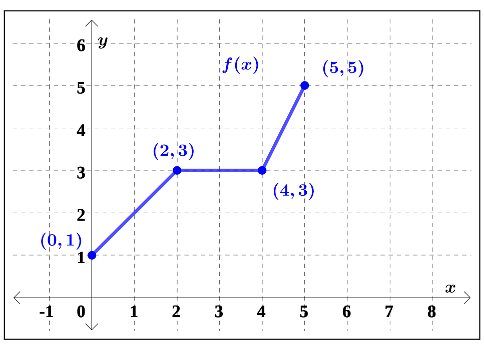


Figure 5.7.14: The graph of $f(x)$.

multiply each y–coordinate by -1 $\longrightarrow$ reflect across y–axis

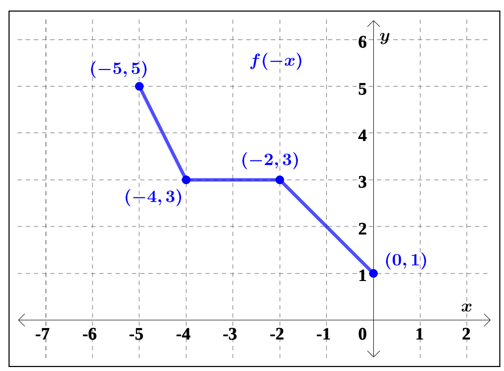


Figure 5.7.15: The graph of $k(x)$.

▪ **Example 2**

a. Graph $f(x) = 2^x$. Plot at least three points.
b. Use your graph of $f(x)$ to graph $g(x) = -2^x$.
c. Use your graph of $f(x)$ to graph $j(x) = 2^{-x}$.

Solution:

a. We know the domain of $f(x)$ is $(-\infty, \infty)$, as it is an exponential growth function. We choose integer values from -2 to 2 to build **Table 5.25**, in order to graph $f(x)$ in **Figure 5.7.16**, below.

x	$f(x) = 2^x$	$(x, f(x))$
-2	$\frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$\frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	4	$(2, 4)$

Table 5.25: A chart of ordered pairs on the graph of $f(x) = 2^x$.

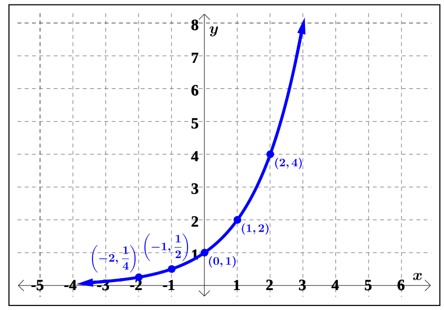


Figure 5.7.16: The graph of $f(x) = 2^x$.

b. $g(x) = -2^x = -1 \cdot 2^x = -1 \cdot f(x)$ indicates a reflection of $f(x)$ across the x-axis.

The point $\left(-2, \dfrac{1}{4}\right)$ is transformed by multiplying the y-coordinate by -1.

$$\left(-2, \frac{1}{4}\right) \rightarrow \left(-2, -\frac{1}{4}\right)$$

Similarly,

$$\left(-1, \frac{1}{2}\right) \rightarrow \left(-1, -\frac{1}{2}\right)$$
$$(0, 1) \rightarrow (0, -1)$$
$$(1, 2) \rightarrow (1, -2)$$
$$(2, 4) \rightarrow (2, -4)$$

Plotting the new points and connecting them in the same manner as the graph of $f(x)$ results in **Figure 5.7.17**.

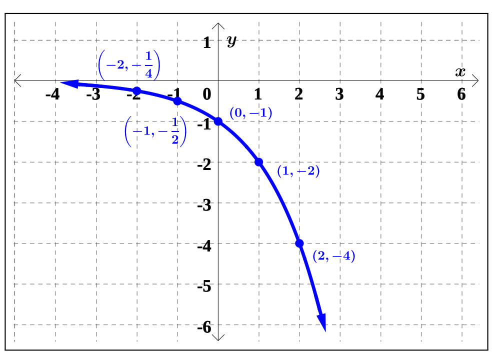


Figure 5.7.17: The graph of $g(x) = -2^x$.

c. $j(x) = 2^{-x} = f(-x)$ indicates a reflection of $f(x)$ across the y-axis.

The point $\left(-2, \dfrac{1}{4}\right)$ is transformed by multiplying the x-coordinate by -1.

$$\left(-2, \frac{1}{4}\right) \to \left(2, \frac{1}{4}\right)$$

Similarly,

$$\left(-1, \frac{1}{2}\right) \to \left(1, \frac{1}{2}\right)$$
$$(0,1) \to (0,1)$$
$$(1,2) \to (-1,2)$$
$$(2,4) \to (-2,4)$$

Plotting the new points and connecting them in the same manner as the graph of $f(x)$ results in **Figure 5.7.18**.

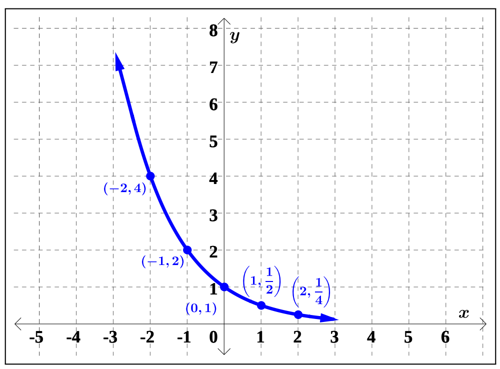


Figure 5.7.18: The graph of $j(x) = 2^{-x}$.

N *For $g(x)$ the domain remains $(-\infty,\infty)$, but the range changes to $(-\infty,0)$, as changes were made to the outputs. For $j(x)$ the domain remains $(-\infty,\infty)$ and the range remains the same; although changes were made to the inputs, the set of real numbers $(-\infty,\infty)$ does not change when multiplied by -1.*

▪

Try It # 2:

a. Graph $f(x) = |x|$. Plot at least three points.

b. Use the graph of $f(x)$ to graph $g(x) = -|x|$.

c. Use the graph of $f(x)$ to graph $j(x) = |-x|$.

Scalings (Stretches and Compressions)

Our last class of transformations are known as **scalings**. The transformations discussed thus far are known as **rigid transformations**. Simply put, they do not change the *shape* of the graph, only its position and orientation in the plane. If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be changing the shape of the graph. This type of transformation is called **non-rigid**, where not only will it be important for us to differentiate between modifying inputs versus outputs, we must also pay attention to the magnitude of the changes we make. As you will see shortly, the mathematics turns out to be easier than the associated grammar.

Suppose we wish to graph the function $m(x) = 2f(x)$, where $f(x)$ is the 'parent' function given at the beginning of the section. (See **Figure 5.7.2**.) From its graph, we can build a table of values for $m(x)$, similarly as before.

x	$(x, f(x))$	$f(x)$	$m(x) = 2f(x)$	$(x, m(x))$
0	(0,1)	1	2	(0,2)
2	(2,3)	3	6	(2,6)
4	(4,3)	3	6	(4,6)
5	(5,5)	5	10	(5,10)

Table 5.26: A chart of ordered pairs on the graphs of $f(x)$ and $m(x) = 2f(x)$.

In general, if (a,b) is on the graph of $f(x)$, then $f(a) = b$, so $m(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of $m(x)$. In other words, to obtain the graph of $m(x)$, we multiply all of the y-coordinates of the points on the graph of $f(x)$ by 2. Multiplying all of the y-coordinates of all of the points on the graph of $f(x)$ by 2 causes what is known as a "vertical scaling by a factor of 2," and the results are shown in **Figure 5.7.20**.

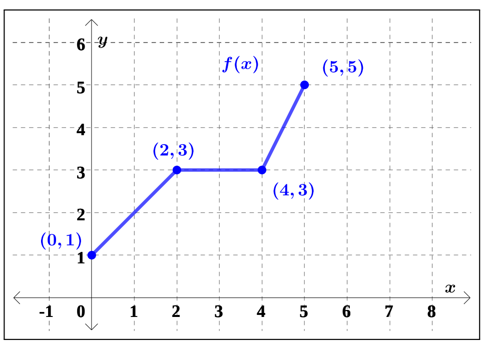


Figure 5.7.19: The graph of $f(x)$.

vertical scaling by a factor of 2 $\longrightarrow$ multiply each y−coordinate by 2

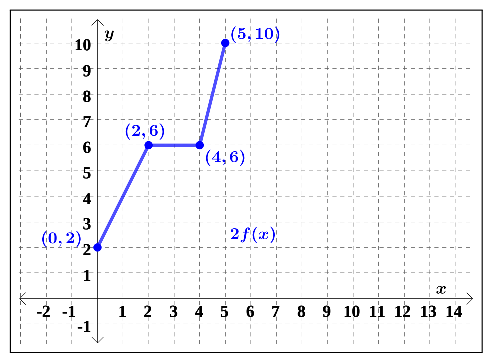


Figure 5.7.20: The graph of $m(x)$.

If we wish to graph $n(x) = \frac{1}{2}f(x)$, we multiply all of the y-coordinates of the points on the graph of $f(x)$ by $\frac{1}{2}$. This creates a "vertical scaling by a factor of $\frac{1}{2}$," as seen in **Figure 5.7.22**.

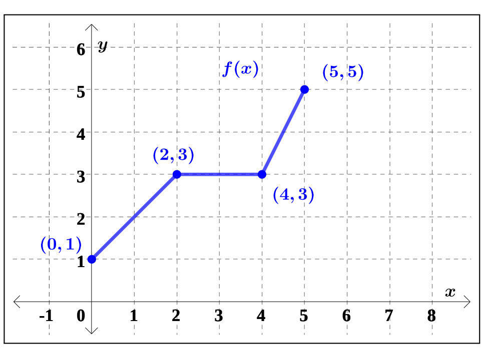


Figure 5.7.21: The graph of $f(x)$.

vertical scaling by a factor of $\frac{1}{2}$ $\longrightarrow$ multiply each y−coordinate by $\frac{1}{2}$

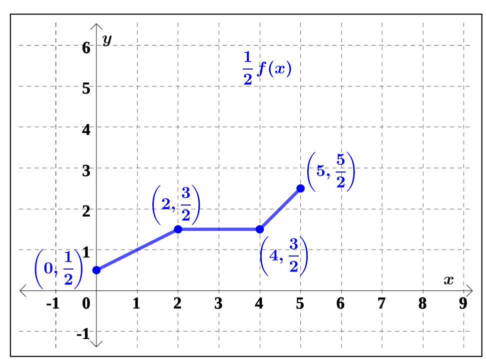


Figure 5.7.22: The graph of $n(x)$.

These results are generalized in the following theorem.

Theorem 5.10 Vertical Scalings

Suppose f is a function and $a > 0$. To graph $y = af(x)$, multiply all of the y-coordinates of the points on the graph of $f(x)$ by a. We say the graph of $f(x)$ has been vertically scaled by a factor of a.

- If $a > 1$, we say the graph of $f(x)$ has undergone

  a vertical stretching (expansion, dilation) by a factor of a.

- If $0 < a < 1$, we say the graph of $f(x)$ has undergone

  a vertical shrinking (compression, contraction) by a factor of $\frac{1}{a}$.

A few remarks about **Theorem 5.10** are in order. First, a note about the verbiage. To the authors, the words 'stretching,' 'expansion,' and 'dilation' all indicate something getting bigger. Hence, "stretched by a factor of 2" makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrinking,' 'compression,' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it "shrinks by a factor of 2" – not "shrinks by a factor of $\frac{1}{2}$." This is why we have written the descriptions "stretching by a factor of a" and "shrinking by a factor of $\frac{1}{a}$" in the statement of the theorem. Second, in terms of inputs and outputs, **Theorem 5.10** says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a.

Example 3

a. Graph $f(x) = x^3$. Plot at least three points.

b. Use the graph of $f(x)$ to graph $g(x) = 4x^3$.

c. Use the graph of $f(x)$ to graph $j(x) = \frac{1}{4}x^3$.

Solution:

a. We know the domain of $f(x)$ is $(-\infty, \infty)$, as it is the parent cubic polynomial function. We choose real number values from -1 to 1 to build **Table 5.27**, in order to graph $f(x)$ in **Figure 5.7.23**, below.

x	$f(x)$	$(x, f(x))$
-1	-1	$(-1,-1)$
$-\frac{1}{2}$	$-\frac{1}{8}$	$\left(-\frac{1}{2},-\frac{1}{8}\right)$
0	0	$(0,0)$
$\frac{1}{2}$	$\frac{1}{8}$	$\left(\frac{1}{2},\frac{1}{8}\right)$
1	1	$(1,1)$

Table 5.27: A chart of ordered pairs on the graph of $f(x) = x^3$.

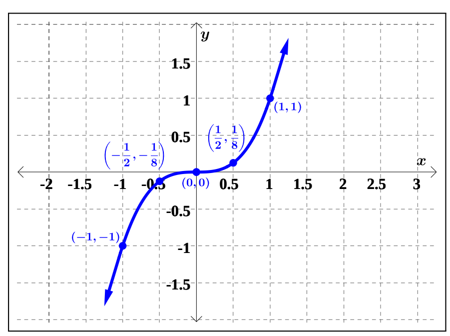


Figure 5.7.23: The graph of $f(x) = x^3$.

b. $g(x) = 4x^3 = 4f(x)$ indicates a vertical scaling by a factor of 4. Because $a = 4 > 1$, then the graph of $f(x)$ undergoes a vertical *stretch* by a factor of 4.

The point $(-1,-1)$ is transformed by multiplying the y-coordinate by 4.

$$(-1,-1) \rightarrow (-1,-4)$$

Similarly,

$$\left(-\frac{1}{2},-\frac{1}{8}\right) \rightarrow \left(-\frac{1}{2},-\frac{1}{2}\right)$$

$$(0,0) \rightarrow (0,0)$$

$$\left(\frac{1}{2},\frac{1}{8}\right) \rightarrow \left(\frac{1}{2},\frac{1}{2}\right)$$

$$(1,1) \rightarrow (1,4)$$

Plotting the new points and connecting them in the same manner as the graph of $f(x)$ results in **Figure 5.7.24**.

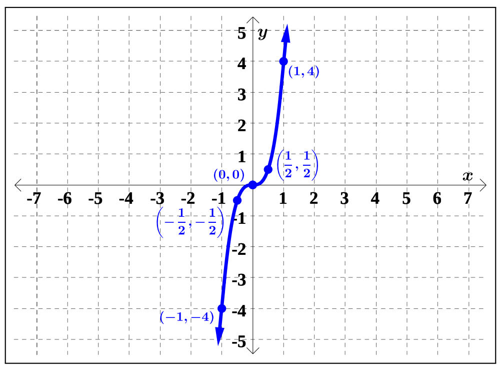


Figure 5.7.24: The graph of $g(x) = 4x^3$.

c. $j(x) = \frac{1}{4}x^3 = \frac{1}{4}f(x)$ indicates a vertical scaling by a factor of $\frac{1}{4}$. Because $0 < a = \frac{1}{4} < 1$, then the graph of $f(x)$ undergoes a vertical *compression* by a factor of *4*.

The point $(-1,-1)$ is transformed by multiplying the y-coordinate by $\frac{1}{4}$.

$$(-1,-1) \rightarrow \left(-1,-\frac{1}{4}\right)$$

Similarly,

$$\left(-\frac{1}{2},-\frac{1}{8}\right) \rightarrow \left(-\frac{1}{2},-\frac{1}{32}\right)$$

$$(0,0) \rightarrow (0,0)$$

$$\left(\frac{1}{2},\frac{1}{8}\right) \rightarrow \left(\frac{1}{2},\frac{1}{32}\right)$$

$$(1,1) \rightarrow \left(1,\frac{1}{4}\right)$$

Plotting the new points and connecting them in the same manner as the graph of $f(x)$ results in **Figure 5.7.25**.

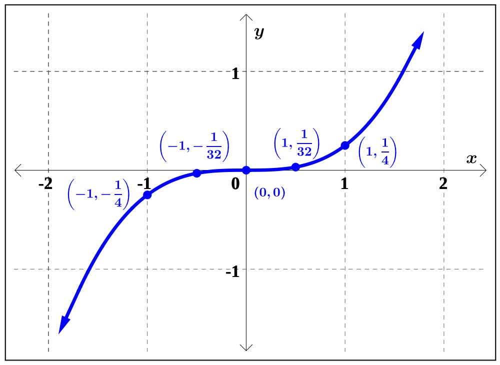


Figure 5.7.25: The graph of $j(x) = \frac{1}{4}x^3$.

N *Although changes were made to the outputs in both parts **b** and **c**, the set of real numbers $(-\infty, \infty)$ does not change when multiplying by a nonzero scalar. Thus, the range remains the same in the graphs of both transformations.*

■

Try It # 3:

a. Graph $f(x) = \sqrt[3]{x}$. Plot at least three points.

b. Use the graph of $f(x)$ to graph $g(x) = 5\sqrt[3]{x}$.

c. Use the graph of $f(x)$ to graph $j(x) = \dfrac{1}{6}\sqrt[3]{x}$.

It is natural to ask what would happen if we multiply the inputs of a function by a positive number. As previously mentioned changes to inputs result in horizontal changes, and, in this instance, the result is a horizontal scaling. The authors leave the topic of horizontal scalings to the reader to explore outside of this text.

Combining Transformations

As with evaluating functions, where order of operations is important, so is the order of transformations.

Order of Transformations

Suppose $f(x)$ is a function. If $A \neq 0$, then to graph

$$g(x) = Af(x+H)+K$$

1. Subtract H from each of the x-coordinates of the points on the graph of $f(x)$. This results in a horizontal shift to the left if $H > 0$ or right if $H < 0$.
2. Multiply the y-coordinates of the points on the graph obtained in Step 1 by A. If $A > 0$, this results in a vertical scaling. If $A < 0$, this results in a vertical scaling and then a reflection across the x-axis.
3. Add K to each of the y-coordinates of the points on the graph obtained in Step 2. This results in a vertical shift up if $K > 0$ or down if $K < 0$.

Due to the fact that the authors have chosen to exclude the discussion of horizontal scalings from this text, reflections across the y-axis are not considered in the above order of transformations.

The steps above are logical, as they follow the order in which we would use to evaluate a function at a given value of x.

When asked to graph $g(x)$ where $g(x) = Af(x+H)+K$, the reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of $f(x)$ into the graph of $g(x)$, in the order outlined above.

If asked to graph the function $g(x) = -|x+3|-8$, how would you use the previously discussed procedure?

First, we identify the parent function, $f(x)$, and graph it. The parent function for $g(x)$ is $f(x) = |x|$.

Second, we identify the horizontal shift, if one exists. There is a horizontal shift of $f(x)$ left 3 units ($H = 3$). Thus, the next graph in the series would be $f_1(x) = f(x+3) = |x+3|$.

Third, we identify the scalar, if one exists. The scalar is $A = -1$. Given that $|A| = 1$, there is no vertical scaling, but because $A < 0$, $f_1(x)$ is reflected across the x-axis. The subsequent graph in the series would be $f_2(x) = -f_1(x) = -|x+3|$.

Last, we identify the vertical shift, if one exists. There is a vertical shift of $f_2(x)$ down 8 units ($K = -8$). The final graph in the series would be $f_3(x) = f_2(x) - 8 = -|x+3|-8$.

Example 4 Given $g(x) = \frac{3}{4}(x-1)^2+5$,

a. Identify the parent function, $f(x)$.

b. Describe, step-by-step, how the graph of $f(x)$ transforms into the graph of $g(x)$.

Solution:

a. $f(x) = x^2$

b. Notice from the rule of $g(x)$, $A = \dfrac{3}{4}, H = -1$, and $K = 5$. Thus, in order, the list of transformations to the graph of $f(x)$ are:

1. Shift right 1 unit, as $H = -1$.
2. As $A = \dfrac{3}{4}$, vertically compress by a factor of $\frac{4}{3}$.
3. Shift up 5 units, as $K = 5$.

■

■ **Example 5** Given $g(x) = 2 - e^x$,

a. Identify the parent function, $f(x)$.

b. Describe, step-by-step, how the graph of $f(x)$ transforms into the graph of $g(x)$.

Solution:

a. $f(x) = e^x$

b. Before we determine the list of transformations, let's rewrite $g(x)$ in a more standard form: $g(x) = -e^x + 2$.

Notice from the rule of $g(x)$, $A = -1, H = 0$, and $K = 2$. Thus, in order, the list of transformations to the graph of $f(x)$ are:

1. Reflection across the x-axis, as $A = -1$. (There is no vertical scaling, as $|A| = 1$.)
2. Shift up 2 units, as $K = 2$.

■

Try It # 4:
Given $g(x) = -3\sqrt{x-2} - 1$,

a. Identify the parent function, $f(x)$.

b. Describe, step-by-step, how the graph of $f(x)$ transforms into the graph of $g(x)$.

Now let's assume we are given a list of transformations to be performed on the graph of a specific 'parent' function. We can use the order of transformations to write the equation of the resulting function.

■ **Example 6** Let $f(x) = x^3$. Find the equation of the function $g(x)$ whose graph is the result of $f(x)$ undergoing the following transformations.

1. Horizontal shift right 1 unit.
2. Reflection across the x-axis.
3. Vertical shift up 8 units.

Solution:

1. A horizontal shift right 1 unit means $H = -1$ and
$$g_1(x) = f(x-1) = (x-1)^3$$
2. A reflection across the x-axis means $A < 0$. Seeing as no vertical scaling is described, $|A| = 1$ and thus, $A = -1$.
$$g_2(x) = -g_1(x) = -(x-1)^3$$
3. A vertical shift up 8 units means $K = 8$, and so the equation of $g(x)$ is
$$\begin{aligned} g(x) &= g_2(x) + 8 \\ g(x) &= -(x-1)^3 + 8 \end{aligned}$$

■

■ **Example 7** If the graph of $f(x) = \dfrac{1}{x}$ is shifted left 5 units, vertically stretched by a factor of 9, reflected across the x-axis, and then shifted down $\frac{1}{2}$ a unit, what is the equation of the resulting graph?

Solution:
We begin by interpreting the meaning of each transformation described.

A shift left 5 units means $H = 5$ and
$$g_1 = f(x+5) = \frac{1}{x+5}$$

A vertical stretch by a factor of 9, and then a reflection across the x-axis means $A = -9$, so
$$g_2 = -9g_1(x) = -9\left(\frac{1}{x+5}\right) = \frac{-9}{x+5}$$

Last, a shift down $\frac{1}{2}$ a unit means $K = -\frac{1}{2}$ and so the equation of $g(x)$ is
$$\begin{aligned} g(x) &= g_2(x) - \frac{1}{2} \\ g(x) &= -\frac{9}{x+5} - \frac{1}{2} \end{aligned}$$

■

Try It # 5:
If the graph of $f(x) = \sqrt{x}$ is shifted right $\frac{2}{3}$ of a unit, vertically compressed by a factor of $\frac{7}{6}$, and then shifted down $\frac{5}{4}$ units, what is the equation of the resulting graph?

Performing Function Arithmetic

In the section on Relations and Functions, we used function notation to make sense of expressions such as "$f(x)+2$" and "$2f(x)$" for a given function $f(x)$. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply, and divide functions, using the arithmetic we already know for real numbers.

Function Arithmetic

Suppose f and g are functions and x is in both the domain of $f(x)$ and the domain of $g(x)$, that is, x is in the intersection of the two domains.

- The **sum** of f and g, denoted $\boldsymbol{f+g}$, is the function defined by
$$(f+g)(x) = f(x)+g(x)$$
- The **difference** of f and g, denoted $\boldsymbol{f-g}$, is the function defined by
$$(f-g)(x) = f(x)-g(x)$$
- The **product** of f and g, denoted $\boldsymbol{fg}$, is the function defined by
$$(fg)(x) = (f\cdot g)(x) = f(x)g(x)$$
- The **quotient** of f and g, denoted $\boldsymbol{\frac{f}{g}}$, is the function defined by
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$
provided $g(x) \neq 0$.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Notice that while the formula $(f+g)(x) = f(x)+g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function, $f(x)+g(x)$, as the sum of the real number outputs of $f(x)$ and $g(x)$.

■ **Example 8** Given $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \frac{1}{x}$, compute the following.

a. $(f+g)(-1)$

b. $(fg)(2)$

c. $(g-f)(x)$

d. $\left(\frac{g}{f}\right)(x)$

Solution:
The domain of $f(x)$ is $(-\infty,\infty)$, because $f(x)$ is a polynomial function. The domain of $g(x)$ is $(-\infty,0)\cup(0,\infty)$, as $g(x)$ has a denominator which cannot be zero. Thus, the intersection of these domains is $(-\infty,0)\cup(0,\infty)$.

a. $x=-1$ is in the intersection of the domains of $f(x)$ and $g(x)$, therefore $(f+g)(-1)$ is defined.

$$\begin{aligned}(f+g)(-1) &= f(-1)+g(-1)\\ &= \left[6(-1)^2-2(-1)\right]+\left[3-\frac{1}{(-1)}\right]\\ &= 8+4\\ &= 12\end{aligned}$$

b. $x=2$ is in the intersection of the domains of $f(x)$ and $g(x)$, therefore $(fg)(2)$ is defined.

$$\begin{aligned}(fg)(2) &= f(2)\cdot g(2)\\ &= \left[6(2)^2-2(2)\right]\cdot\left[3-\frac{1}{(2)}\right]\\ &= [20]\cdot\left[\frac{5}{2}\right]\\ &= 50\end{aligned}$$

c. In order for $(g-f)(x)$ to be defined, x must be in the intersection of the domains of $f(x)$ and $g(x)$.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= \left[3-\frac{1}{x}\right]-\left[6x^2-2x\right]\\ &= 3-\frac{1}{x}-6x^2+2x\\ &= \left(\frac{3}{1}\right)\left(\frac{x}{x}\right)-\frac{1}{x}-\left(\frac{6x^2}{1}\right)\left(\frac{x}{x}\right)+\left(\frac{2x}{1}\right)\left(\frac{x}{x}\right)\\ &= \frac{3x}{x}-\frac{1}{x}-\frac{6x^3}{x}+\frac{2x^2}{x}\\ &= \frac{3x-1-6x^3+2x^2}{x}\\ &= \frac{-6x^3+2x^2+3x-1}{x},\ \text{where } x \text{ is in } (-\infty,0)\cup(0,\infty)\end{aligned}$$

d. In order for $\left(\frac{g}{f}\right)(x)$ to be defined, x must be in the intersection of the domains of $f(x)$ and $g(x)$ AND $f(x) \neq 0$. We know the intersection is $(-\infty, 0) \cup (0, \infty)$, but we need to find where $f(x) \neq 0$. Thus,

$$6x^2 - 2x \neq 0$$
$$2x(3x-1) \neq 0$$
$$2x \neq 0 \qquad \text{AND} \qquad 3x - 1 \neq 0$$
$$x \neq 0 \qquad \text{AND} \qquad x \neq \frac{1}{3}$$

Therefore, for $\left(\frac{g}{f}\right)(x)$ to be defined x must be in $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x}$$

If simplified, we have

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \\
&= \frac{\left(\frac{x}{x}\right)\left(\frac{3}{1}\right) - \left(\frac{1}{x}\right)\left(\frac{1}{1}\right)}{6x^2 - 2x} \\
&= \frac{\frac{3x-1}{x}}{6x^2 - 2x} \\
&= \left(\frac{3x-1}{x}\right) \cdot \left(\frac{1}{6x^2 - 2x}\right) \\
&= \frac{(3x-1)(1)}{x(6x^2 - 2x)} \\
&= \frac{3x-1}{2x^2(3x-1)} \\
&= \frac{\cancel{3x-1}}{2x^2\cancel{(3x-1)}} \\
&= \frac{1}{2x^2}, \text{ where } x \text{ is in } (-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)
\end{aligned}$$

Please note the importance of identifying the domain of a function *before* simplifying its expression. If we had waited to compute the domain of $\dfrac{g(x)}{f(x)}$ until after simplifying, we would just have the formula $\dfrac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty,0)\cup(0,\infty)$, because the other troublesome number, $x=\frac{1}{3}$, was 'divided out.'

Because $\left(\dfrac{g}{f}\right)(x)$ is a rational function, recall there will be a hole in the graph at $x=\frac{1}{3}$ and a vertical asymptote at $x=0$.

∎

Throughout this chapter we have seen function arithmetic in action. Thus, the authors leave it to the reader to practice function arithmetic with no further examples.

Try It # 6:
Given $f(x)=\sqrt{x+3}$ and $g(x)=2x-1$, compute the following.

a. $(f-g)(6)$

b. $\left(\dfrac{f}{g}\right)(0)$

c. $(g+f)(x)$

d. $(gf)(x)$

Finding the Composition of Functions

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: the cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two relationships as functions. The function $C(T)$ gives the cost, C, of heating a house for a given average daily temperature in T degrees Fahrenheit. The function $T(d)$ gives the average daily temperature on day d of the year. Then, for any given day, Cost $=C(T(d))$ means that the cost depends on the temperature, which in turn depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C(T(5))$.

$$\overbrace{C(\underset{\substack{\uparrow\\ \text{Temperature on day 5}}}{T(5)})}^{\text{Cost for the temperature}}$$

Performing algebraic operations on functions combines them into a new function, but we can also create functions by **composing** functions. The process of combining functions so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**.

Definition

Suppose f and g are two functions. The **composition** of f with g, denoted $\boldsymbol{f \circ g}$, is defined by

$$(f \circ g)(x) = f(g(x)),$$

provided x is an element of the domain of the function g and $g(x)$ is an element of the domain of the function f. ▪

In this definition the left-hand side is read as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number.

It is also important to understand order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then, the function f takes $g(x)$ as its input and yields an output $f(g(x))$.

$g(x)$, the output of g
is the input of f

$$(f \circ g)(x) = f\left(\overline{g(\underline{x})}\right)$$

x is the input of g

In the expression $f(g(x))$, the function g is often called the 'inside' function, while the function f is often called the 'outside' function.

Abstractly, we have

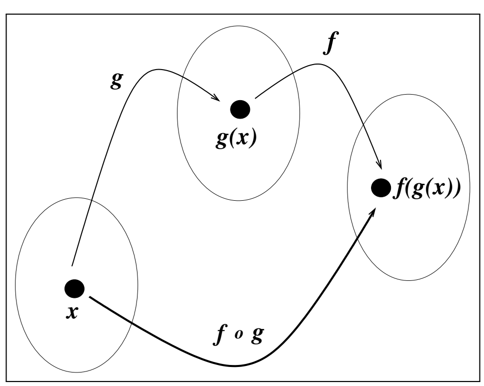


Figure 5.7.26: A visual representation of the composition of two functions, $(f \circ g)(x)$.

Multiplication of functions is not the same as composition of functions. Pay close attention to the operator given.

$$(f \cdot g)(x) \neq (f \circ g)(x)$$

For the purposes of this text, when composing functions x is assumed to be in the domain of the composition. The authors will not compute the domain of a composition at this time, as similar domains have been discussed in previous sections.

Example 9 Using the functions provided, compute $f(g(x))$ and $g(f(x))$.

$$f(x) = 2x + 1 \qquad\qquad g(x) = -x + 3$$

Solution:
Let's begin with $f(g(x))$, by substituting $g(x)$ into $f(x)$ for x.

$$\begin{aligned} g(x) &= -x + 3 \\ f(g(x)) &= 2(-x + 3) + 1 \\ &= -2x + 6 + 1 \\ &= -2x + 7 \end{aligned}$$

Now, we move to $g(f(x))$, by substituting $f(x)$ into $g(x)$ for x.

$$\begin{aligned} f(x) &= 2x + 1 \\ g(f(x)) &= -(2x + 1) + 3 \\ &= -2x - 1 + 3 \\ &= -2x + 2 \end{aligned}$$

■

In general, $(f \circ g)(x)$ and $(g \circ f)(x)$ are different. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x.

Because $g(f(x)) \neq f(g(x))$ for all x, the operation of function composition is not commutative.

Example 10 Given $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$, compute the following.

a. $(g \circ f)(1)$

b. $(g \circ g)(6)$

c. $(h \circ g)(x)$

d. $(f \circ f)(x)$

Solution:

a. Using the definition, $(g \circ f)(1) = g(f(1))$. We first calculate $f(1)$:

$$\begin{aligned} f(1) &= (1)^2 - 4(1) \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

So,

$$\begin{aligned} (g \circ f)(1) &= g(f(1)) \\ &= g(-3) \\ &= 2 - \sqrt{-3+3} \\ &= 2 - \sqrt{0} \\ (g \circ f)(1) &= 2 \end{aligned}$$

b. Again, the definition tells us $(g \circ g)(6) = g(g(6))$. That is, we evaluate $g(x)$ at $x = 6$, and then plug the result back into $g(x)$. Given

$$\begin{aligned} g(6) &= 2 - \sqrt{6+3} \\ &= 2 - \sqrt{9} \\ &= -1, \end{aligned}$$

then

$$\begin{aligned} (g \circ g)(6) &= g(g(6)) \\ &= g(-1) \\ &= 2 - \sqrt{-1+3} \\ (g \circ g)(6) &= 2 - \sqrt{2} \end{aligned}$$

c. We know $(h \circ g)(x) = h(g(x))$. Thus, we insert the expression $g(x)$ into h for x to get

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) \\ &= h(2 - \sqrt{x+3}) \\ &= \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1} \\ (h \circ g)(x) &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}} \end{aligned}$$

d. We know $(f \circ f)(x) = f(f(x))$. Thus, we insert the expression $f(x)$ into f for x to get

$$\begin{aligned}
(f \circ f)(x) &= f(f(x)) \\
&= f(x^2 - 4x) \\
&= \left(x^2 - 4x\right)^2 - 4\left(x^2 - 4x\right) \\
&= (x^2 - 4x)(x^2 - 4x) - 4(x^2 - 4x) \\
&= x^4 - 4x^3 - 4x^3 + 16x^2 - 4x^2 + 16x \\
(f \circ f)(x) &= x^4 - 8x^3 + 12x^2 + 16x
\end{aligned}$$

■

Try It # 7:
Given the pair of functions $f(x) = 2x + 3$ and $g(x) = x^2 - 9$, compute and simplify the following.

a. $(f \circ g)(8)$

b. $(g \circ f)(x)$

c. $(f \circ f)(-10)$

d. $(g \circ g)(x)$

When working tables of function values, we read input and output values of the composition from the table entries. We evaluate the inside function first, and then use the output of the inside function as the input of the outside function.

■ **Example 11** Using **Table 5.28**, evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
0	2	11
1	6	3
2	8	5
3	4	2
4	1	7
5	9	0

Table 5.28: A chart of x-values, with corresponding output values from $f(x)$ and $g(x)$.

Solution:
To evaluate $f(g(3))$, we start from the inside with the input value, $x = 3$. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3) = 2$. We then use this result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Last, using the table that defines the function f, we determine that $f(2) = 8$.

$$g(3) = 2$$
$$\text{and}$$
$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression, $f(3)$, using the table: $f(3) = 4$. Then, using the table for function g, we can evaluate $g(f(3))$.

$$g(f(3)) = g(4) = 7$$

■

Try It # 8:
Using **Table 5.28**, evaluate each of the following.

a. $f(g(5))$

b. $g(f(4))$

c. $(f \circ f)(0)$

d. $(g \circ g)(2)$

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we used when given functions as tables. We read the input and output values, but this time, from the points on the coordinate plane.

■ **Example 12** Using **Figure 5.7.27**, evaluate $f(g(1))$.

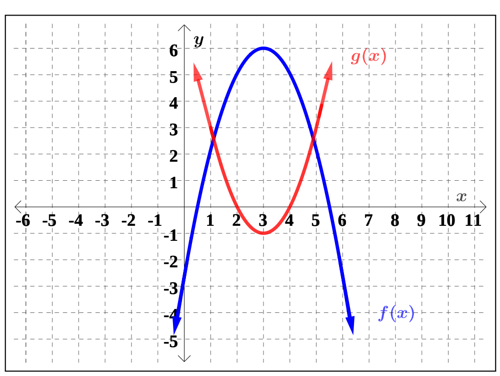


Figure 5.7.27: The coordinate plane with two curves $f(x)$ and $g(x)$.

Solution:

To evaluate $f(g(1))$, we start with the inside evaluation. We evaluate $g(1)$ using the graph of $g(x)$, finding the point where $x = 1$, and noting the output value of the graph at that input value. Here, $g(1) = 3$. (See **Figure 5.7.28**.)

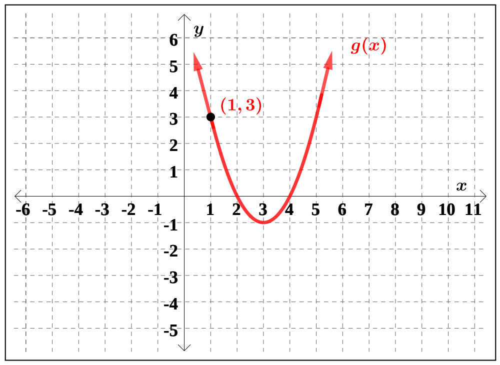


Figure 5.7.28: The coordinate plane with the curve $g(x)$. The point $(1, 3)$ is indicated.

We use this function value as the input for function f: $f(g(1)) = f(3)$.

We evaluate $f(3)$ using the graph of $f(x)$, finding the point where $x = 3$, and noting the output value of the graph at that input value. Here, $f(3) = 6$. (See **Figure 5.7.29**.)

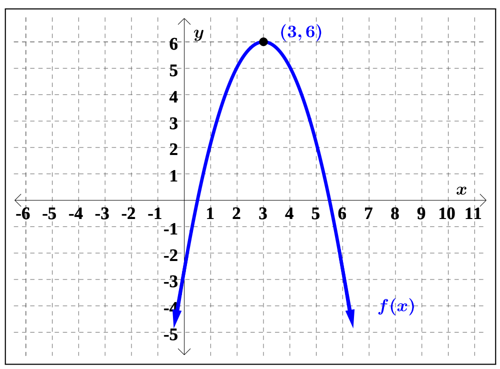


Figure 5.7.29: The coordinate plane with the curve $f(x)$. The point $(3, 6)$ is indicated.

Thus, $f(g(1)) = 6$.

■

Try It # 9:
Using **Figure 5.7.27**, evaluate $g(f(2))$.

Example 13 Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution:
The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven. This means:

$$\text{number of miles} = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons} = g(\text{number of miles})$$

$\underline{f(g(y))}$:

The expression $g(y)$ takes miles as the input and gives a number of gallons as the output. The function $f(x)$ requires a number of hours as the input, so trying to input a number of gallons does not make sense. Therefore, the expression $f(g(y))$ is meaningless.

$\underline{g(f(x))}$:

The expression $f(x)$ takes hours as the input and gives a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. Therefore, the expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Try It # 10:
The function $c(s)$ gives the number of calories burned while completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

In calculus, it is sometimes necessary to decompose a complicated function. In other words, we need to write the function as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Example 14 Write $f(x) = \sqrt{5 - x^2}$ as the composition of two functions.

Solution:
We are looking for two functions, $g(x)$ and $h(x)$, such that $f(x) = g(h(x))$. To do this, we look for a function inside of another function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

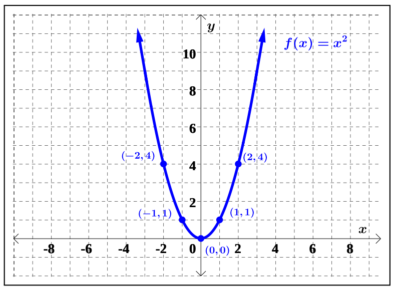
$$g(h(x)) = g\left(5 - x^2\right) = \sqrt{5 - x^2} \checkmark$$

Try It # 11:

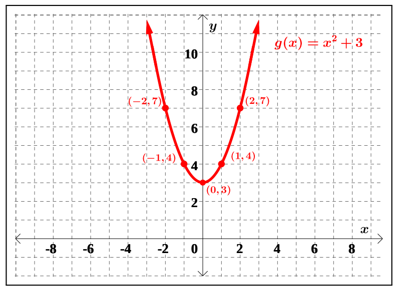
Write $f(x) = \dfrac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions.

Try It Answers

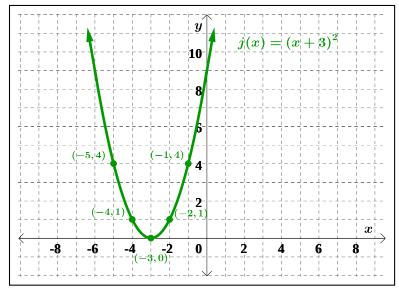
1. a.



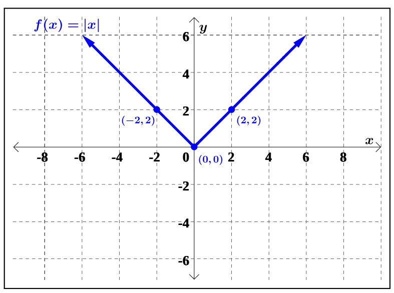
b.



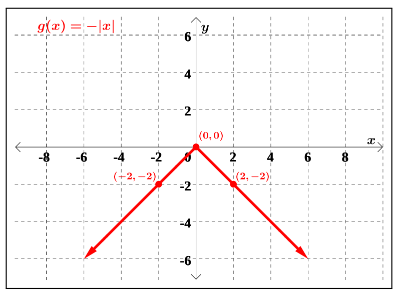
c.



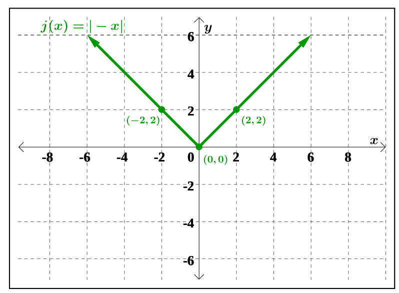
2. a.



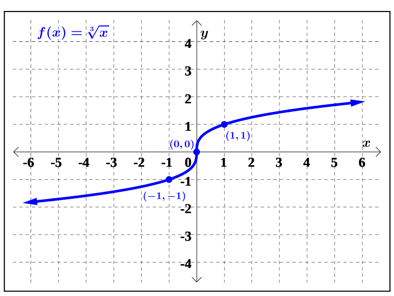
b.



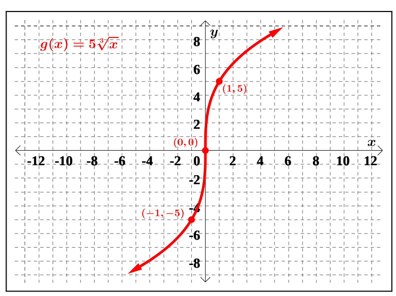
c.



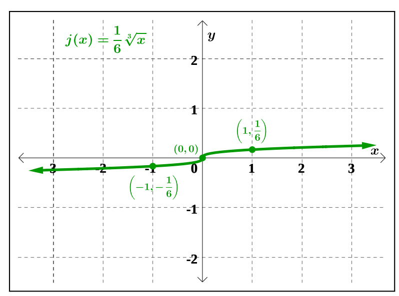
3. a.



b.



c.



4. **a.** $f(x) = \sqrt{x}$

b. 1. Horizontal shift right 2 units.

2. Vertically stretch by a factor of 3, and then reflect across the x-axis.

3. Vertical shift down 1 unit.

5. $g(x) = \frac{6}{7}\sqrt{x - \frac{2}{3}} - \frac{5}{4}$

6. **a.** -8

b. $-\sqrt{3}$

c. $2x - 1 + \sqrt{x+3}$, where x is in $[-3, \infty)$

d. $(2x-1)\left(\sqrt{x+3}\right)$, where x is in $[-3, \infty)$

7. **a.** 113

b. $4x^2 + 12x$

c. -31

d. $x^4 - 18x^2 + 72$

8. **a.** 2

b. 3

c. 8

d. 0

9. 3

10. The number of calories burned by doing sit-ups for 3 minutes.

11. One decomposition is $f(x) = g(h(x))$ where $h(x) = 3 - \sqrt{4 + x^2}$ and $g(x) = \frac{4}{x}$.

EXERCISES

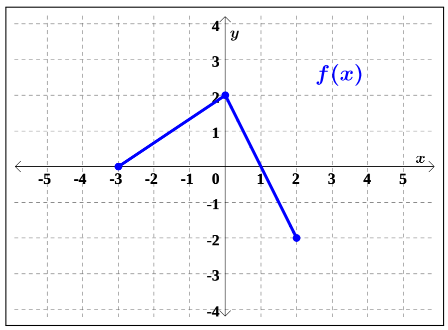
BASIC SKILLS PRACTICE (Answers)

For Exercises 1 - 4, write the equation for the parent function described.

1. Linear Function
2. Absolute Value Function
3. Square Root Function
4. Exponential Decay Function

Vertical and Horizontal Shifts

For Exercises 5 - 8, given the graph of $f(x)$ below, sketch a graph of each of the following.



5. $f(x)+3$
6. $f(x+3)$
7. $f(x)-2$
8. $f(x-2)$

For Exercises 9 - 12, state the appropriate parent function, $f(x)$, and the list of transformations (in the correct order) needed to graph the function.

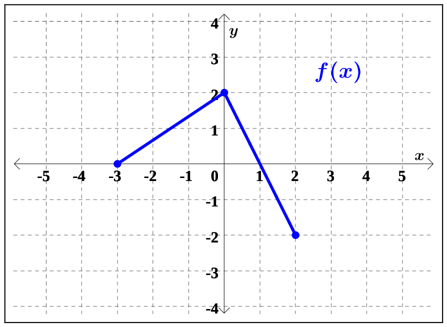
9. $g(x) = \sqrt[3]{x} - 6$
10. $h(x) = e^x + 1$
11. $j(x) = (x-7)^3$
12. $k(x) = |x+4|$

For Exercises 13 - 16, write the equation of the function, $g(x)$, whose graph is the result of $f(x)$ undergoing the given transformation.

13. $f(x) = x^2$ shifted down 8 units
14. $f(x) = |x|$ shifted up 20 units
15. $f(x) = 3^x$ shifted right 5 units
16. $f(x) = x^3$ shifted left 8 units

Reflections

For Exercises 17 - 18, given the graph of $f(x)$ below, sketch a graph of each of the following.



17. $f(-x)$

18. $-f(x)$

For Exercises 19 - 20, state the appropriate parent function, $f(x)$, and the list of transformations (in the correct order) needed to graph the function.

19. $g(x) = -x^2$

20. $h(x) = \sqrt{-x}$

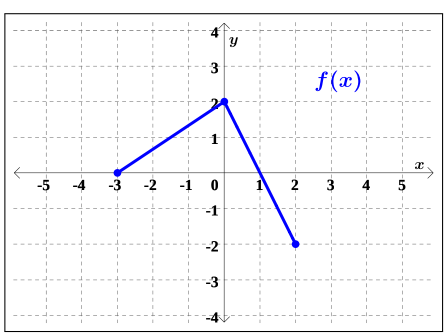
For Exercises 21 - 22, write the equation of the function, $g(x)$, whose graph is the result of $f(x)$ undergoing the given transformation.

21. $f(x) = \dfrac{1}{x^2}$ reflected across the x-axis

22. $f(x) = 10^x$ reflected across the y-axis

Vertical Scalings

For Exercises 23 - 26, given the graph of $f(x)$ below, sketch a graph of each of the following.



23. $2f(x)$

24. $\dfrac{1}{2}f(x)$

25. $0.25f(x)$

26. $\dfrac{3}{2}f(x)$

For Exercises 27 - 30, state the appropriate parent function, $f(x)$, and the list of transformations (in the correct order) needed to graph the function.

27. $g(x) = 6|x|$
28. $h(x) = \frac{5}{3} \cdot 10^x$
29. $j(x) = 3.8x^3$
30. $k(x) = \frac{1}{7}x$

For Exercises 31 - 32, write the equation of the function, $g(x)$, whose graph is the result of $f(x)$ undergoing the given transformation.

31. $f(x) = \frac{1}{x}$ vertically compressed by a factor of 6
32. $f(x) = e^x$ vertically stretched by a factor of 8

Function Arithmetic

For Exercises 33 - 40, use the given functions to calculate each operation and simplify, if possible.

$f(x) = 3x + 9$ $\quad g(x) = -x^2 + 4$ $\quad h(x) = x^3 + x - 1$ $\quad j(x) = |x|$

33. $(f + g)(3)$
34. $(h - j)(-2)$
35. $(gh)(0)$
36. $\left(\frac{h}{f}\right)(-1)$
37. $(g + h)(x)$
38. $(j - f)(x)$
39. $(fg)(x)$
40. $\left(\frac{j}{g}\right)(x)$

Compositions of Functions

For Exercises 41 - 44, given $f(x) = 2^x$ and $g(x) = x - 8$, compute each of the following.

41. $(f \circ g)(7)$
42. $(g \circ f)(3)$
43. $(f \circ f)(2)$
44. $(g \circ g)(-5)$

For Exercises 45 - 50, given the values in the table, compute each of the following.

x	$f(x)$	$g(x)$
-2	2	-1
-1	0	1
0	1	-2
1	-2	2
2	-1	0

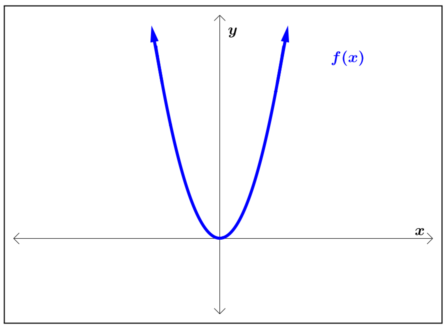
45. $(f \circ g)(-2)$
46. $(f \circ g)(0)$
47. $(g \circ f)(1)$
48. $(g \circ f)(2)$
49. $(f \circ f)(-1)$
50. $(g \circ g)(1)$

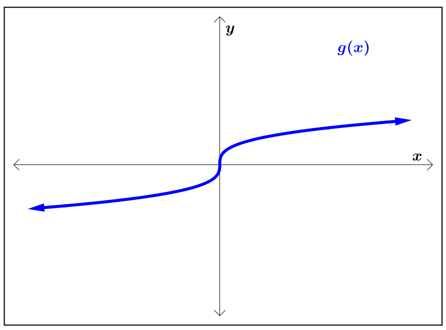
For Exercises 51 - 54, write $f(x)$ as the composition of two functions.

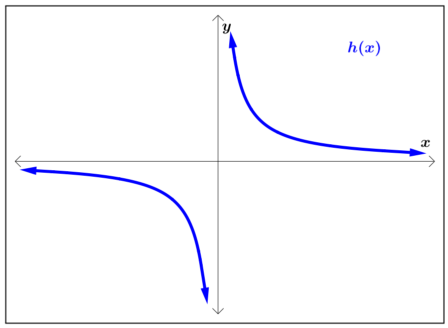
51. $f(x) = \sqrt[4]{x-9}$
52. $f(x) = \dfrac{3}{x+6}$
53. $f(x) = e^{7x+10}$
54. $f(x) = \dfrac{1}{(2x+1)^2}$

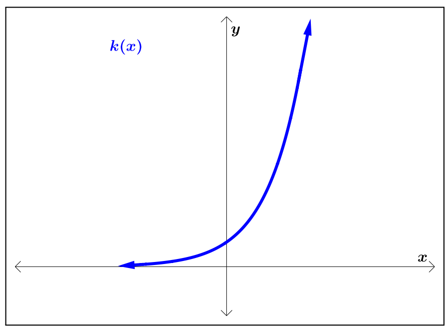
Intermediate Skills Practice (Answers)

For Exercises 55 - 58, write the equation and name for the graphed parent function.

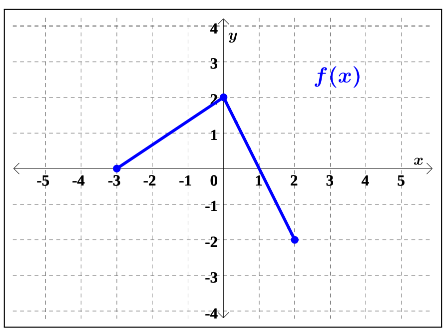
55. 

56. 

57. 

58. 

For Exercises 59 - 62, given the graph of $f(x)$ below, sketch a graph of each of the following.



59. $f(x+1)-2$
60. $f(x-3)+1$
61. $2f(x)-1$
62. $-0.5f(x)$

For Exercises 63 - 66, state the appropriate parent function, $f(x)$, and the list of transformations (in the correct order) needed to graph the function.

63. $g(x) = -2^x + 3$
64. $h(x) = \sqrt{x+4} - 7$
65. $j(x) = 3e^{x-6}$
66. $k(x) = 0.25(x-1)^3$

For Exercises 67 - 70, write the equation of the function, $g(x)$, whose graph is the result of $f(x)$ undergoing the given transformation.

67. $f(x) = |x|$ shifted left 3 units and then shifted down 8 units
68. $f(x) = 5^x$ shifted right 7 units and then reflected over the x-axis
69. $f(x) = \sqrt[3]{x}$ vertically stretched by a factor of 9 and then shifted up 1 unit
70. $f(x) = x^3$ reflected over the x-axis and then vertically compressed by a factor of 2

For Exercises 71 - 78, use the given functions to calculate each operation and simplify, if possible.

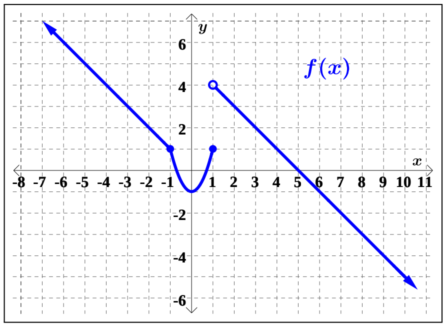
$f(x) = e^{-x}$ $\quad g(x) = 3e^x$ $\quad h(x) = \sqrt[3]{6-x}$ $\quad j(x) = \sqrt{x+1}$

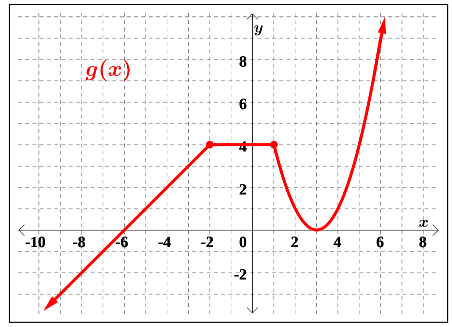
71. $(f+g)(0)$
72. $(h-j)(3)$
73. $(gh)(0)$
74. $\left(\dfrac{h}{f}\right)(-2)$
75. $(g+h)(x)$
76. $(j-f)(x)$
77. $(fg)(x)$
78. $\left(\dfrac{j}{g}\right)(x)$

For Exercises 79 - 82, given $f(x) = 5x^2 + x$ and $g(x) = x+1$, compute each of the following.

79. $(f \circ g)(x)$
80. $(g \circ f)(x)$
81. $(f \circ f)(x)$
82. $(g \circ g)(x)$

For Exercises 83 - 88, given the graphs of $f(x)$ and $g(x)$, compute each of the following.





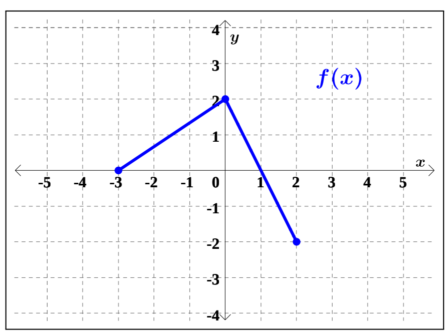
83. $(f \circ g)(3)$
84. $(f \circ g)(0)$
85. $(g \circ f)(4)$
86. $(g \circ f)(0)$
87. $(f \circ f)(-2)$
88. $(g \circ g)(-3)$

For Exercises 89 - 92, write $f(x)$ as the composition of two functions.

89. $f(x) = 3\sqrt{8-x}$
90. $f(x) = \dfrac{7x^2+11}{9}$
91. $f(x) = e^{x^2} + 100$
92. $f(x) = \dfrac{17}{4}(x-2)^3 - 11$

Mastery Practice (Answers)

93. Given the graph of $f(x)$ shown below, sketch the graph of $-f(x-1)+2$.



(

94. State the appropriate parent function, $f(x)$, and the list of transformations (in the correct order) needed to graph the function $g(x) = -\dfrac{1}{8}\left(\dfrac{1}{x-5}\right) + 6$.

95. State the appropriate parent function, $f(x)$, and the list of transformations (in the correct order) needed to graph the function $g(x) = 2 \cdot 10^{x+4} - 1$.

96. Write the equation of the function, $g(x)$, whose graph is the result of $f(x) = \dfrac{1}{x^2}$ shifted left 3 units, vertically stretched by a factor of 6, reflected over the x-axis, and then shifted down 2 units.

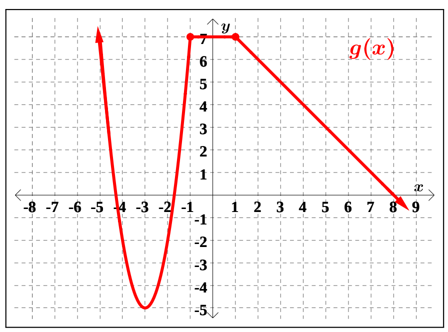
97. Write the equation of the function, $g(x)$, whose graph is the result of $f(x) = |x|$, shifted right 2 units, vertically compressed by a factor of 12, and then shifted up 1 unit.

For Exercises 98 - 109, use the given functions to calculate each operation and simplify, if possible.

$f(x) = 7 - 4x$ $\quad g(x) = -3x^2 + 2x$ $\quad h(x) = 5e^{x-2}$ $\quad j(x) = \dfrac{1}{x+6}$

98. $(j + f)(-1)$
99. $(h - g)(2)$
100. $(fj)(19)$
101. $\left(\dfrac{g}{h}\right)(0)$
102. $f(h(3))$
103. $(j \circ j)(0)$
104. $(f + g)(x)$
105. $(j - h)(x)$
106. $(fh)(x)$
107. $\left(\dfrac{j}{g}\right)(x)$
108. $(g \circ f)(x)$
109. $g(g(x))$

110. Given $f(x) = x^2 + 3$ and $g(x)$ is given by the graph below, compute $g(f(0))$ and $(f \circ g)(0)$.



111. Write $f(x) = \dfrac{8\sqrt{7x+6}}{3\sqrt{7x+6} - 32}$ as the composition of two functions.

112. The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in their system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would determine when the patient will be at a pain level of 4?
- Evaluate $A(m(4))$.
- Evaluate $m(A(4))$.
- Evaluate $A(m(t)) = 4$.
- Evaluate $m(A(d)) = 4$.

Communication Practice (Answers)

113. Explain when the order matters in graphed transformations.

114. Explain the difference between $(f \cdot g)(x)$ and $(f \circ g)(x)$, mathematically.

5.8 Inverse Functions and Logarithms

© Photo by Susan Giefer, 2019

If we think of a function as a process, then we can seek another function which might reverse that process. As in real life, we will determine that some processes (like putting on socks and shoes) are reversible, while other processes (like cooking a steak) are not.

Learning Objectives:
In this section, you will learn about the properties and characteristics of logarithmic functions. Upon completion you will be able to:

- Justify whether or not a function is one-to-one.
- Identify a logarithmic function.
- Convert equations between exponential and logarithmic form.
- Apply the laws and properties of logarithms to expand, condense, and simplify logarithmic expressions.
- Memorize the graph of the parent logarithmic function base a.
- Determine the domain of a logarithmic function, using interval notation.
- Solve equations involving exponential functions with different bases.
- Solve equations involving logarithms.
- Use exponential and logarithmic functions to model and solve real-world applications.

Defining an Inverse Function

Let's begin by discussing a very basic function which is reversible, $f(x) = 3x+4$. Thinking of f as a process, we start with an input, x, and apply two steps:

1. Multiply by 3
2. Add 4

To reverse this process, we seek a function, g, which will undo each of these steps and take the output from f, $3x+4$, and return the input, x. If we think of the real-world reversible two step process of first putting on socks, and then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the second step of f first. That is, function g should

1. *Subtract* 4
2. *Divide* by 3

Following this procedure, we get $g(x) = \dfrac{x-4}{3}$.

Let's check to see if the function g reverses the process of $f(x)$. If $x = 5$, then $f(5) = 3(5)+4 = 15+4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f.

To check that g reverses the process of $f(x)$ for all x in the domain of f, we take the generic output from f, $f(x) = 3x+4$, and substitute into g. That is $g(f(x)) = f(3x+4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \dfrac{x-4}{3}$, and substitute that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right)+4 = (x-4)+4 = x$. Using the language of composition, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively.

Abstractly, we can visualize the relationship between f and g in **Figure 5.8.2** below.

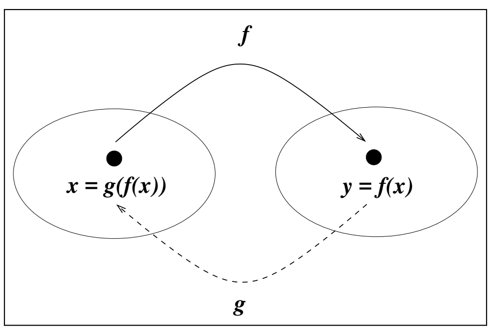


Figure 5.8.2: A visual representation of the functions f and g 'undoing' each others processes.

From the diagram above we see that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the following definition.

Definition

Suppose f and g are two function such that

1. $(g \circ f)(x) = x$ for all x in the domain of f and
2. $(f \circ g)(x) = x$ for all x in the domain of g,

then f and g are **inverses** of each other, and the functions f and g are said to be **invertible**. ▪

Properties of Inverse Functions

Suppose $f(x)$ and $g(x)$ are inverse functions.

1. There is exactly one inverse function for $f(x)$, denoted $f^{-1}(x)$, which is equal to $g(x)$.
2. $f(a) = b$ if and only if $g(b) = a$.
3. (a,b) is on the graph of $f(x)$ if and only if (b,a) is on the graph of $g(x)$.
4. The range of $f(x)$ is the domain of $g(x)$, and the domain of $f(x)$ is the range of $g(x)$.
5. The graph of $g(x) = f^{-1}(x)$ is the reflection of the graph of $f(x)$ across the line $y = x$.

The notation f^{-1} is an unfortunate choice, as you might think of this as $\frac{1}{f}$. This is definitely not the case. Recall $f(x) = 3x+4$ has as its inverse $f^{-1}(x) = \dfrac{x-4}{3}$, which is certainly different than $\dfrac{1}{f(x)} = \dfrac{1}{3x+4}$.

Besides using compositions, one way to determine if a function is invertible is to determine if the function is **one-to-one**.

Definition

A function $f(x)$ is said to be **one-to-one** if $f(x)$ matches different inputs to different outputs. Equivalently, $f(x)$ is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$ (or if $c \neq d$ then $f(c) \neq f(d)$). ▪

Graphically, we detect one-to-one functions using the test below.

Theorem 5.11 The Horizontal Line Test

A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function *passes* the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise we say the graph of the function *fails* the Horizontal Line Test.

Theorem 5.12 Equivalent Conditions for Invertibility

Suppose $f(x)$ is a function. The following statements are equivalent.

- $f(x)$ is invertible.
- $f(x)$ is one-to-one.
- The graph of $f(x)$ passes the Horizontal Line Test.

▪ **Example 1** Determine if the following functions are one-to-one, using the Horizontal Line Test.

a. $g(x) = \dfrac{2x}{1-x}$

b. $h(x) = x^2 - 2x + 4$

Solution:

a. We begin by graphing $g(x) = \dfrac{2x}{1-x}$ in our calculator, as seen in **Figure 5.8.3**.

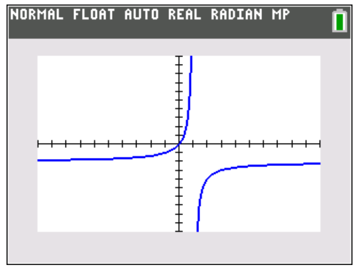


Figure 5.8.3: Calculator screenshot of $g(x)$.

One can imagine any horizontal line drawn through the graph will only intersect the curve at most once. Therefore, the graph of $g(x)$ passes the Horizontal Line Test, and $g(x)$ is a one-to-one function. Thus, $g(x)$ is also invertible.

b. Again, we begin by graphing $h(x) = x^2 - 2x + 4$ in our calculator. (See **Figure 5.8.4**.)

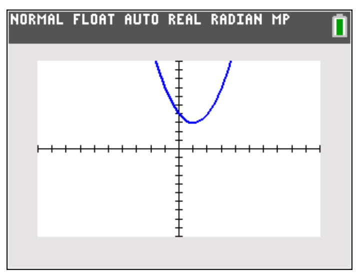


Figure 5.8.4: Calculator screenshot of $h(x)$.

We see immediately from the graph that $h(x)$ fails the Horizontal Line Test, as there are several horizontal lines which cross the graph more than once (and all we need is one horizontal line to cross more than once). Thus, $h(x)$ is NOT a one-to-one function and, therefore, not invertible.

▪

▪ **Example 2** Determine whether or not $f(x) = \dfrac{1-2x}{5}$ and $g(x) = -\dfrac{5}{2}x + \dfrac{1}{2}$ are inverses of each other.

Solution:
To determine if f and g are inverses, we must compute $(g \circ f)(x)$ and $(f \circ g)(x)$. We first check that $(g \circ f)(x) = x$ for all x in the domain of $f(x)$, which is $(-\infty, \infty)$.

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g\left(\frac{1-2x}{5}\right) \\
&= -\frac{5}{2}\left(\frac{1-2x}{5}\right) + \frac{1}{2} \\
&= -\frac{5}{2}\left(\frac{1}{5} - \frac{2}{5}x\right) + \frac{1}{2} \\
&= -\frac{1}{2} + x + \frac{1}{2} \\
&= x \checkmark
\end{aligned}$$

We now check that $(f \circ g)(x) = x$ for all x in the domain of $g(x)$, which is also $(-\infty, \infty)$.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(-\frac{5}{2}x + \frac{1}{2}\right) \\
&= \frac{1 - 2\left(-\frac{5}{2}x + \frac{1}{2}\right)}{5} \\
&= \frac{1 + 5x - 1}{5} \\
&= \frac{5x}{5} \\
&= x \checkmark
\end{aligned}$$

Thus, $f(x)$ and $g(x)$ are inverses of each other.

To check our result, we could graph $f(x)$ and $g(x)$ on the same set of axes and confirm their graphs are reflections of each other across the line $y = x$. Another way to verify our results is to make a table of values for each function and confirm that if (a, b) is on $f(x)$, then (b, a) is on $g(x)$.

Instead of graphing, we will first choose values for x to build **Table 5.29** with $f(x)$ values. Then, we will use the function values found in **Table 5.29** as x-values to build **Table 5.30** with $g(x)$ values, and verify $f(x)$ and $g(x)$ are indeed inverses.

x	$f(x)$	$(x, f(x))$
-5	$\frac{11}{5}$	$\left(-5, \frac{11}{5}\right)$
-3	$\frac{7}{5}$	$\left(-3, \frac{7}{5}\right)$
-1	$\frac{3}{5}$	$\left(-1, \frac{3}{5}\right)$
0	$\frac{1}{5}$	$\left(0, \frac{1}{5}\right)$
2	$-\frac{3}{5}$	$\left(2, -\frac{3}{5}\right)$
4	$-\frac{7}{5}$	$\left(4, -\frac{7}{5}\right)$
6	$-\frac{11}{5}$	$\left(6, -\frac{11}{5}\right)$

Table 5.29: A chart of inputs, x, and their corresponding outputs, $f(x)$.

x	$g(x)$	$(x, g(x))$
$\frac{11}{5}$	$-\frac{10}{2}$	$\left(\frac{11}{5}, -5\right)$
$\frac{7}{5}$	$-\frac{6}{2}$	$\left(\frac{7}{5}, -3\right)$
$\frac{3}{5}$	$-\frac{2}{2}$	$\left(\frac{3}{5}, -1\right)$
$\frac{1}{5}$	0	$\left(\frac{1}{5}, 0\right)$
$-\frac{3}{5}$	$\frac{4}{2}$	$\left(-\frac{3}{5}, 2\right)$
$-\frac{7}{5}$	$-\frac{8}{2}$	$\left(-\frac{7}{5}, 4\right)$
$-\frac{11}{5}$	$\frac{12}{2}$	$\left(-\frac{11}{5}, 6\right)$

Table 5.30: A chart of inputs, x, and their corresponding outputs, $g(x)$. The inputs come from the outputs in Table 5.29.

▪

Try It # 1:
Show that $f(x) = \dfrac{x+5}{3}$ and $g(x) = 3x - 5$ are inverses, using compositions.

In the previous two examples, we used the Horizontal Line Test to show a function has an inverse and compositions to verify functions are inverses of each other. While it is common to next discuss techniques for computing an inverse function, $f^{-1}(x)$, the topic is not a focus of this text. The authors will leave it to the reader to explore computing inverse functions outside of this text.

Defining Logarithmic Functions

According to http://earthquake.usgs.gov/earthquakes/eqinthenews/, there were many earthquakes in 2010 and 2011. In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Richter Scale uses powers of 10 to classify the strength of an earthquake. The Haitian earthquake registered a 7.0 on the Richter Scale, whereas the Japanese earthquake registered a 9.0.

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert from exponential forms to a new form called **logarithmic** form. For example, suppose the amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. To calculate the difference in magnitude involves the equation $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving an exponential equation of this type. None of the algebraic tools discussed so far are sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, as $y = f(x) = 10^x$ is increasing. We can examine a graph, as in **Figure 5.8.5**, to better estimate the solution.

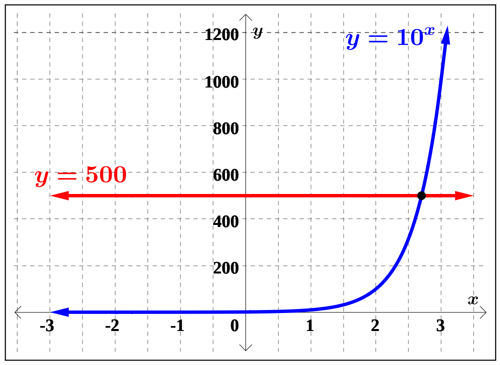


Figure 5.8.5: A graph of $y = 10^x$ and $y = 500$ on the same coordinate plane. Their intersection point is indicated.

Estimating from a graph, however, is imprecise. To identify an algebraic solution, we must introduce a new function. Observe that the graph of $y = 10^x$ in **Figure 5.8.5** passes the Horizontal Line Test. Thus, the exponential function $y = 10^x$ is one-to-one, and it has an inverse. Although not discussed in detail by the authors, it is the case with all invertible functions that you can simply interchange the x and y to find the function's inverse. However, to represent the inverse function as a function of x, we must then solve the new equation for y. For $y = 10^x$, after interchanging the variables we would get $x = 10^y$ and then need to solve for y.

To solve for y requires the removal of a variable from the exponent. A new function type, called a **logarithmic function** (also called a 'log'), is necessary to perform this operation. So for $y = 10^x$, the following work would be necessary to compute the inverse function, $f^{-1}(x)$.

$$\begin{aligned} x &= 10^y \\ \log_{10}(x) &= y \\ f^{-1}(x) &= \log_{10}(x) \end{aligned}$$

Graphing this function, along with $f(x) = 10^x$ (as shown in **Figure 5.8.6**), we see the graph of $f^{-1}(x)$ is, as it should be, the graph of $f(x)$ reflected over the line $y = x$.

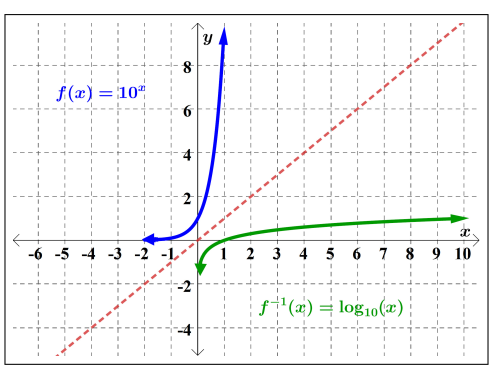


Figure 5.8.6: A graph of $f(x)$ and its inverse, $f^{-1}(x)$.

Definition

In general, the inverse of the exponential function $f(x) = b^x$ is called the **base b logarithm function**, and is denoted $f^{-1}(x) = \log_b(x)$. ▪

Definition

We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

- The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written **log(x).**
- The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written **ln(x).**

▪

On a procedural level, as they are inverses, logs 'undo' exponentials. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the real number 3 (which was the exponent of 2) as its output. In symbols, $\log_2(8) = 3$. More generally, $\log_2(x)$ is the exponent you raise 2 to in order to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$.

The following defines a logarithmic function using exponent notation, instead of inverse notation.

Definition

A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0$, $b > 0$, $b \neq 1$,

$$y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x$$

- We read $\log_b(x)$ as the "logarithm with base b of x" or the "log base b of x."
- The logarithm, y, is the exponent to which b must be raised to get x.

▪

Using Algebraic Properties of Logarithms

We introduced logarithmic functions as inverses of exponential functions. Now, we will explore the algebraic properties of logarithms.

Converting Equations between Exponential and Logarithmic Forms

Recall, for $b > 1$, $b \neq 1$, and $x > 0$

$$\log_b(x) = y \text{ if and only if } b^y = x.$$

We can illustrate this as follows:

$$\log_b(x) = y \quad \text{means} \quad b^y = x$$

to

To convert in either direction, we identify the values of b, x, and y.

Example 3 Write the following logarithmic equations in exponential form.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

b. $\log_3(9) = 2$

Solution:

First, identify the values of b, x, and y. Then, write the equation in the form $b^y = x$.

a. For $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$: $\quad b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$

Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}$, which is illustrated as

$$\log_6\left(\sqrt{6}\right) = \frac{1}{2} \quad \text{means} \quad 6^{\frac{1}{2}} = \sqrt{6}$$

to

b. For $\log_3(9) = 2$: $\quad b = 3$, $y = 2$, and $x = 9$

Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$, which is illustrated as

$$\log_3(9) = 2 \quad \text{means} \quad 3^2 = 9$$

to

■

Try It # 2:

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1000000) = 6$

b. $\log_5(25) = 2$

Example 4 Write the following exponential equations in logarithmic form.

a. $2^3 = 8$

b. $5^2 = 25$

c. $10^{-4} = \dfrac{1}{10000}$

Solution:

First, identify the values of b, y, and x. Then, write the equation in the form $y = \log_b(x)$.

a. For $2^3 = 8$: $b = 2$, $y = 3$, and $x = 8$

Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. For $5^2 = 25$: $b = 5$, $y = 2$, and $x = 25$

Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. For $10^{-4} = \dfrac{1}{10000}$: $b = 10$, $y = -4$, and $x = \dfrac{1}{10000}$

Therefore, the equation $10^{-4} = \dfrac{1}{10000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10000}\right) = -4$.

Try It # 3:

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$

b. $5^3 = 125$

c. $2^{-1} = \dfrac{1}{2}$

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2(16)$. We ask, "To what exponent must 2 be raised in order to get 16?" This is equivalent to $\log_2(16) = y$, which means $2^y = 16$. Because we already know $16 = 2^4$, then

$$\begin{aligned} 2^y &= 16 \\ 2^y &= 2^4 \\ y &= 4 \end{aligned}$$

So, $\log_2(16) = 4$.

Example 5 Evaluate $\log_4(64)$, without a calculator.

Solution:
First, we rewrite the logarithm, $y = \log_4(64)$, in exponential form.

$$4^y = 64$$

Then, we find a common base, and solve for y.

$$\begin{aligned} 4^y &= 4^3 \\ y &= 3 \end{aligned}$$

So, $\log_4(64) = 3$.

Try It # 4:
Evaluate $\log_2\left(\frac{1}{32}\right)$, without a calculator.

Suppose we wanted to evaluate $\log_2(7)$. Proceeding in the manner just described gives us

$$\begin{aligned} y &= \log_2(7) \\ 2^y &= 7 \\ &\text{hmmmm?} \end{aligned}$$

The answer should be a little less than 3 given $7 < 8$ and $8 = 2^3$, but can we be more precise? Because 7 is not an integer power of 2, we can use the calculator to tell us a more accurate answer. The two logarithm buttons commonly found on calculators are the "LOG" and "LN" buttons, which correspond to the common and natural logs, respectively. As we have base 2 here, and not base 10 or base e, in order to use the calculator to give us a more accurate evaluation, we need the following theorem.

Theorem 5.13 Change of Base Formula

Let $a, b > 0$, $a \neq 1$, $b \neq 1$, and $x > 0$.
$\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$. In particular,

$$\log_a(x) = \frac{\log(x)}{\log(a)} = \frac{\ln(x)}{\ln(a)}$$

Returning to our evaluation of $\log_2(7)$, we have

$$\log_2(7) = \frac{\log(7)}{\log(2)} \quad \text{or} \quad \log_2(7) = \frac{\ln(7)}{\ln(2)}$$

Using the calculator and the change of base formulas shown above, we can now obtain a closer approximation. (See **Figures 5.8.7** and **5.8.8**.)

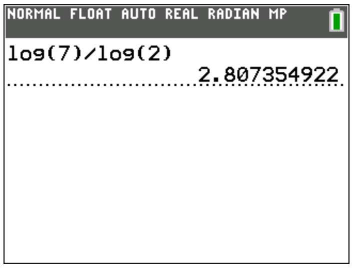


Figure 5.8.7: A calculator approximation of $\log_2(7)$ using the "LOG" button.

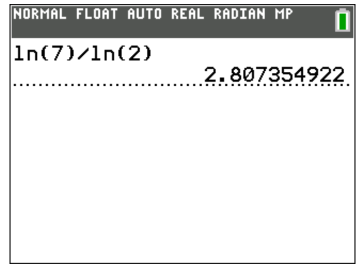


Figure 5.8.8: A calculator approximation of $\log_2(7)$ using the "LN" button.

From the calculator we see that $\log_2(7) \approx 2.80735$, which is indeed "a little less than 3."

Condensing and Expanding Logarithms

As logarithmic and exponential functions 'undo' each other, logarithms have similar algebraic properties to exponentials. Some important properties of logarithms are given here.

Properties of Logarithms

Let $g(x) = \log_b(x)$ be a logarithmic function $(b > 0, b \neq 1)$ and let M and N be positive real numbers. Then,

- $g(1) = \log_b(1) = 0$
- $g(b) = \log_b(b) = 1$
- **Product Rule**: $\log_b(MN) = \log_b(M) + \log_b(N)$
- **Quotient Rule**: $\log_b\left(\dfrac{M}{N}\right) = \log_b(M) - \log_b(N)$
- **Power Rule**: $\log_b(M^N) = N\log_b(M)$

In the Product and Quotient Rules, the base of the logarithms must be the same in order to condense them into a single logarithm of the same base.

Due to the fact that $\log_b(b) = 1$, *it follows that* $\ln(e) = 1$ *and* $\log(10) = 1$.

Here are some common mistakes to be aware of as you use these rules.

$$\log_b\left(\frac{M}{N}\right) \neq \frac{\log_b(M)}{\log_b(N)} \qquad (\log_b(M))^N \neq N\log_b(M) \qquad (\log_b(ax^n)) \neq n\log_b(ax)$$

Example 6 For x, y, $z > 0$, use the properties of logarithms to write the following as a single logarithm.

$$\log(x) - 2\log(y) + \log(z)$$

Solution:
In the expression $\log(x) - 2\log(y) + \log(z)$, we have both a difference and a sum of logarithms, all in the same base. However, before we use the Quotient Rule to combine $\log(x) - 2\log(y)$, we need to address the coefficient, 2, of $\log(y)$. This can be handled using the Power Rule. We can then apply the Quotient and Product Rules, as we move from left to right. Putting this all together, we have

$$\begin{aligned}
\log(x) - 2\log(y) + \log(z) &= \log(x) - \log\left(y^2\right) + \log(z) \\
&= \log\left(\frac{x}{y^2}\right) + \log(z) \\
&= \log\left(\frac{x}{y^2} \cdot z\right) \\
&= \log\left(\frac{xz}{y^2}\right)
\end{aligned}$$

■

Example 7 For x, y, $z > 0$, use the properties of logarithms to write the following as a single logarithm.

$$\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right)$$

Solution:
All three logarithms have the same base, 2, so we can use the given rules to condense them into a single logarithm. Remember to use the Power Rule to address any coefficients, before using the Product and Quotient Rules.

$$\begin{aligned}
\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right) &= \log_2(x^2) + \log_2\left((x-1)^{\frac{1}{2}}\right) - \log_2\left(((x+3)^2)^3\right) \\
&= \log_2(x^2) + \log_2\left((x-1)^{\frac{1}{2}}\right) - \log_2\left((x+3)^6\right) \\
&= \log_2\left(x^2(x-1)^{\frac{1}{2}}\right) - \log_2\left((x+3)^6\right) \\
&= \log_2\left(\frac{x^2(x-1)^{\frac{1}{2}}}{(x+3)^6}\right) \\
&= \log_2\left(\frac{x^2\sqrt{x-1}}{(x+3)^6}\right)
\end{aligned}$$

■

It is sometimes necessary to use the facts that $\ln(e) = 1$ and $\log(10) = 1$ when condensing or simplifying logarithms.

Example 8 Use the properties of logarithms to write the following as a single logarithm. Assume, when necessary, that all variables represent positive real numbers.

$$-\ln(x)-\frac{1}{2}$$

Solution:
First, we use the Power Rule to move the coefficient of the logarithm to the exponent of the variable.

$$\begin{aligned}-\ln(x)-\frac{1}{2} &= (-1)\ln(x)-\frac{1}{2}\\ &= \ln\left(x^{-1}\right)-\frac{1}{2}\end{aligned}$$

In order to simplify further $\frac{1}{2}$ must be rewritten as a logarithm. Because $\ln(e)=1$,

$$\begin{aligned}&= \ln\left(x^{-1}\right)-\frac{1}{2}(1)\\ &= \ln\left(x^{-1}\right)-\frac{1}{2}(\ln(e))\\ &= \ln\left(x^{-1}\right)-\ln\left(e^{\frac{1}{2}}\right)\end{aligned}$$

Finally, the Quotient Rule gives us

$$\begin{aligned}&= \ln\left(\frac{x^{-1}}{e^{\frac{1}{2}}}\right)\\ &= \ln\left(\frac{1}{x\sqrt{e}}\right)\end{aligned}$$

■

Try It # 5:
Rewrite $\log(5)+0.5\log(x)-\log(7x-1)+3\log(x-1)$ as a single logarithm, for $x>1$.

Example 9 Use the properties of logarithms to fully expand and simplify the following expression. Assume, when necessary, that all arguments represent positive real numbers.

$$\log_2\left(\frac{8}{x}\right)$$

Solution:
To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule.

$$\log_2\left(\frac{8}{x}\right)=\log_2(8)-\log_2(x)$$

Due to the fact that $8=2^3$ and $\log_2(2)=1$, we can simplify and have

$$\begin{aligned}\log_2(8)-\log_2(x) &= \log_2\left(2^3\right)-\log_2(x)\\ &= 3-\log_2(x)\\ &= -\log_2(x)+3\end{aligned}$$

■

■ **Example 10** Use the properties of logarithms to fully expand and simplify the following expression. Assume, when necessary, that all arguments represent positive real numbers.

$$\ln\left(\frac{3}{ex}\right)^2$$

Solution:

When expanding logarithms, before using the Quotient or Product Rules, we must first address any powers on the arguments of the logarithms.

$$\ln\left(\frac{3}{ex}\right)^2 = 2\ln\left(\frac{3}{ex}\right)$$

Then, we can continue with the Quotient and Product Rules, as follows.

$$\begin{aligned} &= 2[\ln(3) - \ln(ex)] \\ &= 2\left[\ln(3) - \{\ln(e) + \ln(x)\}\right] \end{aligned}$$

To simplify this expression, distribute and remember $\ln(e) = 1$.

$$\begin{aligned} &= 2\ln(3) - 2[\ln(e) + \ln(x)] \\ &= 2\ln(3) - 2\ln(e) - 2\ln(x)] \\ &= 2\ln(3) - 2 - 2\ln(x) \end{aligned}$$

■

■ **Example 11** Use the properties of logarithms to fully expand and simplify $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Assume, when necessary, that $x > \frac{1}{2}$.

Solution:

$$\begin{aligned} \log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6\left(64x^3(4x+1)\right) - \log_6(2x-1) \\ &= \log_6\left(64x^3\right) + \log_6(4x+1) - \log_6(2x-1) \\ &= \log_6(64) + \log_6\left(x^3\right) + \log_6(4x+1) - \log_6(2x-1) \\ &= \log_6\left(2^6\right) + \log_6\left(x^3\right) + \log_6(4x+1) - \log_6(2x-1) \\ &= 6\log_6(2) + 3\log_6(x) + \log_6(4x+1) - \log_6(2x-1) \end{aligned}$$

■

Try It # 6:

Use the properties of logarithms to fully expand and simplify $\log_b\left(\frac{x^2y^3}{z^4\sqrt{b}}\right)$.

Assume, when necessary, that all arguments represent positive real numbers.

Properties of Logarithmic Functions

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

We just learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as the inverse function

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

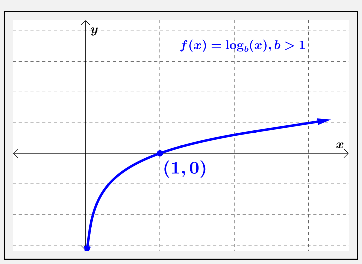
The following summarizes this and other basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions and are exponents themselves.

Properties of Logarithmic Functions

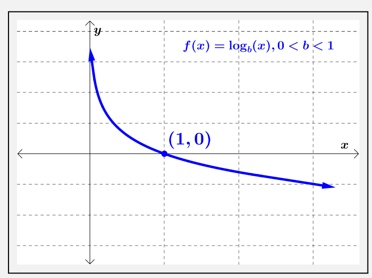
Inverse Properties:

Suppose $f(x) = \log_b(x)$.

- The domain of $f(x)$ is $(0, \infty)$.
- The range of $f(x)$ is $(-\infty, \infty)$.
- The x-intercept is $(1, 0)$.
- There is no y-intercept.
- $f(x)$ is one-to-one.
- End Behavior
  - If $b > 1$
    As $x \to 0$ from the right,
    $f(x) \to -\infty$ (V.A.: $x = 0$)
    As $x \to \infty$, $f(x) \to \infty$
  - If $0 < b < 1$
    As $x \to 0$ from the right,
    $f(x) \to \infty$ (V.A.: $x = 0$)
    As $x \to \infty$, $f(x) \to -\infty$
- The graph of $f(x)$ resembles:
  - If $b > 1$



  - If $0 < b < 1$



Exponent Properties:

- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent to which you raise b to get c.
- $\log_b(b^x) = x$ for all x
- $b^{\log_b(x)} = x$ for all $x > 0$

We will add our final two parent functions to our list of parent functions, without including a table of values for each.

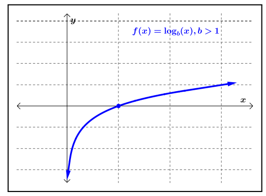
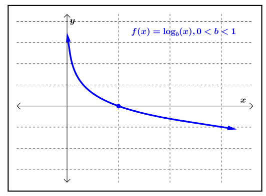
Name	Function	Graph	Domain
Logarithmic Growth	$f(x) = \log_b(x)$ $b > 1$		$(0, \infty)$
Logarithmic Decay	$f(x) = \log_b(x)$ $0 < b < 1$		$(0, \infty)$

Table 5.31: Parent Logarithmic Functions

▪ **Example 12** Given $h(x) = \log_5(x)$, state each of the following, if it exists.

a. Domain
b. Range
c. End behavior
d. x-intercept(s)
e. y-intercept(s)

Solution:
We begin by recognizing $b = 5$, $(b > 1)$.

a. The domain of $h(x)$ is $(0, \infty)$.

b. The range of $h(x)$ is $(-\infty, \infty)$.

c. The end behavior of $h(x)$ is
As $x \to 0$ from the right, $f(x) \to -\infty$. (There is a vertical asymptote at $x = 0$.)
As $x \to \infty$, $f(x) \to \infty$.

d. $h(x)$ has an x-intercept at (1,0).

e. $h(x)$ has no y-intercept, as $x = 0$ is a vertical asymptote of $h(x)$.

▪

Try It # 7:
Given $f(x) = \log_{\frac{1}{5}}(x)$, state each of the following, if it exists.

a. Domain
b. Range
c. End behavior
d. x-intercept(s)
e. y-intercept(s)

Computing the Domain of a Logarithmic Function

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even radicals. With the introduction of logarithms, we now have another restriction. Due to the fact that the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the input, or **argument**, of the logarithm must be strictly positive.

Generally speaking, in order for $f(x) = \log_b(g(x))$ to be defined, $g(x)$ must be defined AND $g(x)$ must be greater than zero.

Consider $f(x) = \log_4(2x-3)$. This function is defined for any values of x such that the argument, $2x-3$, is defined AND greater than zero. As $g(x)$ is a linear polynomial, it is defined for all real numbers. We now set the argument greater than zero and solve for x to find any domain restrictions.

$$\begin{aligned} 2x-3 &> 0 \\ 2x &> 3 \\ x &> \frac{3}{2} \end{aligned}$$

So, the domain of $f(x)$ is $\left(\frac{3}{2}, \infty\right)$.

Example 13 State the domain of the following functions, using interval notation.

a. $f(x) = 2\log(3-x) - 1$

b. $h(x) = \ln\left(\dfrac{4}{x-1}\right)$

Solution:

a. For $f(x)$ to be defined, the argument, $g(x) = 3-x$, must be defined AND greater than zero. Given that $g(x) = 3-x$ is a linear polynomial, $g(x)$ is defined for all real numbers. So, we set the argument greater than zero and solve for x, to find any domain restrictions.

$$\begin{aligned} 3-x &> 0 \\ -x &> -3 \\ x &< 3 \end{aligned}$$

Therefore, the domain of $f(x) = 2\log(3-x) - 1$ is $(-\infty, 3)$.

b. For $h(x)$ to be defined, the argument, $g(x) = \dfrac{4}{x-1}$, must be defined AND greater than zero. Seeing as $g(x) = \dfrac{4}{x-1}$ is a rational function, $g(x)$ is defined for $x \neq 1$. For $\dfrac{4}{x-1}$ to be greater than 0 (positive), $x-1$ must be positive, as 4 is always positive:

$$\begin{aligned} x-1 &> 0 \\ x &> 1 \end{aligned}$$

Therefore, the domain of $h(x)$ is the overlapping intervals of $x \neq 1$ and $x > 1$. So, the domain of $h(x) = \ln\left(\dfrac{4}{x-1}\right)$ is $(1, \infty)$.

■

Try It # 8:
State the domain of $f(x) = \log(7x-2)+1$, using interval notation.

Solving Equations Involving Exponentials

Recall the Common Base Property of Exponents (One-to-One Property of Exponents)

$$b^S = b^T \quad \text{if and only if} \quad S = T$$

for all real numbers S and T and $b > 0$, $b \neq 1$. So when the bases are the same, the exponents are equal.

Similarly, there is a One-to-One Property for Logarithms.

Theorem 5.14 One-to-One Property of Logarithms

For all real numbers $M > 0$ and $N > 0$, $\log_b(M) = \log_b(N)$ if and only if $M = N$.

Sometimes the terms of an equation involving exponentials cannot be rewritten with a common base. By the One-to-One Property of Logarithms, we solve such equations by applying the logarithm to each side. While you can use a logarithm with any base, the authors will choose to use the natural logarithm, unless the base of the exponential expression is 10. In this case, the authors will use the common logarithm. The authors make these choices in the text so that readers can quickly verify their answers using most calculators.

■ **Example 14** Solve $5^{(x+2)} = 4^x$ for x. Leave all answers as exact values.

Solution:
There is no easy way to rewrite all exponential expressions in this equation with the same base. Thus, we will use the One-to-One Property of Logarithms and apply the natural log to both sides of the equation.

$$\begin{aligned} 5^{(x+2)} &= 4^x \\ \ln\left(5^{(x+2)}\right) &= \ln\left(4^x\right) \end{aligned}$$

From here we will use the properties of logarithms to solve for x.

$$\begin{aligned} (x+2)\ln(5) &= x\ln(4) \\ x\ln(5)+2\ln(5) &= x\ln(4) \\ x\ln(5)-x\ln(4) &= -2\ln(5) \\ x[\ln(5)-\ln(4)] &= \ln\left(5^{-2}\right) \\ x\ln\left(\frac{5}{4}\right) &= \ln\left(\frac{1}{25}\right) \\ x &= \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)} \end{aligned}$$

The solution to $5^{(x+2)} = 4^x$, $x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$, is NOT the same as $x = \ln\left(\frac{1}{25}\right) - \ln\left(\frac{5}{4}\right)$. A quick check on your calculator will show the difference between the two.

■

■ **Example 15** Solve $4e^{2x} + 5 = 12$ for x. Leave all answers as exact values.

Solution:
We begin by moving and combining like terms so that we have a single term on each side of the equation.

$$\begin{aligned} 4e^{2x} + 5 &= 12 \\ 4e^{2x} &= 7 \end{aligned}$$

Before applying the natural log of both sides, it is a best practice to divide both sides of the equation by the coefficient of the exponential expression.

$$e^{2x} = \frac{7}{4}$$

Now, take the natural log of both sides and simplify to solve for x.

$$\begin{aligned} \ln\left(e^{2x}\right) &= \ln\left(\frac{7}{4}\right) \\ 2x(\ln(e)) &= \ln\left(\frac{7}{4}\right) \\ 2x &= \ln\left(\frac{7}{4}\right) \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) \end{aligned}$$

■

Try It # 9:
Solve $3 + 10^{2t} = 7 \cdot 10^{2t}$ for t. Leave all answers as exact values.

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically, but *does not satisfy the conditions of the original equation*. One such situation arises when the logarithm is applied to both sides of the equation. Remember that the argument of the logarithm must be positive, so if the number we are evaluating in a logarithmic function is non-positive, there is no output.

■ **Example 16** Solve $e^{2x} - e^x = 56$ for x. Leave all answers as exact values.

Solution:
Because we cannot move and combine like terms to have a single term on each side of the equals sign, we will move

terms so that we produce an equation that is equal to zero.

$$\begin{aligned} e^{2x} - e^x &= 56 \\ e^{2x} - e^x - 56 &= 0 \end{aligned}$$

While it may not be immediately obvious, $e^{2x} - e^x - 56$ is in quadratic form, $(e^x)^2 - e^x - 56$. Recognizing this allows us to factor, and solve for x.

$$\begin{aligned} (e^x)^2 - e^x - 56 &= 0 \\ (e^x + 7)(e^x - 8) &= 0 \\ e^x + 7 = 0 \quad \text{or} \quad e^x - 8 &= 0 \\ e^x = -7 \quad \text{or} \quad e^x &= 8 \end{aligned}$$

Given $e^x > 0$ for all values of x, then $e^x \neq -7$, and we only have that $e^x = 8$. Solving for x, we have

$$\begin{aligned} e^x &= 8 \\ \ln(e^x) &= \ln(8) \\ x &= \ln(8) \end{aligned}$$

■

If you do not reject the equation $e^x = -7$, as was done in the solution, then you should see the extraneous solution when you get $\ln(e^x) = \ln(-7)$ as part of the process, which is undefined.

Try It # 10:
Solve $8^{2x} = 8^x + 12$ for x. Leave all answers as exact values.

Solving Equations Involving Logarithms

As with exponential equations, we can use the One-to-One Property to solve logarithmic equations.

For example, if

$$\log_2(x-1) = \log_2(8),$$

then by the One-to-One Property

$$\begin{aligned} x - 1 &= 8 \\ x &= 9. \end{aligned}$$

While the reader has been encouraged to check their solutions in previous sections, it is *imperative* to do so when solving equations involving logarithms, as these equations often have extraneous solutions. When given an equation with logarithms of the same base on each side, we can use the rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2)-\log(2)=\log(x+4)$.

To solve this equation for x, we can use the rules of logarithms to rewrite the left-hand side as a single logarithm, and then apply the One-to-One Property.

$$\begin{aligned}
\log(3x-2)-\log(2) &= \log(x+4)\\
\log\left(\frac{3x-2}{2}\right) &= \log(x+4)\\
\frac{3x-2}{2} &= x+4\\
3x-2 &= 2(x+4)\\
3x-2 &= 2x+8\\
x &= 10
\end{aligned}$$

Our original equation contains logarithms, so we *must* check to make sure this answer is not extraneous.

To check, substitute $x=10$ into the *original* equation: $\log(3x-2)-\log(2)=\log(x+4)$.

$$\begin{aligned}
\log(3(10)-2)-\log(2) &\stackrel{?}{=} \log(10+4)\\
\log(28)-\log(2) &\stackrel{?}{=} \log(14)\\
\log\left(\frac{28}{2}\right) &\stackrel{?}{=} \log(14)\\
\log(14) &= \log(14)\ \checkmark
\end{aligned}$$

The solution checks and all arguments of logs in the original equation (28, 2, and 14) are positive.

So, the solution to $\log(3x-2)-\log(2)=\log(x+4)$ is $x=10$.

N *For x to be a solution to the equation $\log(3x-2)-\log(2)=\log(x+4)$, x must be in the intersection of the domains of the logarithms.*

The domain for each logarithm in the equation is

$$\left.\begin{array}{ll}
\log(3x-2): & x>\frac{2}{3}\\
\log(2): & -\infty<x<\infty\\
\log(x+4): & x>-4
\end{array}\right\}\ \textit{Intersection of the Domains:}\quad x>\tfrac{2}{3}\ \textit{or}\ \left(\tfrac{2}{3},\infty\right)$$

Our solution $x=10>\frac{2}{3}$. Therefore, we know $x=10$ is a valid solution to the equation.

▪ **Example 17** Solve $\ln\left(x^2\right) = \ln(2x+3)$ for x. Leave all answers as exact values.

Solution:
We already have a single logarithm, in the same base, on each side of the equation. Thus, we can apply the One-to-One Property, and solve for x.

$$\begin{aligned} \ln\left(x^2\right) &= \ln(2x+3) \\ x^2 &= 2x+3 \\ x^2-2x-3 &= 0 \\ (x-3)(x+1) &= 0 \\ x-3=0 \quad &\text{or} \quad x+1=0 \\ x=3 \quad &\text{or} \quad x=-1 \end{aligned}$$

To check, first, substitute $x=3$ into the original equation: $\ln\left(x^2\right) = \ln(2x+3)$.

$$\begin{aligned} \ln\left(3^2\right) &\stackrel{?}{=} \ln(2(3)+3) \\ \ln(9) &= \ln(9) \checkmark \end{aligned}$$

Next, substitute $x=-1$ into the original equation: $\ln\left(x^2\right) = \ln(2x+3)$.

$$\begin{aligned} \ln\left((-1)^2\right) &\stackrel{?}{=} \ln(2(-1)+3) \\ \ln(1) &= \ln(1) \checkmark \end{aligned}$$

While the solution $x=-1$ is negative, it checks when substituted into the original equation because the arguments of the logarithmic functions are still positive, as is also the case with $x=3$.

Thus, $x=3$ and $x=-1$ are both solutions to the equation $\ln\left(x^2\right) = \ln(2x+3)$.

▪

▪ **Example 18** Solve $\log_7(1-2x) = 1-\log_7(3-x)$ for x. Leave all answers as exact values.

Solution:
We begin by rewriting the equation with the logarithms of the same base on the same side of the equals sign, which we can then condense to a single logarithm in the same base.

$$\begin{aligned} \log_7(1-2x) &= 1-\log_7(3-x) \\ \log_7(1-2x)+\log_7(3-x) &= 1 \\ \log_7[(1-2x)(3-x)] &= 1 \end{aligned}$$

Rewriting the logarithmic equation in its equivalent exponential form gives

$$\begin{aligned} 7^1 &= (1-2x)(3-x) \\ (1-2x)(3-x) &= 7 \end{aligned}$$

Now, we can multiply the binomial factors and solve the quadratic equation that results.

$$\begin{aligned} 2x^2 - 7x + 3 &= 7 \\ 2x^2 - 7x - 4 &= 0 \\ (2x+1)(x-4) &= 0 \\ 2x+1 = 0 \quad &\text{or} \quad x-4=0 \\ x = -\frac{1}{2} \quad &\text{or} \quad x = 4 \end{aligned}$$

To check the results, first substitute $x = -\frac{1}{2}$ into the original equation: $\log_7(1-2x) = 1 - \log_7(3-x)$.

$$\begin{aligned} \log_7\left(1 - 2\left(-\frac{1}{2}\right)\right) &\stackrel{?}{=} 1 - \log_7\left(3 - \left(-\frac{1}{2}\right)\right) \\ \log_7(1+1)) &\stackrel{?}{=} 1 - \log_7\left(\frac{7}{2}\right) \\ \log_7(2) &\stackrel{?}{=} 1 - [\log_7(7) - \log_7(2)] \\ \log_7(2) &\stackrel{?}{=} 1 - [1 - \log_7(2)] \\ \log_7(2) &\stackrel{?}{=} 1 - 1 + \log_7(2) \\ \log_7(2) &= \log_7(2) \checkmark \end{aligned}$$

The solution $x = -\frac{1}{2}$ checks and all the logarithms are defined.

Next, substitute $x = 4$ into the original equation: $\log_7(1-2x) = 1 - \log_7(3-x)$.

$$\begin{aligned} \log_7(1 - 2(4)) &\stackrel{?}{=} 1 - \log_7(3-4) \\ \log_7(-7) &\stackrel{?}{=} 1 - \log_7(-1) \end{aligned}$$

We can stop checking at this point, as logarithms cannot have a negative argument, which occurs with $\log_7(-7)$ and $\log_7(-1)$. Thus, even though $x = 4$ is positive, it is an extraneous solution for the original equation.

Therefore, the solution to $\log_7(1-2x) = 1 - \log_7(3-x)$ is $x = -\frac{1}{2}$.

■

! *When checking a result for validity, if you get, for example,* $\log_b(-8) \stackrel{?}{=} \log_b(-8)$ *or* $\log_b(0) \stackrel{?}{=} \log_b(0)$*, the result is NOT a solution, even though the left-hand and right-hand sides of the equation appear equal. The argument of a logarithm must be positive in order for a solution to be valid.*

Try It # 11:
Solve $6 + 4\ln(x) = 10$ for x. Leave all answers as exact values.

Try It # 12:
Solve $\log_2(x+3) = \log_2(6-x)+3$ for x. Leave all answers as exact values.

Applying Exponential and Logarithmic Functions

Computing Domains of Algebraic Functions

When computing the domain of an algebraic function, it may be necessary to solve equations involving exponentials and/or logarithms.

▪ **Example 19** State the domain of the following functions, using interval notation.

a. $f(x) = \dfrac{x}{1-e^{2x}}$

b. $h(x) = \dfrac{5}{\log(3x+1)-2}$

Solution:

a. $f(x)$ is the quotient of two functions, so both the numerator and denominator must be defined AND the denominator cannot be zero.

The numerator, x, is a linear polynomial and is defined for all real numbers.

The denominator, $1-e^{2x}$, is an exponential function and is defined for all real numbers.

As the denominator, $1-e^{2x}$, cannot be zero. Thus,

$$\begin{aligned}
1-e^{2x} &\neq 0\\
-e^{2x} &\neq -1\\
e^{2x} &\neq 1\\
\ln(e^{2x}) &\neq \ln(1)\\
2x\ln(e) &\neq \ln(1)\\
2x &\neq 0\\
x &\neq 0
\end{aligned}$$

So, the domain of $f(x)$ is the intersection of $(-\infty,\infty)$ and $(-\infty,0)\cup(0,\infty)$.

If we drew $(-\infty,\infty)$ and $(-\infty,0)\cup(0,\infty)$ on a number line (using techniques from **Section 5.1**), the overlapping segments of these intervals would show that the domain of $f(x)$ is $(-\infty,0)\cup(0,\infty)$.

b. $h(x)$ is the quotient of two functions, so both the numerator and denominator must be defined AND the denominator cannot be zero.

The numerator, 5, is a constant polynomial and is defined for all real numbers.

The denominator, $\log(3x+1)-2$, is a logarithmic function and is defined when $3x+1$ is defined AND when $3x+1>0$. As $3x+1$ is a linear polynomial which is defined for all real numbers, we must only determine where $3x+1$ is positive for the logarithm to be defined.

$$\begin{aligned} 3x+1 &> 0 \\ 3x &> -1 \\ x &> -\frac{1}{3} \end{aligned}$$

As the denominator, $\log(3x+1)-2$ cannot be zero. Thus,

$$\begin{aligned} \log(3x+1)-2 &\neq 0 \\ \log(3x+1) &\neq 2 \\ 3x+1 &\neq 10^2 \\ 3x+1 &\neq 100 \\ 3x &\neq 99 \\ x &\neq 33 \end{aligned}$$

So, the domain of $h(x)$ is the intersection of these intervals: $(-\infty,\infty)$, $\left(-\frac{1}{3},\infty\right)$, and $(-\infty,33)\cup(33,\infty)$.

If we drew $(-\infty,\infty)$, $\left(-\frac{1}{3},\infty\right)$, and $(-\infty,33)\cup(33,\infty)$ on a number line (using techniques from **Section 5.1**), the overlapping segments of these intervals would show that the domain of $h(x)$ is $\left(-\dfrac{1}{3},33\right)\cup(33,\infty)$.

■

Real-World Applications

Historically, logarithms have played a huge role in the scientific development of our society. They were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing.

In a previous section, we learned that exponential functions are used to model exponential growth and decay, including compound interest. When solving an exponential model for a variable in the exponent, we use logarithms.

Due to the applied nature of the problems we will examine here, the calculator is often used to express our answers as decimal approximations, after calculating the exact answers.

Example 20 Suppose P dollars is invested in an account which offers interest of 7.125% per year. How long does it take the initial investment to double if

a. The interest is compounded monthly?
b. The interest is compounded continuously?

Solution:

a. When interest is compounded m times per year, recall that the amount of money in the account is given by

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$$

The initial amount is given to be P. The annual interest rate is 7.125%, so $r = 0.07125$. Because we are compounding monthly, $m = 12$. We want to calculate the amount of time it takes for the account to double, so we are solving for the value of t when $A = 2P$. Using this information in the formula and solving for t gives us

$$\begin{aligned}
2P &= P\left(1 + \frac{0.07125}{12}\right)^{12t} \\
2P &= P(1.0059375)^{12t} \\
2\cancel{P} &= \cancel{P}(1.0059375)^{12t} \\
2 &= (1.0059375)^{12t} \\
\ln(2) &= \ln(1.0059375)^{12t} \\
\ln(2) &= 12t\ln(1.0059375) \\
t &= \frac{\ln(2)}{12\ln(1.0059375)} \\
t &\approx 9.76
\end{aligned}$$

Thus, after approximately 9.76 years the account will double.

b. When interest is compounded continuously, recall the amount of money in the account is given by

$$A(t) = Pe^{rt}$$

Using the given values of A and r (the same as part **a**) and solving for t, we have

$$\begin{aligned}
2P &= Pe^{0.07125t} \\
2\cancel{P} &= \cancel{P}e^{0.07125t} \\
2 &= e^{0.07125t} \\
\ln(2) &= \ln\left(e^{0.07125t}\right) \\
\ln(2) &= 0.07125t\left(\cancelto{1}{\ln(e)}\right) \\
t &= \frac{\ln(2)}{0.07125} \\
t &\approx 9.73
\end{aligned}$$

Thus, after approximately 9.73 years the account will double. (A little less time than when only compounding monthly).

■ **Example 21** If the number of people, $N(t)$, in hundreds, at a local community college who have heard the rumor "Adam is afraid of Virginia Woolf," after t days can be modeled by

$$N(t) = \frac{84}{1+2799e^{-t}},$$

how long will it be before 4200 people have heard the rumor?

Solution:
To determine how long it takes until 4200 people have heard the rumor, we need to solve for t, after substituting the appropriate value for $N(t)$. $N(t)$ is written in *hundreds*, so if 4200 people have heard the rumor, then $N(t) = \frac{4200}{100} = 42$.

Now, we are ready to solve for t.

$$\begin{aligned}
42 &= \frac{84}{1+2799e^{-t}} \\
42(1+2799e^{-t}) &= 84 \\
1+2799e^{-t} &= 2 \\
2799e^{-t} &= 1 \\
e^{-t} &= \frac{1}{2799} \\
\ln\left(e^{-t}\right) &= \ln\left(\frac{1}{2799}\right) \\
-t\cancelto{1}{\ln(e)} &= \ln\left(\frac{1}{2799}\right) \\
-t &= \ln\left(\frac{1}{2799}\right) \\
t &= -\ln\left(\frac{1}{2799}\right) \\
t &= -[\ln(1) - \ln(2799)] \\
t &= -[0 - \ln(2799)] \\
t &= \ln(2799) \\
t &\approx 7.94
\end{aligned}$$

Thus, after approximately 8 days, 4200 people would have heard "Adam is afraid of Virginia Woolf."

■

Try It # 13:
Suppose you have \$3000 invested in an account compounded continuously. How long will it take for the account to reach \$9000, if the annual interest rate on the account is 2.85%?

Try It Answers

1. Check $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= \tfrac{(3x-5)+5}{3} \\ &= \tfrac{3x}{3} \\ &= x\,\checkmark\end{aligned} \qquad \begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= 3\left(\tfrac{(x+5)}{3}\right)-5 \\ &= x+5-5 \\ &= x\,\checkmark\end{aligned}$$

2. **a.** $10^6 = 1{,}000{,}000$

b. $5^2 = 25$

3. **a.** $\log_3(9) = 2$

b. $\log_5(125) = 3$

c. $\log_2\left(\frac{1}{2}\right) = -1$

4. -5

5. $\log\left(\frac{5\sqrt{x}\,(x-1)^3}{(7x-1)}\right)$

6. $2\log_b(x) + 3\log_b(y) - 4\log_b(z) - \frac{1}{2}$

7. **a.** $(0, \infty)$

b. $(-\infty, \infty)$

c. As $x \to 0$ from the right, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$

d. $(1, 0)$

e. No y-intercept

8. $\left(\frac{2}{7}, \infty\right)$

9. $t = \frac{1}{2}\log\left(\frac{1}{2}\right)$

10. $x = \frac{\ln(4)}{\ln(8)}$

11. $x = e$

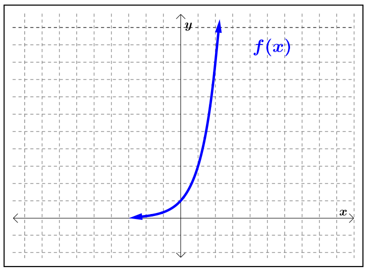
12. $x = 5$

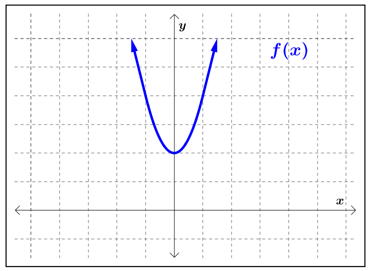
13. Approximately 38.55 years

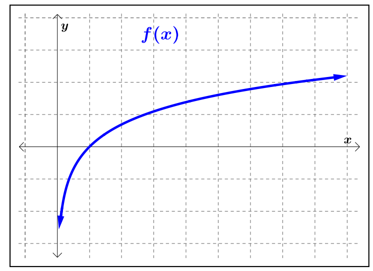
EXERCISES

Basic Skills Practice (Answers)

For Exercises 1 - 3, determine if the graph of $f(x)$ is a one-to-one function.

1. 

2. 

3. 

For Exercises 4 - 5, determine if $f(x)$ and $g(x)$ are inverse functions.

4. $f(x) = 6x - 2$ and $g(x) = \frac{1}{6}x + \frac{1}{3}$
5. $f(x) = 42 - x$ and $g(x) = 42 + x$

For Exercises 6 - 9, write the logarithmic equation in exponential form.

6. $\log_4(16) = 2$
7. $\log\left(\frac{1}{100}\right) = -2$
8. $\log_3(81) = 4$
9. $\ln\left(\sqrt{e}\right) = \frac{1}{2}$

For Exercises 10 - 13, write the exponential equation in logarithmic form.

10. $e^0 = 1$
11. $5^{-3} = \frac{1}{125}$
12. $10^5 = 100000$
13. $2^{-6} = \frac{1}{64}$

For Exercises 14 - 17, express each as a single logarithm. Assume when necessary that all variables represent positive real numbers.

14. $2\log_8(x) - 4\log_8(y)$
15. $3\ln(2) + 7\ln(x)$
16. $-\log_5(2) + 11\log_5(y)$
17. $\sqrt{2}\log(y) - 3\log(x)$

For Exercises 18 - 21, use the properties of logarithms to fully expand and simplify each expression. Assume when necessary that all variables represent positive real numbers and $y > z$.

18. $\log(100xyz)$
19. $\log_7\left(\dfrac{x^2+y}{y-z}\right)$
20. $\ln\left(\sqrt{x}y^4\right)$
21. $\log_{13}\left(\dfrac{13x^2y}{z^3}\right)$

For Exercises 22 - 23, state the following properties of the logarithmic function, if they exist.

a. Domain

b. Range

c. End behavior

d. x - intercept(s)

e. y - intercept(s)

22. $f(x) = \log_8(x)$
23. $g(x) = \log_{\frac{1}{8}}(x)$

For Exercises 24 - 27, state the domain of each logarithmic function, using interval notation.

24. $f(x) = \log(x-5)$
25. $g(x) = \log_3(x+9)$
26. $h(x) = \ln(2x-7)$
27. $j(x) = \log_{\frac{1}{7}}(3x+8)$

For Exercises 28 - 31, solve each of the following for x. Leave your answers in exact form.

28. $5^x = 7$
29. $e^x = 2$
30. $10^{2x-9} = 3$
31. $0.4^{8-3x} = 11$

For Exercises 32 - 37, solve each of the following for x. Leave your answers in exact form.

32. $\log_2(x) = 4$
33. $\log(4x-5) = 3$
34. $\log_2(x^2-8x-25) = 3$
35. $5\ln(x) = 1$
36. $\ln(2-3x) - 1 = 0$
37. $\log_4(x+6) - \log_4(x-2) = 1$

For Exercises 38 - 39, the model for continuous (exponential) growth/decay is given by $y = ce^{kt}$, where c is the initial amount, k is the relative growth rate (as a decimal), t is time (in years), and y is the future amount after t years.

38. At the beginning of the year 2000, the world population was about six billion people. Assuming the population increases continuously at a rate of 1.3% per year, how many years later will the population reach eight billion people? Round your answer to the nearest tenth of a year.

39. A new piece of equipment worth \$75,000 depreciates continuously at a relative rate of 8.2% per year. How long will it take for the equipment to reach its scrap value of \$23,000? Round your answer to three decimal places.

Intermediate Skills Practice (Answers)

For Exercises 40 - 43, determine if $f(x)$ is a one-to-one function, using the Horizontal Line Test.

40. $f(x) = 3x+5$
41. $f(x) = 5x^3 - x^2 + 9$
42. $f(x) = 2|x-7|$
43. $f(x) = e^{-x}$

For Exercises 44 - 45, determine if $f(x)$ and $g(x)$ are inverse functions.

44. $f(x) = \dfrac{3}{4-x}$ and $g(x) = \dfrac{4x-3}{x}$
45. $f(x) = \dfrac{x}{1-3x}$ and $g(x) = \dfrac{x}{4}$

For Exercises 46 - 51, evaluate each expression, without using a calculator.

46. $\ln\left(e^3\right)$
47. $\log_{19}\left(\dfrac{1}{361}\right)$
48. $\log_5\left(\sqrt{5}\right)$
49. $10^{\log\left(\sqrt{3}\right)}$
50. $\sqrt{6^{\log_6(\pi)}}$
51. $7^{2\log_7(4)}$

For Exercises 52 - 54, express each as a single logarithm. Assume when necessary that all variables represent positive real numbers.

52. $\log(x) - 3\log(y) - \dfrac{1}{2}\log(z)$
53. $\dfrac{1}{3}\ln(x) - \ln(x+y) + 4\ln(2z)$
54. $\dfrac{1}{7}\log_4\left(3x^2+5\right) - 2\log_4\left(x^3\right)$

For Exercises 55 - 58, use the properties of logarithms to fully expand and simplify each expression. Assume when necessary that all variables represent positive real numbers greater than 2.

55. $\log_3\left(\dfrac{x+5}{y^8\sqrt{z}}\right)$
56. $\log_5\sqrt{625xy^3}$
57. $\ln\left(\sqrt[3]{\dfrac{x^3}{e^2z^4}}\right)$
58. $\log_2(x(x+4)(2x-3))$

For Exercises 59 - 60, state the following properties of the logarithmic function, if they exist.

a. Domain

b. Range

c. End behavior

d. x - intercept(s)

e. y - intercept(s)

59. $f(x) = \log(x)$
60. $g(x) = \ln(x)$

For Exercises 61 - 66, state the domain of the algebraic function, using interval notation.

61. $f(x) = \ln(-6x+11)$
62. $g(x) = \log_{\frac{2}{3}}\left(\sqrt[3]{x+4}\right)$
63. $h(x) = \dfrac{\log(9-x)}{x^2-49}$
64. $j(x) = \log_2(10+33x)+45$
65. $k(x) = \ln(e^x)-35$
66. $m(x) = \dfrac{\log(x+1)}{\sqrt[11]{x-8}}$

For Exercises 67 - 72, solve each of the following for x. Leave your answers in exact form.

67. $4e^{-3x} = 5$
68. $2 \cdot 3^{-x} = 16$
69. $11^{\sqrt{x}} = 21$
70. $6^{3+5x} - 12 = 0$
71. $e^{4x} = 5e^{7x}$
72. $e^{x^2} = 14$

For Exercises 73 - 76, solve each of the following for x. Leave your answers in exact form.

73. $(2^x-5)(2^x-13) = 0$
74. $(3^x-9)(3^x-4) = 0$
75. $\left(7^{8x}+14\right)\left(9^{2x}-36\right) = 0$
76. $(e^x+1)(3e^x-2) = 0$

For Exercises 77 - 82, solve each of the following for x. Leave your answers in exact form.

77. $\log(x)+\log(x-5) = \log(24)$
78. $\log_5(2x-6)-3 = 0$
79. $\log_8(4-x) = \log_8(12)-\log_8(-x)$
80. $\log(14x+3) = 2+\log(x-1)$
81. $\ln(x-1) = \ln(1)-\ln(3x)$
82. $\log_7(2x-3)-\log_7(x+1) = \log_7(10)$

83. If \$3600 is invested in an account that earns interest at an annual rate of 4.7%, compounded continuously, how long will it take for the account to reach \$12,000? Round your answer to the nearest year.

84. At what annual interest rate must you invest \$1000 in order to have \$100,000 in 45 years, assuming the account is compounded continuously? Round your percentage to four decimal places.

Mastery Practice (Answers)

85. Given $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ -2x-3 & \text{if } x > 0 \end{cases}$, determine if $f(x)$ is a one-to-one function.

86. Given $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ -2x-3 & \text{if } x > 0 \end{cases}$, determine if $f(x)$ is a one-to-one function.

87. Determine if $f(x) = x^2 - 10x$, for $x \geq 5$, and $g(x) = 5 + \sqrt{x+25}$ are inverse functions.

88. Express the given expression as a single logarithm, assuming all variables represent positive real numbers.

$$-a\log_3(w) + \log_3(9+z) + b\log_3(x) - c\log_3(y)$$

89. Expand and simplify the given expression. Assume when necessary that all variables represent positive real numbers greater than 3.

$$\log_b\left(\frac{(x-3)y^{7/8}}{bz}\right)$$

90. State the following properties of the logarithmic function $f(x) = \log_a(x-4)+2$ for real number $a > 1$.

    a. Domain

    b. Range

    c. End behavior

    d. x - intercept(s)

    e. y - intercept(s)

For Exercises 91 - 94, state the domain of each algebraic function, using interval notation.

91. $f(x) = \dfrac{e^{\sqrt{x+2}}}{\log_{13}(4-x)}$

92. $g(x) = \dfrac{\ln(-x)}{2 - e^{3x+9}}$

93. $h(x) = \dfrac{\sqrt[6]{4-3x}}{\log_5(x+7)-1}$

94. $j(x) = \dfrac{\log(x+2) - \log(8-x)}{-7+10^{2x-1}}$

95. Solve $e^{x-6} = 3^x$ for x. Leave your answers in exact form.

96. Solve $e^{2x} - e^x = 72$ for x. Leave your answers in exact form.

97. Solve $\log_b(x) = \frac{2}{3}\log_b(8) + 3\log_b(2) - \log_b(16)$ for x, if b is a real number greater than zero and not equal to 1. Leave your answers in exact form.

98. Solve $\log(\log x) = 1$ for x. Leave your answers in exact form.

99. Solve $2\log_5(x+4)+3 = \log_5(1)$ for x. Leave your answers in exact form.

100. Solve $\ln(x) + \ln(x - 2) = 1$ for x. Leave your answers in exact form.

101. If you invest \$500 in an account that earns interest at a rate of 5.6% per year, compounded quarterly, how long will it take the account to reach \$20,000. Round your answer to the nearest year.

102. If the population of an organism can be modeled by the function $P(t) = 350.2(1.045)^t$, where t is time in years, after how many years will the population reach 2000. Round your answer to three decimal places.

Communication Practice (Answers)

103. Describe what it means for a function to be one-to-one.

104. Explain why it is important to state the domain of each expression in an equation, when solving for x.

Chapter Review

Directions: Read each reflection question. If you are able to answer the question, then move on to the next question. If not, use the mathematical questions included to help you review. (Answers)

1. Can you explain what properties make a relation a function?

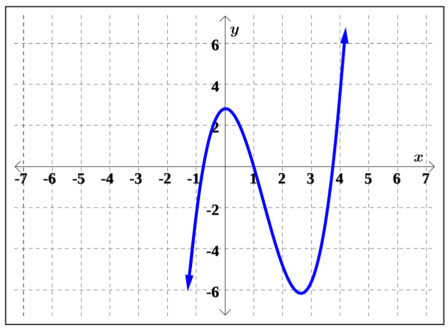
   a. Explain to someone outside of a math class how you can tell if a graph of a relation represents a function.

   b. Determine whether the relation "the input is a state and the output is the state's capital city" is a function. Justify your answer.

   c. Determine whether the relation below represents a function. Justify your answer.

   $$R_a = \{(-3,0),(1,4),(2,-5),(4,2),(-5,6),(3,6),(0,-1),(4,-5),(6,1)\}$$

   d. Determine whether the graph of the relation below represents a function. Justify your answer.



2. What does $f(*)$ represent?

   a. If the number of hours a toy boat will run, H, is a function of how much gas is in the tank, g, in gallons, explain what $H(3) = 8$ means.

   b. If the height (in meters) of the tides in the Bay of Fundy, H, is a function of the time of day, t (in hours after 12 am), explain what $H(9) = 1.76$ means.

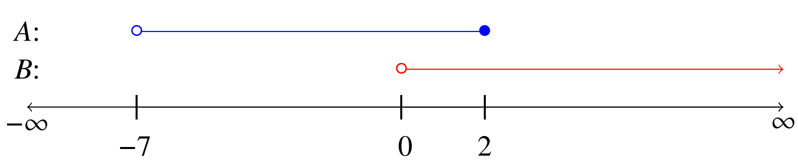
   c. Write $f(3) = 19$ as an ordered pair.

   d. Write $P(20) = 15000$ as an ordered pair.

3. Can you represent a portion of the real number line using interval notation?

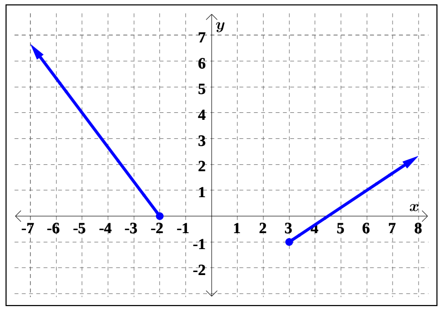
   a. Draw the interval $\left(\frac{3}{4}, 7\right)$ on a number line.

   b. Use interval notation to represent both intervals shown on the number line below.

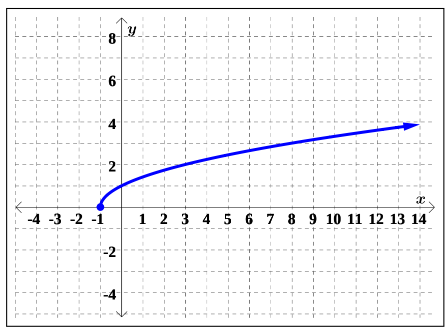


4. Can you list the domain and range of a function from a graph?

   a. State the domain and range of the function given in the graph below, using interval notation.



   b. State the domain and range of the function given in the graph below, using interval notation.



5. Can you describe the properties of a given polynomial, without the use of technology?

   a. Determine if $g(n) = 3n^{20} - \frac{2}{3}n^{197} + 6^{12}$ is a polynomial function or not. If the function is a polynomial, state the degree, leading coefficient, and constant term.

   b. Determine if $h(x) = 9x^{128} - 7^{2x} + 82x^{0.4} - 60$ is a polynomial function or not. If the function is a polynomial, state the degree, leading coefficient, and constant term.

   c. State the end behavior of the polynomial function $p(x) = -2x^4 + x^3 + 7x^2 - x + 9$.

   d. State the end behavior of the polynomial function $q(x) = 7x + 11x^3 - 16x + 2$.

   e. State any x- and y-intercept(s) of the polynomial function $f(x) = x^3 - x^2 - 30x$.

   f. State any x- and y-intercept(s) of the polynomial function $f(x) = 4(x-1)^3(x+2)(x-7)$.

6. Can you select the graphs of the parent polynomial functions from a group of graphs?
   a. Sketch a graph of the constant parent function.

   b. Sketch a graph of the linear parent function.

   c. Sketch a graph of the quadratic parent function.

   d. Sketch a graph of the cubic parent function.

7. Can you explain how to find the domain of a polynomial function, algebraically?
   a. State the domain of the polynomial function $f(x) = \frac{3}{7}x^4 + 10x^2 - 0.5$, using interval notation.

   b. State the domain of the polynomial function $h(x) = (x+1)^3(x-2)(x-5)$, using interval notation.

8. Can you explain what a real root/zero of a function is, in terms a person outside of a mathematics class could understand?
   a. If you are told $x = 3, -2$, and 5 are the real zeros of a polynomial $f(x)$, what else can you say about $f(x)$?

   b. Given the only x-intercepts of $g(x)$ are $(-8, 0)$ and $(-2, 0)$, state all real zeros of $g(x)$.

9. Given $f(x) = ax^2 + bx + c$, what can you say about the graph of $f(x)$, without the use of technology?

   State the domain, range, vertex, y-intercept, and any x-intercepts for each quadratic function.
   a. $f(x) = -4x^2 + 32x - 28$

   b. $g(x) = x^2 - 2x + 1$

   c. $f(x) = \frac{1}{5}x^2 + \frac{4}{5}x + 1$

10. Given a quadratic function, can you differentiate when to apply the quadratic formula and when to factor?

    a. Determine if the quadratic function $g(x) = -2x^2 + 9x - 7$ can be factored.
    i. If the quadratic function factors, then factor and use the factorization to compute the real zeros of function.
    ii. If the quadratic function does not factor, then use the quadratic formula to compute the real zeros of the function.

    b. Determine if the quadratic function $f(x) = 3x^2 + 12x + 5$ can be factored.
    i. If the quadratic function factors, then factor and use the factorization to compute the real zeros of function.
    ii. If the quadratic function does not factor, then use the quadratic formula to compute the real zeros of the function.

    c. Determine if the quadratic function $h(x) = \frac{1}{2}x^2 + 2x - 5$ can be factored.
    i. If the quadratic function factors, then factor and use the factorization to compute the real zeros of function.
    ii. If the quadratic function does not factor, then use the quadratic formula to compute the real zeros of the function.

11. Can you graph a quadratic function without the use of technology?
    - **a.** Graph $f(x) = 0.5x^2 + 3x + 7$. Make sure you note the intercepts and vertex on the graph.
    - **b.** Graph $f(x) = -16x^2 + 240x + 544$. Make sure you note the intercepts and vertex on the graph.

12. Can you explain a business or social science situation where you would develop and use a quadratic function?
    - **a.** A company's profit (in dollars) for selling x thousand items is given by $P(x) = -0.01x^2 + 3.1x - 74$. How many items must be produced and sold to maximize the company's profit? What is their maximum profit?
    - **b.** The total cost (in dollars) of producing x items is given by the function $C(x) = 2x + 35$. The revenue (in dollars) from x items being sold is $R(x) = -x^2 + 122x$. What is the maximum revenue and how many items should be sold to have a maximum revenue? How many items should be produced and sold to maximize the profit?
    - **c.** A college organization is holding a pumpkin chunkin contest. You and a friend build a catapult to toss your pumpkin. The pumpkin flung from the catapult follows a path modeled by $H(d) = -d^2 + 82d + 72$, where $H(d)$ is the height (in meters) of the pumpkin d meters from the catapult. How far did your pumpkin land from the catapult? What was the maximum height your pumpkin reached during its flight?

13. Can you recognize a rational function and explain how to find its domain?
    - **a.** Determine if $f(x) = \dfrac{4x - 3x^2}{2 - x}$ is a rational function. If $f(x)$ is a rational function, state its domain using interval notation.
    - **b.** Determine if $f(x) = \dfrac{7}{x^2 + 5x + 6}$ is a rational function. If $f(x)$ is a rational function, state its domain using interval notation.
    - **c.** Determine if $f(x) = \dfrac{x^{0.5} - 8x^2}{9x}$ is a rational function. If $f(x)$ is a rational function, state its domain using interval notation.

14. Can you describe the properties of a rational function?

    State the domain, y-intercept (if it exists), any x-intercepts, any vertical asymptotes, and any holes for each rational function.
    - **a.** $f(x) = \dfrac{4x - 3x^2}{2 - x}$
    - **b.** $g(x) = \dfrac{7}{x^2 + 5x + 6}$
    - **c.** $h(x) = \dfrac{3x^2 - 48}{(x-2)(x+4)}$

15. When are common denominators necessary when working with rational expressions?
   a. Perform the indicated operation, and write your answer in lowest terms.
$$\frac{x^2+4x+4}{x^2-9}\cdot\frac{x^2-2x-3}{x^2-4}$$
   b. Perform the indicated operation, and write your answer in lowest terms.
$$\frac{a^2-16}{a^3}\div\frac{7a^2+14a-56}{2a-a^2}$$
   c. Perform the indicated operation, and write your answer in lowest terms.
$$\frac{x+1}{x^2+x-30}+\frac{2}{x+7}$$
   d. Perform the indicated operation, and write your answer in lowest terms.
$$\frac{y-4}{3y^3-11y^2+6y}-\frac{y}{y^2-y-6}+\frac{1}{y}$$

16. Can you write the formula for the difference quotient of $f(x)$ and apply it to any previously discussed function?
   a. Compute the difference quotient $\dfrac{f(7+h)-f(7)}{h}$ for $f(x)=\dfrac{7}{9x-4}$.
   b. Compute the difference quotient $\dfrac{g(x+h)-g(x)}{h}$ for $g(x)=\dfrac{3x}{x+5}$.
   c. Compute the difference quotient $\dfrac{k(x+h)-k(x)}{h}$ for $k(x)=\dfrac{8x-1}{x^2}$.

17. Can you recognize a power/radical function and transition between the two forms of the same function?
   a. Given $f(x)=11x^{\frac{6}{5}}$ state the second form of the function. (i.e. If $f(x)$ is in power form, write its corresponding radical form and visa versa.)
   b. Given $f(x)=\sqrt[6]{3x-4}$ state the second form of the function. (i.e. If $f(x)$ is in power form, write its corresponding radical form and visa versa.)
   c. Given $f(x)=\dfrac{9}{\sqrt[3]{x^2-8}}$ state the second form of the function. (i.e. If $f(x)$ is in power form, write its corresponding radical form and visa versa.)

18. Can you select the graphs of the parent power/radical functions from a group of graphs?
   a. Sketch the graph of the parent square root function.
   b. Sketch the graph of the parent cube root function.

19. Can you explain how to find the domain of a power/radical function based on the index?
    a. State the domain of $f(x) = \sqrt[7]{3x-14}$, using interval notation.

    b. State the domain of $f(x) = \sqrt[4]{9-2x}$, using interval notation.

    c. State the domain of $f(x) = \dfrac{(3x+4)^{\frac{3}{2}}}{(x+9)(x+5)}$, using interval notation.

    d. State the domain of $f(x) = \dfrac{5-x}{(3x+4)^{\frac{2}{3}}}$, using interval notation.

20. Can you explain how a conjugate is used to rationalize an expression?
    a. Rationalize $\dfrac{x-4}{\sqrt{6-x}+9}$.

    b. Compute the difference quotient $\dfrac{g(x+h)-g(x)}{h}$ for $g(x) = 3\sqrt{x}$.

    c. Compute the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for $f(x) = \sqrt{2x-5}+1$.

21. Can you use the values of a piecewise-defined function to graph the function, without the use of technology?
    a. Graph $f(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0 \\ 4 & \text{if} \quad x = 0 \end{cases}$

    b. Graph $f(x) = \begin{cases} -x+1 & \text{if} \quad x < -3 \\ \sqrt{x+4} & \text{if} \quad -3 \leq x \leq 5 \\ x^2-36 & \text{if} \quad x > 5 \end{cases}$

22. Can you explain how to find the domain of a piecewise-defined function?
    a. State the domain of $f(x)$, using interval notation.

    $$f(x) = \begin{cases} 14 & \text{if} \quad x \leq -5 \\ x^2+8x+7 & \text{if} \quad 0 < x \leq 4 \\ \sqrt{x+16} & \text{if} \quad x \geq 4.5 \end{cases}$$

    b. State the domain of $f(x)$, using interval notation.

    $$f(x) = \begin{cases} \frac{x-1}{x+1} & \text{if} \quad x < 1 \\ 4 & \text{if} \quad x \geq 1 \end{cases}$$

    c. State the domain of $f(x)$, using interval notation.

    $$f(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \leq -8 \\ x-3 & \text{if} \quad -8 < x \leq 4 \\ x^2+4x+4 & \text{if} \quad x > 4 \end{cases}$$

23. Can you dissect a real-world scenario to define an equivalent piecewise-defined function?
    a. The admission to a local medieval festival is charged according to age. Children under 3 get in free. Children aged 3 to 14 are charged 2 times their age. All others entering the festival are charged \$29.95. Write a piecewise-defined function to represent what the festival charges a person of age a to enter.

    b. A factory worker earns \$12.50 an hour for their normal weekly shift hours. Any time over 40 hours in a week is considered overtime. Overtime is paid at time and a half for hours over 40. If the factory worker works more than 60 hours during a week, they receive double their normal hourly pay for those additional hours. Write a piecewise-defined function to represent a factory worker's pay in terms of the number of hours, h, the worker works per week. If a worker works 71 hours, how much will their paycheck be for that week?

    c. The price of a sporting event ticket is dependent on the number of tickets sold. If less than 1000 tickets are sold, tickets are \$120 each. Ticket sales of at least 1000, but less than 4000, mean ticket prices will drop to \$100. If the sporting event has more than 4000 tickets sold, the price drops to \$75. Note the stadium can only hold 4675 people. Write the price per ticket as a function of the number of tickets, t, sold. What is the price per ticket when 900, 2300, and 4500 tickets are sold? What is the price per ticket if the event is sold out? (A ticket for each seat is sold.)

24. Can you select the graph of the parent absolute value function from a group of graphs?
    a. Sketch a graph of the parent absolute value function.

25. Can you describe the correspondence between an absolute value function and an equivalent piecewise-defined function?
    a. Write $f(x) = |x+3|$ as a piecewise-defined function.

    b. Write $f(x) = |2-x|$ as a piecewise-defined function.

    c. Write $f(x) = 3|8-5x|$ as a piecewise-defined function.

26. Can you differentiate between an exponential function and a power function?
    a. Determine if $f(x) = 9e^{3x-5}$ is a power function or an exponential function. Explain your reasoning.
    b. Determine if $f(x) = x^3$ is a power function or an exponential function. Explain your reasoning.
    c. Determine if $f(x) = \left(\frac{3}{4}\right)^{x-8}$ is a power function or an exponential function. Explain your reasoning.

27. What component(s) of an exponential function determine if the function grows or decays?
    a. Determine if $f(c) = 5^{2c}$ is an exponential growth or exponential decay function. Explain your reasoning.
    b. Determine if $f(x) = \left(\frac{5}{9}\right)^{x}$ is an exponential growth or exponential decay function. Explain your reasoning.
    c. Determine if $f(t) = \pi^t$ is an exponential growth or exponential decay function. Explain your reasoning.
    d. State the values of a and k for which $f(t) = ae^{kt}$ is an exponential growth function. Explain your reasoning.

28. Can you select the graph of the parent exponential functions from a group of graphs?
   a. Sketch the graph of the parent exponential growth function.

   b. Sketch the graph of the parent exponential decay function.

29. Can you explain how to find the domain of an exponential function?
   a. State the domain of $f(x) = 2^{x-1}$, using interval notation.

   b. State the domain of $g(x) = 7e^{\sqrt{2x+1}}$, using interval notation.

   c. State the domain of $k(x) = \dfrac{4x}{\left(\frac{1}{5}\right)^x}$, using interval notation.

30. Can you simplify an exponential expression fully?
   a. Simplify $\left(\dfrac{3x^{-4}y^8}{3^{-2}x^5y}\right)^3$. Write your answer using positive exponents only, including no radicals.

   b. Simplify $8\left(\sqrt{x} - y^{-\frac{1}{2}}\right)^2$. Write your answer using positive exponents only, including no radicals.

   c. Simplify $\left(xy^{-1} - 3y\right)^0$. Write your answer using positive exponents only, including no radicals.

31. What techniques are necessary to solve an equation involving exponential expressions containing like bases?
   a. Algebraically solve $4^{2x-1} = 64$ for x.

   b. Algebraically solve $e^x e^2 = \dfrac{e^4}{e^{x+1}}$ for x.

   c. Algebraically solve $2\left(\dfrac{1}{3}\right)^{3x} + 8 = 62$ for x.

32. Can you explain a real-world situation where you would solve an exponential equation?
   a. The copy machines in a large department depreciate at a nonlinear rate due to their massive use. Suppose a copier loses its value modeled by the function $V(t) = P\left(\dfrac{4}{7}\right)^t$, where P represents the purchase price and t represents the number of years since purchase. If a copier is purchased for \$260,000, how much is the copier worth in 4 years?

   b. You place \$2500 in a savings account earning annual interest at a rate of 0.6%. How much is in the account after 5 years, if the account is compounded continuously?

   c. The population, in thousands, in a given city is modeled by the function $P(t) = 100.2(1.04)^t$, where t is the number of years since 2010. What is the population of the city in 2019?

33. Can you recognize the basic transformations to a parent function?
   a. State the parent function, $f(x)$, for the transformed function $g(x) = -2\sqrt{x-1} + 3$. Describe the transformations of $f(x)$ to $g(x)$, using the order of transformations discussed.

   b. State the parent function, $f(x)$, for the transformed function $g(x) = \dfrac{1}{5}|x+4| - 2$. Describe the transformations of $f(x)$ to $g(x)$, using the order of transformations discussed.

   c. State the parent function, $f(x)$, for the transformed function $g(x) = 4e^{x+8} + 7$. Describe the transformations of $f(x)$ to $g(x)$, using the order of transformations discussed.

34. How would you graph a transformed parent function and express the transformations symbolically, without the use of technology?
    a. If the graph of $f(x) = x^2$ is shifted right 3 units, vertically stretched by a factor of 2, reflected across the x-axis, and then shifted down 1 unit, what is the equation of the resulting graph?

    b. If the graph of $f(x) = \dfrac{1}{x}$ is shifted left 4 units, reflected across the x-axis, and then shifted up 3 units, what is the equation of the resulting graph?

    c. If the graph of $f(x) = 2^x$ is vertically compressed by a factor of $\dfrac{4}{3}$, and then shifted up $\dfrac{5}{2}$ units, what is the equation of the resulting graph?

35. Can you combine functions?

    Given $f(x) = 4x - 3$, $g(x) = 6x^2 - x - 7$, $h(x) = 7 - 3\sqrt[3]{2x+1}$, and $j(x) = \dfrac{x+1}{2x+5}$, compute each of the following.

    a. $(g - f)(x)$

    b. $(hj)(x)$

    c. $\left(\dfrac{g}{j}\right)(x)$

36. Can you compose two or more functions?

    Let $f(x) = 4x - 3$, $g(x) = 6x^2 + 3x + 8$, $h(x) = 7 - 3\sqrt[3]{2x+1}$, and $j(x) = \dfrac{x+1}{2x+5}$. Compute each of the following.

    a. $(g \circ f)(x)$

    b. $(h \circ g)(x)$

    c. $j(f(x))$

    d. $j(j(x))$

37. How would you know when to combine or compose functions?
    a. Explain the difference between the notation $f(x)g(x)$ and $f(g(x))$.

38. How do you determine if a function has an inverse?
    a. Use the Horizontal Line Test to determine if $f(x) = \dfrac{4x+3}{6x^2-1}$ has an inverse.

39. Can you recognize both the symbolic and graphical representations of a logarithmic function?
    a. Determine if the function $f(x) = \log_3(x^2 - 4)$ is a logarithmic function. Explain your reasoning.

    b. Determine if the function $f(x) = \log_{\frac{4}{5}}(6 - x) + 8$ is a logarithmic function. Explain your reasoning.

    c. Determine if the function $f(x) = \log_{-9}(x^2 - 4)$ is a logarithmic function. Explain your reasoning.

40. Can you transition between exponential and logarithmic forms?
   a. Write the equation $3 = \log_2(8)$ in exponential form.

   b. Write the equation $\ln(x) = 0$ in exponential form.

   c. Write the equation $\log(12) = x$ in exponential form.

   d. Write the equation $5^3 = 125$ in logarithmic form.

   e. Write the equation $7^x = 98$ in logarithmic form.

   f. Write the equation $\sqrt{3^x} = 24$ in logarithmic form.

41. Can you select the graph of a parent logarithmic function from a group of graphs?
   a. Sketch a graph of the parent logarithmic function with $0 < b < 1$.

   b. Sketch a graph of the parent logarithmic function with $b > 1$.

42. Can you explain how to find the domain of a logarithmic function?
   a. State the domain of $f(x) = \ln(x-3)+9$, using interval notation.

   b. State the domain of $g(x) = \log_5(6-x)$, using interval notation.

   c. State the domain of $h(x) = \log\left(\sqrt{5-3x}\right) - 10$, using interval notation.

   d. State the domain of $k(x) = \ln\left(\frac{3}{2}x+9\right)$, using interval notation.

   e. State the domain of $m(x) = \dfrac{9x}{\log_2(x+7)}$, using interval notation.

43. Can you state and use the properties of logarithms?
   a. Use properties of logarithms to write $\log_2(x-5)+\log_2(x)-\log_2(3)$ as a single logarithm.

   b. Use properties of logarithms to write $7+4\log(2x+1)-2\log(3x)-\log(x-6)$ as a single logarithm.

   c. Use the properties of logarithms to write $\ln\left(\dfrac{x(3x+4)}{6}\right)^{\frac{1}{3}}$ in fully expanded form.

   d. Use the properties of logarithms to write $\ln(x-4)^2$ in fully expanded form.

44. How would you solve an equation involving exponential functions with different bases?
   a. Algebraically solve $2+10^{2x}+180 = 245$ for x. Leave all answers in exact form.

   b. Algebraically solve $2e^{0.05t}-4 = 16$ for x. Leave all answers in exact form.

   c. Algebraically solve $7^x = 4^{2x-1}$ for x. Leave all answers in exact form.

45. How would you solve an equation involving logarithms and verify the solutions?
   a. Algebraically solve $3\ln(x+4)-5=9$ for x. Leave all answers in exact form.

   b. Algebraically solve $-22=-14\log(4x+1)-8$ for x. Leave all answers in exact form.

   c. Algebraically solve $\ln\left(x-\frac{3}{2}\right)=-\ln(x)$ for x. Leave all answers in exact form.

   d. Algebraically solve $\log_2(8)=\log_2(x+4)+3$ for x. Leave all answers in exact form.

46. Can you explain a real-world situation where you would solve an exponential equation using logarithms?
   a. The amount in a savings account, compounded annually, is computed by $A=P(1+r)^t$, where A is the accumulated amount, P is the amount of the original deposit, r is the annual interest rate (in decimal form) and t is time since deposit, in years.
      i. If you deposit \$12,000 in this savings account and 3 years later the account has \$15,000, what was the annual interest rate on the account?
      ii. How long would it take to reach \$25,000, if the annual interest rate was only 7%?

   b. You place \$2500 in a savings account earning interest at a rate of 0.6% per year, compounded continuously.
      i. How long will it take the account to double?
      ii. How long would it take the account to triple?

   c. The population, in thousands, in a given city is modeled by the function $P(t)=100.2(1.04)^t$, where t is the number of years since 2010.
      i. How long will it take the city to reach a population of 200,000, if the current model holds?
      ii. How long will it take the city to triple in size, if the current model holds?

VI

Chapter 6

6. Mathematical Finance

In this chapter we are going to discuss interest, effective rates, and compute values in an annuity or sinking fund.

The authors recommend all students are comfortable with the topics below prior to beginning this chapter. If you would like additional support on any of these topics, please refer to the Appendix.

A.1 - Integers

A.1 - Fractions

A.1 - Simplifying Fractions

A.1 - Decimals

A.1 - Integer Exponents and Scientific Notation

A.1 - Properties of Real Numbers

A.1 - Systems of Time Measurement

A.2 - Using Variables and Algebraic Symbols

A.2 - Evaluating an Expression

A.2 - Translating an English Phrase to an Algebraic Expression or Equation

A.2 - Solving Linear Equations with One Variable

A.2 - Using Problem-Solving Strategies

A.3 - Simplifying Expressions with Exponents

6.1 Interest and Effective Rates

© Photo by MarkyB, 2020; Edits by Jason Allen, 2020

Most adults have to work with money every day, regardless of their job or income. While balancing your checkbook or calculating your monthly expenditures on Aggie t-shirts requires only arithmetic, when we start saving, planning for retirement, or need a loan, we need more sophisticated mathematics. Understanding the financial situations helps us to make decisions personally, as well as professionally.

Learning Objectives:
In this section, you will learn about various ways to solve simple and compound interest problems related to bank accounts and to calculate the effective rate of interest. Upon completion you will be able to:

- Apply the simple interest formula to various financial scenarios.
- State the difference between simple interest and compound interest.
- Use technology to solve compound interest problems, not involving continuously compounded interest.
- Apply the continuously compounded interest formula to various financial scenarios.
- Compute the effective rate of interest, using technology when possible.
- Compare multiple accounts, using the effective rates of interest/effective annual yields.

Working with Simple Interest

It costs money to borrow money. The rent one pays for the use of money is called **interest**. The amount of money that is being borrowed or loaned is called the **principal** or **present value**.

Interest, in its simplest form, is called **simple interest** and is paid only on the original amount borrowed. When the money is loaned out, the person who borrows the money generally pays a fixed rate of interest on the principal for the time period the money is kept. Although the interest rate is often specified for a year (**annual percentage rate**), it may be specified for a week, a month, a quarter, etc. When a person pays back the money owed, they pay back the original amount borrowed plus the interest owed on the loan, which in total is called the **accumulated amount** or **future value**.

Definition

Simple interest is the interest that is paid only on the principal, and is given by

$$I = Prt$$

where,

I = Interest earned or paid,

P = Present Value or Principal,

r = *Annual** percentage rate (APR) changed to a decimal, and

t = Number of *years**.

*The units of time for r and t must be the same.

Formula for Total Owed/Paid with Simple Interest

$$\begin{aligned} A &= P + I \\ &= P + Prt \\ A &= P(1 + rt) \end{aligned}$$

where,

A = Future Value or Accumulated Amount,

P = Present Value or Principal,

r = *Annual** percentage rate (APR) changed to a decimal, and

t = Number of *years**.

*The units of time for r and t must be the same.

Example 1 Ursula borrows $600 for 5 months at a simple interest rate of 15% per year.

a. Compute the interest paid on the borrowed amount.

b. Calculate the total amount Ursula is obligated to pay back at the end of the 5 months.

Solution:

Because Ursula is borrowing $600, $P = 600$. The annual interest rate is 15%, so $r = 0.15$. We are looking for the interest after 5 months, so $t = \frac{5}{12}$, as the rate is given in *years* and time units must match.

a. Because the interest calculated is *simple* interest,

$$\begin{aligned} I &= Prt \\ &= 600(0.15)\left(\frac{5}{12}\right) \\ I &= \$37.50 \end{aligned}$$

b. The total amount Ursula is obligated to pay back is given by

$$\begin{aligned} A &= P + I \\ &= 600 + 37.50 \\ A &= \$637.50 \end{aligned}$$

Alternatively, the total amount Ursula will pay back can be computed directly as follows:

$$\begin{aligned} A &= P(1+rt) \\ &= 600\left[1+(0.15)\left(\frac{5}{12}\right)\right] \\ &= 600(1+0.0625) \\ &= 600(1.0625) \\ A &= \$637.50 \end{aligned}$$

■

■ **Example 2** Ben wants to buy a used car worth \$3500, but only has \$3000. Ben decides to invest his \$3000 in an account earning 6% annual simple interest. How long will Ben need to leave the \$3000 in the account to accumulate the \$3500 he wants to spend?

Solution:

> "Ben wants to buy a *used car worth \$3500*, but *only has \$3000*. Ben decides to invest his \$3000 in an account earning *6% annual* **simple** *interest*."

From the given statement, $P = 3000$, $A = 3500$, and $r = 0.06$.
Because the interest earned is **simple interest**, we use $A = P(1+rt)$ to determine how long Ben will need to leave his money in the account.

$$\begin{aligned} \mathbf{A} &= \mathbf{P(1+rt)} \\ 3500 &= 3000(1+0.06t) \\ \frac{3500}{3000} &= 1+0.06t \\ \frac{3500}{3000} - 1 &= 0.06t \\ \frac{\frac{3500}{3000}-1}{0.06} &= t \\ t &\approx 2.8 \text{ years} \end{aligned}$$

Ben would need to invest his \$3000 for about 2.8 years until he would have the \$3500 he would like to spend on a used car.

■

N *It is a sound practice to wait to round your answer until the very last step, in order to get the most accurate answer. For answers involving dollars, your answer should be rounded to the nearest cent, unless given directions otherwise.*

Try It # 1:
Darnel borrowed some money from a friend. Darnel now owes the friend a total of \$3060, which includes 12% annual simple interest for the three years over which the money was borrowed. How much did Darnel originally borrow?

Working with Compound Interest

Simple interest is normally charged when the lending period is short and often less than a year. When the money is loaned, borrowed, or invested for a longer time period (mortgages, auto loans, savings), the interest is paid (or charged) not only on the principal, but also on the past interest, and we say the interest is **compounded**. Interest on a mortgage or auto loan is usually compounded monthly. Interest on a savings account can be compounded quarterly. Interest on a credit card can be compounded weekly or daily.

Suppose that we deposit \$1000 in a bank account offering 3% annual interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) - the total interest to be paid during the year. Due to the fact that interest is being paid monthly, each month we will earn $\frac{3\%}{12} = 0.25\%$. The interest is calculated based on the balance of the account at the beginning of each month.

In the first month, $P = \$1000$, $r = 0.0025$ (0.25%), and $t = 1$.

$$\begin{aligned} I &= Prt \\ &= 1000(0.0025)(1) \\ I &= \$2.50 \end{aligned}$$

and then

$$\begin{aligned} A &= P + I \\ &= 1000 + 2.50 \\ A &= \$1002.50 \end{aligned}$$

So, in the first month, we will earn \$2.50 in interest, raising our account balance to \$1002.50.

In the second month, now $P = \$1002.50$, and

$$\begin{aligned} I &= (1002.50)(0.0025)(1) \\ &= 2.50625 \\ I &= \$2.51 (\text{rounded}) \end{aligned}$$

meaning

$$\begin{aligned} A &= \$1002.50 + 2.51 \\ A &= \$1005.01 \end{aligned}$$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1000 we deposited, but also on the \$2.50 of interest we earned the first month.

Calculating what happens month-by-month for a year gives us the information in **Table 6.1**.

Month	Starting Balance	Interest Earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

Table 6.1: The monthly account growth over the first year. All values are rounded to the nearest cent.

To develop an equation to represent this process, if P_N represents the amount of money after N months (so P_{N-1} represents the amount of money at the end of the previous month), then the following relationship is true for this scenario:

P_0 = the initial amount = \$1000
$P_N = (1 + 0.0025)P_{N-1}$

And,

$$P_1 = 1.0025P_0 = 1.0025(1000)$$
$$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = (1.0025)^2(1000)$$
$$P_3 = 1.0025P_2 = 1.0025\Big((1.0025)^2(1000)\Big) = (1.0025)^3(1000)$$
$$P_4 = 1.0025P_3 = 1.0025\Big((1.0025)^3(1000)\Big) = (1.0025)^4(1000)$$

Observing a pattern, we could conclude

$$P_N = (1.0025)^N(1000)$$

Notice that the \$1000 in this equation is equal to P_0 (the starting amount) and 1.0025 is equal to 1 plus the monthly interest rate, 0.0025.

Generalizing our result, we could write $P_N = P_0\left(1 + \frac{r}{m}\right)^N$,
where r is the annual interest rate (in decimal form) and m is the number of compounding periods per year (months in our scenario).

While this formula works fine, it is more common to use a formula that involves the number of years the money is invested. If t is the number of years of the investment, then $N = mt$. Making this change gives us the following standard formula for compound interest.

Formula for Total Owed/Paid with Compound Interest

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

where,
A = Future Value or Accumulated Amount,
P = Present Value or Principal,
r = Annual percentage rate (APR) changed to a decimal,
t = Number of years, and
m = Number of compounding periods per year.

Table 6.2 summarizes different compounding types we might encounter and their corresponding number of compounding periods each year, which will be used in the formula above.

Compounding Type	**Number of Compounding Periods per Year,** m
Annually	1
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

Table 6.2: Compounding Period Information

The formula for compound interest should look familiar, as it was previously introduced as an example of an exponential function.

The most important thing to remember about using this compound interest formula is that it assumes that we put money into an account **once** and leave it there to earn interest.

Example 3 Let's compare a savings plan that pays 6% annual simple interest versus another plan that pays 6% annual interest, compounded quarterly. If we deposit \$8000 into each savings account, how much money will we have in each account after three years? How much interest is earned on each account?

Solution:
For "a savings plan that pays 6% annual **simple** interest ... If we deposit \$8000 into [the] savings account, how much money will we have in [the] account after three years?" Using the simple interest formula, with $P = 8000$, $r = 0.06$, and $t = 3$,

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{P(1+rt)} \\ &= 8000(1+0.06\cdot 3) \\ A &= 9440 \end{aligned}$$

Thus, we have \$9440.00 in the simple interest bearing account after three years. Using the fact that $A = P + I$, in this account we earned interest in the amount of

$$\begin{aligned} I &= A - P \\ &= 9440 - 8000 \\ I &= \$1440.00 \end{aligned}$$

For "another plan that pays 6% annual interest **compounded** quarterly. If we deposit \$8000 into [the] savings account, how much money will we have in [the] account after three years?" Using the compound interest formula, with $P = 8000$, $r = 0.06$, $t = 3$, and $m = 4$,

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{P}\left(\mathbf{1} + \frac{\boldsymbol{r}}{\boldsymbol{m}}\right)^{\boldsymbol{mt}} \\ &= 8000\left(1 + \frac{0.06}{4}\right)^{4\cdot 3} \\ &= 8000\left(1 + \frac{0.06}{4}\right)^{12} \\ A &\approx 9564.95 \end{aligned}$$

So, we have \$9564.95 in the account compounded quarterly, after three years. In this account, we earned interest in the amount of

$$\begin{aligned} I &= A - P \\ &= 9564.95 - 8000 \\ I &= \$1564.95 \end{aligned}$$

Therefore, we receive more money from accounts earning compound interest than from accounts earning simple interest.

■

When comparing compound and simple interest bearing accounts with the same annual rate, the account earning compound interest will always accumulate more money.

While we could continue to use the formula for compound interest in problems where money is only put into an account once and interest is compounded a finite number of times per year, the authors choose to turn their focus to the use of technology for such problems.

Using a TI-84 calculator, there exists a Finance Application called the TVM Solver. **Figures 6.1.2** and **6.1.3** show the steps to reach the TVM Solver.

First, select "Applications" by pressing the $\boxed{\text{APPS}}$ button and then press $\boxed{1}$ for the "Finance" option.

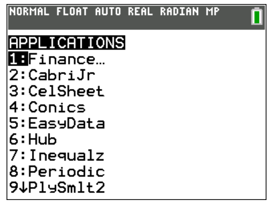


Figure 6.1.2: Calculator screenshot showing the Applications menu.

Now, press $\boxed{1}$ to select "TVM Solver...".

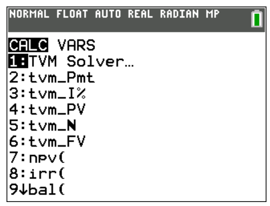


Figure 6.1.3: Calculator screenshot showing the Finance menu.

When using the TVM Solver (shown in **Figure 6.1.4**),

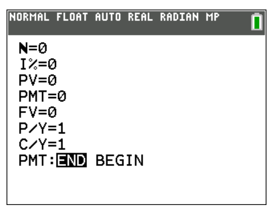


Figure 6.1.4: Calculator screenshot with the TVM Solver menu.

you fill in all but the one entry you are solving for, according to the following.

N = $m * t$ (the total number of compounding periods or the total number of payments (assume P/Y = C/Y below))

I% = the *annual* interest rate (*as a %*)

PV = P (Present Value or Principal)

PMT = regular payment amount per period ($0 in this section)

FV = A (Accumulated Amount or Future Value)

P/Y = the number of payments made per year (autofills as m)

C/Y = m (the number of compounding periods per year)

PMT: **END** BEGIN (the payments are made at the end of the period, even if no payments are made)

Then, you move the cursor to the entry in question. At this point it does not matter what value this entry currently contains, as the application will overwrite the entry with the result when you press $\boxed{\text{ALPHA}}$ $\boxed{\text{ENTER}}$. For all futures examples involving the TVM Solver, we will use "?" to denote the entry in question.

Caution: When using the TVM Solver all 'money' entries (PV, PMT, FV) should be from the perspective of the investor or borrower.

- A negative monetary value indicates money "leaving" the investor or borrower, such as deposits, payments, or investments.
- A positive monetary value indicates money "coming to" the investor or borrower, such as receiving a loan or money returned from an investment.

! *If 'non-money' entries are negative (N, $I\%$, P/Y, C/Y), then usually some entry was entered incorrectly.*

Let's return to our last example:

> If we deposit \$8000 in a savings plan that pays 6% annual interest, compounded quarterly, how much money will we have in the account after three years?

Using the calculator, instead of the formula, to determine how much money will be in the account, the entries for the TVM Solver would be as follows and can be seen in **Figure 6.1.5**.

N = $m \cdot t = 4 \cdot 3$
I% = 6
PV = −8000 (negative – money is deposited)
PMT = 0
FV = ? (this is what we are solving for)
P/Y = 4
C/Y = m = 4
PMT: **END** BEGIN

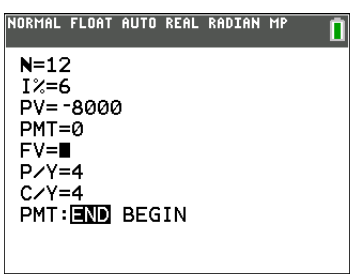


Figure 6.1.5: Calculator screenshot of the TVM Solver menu. All entries of the example, except the FV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

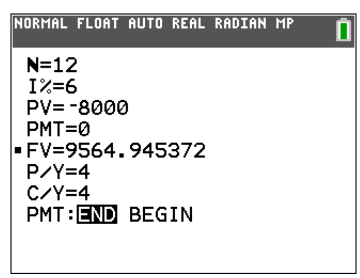


Figure 6.1.6: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for FV is now shown.

Solving for FV, we can see in **Figure 6.1.6** that FV = 9564.945372, which means we have \$9564.95 after three years. Notice that this is the same value found when using the compound interest formula.

Example 4 How much should be invested into an account paying 9% annual interest, compounded daily, for it to accumulate to \$5000 in five years?

Solution:

> "How much should be invested into an account paying 9% annual interest, compounded daily, for it to accumulate to \$5000 in five years?

So, $P = ?$, $r = 0.09$ or 9%, $m = 365$, $t = 5$, and $A = 5000$.

The entries for the TVM Solver (shown in **Figure 6.1.7**) will be:

N = $365 \cdot 5$
I% = 9
PV = ? (this is what we're solving for)
PMT = 0
FV = 5000 (positive – money is returned from an investment)
P / Y = 365
C / Y = $m = 365$
PMT: **END** BEGIN

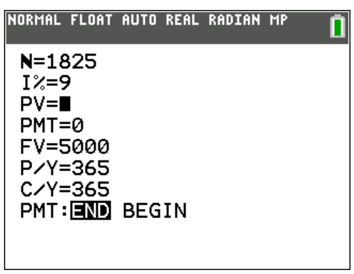


Figure 6.1.7: Calculator screenshot of the TVM Solver menu. All entries, except the PV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

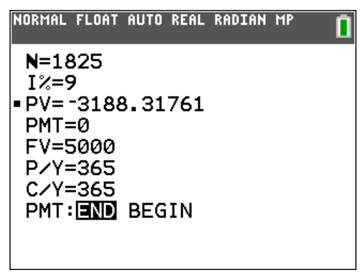


Figure 6.1.8: The calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PV is now shown.

Solving for PV, we can see in **Figure 6.1.8** that PV = −3188.31761, which means that we should invest \$3188.32 into the account.

N *Remember, while PV was negative in the calculator, the final answer is not negative. The negative indicated that \$3188.32 was invested and "left" us.*

■

Example 5 A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying annual interest, compounded monthly. If after 20 years, the CD has \$9930.62, at what rate is the account earning interest?

Solution:

> "Suppose you deposit \$3000 in a CD paying annual interest, compounded monthly. If after 20 years, the CD has \$9930.62, at what rate is the account earning interest? "

So, $P = 3000$, $r = ?$, $m = 12$, $t = 20$, and $A = 9930.62$.

The entries for the TVM Solver (shown in **Figure 6.1.9**) will be:

N = $12 \cdot 20$
I% = ? (this is what we are solving for)
PV =-3000 (negative – money is deposited)
PMT = 0
FV = 9930.62 (positive – money is returned from an investment)
P / Y = 12
C / Y = 12
PMT: **END** BEGIN

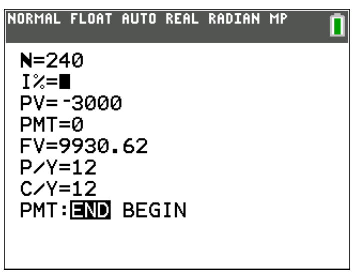


Figure 6.1.9: The calculator screenshot of the TVM Solver menu. All entries, except the I% entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

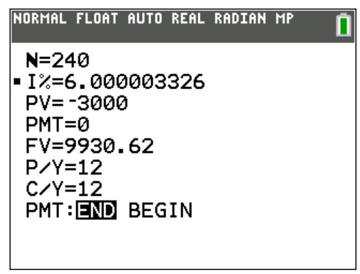


Figure 6.1.10: The calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for I% is now shown.

Solving for I%, we can see in **Figure 6.1.10** that I% = 6.000003326, which means the CD has an interest rate of 6%. ■

■ **Example 6** Geneva wants to save \$12,000 to buy a new car. She just received an \$8000 bonus and plans to invest it into an account earning 7% annual interest, compounded monthly. How long will Geneva need to leave the \$8000 in the account to accumulate the \$12,000 she needs?

Solution:

> "Geneva wants to save \$12,000 to buy a new car. She just received an \$8000 bonus and plans to invest it into an account earning 7% annual interest, compounded monthly. How long will she need to leave her money in the account to accumulate the \$12,000 she needs?"

So, $P = 8000$, $r = 0.07$ or 7%, $m = 12$, $t = ?$, and $A = 12000$.

The entries for the TVM Solver (shown in **Figure 6.1.11**) will be:

N = ? (while we know m, we do not know t)
I% = 7
PV = −8000 (negative – money is deposited)
PMT = 0
FV = 12000 (positive – money is returned from an investment)
P / Y = 12
C / Y = 12
PMT: **END** BEGIN

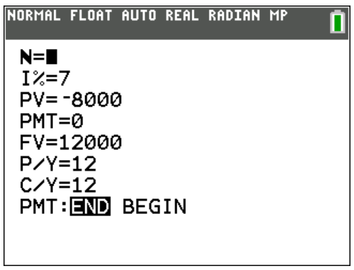


Figure 6.1.11: Calculator screenshot of the TVM Solver menu. All entries, except the N entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

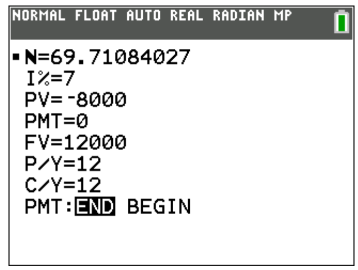


Figure 6.1.12: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for N is now shown.

Solving for N, we can see in **Figure 6.1.12** that N = 69.71084027. N represents the total number of compounding periods, in months here, so it will take 69.7 months for the account to reach \$12,000. Because compounding is done at the end of each month, we round up to 70 months. Therefore, the account will reach \$12,000 in 5 years and 10 months.

▪

Try It # 2:
Sophia's grandparents bought her a savings bond for \$200 when she was born. The interest rate was 3.28% compounded semiannually, and the bond would mature in 30 years. How much will Sophia's bond be worth when she turns 30?

Interest can be compounded any finite number of times per year: annually, semiannually, quarterly, monthly, weekly, daily, hourly, every minute, and even every second. However, what do we mean when we say the interest is compounded **continuously**, and how do we compute such amounts? When interest is compounded "infinitely many times," we say that the interest is **compounded continuously**. As infinity is not a finite number, we cannot use the TVM Solver. Instead, we will use the same formula discussed in the Exponential Functions section, which can be derived from the compound interest formula, using techniques from calculus.

Continuously Compounded Interest Formula

$$A = Pe^{rt}$$

where,
A = Future Value or Accumulated Amount,
P = Present Value or Principal,
r = Annual percentage rate (APR) changed to a decimal, and
t = Number of years.

Example 7 The Isabel family needs $100,000 to buy a house in 5 years. How much should the Isabel family deposit into an account now, earning 5.7% annual interest, compounded continuously, in order to reach their goal?

Solution:

> "The Isabel family needs $100,000 to buy a house in 5 years. How much should the Isabel family deposit into an account now, earning 5.7% interest, **compounded continuously**, in order to reach their goal?"

So, $P = ?$, $r = 0.057$, $t = 5$, and $A = 100000$.

Using the continuously compounded interest formula,

$$\begin{aligned} A &= Pe^{rt} \\ 100000 &= Pe^{(0.057)(5)} \\ \frac{100000}{e^{(0.057)(5)}} &= P \\ \$75,201.43 &\approx P \end{aligned}$$

Thus, the Isabel family needs to deposit $75,201.43 in the account now to reach their goal.

■

Try It # 3:
Suppose you are investing $1000 in an account paying 5% annual interest, compounded continuously. How much would be in the account after 30 years, assuming there are no additional deposits or withdrawals made?

Comparing Interest Rates

For comparison purposes, the government requires the bank to state their interest rate in terms of an **effective interest rate**. This is also known as **effective yield** or **annual percentage yield** (**APY**). The effective interest rate gives the actual percentage by which a balance increases in one year.

Let's suppose you invest a dollar, for one year, in each of the following accounts.

a. 7.7% annual interest, compounded monthly
b. 7.7% annual interest, compounded daily
c. 7.7% annual interest, compounded continuously

What is the annual percentage yield for each account?

a. Given a dollar is invested for one year at 7.7% annual interest, **compounded monthly**, we have $P = 1$, $r = 0.077$, $t = 1$, $m = 12$, and get

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{P}\left(\mathbf{1} + \frac{\boldsymbol{r}}{\boldsymbol{m}}\right)^{\boldsymbol{mt}} \\ &= 1\left(1 + \frac{0.077}{12}\right)^{12\cdot 1} \\ A &\approx 1.079776 \end{aligned}$$

Thus, the \$1 grew to \$1.079776, so the percentage the \$1 increased by was 7.9776%. This means, the APY is 7.9776%.

b. Given a dollar is invested for one year at 7.7% annual interest, **compounded daily**, we have $P = 1$, $r = 0.077$, $t = 1$, $m = 365$, and get

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{P}\left(\mathbf{1} + \frac{\boldsymbol{r}}{\boldsymbol{m}}\right)^{\boldsymbol{mt}} \\ &= 1\left(1 + \frac{0.077}{365}\right)^{365\cdot 1} \\ A &\approx 1.080033 \end{aligned}$$

Thus, the \$1 grew to \$1.080033, so the percentage the \$1 increased by was 8.0033%. This means, the APY is 8.0033%.

c. Given a dollar is invested for one year at 7.7% annual interest, **compounded continuously**, we have $P = 1$, $r = 0.077$, $t = 1$, and get

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{P}\boldsymbol{e}^{\boldsymbol{rt}} \\ &= 1e^{(0.0077)(1)} \\ A &\approx 1.080042 \end{aligned}$$

Thus, the \$1 grew to \$1.080042, so the percentage the \$1 increased by was 8.0042%. This means, the APY is 8.0042%.

Because all the annual interest rates were the same (7.7%), we found that the APY increases as the number of compounding periods increases, and that the account compounded continuously had the highest APY.

N *For a fixed annual interest rate, as the number of compounding periods increases, the annual percentage yield/effective interest rate increases, too.*

The effective interest rate is often used to compare accounts that have different compounding intervals and have different stated annual interest rates.

Definition

The **effective interest rate** or **annual percentage yield** is given by

$$\boldsymbol{r_{eff} = \left(1+\frac{r}{m}\right)^m - 1}$$

when interest is compounded m times a year, and

$$\boldsymbol{r_{eff} = e^r - 1}$$

when interest is compounded continuously,

where,

r = Annual percentage rate changed to a decimal, and

m = Number of compounding periods per year.

r_{eff} is computed as a decimal, but compared as a percent, therefore the decimals must be converted back to percentages. ▪

▪ **Example 8** Joe had fallen asleep watching TV, and awoke to an advertisement describing the following accounts.

Account A: 3.62% APR, compounded semiannually
Account B: 3.6% APR, compounded weekly

Joe missed whether these accounts were for investment or borrowing purposes. Explain which account is better for each purpose.

Solution:
Due to the fact that the annual interest rates and compounding periods are different for both accounts, we will compute the effective interest rate of each account in order for us to accurately compare the two accounts.

<u>Account A</u>:

We know $r = 0.0362$ and $m = 2$. Thus,

$$\begin{aligned} r_{eff} &= \left(1+\frac{r}{m}\right)^m - 1 \\ &= \left(1+\frac{0.0362}{2}\right)^2 - 1 \\ &= (1+0.0181)^2 - 1 \\ &\approx 0.03652761 \end{aligned}$$

Therefore, the APY for Account A is approximately 3.6528%.

Account B:

We know $r = 0.036$ and $m = 52$. Thus,

$$\begin{aligned} r_{eff} &= \left(1 + \frac{r}{m}\right)^m - 1 \\ &= \left(1 + \frac{0.036}{52}\right)^{52} - 1 \\ &\approx 0.0366429342 \end{aligned}$$

Therefore, the APY for Account B is approximately 3.6643%.

For investment purposes, in general, you want to make/earn the most money possible, and so you want a higher APY. Given this reasoning you would invest in Account B.

For loan/borrowing purposes, in general, you want to owe/pay the least amount of money possible, thus you want a lower APY. So you would borrow money using Account A.

■

We can return to our calculator when computing effective rates of interest, as long as the number of compounding periods is finite.

The TI-84 calculator has an effective interest rate option under the Finance Application called Eff. See **Figure 6.1.13** for the first nine options under the Finance Application and **Figure 6.1.14** for the remaining options, which includes effective interest rate.

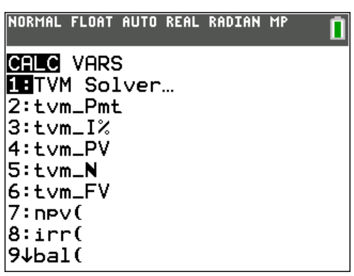


Figure 6.1.13: Calculator screenshot showing the Finance menu.

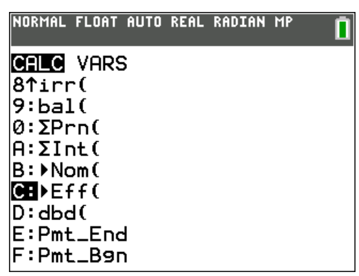


Figure 6.1.14: Calculator screenshot showing the Finance menu with option C:Eff highlighted.

The Eff application requires two arguments, separated by a comma:

$$\text{Eff(annual interest rate as \%, number of compounding periods)} = \text{Eff}(I\%, m)$$

The result of the Eff application is a percentage.

For **Example 8**, we could compute the APY using the calculator, as shown in **Figures 6.1.15** and **6.1.16**.

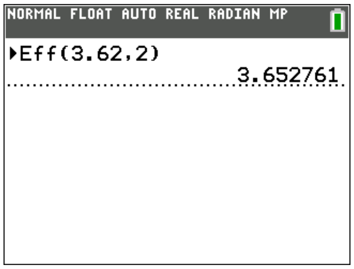


Figure 6.1.15: Calculator screenshot displaying the Eff application with I% = 3.62% and m = 2 for Account A.

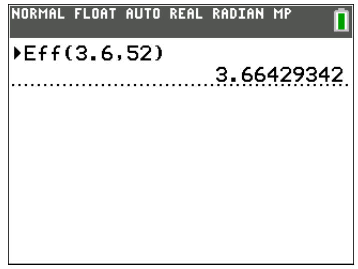


Figure 6.1.16: Calculator screenshot displaying the Eff application with I% = 3.6% and m = 52 for Account B.

The Eff application only works when m is finite. If an account is compounded continuously, the formula, $r_{eff} = e^r - 1$, *must be used for an accurate result.*

Try It # 4:
For the following accounts, state the better investment option and better loan option.

Account A: 5.23% APR, compounded continuously
Account B: 5.25% APR, compounded daily
Account C: 5.27% APR, compounded semiannually

Try It Answers

1. \$2250
2. \$530.77
3. \$4481.69
4. Investment: Account B
   Loan: Account C

EXERCISES

Basic Skills Practice (Answers)

Simple Interest

1. You invest \$85 in an account paying simple interest at a rate of 2.5% per year. If you make no additional deposits or withdrawals, how much interest does the account earn in 15 years?
2. You borrow \$400 from a payday loan company which charges simple interest at a rate of 175% per year. How much interest do you owe on the loan after 2 months (1/6 of a year)?
3. You invest \$625 in an account paying simple interest at a rate of 1.35% per year. If you make no additional deposits or withdrawals, how much money is in the account after 5 years?
4. You borrow \$1000 from a friend for 3 years who charges you simple interest at a rate of 1% per year. How much money do you owe your friend?

Compound Interest

5. Given $P = \$100$, $r = 0.08$, $m = 4$, $t = 5$, compute A.
6. Given $P = \$19{,}000$, $r = 0.035$, $m = 365$, $t = 2$, compute A.
7. Given $A = \$100$, $r = 0.08$, $m = 4$, $t = 5$, compute P.
8. Given $A = \$835$, $r = 0.0127$, $m = 12$, $t = 50$, compute P.

Continuously Compounded Interest

9. You invest \$30 in an account with an annual interest rate of 4.06%, compounded continuously. How much money is in the account after 20 years, assuming no other deposits or withdrawals are made?
10. An account currently has \$250,000 in it. How much money was originally placed in the account 10 years ago, if the account has an annual interest rate of 2.3%, compounded continuously? (Assume no other deposits or withdrawals were made.)

For Exercises 11 - 16, compute the effective interest rate, given the nominal interest rate and compounding period.

11. 2.6%, quarterly
12. 1.07%, weekly
13. 4.5%, semiannually
14. 6.75%, monthly
15. 3.4%, daily
16. 8.95%, continuously

Intermediate Skills Practice (Answers)

17. You want to save \$12,000 to buy a new car. You just received an \$8000 bonus and plan to invest it in an account earning 7% annual simple interest. How long will you need to leave your money in the account to accumulate the \$12,000 you need?
18. If you deposit \$1500 into a savings account and earn \$180 of simple interest in 3 years, what was the annual simple interest rate on the account?

19. How much money was borrowed at a simple interest rate of 2.8% per year, if after 1.5 years \$521 is owed?

20. You deposited \$1000 into a savings account earning 4.6% APR, compounded quarterly. How much will you have in your account after 15 years?

21. Suppose you need \$1230 to purchase a new TV in three years. If the interest rate of a savings account is 3.8% APR, compounded monthly, how much money do you need to deposit in the savings account today to have accumulated enough money to buy the TV?

22. You invest \$15,000 in an account earning 1.9% APR, compounded daily.
    a. How much money will you have in the account after 8 years, assuming no other deposits or withdrawals were made?
    b. If you then move the money in this account to an account earning 4.7% per year, compounded monthly, and leave the money for 8 more years, how much money will you have in the second account?

23. You inherited \$35,000 from a family member. You decide to put the money in a savings account with an interest rate of 7.6% per year, compounded semiannually.
    a. How much money is in the account after 29 years, assuming you make no additional deposits or withdrawals?
    b. How much interest did the account earn in 29 years?

24. How long will it take an account with an interest rate of 3.5% per year, compounded continuously, to double?

25. What is the interest rate, as a percent, on an account that is compounded continuously, if \$1000 grows to \$2149 in 17 years? Round your percentage to the nearest two decimal places.

26. The bank is offering a loan with an APR of 3.79%, compounded monthly. What is the effective interest rate on the loan?

27. What is the effective yield on a savings account with an interest rate of 2.875% per year, compounded semiannually?

28. You are told the *effective interest rate* on an account is 2.5%, if the interest is compounded continuously. What is the *annual percentage rate* for the account?

Mastery Practice (Answers)

29. You received a student loan of \$48,000 at 5.4% annual simple interest, with the expectation you would pay the loan back, with interest, at the end of one year. How much do you owe at the end of one year? How much total interest will you pay?

30. You borrow \$7500 from your parents. They charge you simple interest at a rate of 0.7% per year. If you wait 6 years to pay them back, how much money will you owe them?

31. You deposit \$50,000 in a certificate of deposit, CD, for 9 months. The CD pays interest at a simple interest rate of 2% per year. How much money is in the CD at the end of the 9 months?

32. Suppose you would like to take six months off and cycle the Great Divide Mountain Bike Route from Jasper, Alberta, Canada to Antelope Wells, New Mexico. You estimate the trip will cost you $8000. The bank is currently offering a savings account with an interest rate of 2.25%, compounded monthly.
    a. If you would like to take this trip 5 years from now, how much money do you need to deposit in the savings account now to reach your goal?
    b. If you only have $2400 to invest now, how long will it be before you reach your goal? Assume you make no additional deposits or withdrawals.
    c. What interest rate would the bank need to offer you to meet your goal in 5 years, if you invest $2400 now?

33. You invest $40,000 in an account earning 1.56% APR, compounded weekly. How much total interest is earned on the account after 72 weeks? Round your answer to the nearest cent.

34. If the total interest earned on a $700 investment was $48 over 5 years and interest on the account is compounded continuously, what was the interest rate, as a percent, on the account? Round your percentage to two decimal places.

35. You are discussing your options for a retirement account with the human resources department at your new job. Which account option below is best for you?
    a. Account A: 5.22% APR, compounded continuously
    b. Account B: 5.23% APR, compounded daily
    c. Account C: 5.24% APR, compounded monthly
    d. Account D: 5.26% APR, compounded semiannually

36. You are discussing your loan options for a car with the dealership. Which account option below is best for you?
    a. Account A: 3.65% APR, compounded continuously
    b. Account B: 3.66% APR, compounded weekly
    c. Account C: 3.68% APR, compounded monthly
    d. Account D: 3.69% APR, compounded quarterly

Communication Practice (Answers)

37. Give examples of words in an application that would indicate the TVM Solver can be used to solve the application.

38. Explain why you should look for a higher effective interest rate when you are trying to save money and a lower effective interest rate when you are borrowing money, in general.

39. For a fixed interest rate, explain what happens to an account as the number of compounding periods increases.

6.2 Annuities, Sinking Funds, and Amortization

© Photo by Greg Klein, 2020

In 1975, the Jackson family started to save money to buy a house, and in 1980 used the money they had saved as a down payment on the house. They financed the remainder of the price of the house with a 30-year loan at an annual interest rate of 9.8%, compounded monthly.

Scenarios like buying a car, investing for the future, and the Jackson family buying a home are all examples of multi-payment finance.

Learning Objectives:
In this section, you will learn about the future value of an ordinary annuity, the present value of an ordinary annuity, amortizations, and equity. Upon completion you will be able to:

- Use technology to compute the future value (FV) of an account in which payments are made on a regular basis.
- Use technology to compute the payments (PMT) needed for a given account, including retirement funds.
- Use technology to compute the time required for an account receiving regular payments to reach a specified dollar amount or to pay off a specified dollar amount of debt.
- Use technology to compute the present value (PV) of an account required to make specified payments over a given period of time.
- Calculate a down payment and understand how it relates to the amount of a loan.
- Calculate the total interest earned or paid on an account.
- Use technology to compute the outstanding principal on a debt.
- Calculate the equity of a purchased item.
- Specify how much of a monthly payment is applied toward interest and how much is applied toward principal.

Understanding Annuities Involving Deposits

In the previous section, we discussed problems where a single deposit of money was made into an account and was left there for a specified time period. Now we will examine situations where regular periodic payments are made to, or taken from, an account. When a sequence of payments of some fixed amount is made into or taken out of an account at equal intervals of time, we call this an **annuity**. When the payments are made at the end of each period, we call it an **ordinary annuity**. If the payments are made at the beginning of each period, we call it an **annuity due**.

For the purpose of this text, we will assume all annuities are *ordinary* and, furthermore, are **certain**, meaning the payments are made over a set period of time. All annuities discussed in this text will also have the number of payments equal to the number of compounding periods, and all deposits/withdrawals will be stated.

Ordinary Annuity Future Value Formula

$$A = PMT\left[\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right]$$

where,

A = Future Value or Accumulated Amount,

PMT = Periodic payment,

r = Annual percentage rate (APR) changed to a decimal,

t = Number of years, and

m = Number of payments made per year.

Consider the following scenario:

> The Henry family decides to save up for a big vacation by depositing \$100 every month into an account earning interest at 4% per year, compounded monthly. How much money will they have saved at the end of two years?

Reading the given information,

> "decides to save up for a big vacation by depositing \$100 **every month** into an account earning interest at 4% per year, **compounded monthly**. How much money will they have saved at the end of two years?"

we can see that $PMT = 100$, $r = 0.04$, $t = 2$, and $m = 12$. Substituting these values into the formula above, we have

$$\begin{aligned} A &= PMT\left[\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right] \\ &= 100\left[\frac{\left(1+\frac{0.04}{12}\right)^{(12)(2)}-1}{\frac{0.04}{12}}\right] \\ &\approx 2494.288775 \end{aligned}$$

Therefore, the Henry family will have \$2,494.29 saved for their vacation.

As in the previous section, the authors will turn to technology and the TVM Solver when working with annuities. Because the annuities in this text all have an equal number of payments and compounding periods, P/Y = C/Y. As a reminder, when using the TVM Solver, deposits, investments, and payments made will be entered as a negative value, while money returned from an investment or money received as a loan will be entered as a positive value.

For the Henry family scenario above, the entries for the TVM Solver (shown in **Figure 6.2.2**) will be:

N= $12 \cdot 2$
I% = 4
PV= 0 (Nothing is deposited until the first payment is made)
PMT= -100 (Money deposited *every* month)
FV= ? (what we are solving for)
P/Y= 12 (deposits every month = 12 payments per year)
C/Y= 12
PMT: **END** BEGIN

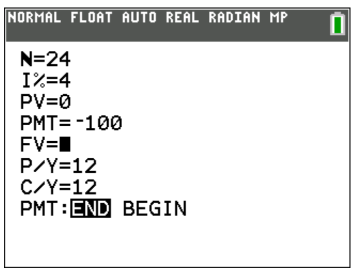


Figure 6.2.2: Calculator screenshot of the TVM Solver menu. All entries, except the FV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

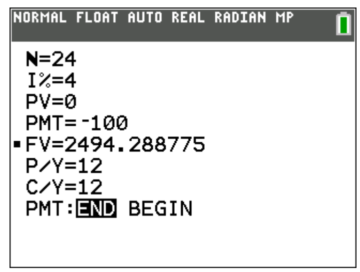


Figure 6.2.3: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for FV is now shown.

Solving for FV, we can see in **Figure 6.2.3** that FV= 2494.288775 and verify the Henry family will have \$2,494.29 saved for their vacation.

The difference between how much money is in the account at the end of the life of an investment and how much money is deposited in total into the account is the **total interest earned**.

In this case, the Henry family deposits a total of \$2400 over 2 years into the account (PMT = \$100 a month for N= 24 months), and the total interest earned is \$2494.29 − \$2400 = \$94.29.

▪ **Example 1** A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$250 each month into an IRA earning 6% annual interest, compounded monthly, how much will you have in the account after 20 years? How much of this future value is earned interest?

Solution:

> "If you deposit \$250 **each month** into an IRA earning 6% annual interest, compounded **monthly**, how much will you have in the account after 20 years?"

So, the entries for the TVM Solver (shown in **Figure 6.2.4**) will be:

N= $12 \cdot 20$
I% = 6
PV= 0 (Nothing is deposited until the first payment is made)
PMT= -250 (Money deposited each month)
FV=? (what we are solving for)
P/Y= 12 (deposits every month = 12 payments per year)
C/Y= 12
PMT: **END** BEGIN

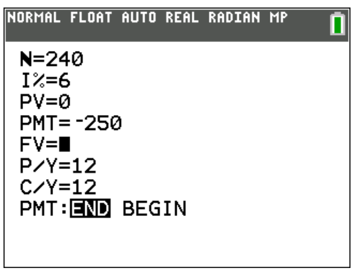


Figure 6.2.4: Calculator screenshot of the TVM Solver menu. All entries, except the FV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

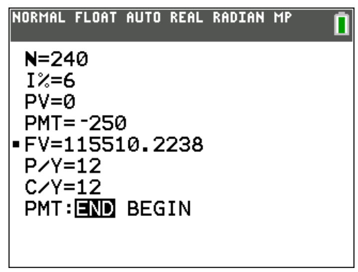


Figure 6.2.5: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for FV is now shown.

Solving for FV, we can see in **Figure 6.2.5** that FV= 115510.2238, which means you will have \$115,510.22 in your IRA after 20 years.

The interest earned on the IRA is

$$\begin{aligned}
\text{Interest Earned} &= \text{Accumulated Amount} - \text{Money Deposited} \\
&= \text{FV} - (\text{PMT})(\text{N}) \\
&= 115510.22 - 250(240) \\
&= 115510.22 - 60000 \\
&= \$55,510.22
\end{aligned}$$

■

Try It # 1:
A conservative investment account pays 3% annual interest, compounded daily.
a. If you deposit \$5 a day into this account, how much will you have after 10 years?
b. How much of the accumulated amount is from interest?

For some problems, you will compute the payment needed to accumulate a desired amount of money, instead of knowing the payment and determining a future value. If this is the case, you can simply solve the ordinary annuity formula for the payment, or you may continue to use the TVM Solver. The authors will use the TVM Solver.

▪ **Example 2** You would like \$200,000 in your retirement account when you retire in 30 years. Your retirement account earns 8% annual interest, compounded monthly. How much do you need to deposit each month to meet your retirement goal?

Solution:

> "You would like \$200,000 in your retirement account when you retire in 30 years. Your retirement account earns 8% annual interest, compounded monthly. How much do you need to deposit each month to meet your retirement goal?"

The entries for the TVM Solver (shown in **Figure 6.2.6**) will be:

N= $12 \cdot 30$
I% = 8
PV= 0 (Nothing is deposited until the first payment is made)
PMT= ? (what we are solving for)
FV= 200000
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

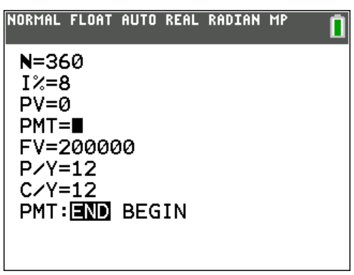


Figure 6.2.6: Calculator screenshot of the TVM Solver menu. All entries, except the PMT entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

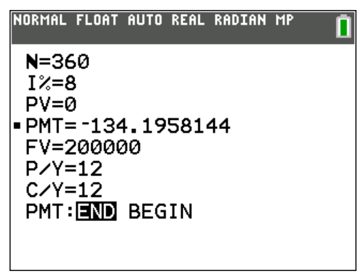


Figure 6.2.7: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PMT is now shown.

Solving for PMT, we can see in **Figure 6.2.7** that PMT= $-134.19581\ldots$, which means the monthly deposit necessary to meet your retirement goal is \$134.20.

▪

Try It # 2:
You want to buy a pop-up trailer that costs \$9000. You want to pay in cash, so you save money by making monthly deposits into an account earning 3.2% APR, compounded monthly. How much should your monthly payments be to save up the \$9000 in 3 years?

For other problems, you may need to know how long it will take to accumulate a certain amount of money when depositing a regular specified amount of money. Again, you can solve the ordinary annuity future value formula for a variable other than A, but solving for time requires solving an exponential equation, using logarithms. The authors will choose to continue using the TVM Solver.

▪ **Example 3** You have \$300 a month you can deposit into a savings account earning 6.8% APR, compounded monthly. How long will it take for you to save \$10,000?

Solution:

> "You have \$300 a **month** you can deposit into a savings account earning 6.8% APR, compounded monthly. How long will it take for you to save \$10,000?"

The entries of the TVM Solver (shown in **Figure 6.2.8**) will be:

N= $12 \cdot t$ = ? (what we are solving for)
I% = 6.8
PV= 0 (Nothing is deposited until the first payment is made)
PMT= −300
FV= 10000
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

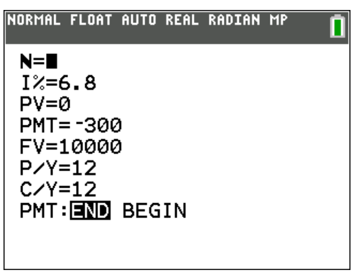


Figure 6.2.8: Calculator screenshot of the TVM Solver menu. All entries, except the N entry, have been entered.

ALPHA → ENTER

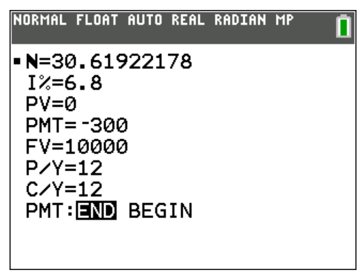


Figure 6.2.9: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for N is now shown.

Solving for N, we can see in **Figure 6.2.9** that N= 30.61922178. Considering N represents the total number of compounding periods (in months here), it takes you 30.62 months to reach \$10,000. Because the payments are made at the end of the month, it will actually take you 31 months of payments to reach your savings goal.

▪

Try It # 3:
If you invest \$25 each month into an account earning 3% annual interest, compounded monthly, how long will it take the account to grow to \$8000?

Understanding Annuities Involving Withdrawals

In the ordinary annuities discussed thus far, we used the ordinary annuity future value formula, because each problem started with no initial deposit and then money was deposited into an account on a regular basis to accumulate a desired amount of money.

Now we will learn about ordinary annuities where we begin with money in an account, and we withdraw money out of the account on a regular basis. As money is withdrawn from the account, any remaining money in the account earns interest; after a fixed amount of time, the account will be empty. Situations like these typically occur when discussing retirement income. For these types of accounts, we need a formula involving the present value of an ordinary annuity.

Ordinary Annuity Present Value Formula

$$P = PMT\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right]$$

where,

P = Present Value or the Principal,

PMT = Periodic payment,

r = Annual percentage rate (APR) changed to a decimal,

t = Number of years, and

m = Number of payments made per year.

After retiring, suppose you want to be able to take \$4000 every month, over a total of 20 years, from your retirement account. The account earns 6% annual interest, compounded monthly. How much will you need in your account when you retire, in order to withdraw your desired monthly amount?

Reading through the given information,

> "you want to be able to take \$4000 every month for a total of 20 years from your retirement account. The account earns 6% annual interest, compounded monthly. How much will you need in your account when you retire, in order to withdraw your discussed monthly amount?"

we can see that $PMT = 4000$, $r = 0.06$, $t = 20$, and $m = 12$. Substituting these values into the formula above, we have

$$\begin{aligned} P &= PMT\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right] \\ &= 4000\left[\frac{1-\left(1+\frac{0.06}{12}\right)^{-(12)(20)}}{\frac{0.06}{12}}\right] \\ &\approx 558323.0867 \end{aligned}$$

Thus, your account should have \$558,323.09 upon retirement, in order for you to withdraw \$4000 a month for 20 years.

Using the TVM Solver (shown in **Figure 6.2.10**) to check our work, we have,

N= $12 \cdot 20$
I% = 6
PV= ? (what we are solving for)
PMT= 4000 (you receive this payment amount)
FV= 0 (the account should be empty after 20 years)
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

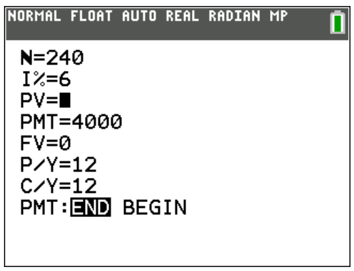


Figure 6.2.10: Calculator screenshot of the TVM Solver menu. All entries, except the PV entry, have been entered.

ALPHA $\longrightarrow$ ENTER

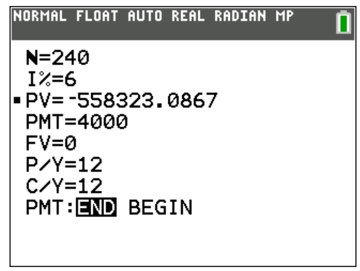


Figure 6.2.11: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PV is now shown.

Solving for PV, we can see in **Figure 6.2.11** that PV= -558323.0867 and verify the present value of the account needs to be \$558,323.09.

While we could continue to use the present value formula, the authors choose to use the TVM Solver instead.

▪ **Example 4** Suppose you have won a lottery that pays \$1000 per month for the next 20 years, but you would prefer to have the entire amount given to you as a lump sum now. If the lottery has its money in an account earning annual interest at a rate of 8%, compounded monthly, how much will you be given as a lump sum?

Solution:
The lump sum you will be given should be equal to the amount of money the lottery currently has to fulfill its \$1000/month payment to you for 20 years.

> “Suppose you have won a lottery that pays \$1000 per month for the next 20 years, but you would prefer to have the entire amount given to you as a lump sum now. If the lottery has its money in an account earning annual interest at a rate of 8%, compounded monthly, how much will you be given as a lump sum?”

So, the entries for the TVM Solver (shown in **Figure 6.2.12**) will be:

N= $12 \cdot 20$
I% = 8
PV= ?
PMT= 1000
FV= 0 (the account should be empty after 20 years)
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

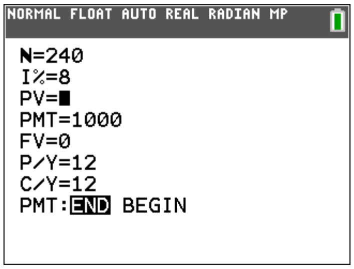


Figure 6.2.12: Calculator screenshot of the TVM Solver menu. All entries, except the PV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

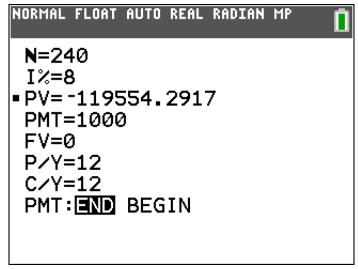


Figure 6.2.13: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PV is now shown.

So solving for PV, we can see in **Figure 6.2.13** that PV= -119554.2917. Thus, the account from the lottery must contain \$119,554.29 in order to fulfill the monthly payments over 20 years, which is the lump sum amount you would be given now.

N *If you accept the monthly installments of \$1000 for 20 years, you will receive a greater total of \$240,000. However, if you were to invest the entire lump sum amount, \$119,554.29, into an account earning what the lottery's account earns (8% annual interest, compounded monthly, for 20 years), you will have received an even greater total of \$589,020.41 (assuming you withdraw no money during the 20 years).*

■

Try It # 4:
You received an inheritance and invested it at 6% annual interest, compounded quarterly. You are going to use the investment for college, withdrawing money for tuition and expenses each of your remaining quarters. How much money did you inherit (to the nearest dollar), if you can take out \$2000 each quarter during your remaining 3 years of school?

Understanding Loans

We will now learn about conventional loans (also called **amortized loans** or installment loans). Examples include auto loans and home **mortgages**. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use the present value formula for an ordinary annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of an ordinary annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a lender is acting as you. The lender invests \$10,000 in you. While you're acting as the bank, you pay interest. The lender takes payments until the balance is zero.

Suppose a college freshman needs an electronic tablet and finds one online for \$500. Because they do not have \$500, they decide to purchase the same tablet from a rent-to-own business, using a loan by credit with no initial payment, but payments of \$30 a month for four years at 14.5% APR, compounded monthly. What is the price of the tablet at the rent-to-own business? How much interest was paid by the freshman to the rent-to-own business?

Reading through the information given about the rent-to-own purchase,

> "using a loan by credit with no initial payment, but payments of \$30 a month for four years at 14.5% APR, compounded monthly"

we can see that $PMT = 30$, $r = 0.145$, $t = 4$, and $m = 12$. Substituting these values into the present value formula for an ordinary annuity, we have

$$\begin{aligned} P &= PMT\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right] \\ &= 30\left[\frac{1-\left(1+\frac{0.145}{12}\right)^{-(12)(4)}}{\frac{0.145}{12}}\right] \\ &\approx 1087.825487 \end{aligned}$$

Thus, the price of the tablet was \$1,087.83, which is a lot more than the \$500 the freshman could have paid by purchasing it online.

Using the TVM Solver (shown in **Figure 6.2.14**) to check our work, we have,

N= $12 \cdot 4$
I% = 14.5
PV= ? (loan amount we are looking for)
PMT= −30
FV= 0 (after all payments are made, \$0 is owed)
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

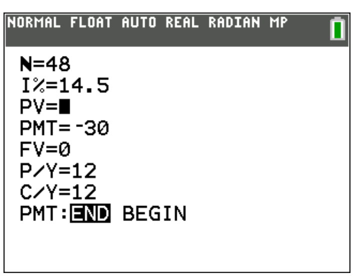


Figure 6.2.14: Calculator screenshot of the TVM Solver menu. All entries, except the PV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

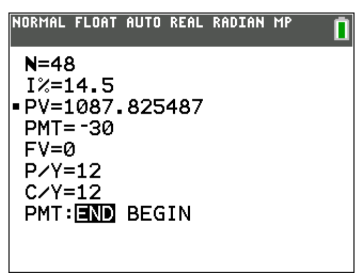


Figure 6.2.15: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PV is now shown.

Solving for PV, we can see in **Figure 6.2.15** that PV= 1087.825487, and we verify the price of the tablet at the rent-to-own business was \$1087.83.

The **total amount they paid** over the course of the loan is equal to

$$\begin{aligned}\text{Total amount paid} &= (\text{Monthly payment amount})(\text{\# of compounding periods per year})(\text{\# of years})\\ &= \text{PMT}\cdot \text{m}\cdot \text{t}\\ &= 30\cdot 12\cdot 4\\ &= \$1440\end{aligned}$$

Therefore, the **total amount of interest paid** by the freshman over the course of the loan is given by

$$\begin{aligned}\text{Total amount of interest} &= \text{Total amount paid} - \text{Loan amount}\\ &= 1440 - 1087.83\\ &= \$352.17\end{aligned}$$

The authors choose to work the remaining problems only using the TVM Solver.

▪ **Example 5** You go to a car dealership to buy a new car. You purchase the car with a five-year loan of \$18,000 from the dealer, financed at 2% APR, compounded monthly. The dealer quotes you a monthly payment of \$425. Is the dealer correct?

Solution:
To determine whether or not the dealer is correct, we need to determine the payment amount required with the given financing information.

> "You purchase the car with a five-year loan of \$18,000 from the dealer, financed at 2% APR, compounded monthly."

The entries of the TVM Solver (shown in **Figure 6.2.16**) will be:

N= $12\cdot 5$
I% = 2
PV= 18000 (positive, because this amount is loaned *to you*)
PMT= ?
FV= 0 (the loan will be paid back after 5 years)
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

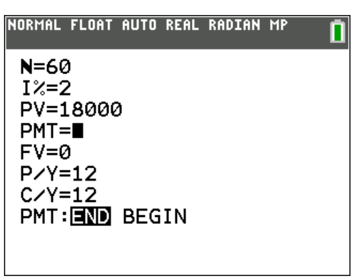


Figure 6.2.16: Calculator screenshot of the TVM Solver menu. All entries, except the PMT entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

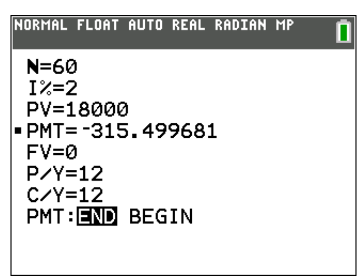


Figure 6.2.17: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PMT is now shown.

Solving for PMT, we can see in **Figure 6.2.17** that PMT= -315.499681, which means your monthly car payments should be \$315.50, which is not the amount quoted to you by the dealer.

Let's use the quoted payment to determine how much the dealer is trying to get you to pay in total for the car.

$$425 \cdot 12 \cdot 5 = \$25,500$$

The dealer is trying to sell you the car for a total of \$25,500, with principal and interest combined. What should the total principal and interest be with the calculated \$315.50 monthly payments?

$$315.50 \cdot 12 \cdot 5 = \$18,930$$

Therefore, the dealer is trying to get you to pay $\$25,500 - \$18,930 = \$6,570$ in additional principal and interest charges. The dealer quoted payment would mean that the quoted rate of 2% APR is not accurate, or the quoted price of \$18,000 is not accurate, or both.

■

The reader is encouraged to have their calculator with them when applying for a loan.

When making 'large' purchases (car, house, furniture, ...), you may be required to pay some money towards the purchase and then finance the remaining balance. The amount of money you pay towards the purchase, prior to financing, is called a **down payment**. This leads to the following relationship:

$$\text{Purchase Price} = \text{Down Payment Amount} + \text{Loan Amount}$$

■ **Example 6** The VanKats are going to buy a house for \$290,000. The bank requires a 20% down payment to obtain a mortgage. With their credit score, the bank offers the VanKats a 30-year mortgage with a 3.6% APR, compounded monthly. What monthly payment will the VanKats pay towards their new home?

Solution:

> "The VanKats are going to buy a house for \$290,000. The bank requires a 20% down payment to obtain mortgage. With their credit score, the bank offers the VanKats a 30-year mortgage with a 3.6% APR, compounded monthly."

We have the following entries for the TVM Solver.

N= $12 \cdot 30$
I% = 3.6
PV= ?
PMT= ?
FV= 0 (the loan will be paid back after 30 years)
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

We are currently unable to solve for the monthly payment, as we do not know the amount of the loan. To determine the amount of the loan, we need to subtract the amount of the down payment from the purchase price of the house.

Purchase Price: 290000

Down Payment Amount: 0.20(290000) = 58000

$$\begin{aligned}\text{Loan Amount} &= \text{Purchase Price} - \text{Down Payment Amount}\\ &= 290000 - 58000\\ &= 232000\end{aligned}$$

With this information, we now know PV= 232000.

When the down payment is given as a percentage of the purchase price, d%, the loan amount can be found by directly computing

$$(100 - d)\% \cdot (\textit{Purchase Price})$$

Here the loan amount would be calculated by

$$\begin{aligned}(100 - 20)\% \cdot (290000) &= 80\% \cdot (290000)\\ &= 0.80 \cdot 290000\\ &= 232000\end{aligned}$$

We return to the TVM Solver, insert 232000 for PV (shown in **Figure 6.2.18**), and solve for PMT.

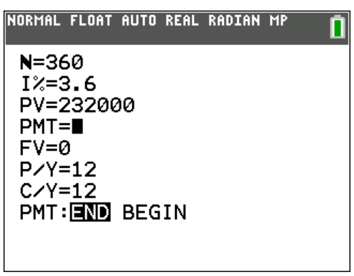


Figure 6.2.18: Calculator screenshot of the TVM Solver menu. All entries, except the PMT entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

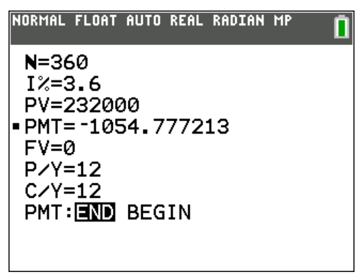


Figure 6.2.19: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PMT is now shown.

Solving for PMT, we can see in **Figure 6.2.19** that PMT= −1054.777213, so the VanKats' monthly mortgage payment will be \$1054.78.

■

■ **Example 7** The Bollelt family purchased a new car. They made a down payment of \$4000 towards the purchase and financed the rest at an annual rate of 2.9%, compounded monthly. The terms of their loan stated the Bollelts will make payments of \$378/month for 60 months.

a. What was the cash price (purchase price) of the car?

b. How much will the Bollelt family pay in total interest on the loan?

Solution:

a. Cash Price = Down Payment Amount + Loan Amount

While we know the down payment amount, we do not know the loan amount. We can solve for the loan amount, PV, using the given information and the TVM Solver.

> "The Bollelt family purchased a new car. They made a down payment of \$4000 towards the purchase and financed the rest at an annual rate of 2.9%, compounded monthly. The terms of their loan stated the Bollelts will make payments of \$378/month for 60 months."

The entries of the TVM Solver (shown in **Figure 6.2.20**) will be:

N= 60
I% = 2.9
PV= ?
PMT= −378
FV= 0 (the loan will be paid back after 60 months)
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

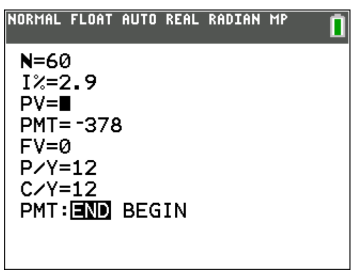


Figure 6.2.20: Calculator screenshot of the TVM Solver menu. All entries, except the PV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

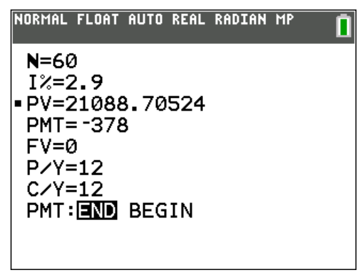


Figure 6.2.21: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PV is now shown.

Solving for PV, we can see in **Figure 6.2.21** that PV= 21088.70524, so the loan amount is \$21,088.71. Thus,

$$\begin{aligned}\text{Cash Price} &= \text{Down Payment Amount} + \text{Loan Amount}\\ &= 4000 + 21088.71\\ &= 25088.71\end{aligned}$$

Therefore, the cash price of the Bollelt's car was \$25,088.71.

b.
$$\begin{aligned}\text{Total Interest on Loan} &= (\text{Payment Amount})(\text{Number of Payments}) - \text{Loan Amount}\\ &= (378)(60) - 21088.71\\ &= 1591.29\end{aligned}$$

The Bollets will pay a total of \$1591.29 in interest.

■

Total interest owed on a loan does not involve the cash price or the down payment. Interest is only paid on money borrowed, the loan amount.

Try It # 5:
You recently purchased \$3000 of new furniture on credit. Because your credit score isn't very good, the store is charging you a fairly high interest rate on the loan – 16% APR, compounded monthly. If you agree to pay off the furniture over 2 years,

a. How much will you have to pay each month?
b. How much total interest will you pay?

Try It # 6:
Friendly Auto offers you a car for \$2000 down and \$300 per month for 5 years. You want to buy the same car, but you want to pay cash. How much must you pay, if the interest rate for buying the car with a loan is stated as 9.4% per year, compounded monthly?

With loans, it is often desirable to determine what the remaining loan balance will be after a specified number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you still have to pay from the sale. Or, if in the middle of the loan, interest rates significantly drop, you might want to **refinance** the loan.

To determine the remaining loan balance after a specified number of years, we first need to determine the periodic loan payment, if we do not already know it. Note that only a portion of your loan payment goes towards the loan balance; a portion is also going to go towards interest. This means, for example, if your payments were \$1000 a month, after a year you will not have paid off \$12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?" In other words, we compute the present value of the loan, where N is the number of payments *remaining*.

Example 8 After 10 years of the VanKats paying on their home, mortgage rates dropped down to 1.8% APR, compounded monthly, for a 15-year mortgage. The VanKats decide to take advantage of this new rate and refinance their remaining loan.

a. How much of the loan are the VanKats going to refinance?
b. How much equity do the VanKats have in their home after 10 years (before refinancing)?
c. What is their new monthly payment?
d. How much will the VanKats save by refinancing?

Solution:
Recall from **Example 6**; $m = 12$, $t = 30$, I% = 3.6, PV= 232000, and PMT= −1054.78.

a. To determine the amount the VanKats will refinance after 10 years, we compute the present value, or **outstanding principal**, of the loan, with **20 years remaining**.

The entries in the TVM Solver (shown in **Figure 6.2.22**) will be:

N= 12(30 − 10) = 12(**20**)
I% = 3.6 (current rate)
PV= ?
PMT= −1054.78 (current monthly payment)
FV= 0
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

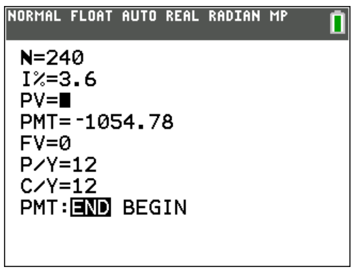


Figure 6.2.22: Calculator screenshot of the TVM Solver menu. All entries, except the PV entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

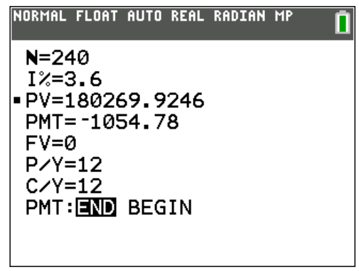


Figure 6.2.23: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PV is now shown.

Solving for PV, we can see in **Figure 6.2.23** that PV= 180269.9246, which means the loan amount remaining after 10 years is $180,269.92; this is the amount the VanKats will refinance.

b. **Equity** is the amount of an item a purchaser owns.

$$\begin{aligned}\text{Equity} &= \text{Total Value} - \text{Outstanding Debt}\\ &= \textit{Cash Price} - \text{Loan Balance}\\ &= 290000 - 180269.92\\ &= 109{,}730.08\end{aligned}$$

After 10 years, the VanKats have $109,730.08 of equity in their home.

c. When refinancing a loan, the new loan *pays off* the old loan and the borrower pays back the new loan under the newly given terms. So, on the new loan, $m = 12$, $t = 15$, PV= 180269.92, and I% = 1.8.

The entries of the TVM Solver for the new loan (shown in **Figure 6.2.24**) will be:

N= 12(15)
I% = 1.8
PV= 180269.92
PMT= ?
FV= 0
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

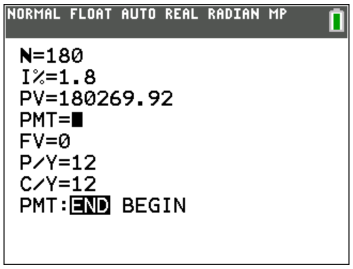


Figure 6.2.24: Calculator screenshot of the TVM Solver menu. All entries, except the PMT entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

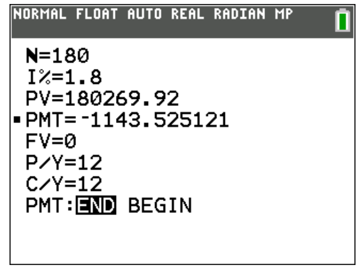


Figure 6.2.25: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PMT is now shown.

Solving for PMT, we can see in **Figure 6.2.25** that PMT= -1143.525121, so the VanKats' new monthly mortgage payment is \$1143.53.

d. While the VanKats' new monthly mortgage payment is higher, it is being repaid over less time at a lower interest rate, so they should save money. If the VanKats had not refinanced, they would have paid

$$(1054.78)(12)(20) = \$253,147.20$$

over the remaining 20 years of their original loan.

Seeing as the VanKats did refinance, they will instead pay

$$(1143.53)(12)(15) = \$205,835.40$$

over the remaining 15 years of their new loan.

Thus, by refinancing, the VanKats will save

$$\$253147.20 - \$205835.40 = \$47,311.80$$

▪

N *When refinancing you may encounter fees called refinancing fees. These fees can be either paid out of pocket or rolled into the amount of the new loan. Due to the variation, the authors choose to ignore these fees in our examples.*

Try It # 7:
You have decided to refinance your home mortgage with a 15-year loan at 4.0% APR, compounded monthly. The outstanding balance on your current loan is \$150,000. Under your current loan, your monthly mortgage payment is \$1610, which you must continue to pay for the next 20 years if you do not refinance.

a. What is the new monthly payment if you refinance?
b. How much will you save by refinancing?
c. How much total interest will you pay on this new loan?

Understanding Amortization

Based on a new federal law, when receiving the paperwork for a loan, the borrower will find a table showing where the portions of each payment goes. This table is called an **amortization schedule**. A portion of a sample schedule is shown in **Table 6.3** below.

Payment Number	Beginning Principal	Payment	Principal Portion	Interest Portion	Ending Principal
1	\$100,000.00	\$536.82	\$120.15	\$416.67	\$99,879.85
2	\$99,879.85	\$536.82	\$120.66	\$416.17	\$99,759.19
3	\$99,759.19	\$536.82	\$121.16	\$415.66	\$99,638.03
4	\$99,638.03	\$536.82	\$121.66	\$415.16	\$99,516.37
5	\$99,516.37	\$536.82	\$122.17	\$414.65	\$99,394.20
6	\$99,394.20	\$536.82	\$122.68	\$414.14	\$99,271.52
7	\$99,271.52	\$536.82	\$123.19	\$413.63	\$99,148.33
8	\$99,148.33	\$536.82	\$123.70	\$413.12	\$99,024.62
9	\$99,024.62	\$536.82	\$124.22	\$412.60	\$98,900.41
10	\$98,900.41	\$536.82	\$124.74	\$412.09	\$98,775.67
11	\$98,775.67	\$536.82	\$125.26	\$411.57	\$98,650.41
12	\$98,650.41	\$536.82	\$125.78	\$411.04	\$98,524.63

Table 6.3: A portion of a sample amortization schedule.

Definition

A table showing how much of each loan payment goes to the principal is called a **amortization schedule**. ▪

An amortization schedule always includes the following loan information:

- Outstanding Principal: the present value of the loan after the i^{th} payment, PV_i
- Regular Payment Amount: the amount of the periodic payment, PMT
- Periodic Interest Rate: the percentage of interest accrued on the outstanding principal during a given period, $\frac{r}{m}$
- Payment Amount Paid to Interest: the amount of each periodic payment that goes towards interest accrued on the outstanding principal from the previous period, $\mathrm{I}_i = \left(\frac{r}{m}\right) \cdot \mathrm{PV}_{i-1}$
- Payment Amount Paid to the Debt: the amount of each periodic payment that goes towards the outstanding principal from the previous period (the remaining portion of a periodic payment after the interest has been deducted), $\mathrm{PMT} - \mathrm{I}_i$

The process for creating an amortization schedule is easier seen with an example.

▪ **Example 9** An amount of \$500 is borrowed for 6 months at an annual rate of 12%, compounded monthly. Construct an amortization schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

Solution:

To create an amortization schedule, we must first determine the monthly payment.

> "An amount of $500 is borrowed for 6 months at an annual rate of 12%, compounded monthly."

Thus, we know
$P_0 = 500$ (the original outstanding principal)
$r = 0.12$
$m = 12$
$t = 1/2$ (as 6 months/12 months = 1/2 year)

To calculate the periodic (monthly) payment, we use the TVM Solver (shown in **Figure 6.2.26**) with the following entries:

N= $12(1/2)$
I% = 12
PV= 500
PMT= ?
FV= 0
P/Y= 12
C/Y= 12
PMT: **END** BEGIN

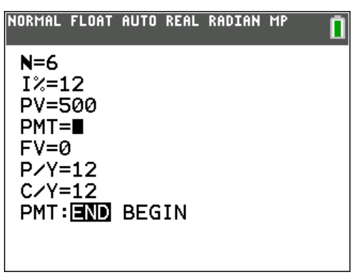


Figure 6.2.26: Calculator screenshot of the TVM Solver menu. All entries, except the PMT entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

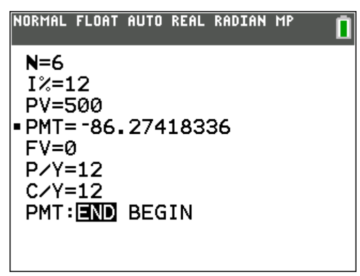


Figure 6.2.27: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PMT is now shown.

Solving for PMT, we can see in **Figure 6.2.27** that PMT= −86.27418336, which means the monthly payment for the loan is $86.27.

Prior to the first payment being made, the outstanding principal is $500, and the first row of the amortization schedule is shown below in **Table 6.4**.

Payment Number	Payment Amount	Payment Amount to Interest	Payment Amount to Debt	Outstanding Principal
0	–	–	–	500

Table 6.4: The 0th payment row of the amortization schedule.

During the first month, the outstanding principal is \$500, and therefore, the monthly interest accrued on the outstanding principal is

$$\begin{aligned} I_1 &= \text{(monthly interest rate)(outstanding principal)} \\ &= \left(\frac{r}{m}\right) \cdot PV_0 \\ &= \left(\frac{0.12}{12}\right) \cdot (500) \\ &= (0.01)(500) \\ &= \$5 \end{aligned}$$

This means for the first month, when the payment of \$86.27 is made, \$5 goes towards the interest and the remaining amount, $86.27 - 5 = \$81.27$, goes towards the debt. Thus, at the end of the first month, the outstanding principal is $500 - 81.27 = \$418.73$. The second row of the schedule represents this information, as seen in **Table 6.5**.

Payment Number	Payment Amount	Payment Amount to Interest	Payment Amount to Debt	Outstanding Principal
0	–	–	–	500
1	86.27	5.00	81.27	418.73

Table 6.5: The 1st payment row of the amortization schedule.

During the second month, the outstanding principal is \$418.73, and therefore, the monthly interest accrued on the outstanding principal is

$$\begin{aligned} I_2 &= \text{(monthly interest rate)(outstanding principal)} \\ &= \left(\frac{r}{m}\right) \cdot PV_1 \\ &= (0.01) \cdot (418.73) \\ &= 4.1873 \\ &= \$4.19 \end{aligned}$$

This means for the second month, when the payment of \$86.27 is made, \$4.19 goes towards the interest and the remaining amount, $86.27 - 4.19 = \$82.08$, goes towards the debt. Thus, at the end of the second month, the outstanding principal is $418.73 - 82.08 = \$336.65$. The third row of the schedule represents this information, as seen in **Table 6.6**.

Payment Number	Payment Amount	Payment Amount to Interest	Payment Amount to Debt	Outstanding Principal
0	–	–	–	500
1	86.27	5.00	81.27	418.73
2	86.27	4.19	82.08	336.65

Table 6.6: The 2nd payment row of the amortization schedule.

Continuing this process through the sixth and final payment gives the complete amortization schedule for the loan. (See **Table 6.7**.)

Payment Number	Payment Amount	Payment Amount to Interest	Payment Amount to Debt	Outstanding Principal
0	–	–	–	500
1	86.27	5.00	81.27	418.73
2	86.27	4.19	82.08	336.65
3	86.27	3.37	82.90	253.75
4	86.27	2.54	83.73	170.02
5	86.27	1.70	84.57	85.45
6	86.27	0.85	85.42	0.03

Table 6.7: The complete amortization schedule.

In **Table 6.7**, notice while the payment amount remains unchanged, the payment amount to interest decreases, but the payment amount to the debt increases, over the life of the loan.

The outstanding principal should decrease to 0, as the number of payments reaches N. However, due to rounding the interest paid to the nearest cent, the outstanding principal in this example is \$0.03 and not \$0. In the real-world rounding will always occur, so the final payment (the **payoff amount**) is normally different from all previous periodic payments to ensure the debt is completely paid off.

■

■ **Example 10** The Lynnleas purchased a new gaming computer for \$7000 by paying \$800 down and using store financing for the remaining balance. The financing agreement requires quarterly payments for 3 years and charges interest at an annual rate of 24%, compounded quarterly.

a. What is the Lynnleas' required quarterly payment?
b. How much of their first payment goes towards interest?
c. How much of their first payment goes towards the debt?

Solution:

"The Lynnleas purchased a new gaming computer for \$7000 by paying \$800 down and using store financing for the remaining balance. The financing agreement requires quarterly payments for 3 years and charges interest at an annual rate of 24%, compounded quarterly."

a. To determine the quarterly payment, the TVM Solver (shown in **Figure 6.2.28**) entries will be:
N= $4 \cdot 3 = 12$
I% = 24
PV= $7000 - 800 = 6200$
PMT= ?
FV= 0
P/Y= 4
C/Y= 4
PMT: **END** BEGIN

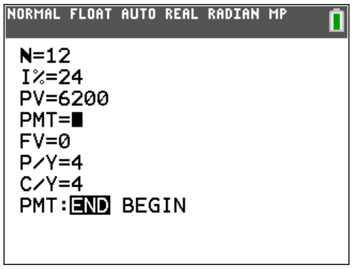


Figure 6.2.28: Calculator screenshot of the TVM Solver menu. All entries, except the PMT entry, have been entered.

$\xrightarrow[\text{ENTER}]{\text{ALPHA}}$

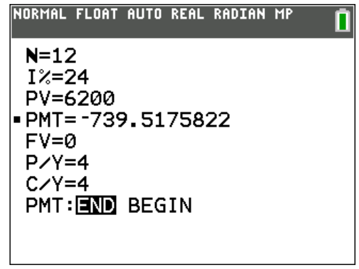


Figure 6.2.29: Calculator screenshot of the TVM Solver menu. All entries from the previous figure are the same. The solution for PMT is now shown.

Solving for PMT, we can see in **Figure 6.2.29** that PMT= -739.5175822, so the periodic (quarterly) payment is \$739.52.

b. Because $r = 0.24$, $m = 4$, and $\text{PV}_0 = 6200$, the amount of interest accrued during the first quarter is given by

$$\begin{aligned} I_1 &= \left(\frac{0.24}{4}\right)(6200) \\ &= (0.06)(6200) \\ &= \$372.00 \end{aligned}$$

c. From part **a**, the first payment is \$739.52, and from part **b**, we know \$372 of this payment goes towards interest. This leaves

$$739.52 - 372 = \$367.52$$

of the first payment to go towards the debt.

If we were asked about the second, third, or any payment, we could use the process lined out in the amortization schedule discussion, continuing from the information found here.

■

Try It # 8:
You purchase a party boat for \$68,000. You finance 80% of the purchase price with a boat loan at 6% APR, compounded monthly, requiring monthly payments over 5 years. Create the first five rows of the amortization schedule.

Summarizing Finance Problems

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or how to find a solution. Here are some hints on deciding how to begin a solution based on the wording of the problem.

- The easiest types of problems to identify are loans. Loan problems almost always include words like: "loan," "amortize," "finance (a car)," or "mortgage" (a home loan). Look for these words. If they are there, you are probably looking at a loan problem. To make sure, see if you are given what your payment amount is, or if you are trying to find a payment amount. For these types of problems you can use the TVM Solver.
- If the problem is not a loan, the next question you want to ask is: "Am I depositing money in an account and letting it sit, or am I making regular payments or withdrawals?"
  - If you're letting money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. For these types of problems you can use the TVM Solver and PMT will equal zero unless the compounding occurs continuously where you will need to use the formula for continuously compounded interest. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest, where you will need to use the simple interest formulas to find a solution.
  - If you are making regular payments or withdrawals, the next question is: "Am I depositing money into the account, or am I pulling money out?" If you're depositing money into the account on a regular basis then you're looking at using the formula for the future value of an ordinary annuity. If you're pulling money out of the account on a regular basis, then you are using the formula for the present value of an ordinary annuity. In either case, you can use the TVM Solver to find a solution to your problem.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

In the box below we have summarized some of the important verbal equations used throughout this section.

- Interest Earned on an Investment = Accumulated Amount − Money Deposited.
- Total Amount Paid on a Loan = (Periodic Payment Amount)(Total Number of Payments)
- Total Amount of Interest Paid on a Loan = Total Amount Paid on the Loan − Loan Amount
- Purchase Price = Down Payment Amount + Loan Amount
- Equity = Cash Price − Outstanding Debt

Purchase Price = Cash Price

Try It Answers

1. **a.** \$21,282.07

   b. \$3032.07

2. \$238.52

3. 236 months = 19 years and 8 months

4. \$21,815

5. **a.** \$146.89

   b. \$525.36

6. \$16,317.74

7. **a.** \$1109.53

   b. \$186,684.60

   c. \$49,715.40

8.

Payment Number	Payment Amount	Payment Amount to Interest	Payment Amount to Debt	Outstanding Principal
0	–	–	–	54400
1	1051.70	272.00	779.70	53620.30
2	1051.70	268.10	783.60	52836.70
3	1051.70	264.18	787.52	52049.18
4	1051.70	260.25	791.45	51257.73

EXERCISES

Basic Skills Practice (Answers)

Compound Interest with Payments

For Exercises 1 - 6, calculate the indicated monetary value.

1. Given $P = \$0$, $r = 0.032$, $m = 52$, $t = 10$, $PMT = \$275$, compute A.
2. Given $P = \$0$, $r = 0.07$, $m = 12$, $t = 15$, $PMT = \$60$, compute A.
3. Given $A = \$0$, $r = 0.098$, $m = 4$, $t = 25$, $PMT = \$135$, compute P.
4. Given $A = \$0$, $r = 0.041$, $m = 365$, $t = 3$, $PMT = \$1.50$, compute P.
5. Given $A = \$0$, $P = \$15{,}000$, $r = 0.0165$, $m = 12$, $t = 18$, compute PMT.
6. Given $A = \$50{,}000$, $P = \$0$, $r = 0.0625$, $m = 2$, $t = 30$, compute PMT.

Loans

For Exercises 7 - 10, calculate the loan amount, given the cash price and information about the down payment.

7. Cash Price = \$450,000 with a down payment of \$90,000
8. Cash Price = \$22,500 with a down payment of \$3000
9. Cash Price = \$180,000 with a 20% down payment
10. Cash Price = \$57,000 with a 15% down payment

For Exercises 11 - 13, use the fact that Cash Price = Equity + Outstanding Debt.

11. Compute the cash price of an item, where the owner has an outstanding debt of \$1300 and equity of \$5000 in the item.
12. Compute the equity in an item, where the owner has an outstanding debt of \$94,000 on an item with a cash price of \$110,000.
13. Compute the outstanding debt on an item, where the owner has \$65,000 of equity in an item worth \$315,000.

For Exercises 14 - 16, assume you have taken out a 25-year loan with an annual interest rate of 4.83%, compounded monthly. Determine

a. The payment amount, to the nearest cent, on the given loan amount, and then
b. The outstanding balance, to the nearest dollar, after the given time.

14. Loan = \$63,600; After 6 years
15. Loan = \$837,200; After 19 years
16. Loan = \$173,525; After 13 years

For Exercises 17 - 22, compute the periodic interest rate, as a decimal, given the nominal interest rate and compounding period.

17. 8%, semiannually
18. 13%, weekly
19. 3.768%, annually
20. 6.3%, monthly
21. 10.95%, daily
22. 7.5%, quarterly

Intermediate Skills Practice (Answers)

23. You are saving your \$50 monthly allowance by depositing it into an account earning 4.5% APR, compounded monthly. How much money will you have at the end of five years?
24. You would like to save up \$2500 to buy a new laptop. You prefer to pay in cash, so you make weekly deposits into a savings account earning 3.8% APR, compounded weekly. How much do your weekly deposits need to be, in order to save up the \$2500 in two years?
25. You are able to save \$12,194 by depositing \$200 in an account each quarter for 11 years. What was the annual interest rate, as a percent, compounded quarterly, for the account? Round your percentage to one decimal place.
26. You purchase a home for \$575,000. You make a down payment of 25% and finance the remaining amount with a 30-year mortgage having an annual percentage rate of 5.25%, compounded monthly. Determine your monthly mortgage payment.
27. You purchase a car for \$32,000. You make a down payment of \$4750 and finance the remaining amount with a 60-month loan having an annual percentage rate of 2.89%, compounded monthly. Determine your monthly car payment.
28. Recently, you made multiple large purchases on your credit card totaling \$15,000. The interest rate on your credit card is 19.8% per year, compounded monthly, and your statement says your minimum payment is \$250 per month.
    a. How many minimum payments will you need to make to pay off your purchases, assuming you cut up your credit card and do not make any additional purchases?
    b. How much total interest will you pay by making the minimum monthly payment?
29. You purchase a car by making a down payment of \$6500 and financing the remaining balance. The terms of your finance agreement require you to make 36 monthly payments of \$1012.95, with an annual interest rate of 1.75%, compounded monthly. What is the cash price of your car? Round your answer to the nearest dollar.
30. You bought an RV worth \$85,000 by securing a 30-year loan charging interest at 8.23% APR, compounded monthly, and requiring monthly payments of \$405.
    a. What was your original loan amount, to the nearest dollar?
    b. What is the outstanding balance on your loan after making 10 years of payments?
    c. How much equity do you have in the RV after 10 years?
31. Five years ago you acquired a 30-year loan of \$165,500, charging 4.3% annual interest, compounded monthly, and requiring monthly payments. At this time, interest rates on 30-year loans have dropped to 2.5% APR, compounded monthly, and you wish to refinance your loan at this new rate.
    a. How much will you be refinancing?
    b. How much will your new monthly payment be after refinancing?

32. Fill in an amortization table for a loan of \$3000 to be paid back over 3 years at an annual interest rate of 2.4%, compounded semiannually.

33. Fill in an amortization table for a loan of \$850 to be paid back over 1 year at an annual interest rate of 12%, compounded quarterly.

Mastery Practice (Answers)

34. Business Enterprises needs to save up \$150,000 for a planned expansion. They make an initial deposit of \$25,000 and plan on depositing \$450 at the end of each month in an account earning 7.2% APR, compounded monthly, to reach their goal. How many months will it take Business Enterprises to save up the \$150,000 they need?

35. At age 30, you start an IRA to save for retirement by depositing \$100 at the end of each month. If you can count on an APR of 6%, compounded monthly, how much total interest will you have earned when you retire at age 65?

36. You want to have saved \$100,000 in a college fund by the end of 18 years for your child. You plan to make regular, monthly deposits in order to reach this amount. Assuming the college fund has an APR of 7%, compounded monthly, how much of the \$100,000 comes from your actual deposits and how much comes from interest earned?

37. You purchase a home for \$325,000. You make a down payment of 20% and finance the remaining amount with a 15-year mortgage having an annual percentage rate of 4.34%, compounded monthly. How much total interest will you pay on this loan after 15 years?

38. You purchase a cruise, as motivation, for your graduation in 4 years. The cruise line requires you to pay 20% of the total price as a security deposit upon purchase. You then pay \$170 per month for the next 4 years, with additional annual interest rate of 0.85%, compounded monthly, charged on the remaining balance.
    a. What was the list price of the cruise, when you purchased it?
    b. How much extra money were you required to pay in interest, by not purchasing the cruise outright?

39. You bought a new \$45,750 boat by making a down payment of \$10,000 and financing the remaining balance with an 8-year loan charging annual interest of 6.99%, compounded monthly. After making monthly payments for 3 years, how much equity will you have in the boat?

40. You acquired a \$250,000 loan to help pay for your new house. Under the terms of your 30-year loan, you were required to make monthly payments of \$1250. After 10 years, you decide to refinance the outstanding balance with a 15-year loan charging interest at an annual rate of 3%, compounded monthly. How much money, to the nearest dollar, will you save by refinancing?

41. You purchase a car worth \$10,500 by making a \$1000 down payment and financing the remainder with a 4-year loan charging annual interest of 6%, compounded monthly, and requiring monthly payments.
    a. What is your required monthly payment?
    b. How much of your first payment goes towards interest?
    c. How much of the third payment goes towards principal?
    d. How much equity do you have in the car after 2 years worth of payments?

Communication Practice (Answers)

42. Explain what a negative monetary value represents in the TVM Solver.
43. Explain whether or not interest is paid on down payments when a loan is acquired.
44. Explain the process of refinancing, in your own words.
45. Explain why the last payment on a loan is always different from all other periodic payments.

Chapter Review

Directions: Read each reflection question. If you are able to answer the question, then move on to the next question. If not, use the mathematical questions included to help you review. (Answers)

1. Given a financial scenario, can you determine the appropriate formula needed?

   For each problem determine what type of problem is given: Simple Interest, Compound Interest, Continuously Compounded Interest, Effective Rate, or Ordinary Annuity.

   a. You deposited \$450 in an investment account earning 2.8% annual interest, compounded semiannually. If the investment was left in the account for 15 years, how much interest was earned on the investment?

   b. At the end of every week for 8 years you deposit \$50 in a savings account earning 0.9% annual interest, compounded weekly. What is the accumulated value of the account at the end of the 8 years? How much money did you deposit in the account during the 8 years, and how much interest was earned on the account?

   c. You borrow \$4000 from a payday loan business at a simple interest rate of 135%. If the loan term is 2 months, how much do you owe the business at the end of the loan?

   d. You take out a mortgage on a new \$200,000 home, after paying 15% down. The terms of the mortgage are 30 years at a fixed annual interest rate of 4.25%, compounded monthly. How much of the second monthly mortgage payment is going towards the principal?

   e. When looking at retirement investment options you have narrowed down your options to

   - Account A: 3.15% APR, compounded continuously
   - Account B: 3.16% APR, compounded quarterly

   Which account should you invest in to have the most money when you retire?

   f. You invest \$1000 into an account paying 1.02% annual interest, compounded continuously. How much money will you have in the account after 20 years, if no additional money is deposited or withdrawn?

2. When is it appropriate to use the TVM Solver in a financial scenario without payments?

   For each problem determine whether or not the TVM Solver may be used.

   a. A friend lends you \$100 for a week, which you agree to repay with 8% simple interest. How much does your friend expect you to pay them at the end of the week?

   b. How much do you need to deposit in an account now in order to have \$125,000 at the end of 25 years, if the account has a fixed annual interest rate of 3.4%, compounded monthly?

   c. You deposit \$675 in account earning 4.3%, compounded continuously, for 10 years. How much interest was earned on the account?

3. How do you determine the best rate of return for financial gain?

   a. In general, which of the following account options is best to accept for a student education loan? Explain your reasoning.

   Account A: 2.165% APR, compounded semiannually

   Account B: 2.16% APR, compounded monthly

   Account C: 2.15% APR, compounded daily

   Account D: 2.15% APR, compounded continuously

b. In general, which of the following account options is best to accept for a rainy day emergency fund? Explain your reasoning.

Account A: 4.38% APR, compounded annually

Account B: 4.39% APR, compounded quarterly

Account C: 4.385% APR, compounded weekly

Account D: 4.38% APR, compounded continuously

4. Can you interpret a financial scenario without payments and solve for the correct variable within the TVM Solver?
   a. You and a friend are planning a big trip to celebrate graduating from college in 4 years. The trip is estimated to cost $8000. If you have $900 you can invest now, what annual interest rate, compounded monthly, will you need on the investment to reach your goal?

   b. A lawyer has contacted you and told you that you just inherited $10,250 from a family friend. The lawyer says the family friend invested $4000 in an account earning 2.1% annual interest, compounded quarterly. How long ago did the family friend invest the $4000 in the account, for the account to grow to $10,250?

   c. You invest $10,000 in a stock paying 4% APR, compounded monthly. How much will the stock be worth in 25 years, if no additional money is invested or withdrawn and the stock's APR remains unchanged?

5. Can you interpret a financial scenario with payments and solve for the correct variable within the TVM Solver?
   a. You deposit $5 into an account each month which earns 8% annual interest, compounded monthly. How much will you have in the account at the end of 85 months?

   b. You believe you should have $1,000,000 in your retirement account when you retire in 45 years. If you deposit $1000 in your retirement account now, and it earns 3.75% annual interest, compounded monthly, how much do you need to deposit in the account at the end of each month in order to reach your goal?

   c. You can afford a $700 per month mortgage payment on a new home. The bank has offered you a 15-year mortgage at 4.8% APR, compounded monthly. What is the most expensive home you could buy using this information, if you do not plan on making a down payment?

6. What is a down payment and how does it affect the loan amount?
   a. You want to buy a $22,000 vehicle. The dealership is offering several incentives. Which is the best financing option for you?

   Option A: 6% APR (compounded monthly) for 60 months

   Option B: 10% down and 5.5% APR (compounded monthly) for 60 months

   Option C: $3000 down and 1.4% APR (compounded monthly) for 48 months

   b. You are purchasing new furniture from a local retailer. The furniture price is $12,500, and the retailer offers financing options. Which option is the best option for you?

   Option A: 9.7% APR (compounded monthly) for 60 months

   Option B: 5% down and 4.6% APR (compounded monthly) for 30 months

   c. Explain how to calculate a down payment and why someone might chose to make a down payment on a large purchase.

7. Can you explain how to find the total interest earned or paid on an account in a financial scenario?
   a. From the scenario in **Question 4c**, how much interest did the stock earn over the 25 years?

   b. From the scenario in **Question 5a**, how much will you have deposited in the account over the 85 months? What is the total interest earned on the account?

   c. When purchasing a new home for $325,000, your bank offers you a 15-year loan or a 30-year loan at 3.5% annual interest, compounded monthly. Both loans require a 10% down payment, with payments being made monthly. How much less will you pay in total interest on the 15-year loan?

   d. You want to be able to withdraw $36,000 each year when you retire. Based on family history, you believe you will live 25 years after you retire. If your retirement fund earns 6% annual interest, compounded annually, how much do you need in the account at the beginning of your retirement? How much money will you pull out of the account over the 25 years after you retire? How much of the money you withdrew after retiring was money from interest earned on the account?

8. Can you interpret a debt scenario and solve for the correct variable within the TVM Solver?
   a. You have $4500 in credit card debt, which charges 18% annual interest, compounded monthly. How long will it take to pay off the card, if you make the minimum monthly payments of $75 and do not put any other purchases on the card? What monthly payment should you make to pay off the card in a year?

   b. You purchased a home 15 years ago with a $135,000 mortgage charging 7.2% annual interest, compounded monthly, for 30 years of monthly payments. How much do you still owe on your home?

   c. You bought a car for $16,577 with no money down and a dealer loan at 4.9% annual interest, compounded monthly, for 60 months. After the twelfth payment you decide to increase the amount you are paying to the dealership each month by $200. How long will it take you to pay off the remainder of the loan?

   d. You purchased a new boat for $49,675, five years ago, using a 10-year loan requiring monthly payments and charging interest at 16% APR, compounded monthly. The bank offers you the option to refinance your remaining loan balance for another 8 years with monthly payments and charging interest at 12% APR, compounded monthly. What is the amount you would refinance? How much are your new payments?

9. Can you describe equity in your own words?
   a. A small boat is priced at $38,000. You pay $5000 down and finance the remaining balance for 7 years at 6.2% annual interest, compounded monthly. After 3 years of monthly payments you decide you are tired of the boat. How much equity do you have in the boat at this time?

   b. After a 20% down payment, you take out a mortgage on a new home for 30 years charging interest at 3.45% APR, compounded monthly. If your mortgage payments are $1695.78 each month and after 10 years you have $181,304.39 of equity in the home, what was the original purchase price of the home?

10. Can you describe, payment by payment, how a loan is paid off?
    a. You purchase a car for $15,000 with a bank loan charging 8.25% annual interest, compounded monthly, and requiring monthly payments for 48 months.

      i. How much is your monthly payment?

      ii. How much of the sixth payment goes towards the principal?

      iii. How much of the sixth payment goes towards interest on the loan?

      iv. How much interest will you pay over the life of the loan?

b. Suppose that 10 years ago you bought a home for \$195,000 by paying 15% down and financing the rest with a mortgage charging 4.8% annual interest, compounded monthly, and requiring monthly payments for 30 years.

- **i.** How much money did you pay as the down payment?
- **ii.** As a dollar amount, how much of the purchase price did you mortgage?
- **iii.** What is your monthly payment?
- **iv.** How much of the third payment went towards the principal?
- **v.** How much of the third payment went towards interest on the loan?

c. Interest rates have hit a new low and you decide to refinance your remaining mortgage (from part **b**). The new mortgage has a 15-year term, charges 2.7% annual interest, compounded monthly, and requires monthly payments.

- **i.** How much of your original loan did you refinance (what amount of money are you loaned for your new mortgage)?
- **ii.** What is your new monthly payment?
- **iii.** How much of the third new payment goes towards the principal?
- **iv.** How much of the third new payment goes towards interest on the loan?
- **v.** How much will you save in interest by refinancing?

VII Appendices

Appendix

A. Appendix

A.1 Number Sense

Just like a building needs a firm foundation to support it, your study of mathematics needs one too. This section will focus on the first building blocks of this foundation, with whole numbers, integers, fractions, decimals, and real numbers. We will also introduce the use of algebraic notation and vocabulary.

Introduction to Whole Numbers

Using Place Value with Whole Numbers

The most basic numbers used in algebra are the numbers we use to count objects in our world: 1, 2, 3, 4, and so on. These are called the **counting numbers**. Counting numbers are also called **natural numbers**. If we add zero to the counting numbers, we produce the set of **whole numbers.**

Counting Numbers: 1, 2, 3, ...

Whole Numbers: 0, 1, 2, 3, ...

The notation "..." is called an ellipsis and means "and so on," or that the pattern continues endlessly. We can visualize the whole numbers, including the natural numbers, on a **number line**, as shown in **Figure A.1.1**. Notice the numbers on the number line get larger as they go from left to right and smaller as they go from right to left. While this number line shows only the whole numbers 0 through 6, the whole numbers keep going, without end, to the right.

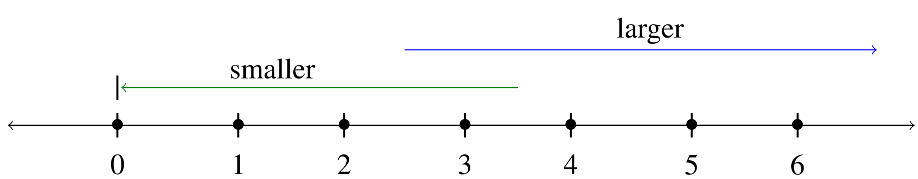


Figure A.1.1: The number line illustrating some whole numbers and natural numbers.

Our number system is called a place value system, because the value of a digit depends on its position in a number. The place values are separated into groups of three, with each group separated by commas. The number $5,278,194$ is shown in a place value chart in **Figure A.1.2**, with each digit in its respective place value.

Place Value														
Trillions			**Billions**			**Millions**			**Thousands**					
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

Figure A.1.2: The number $5,278,194$ shown in the place value chart.

When you write a check, you write out the number in words, as well as in digits. To write a number in words, write the number in each group of three, followed by the name of the group, without the *s* at the end. Start at the left, where the groups have the largest value; the last group of three is not named. The commas separate the groups, so wherever there is a comma in the number, place a comma between the words. The number $74,218,369$ is written as seventy-four million, two hundred eighteen thousand, three hundred sixty-nine, as illustrated below.

$$\underbrace{74}_{\text{millions}},\underbrace{218}_{\text{thousands}},\underbrace{369}$$

$$74 \longrightarrow \text{Seventy-four million,}$$

$$218 \longrightarrow \text{two hundred eighteen thousand,}$$

$$369 \longrightarrow \text{three hundred sixty-nine}$$

We are now going to reverse the process by writing the digits from the name of the number. To write the number in digits, we first look for the clue words that indicate the groups. It is helpful to draw three blanks for the needed groups and then fill in the blanks with the numbers, separating the groups with commas.

To write *nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine* as a whole number, using digits, first identify the words that indicate groups. Except for the first group, all other groups must have three places. Draw three blanks to indicate the number of places needed in each group, and separate the groups by commas. Then, write the digits in each group.

$$\overbrace{\text{nine billion}}^{\text{billions}} \quad \overbrace{\text{two hundred forty-six million}}^{\text{millions}} \quad \overbrace{\text{seventy-three thousand}}^{\text{thousands}} \quad \overbrace{\text{one hundred eighty-nine}}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\_\,\_\,\underline{9}\,, \qquad \underline{2}\,\underline{4}\,\underline{6}\,, \qquad \underline{0}\,\underline{7}\,\underline{3}\,, \qquad \underline{1}\,\underline{8}\,\underline{9}$$

The number is written $9,246,073,189$.

Rounding Whole Numbers

In 2019, the U.S. Census Bureau (https://census.gov/quickfacts/NY) estimated the population of the state of New York as 19,453,561. We could say the population of New York was approximately 19 million. In many cases, you don't need the exact value; an approximate number is good enough.

The process of approximating a number is called **rounding**. Numbers are rounded to a specific place value, depending on how much accuracy is needed. Saying that the population of New York is approximately 19 million means that we rounded to the nearest million.

■ **Example 1** Round 23,658 to the nearest hundred.

Solution:
Locate the hundreds place, and mark it with an arrow. All digits to the left of the hundreds place do not change.

$$\begin{array}{c}\text{hundreds place}\\ \downarrow \\ 23{,}658\end{array}$$

Underline the digit to the right of the hundreds place. The underlined digit is in the tens place.

$$23{,}6\underline{5}8$$

Is the underlined digit greater than or equal to 5?

- If so, add 1 to the digit in the given place value and replace all digits to the right of the given place value with zeros.
- If not, do not change the digit in the given place value, but replace all digits to the right of the given place value with zeros.

Given the underlined digit is a 5, we add 1 to the hundreds place and replace all the digit to the right with zeros.

$$23{,}700$$

So 23,658 rounded to the nearest hundred is 23,700.

■

Identifying Multiples and Applying Divisibility Tests

The numbers 2, 4, 6, 8, 10, and 12 are called **multiples** of 2. A multiple of 2 can be written as the product of a counting number and 2.

$$\begin{array}{ccccccc} 2, & 4, & 6, & 8, & 10, & 12, & \ldots \\ 2\cdot 1, & 2\cdot 2, & 2\cdot 3, & 2\cdot 4, & 2\cdot 5, & 2\cdot 6, & \ldots \end{array}$$

Similarly, a multiple of 3 would be the product of a counting number and 3.

$$\begin{array}{ccccccc} 3, & 6, & 9, & 12, & 15, & 18, & \ldots \\ 3\cdot 1, & 3\cdot 2, & 3\cdot 3, & 3\cdot 4, & 3\cdot 5, & 3\cdot 6, & \ldots \end{array}$$

We could identify the multiples of any number by continuing this process.

Table A.1 shows the multiples of 2 through 10 using the first 12 counting numbers.

Counting Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108
Multiples of 10	10	20	30	40	50	60	70	80	90	100	110	120

Table A.1: Multiples of Counting Numbers

Now, we will define a multiple using this product property.

Definition

A number is a **multiple** of n if it is the product of a counting number and n. ▪

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, the result is a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

Definition

If a number m is a multiple of n, then m is **divisible** by n. ▪

Notice that all of the multiples of 5 in **Table A.1** end in 5 or 0. We can conclude that numbers with the last digit of 5 or 0 are divisible by 5. Looking for other patterns in **Table A.1**, showing multiples of the numbers 2 through 10, we can discover the following divisibility tests.

Divisibility Tests

A number is divisible by

- 2, if the last digit is 0, 2, 4, 6, or 8.
- 3, if the sum of the digits is divisible by 3.
- 5, if the last digit is 5 or 0.
- 6, if it is divisible by both 2 and 3.
- 10, if it ends with 0.

▪ **Example 2** Is 5,625 divisible by

a. 2?
b. 3?
c. 6?
d. 5 or 10?

Solution:

a. Is 5,625 divisible by 2? In other words, does it end in 0, 2, 4, 6, or 8?

$\Longrightarrow$ No, thus 5,625 is not divisible by 2.

b. Is 5,625 divisible by 3? In other words, is the sum of the digits, $5+6+2+5 = 18$, divisible by 3?

$\Longrightarrow$ Yes, thus 5,625 is divisible by 3.

c. Is 5,625 is divisible by 6? In other words, is it divisible by both 2 and 3?

$\Longrightarrow$ No, 5,625 is not divisible by 2, so 5,625 is not divisible by 6.

d. Is 5,625 divisible by 5 or 10? In other words, what is the last digit? It is 5, not 0.

$\Longrightarrow$ 5,625 is divisible by 5, but not by 10, as the last digit is 5, not 0.

▪

Finding Prime Factorizations and Least Common Multiples

In mathematics, there are often several ways to talk about the same ideas. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. For example, 72 is a multiple of 8, so we say 72 is divisible by 8. Also 72 is a multiple of 9, so we say 72 is also divisible by 9. We can express this still another way.

As $8 \cdot 9 = 72$, we say that 8 and 9 are **factors** of 72. When we write $72 = 8 \cdot 9$, we say we have factored 72.

$$\underbrace{8 \cdot 9}_{factors} = \underbrace{72}_{product}$$

Other ways to factor 72 are $1 \cdot 72$, $2 \cdot 36$, $3 \cdot 24$, $4 \cdot 18$, and $6 \cdot 12$. Seventy-two has many factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 36, and 72.

Definition

If $a \cdot b = m$, then a and b are **factors** of m. ▪

Some numbers, like 72, have many factors. Other numbers have only two factors.

Definition

A **prime number** is a counting number *greater than 1*, whose only factors are 1 and itself.

A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and itself. ▪

The natural numbers from 2 to 19 and their factors are listed in **Table A.2**. The authors leave it to the reader to verify the classification of each number.

Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite

Number	Factors	Prime or Composite?
11	1,11	Prime
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime

Table A.2: Prime or Composite Classifications

The prime numbers less than 20 are 2,3,5,7,11,13,17, and 19. Notice that the only even prime number is 2.

Prime numbers are building blocks, in some sense, so it can be help to write numbers in terms of their prime factors.

Definition

The **prime factorization** of a number is the product of prime numbers that equals the number. ▪

To determine the prime factorization of a composite number, identify any two factors of the number and use them to create two branches.

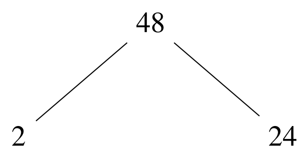
- If a factor is prime, that branch is complete, and we circle that prime.
- If the factor is not prime, identify two factors of that number and continue the process.

Once all the branches end with circled primes, the factorization is complete. The composite number can now be written as a product of the circled prime numbers.

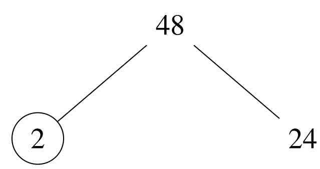
▪ **Example 3** Factor 48 into its prime factorization.

Solution:

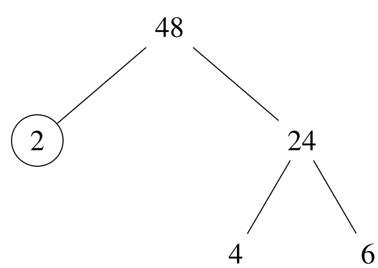
Identify two factors whose product is 48. As $48 = 2 \cdot 24$, we use these numbers to create two branches.



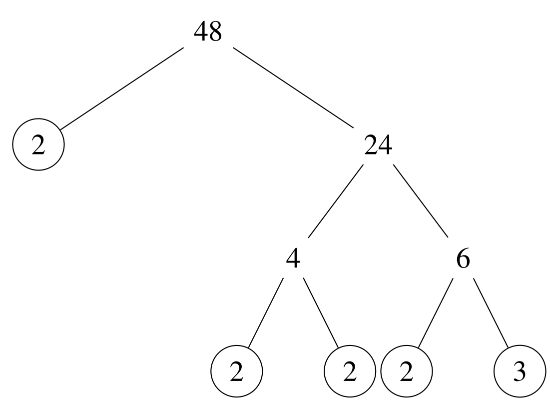
Given that 2 is prime, that branch is complete, and we circle the 2.



Given that 24 is not prime, we write 24 as the product of two factors (we will use $24 = 4 \cdot 6$).



As neither 4 nor 6 are prime, we continue the process with each composite number.



Now that each branch ends with a circled prime, we can write the composite number, 48, as the product of all the circled primes.

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer.

If we first factored 48 in a different way, for example as $6 \cdot 8$, the final result would still be the same. The authors leave it to the reader to verify.

▪

One of the reasons we discuss multiples and primes is to use these techniques to compute the **least common multiple** of two numbers. This will be useful when we add and subtract fractions with different denominators. Two methods are used most often to determine the least common multiple of two numbers, and we will look at both of them.

The first method is the Listing Multiples Method. For example, to identify the least common multiple of 12 and 18, we list the first few multiples of 12 and 18:

$$\begin{aligned}&12\text{: } 12, 24, 36, 48, 60, 72, 84, 96, 108, \ldots\\&18\text{: } 18, 36, 54, 72, 90, 108, \ldots\end{aligned}$$

Notice that some numbers appear in both lists; they are **common multiples** of 12 and 18.

Common Multiples of 12 and 18: 36, 72, 108, ...

As 36 is the smallest of the common multiples, we call it the *least* common multiple.

Definition

The **least common multiple (LCM)** of two numbers is the smallest number that is a multiple of both numbers. ■

The Listing Multiples Method for the Least Common Multiple

1. List several multiples of each number.
2. Look for the smallest number that appears in both lists.
3. This number is the LCM.

■ **Example 4** Determine the least common multiple of 15 and 20, using the Listing Multiples Method.

Solution:
Begin by making lists of the first few multiples of 15 and 20.

$$\mathbf{15:}\ 15, 30, 45, 60, 75, 90, 105, 120, \ldots$$

$$\mathbf{20:}\ 20, 40, 60, 80, 100, 120, 140, 160, \ldots$$

Now, look for the smallest number that appears in both lists. The first number to appear in both lists is 60, so 60 is the least common multiple of 15 and 20.

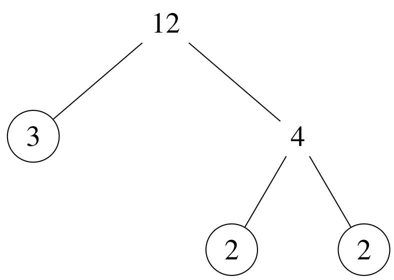
N *Notice that 120 is in both lists, too. It is a common multiple, but it is not the least common multiple of 15 and 20.*

■

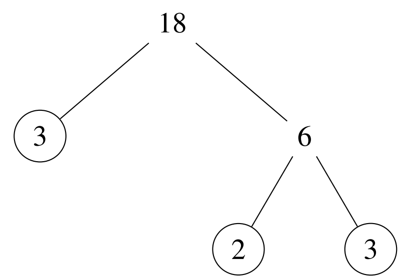
Our second method to compute the least common multiple of two numbers is the Prime Factors Method. Let's determine the LCM of 12 and 18 again, this time using their prime factorizations.

■ **Example 5** Calculate the Least Common Multiple (LCM) of 12 and 18, using the Prime Factors Method.

Solution:
First, write each number as its product of primes.



$12 = 2 \cdot 2 \cdot 3$



$18 = 2 \cdot 3 \cdot 3$

Now, list each number as its product of primes, matching primes vertically when possible.

$$\begin{array}{rl} 12 = & 2 \cdot 2 \cdot 3 \\ 18 = & 2 \cdot 3 \cdot 3 \\ \hline \end{array}$$

Bring down the prime from each column, and multiply the factors

$$\begin{array}{rl} 12 = & 2 \cdot 2 \cdot 3 \\ 18 = & 2 \cdot 3 \cdot 3 \\ \hline \text{LCM} = & 2 \cdot 2 \cdot 3 \cdot 3 = 36 \end{array}$$

So, 36 is the least common multiple of 12 and 18.

N *By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the least common multiple.*

■

The Prime Factors Method for the Least Common Multiple

1. Write each number as its product of primes.
2. List each number as its product of primes, and match primes vertically when possible.
3. Bring down the prime from each column and multiply the factors.

■ **Example 6** Compute the Least Common Multiple (LCM) of 30 and 36, using the Prime Factors Method.

Solution:
Rewrite 30 and 36 in their prime factorizations, matching primes vertically when possible. Bring down the prime in each column, and multiply the factors.

$$\begin{array}{rl} 30= & 2\cdot\quad\ 3\cdot\quad\ 5 \\ 36= & 2\cdot 2\cdot 3\cdot 3 \\ \hline \text{LCM}= & 2\cdot 2\cdot 3\cdot 3\cdot 5 = 180 \end{array}$$

The LCM of 30 and 36 is 180.

The authors leave it to the reader to verify the LCM, by using the Listing Multiples Method.

■

EXERCISES

Skills Practice (Answers)

For Exercises 1 - 2, state the place value of each digit in the given number.

1. $51,493$
   - **a.** 1
   - **b.** 4
   - **c.** 9
   - **d.** 5
   - **e.** 3

2. $36,084,215$
   - **a.** 8
   - **b.** 6
   - **c.** 5
   - **d.** 4
   - **e.** 3

For Exercises 3 - 4, write the number, using words.

3. $5,902$

4. $364,510$

For Exercises 5 - 6, write the number, using digits.

5. four hundred twelve

6. eighteen million, one hundred two thousand, seven hundred eighty-three

For Exercises 7 - 10, round to the indicated place value.

7. Round to the nearest ten.
   - **a.** 386
   - **b.** $2,931$

8. Round to the nearest ten.
   - **a.** 792
   - **b.** $5,647$

9. Round to the nearest hundred.
   - **a.** $13,748$
   - **b.** $391,794$

10. Round to the nearest hundred.
    - **a.** $28,166$
    - **b.** $481,628$

For Exercises 11 - 14, use divisibility tests to determine whether the number is divisible by 2, 3, 5, 6, or 10.

11. $9,696$
12. 75
13. 78
14. 350

For Exercises 15 - 18, determine the number's prime factorization.

15. 86
16. 132
17. 627
18. $2,520$

For Exercises 19 - 20, identify the least common multiple of each pair of numbers, using the Listing Multiples Method.

19. $12,16$
20. $44,55$

For Exercises 21 - 22, compute the least common multiple of each pair of numbers, using the Prime Factors Method.

21. $28,40$
22. $84,90$

INTEGERS

Using Negatives and Opposites

Our work so far has only included the natural numbers, or counting numbers, and the whole numbers. However, if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with **negative numbers**. Negative numbers are numbers less than 0. The negative numbers are to the left of zero on the number line, as shown in **Figure A.1.3**.

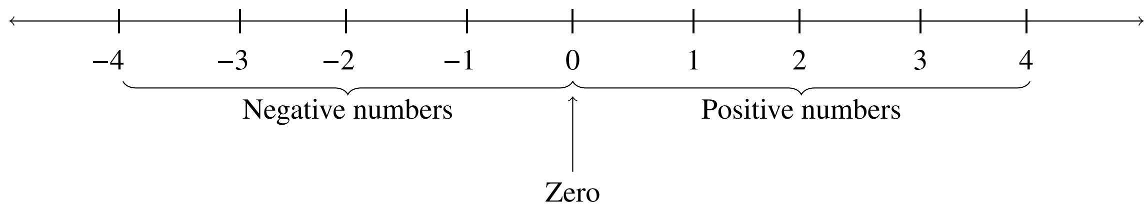


Figure A.1.3: A number line showing the location of positive numbers, negative numbers, and zero.

The arrows on the ends of the number line indicate that the numbers keep on going forever. There is no biggest positive number, and there is no smallest negative number.

Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. (See **Figure A.1.4**.)

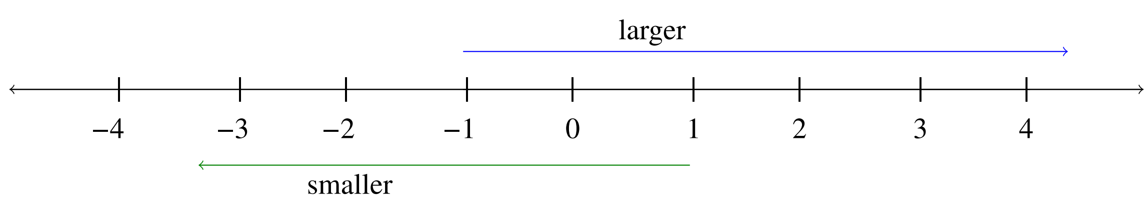


Figure A.1.4: Number Line

- When a is to the left of b on the number line, we write $a < b$, which is read "a is less than b."
- When a is to the right of b on the number line, we write $a > b$, which is read "a is greater than b."

Example 7 Order each of the pairs of number, using $<$ or $>$.

a. $14\_\_\_\_6$

b. $-1\_\_\_\_9$

c. $-1\_\_\_\_-4$

d. $0\_\_\_\_-20$

Solution:

We will begin by drawing a number line and indicating the numbers 14, 6, −1, 9, −4, 0, and −20 on the line. (See **Figure A.1.5**)

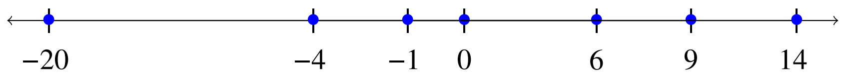


Figure A.1.5: Number Line

a. 14 is to the right of 6 on the number line, thus 14 is greater than 6.

$$14 \underline{\qquad} 6$$
$$14 > 6$$

b. −1 is to the left of 9 on the number line, so −1 is less than 9.

$$-1 \underline{\qquad} 9$$
$$-1 < 9$$

c. −1 is to the right of −4 on the number line, therefore −1 is greater than −4.

$$-1 \underline{\qquad} -4$$
$$-1 > -4$$

d. 0 is to the right of −20 on the number line, which means 0 is greater than −20.

$$0 \underline{\qquad} -20$$
$$0 > -20$$

■

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 3 and −3 are the same distance from zero, they are called **opposites**.

Definition

The **opposite** of a number is the number that is the same distance from zero on the number line, but on the opposite side of zero. ■

The opposite of 3 is −3, and the opposite of −3 is 3. **Figure A.1.6** illustrates the definition by showing both −3 and 3 are the same distance from 0.

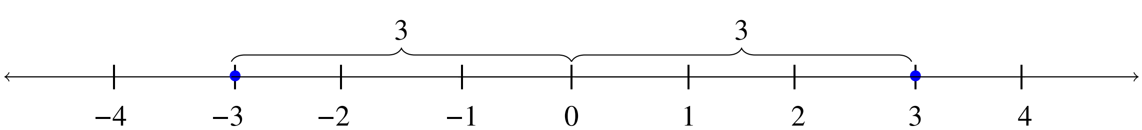


Figure A.1.6: Visual Representation of Opposite Numbers

Sometimes in algebra the same symbol has different meanings. Just like with some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol "$-$" used in three different ways.

$10-4$ Between two numbers, it indicates the operation of *subtraction.*
We read $10-4$ as "10 minus 4."

-8 In front of a number, it indicates a *negative* number.
We read -8 as "negative eight."

$-x$ In front of a variable, it indicates the *opposite.*
We read $-x$ as "the opposite of x."

$-(-2)$ Here there are two "$-$" signs. The one in the parentheses tells us the number is negative 2.
The one outside the parentheses tells us to take the *opposite* of -2.
We read $-(-2)$ as "the opposite of negative two."

Example 8 Determine

a. the opposite of 7
b. the opposite of -10
c. $-(-6)$

Solution:

a. The opposite of 7 is -7, because -7 is the same distance from 0 as 7, but on the opposite side of 0.

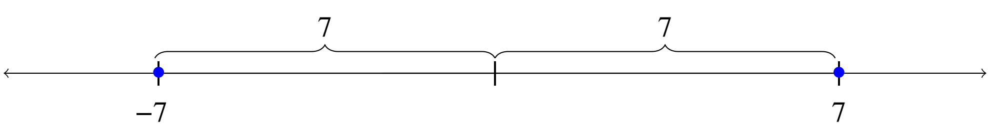


Figure A.1.7: Visual Representation of the Opposite of 7

b. The opposite of -10 is 10, because 10 is the same distance from 0 as -10, but on the opposite side of 0.

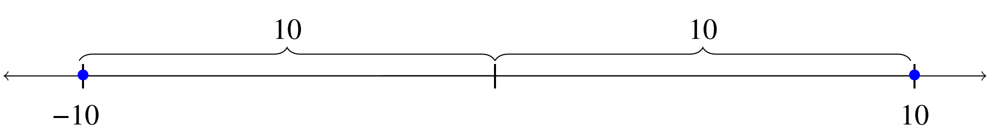


Figure A.1.8: Visual Representation of the Opposite of -10

c. $-(-6)$ is the opposite of -6, which is 6, because 6 is the same distance from 0 as -6, but on the opposite side of 0.

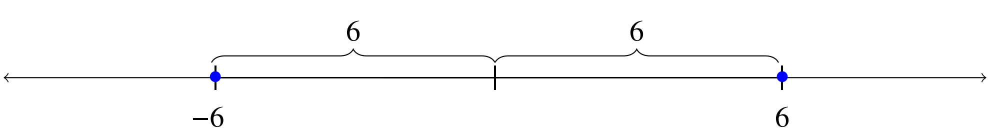


Figure A.1.9: Visual Representation of $-(-6)$

■

When determining the opposite of a variable, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether $-x$ is positive or negative. We can see this in the next example.

Example 9 Determine $-x$ at

a. $x = 8$
b. $x = -8$

Solution:

a. Substituting $x = 8$ into $-x$ for x, gives us

$$\begin{aligned} -x &= -(8) \\ &= -8 \end{aligned}$$

In other words, the opposite of 8 is -8.

b. Substituting $x = -8$ into $-x$ for x, gives us

$$\begin{aligned} -x &= -(-8) \\ &= 8 \end{aligned}$$

In other words, the opposite of -8 is 8.

■

Expressions with Absolute Value

We saw that numbers such as 3 and -3 are opposites, because they are the same distance from 0 on the number line; they are both three units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Definition

The **absolute value** of a number is its distance from 0 on the number line.

The absolute value of a number n is written as $|n|$. ■

For example,

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure A.1.10 illustrates these distances, using a number line.

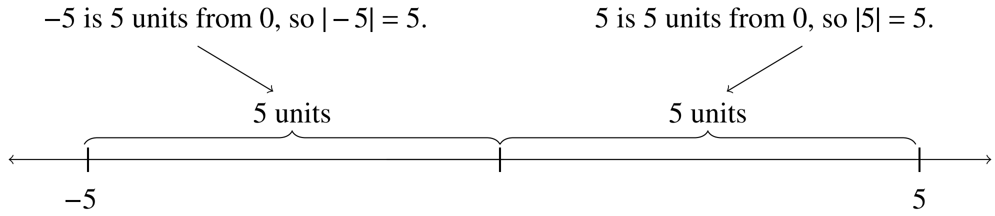


Figure A.1.10: Visual Representation of Absolute Value

The absolute value of a number is never negative, because distance cannot be negative. The only number with absolute value equal to zero is the number zero itself (because the distance from 0 to 0 on the number line is zero units). Thus,

$$|n| \geq 0.$$

Due to the fact that absolute values are always greater than or equal to zero, we say *absolute values* are *non-negative*.

▪ **Example 10** Simplify each absolute value.

a. $|3|$
b. $|-44|$
c. $|0|$

Solution:
The absolute value of a number is the distance between the number and zero.

a. $|3| = 3$

b. $|-44| = 44$

c. $|0| = 0$

▪

▪ **Example 11** Substitute the given number into the absolute value expression and simplify.

a. $|x|$ when $x = -35$
b. $|-y|$ when $y = -20$
c. $-|u|$ when $u = 12$
d. $-|p|$ when $p = -14$

Solution:

a. Substituting $x = -35$ into $|x|$ for x, gives us

$$|-35| = 35$$

b. Substituting $y = -20$ into $|-y|$ for y, gives us

$$|-(-20)| = |20| = 20$$

c. Substituting $u = 12$ into $-|u|$ for u, gives us

$$-|12| = -(12) = -12$$

d. Substituting $p = -14$ into $-|p|$ for p, gives us

$$-|-14| = -(14) = -14$$

■

Our work with opposites gives us a way to define the **integers**.

Definition

The whole numbers and their opposites are called the **integers**.

The integers are the numbers $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$ ■

Operations with Integers

Here, we will give properties of adding, subtracting, multiplying, and dividing integers, but will not explain the process. We encourage the reader to seek out additional resources on these techniques, if deemed necessary.

Addition of Positive and Negative Integers

When the signs are the same, we add the absolute values of the numbers and keep the sign.

$5+3$	$-5+(-3)$
8	-8
both positive, sum positive	both negative, sum negative

When the signs are different, subtract the absolute values of the numbers and take the sign of the number with the bigger absolute value.

$-5+3$	$5+(-3)$
-2	2
different signs, more negatives, sum negative	different signs, more positives, sum positive

Example 12 Simplify each sum.

a. $-14+(-36)$
b. $19+(-47)$

Solution:

a. The signs of both numbers are the same, so we add $|-14| = 14$ and $|-36| = 36$. The answer will be negative, because both numbers are negative.

$$-14+(-36) = -50$$

b. The signs are different, so we subtract $|19| = 19$ from $|-47| = 47$. The answer will be negative, because $|-47| > |19|$.

$$19+(-47) = -28$$

■

Subtraction of signed numbers can be done by adding the opposite of the second number. For instance, $-3-1$ is the same as $-3+(-1)$, and $3-(-1)$ is the same as $3+1$. You will often see this concept, the **subtraction property**, written in the following manner.

Subtraction Property

$$a-b = a+(-b)$$

Example 13 Simplify each pair of operations and verify they are equal.

a. $13-8$ and $13+(-8)$
b. $-17-9$ and $-17+(-9)$

Solution:

a. Using the property of subtraction of whole numbers, $13-8 = 5$. We can visualize this with a number line. Starting at 13, we move 8 units to the *left*, to illustrate we are *subtracting* 8.

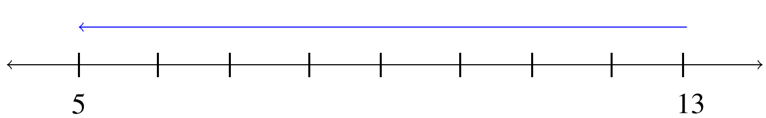


Figure A.1.11: A visual representation of subtracting 8 from 13.

When adding $13+(-8)$, because the signs are different, we subtract $|-8| = 8$ from $|13| = 13$. The answer will be positive, as $|13| > |-8|$.

$$13+(-8) = 5$$

Thus, we have shown $13-8$ and $13+(-8)$ are equal.

b. We can visualize $-17-9$ by using a number line. Starting at -17, we move 9 units to the *left*, to illustrate we are *subtracting* 9.

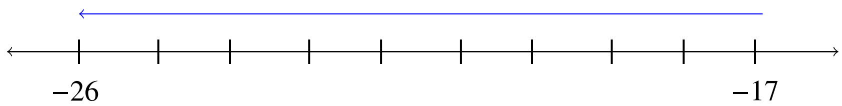


Figure A.1.12: A visual representation of subtracting 9 from -17.

Thus, $-17-9=-26$.

When adding $-17+(-9)$, because the signs are the same, we add the absolute values of the numbers, $|-17|=17$ and $|-9|=9$, and keep the sign, which is negative.

$$-17+(-9)=-26$$

■

Multiplication of Signed Numbers

Multiplication is mathematical shorthand for repeated addition. For example, we can rewrite $5\cdot 2$ as $5+5$, which is 10. Similarly, we can rewrite $-4\cdot 3$ as $(-4)+(-4)+(-4)$. Then, using the techniques for adding negative integers we see $-4\cdot 3=(-4)+(-4)+(-4)=-12$. We can generalize the sign resulting from the multiplication of two signed numbers as follows.

Same Signs	Product	Example
Two Positives	Positive	$7\cdot 4=28$
Two Negatives	Positive	$-8(-6)=48$

Different Signs	Product	Example
Positive · Negative	Negative	$7(-9)=-63$
Negative · Positive	Negative	$-5\cdot 10=-50$

■ **Example 14** Compute each product.

a. $-9\cdot 3$
b. $(-2)(-5)$
c. $4(-8)$
d. $7\cdot 6$

Solution:

a. We see that the signs are different, so the product is negative.

$$-9\cdot 3=-27$$

b. We see that the signs are the same, so the product is positive.

$$-2(-5)=10$$

c. We see that the signs are different, so the product is negative.

$$4(-8) = -32$$

d. We see that the signs are the same, so the product is positive.

$$7 \cdot 6 = 42$$

■

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number, and then a negative number, by -1, to see what the results are.

$$\begin{array}{cc} -1 \cdot 4 & -1(-3) \\ -4 & 3 \\ -4 \text{ is the opposite of } 4 & 3 \text{ is the opposite of } -3 \end{array}$$

Each time we multiply a number by -1, the result is its opposite.

Multiplication by -1

$$-1a = -a$$

Multiplying a number by -1 gives its opposite.

Dividing signed numbers follows the same rules as multiplying signed numbers.

For division of two signed numbers, when the

- Signs are the *same*, the quotient is *positive*.
- Signs are *different*, the quotient is *negative*.

Remember that we can always check the answer of a division problem by multiplying.

■ **Example 15** Compute each quotient.

a. $-27 \div 3$
b. $-100 \div (-4)$

a. As the signs are different, the quotient is negative.

$$-27 \div 3 = -9$$

b. As the signs that are the same, the quotient is positive.

$$-100 \div (-4) = 25$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 23 - 24, order the pairs of numbers, using < or >.

23. **a.** $9\_\_\_\_\_4$
    b. $-3\_\_\_\_\_6$
    c. $-8\_\_\_\_\_-2$
    d. $1\_\_\_\_\_-10$

24. **a.** $-7\_\_\_\_\_3$
    b. $-10\_\_\_\_\_-5$
    c. $2\_\_\_\_\_-6$
    d. $8\_\_\_\_\_9$

For Exercise 25, simplify the absolute value.

25. **a.** $|-32|$
    b. $|0|$
    c. $|16|$

For Exercises 26 - 27, fill in the blank with <, >, or = for the given pair of numbers.

26. **a.** $-6\_\_\_\_\_|-6|$
    b. $-|-3|\_\_\_\_\_-3$

27. **a.** $|-5|\_\_\_\_\_-|-5|$
    b. $9\_\_\_\_\_-|-9|$

For Exercises 28 - 29, determine the absolute value at the given number.

28. **a.** $-|p|$ when $p = 19$
    b. $-|q|$ when $q = -33$

29. **a.** $-|a|$ when $a = 160$
    b. $-|b|$ when $b = -12$

For Exercises 30 - 33, simplify the expression.

30. $-21 + (-59)$
31. $-34 + 19$
32. $-35 + (-47)$
33. $48 + (-16)$

For Exercises 34 - 37, simplify the expression.

34. $8-2$

35. $-6-(-4)$

36. $-5-4$

37. $7-(-3)$

For Exercises 38 - 43, compute the product.

38. $-4\cdot 8$

39. $-13\cdot(-5)$

40. $-1(-14)$

41. $-3(-9)$

42. $-1(19)$

43. $9(-7)$

For Exercises 44 - 47, compute the quotient.

44. $-24\div 6$

45. $-84\div(-6)$

46. $35\div(-7)$

47. $-192\div 12$

Fractions

Finding Equivalent Fractions

Fractions are a way to represent parts of a whole. The fraction $\frac{1}{3}$ means that one whole has been divided into 3 equal parts. The fraction $\frac{2}{3}$ represents two of the three equal parts. (See **Figure A.1.13**.)

Figure A.1.13: Visual Representations for Parts of a Whole

In the fraction $\frac{2}{3}$, the 2 is called the **numerator** and the 3 is called the **denominator**.

Definition

A **fraction** is written $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

- The number a is the **numerator** and the number b is the **denominator**.
- The line that separates the numerator from the denominator is called the **fraction bar**.

A fraction represents parts of a whole. The denominator, b, is the number of equal parts the whole has been divided into, and the numerator, a, indicates how many parts are included. ▪

If a whole pie has been cut into six pieces and we eat all six pieces, then we will have eaten $\frac{6}{6}$ pieces, or, in other words, one whole pie, as shown in **Figure A.1.14**.

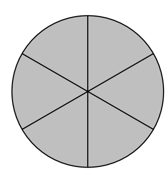


Figure A.1.14: A pie cut into six equal pieces.

If the pie was cut into eight pieces and we ate all eight pieces, then we ate $\frac{8}{8}$ pieces, or one whole pie. No matter the total number of pieces, we ate the same amount – one whole pie.

Property of One

Any integer a, where $a \neq 0$, divided by itself is one.

$$\frac{a}{a} = 1 \qquad (a \neq 0)$$

The fractions $\frac{6}{6}$ and $\frac{8}{8}$ represent the same value, 1, and so they are called **equivalent fractions.**

Let's think of pizzas this time. **Figure A.1.15** shows two images: a single pizza on the left, cut into two equal pieces, and a second pizza of the same size, cut into eight equal pieces, on the right. As the same amount of each pizza is shaded, we see that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$; they are equivalent fractions.

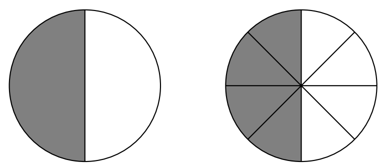


Figure A.1.15: Visual Representations of $\frac{1}{2}$

Definition

Equivalent fractions are fractions that represent the same value. ▪

How could we take a pizza that is cut into 2 pieces and cut it into 8 pieces? We could cut each of the 2 larger pieces into 4 smaller pieces. The whole pizza would then be cut into 8 pieces instead of just 2, as shown in **Figure A.1.16**.

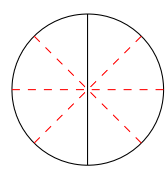


Figure A.1.16: Cutting each half of the pizza into 4 pieces

Mathematically, what we've described could be written as $\dfrac{1\cdot 4}{2\cdot 4} = \dfrac{4}{8}$. This model demonstrates the following property.

Equivalent Fractions Property

If a, b, and c are integers, where $b \neq 0$ and $c \neq 0$, then

$$\frac{a}{b} = \frac{a\cdot c}{b\cdot c}$$

If we had started with the pizza cut into two pieces, and then cut each half into smaller pieces, we could have

$$\frac{1\cdot 2}{2\cdot 2} = \frac{2}{4} \quad \text{so} \quad \frac{1}{2} = \frac{2}{4}$$
$$\frac{1\cdot 3}{2\cdot 3} = \frac{3}{6} \quad \text{so} \quad \frac{1}{2} = \frac{3}{6}$$
$$\frac{1\cdot 10}{2\cdot 10} = \frac{10}{20} \quad \text{so} \quad \frac{1}{2} = \frac{10}{20}$$

So, we say $\dfrac{1}{2}, \dfrac{2}{4}, \dfrac{3}{6}, \dfrac{4}{8}$, and $\dfrac{10}{20}$ are all equivalent fractions.

Example 16 Compute three fractions equivalent to $\dfrac{2}{5}$.

Solution:
To compute a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number; we can choose any number, except for zero. Choosing to multiply by $2, 3$, or 5 gives us

$$\frac{2\cdot 2}{5\cdot 2}=\frac{4}{10} \qquad \frac{2\cdot 3}{5\cdot 3}=\frac{6}{15} \qquad \frac{2\cdot 5}{5\cdot 5}=\frac{10}{25}$$

So, $\dfrac{4}{10}, \dfrac{6}{15}$, and $\dfrac{10}{25}$ are all equivalent to $\dfrac{2}{5}$.

Usually the negative sign is written in front of a fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$$\frac{-1}{3}=-\frac{1}{3} \qquad \frac{\text{negative}}{\text{positive}}=\text{negative}$$

$$\frac{1}{-3}=-\frac{1}{3} \qquad \frac{\text{positive}}{\text{negative}}=\text{negative}$$

Placement of Negative Sign in a Fraction

For any positive integers a and b,

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$$

Simplifying Fractions

While we have seen $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}$, and $\frac{10}{20}$ are equivalent fractions, $\frac{1}{2}$ is different from the rest as its numerator and denominator have no common factors, other than 1.

Definition

A fraction is considered **simplified**, or in **reduced form**, if there are no common factors, other than 1, of its numerator *and* denominator.

For example,

- $\dfrac{2}{3}$ is in reduced form, because there are no common factors of 2 and 3.
- $\dfrac{10}{15}$ is not in reduced form, because 5 is a common factor of 10 and 15.

The phrase *reduce a fraction* means to simplify the fraction. We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. In Example 16, we used the Equivalent Fractions Property to determine equivalent fractions. Now we'll use the Equivalent Fractions Property, in reverse, to simplify fractions.

Example 17 Simplify $-\frac{32}{56}$.

Solution:
Rewrite the numerator and denominator, showing the common factors.

$$-\frac{32}{56} = -\frac{4\cdot 8}{7\cdot 8}$$

Simplify, using the Equivalent Fractions Property.

$$= -\frac{4}{7}$$

Notice that the fraction $-\frac{4}{7}$ is simplified, because there are no more common factors.

Sometimes it may not be easy to identify common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors, using the Equivalent Fractions Property.

Example 18 Simplify $-\frac{210}{385}$.

Solution:
First, factor the numerator and denominator into their prime factorizations.

$$-\frac{210}{385} = -\frac{2\cdot 3\cdot 5\cdot 7}{5\cdot 7\cdot 11}$$

Next, simplify by dividing out common factors.

$$= -\frac{2\cdot 3\cdot \cancel{5}\cdot \cancel{7}}{\cancel{5}\cdot \cancel{7}\cdot 11}$$
$$= -\frac{2\cdot 3}{11}$$

Last, multiply the remaining factors, if necessary.

$$= -\frac{6}{11}$$

So, $-\frac{210}{385}$ is simplified to $-\frac{6}{11}$.

Multiplying Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions, so we will start with fraction multiplication.

Fraction Multiplication Property

If a, b, c, and d are integers, where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step.

■ **Example 19** Multiply $-\frac{11}{12} \cdot \frac{5}{7}$ and completely simplify.

Solution:
The first step is to determine the sign of the product. As the signs of the numbers are different, the product is negative.

$$-\frac{11}{12} \cdot \frac{5}{7} = -\left(\frac{11 \cdot 5}{12 \cdot 7}\right) = -\frac{55}{84}$$

Because there are not any common factors in the numerator and denominator, the product is completely simplified.

■

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

■ **Example 20** Multiply $-\frac{12}{5}(-20)$ and completely simplify.

Solution:
The signs of the numbers are the same, so the product is positive.

$$-\frac{12}{5}(-20) = \frac{12}{5}(20)$$

Rewrite 20 as a fraction, $\frac{20}{1}$, and multiply.

$$\frac{12}{5}(20) = \frac{12 \cdot 20}{5 \cdot 1} = \frac{240}{5}$$

Because 240 ends in a 0, we know it has a factor of 5, so we factor and divide the common factors out.

$$\frac{12 \cdot 4 \cdot \cancel{5}}{\cancel{5} \cdot 1} = \frac{48}{1}$$

Thus, $-\frac{12}{5}(-20) = 48$ in its simplified form.

■

Dividing Fractions

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, we need some vocabulary.

Notice that $\frac{2}{3} \cdot \frac{3}{2} = 1$. If the product of two numbers is equal to 1, then the two numbers are **reciprocals** of each other. A reciprocal of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator.

Definition

The **reciprocal** of $\frac{a}{b}$, where a and b are nonzero, is $\frac{b}{a}$ and has the sign of the original fraction.

A number and its reciprocal multiply to 1.

$$\frac{a}{b} \cdot \frac{b}{a} = 1$$

■

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Fraction Division Property

If a, b, c, and d are integers, where $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

N *We must include $b \neq 0$, $c \neq 0$, and $d \neq 0$ to ensure we do not divide by zero.*

■ **Example 21** Compute $-\frac{7}{8} \div \left(-\frac{14}{27}\right)$ and completely simplify.

Solution:
To divide, multiply the first fraction by the reciprocal of the second.

$$-\frac{7}{8} \div \left(-\frac{14}{27}\right) = -\frac{7}{18} \cdot \left(-\frac{27}{14}\right)$$

Determine the sign of the product, and then multiply. As both fractions are negative, the product is positive.

$$\begin{aligned} &= \frac{7 \cdot 27}{18 \cdot 14} \\ &= \frac{189}{252} \end{aligned}$$

To simplify, start by rewriting the numerator and denominator, showing common factors.

$$= \frac{7 \cdot 9 \cdot 3}{9 \cdot 2 \cdot 7 \cdot 2}$$

Divide out any common factors.

$$\begin{aligned} &= \frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2} \\ &= \frac{3}{2 \cdot 2} \end{aligned}$$

Multiplying the remaining factors, we have

$$-\frac{7}{18} \div \left(-\frac{14}{27}\right) = \frac{3}{4}$$

■

The numerators or denominators of some fractions contain fractions themselves.

Definition

A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction. ■

Some examples of complex fractions are:

$$\frac{\left(\frac{6}{7}\right)}{3} \qquad \frac{\left(\frac{3}{4}\right)}{\left(\frac{5}{8}\right)} \qquad \frac{4}{\left(\frac{1}{7}\right)}$$

To simplify a complex fraction, we remember that the fraction bar means division. For example, the complex fraction $\frac{\left(\frac{3}{4}\right)}{\left(\frac{5}{8}\right)}$ means $\frac{3}{4} \div \frac{5}{8}$.

Example 22 Compute $\dfrac{\left(\frac{3}{4}\right)}{\left(\frac{5}{8}\right)}$ and completely reduce.

Solution:
First, rewrite the complex fraction using the division symbol.

$$\frac{\left(\frac{3}{4}\right)}{\left(\frac{5}{8}\right)} = \frac{3}{4} \div \frac{5}{8}$$

Multiply the first fraction by the reciprocal of the second.

$$\frac{3}{4} \div \frac{5}{8} = \frac{3}{4} \cdot \frac{8}{5}$$

Multiply.

$$\frac{3 \cdot 8}{4 \cdot 5} = \frac{24}{20}$$

Factor, divide out common factors, and simplify.

$$\begin{aligned} &= \frac{3 \cdot 4 \cdot 2}{4 \cdot 5} \\ &= \frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 5} \\ &= \frac{6}{5} \end{aligned}$$

Therefore, $\dfrac{\left(\frac{3}{4}\right)}{\left(\frac{5}{8}\right)} = \dfrac{6}{5}$.

■

Adding or Subtracting Fractions with a Common Denominator

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators. To add or subtract fractions, they *must* have a common denominator.

Fraction Addition and Subtraction Properties

If a, b, and c are integers, where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

In other words, to add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

*When adding or subtracting fractions, we **only** add or subtract the **numerators** after determining a common denominator.*

▪ **Example 23** Compute $\frac{5}{3}+\frac{2}{3}$ and completely simplify.

Solution:
Add the numerators and place the sum over the common denominator.

$$\frac{5}{3}+\frac{2}{3}=\frac{5+2}{3}=\frac{7}{3}$$

▪

▪ **Example 24** Compute $-\frac{23}{24}-\frac{13}{24}$ and completely simplify.

Solution:
Subtract the numerators and place the difference over the common denominator.

$$-\frac{23}{24}-\frac{13}{24}=\frac{-23-13}{24}=\frac{-36}{24}$$

Simplify.

$$\frac{-36}{24}=-\frac{36}{24}=-\frac{\cancel{2}\cdot 3\cdot \cancel{6}}{\cancel{2}\cdot 2\cdot \cancel{6}}=-\frac{3}{2}$$

▪

Adding or Subtracting Fractions with Different Denominators

Thus far, when adding or subtracting fractions, the denominators have been the same. When the denominators are not the same, we must create a common denominator prior to adding or subtracting. The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions.

Definition

The **least common denominator (LCD)** of two fractions is the least common multiple (LCM) of their denominators. ▪

After we determine the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD, allowing us to add and subtract the fractions, because their denominators will be the same.

Example 25 Compute $\dfrac{7}{12}+\dfrac{5}{18}$ and completely simplify.

Solution:
Given that the fractions do not have a common denominator, to determine the LCD (least common denominator) we will use the Prime Factorization Method to calculate the LCM of the denominators.

$$\begin{array}{r l}
12= & 2\cdot 2\cdot 3 \\
18= & 2\cdot \quad\; 3\cdot 3 \\
\hline
\text{LCM}= & 2\cdot 2\cdot 3\cdot 3 = 36
\end{array}$$

Rewrite each fraction with the LCD, 36, and add the fractions

$$\begin{aligned}
\frac{7}{12}+\frac{5}{18} &= \frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2} \\
&= \frac{21}{36}+\frac{10}{36} \\
&= \frac{31}{36}
\end{aligned}$$

Because 31 is a prime number, it has no factors in common with 36. The answer is simplified.

Do not simplify the equivalent fractions before adding or subtracting. If you do, the result will be the original fractions and you will lose the common denominator.

When calculating the equivalent fractions needed to create a common denominator, there is a quick way to identify the number we need to multiply both the numerator and denominator. This method works if we found the LCD by factoring into primes.

Look at the factors of the LCD and then at each column above those factors. The "missing" factors of each denominator are the numbers we need.

$$\begin{array}{r l}
12= & 2\cdot 2\cdot 3\,\underline{\quad} \\
18= & 2\cdot \underline{\quad}\, 3\cdot 3 \\
\hline
\text{LCD}= & 2\cdot 2\cdot 3\cdot 3 = 36
\end{array}$$

In the last example, the LCD, 36, has two factors of 2 and two factors of 3.

The denominator of the first fraction, 12, has two factors of 2 but only one of 3. It is "missing" one 3, so we multiply the numerator and denominator of the first fraction by 3.

The denominator of the second fraction, 18, is "missing" one factor of 2, so we multiply the numerator and denominator of the second fraction by 2.

We will apply this method as we subtract the fractions in the next example.

▪ **Example 26** Compute $\frac{7}{15}-\frac{19}{24}$ and completely simplify.

Solution:
Seeing as the fractions do not have a common denominator, we need to determine the LCD.

$$\begin{array}{rl} 15= & \qquad\quad 3\cdot 5 \\ 24= & 2\cdot 2\cdot 2\cdot 3 \\ \hline \text{LCM}= & 2\cdot 2\cdot 2\cdot 3\cdot 5=120 \end{array}$$

Notice, 15 is "missing" three factors of 2 and 24 is "missing" the 5 from the factors of the LCD, 120. So we multiply the numerator and denominator by $2\cdot 2\cdot 2=8$ in the first fraction and by 5 in the second fraction to product equivalent fractions with the LCD.

$$\begin{aligned} \frac{7}{15}-\frac{19}{24} &= \frac{7\cdot 8}{15\cdot 8}-\frac{19\cdot 5}{24\cdot 5} \\ &= \frac{56}{120}-\frac{95}{120} \\ &= -\frac{39}{120} \end{aligned}$$

Both 39 and 120 have a factor of 3, so divide out the common factor and simplify.

$$-\frac{39}{120}=-\frac{13\cdot \cancel{3}}{40\cdot \cancel{3}}=-\frac{13}{40}$$

▪

We will summarize all the basic operations with fractions that we have explored thus far.

Summary of Operations with Fractions

Given a, b, c, and d are integers, with $b\neq 0$, $c\neq 0$, and $d\neq 0$, then we can say:

Fraction Multiplication	**Fraction Division**
$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ Multiply the numerators and multiply the denominators.	$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$ Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ Add the numerators and place the sum over the common denominator.	$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$ Subtract the numerators and place the difference over the common denominator.
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

EXERCISES

Skills Practice (Answers)

For Exercises 48 - 49, state three fractions equivalent to the given fraction.

48. $\frac{5}{9}$

49. $\frac{1}{8}$

For Exercises 50 - 55, reduce the fraction.

50. $-\frac{40}{88}$

51. $-\frac{104}{48}$

52. $-\frac{63}{99}$

53. $\frac{120}{252}$

54. $-\frac{108}{63}$

55. $\frac{182}{294}$

For Exercises 56 - 61, compute the product and simplify completely.

56. $\frac{3}{4} \cdot \frac{9}{10}$

57. $-\frac{3}{4}\left(-\frac{4}{9}\right)$

58. $-\frac{5}{9} \cdot \frac{3}{10}$

59. $\left(\frac{63}{60}\right)\left(-\frac{40}{88}\right)$

60. $(-1)\left(-\frac{6}{7}\right)$

61. $\frac{5}{6} \cdot 30$

For Exercises 62 - 67, compute the quotient and simplify completely.

62. $\frac{3}{4} \div \frac{2}{3}$

63. $-\frac{5}{6} \div \left(-\frac{5}{6}\right)$

64. $-\frac{7}{9} \div \left(-\frac{7}{4}\right)$

65. $\frac{5}{18} \div \left(-\frac{15}{24}\right)$

66. $\frac{3}{4} \div (-12)$

67. $-3 \div \frac{1}{4}$

For Exercises 68 - 69, compute the quotient and simplify completely.

68. $\dfrac{\left(-\frac{8}{21}\right)}{\left(\frac{12}{35}\right)}$

69. $\dfrac{\left(-\frac{4}{5}\right)}{2}$

For Exercises 70 - 75, perform the given operation and simplify completely.

70. $\dfrac{6}{13}+\dfrac{5}{13}$

71. $-\dfrac{3}{16}+\left(-\dfrac{7}{16}\right)$

72. $-\dfrac{8}{17}+\dfrac{15}{17}$

73. $\dfrac{11}{12}-\dfrac{5}{12}$

74. $-\dfrac{29}{14}-\dfrac{26}{14}$

75. $-\dfrac{3}{5}-\left(-\dfrac{4}{5}\right)$

For Exercises 76 - 80, add or subtract and simplify completely.

76. $\dfrac{1}{2}+\dfrac{1}{7}$

77. $\dfrac{1}{4}-\left(-\dfrac{1}{8}\right)$

78. $\dfrac{7}{12}-\dfrac{9}{16}$

79. $-\dfrac{11}{30}+\dfrac{27}{40}$

80. $-\dfrac{2}{3}-\left(-\dfrac{3}{4}\right)$

Decimals

Naming and Writing Decimals

Decimals are another way of writing fractions whose denominators are powers of 10.

$$0.1 = \frac{1}{10} \qquad 0.1 \text{ is “one tenth”}$$

$$0.01 = \frac{1}{100} \qquad 0.01 \text{ is “one hundredth”}$$

$$0.001 = \frac{1}{1000} \qquad 0.001 \text{ is “one thousandth”}$$

$$0.0001 = \frac{1}{10000} \qquad 0.0001 \text{ is “one ten-thousandth”}$$

Notice that “ten thousand” is a number larger than one, but “one ten-thousand**th**” is a number smaller than one. The “**th**” at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value, based on the powers of ten. We read 10,000 as “ten thousand” and 10,000,000 as “ten million.” Likewise, the names of the decimal places correspond to their fraction values. **Figure A.1.17** shows the names of the place values to the left and right of the decimal point.

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						.					

Figure A.1.17: Place value of numbers are shown to the left and right of the decimal point.

■ **Example 27** Name the decimal 4.3.

Solution:

First, name the number to the left of the decimal point.	four ________________
Next, write "and" for the decimal point.	four and ______________
Then, name the 'number' part to the right of the decimal point, as if it were a whole number.	four and three __________
Complete the name with the decimal place.	four and three tenths

■

■ **Example 28** Name the decimal -15.571.

Solution:

Name the number to the left of the decimal point.	negative fifteen ___________________________
Write "and" for the decimal point.	negative fifteen and _________________________
Name the number to the right of the decimal point.	negative fifteen and five hundred seventy-one ___________
The last digit, 1, is in the thousandths place.	negative fifteen and five hundred seventy-one thousandths

■

Rounding Decimals

Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers.

■ **Example 29** Round 18.379 to the nearest hundredth.

Solution:
Locate the hundredths place, and mark it with an arrow. All digits to the left of the hundredths place do not change.

$$\begin{array}{c}\text{hundredths place}\\ \downarrow\\ 18.3\color{red}{7}9\end{array}$$

Underline the digit to the right of the hundredths place. The underlined digit is in the thousandths place.

$$18.37\underline{9}$$

Is the underlined digit greater than or equal to 5?

- If so, add 1 to the digit in the given place value, and remove all digits to the right of the given place value.
- If not, do not change the digit in the given place value, but remove all digits to the right of the given place value.

Seeing as the underlined digit is greater than 5, we add 1 to the hundredths place and remove all digits to the right.

$$18.37\underbrace{9}_{\text{delete}}$$

$$18.38$$

So, 18.379 rounded to the nearest hundredth is 18.38.

■

■ **Example 30** Round \$18.379 to the nearest dollar.

Solution:
To round to the nearest dollar means to round to the nearest whole number. Locate the ones place with an arrow.

$$\overset{\text{ones place}}{1\overset{\downarrow}{8}.379}$$

Underline the digit to the right of the ones place.

$$18.\underline{3}79$$

As 3 is not greater than or equal to 5, do not add 1 to the 8, and rewrite the number, deleting all digits to the right of the rounding digit.

$$18$$

So, \$18.379 rounded to the nearest whole dollar is \$18.

■

Converting Decimals, Fractions, and Percents

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03 the 3 is in the *hundredths* place, so 3 is the numerator and *100* is the denominator of the fraction equivalent to 0.03.

$$0.03 = \frac{3}{100}$$

Notice, when the number to the left of the decimal is zero, we obtain a fraction whose numerator is less than its denominator. Fractions like this are called **proper fractions.**

Example 31 Write 0.374 as a fraction in reduced form.

Solution:
When writing 0.374 as an equivalent fraction, we must identify the numerator and denominator of the fraction.

- The numerator is 374.

To determine the denominator, we identify the place value of the final digit.

0.374

0.3	7	4
tenths	hundredths	thousandths

- The denominator is 1000, based on the place value of the last digit.

Thus, $0.374 = \dfrac{374}{1000}$.

We can simplify the fraction by dividing out the common factors.

$$\frac{374}{1000} = \frac{\cancel{2}\cdot 187}{\cancel{2}\cdot 500}$$

$$= \frac{187}{500}$$

So, $0.374 = \dfrac{187}{500}$ in reduced form.

■

We've learned to convert decimals to fractions. Now we will do the reverse and convert fractions to decimals. Remember that the fraction bar means division, so $\frac{4}{5}$ can be written as a decimal by dividing 4 by 5 $\left(4 \div 5 \text{ or } 5\overline{)4}\right)$.

Example 32 Write $-\dfrac{5}{8}$ as a decimal.

Solution:
We will ignore the negative sign for now and focus on converting $\dfrac{5}{8}$ to a decimal. As the fraction bar means division, we begin by writing $\dfrac{5}{8}$ as $8\overline{)5}$, and divide.

$$\begin{array}{r}
0.625 \\
8\overline{)5.000} \\
-\ 4\ 8 \\
\hline
20 \\
-16 \\
\hline
40 \\
-40 \\
\hline
0
\end{array}$$

As the original fraction is negative, $-\dfrac{5}{8} = -0.625$.

■

Definition

A **percent** is a ratio whose denominator is 100.

- Percent means per hundred.
- We use the percent symbol, %, to show percent.

Considering a percent is a ratio, it can easily be expressed as a fraction. Because percent means per 100, the denominator of the fraction is 100. We can then change the fraction to a decimal by dividing the numerator by the denominator. For example,

$$\begin{array}{ccc} 6\% & 78\% & 135\% \\ \\ \dfrac{6}{100} & \dfrac{78}{100} & \dfrac{135}{100} \\ \\ 0.06 & 0.78 & 1.35 \end{array}$$

Therefore, to convert a percent number to a decimal number, we drop the percent symbol and simply move the decimal point two places to the *left*.

Example 33 Convert the percent to a decimal.

a. 62%
b. 5.7%
c. 0.2%

Solution:

a. We drop the percent symbol and move the decimal point two places to the left.

$$0.62$$

b. We convert the percent to a decimal by moving the decimal point two places to the left.

$$0.057$$

c. We move the decimal point two places to the left.

$$0.002$$

Similarly, to convert a decimal to a percent, we move the decimal point two places to the *right*, and then add the percent sign. For example,

$$\begin{array}{ccccc} 0.05 & 0.83 & 1.05 & 0.075 & 0.003 \\ 5\% & 83\% & 105\% & 7.5\% & 0.3\% \end{array}$$

Adding and Subtracting Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers, and keep the decimal point in the same place in the sum.

■ **Example 34** Compute $23.5 + 41.38$.

Solution:
Write the numbers so the decimal points line up vertically.

$$\begin{array}{r} 23.5 \\ +41.38 \\ \hline \end{array}$$

Insert 0 as a placeholder after the 5 in 23.5. (Remember, $\frac{5}{10} = \frac{50}{100}$ so $0.5 = 0.50$.)

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline \end{array}$$

Add the numbers as if they were whole numbers. Then, bring down the decimal point in the sum.

$$\begin{array}{r} 23.50 \\ +41.38 \\ \hline 64.88 \end{array}$$

■

■ **Example 35** Compute $20 - 14.65$.

Solution:
Write the numbers so the decimal points line up vertically. Remember, 20 is a whole number, so place the decimal point after the 0.

$$\begin{array}{r} 20. \\ -14.65 \\ \hline \end{array}$$

Write in zeros to the right of the decimal point as placeholders.

$$\begin{array}{r} 20.00 \\ -14.65 \\ \hline \end{array}$$

Subtract and bring down the decimal point in the answer.

$$\begin{array}{r} \overset{1}{\cancel{2}}\,\overset{\overset{9}{\cancel{10}}}{\cancel{0}}.\overset{\overset{9}{\cancel{10}}}{\cancel{0}}\,\overset{10}{\cancel{0}} \\ -14.\,6\,5 \\ \hline 5.\,3\,5 \end{array}$$

■

Multiplying and Dividing Decimals

Multiplying decimals is very much like multiplying whole numbers; we just have to determine where to place the decimal point. We will multiply the numbers just as we do whole numbers, temporarily ignoring the decimal point. We then count and add the number of decimal points in the factors and that sum tells us the number of decimal places in the product. This procedure for multiplying decimals will make sense if we first convert the decimals to fractions and then multiply.

We will illustrate this process two examples, side-by-side.

	$(0.\underbrace{3}_{\text{1 place}})\,(0.\underbrace{7}_{\text{1 place}})$	$(0.\underbrace{2}_{\text{1 place}})\,(0.\underbrace{46}_{\text{2 places}})$
Convert to fractions:	$\dfrac{3}{10}\cdot\dfrac{7}{10}$	$\dfrac{2}{10}\cdot\dfrac{46}{100}$
Multiply:	$\dfrac{21}{100}$	$\dfrac{92}{1000}$
Convert to a decimal:	$0.\underbrace{21}_{\text{2 places}}$	$0.\underbrace{092}_{\text{3 places}}$

Notice, in the first example, we multiplied two numbers that each had one digit after the decimal point, and the product had $1+1=2$ decimal places. In the second example, we multiplied a number with one decimal place by a number with two decimal places, and the product had $1+2=3$ decimal places.

The rules for multiplying positive and negative numbers apply to decimals, as well.

Recall when *multiplying* two numbers,

- If their signs are the *same*, the product is *positive*.
- If their signs are *different*, the product is *negative*.

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. Finally, we write the product with the appropriate sign.

■ **Example 36** Compute $(-3.9)(4.075)$.

Solution:
Due to the fact that the signs of the two numbers are different, the product will be negative.

Write the product in vertical format (as if the numbers were both positive), lining up the numbers on the right.

$$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline \end{array}$$

Multiply the numbers, while ignoring the decimal points.

$$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ +\ 12225 \\ \hline 158925 \end{array}$$

To determine the placement of the decimal point, add the number of decimal places in the factors: $(-3.\underbrace{9}_{\text{1 place}})(4.\underbrace{075}_{\text{3 places}})$.

Place the decimal point 1+3 = 4 places from the right.

$$\begin{array}{r} 4.075 \\ \times\ 3.9 \\ \hline 36675 \\ +\ 12225 \\ \hline 15.\underbrace{8925}_{\text{4 places}} \end{array}$$

Remembering that the product is negative, we have

$$(-3.9)(4.075) = -15.8925$$

■

Just as with multiplication, division of decimals is very much like dividing whole numbers. The only difference is determining where the decimal point must be placed.

To divide decimals, determine what power of 10 to multiply the denominator by in order for the product to be a whole number. Then multiply the numerator by that same power of 10; because of the Equivalent Fractions Property, we haven't changed the value of the fraction. The effect is to move the decimal points in the numerator and denominator the same number of places to the right. For example,

$$\begin{aligned} \frac{0.8}{0.4} &= \frac{0.8(10)}{0.4(10)} \\ &= \frac{8}{4} \\ &= 2 \end{aligned}$$

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient, and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

Sometimes we may have to simplify expressions which contain both fractions and decimals.

Example 37 Compute $\dfrac{7}{8}+6.4$ and simplify.

Solution:
First, we must change one number so both numbers are in the same form. We can change the fraction to a decimal, or change the decimal to a fraction. Here, we will choose to change the fraction, $\frac{7}{8}$, to a decimal by dividing 7 by 8.

$$\begin{array}{r@{}l}
 & 0.875 \\
8 & \overline{)7.000} \\
- & 6\,4 \\ \hline
 & 60 \\
 & -56 \\ \hline
 & 40 \\
 & -40 \\ \hline
 & 0
\end{array}$$

Knowing $\frac{7}{8} = 0.875$, then

$$\begin{aligned}
\frac{7}{8}+6.4 &= 0.875+6.4 \\
&= 7.275
\end{aligned}$$

EXERCISES

Skills Practice (Answers)

For Exercises 81 - 82, name the decimal.

81. 2.64

82. -31.4

For Exercises 83 - 84, round the number to the nearest hundredth.

83. 0.845

84. 0.761

For Exercises 85 - 88, round the dollar amount to the nearest
 a. dollar.
 b. cent (hundredth).

85. \$5.781

86. \$1.6381

87. \$84.281

88. \$63.479

For Exercises 89 - 96, compute the given operation.

89. $38.6+13.67$

90. $-16.53-24.38$

91. $94.69-(-12.678)$

92. $72.5-100$

93. $(-5.18)(-65.23)$

94. $(-4.3)(2.71)$

95. $1.25\div(-0.5)$

96. $117.25\div 48$

For Exercises 97 - 98, write the decimal as a fraction in reduced form.

97. 1.35

98. 0.085

For Exercises 99 - 100, convert the fraction to a decimal.

99. $\frac{17}{20}$

100. $-\frac{284}{25}$

For Exercises 101 - 102, convert the percent to a decimal.

101. 1%

102. 250%

For Exercises 103 - 104, convert the decimal to a percent.

103. 0.0625

104. 4

105. Hyo Jin lives in San Diego. She bought a refrigerator for \$1,624.99, and when the clerk calculated the sales tax it came out to exactly \$142.186625. Round the sales tax to the nearest

   a. penny.
   b. dollar.

106. Jennifer bought a \$1,038.99 dining room set for her home in Cincinnati. She calculated the sales tax to be exactly \$67.53435. Round the sales tax to the nearest

   a. penny.
   b. dollar.

Integer Exponents and Scientific Notation

Using Integer and Whole Number Exponents

An exponent indicates repeated multiplication of the same quantity. For example, 3^5 means to multiply 3 by itself five times, so 3^5 means $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.

Definition

Exponential notation is defined as

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}$$

and is read as "a to the mth power." In the expression, a is called the **base** and m is a whole number called the **exponent**. The exponent tells us how many times we multiply the base by itself. ▪

N *For any nonzero base, a, then $a^0 = 1$.*

▪ **Example 38** Simplify the expression completely.

a. $(-7)^2$

b. $\left(\frac{3}{2}\right)^4$

Solution:

a. Considering the exponent is 2, we multiply the base, -7, by itself twice and simplify.

$$\begin{aligned}(-7)^2 &= (-7)(-7)\\ &= 49\end{aligned}$$

b. With the exponent being 4, we multiply the base, $\frac{3}{2}$, by itself four times and simplify.

$$\begin{aligned}\left(\frac{3}{2}\right)^4 &= \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\\ &= \frac{3 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2}\\ &= \frac{81}{16}\end{aligned}$$

▪

We will derive the properties of exponents, by looking for patterns in several examples.

Let's first look at what happens when we multiply two numbers with the same base, written in exponential notation. For example,

$$(5)^2 \cdot (5)^4$$

If we expand each power of 5, we have

$$(5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5)$$

and then we drop the parenthesis and rewrite the resulting product in exponential notation.

$$5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = (5)^6$$

So, $(5)^2 \cdot (5)^4 = (5)^6$.

This result leads us to the Product Property for Exponents.

Product Property for Exponents

If a is a given nonzero base and m and n are whole numbers, then

$$a^m \cdot a^n = a^{m+n}$$

Let's look at what happens when we raise a number written in exponential notation to a power. For example,

$$\left(4^3\right)^2$$

We will approach this in two ways.

First, we will rewrite the number to the second power as multiplication.

$$\left(4^3\right) \cdot \left(4^3\right)$$

Next, we can expand each number.

$$(4 \cdot 4 \cdot 4) \cdot (4 \cdot 4 \cdot 4)$$

So, we see that we have

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^6$$

Another approach would be to expand 4^3 first.

$$(4\cdot 4\cdot 4)^2$$

This expression can then be rewritten as

$$(4\cdot 4\cdot 4)\cdot(4\cdot 4\cdot 4)$$

So, we have again

$$4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4 = 4^6$$

This result leads us to the Power Property for Exponents.

Power Property for Exponents

If a is a given nonzero base and m and n are whole numbers, then

$$(a^m)^n = a^{m\cdot n}$$

Now, let's look at what happens when we divide two numbers with the same base, written in exponential notation. For example,

$$\frac{(2)^7}{(2)^3}$$

If we expand the numerator and denominator we have

$$\frac{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}{2\cdot 2\cdot 2}$$

Dividing out common factors leaves us with

$$\frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2\cdot 2\cdot 2}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 2\cdot 2\cdot 2\cdot 2 = 2^4$$

Let's look at another quotient.

$$\frac{(8)^3}{8}$$

While the denominator does not appear to be written in exponential notation, $8 = 8^1$, and we have

$$\frac{(8)^3}{8} = \frac{8^3}{8^1} = \frac{8\cdot 8\cdot 8}{8}$$

Dividing out the common factor gives us

$$\frac{\cancel{8}\cdot 8\cdot 8}{\cancel{8}} = 8\cdot 8 = 8^2$$

Notice in the two quotients above, we found

$$\frac{2^7}{2^3} = 2^4 \quad \text{and} \quad \frac{8^3}{8^1} = 8^2$$

This leads us to the Quotient Property for Exponents.

Quotient Property for Exponents

If a is a given nonzero base and m and n are whole numbers where $m > n$, then

$$\frac{a^m}{a^n} = a^{m-n}$$

So far we have looked at examples where all numbers written in exponential notation involved the same base. Now let's look at what happens when we multiply and divide numbers written in exponential notation where the bases are different.

We know $100 = 10^2$. We also know that $10 = 2 \cdot 5$. Combining these facts, we have

$$\begin{aligned} 100 &= 10^2 \\ &= (2 \cdot 5)^2 \\ &= (2 \cdot 5) \cdot (2 \cdot 5) \\ &= 2 \cdot 5 \cdot 2 \cdot 5 \\ &= 2 \cdot 2 \cdot 5 \cdot 5 \\ &= (2 \cdot 2) \cdot (5 \cdot 5) \\ &= (2)^2 \cdot (5)^2 \end{aligned}$$

This leads us to the Product to a Power Property for Exponents.

Product to a Power Property for Exponents

If a and b are two given nonzero bases and m is a whole number, then

$$(ab)^m = a^m \cdot b^m$$

Recall from Example 38,

$$\begin{aligned} \left(\frac{3}{2}\right)^4 &= \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) \\ &= \frac{3 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2} \\ &= \frac{3^4}{2^4} \end{aligned}$$

This leads us to the Quotient to a Power Property for Exponents.

Quotient to a Power Property for Exponents

If a and b are two given nonzero bases and m is a whole number, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

In the Quotient Property for Exponents, we considered the case where $m > n$. Now let's consider a case where $m < n$,

$$\frac{(8)^2}{(8)^5}$$

we have, in expanded form,

$$\frac{8 \cdot 8}{8 \cdot 8 \cdot 8 \cdot 8 \cdot 8}$$

which simplifies after dividing out common factors to

$$\frac{\cancel{8} \cdot \cancel{8}}{\cancel{8} \cdot \cancel{8} \cdot 8 \cdot 8 \cdot 8} = \frac{1}{8 \cdot 8 \cdot 8}$$

$$= \frac{1}{8^3}$$

If we simply subtract the exponents, as we did when $m > n$, we get

$$\frac{8^2}{8^5} = 8^{2-5} = 8^{-3}$$

This implies that $8^{-3} = \frac{1}{8^3}$, and it leads us to the definition of a *negative exponent.*

Definition

If a is a given nonzero base and n is an integer, then

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$$

▪

A negative exponent tells us we can rewrite the number by taking the reciprocal of the base and then changing the sign of the exponent.

Any number that has a negative exponent is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write numbers with only positive exponents. For example, if after simplifying a number we end up with a^{-3}, we will take one more step and write $\frac{1}{a^3}$, to *fully* simplify.

Example 39 Simplify the expression completely.

a. 4^{-2}
b. 10^{-3}

Solution:

a. Use the definition of a negative exponent, $a^{-n} = \dfrac{1}{a^n}$, and simplify.

$$\begin{aligned} 4^{-2} &= \frac{1}{4^2} \\ &= \frac{1}{4 \cdot 4} \\ &= \frac{1}{16} \end{aligned}$$

b. Use the definition of a negative exponent, $a^{-n} = \dfrac{1}{a^n}$, and simplify.

$$\begin{aligned} 10^{-3} &= \frac{1}{10^3} \\ &= \frac{1}{1000} \end{aligned}$$

■

Previously, we raised an integer to a negative exponent. We now look at what happens when we raise a fraction to a negative exponent. We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent, $\frac{1}{a^{-n}}$.

Using the definition of a negative exponent, $a^{-n} = \dfrac{1}{a^n}$, we have

$$\frac{1}{a^{-n}} = \frac{1}{\left(\frac{1}{a^n}\right)}$$

Simplifying the complex fraction gives us

$$\begin{aligned} &= 1 \cdot \frac{a^n}{1} \\ &= a^n \end{aligned}$$

This leads us to the Property of Negative Exponents.

Property of Negative Exponents

If a is a given nonzero base and n is an integer, then

$$\frac{1}{a^{-n}} = a^n$$

■ **Example 40** Completely simplify $\frac{1}{3^{-2}}$.

Solution:
Use the Property of a Negative Exponent, $\frac{1}{a^{-n}} = a^n$, and simplify.

$$\frac{1}{3^{-2}} = 3^2$$
$$= 9$$

■

Suppose we now have a fraction, whose numerator is not one, raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

Consider

$$\left(\frac{3}{4}\right)^{-2}$$

Using the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$, we have

$$\left(\frac{3}{4}\right)^{-2} = \frac{1}{\left(\frac{3}{4}\right)^2}$$

Simplifying the denominator and dividing,

$$= \frac{1}{\left(\frac{9}{16}\right)}$$

$$= \frac{16}{9}$$

We know that $\frac{16}{9} = \frac{4\cdot 4}{3\cdot 3} = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$, which tells us that

$$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$$

To transition from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base, the fraction, and changed the sign of the exponent.

This leads us to the Quotient to a Negative Power Property.

Quotient to a Negative Power Property

If a and b are nonzero given bases, and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

■ **Example 41** Completely simplify $\left(\frac{5}{7}\right)^{-2}$.

Solution:

Using the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$, take the reciprocal of the base and change the sign of the exponent. Remember to simplify your final answer.

$$\begin{aligned}\left(\frac{5}{7}\right)^{-2} &= \left(\frac{7}{5}\right)^{2} \\ &= \frac{7^2}{5^2} \\ &= \frac{49}{25}\end{aligned}$$

■

Converting from Decimal Notation to Scientific Notation

Our number system is based on powers of 10: tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens: tenths, hundredths, thousandths, and so on. Consider the numbers 4000 and 0.004. We know that 4000 means 4×1000 and 0.004 means $4 \times \frac{1}{1000}$.

If we write 1000 as a power of ten in exponential form, we can rewrite these numbers in the following way:

4000	0.004
4×1000	$4 \times \frac{1}{1000}$
4×10^3	$4 \times \frac{1}{10^3}$
	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10, written in exponential form, it is said to be in **scientific notation.**

Definition

A number is expressed in **scientific notation** when it is of the form

$$a \times 10^n,$$

where $1 \leq a < 10$ and n is an integer. ■

It is customary in scientific notation to use "×" as the multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$$\begin{aligned} &4000. \\ &= 4000. \\ &= 4 \times 10^3 \end{aligned}$$

Moved the decimal point 3 places to the left.

$$\begin{aligned} &0.004 \\ &= 0.004 \\ &= 4 \times 10^{-3} \end{aligned}$$

Moved the decimal point 3 places to the right.

Thus, to convert from decimal notation to scientific notation,

- When the number is larger than 1, the decimal point is moved to the *left* and the power of 10 is *positive*.

$$4,000 = 4 \times 10^3$$

- When the number is between 0 and 1, the decimal point is moved to the *right* and the power of 10 is *negative*.

$$0.004 = 4 \times 10^{-3}$$

■ **Example 42** Write $37,000$ using scientific notation.

Solution:
First, we move the decimal point so that the first factor is greater than or equal to 1, but less than 10.

$$37,000. \longrightarrow 3.7000$$

Next, we count the number of places, n, that the decimal point was moved. Here we moved the decimal point 4 places to the left, so $n = 4$.

As 37,000 is greater than 1, the power of 10 will be positive, 10^4, and the result is

$$37,000 = 3.7 \times 10^4$$

■

When converting from scientific notation to decimal notation, move the decimal in the opposite direction as before.

- If the power of 10 is positive, move the decimal point to the right.
- If the power of 10 is negative, move the decimal point to the left.

■ **Example 43** Write 4.789×10^{-4} as a decimal.

Solution:
Because the power of 10 is negative, move the decimal 4 places to the left.

$$4.789 \times 10^{-4} = 0.0004789$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 107 - 120, simplify the expression, and write the final answer using only positive exponents.

107. 10^{-3}

108. 3^{-4}

109. $\dfrac{1}{5^{-2}}$

110. $\dfrac{1}{7^{-9}}$

111. $\left(\dfrac{3}{10}\right)^{-2}$

112. $\left(\dfrac{4}{9}\right)^{-3}$

113. $\left(\dfrac{7}{2}\right)^{-3}$

114. $\left(\dfrac{8}{5}\right)^{5}$

115. $\left(-\dfrac{1}{3}\right)^{-3}$

116. $(-3)^3$

117. $\dfrac{9^7}{9^2}$

118. $(3 \cdot 5)^4$

119. $5^2 \cdot 5^{10}$

120. $(6^2)^4$

For Exercises 121 - 123, write the number in scientific notation.

121. 0.026

122. 0.00000103

123. $8,750,000$

For Exercises 124 - 126, convert the number to decimal form.

124. 5.2×10^2

125. 4.13×10^{-5}

126. 3.9×10^{-2}

The Real Numbers

Simplifying Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2, which we read as "n squared." The simplified result is called the **square** of n. For example, 8^2 is read as "8 squared" and simplifies to 64, which is called the *square* of 8.

Similarly, 121 is the square of 11, because $11^2 = 121$.

Definition

If $n^2 = m$, then m is the **square** of n. ▪

Table A.3, below, shows the squares of the natural numbers 1 through 15.

Number	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square	n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Table A.3: Perfect Squares

The numbers in the second row are called **perfect square numbers**; it will be helpful to learn to recognize these.

We can see from **Table A.3** that the squares of the natural numbers are positive numbers. When looking at negative numbers, we know that when the signs of two numbers are the same, their product is positive. Thus, the square of any negative number is also positive, as illustrated below.

$$(-3)^2 = 9 \qquad (-8)^2 = 64 \qquad (-11)^2 = 121 \qquad (-15)^2 = 225$$

Notice that these squares are the same as the squares of the corresponding positive numbers.

Sometimes we will need to look at the relationship between numbers and their squares, in reverse. Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a *square root* of 100. A number whose square is m is called a **square root** of m.

Definition

If $n^2 = m$, then n is a **square root** of m. ▪

Notice $(-10)^2 = 100$, meaning -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100.

Every positive number has two square roots, one positive and one negative. If we only want the positive square root of a positive number, then we use the **radical sign**, $\sqrt{m}$, to denote the positive square root. The positive square root is called the **principal square root**.

Although zero is neither positive nor negative, $0^2 = 0$, and we use the radical sign for the only square root of zero, $\sqrt{0} = 0$.

Definition

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

- The $\sqrt{\ }$ is called **the radical sign** and m is called the **radicand.**
- The **principal square root** of m, $\boldsymbol{\sqrt{m}}$, is the positive number whose square is m.

As 10 is the principal square root of 100, we write $\sqrt{100} = 10$. **Table A.4** gives some principal square roots; it may be helpful to learn to recognize these.

$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Table A.4: Principal Square Roots

We know that every positive number has two square roots and the radical sign indicates the positive one. If we want the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$. We read $-\sqrt{100}$ as "the opposite of the square root of 10."

Example 44 Simplify the square root.

a. $-\sqrt{9}$

b. $-\sqrt{144}$

Solution:

a. The principal square root of 9 is 3, so

$$-\sqrt{9} = -3.$$

b. The principal square root of 144 is 12, so

$$-\sqrt{144} = -12.$$

Thus far we have discussed the square roots of positive numbers and zero, now let's turn our attention to negative numbers. Consider $\sqrt{-49}$; is there a number whose square is -49?

$$(?)^2 = -49$$

Any positive number squared is positive, and any negative number squared is also positive. Therefore, there are no numbers in our discussions up to this point that equal $\sqrt{-49}$. In fact, there are is no number that we will discuss in this text that equals the square root of a negative number.

Until now we have only talked about squares and square roots. Let's extend our work to include higher powers and higher roots. To do so, we need to introduce some vocabulary first.

We write:	We say:
a^2	a squared
a^3	a cubed
a^4	a to the fourth power
a^5	a to the fifth power

The terms 'squared' and 'cubed' come from the formulas for the area of a square and the volume of a cube. **Table A.5** is a table of powers $(2-5)$ of the nonzero integers from -5 to 5.

Number	**Square**	**Cube**	**Fourth power**	**Fifth power**
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125

Number	**Square**	**Cube**	**Fourth power**	**Fifth power**
n	n^2	n^3	n^4	n^5
-1	1	-1	1	-1
-2	4	-8	16	-32
-3	9	-27	81	-243
-4	16	-64	256	-1024
-5	25	-125	625	-3125

Table A.5: Odd and Even Powers of Integers

Notice the signs in **Table A.5**; all powers of positive numbers are positive, of course, but when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We will zoom in on the row with the powers of -2 to illustrate the resulting signs.

n	n^2	n^3	n^4	n^5
-2	4	-8	16	-32

Even **Power** — **Positive** Result

Odd **Power** — **Negative** Result

Using the information above, we extend the square root definition to higher roots.

Definition

If $a^n = b$, then a is an $\boldsymbol{n^{th}}$ **root** of b.

The **principal** $\boldsymbol{n^{th}}$ **root** of b is $\boldsymbol{\sqrt[n]{b}}$, where n is called the **index** of the radical.

Just like we say the word "cubed" for b^3, we say the words "cube root" for $\sqrt[3]{b}$.

As we previously discussed, we know that there is not a square root of a negative number. The same is true for any even root. However, *odd* roots of negative numbers do exist, as a negative number raised to an odd power results in a negative number.

Existence Properties of $\sqrt[n]{a}$

When n is an even natural number and

- $a \geq 0$, then $\sqrt[n]{a}$ exists.
- $a < 0$, then $\sqrt[n]{a}$ does not exist, with the numbers under discussion currently.

When n is an odd natural number greater than 1, $\sqrt[n]{a}$ exists for all values of a.

Example 45 Simplify the root.

a. $\sqrt[3]{64}$
b. $\sqrt[4]{81}$
c. $\sqrt[5]{32}$

Solution:

a. $(4)^3 = 64$, so $\sqrt[3]{64} = 4$.

b. $(3)^4 = 81$, so the principal fourth root is $\sqrt[4]{81} = 3$.

c. $(2)^5 = 32$, so $\sqrt[5]{32} = 2$.

Example 46 Simplify the root.

a. $\sqrt[3]{-125}$
b. $\sqrt[4]{-16}$
c. $\sqrt[5]{-243}$

Solution:

a. $(-5)^3 = -125$, so $\sqrt[3]{-125} = -5$.

b. Using the fact that $a = -16 < 0$, $\sqrt[4]{-16}$ does not exist. The number in question must satisfy $(?)^4 = -16$, and no number we have discussed raised to the fourth power is negative.

c. $(-3)^5 = -243$, so $\sqrt[5]{-243} = -3$.

Estimating Roots

When we see a number with a radical sign, we often do not think about its equivalent numerical value. While we recognize that $\sqrt{4} = 2$, we may not know the value of $\sqrt{11}$ or $\sqrt[3]{91}$, without the use of technology. In some situations a quick estimate is meaningful.

To numerically estimate a square root, we look for perfect square numbers closest to the radicand. To determine an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

	Perfect Square Number	Square Root	
	4	2	
	9	3	
$11 \rightarrow$			$\leftarrow \sqrt{11}$
	16	4	
	25	5	

$$9 < 11 < 16$$

$$3 < \sqrt{11} < 4$$

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

	Perfect Cube Number	Cube Root	
	8	2	
	27	3	
	64	4	
$91 \rightarrow$			$\leftarrow \sqrt[3]{91}$
	125	5	

$$64 < 91 < 125$$

$$4 < \sqrt[3]{91} < 5$$

Example 47 Determine the two consecutive whole numbers that the root is between.

a. $\sqrt{105}$

b. $\sqrt[3]{43}$

Solution:

a. Consider the perfect square numbers closest to 105, by making a small table (**A.6**) of these perfect squares and their square roots. Locate 105 between two consecutive perfect squares.

	Perfect Square Number	Square Root	
	81	9	
	100	10	
$105 \rightarrow$			$\leftarrow \sqrt{105}$
	121	11	
	144	12	

Table A.6: Perfect Squares Close to 105 and Their Square Roots

Seeing as $100 < 105 < 121$, then $10 < \sqrt{105} < 11$.

b. Similarly, we locate 43 between the perfect cube numbers 27 and 64. (**See A.7**.)

Perfect Cube Number	Cube Root
8	2
27	3
64	4
125	5

$43 \rightarrow$ $\leftarrow \sqrt[3]{43}$

Table A.7: Perfect Cubes Close to 43 and Their Cube Roots

Seeing as $27 < 43 < 64$, then $3 < \sqrt[3]{43} < 4$.

■

Identifying Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *natural (counting) numbers, whole numbers,* and *integers*. What is the difference between these types of numbers?

Natural numbers	1, 2, 3, 4, ...
Whole numbers	0, 1, 2, 3, 4, ...
Integers	$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$

What type of numbers would we get if we started with all the integers and then included all the fractions? The set of all numbers which include both integers and fractions form the set of numbers called the **rational numbers.** Remember that the natural numbers and the whole numbers are also integers, and so they, too, are rational numbers.

Definition

A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

In other words, a rational number can be written as a ratio of two integers. ■

Let's consider a few decimals to see if we can write each of them as the ratio of two integers.

We know that integers are rational numbers and that integers can be written as decimals. For instance, the integer -8 could be written as the decimal -8.0. So, clearly, some decimals are rational numbers.

Now consider the decimal 7.3. We can write it as a ratio of two integers, because $7.3 = 7.3\left(\frac{10}{10}\right) = \frac{73}{10}$. So 7.3 is the ratio of the integers 73 and 10; it is a rational number.

In general, any decimal that 'terminates' (such as 7.3 or -1.2684) is a rational number, because we can use the place value of the last digit as the denominator to convert the decimal to a fraction.

Let's look at the decimal form of the numbers we know are rational.

We know that *every integer is a rational number*, as $a = \frac{a}{1}$ for any integer, a. We can also change any integer to a decimal by adding a decimal point and a zero.

Integer	-2	-1	0	1	2	3
Decimal form	-2.0	-1.0	0.0	1.0	2.0	3.0

These decimal numbers terminate.

By definition, *every fraction is a rational number*, and so let's examine the decimal form of some fractions.

Ratio of integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$
The decimal form	0.8	-0.875	3.25	$-6.666...$

These decimals either terminate or repeat.

These examples help to show that *every rational number can be written both as a ratio of integers* $\left(\frac{p}{q}\right)$*, where p and q are integers and $q \neq 0$, and as a decimal that either terminates or repeats.*

However, there are decimals that do not terminate or repeat. The number e, which is very important in describing nature and some financial situations, has a decimal form that does not terminate or repeat.

$$e = 2.7182818285...$$

We can even create a decimal pattern that does not terminate or repeat, such as

$$2.01001000100001....$$

Numbers whose decimal form does not terminate or repeat form the set of numbers called the **irrational numbers**.

Definition

An **irrational number** is a number that cannot be written as the ratio of two integers.

Its decimal form does not terminate and does not repeat. ▪

▪ **Example 48** For the number given, identify whether it is rational or irrational.

a. $\sqrt{36}$

b. $\sqrt{44}$

Solution:

a. We recognize that 36 is a perfect square, as $6^2 = 36$. So $\sqrt{36} = 6$ (an integer), therefore $\sqrt{36}$ is rational.

b. Remember that $6^2 = 36$ and $7^2 = 49$, so 44 is not a perfect square and thus $\sqrt{44}$ is not an integer. Therefore, if we use technology, we see the decimal form of $\sqrt{44} \approx 6.633249581\ldots$ which will never repeat and never terminate. So, $\sqrt{44}$ is irrational.

■

We have seen that all counting (natural) numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not terminate and does not repeat. When we combine the set of all rational numbers and the set of all irrational numbers, the result is the set of **real numbers.**

Definition

A **real number** is a number that is either rational or irrational. ■

All the numbers we use in this course are real numbers. **Figure A.1.18** illustrates how the number sets we have discussed in this section fit together.

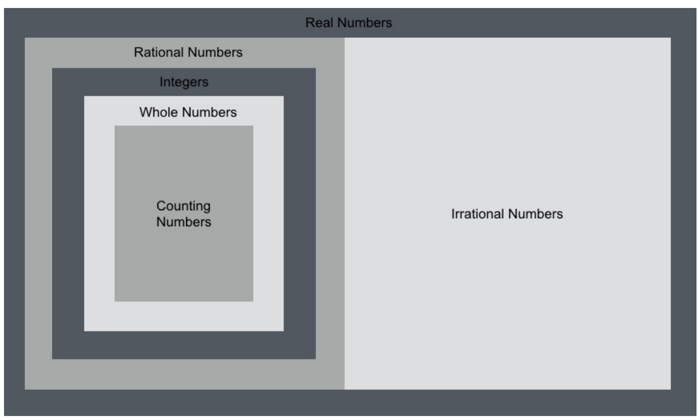


Figure A.1.18: The Set of Real Numbers

None of the numbers that we have dealt with so far has a square that is -25, because any positive number squared is positive, and any negative number squared is positive. So, we say there is no *real number* equal to $\sqrt{-25}$.

The square root of a negative number is not a real number. Similarly, other even roots of a negative number are also not real numbers.

■ **Example 49** For the number given, identify whether or not it is a real number.

a. $\sqrt{-169}$

b. $-\sqrt{64}$

Solution:

a. There is no real number whose square is -169. Therefore, $\sqrt{-169}$ is not a real number.

b. As the negative is in front of the radical, $-\sqrt{64}$ is -8. Given -8 is a real number, $-\sqrt{64}$ is a real number.

■

The last time we looked at the number line, it only had zero and positive and negative integers on it. We now want to include fractions and decimals on it.

Locating Fractions on the Number Line

Let's start with locating $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$, and $\frac{8}{3}$ on a number line.

We will start with the whole numbers 3 and -5, because they are the easiest to plot.

We know the proper fraction $\frac{1}{5}$ has a value less than one, as the numerator is smaller than the denominator, and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts and plot $\frac{1}{5}$. Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit from 0 to -1 into 5 equal parts, we plot $-\frac{4}{5}$. (See **Figure A.1.19**.)

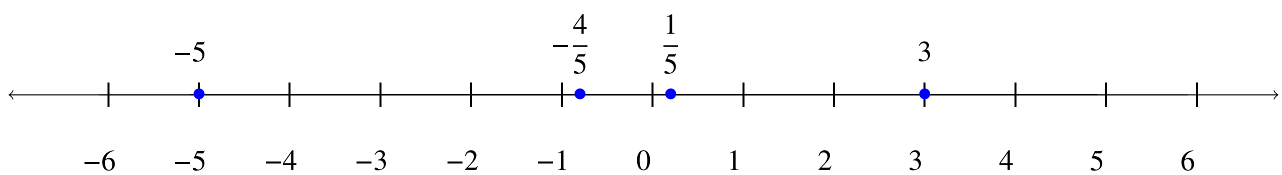


Figure A.1.19: Integers and fractions, with denominators of 5, plotted on a number line.

Now, we look at the fractions in which the numerator is greater than the denominator. Locating these points may be easier if you change how they are written.

$$\begin{aligned}\frac{7}{4} &= \frac{4+3}{4}\\ &= \frac{4}{4}+\frac{3}{4}\\ &= 1+\frac{3}{4}\end{aligned}$$

$$\begin{aligned}-\frac{9}{2} &= -\frac{1+8}{2}\\ &= -\left(\frac{1}{2}+\frac{8}{2}\right)\\ &= -\left(\frac{1}{2}+4\right)\\ &= -\left(4+\frac{1}{2}\right)\end{aligned}$$

$$\begin{aligned}\frac{8}{3} &= \frac{6+2}{3}\\ &= \frac{6}{3}+\frac{2}{3}\\ &= 2+\frac{2}{3}\end{aligned}$$

Figure A.1.20 shows a number line with the points $-5, -\frac{9}{2}, -\frac{4}{5}, \frac{1}{5}, \frac{7}{4}, \frac{8}{3}$, and 3 plotted.

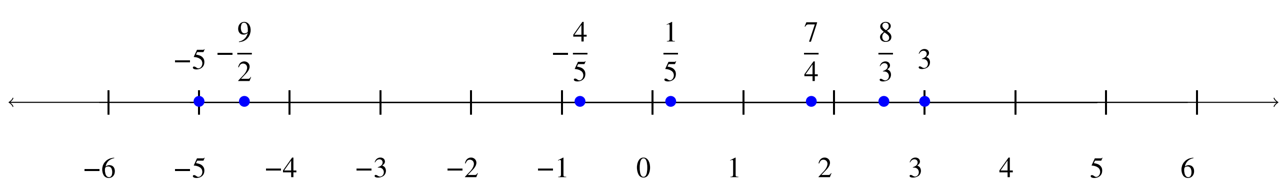


Figure A.1.20: $-5, -\frac{9}{2}, -\frac{4}{5}, \frac{1}{5}, \frac{7}{4}, \frac{8}{3}$, and 3 plotted on a number line.

Locating Decimals on the Number Line

Due to the fact that most decimals can be rewritten as fractions, locating decimals on the number line is similar to locating fractions on the number line.

▪ **Example 50** Locate 0.4 on the number line.

Solution:
The decimal number 0.4 is equivalent to $\frac{4}{10}$, so 0.4 is located between 0 and 1. On a number line, we divide the interval between 0 and 1 into 10 equal parts. Now we label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9. We write 0 as 0.0 and 1 as 1.0, so that the numbers are consistently in tenths. Finally, we mark 0.4 on the number line, as shown in **Figure A.1.21**.

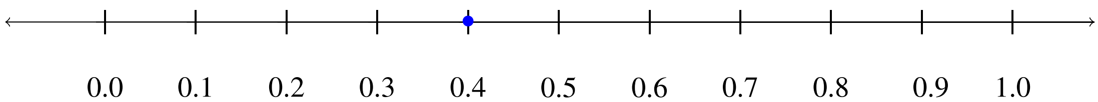


Figure A.1.21: 0.4 plotted on a number line.

▪

▪ **Example 51** Locate −0.74 on a number line.

Solution:
The decimal −0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and −1. On a number line, mark off and label the interval between 0 and −1, and plot −0.74 as appropriate. (See **Figure A.1.22**.)

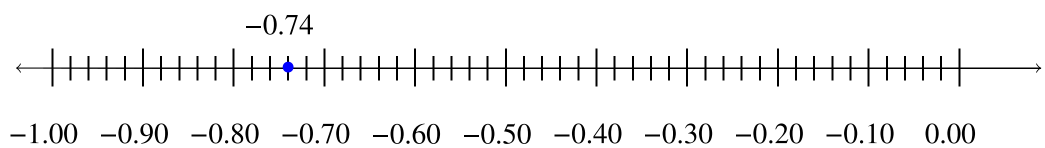


Figure A.1.22: −0.74 plotted on a number line.

▪

When comparing decimals, you can think of the decimals in terms of money. You know that \$0.40 (forty cents) is greater than \$0.04 (four cents), so 0.40 > 0.04.

EXERCISES

SKILLS PRACTICE (Answers)

For Exercises 127 - 132, simplify the expression.

127. $\sqrt{36}$

128. $\sqrt{9}$

129. $-\sqrt{1}$

130. $-\sqrt{121}$

131. $\sqrt[3]{27}$

132. $\sqrt[5]{-32}$

For Exercises 133 - 134, identify whether the number is rational or irrational.

133. **a.** $\sqrt{225}$
 b. $\sqrt{216}$

134. **a.** $\sqrt{164}$
 b. $\sqrt{169}$

For Exercises 135 - 136, identify whether or not the number is a real number.

135. **a.** $-\sqrt{64}$
 b. $\sqrt{-9}$

136. **a.** $\sqrt{-49}$
 b. $-\sqrt{144}$

For Exercises 137 - 138, locate the numbers on a number line.

137. $\frac{3}{4}, -\frac{3}{4}, \frac{5}{3}, -\frac{5}{3}, \frac{5}{2}, -\frac{5}{2}$

138. $\frac{1}{5}, -\frac{2}{5}, \frac{7}{4}, -\frac{7}{4}, \frac{8}{3}, -\frac{8}{3}$

For Exercises 139 - 142, locate the number on a number line.

139. 0.8

140. 3.1

141. -0.9

142. -1.6

Properties of Real Numbers

All of the exponent properties we have already developed apply to real number exponents, too. We will now discuss some additional properties that hold true for all real numbers.

Using the Commutative and Associative Properties

Think about adding two real numbers, say 5 and 3.

$$\begin{array}{ccc} 5+3 & & 3+5 \\ =8 & & =8 \\ \\ 5+3 & = & 3+5 \end{array}$$

The sums are the same. The order we add real numbers does not affect the result.

Think about multiplying 5 and 3.

$$\begin{array}{ccc} 5\cdot 3 & & 3\cdot 5 \\ =15 & & =15 \\ \\ 5\cdot 3 & = & 3\cdot 5 \end{array}$$

Again, the products are the same. The order in which we multiply real numbers does not matter.

These examples illustrate the Commutative Property.

Commutative Property of Addition and of Multiplication

- If a and b are real numbers, then $a+b=b+a$.
- If a and b are real numbers, then $a\cdot b=b\cdot a$.

When adding or multiplying real numbers, changing the *order* produces the same result.

Now, consider the subtraction of two real numbers.

$$\begin{array}{ccc} 7-3 & & 3-7 \\ =4 & & =-4 \\ \\ 4 & \neq & -4 \\ 7-3 & \neq & 3-7 \end{array}$$

The differences are not the same. Changing the order of subtraction did not produce the same result, so we know that *subtraction of real numbers is not commutative*.

Let's consider what happens when we divide two real numbers.

$$\begin{array}{ccc} 12 \div 4 & & 4 \div 12 \\ = \dfrac{12}{4} & & = \dfrac{4}{12} \\ = 3 & & = \dfrac{1}{3} \\ & & \\ 3 & \neq & \dfrac{1}{3} \\ 12 \div 4 & \neq & 4 \div 12 \end{array}$$

The quotients are not the same. Changing the order of division did not produce the same result, so *division of real numbers is not commutative*.

In summary,

- Addition and Multiplication *are* commutative.
- Subtraction and Division *are not* commutative.

Consider simplifying $7+8+2$.

Some people would start by adding the first two numbers, $7+8=15$, and then add to the result the last number, $15+2=17$. Others might start by adding the last two numbers, $8+2=10$, and then add the result to the first number, $7+10=17$. Here, either way gives the same result. We can use parentheses as grouping symbols to indicate which operation was done first.

$$\begin{array}{ccc} (7+8)+2 & & 7+(8+2) \\ 15+2 & & = 7+10 \\ = 17 & & = 17 \\ & & \\ (7+8)+2 & = & 7+(8+2) \end{array}$$

When adding three real numbers, changing the grouping of the numbers produces the same result.

Now, consider simplifying $5 \cdot \dfrac{1}{3} \cdot 3$.

$$\begin{array}{ccc} \left(5 \cdot \dfrac{1}{3}\right) \cdot 3 & & 5 \cdot \left(\dfrac{1}{3} \cdot 3\right) \\ = \dfrac{5}{3} \cdot 3 & & = 5 \cdot 1 \\ = 5 & & = 5 \\ & & \\ \left(5 \cdot \dfrac{1}{3}\right) \cdot 3 & = & 5 \cdot \left(\dfrac{1}{3} \cdot 3\right) \end{array}$$

When multiplying three real numbers, changing the grouping of the numbers produces the same result.

The sums and products, above, illustrate the Associative Property.

Associative Property of Addition and of Multiplication

- If a and b are real numbers, then $(a+b)+c = a+(b+c)$.
- If a and b are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

When adding or multiplying real numbers, changing the *grouping* produces the same result.

Let's think again about multiplying $5 \cdot \frac{1}{3} \cdot 3$. While both groupings give the same result, multiplying $\frac{1}{3}$ and 3 first, as shown above on the right side, eliminates the fraction in the first step.

We saw that subtraction and division were not commutative; they are not associative either.

In summary,

- Addition and Multiplication *are* associative.
- Subtraction and Division *are not* associative.

Using the Identity and Inverse Properties of Addition and Multiplication

Adding 0 to any real number does not change the value. For this reason, we call 0 the **additive identity.**
For example,

$$13+0=13 \qquad -14+0=-14 \qquad 0+(-8)=-8$$

Multiplying any real number by 1 does not change the value, so we call 1 the **multiplicative identity.**
For example,

$$43 \cdot 1 = 43 \qquad -27 \cdot 1 = -27 \qquad 1 \cdot \frac{3}{5} = \frac{3}{5}$$

We summarize the Identity Properties below.

Identity Property of Addition and of Multiplication

- For any real number a: $\quad a+0=a \quad$ and $\quad 0+a=a$
  0 is the **additive identity**.
- For any real number a: $\quad a \cdot 1 = a \quad$ and $\quad 1 \cdot a = a$
  1 is the **multiplicative identity**.

What number added to 5 gives the additive identity? In other words, 5 plus what number results in 0?

$$5 + \_\_\_\_\_ = 0 \qquad \text{We know } 5 + (-5) = 0$$

What number added to -6 gives the additive identity? In other words, -6 plus what number results in 0?

$$-6 + \_\_\_\_\_ = 0 \qquad \text{We know } -6 + 6 = 0$$

Notice that in each case, the missing number was the opposite of the number. We call the opposite of a number, $-a$, the **additive inverse** of a. A number and its opposite add to zero, which is the additive identity.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity? In other words, $\frac{2}{3}$ times what number results in 1?

$$\frac{2}{3} \cdot \_\_\_\_\_ = 1 \qquad \text{We know } \frac{2}{3} \cdot \frac{3}{2} = 1$$

What number multiplied by 2 gives the multiplicative identity? In other words, 2 times what number results in 1?

$$2 \cdot \_\_\_\_\_ = 1 \qquad \text{We know } 2 \cdot \frac{1}{2} = 1$$

Notice that in each case, the missing number was the reciprocal of the number. We call the reciprocal of a number, $\frac{1}{a}$, the **multiplicative inverse** of a. A number and its reciprocal multiply to one, which is the multiplicative identity.

We will formally state the inverse properties below.

Inverse Property

- For any real number a, $\boldsymbol{-a}$ is the **additive inverse** of a.

  A number and its additive inverse add to zero: $a + (-a) = -a + a = 0$

- For any nonzero real number a, $\boldsymbol{\frac{1}{a}}$ is the **multiplicative inverse** of a.

  A number and its multiplicative inverse multiply to one: $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$

Example 52 State the additive inverse of the given number.

a. $\frac{5}{8}$

b. 0.6

c. -8

d. $-\frac{4}{3}$

Solution:
To determine the additive inverse, we identify the opposite of the given number.

a. The opposite of $\frac{5}{8}$ is $-\frac{5}{8}$, so the additive inverse of $\frac{5}{8}$ is $-\frac{5}{8}$.

b. The opposite of 0.6 is -0.6, so the additive inverse of 0.6 is -0.6.

c. The opposite of -8 is written as $-(-8)$ and simplifies to be 8. Therefore, the additive inverse of -8 is 8.

d. The opposite of $-\frac{4}{3}$ is written as $-\left(-\frac{4}{3}\right)$ and simplifies to be $\frac{4}{3}$. Thus, the additive inverse of $-\frac{4}{3}$ is $\frac{4}{3}$.

■

■ **Example 53** State the multiplicative inverse of the given number.

a. 9

b. $-\frac{1}{9}$

c. 0.9

Solution:
To determine the multiplicative inverse, we identify the reciprocal of the given number.

a. The reciprocal of 9 is $\frac{1}{9}$. Therefore, the multiplicative inverse of 9 is $\frac{1}{9}$.

b. The reciprocal of $-\frac{1}{9}$ is $-\frac{9}{1}$. Thus, the multiplicative inverse of $-\frac{1}{9}$ is -9.

c. To determine the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then, we identify the reciprocal of the fraction. The reciprocal of $\frac{9}{10}$ is $\frac{10}{9}$, so the multiplicative inverse of 0.9 is $\frac{10}{9}$.

■

Using the Properties of Zero

The Identity Property of Addition says that when we add 0 to any number, the result is that same number. However, multiplying by 0 results in a product equal to zero.

Multiplication by Zero

For any real number a, $a \cdot 0 = 0$ and $0 \cdot a = 0$.

The product of any real number and 0 is 0.

What about division involving zero?

Consider $0 \div 3$ in terms of a real world example:

> If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get?

As there are no cookies to share, each person gets 0 cookies. So,

$$0 \div 3 = 0$$

We can check division with the related multiplication fact.

$$0 \div 3 = 0 \text{ because } 0 \cdot 3 = 0$$

Division of Zero

For any nonzero real number a, $\frac{0}{a} = 0$ and $0 \div a = 0$.

Zero divided by any real number, except zero, is zero.

Now consider dividing 4 by 0. Think about the related multiplication fact:

$$4 \div 0 = ? \text{ means } ? \cdot 0 = 4.$$

Because any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4. We conclude that there is no answer to $4 \div 0$, and so we say that division by 0 is undefined.

Division by Zero

For any nonzero real number a, $\frac{a}{0}$ and $a \div 0$ are undefined.

Division by zero is undefined.

Example 54 Perform the given operation, if possible, or state the operation is undefined.

a. $-8 \cdot 0$

b. $\frac{0}{-2}$

c. $\frac{-32}{0}$

Solution:

a. The product of any real number and 0 is 0, so $-8 \cdot 0 = 0$.

b. Zero divided by any real number, except itself, is 0, so $\frac{0}{-2} = 0$.

c. Division by 0 is undefined, so $\frac{-32}{0}$ is undefined.

■

Simplifying Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need \$9.25 (that's 9 dollars and 1 quarter) to pay for their ticket. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times 9 dollars so \$27, and 3 times 1 quarter, so 75 cents. Therefore, in total, they need \$27.75. If you think about doing the math in this way, you are using the distributive property.

Distributive Property

If a, b, and c are real numbers, then

$$a(b+c) = ab + ac$$

Also,

$$\begin{aligned}(b+c)a &= ba + ca\\ a(b-c) &= ab - ac\\ (b-c)a &= ba - ca\end{aligned}$$

Back to our friends at the movies, we could calculate the total amount of money they need, using the Distributive Property.

$$\begin{aligned}3(9.25) &= 3(9+0.25)\\ &= 3(9) + 3(0.25)\\ &= 27 + 0.75\\ &= 27.75\end{aligned}$$

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

EXERCISES

Skills Practice (Answers)

For Exercises 143 - 150, perform the given operations and simplify.

143. $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$

144. $\frac{13}{18}\cdot\frac{25}{7}\cdot\frac{18}{13}$

145. $\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15}$

146. $\frac{2}{5}+\frac{5}{12}+\left(-\frac{2}{5}\right)$

147. $\left(\frac{11}{12}+\frac{4}{9}\right)+\frac{5}{9}$

148. $\frac{3}{20}\cdot\frac{49}{11}\cdot\frac{20}{3}$

149. 17(0.25)(4)

150. 360(0.2)(5)

For Exercises 151 - 152, state the additive inverse of the number.

151. **a.** $\frac{5}{9}$
b. 2.1
c. -3
d. $-\frac{9}{5}$

152. **a.** $-\frac{8}{3}$
b. -0.019
c. 52
d. $\frac{5}{6}$

For Exercises 153 - 154, state the multiplicative inverse of the number.

153. **a.** 12
b. $-\frac{9}{2}$
c. 0.13

154. **a.** $\frac{17}{20}$
b. -1.5
c. -3

For Exercises 155 - 160, perform the given operation, if possible, or state that the operation is undefined.

155. $\frac{0}{6}$

156. $0\div\frac{11}{12}$

157. $0\div\frac{8}{15}$

158. (3.14)(0)

159. $\frac{\frac{1}{10}}{0}$

160. $\frac{7}{0}$

Systems of Time Measurement

There are two systems of measurement commonly used around the world. Most countries use the metric system; the U.S. uses a different system of measurement, usually called the **U.S. system**.

However, both the U.S. system and the metric system measure *time* in seconds, minutes, and hours. The equivalencies of time measurements are shown in **Table A.8**. The table also shows, in parentheses, the common abbreviations for each measurement.

Time Measurements		
1 minute (min)	=	60 seconds (sec)
1 hour (hr)	=	60 minutes (min)
1 day	=	24 hours (hr)
1 week (wk)	=	7 days
1 year (yr)	=	365 days

Table A.8: Time Equivalencies

N *While every year does not necessarily have exactly 365 days, 1 yr = 365 days is a standard conversion.*

In many real-world applications, we need to convert between units of measurement. We will use the Identity Property of Multiplication to perform these conversions.

When converting between different units, we use the Identity Property of Multiplication by writing '1' in a form that will help us convert the units. For example, suppose we want to change hours into minutes.

We know that 1 hour is equal to 60 minutes, so we write '1' as the fraction $\frac{1 \text{ hour}}{60 \text{ minutes}}$ or $\frac{60 \text{ minutes}}{1 \text{ hour}}$.

When we multiply by either of these fractions, we do not change the value, but just change the units.

So, how do we decide whether to multiply by $\frac{1 \text{ hour}}{60 \text{ minutes}}$ or $\frac{60 \text{ minutes}}{1 \text{ hour}}$?

We choose the fraction that will make the units we want to convert *from* divide out. Treat the unit words like factors and 'divide out' common units, like we do common factors.

If we want to convert 4 hours to minutes, we must decide which multiplication will eliminate the hours,

$$4 \text{ hours} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \qquad \text{or} \qquad 4 \text{ hours} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}}.$$

Using the second form the "hours" unit word 'divides out,' leaving only minutes.

$$4 \cancel{\text{hours}} \cdot \frac{60 \text{ minutes}}{1 \cancel{\text{hour}}} = \frac{4 \cdot 60 \text{ min}}{1} = 240 \text{ minutes}$$

If we multiplied 4 hours by the first form, no units would divide out.

Sometimes to convert from one unit to another, we may need to multiply by several unit conversions.

▪ **Example 55** The Juliet family is going to their summer home and will be away for 2 weeks. How many minutes will the Juliet family be gone?

Solution:
To convert weeks into minutes, we will convert weeks into days, days into hours, and then hours into minutes. To do this, we begin by writing '1' as $\frac{7 \text{ days}}{1 \text{ week}}, \frac{24 \text{ hours}}{1 \text{ day}}$, and $\frac{60 \text{ minutes}}{1 \text{ hour}}$. We will then multiply 2 weeks by the multiple conversion factors of '1'.

$$\frac{2 \text{ wk}}{1} \cdot \frac{7 \text{ days}}{1 \text{ wk}} \cdot \frac{24 \text{ hr}}{1 \text{ days}} \cdot \frac{60 \text{ min}}{1 \text{ hr}}$$

We divide out the common units and simplify.

$$\begin{aligned} \frac{2 \cancel{\text{wk}}}{1} \cdot \frac{7 \cancel{\text{days}}}{1 \cancel{\text{wk}}} \cdot \frac{24 \cancel{\text{hr}}}{1 \cancel{\text{days}}} \cdot \frac{60 \text{ min}}{1 \cancel{\text{hr}}} &= \frac{2 \cdot 7 \cdot 24 \cdot 60 \text{ min}}{1 \cdot 1 \cdot 1 \cdot 1} \\ &= 20160 \text{ min} \end{aligned}$$

Thus, the Juliets' will be gone for 20,160 minutes.

▪

EXERCISES

Skills Practice (Answers)

For Exercises 161 - 162, convert the units.

161. Rocco waited $1\frac{1}{2}$ hours for his appointment. Convert the time to seconds.

162. Misty's surgery lasted $2\frac{1}{4}$ hours. Convert the time to seconds.

For Exercises 163 - 164, solve the application, by converting to the given units.

163. One day Anya kept track of the number of minutes she spent driving. She recorded 45, 10, 8, 65, 20 and 35. How many hours did Anya spend driving?

164. Last year Eric went on 6 business trips. The number of days of each was 5, 2, 8, 12, 6, and 3. How many weeks did Eric spend on business trips last year?

A.2 Introduction to Algebra

Using Variables and Algebraic Symbols

Suppose this year Greg is 20 years old and Alex is 23. You know that Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more. In algebra, we use letters of the alphabet to represent values that change. So if we call Greg's age g, then we could use $g+3$ to represent Alex's age. (See **Table A.9**.)

Greg's Age	Alex's Age
12	15
20	23
35	38
g	$g+3$

Table A.9: Comparison of Greg's and Alex's Ages

In the language of algebra, we say that Greg's age and Alex's age are **variables** and the 3 years between their ages is a **constant**. The ages change ("vary"), but the 3 years between them always stays the same ("constant").

Definition

A **variable** is a letter that represents a number whose value may change.

A **constant** is a number whose value always stays the same. ▪

The letters most commonly used for variables are x, y, z, a, b, and c, but any letter may appear as a variable.

To translate words into algebraic expressions, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using.

There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We will list the symbols used to indicate these operations below in **Table A.10**. You should recognize most of them from the previous section of the Appendix.

Operation	**Notation**	**Say:**	**The result is...**
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a \cdot b$, ab, $(a)(b)$, $(a)b$, $a(b)$	a times b	the product of a and b
Division	$a \div b$, a/b, $\frac{a}{b}$	a divided by b	the quotient of a and b

Table A.10: Arithmetic Operations and Corresponding Notation

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion.

Does $3xy$ mean $3 \times y$ ("three times y") or $3 \cdot x \cdot y$ ("three times x times y")?

To make it clear, use $\cdot$ or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to English, or from English to symbolic form, pay attention to the words "of" and "and."

- The *difference of* 9 *and* 2 means subtract 9 and 2. In other words, 9 minus 2, which we write symbolically as $9-2$.
- The *product of* 4 *and* 8 means multiply 4 and 8. In other words 4 times 8, which we write symbolically as $4 \cdot 8$.

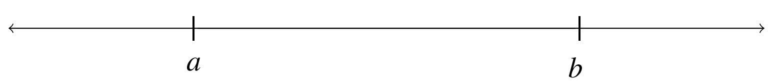
In addition to arithmetic operations, we need symbols to compare the values of two numbers or algebraic expressions.

When two quantities have the same value, we say they are equal and connect them with an **equals sign**, "=".

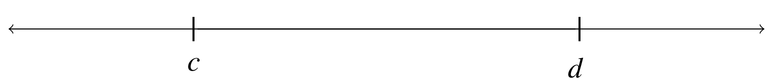
$a = b$ is read "a is equal to b"

On the number line, recall the numbers grow larger as they go from left to right and can be used to explain the symbols "<" and ">".

$a < b$ is read "a is less than b" and a is to the left of b on the number line.



$d > c$ is read "d is greater than c" and d is to the right of c on the number line.



The expressions $a < b$ and $d > c$ can be read from left to right or right to left, though in English we usually read from left to right, as demonstrated in **Table A.11**.

Inequality Symbols	In Words
$a \neq b$	a is *not equal to b*
$a < b$	a is *less than b*
$a \leq b$	a is *less than or equal to b*
$a > b$	a is *greater than b*
$a \geq b$	is *greater than or equal to b*

Table A.11: Reading Inequalities

Example 1 Translate the algebraic expression into English.

a. $17 \leq 26$
b. $8 \neq 17 - 8$
c. $12 > 27 \div 3$
d. $y + 7 < 19$

Solution:

a. $17 \leq 26 \Longrightarrow$ 17 is less than or equal to 26.

b. $8 \neq 17 - 8 \Longrightarrow$ 8 is not equal to 17 minus 8.

c. $12 > 27 \div 3 \Longrightarrow$ 12 is greater than 27 divided by 3.

d. $y + 7 < 19 \Longrightarrow y$ plus 7 is less than 19.

■

Suppose we need to multiply 2 nine times. We could write this as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. This is tedious and it can be hard to keep track of all those 2's, so we use exponents. We write $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ as 2^9. Recall in exponential notation such as 2^9, the 2 is called the base and the 9 is called the exponent. The exponent tells us how many times we need to multiply the base by itself.

$$\text{base} \rightarrow 2^{9 \leftarrow \text{exponent}}$$

Recall from the previous section of the Appendix, the definition of exponential notation.

Definition

Exponential notation is defined as follows:

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$$

and is read as "a to the nth power." In the expression, a is called the **base** and n is a whole number called the **exponent**. The exponent tells us how many times we multiply the base by itself. ■

We read 2^9 as "two to the ninth power."

We say 2^9 is in *exponential notation* and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ is in **expanded notation**.

N *While we read a^n as "a to the nth power," we usually read:*

- *a^2 as "a squared"*
- *a^3 as "a cubed"*

Table A.12 gives some examples of how to read and write exponential notation.

Exponential Notation	Expanded Notation	In Words
7^2	$7 \cdot 7$	7 to the second power or 7 squared
5^3	$5 \cdot 5 \cdot 5$	5 to the third power or 5 cubed
9^4	$9 \cdot 9 \cdot 9 \cdot 9$	9 to the fourth power
12^5	$12 \cdot 12 \cdot 12 \cdot 12 \cdot 12$	12 to the fifth power

Table A.12: Reading and Writing Exponential Notation

▪ **Example 2** Fill in the missing entries of **Table A.13**.

Exponential Notation	Expanded Notation	In Words
3^4		
	$4 \cdot 4 \cdot 4$	
		n squared

Table A.13

Solution:
The missing table entries are filled in below:

Exponential Notation	Expanded Notation	In Words
3^4	$3 \cdot 3 \cdot 3 \cdot 3$	3 to the fourth power
4^3	$4 \cdot 4 \cdot 4$	4 to the third power or 4 cubed
n^2	$n \cdot n$	n squared

▪

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English; they help to make clear which expressions are to be kept together and separate from other expressions. Three types of grouping symbols are parentheses, ( ), brackets, [ ], and braces, { }. Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24 \div \{13-2[1(6-5)+4]\}$$

EXERCISES

Skills Practice (Answers)

For Exercises 1 - 12, translate the expression from algebra to English.

1. $16 - 9$
2. $x + 11$
3. $14 < 21$
4. $6n = 36$
5. $3 \cdot 9$
6. $(2)(7)$
7. $17 < 35$
8. $y - 1 > 6$
9. $a \neq 1 \cdot 12$
10. $28 \div 4$
11. $(-4)(8)$
12. $36 \geq 19$

For Exercises 13 - 16, write the exponential notation in expanded form and using words.

13. 5^3
14. 10^5
15. 8^3
16. 2^8

Simplifying Expressions

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. "Running very fast" is a phrase, but "The football player was running very fast" is a sentence. A sentence has a subject and a verb, whereas phrases are missing the subject or the verb. In algebra, we have a similar distinction: **expressions** and **equations**.

Definition

An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols. ▪

An expression is like an English phrase. Some examples of expressions are given in **Table A.14**.

Expression	Words	English Phrase
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Table A.14: Algebraic Expressions

Notice the English phrases within **Table A.14** do not form complete sentences, because the phrases do not contain a verb. However, when you translate the symbols of an *equation* with words, you form a complete sentence in English; the equals sign gives the verb.

Definition

An **equation** is two expressions connected by an equals sign. ▪

Some examples of equations are given in **Table A.15**.

Expression	Words	English Sentence
$3+5=8$	3 plus 5 equals 8	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals 14	The difference of n and one is equal to fourteen.
$6 \cdot 7 = 42$	6 times 7 equals 42	The product of six and seven is equal to forty-two.
$x=53$	x is 53	x is equal to fifty-three.
$y+9=2y-3$	y plus 9 equals $2y$ minus 3	The sum of y and nine is equal to the difference of two times y and three.

Table A.15: Algebraic Equations

■ **Example 3** Determine if each statement is an expression or equation.

a. $2(x+3) = 10$
b. $4(y-1)+1$
c. $x \div 25$
d. $y+8 = 40$

Solution:

a. This is an *equation* – two expressions are connected with an equals sign.

b. This is an *expression* – there is no equals sign.

c. This is an *expression* – there no equals sign.

d. This is an *equation* – two expressions are connected with an equals sign.

■

To **simplify an expression** means to perform all the mathematical operations possible. For example, to simplify $4 \cdot 2 + 1$ we would first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop, when simplifying expressions, is to work down the page, writing the equivalent result of each step of the process below the previous step. The example just described would look like this:

$$\begin{aligned} &4 \cdot 2 + 1 \\ &= 8 + 1 \\ &= 9 \end{aligned}$$

Notice the equals sign was added once an operation was performed to show equivalence between steps. This does not change the original expression to an equation. We have introduced most of the symbols and notation used in algebra, but now we need to clarify an order of operations. Otherwise, expressions may have different meanings, and they may result in different values. For example, consider the expression:

$$4 + 3 \cdot 7$$

Some students say the expression simplifies to 49, because they perform the operations in order, from left to right.

$$\begin{aligned} &4 + 3 \cdot 7 \\ &= 7 \cdot 7 \\ &= 49 \end{aligned}$$

Others say the expression simplifies to 25, because they perform the operations in order, from right to left.

$$\begin{aligned} &4 + 3 \cdot 7 \\ &= 4 + 21 \\ &= 25 \end{aligned}$$

The same expression should give the same result no matter who is simplifying. Imagine the confusion in our banking system if every problem had several different answers. As a result, mathematicians early on established some guidelines for simplifying expressions that are called the **Order of Operations**.

Performing the Order of Operations

1. **P**arentheses and Other Grouping Symbols
   - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost grouped expressions first.

2. **E**xponents
   - Simplify all expressions with exponents.

3. **M**ultiplication and **D**ivision
   - Perform all multiplication and division in order from left to right. These operations have equal priority.

4. **A**ddition and **S**ubtraction
   - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Students often ask, "How will I remember the order?" A way to help you remember is to take the first letter of each key word and substitute the following silly phrase: "Please Excuse My Dear Aunt Sally."

Parentheses	**P**lease
Exponents	**E**xcuse
Multiplication **D**ivision	**M**y **D**ear
Addition **S**ubtraction	**A**unt **S**ally

It's good that "**M**y **D**ear" goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always perform multiplication before division or always perform division before multiplication. We perform them in order from left to right.

Similarly, "**A**unt **S**ally" goes together and reminds us that **a**ddition and **s**ubtraction also have equal priority, and we perform them in order from left to right.

Example 4 Simplify the given expression.

a. $4+3\cdot 7$
b. $(4+3)\cdot 7$

Solution:

a. Does $4+3\cdot 7$ contain any **p**arentheses? No.

Are there any **e**xponents? No.

Is there any **m**ultiplication or **d**ivision? Yes, there is multiplication that we need to simplify.

$$4+3\cdot 7$$
$$=4+21$$

Is there any **a**ddition or **s**ubtraction? Yes, there is addition that we need to simplify.

$$4+21$$
$$=25$$

So, $4+3\cdot 7=25$.

b. Does $(4+3)\cdot 7$ contain any **p**arentheses? Yes, so we need to simplify inside the parentheses.

$$\begin{aligned}&(4+3)\cdot 7\\&=(7)7\end{aligned}$$

Are there any **e**xponents? No.

Is there any **m**ultiplication or **d**ivision? Yes, there is multiplication that we need to simplify.

$$\begin{aligned}&(7)7\\&=49\end{aligned}$$

So, $(4+3)\cdot 7 = 49$.

■

■ **Example 5** Simplify $18 \div 6 + 4(5-2)$.

Solution:

Parentheses? Yes, therefore we subtract inside the parentheses first.

$$\begin{aligned}&18 \div 6 + 4(5-2)\\&= 18 \div 6 + 4(3)\end{aligned}$$

Exponents? No.

Multiplication or **D**ivision? Yes, we have both. We divide first, because we multiply and divide left to right. Then we multiply.

$$\begin{aligned}&18 \div 6 + 4(3)\\&= 3 + 4(3)\\&= 3 + 12\end{aligned}$$

Addition or **S**ubtraction? Yes, we have addition.

$$\begin{aligned}&3+12\\&=15\end{aligned}$$

So, $18 \div 6 + 4(5-2) = 15$.

■

When there are multiple grouping symbols, we simplify the innermost grouped expressions first and work outward.

■ **Example 6** Simplify: $5 + 2^3 + 3[6 - 3(4-2)]$.

Solution:

There are multiple grouping symbols. We start with the parentheses that are inside the brackets, and subtract.

$$\begin{aligned}&5 + 2^3 + 3[6 - 3(4-2)]\\&= 5 + 2^3 + 3[6 - 3(2)]\end{aligned}$$

We continue inside the brackets and multiply.

$$5+2^3+3[6-3(2)]$$
$$=5+2^3+3[6-6]$$

We finish simplifying inside the brackets by subtracting.

$$5+2^3+3[6-6]$$
$$=5+2^3+3[0]$$

As there are exponents, we simplify these next.

$$5+2^3+3[0]$$
$$=5+8+3[0]$$

Only addition and multiplication remain. Order of operations tells us to multiply first.

$$=5+8+3[0]$$
$$=5+8+0$$

Now we are left only with addition, which we will perform left to right.

$$5+8+0$$
$$=13+0$$
$$=13$$

So, $5+2^3+3[6-3(4-2)]=13$.

■

Evaluating an Expression

In the last few examples, we simplified expressions using the order of operations. Now we will evaluate some expressions, again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

■ **Example 7** Evaluate $7x-4$, when $x=5$.

Solution:
Substitute 5 for x in the expression $7x-4$.

$$7(5)-4$$

Simplify, using the order of operations.

$$=35-4$$
$$=31$$

■

Example 8 Evaluate the given expression at $x = 4$.

a. x^2
b. 3^x

Solution:

a. Substitute 4 for x in the expression x^2.

$$4^2$$

Using the definition of 4 to the second power gives us

$$\begin{aligned} 4^2 &= 4 \cdot 4 \\ &= 16 \end{aligned}$$

b. Substitute 4 for x in the expression 3^x.

$$3^4$$

Using the definition of 3 to the fourth power gives us

$$\begin{aligned} 3^4 &= 3 \cdot 3 \cdot 3 \cdot 3 \\ &= 81 \end{aligned}$$

■

Example 9 Evaluate $2x^2 + 3x + 8$ when $x = -2$.

Solution:
Substitute -2 for x in the expression $2x^2 + 3x + 8$.

$$2(-2)^2 + 3(-2) + 8$$

Simplify, using the order of operations,

$$\begin{aligned} &= 2(4) + 3(-2) + 8 \\ &= 8 - 6 + 8 \\ &= 10 \end{aligned}$$

■

Algebraic expressions are made up of **terms**.

Definition

A **term** is a constant, a single variable, or the product of a constant and one or more variables. ■

Examples of terms are $7, y, 5x^2, 9a$, and b^5.

Adding or subtracting terms forms an expression. In the expression $2x^2+3x+8$, the three terms are $2x^2, 3x$, and 8.

■ **Example 10** Identify the terms in each expression.

a. $9x^2+7x+12$
b. $8x+3y$

Solution:
The terms of an expression are separated by addition and subtraction signs, so

a. The terms of $9x^2+7x+12$ are $9x^2, 7x$, and 12.

b. The terms of $8x+3y$ are $8x$ and $3y$.

■

Definition

The **coefficient** of a term is the constant the variable is multiplied by. ■

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, as $x = 1 \cdot x$.

■ **Example 11** Identify the coefficient of each term.

a. $14y$
b. $15x^2$
c. a

Solution:

a. The coefficient of $14y$ is 14.

b. The coefficient of $15x^2$ is 15.

c. The coefficient of a is1, as $a = 1 \cdot a$.

■

Some terms share common traits. Look at the following six terms. Which ones seem to have traits in common?

$$5x \qquad 7 \qquad n^2 \qquad 4 \qquad 3x \qquad 17n^2$$

The 7 and 4 are both constant terms.

The $5x$ and the $3x$ are both terms with x.

The n^2 and the $17n^2$ are both terms with n^2.

Definition

Terms that are either constants or two terms that have all the same variables raised to all the same powers are called **like terms**. ▪

From the six terms above,

- 7 and 4 are like terms.
- $5x$ and $3x$ are like terms.
- n^2 and $17n^2$ are like terms.

▪ **Example 12** Identify the like terms in the following list.

$$y^3 \qquad 7x^2 \qquad 14 \qquad 23 \qquad 4y^3 \qquad 9x \qquad 5x^2$$

Solution:

- y^3 and $4y^3$ are like terms, because both contain only the variable y raised to the same power, 3.
- $7x^2$ and $5x^2$ are like terms, because both contain only the variable x raised to the same power, 2.
- 14 and 23 are like terms because both are constants.
- There is no other term like $9x$.

▪

If there are like terms in an expression, you can simplify the expression by combining the like terms. This involves combining the coefficients and keeping the common variable.

Consider the expression $4x+7x+x$:

$$\begin{aligned} 4x+7x+x &= \underbrace{x+x+x+x}_{4x} + \underbrace{x+x+x+x+x+x+x}_{7x} + x \\ &= 12x \end{aligned}$$

It does not matter what x is; if you have 4 of something and add 7 more of the same thing and then add 1 more, the result is 12 of them. For example, 4 oranges plus 7 oranges plus 1 orange is 12 oranges. Thus, we can just add the coefficients, $4+7+1 = 12$, and keep the variables the same to get $4x+7x+x = 12x$.

▪ **Example 13** Simplify $2x^2+3x+7+x^2+4x+5$.

Solution:
Begin by identifying the like terms: $2x^2$ and x^2, $3x$ and $4x$, and the constants, 7 and 5.

$$2x^2+3x+7+x^2+4x+5$$

Then, rearrange the expression so the like terms are grouped together.

$$2x^2 + x^2 + 3x + 4x + 7 + 5$$

Combine like terms by combining the coefficients and keeping the variables the same for each group of like terms.

$$3x^2 + 7x + 12$$

■

■ **Example 14** Simplify $8y^2 + 5y - 2 - 4y^2 + 3y + 9$.

Solution:
Begin by identifying the like terms: $8y^2$ and $-4y^2$, $5y$ and $3y$, and the constants, -2 and 9.

$$8y^2 + 5y - 2 - 4y^2 + 3y + 9$$

Then, rearrange the expression so the like terms are grouped together.

$$8y^2 - 4y^2 + 5y + 3y - 2 + 9$$

Combine like terms by combining the coefficients and keeping the variables the same for each group of like terms.

$$4y^2 + 8y + 7$$

■

N *When combining like terms, combine the coefficients and keep the variables raised to the same power.*

$2x^2 + x^2 = 3x^2$, *and not* $2x^4$ *or* $3x^4$.

EXERCISES

Skills Practice (Answers)

For Exercises 17 - 28, simplify the expression, using order of operations.

17. $3+8\cdot 5$
18. $(3+8)\cdot 5$
19. $3^2-18\div(11-5)$
20. $4\cdot 7+3\cdot 5$
21. $2+8(6+1)$
22. $4\cdot 12/8$
23. $33\div 3+8\cdot 2$
24. 3^2+7^2
25. $8^2+1\cdot 12$
26. $(3+7)^2$
27. $5(2+8\cdot 4)-7^2$
28. $2[1+3(10-2)]$

For Exercises 29 - 36, evaluate the expression at the indicated value(s).

29. $8x-6$ when $x=7$
30. x^3 when $x=5$
31. 4^x when $x=2$
32. x^2+3x-7 when $x=4$
33. $(x-y)^2$ when $x=10, y=7$
34. $(x+y)^2$ when $x=6, y=9$
35. a^2+b^2 when $a=6, b=9$
36. r^2-s^2 when $r=10, s=7$

For Exercises 37 - 40, identify the terms in the expression.

37. $15x^2 + 6x + 2$
38. $11x^2 + 8x + 5$
39. $10y^3 + y + 2$
40. $9y^3 + y + 5$

For Exercises 41 - 43, identify the coefficient of the term.

41. $13m$
42. $-5r^2$
43. x^3

For Exercises 44 - 47, identify the like terms in the given list.

44. $x^3, 8x, 14, 8y, 5, 8x^3$
45. $6z, 3w^2, 1, 6z^2, 4z, w^2$
46. $9a, a^2, 16, 16b^2, 4, 9b^2$
47. $3, 25r^2, 10s, 10r, 4r^2, 3s$

For Exercises 48 - 53, simplify the expression by combining like terms.

48. $15x + 4x$
49. $6y + 4y + y$
50. $7u + 2 + 3u + 1$
51. $7c + 4 + 6c - 3 + 9c - 1$
52. $3x^2 + 12x + 11 + 14x^2 + 8x + 5$
53. $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

Translating an English Phrase to an Algebraic Expression or Equation

We have discussed many operation symbols that are used in algebra, and have translated expressions and equations involving them into English phrases and sentences. Now we will reverse the process and translate English phrases into algebraic expressions. **Table A.16** reminds us of the relationships we have between phrases and expressions.

Operation	**Phrase**	**Expression**
Addition	• a plus b • the sum of a and b • a increased by b • b more than a • the total of a and b • b added to a	$a+b$
Subtraction	• a minus b • the difference of a and b • a decreased by b • b less than a • b subtracted from a	$a-b$
Multiplication	• a times b • the product of a and b • twice a	$a \cdot b$ ab $a(b)$ $(a)(b)$ $2a$
Division	• a divided by b • the quotient of a and b • the ratio of a and b • b divided into a	$a \div b$ a/b $\frac{a}{b}$ $b\overline{)a}$

Table A.16: Algebraic Translations of English Phrases

Each phrase in **Table A.16** tells us to operate on two numbers. Again, it is helpful to look for the words "of" and "and" to find the numbers used.

Example 15 Translate the English phrase into an algebraic expression.

a. The difference of $17x$ and 5
b. The quotient of $10x^2$ and 7

Solution:

a. The key word is *difference*, which tells us the operation is subtraction. Look for the words "of" and "and" to find the numbers to subtract.

$$\text{the difference of } 17x \text{ and } 5$$
$$\text{subtract 5 from } 17x$$
$$17x-5$$

b. The key word is *quotient*, which tells us the operation is division.

$$\text{the quotient of } 10x^2 \text{ and } 7$$
$$\text{divide } 10x^2 \text{ by } 7$$
$$10x^2 \div 7$$

This can also be written as $\dfrac{10x^2}{7}$.

■

Example 16 Translate the English phrase into an algebraic expression.

a. Seventeen more than y
b. 8 less than $9x^2$

Solution:

a. The key words are *more than*, which means "added to."

$$\text{Seventeen more than } y$$
$$\text{Seventeen added to } y$$
$$y+17$$

b. The key words are *less than*, which means "subtracted from."

$$\text{Eight less than } 9x^2$$
$$\text{Eight subtracted from } 9x^2$$
$$9x^2-8$$

■

■ **Example 17** Translate the English phrase into an algebraic expression.

a. Five times the sum of m and n
b. The sum of five times m and n

Solution:
In each phrase, there are two operation words; *times* tells us to multiply and *sum* tells us to add.

a. Because we are multiplying 5 times the sum we need parentheses around the sum of m and n, $(m+n)$. This forces us to determine the sum first. (Remember the order of operations.)

$$\text{five times the sum of } m \text{ and } n$$
$$5(m+n)$$

b. To take a sum, we look for the words "of" and "and" to see what is being added. Here we are taking the sum of five times m and n.

$$\text{the sum of five times } m \text{ and } n$$
$$5m+n$$

N *The order of the words in an English phrase determines the order of operations in the algebraic expression.*

Above, "Five times the sum ..." implies the operation of addition comes before multiplication, while "The sum of five times ..." implies the operation of multiplication comes before addition.

■

■ **Example 18** The length of a rectangle is 6 less than the width. Let l represent the length of the rectangle and w represent the width of the rectangle. Write an algebraic equation relating the length and width of the rectangle.

Solution:
The phrase relating the length and width of the rectangle is

"The length of a rectangle is 6 less than the width."

"length of a rectangle" translates to l.

"is" indicates equality and translates to $=$.

"6 less than the width" indicates subtraction and can be rephrased as 6 subtracted from the width, which translates to $w-6$.

Putting this all together we get

$$l = w-6$$

■

■ **Example 19** June has dimes and quarters in her purse. The number of dimes is three more than four times the number of quarters. Let d represent the number of dimes in her purse and q represent the number of quarters in her purse. Write an algebraic equation relating the number of dimes and quarters in June's purse.

Solution:
The phrase relating the number of dimes and quarters in June's purse is

"The number of dimes is three more than four times the number of quarters."

"number of dimes" translates to d.

"is" indicates equality and translates to $=$.

"three more than four times the number of quarters" indicates both addition (more than) and multiplication (times). Translating "the number of quarters" to q we have "3 more than 4 times q." Remembering order of operations, we next translate the multiplication as "3 more than $4q$," which can be then translated to $4q+3$.

Putting this all together we get

$$d = 4q + 3$$

■

■ **Example 20** You have twice as many dogs as cats. Let d represent the number of dogs you own and c represent the number of cats you own. Write an algebraic equation relating the number of dogs and cats you own.

Solution:
The phrase relating the number of dogs and cats you own is

"You have twice as many dogs as cats."

Unlike the previous examples, the phrase does not contain the word "is," so we will reword our statement.

"The number of dogs you have is twice the number of cats you have."

"number of dogs" translates to d.

"is" indicates equality and translates to $=$.

"twice the number of cats" translates to $2c$.

Putting this all together we get

$$d = 2c$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 54 - 65, translate the phrase into an algebraic expression.

54. the difference of 14 and 9
55. the difference of 19 and 8
56. the product of 9 and 7
57. the product of 8 and 7
58. the quotient of 36 and 9
59. the quotient of 42 and 7
60. the sum of $8x$ and $3x$
61. the sum of $13x$ and $2x$
62. the quotient of y and 3
63. the quotient of y and 8
64. eight times the difference of y and nine
65. seven times the difference of y and one

For Exercises 66 - 68, translate the phrase into an algebraic expression.

66. Eric has rock and classical CDs in his car. The number of rock CDs is 3 more than the number of classical CDs. Let r represent the number of rock CDs in Eric's car and c represent the number of classical CDs in Eric's car. Write an algebraic equation relating the number of rock CDs and the number of classical CDs in Eric's car.
67. Greg has nickels and pennies in his pocket. The number of pennies is seven less than twice the number of nickels. Let p represent the number of pennies in Greg's pocket and n represent the number of nickels in Greg's pocket. Write an algebraic equation relating the number of pennies and the number of nickels in Greg's pocket.
68. Jeannette has \$5 and \$10 bills in her wallet. The number of fives is three more than six times the number of tens. Let f represent the number of five dollar bills in Jeannette's wallet and t represent the number of ten dollar bills in Jeannette's wallet. Write an algebraic equation relating the number of five dollar bills and ten dollar bills in Jeannette's wallet.

Solving Linear Equations with One Variable

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same, so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle.

Definition

A **solution of an equation** is the value(s) of the variable(s) that makes a true statement when substituted into the equation. ■

Determining Whether a Number is a Solution to an Equation with One Variable

1. Substitute the number into the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true (the left side is equal to the right side).
   - If it is true, the number is a solution to the equation.
   - If it is not true, the number is not a solution to the equation.

■ **Example 21** Determine whether $x = \frac{3}{2}$ is a solution of $4x - 2 = 2x + 1$.

Solution:
Given that a solution to an equation is a value of the variable that makes the equation true, we begin by substituting the value in question, $\frac{3}{2}$, into the variable, x.

$$4\left(\frac{3}{2}\right) - 2 \stackrel{?}{=} 2\left(\frac{3}{2}\right) + 1$$

Then we simplify the expressions on both sides of the equation.

$$\begin{aligned} 6 - 2 &\stackrel{?}{=} 3 + 1 \\ 4 &= 4 \checkmark \end{aligned}$$

Seeing as $x = \frac{3}{2}$ results in a true equation (4 is in fact equal to 4), $\frac{3}{2}$ IS a solution to the equation $4x - 2 = 2x + 1$. ■

When *solving* a linear equation involving one variable, the goal is to isolate the variable by itself on one side of the equation. To do so, we 'undo' the operation(s) on the variable. If you add, subtract, multiply or divide both sides of an equation by the same number, you still have equality, known as the **Properties of Equality** shown below.

Properties of Equality

Let x, y, and z represent any real numbers. Then the following properties hold:

- **Addition Property**: If $x = y$, then $x + z = y + z$.
- **Subtraction Property**: If $x = y$, then $x - z = y - z$.
- **Multiplication Property**: If $x = y$, then $x \cdot z = y \cdot z$.
- **Division Property**: If $x = y$, then $\frac{x}{z} = \frac{y}{z}$, where $z \neq 0$.

Example 22 Solve $y + 37 = -13$ for y.

Solution:
To isolate the variable, we will undo the addition of 37 by subtracting 37 from both sides and simplify.

$$\begin{aligned} y + 37 &= -13 \\ y + 37 - 37 &= -13 - 37 \\ y &= -50 \end{aligned}$$

We can check our solution by substituting -50 into the variable y in the original equation.

$$\begin{aligned} y + 37 &= -13 \\ -50 + 37 &\stackrel{?}{=} -13 \\ -13 &= -13 \checkmark \end{aligned}$$

$y = -50$ is the solution to $y + 37 = -13$, as it makes the statement true.

Example 23 Solve $a - 28 = -37$ for a.

Solution:
To isolate the variable a, we will undo the subtraction of 28 by adding 28 to both sides and simplify.

$$\begin{aligned} a - 28 &= -37 \\ a - 28 + 28 &= -37 + 28 \\ a &= -9 \end{aligned}$$

Check:

$$\begin{aligned} a - 28 &= -37 \\ -9 - 28 &\stackrel{?}{=} -37 \\ -37 &= -37 \checkmark \end{aligned}$$

We have confirmed $a = -9$ is a solution of $a - 28 = -37$.

If isolating the variable involves addition or subtraction of a fraction or decimal, the process remains the same.

You may have noticed that all of the equations we have solved so far have been of the form $x+a=b$ or $x-a=b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant or divided by a constant.

Example 24 Solve $5x=-27$ for x.

Solution:
To isolate the variable, x, we will undo the multiplication by 5 using division by 5 on both sides and then simplify.

$$\begin{aligned} 5x &= -27 \\ \frac{5x}{5} &= -\frac{27}{5} \\ x &= -\frac{27}{5} \end{aligned}$$

Check:

$$\begin{aligned} 5x &= -27 \\ 5\left(-\frac{27}{5}\right) &\stackrel{?}{=} -27 \\ -27 &= -27 \checkmark \end{aligned}$$

Therefore, $x=-\frac{27}{5}$ is the solution to $5x=-27$.

■

Example 25 Solve $\frac{y}{-7}=-14$ for y.

Solution:
To isolate the variable, y, we will undo the division by -7 using multiplication by -7 on both sides and then simplify.

$$\begin{aligned} \frac{y}{-7} &= -14 \\ -7\left(\frac{y}{-7}\right) &= -7(-14) \\ \frac{-7y}{-7} &= 98 \\ y &= 98 \end{aligned}$$

Check:

$$\begin{aligned}\frac{y}{-7} &= -14\\ \frac{98}{-7} &\stackrel{?}{=} -14\\ -14 &= -14\,\checkmark\end{aligned}$$

So, $y = 98$ is the solution to $\frac{y}{-7} = -14$.

■

Thus far we have checked each solution in the original equation to verify our work. While it is good practice to always check the solution(s), the authors will now leave it to the reader to verify the solutions in the remaining examples.

■ **Example 26** Solve $-n = 9$ for n.

Solution:
Remember $-n$ is equivalent to $-1n$. So to isolate the variable, n, we will undo the multiplication by -1 using division by -1 on both sides and simplify.

$$\begin{aligned}-1n &= 9\\ \frac{-1n}{-1} &= \frac{9}{-1}\\ n &= -9\end{aligned}$$

$n = -9$ is the solution to $-n = 9$.

■

■ **Example 27** Solve $\frac{3}{4}x = 12$ for x.

Solution:
Recall the product of a number and its reciprocal is 1. So, our strategy here will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$ on both sides of the equation.

$$\begin{aligned}\frac{3}{4}x &= 12\\ \frac{4}{3}\cdot\frac{3}{4}x &= \frac{4}{3}\cdot 12\\ 1x &= \frac{4}{3}\cdot\frac{12}{1}\\ x &= 16\end{aligned}$$

$x = 16$ is the solution to $\frac{3}{4}x = 12$.

N *Notice that we could have divided both sides of the equation $\frac{3}{4}x = 12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal of a fraction easier than dividing by a fraction.*

■

In previous examples, we were able to isolate the variable with just one operation. Most of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before isolating the variable; you should always simplify each expression as much as possible beforehand. Remember that to simplify an expression means to perform all the operations in the expression; simplify the expression on each side of the equation one side at a time. Note that simplification is different from the process used to solve an equation in which we apply an operation to both sides at the same time.

■ **Example 28** Solve $14-23 = 12y-4y-5y$ for y.

Solution:
Begin by simplifying the expressions on each side of the equals sign.

$$\begin{aligned} 14-23 &= 12y-4y-5y \\ -9 &= 3y \end{aligned}$$

Now we can divide both sides by 3 to isolate y.

$$\begin{aligned} \frac{-9}{3} &= \frac{3y}{3} \\ -3 &= y \end{aligned}$$

$y = -3$ is the solution to $14-23 = 12y-4y-5y$.

■

In all the equations we have solved so far, all the variable terms were on only one side of the equation, with the constants on the other side. This does not happen all the time. Now we will learn to solve equations in which the variable terms and/or constant terms are on both sides of the equation.

Our strategy will involve choosing one side of the equation to be the "variable side," and the other side of the equation to be the "constant side." Then we will use the Properties of Equality to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that began with variables and constants on both sides into the form $ax = b$, which is an equation we already know how to solve.

Example 29 Solve $7x+8=-13$ for x.

Solution:
In this equation, the variable is found only on the left side. It makes sense to call the left side the "variable side." Therefore, the right side will be the "constant side." We will write labels above the equation to help us remember what goes where.

$$\overset{\text{variable}}{7x+8} = \overset{\text{constant}}{-13}$$

Because the left side is the variable side, the constant, 8, is on the 'wrong' side of the equals sign. We must undo adding 8 by subtracting 8 from both sides and then simplify.

$$\begin{aligned} \overset{\text{variable}}{7x+8} &= \overset{\text{constant}}{-13} \\ 7x+8-8 &= -13-8 \\ 7x &= -21 \end{aligned}$$

Now that all the variables are on the left and the constant is on the right, the equation looks like one we solved earlier. So, we can divide both sides by 7 and then simplify.

$$\begin{aligned} \frac{7x}{7} &= \frac{-21}{7} \\ x &= -3 \end{aligned}$$

$x=-3$ is the solution to $7x+8=-13$.

Example 30 Solve $9x=8x-6$ for x.

Solution:
Here the variable is on both sides of the equation, but the constants only appear on the right side. So, let's make the right side the "constant side," and the left side the "variable side."

To obtain all of the variables on the left side, we must subtract $8x$ from both sides and simplify.

$$\begin{aligned} \overset{\text{variable}}{9x} &= \overset{\text{constant}}{8x-6} \\ 9x-8x &= 8x-8x-6 \\ x &= -6 \end{aligned}$$

We succeeded in getting the variables on one side and the constants on the other, and have obtained the solution, $x=-6$, to the equation $9x=8x-6$.

$$\begin{aligned} 9x &= 8x-6 \\ 9(-6) &\stackrel{?}{=} 8(-6)-6 \\ -54 &\stackrel{?}{=} -48-6 \\ -54 &= -54 \checkmark \end{aligned}$$

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

Solving Linear Equations with Variables and Constants on Both Sides of the Equation

1. Simplify each side of the equals sign as much as possible, and choose which side will be the "variable side;" the other side will be the "constant side."
2. Collect the variable terms on the "variable side" of the equation, and collect all the constant terms on the other side of the equation.
3. Isolate the variable.
4. Check the solution by substituting it into the original equation.

Example 31 Solve $7x+5=6x+2$ for x.

Solution:
First, choose which side will be the "variable side" and which side will be the "constant side." The variable terms are $7x$ and $6x$. Because 7 is greater than 6, we will make the left side the "variable side" so the resulting coefficient of the variable, when simplified, is positive. The right side will be the "constant side."

$$\overset{\text{variable}}{7x+5} = \overset{\text{constant}}{6x+2}$$

Collect the variable terms on the "variable side" of the equation and simplify.

$$7x-6x+5 = 6x-6x+2$$
$$x+5 = 2$$

Collect all the constant terms on the other side of the equation and simplify.

$$x+5-5 = 2-5$$
$$x = -3$$

The variable is already isolated and we have the solution, $x=-3$, to the equation $7x+5=6x+2$.

■

Example 32 Solve $7a-3=13a+7$ for a.

Solution:
First, choose the variable side by comparing the coefficients of the variables on each side. Because $13>7$, we will make the right side the "variable side" so the coefficient of the variables, when simplified, is positive, and the left side the "constant side."

We subtract $7a$ from both sides to collect the variable terms on the right and simplify by combining like terms.

$$\overset{\text{constant}}{7a-3} = \overset{\text{variable}}{13a+7}$$
$$7a-7a-3 = 13a-7a+7$$
$$-3 = 6a+7$$

Then, we subtract 7 from both sides to collect the constant terms on the left and simplify.

$$-3-7=6a+7-7$$
$$-10=6a$$

To isolate the variable, a, we divide both sides by 6 and simplify.

$$\frac{-10}{6}=\frac{6a}{6}$$
$$-\frac{5}{3}=a$$

$a=-\frac{5}{3}$ is the solution to $7a-3=13a+7$.

■

N *In the last example, we could have made the left side the "variable side", but it would have led to a negative coefficient on the variable term. While we could work with the negative, there is less chance of errors when working with positives. The strategy outlined above helps avoid the negatives.*

■ **Example 33** Solve $\frac{5}{4}x+6=\frac{1}{4}x-2$ for x.

Solution:

Due to the fact that $\frac{5}{4}>\frac{1}{4}$, we make the left side the "variable side" and the right side the "constant side."

$$\overset{\text{variable}}{\frac{5}{4}x+6}=\overset{\text{constant}}{\frac{1}{4}x-2}$$

We subtract $\frac{1}{4}x$ from both sides and simplify.

$$\frac{5}{4}x-\frac{1}{4}x+6=\frac{1}{4}x-\frac{1}{4}x-2$$
$$x+6=-2$$

Next, we subtract 6 from both sides and simplify.

$$x+6-6=-2-6$$
$$x=-8$$

$x=-8$ is the solution to $\frac{5}{4}x+6=\frac{1}{4}x-2$.

■

Example 34 Solve $3(2y-1)-5y = 2(y+1)-2(y+3)$ for y.

Solution:
Before choosing which side is the "variable side" and which side is the "constant side," we simplify the expressions on both sides of the equals sign first. To do so, we distribute through the parentheses and combine like terms.

$$\begin{aligned} 3(2y-1)-5y &= 2(y+1)-2(y+3) \\ 6y-3-5y &= 2y+2-2y-6 \\ 6y-5y-3 &= 2y-2y+2-6 \\ y-3 &= -4 \end{aligned}$$

Now that each side is as simplified as possible, we can isolate the variable, y.

$$\begin{aligned} y-3+3 &= -4+3 \\ y &= -1 \end{aligned}$$

The solution to $3(2y-1)-5y = 2(y+1)-2(y+3)$ is $y = -1$.

■

Example 35 Solve $\frac{2}{3}(6m-3) = 8-m$ for m.

Solution:
Again, we distribute to simplify the expression on the left-hand side.

$$\begin{aligned} \frac{2}{3}(6m-3) &= 8-m \\ 4m-2 &= 8-m \end{aligned}$$

As $4 > -1$, we make the left side the "variable side" and the right side the "constant side." To collect the variable terms, add m to both sides and simplify. Then, to collect the constant terms, add 2 to both sides and simplify.

$$\begin{aligned} 4m+m-2 &= 8-m+m \\ 5m-2 &= 8 \\ 5m+2 &= 8+2 \\ 5m &= 10 \end{aligned}$$

To isolate the variable, m, divide both sides by 5 and simplify.

$$\begin{aligned} \frac{5m}{5} &= \frac{10}{5} \\ m &= 2 \end{aligned}$$

$m = 2$ is a solution to the equation $\frac{2}{3}(6m-3) = 8-m$.

■

Example 36 Solve $0.36(100n+5) = 0.6(30n+15)$ for n.

Solution:
First, we multiply to simplify the expressions on both sides of the equals sign.

$$\begin{aligned} 0.36(100n+5) &= 0.6(30n+15) \\ 36n+1.8 &= 18n+9 \end{aligned}$$

Considering $36 > 18$, we make the left side the "variable side" and the right side the "constant side." We subtract $18n$ from both sides to collect the variable terms and 1.8 from both sides to collect the constant terms.

$$\begin{aligned} 36n-18n+1.8 &= 18n-18n+9 \\ 18n+1.8 &= 9 \\ 18n+1.8-1.8 &= 9-1.8 \\ 18n &= 7.2 \end{aligned}$$

Finally, we isolate the variable, n, by dividing and simplify.

$$\begin{aligned} \frac{18n}{18} &= \frac{1.7}{18} \\ n &= 0.4 \end{aligned}$$

The solution to $0.36(100n+5) = 0.6(30n+15)$ is $n = 0.4$.

■

Now let's consider the equation $2y+6 = 2(y+3)$, and see what happens when we solve for y.

$$\begin{aligned} 2y+6 &= 2(y+3) \\ 2y+6 &= 2y+6 \end{aligned}$$

Notice after we multiplied on the right-hand side, we have the equation $2y+6 = 2y+6$. At this point, no matter what value of y we choose, both sides will have the same value. This means the equation is true for any value of y and we say the solution to the equation is all real numbers.

If we continue to solve for y, we have

$$\begin{aligned} 2y-2y+6 &= 2y-2y+6 \\ 6 &= 6 \checkmark \end{aligned}$$

The result, $6 = 6$, is also a true statement for any value of y (as y is no longer on either side of the equals sign). An equation that is true for any value of the variable, like this, is called an **identity**.

Definition

An equation that is true for any value of the variable is called an **identity**. The solution of any identity is all real numbers. ■

Next, let's consider the equation $5z = 5z - 1$. We are looking for a number that when multiplied by 5 is equal to the difference of that same number multiplied by 5 and 1. Logically though this does not make sense. How can we multiply any number by 5 and then subtract 1 and get the same result as when we only multiply by 5?

If we take a more algebraic approach, we have

$$\begin{aligned} 5z &= 5z - 1 \\ 5z - 5z &= 5z - 5z - 1 \\ 0 &= -1 \text{ ✗} \end{aligned}$$

Solving the equation $5z = 5z - 1$ led to a false statement, $0 = -1$. So the equation $5z = 5z - 1$ will not be true for any value of z, as we suspected. Thus, the equation has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

Definition

An equation that is false for all values of the variable is called a **contradiction.** A contradiction has no solution. ▪

EXERCISES

Skills Practice (Answers)

For Exercises 69 - 76, solve the equation and check the solution.

69. $x + 24 = 35$

70. $y + 45 = -66$

71. $b + \frac{1}{4} = \frac{3}{4}$

72. $a - 45 = 76$

73. $a - 30 = 57$

74. $x - \frac{1}{5} = 4$

75. $x + 0.93 = -4.1$

76. $p - \frac{2}{5} = \frac{2}{3}$

For Exercises 77 - 93, solve the equation and check the solution.

77. $8x = 56$

78. $-5c = 55$

79. $-37p = -541$

80. $0.25z = 3.25$

81. $24x = 0$

82. $\frac{x}{4} = 35$

83. $-20 = \frac{q}{-5}$

84. $\frac{y}{9} = -16$

85. $-y = 6$

86. $\frac{3}{5}r = 75$

87. $24 = -\frac{3}{4}x$

88. $-\frac{5}{18} = -\frac{10}{9}u$

89. $100 - 16 = 4p - 10p - p$

90. $\frac{7}{8}n - \frac{3}{4}n = 9 + 2$

91. $0.25d + 0.10d = 6 - 0.75$

92. $c + 31 - 10 = 46$

93. $9x + 5 - 8x + 14 = 20$

For Exercises 94 - 121, solve the equation.

94. $-10(x+4) - 19 = 85$

95. $3m + 9 = -15$

96. $60 = -21x - 24$

97. $9x + 36 = 15x$

98. $5z = 39 - 8z$

99. $21 + 18f = 19f + 14$

100. $\frac{5}{4}a + 15 = \frac{3}{4}a - 5$

101. $\frac{1}{4}y + 7 = \frac{3}{4}y - 3$

102. $13z + 6.45 = 8z + 23.75$

103. $2.4w - 100 = 0.8w + 28$

104. $15(y - 9) = -60$

105. $-9(2n - 1) = 36$

106. $8(22 + 11r) = 0$

107. $-(w - 12) = 30$

108. $8(9b - 4) - 12 = 100$

109. $32 + 3(z + 4) = 41$

110. $51 + 5(4 - q) = 56$

111. $2(9s - 6) - 62 = 16$

112. $\frac{3}{5}(10x-5)=27$

113. $\frac{1}{4}(20d+12)=d+7$

114. $4(a-12)=3(a+5)+6$

115. $3(4n-1)-2=8n+3$

116. $9(2m-3)-8=4m+7$

117. $5(1.2u-4.8)=-12$

118. $0.25(q-6)=0.1(q+18)$

119. $23z+19=3(5z-9)+8z+46$

120. $45(3y-2)=9(15y-6)$

121. $60(2x-1)=15(8x-4)$

Using Problem-Solving Strategies

Previously, we have discussed translating English phrases into algebraic expressions and equations, as well as methods for solving linear equations. Now, we will combine these processes by translating English sentences into algebraic equations and then solving for our variable. Our first step is to look for the word (or words) that would translate to the equals sign. While we have seen the word "is" translate to "=," **Table A.17** shows us some other words that are commonly used.

Equals =
is
is equal to
is the same as
the result is
gives
was
will be

Table A.17: English Phrases Translating to Equality

Example 37 Translate the following phrase into algebraic notation.

Eleven more than x is equal to 54.

Solution:

$$\underbrace{\text{Eleven more than } x}_{x+11} \quad \underbrace{\text{is equal to}}_{=} \quad \underbrace{54}_{54}$$

■

Example 38 Translate the following phrase into algebraic notation.

The difference of $12t$ and $11t$ gives us -14.

Solution:

$$\underbrace{\text{The difference of } 12t \text{ and } 11t}_{12t-11t} \quad \underbrace{\text{gives us}}_{=} \quad \underbrace{-14}_{-14}$$

■

Example 39 Translate the following phrase into algebraic notation.

The number 143 is the same as the product of -11 and y.

Solution:
equal

$$\underbrace{\text{The number 143}}_{143}\quad\underbrace{\text{is the same as}}_{=}\quad\underbrace{\text{the product of} -11 \text{ and } y}_{-11y}$$

■

Example 40 Translate the following phrase into algebraic notation.

n divided by 8 will be -32.

Solution:

$$\underbrace{n \text{ divided by 8}}_{\frac{n}{8}}\quad\underbrace{\text{will be}}_{=}\quad\underbrace{-32}_{-32}$$

■

Example 41 Translate the following phrase into algebraic notation.

The quotient of y and -4 has a result of 68.

Solution:

$$\underbrace{\text{The quotient of } y \text{ and } -4}_{\frac{y}{-4}}\quad\underbrace{\text{has a result of}}_{=}\quad\underbrace{68}_{68}$$

■

Most of the time, an equation that requires an algebraic solution comes out of a real-life question. To begin with, the real-life question is asked in English (or the language of the person asking) and is not in math symbols. Because of this, it is an important skill to be able to translate an everyday situation into algebraic language.

We will start by restating the problem in just one sentence, assigning a variable, and then translating the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for. For example, you might use q for the number of quarters if you were solving a problem about coins.

Using a Problem-Solving Strategy to Solve Word Problems

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for. (Often found in the question being asked.)
3. **Name** what we are looking for. Choose a variable to represent that quantity, including units.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with the important information.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

▪ **Example 42** The MacIntyre family recycled newspapers for two months. The two months of newspapers weighed a total of 57 pounds. The second month, the newspapers weighed 28 pounds. How much did the newspapers weigh the first month?

Solution:
After reading the problem, we know the problem is about the weight of the newspapers. We are asked to find “How much did the newspapers weigh the first month?” We can represent that quantity by letting

w := weight, in pounds, of the newspapers the first month.

We can restate “The two months of newspapers weighed a total of 57 pounds,” as “the weight of the newspapers the first month plus the weight of the newspapers the second month equals 57 pounds.” We know the weight of the newspapers the second month is 28 pounds. So, the weight from the first month plus 28 equals 57, translated into an equation using the variable w, gives us

$$w + 28 = 57$$

Once we have translated the problem into an equation, we are ready to solve for w.

$$\begin{aligned} w + 28 - 28 &= 57 - 28 \\ w &= 29 \end{aligned}$$

Now we check that the answer makes sense. Does the first month’s weight plus the second month’s weight equal 57 pounds?

$$\begin{aligned} 29 + 28 &\stackrel{?}{=} 57 \\ 57 &= 57 \checkmark \end{aligned}$$

So, the first month the newspapers weighed 29 pounds.

▪

▪ **Example 43** Randell paid \$28,675 for his new car, which was \$875 less than the sticker price. What was the sticker price of his car?

Solution:
We are trying to answer the question "What was the sticker price of the car?"

So, we can let $s :=$ the sticker price of the car, in dollars.

We can restate "paid \$28,675 for his new car, which was \$875 less than the sticker price," as "\$28,675 is \$875 less than the sticker price." So, \$28,675 is \$875 less than s. Translating into an equation, and solving we have

$$\begin{aligned} 28675 &= s - 875 \\ 28675 + 875 &= s - 875 + 875 \\ 29550 &= s \end{aligned}$$

Is \$875 less than \$29,550 equal to \$28,675?

$$\begin{aligned} 29550 - 875 &\stackrel{?}{=} 28675 \\ 28675 &= 28675 \checkmark \end{aligned}$$

So, the sticker price of the car was \$29,550.

■

■ **Example 44** Denae bought six pounds of grapes for \$10.74. What was the cost of one pound of grapes?

Solution:
We are asked to find the cost of one pound of grapes, so let $c :=$ the cost of one pound of grapes, in dollars.

Restating the problem, "The cost of six pounds is \$10.74," which means "six times the cost of one pound is \$10.74."

Translating into an equation, we get

$$6c = 10.74$$

Solving the equation, we have

$$\begin{aligned} \frac{6c}{6} &= \frac{10.74}{6} \\ c &= 1.79 \end{aligned}$$

If one pound costs \$1.79, do six pounds cost \$10.74?

$$\begin{aligned} 6(1.79) &\stackrel{?}{=} 10.74 \\ 10.74 &= 10.74 \checkmark \end{aligned}$$

So, the cost of one pound of grapes is \$1.79, or we can say the grapes cost \$1.79 per pound.

■

■ **Example 45** Andreas bought a used car for \$12,000. Because the car was four years old, its price was three-quarters of the original price. What was the original price of the car?

Solution:
We are asked to find the original price of the car, so let $p :=$ the original price of the car, in dollars.

Restating the problem, "\$12,000 is $\frac{3}{4}$ of the original price."

Translating into an equation, we get

$$12000 = \frac{3}{4}p$$

Solving the equation, we have

$$\begin{aligned} \frac{4}{3}(12000) &= \frac{4}{3} \cdot \frac{3}{4}p \\ 16000 &= p \end{aligned}$$

Is $\frac{3}{4}$ of \$16,000 equal to \$12,000?

$$\begin{aligned} \frac{3}{4} \cdot 16000 &\stackrel{?}{=} 12000 \\ 12000 &= 12000 \checkmark \end{aligned}$$

So, the original price of the car was \$16,000.

■

■ **Example 46** Ginny and some classmates formed a study group. The number of freshmen in the study group was three more than twice the number of sophomores. There were 11 freshman in the study group. How many sophomores were in the study group?

Solution:
We are asked to find how many sophomores were in the study group, so let $s :=$ the number of sophomores in the study group.

We can restate the problem in one sentence, with all the important information, and translate into an equation.

$$\underbrace{\text{The number of freshmen, 11,}}_{11} \quad \underbrace{\text{was}}_{=} \quad \underbrace{\text{three more than twice the number of sophomores}}_{2s+3}$$

Solving the equation for s, we have

$$\begin{aligned} 11 &= 2s+3 \\ 11-3 &= 2s+3-3 \\ 8 &= 2s \\ \frac{8}{2} &= \frac{2s}{2} \\ 4 &= s \end{aligned}$$

Is our answer reasonable? Yes, having four sophomores in a study group seems plausible. The problem says the number of freshmen was 3 more than twice the number of sophomores. If there are four sophomores, does that result in eleven freshmen? Twice four sophomores is eight, and three more than eight is 11.

Thus, there were four sophomores in the study group.

■

■ **Example 47** A married couple together earns \$110,000 a year. The wife earns \$16,000 less than twice what her husband earns per year. What does the husband earn annually?

Solution:
We are asked to determine how much money the husband earns in one year, so let $h :=$ the amount the husband earns annually, in dollars.

We can restate "A married couple together earns \$110,000 a year," as "the amount the husband earns annually plus the amount the wife earns annually is \$110,000." While we have already defined a variable for the amount the husband earns, we do not have a variable for the amount the wife earns. We can let $w :=$ the amount the wife earns annually, in dollars, and translate our information as

$$\underbrace{\text{The amount the husband earns annually}}_{h} \quad \underbrace{\text{plus}}_{+} \quad \underbrace{\text{the amount the wife earns annually}}_{w} \quad \underbrace{\text{is}}_{=} \quad \underbrace{\$110{,}000}_{110000}$$

As there are two variables in our equation, we are unable to proceed as we have done previously. This dilemma occurred because we have only use part of the information given. We will now use the remaining information to write our problem as an equation in one variable.

"The wife earns \$16,000 less than twice what her husband earns per year" means "the amount the wife earns annually is \$16,000 less than twice what her husband earns annually." Translating into an equation, we have

$$\underbrace{\text{The amount the wife earns annually}}_{w} \quad \underbrace{\text{is}}_{=} \quad \underbrace{\$16{,}000 \text{ less than twice what her husband earns annually}}_{2h-16000}$$

Given $w = 2h - 16000$, we can substitute $2h - 16000$ for w in our original equation, $h + w = 110000$, and solve for h.

$$\begin{aligned} h + w &= 110000 \\ h + (2h - 16000) &= 110000 \\ h + 2h - 16000 &= 110000 \\ 3h - 16000 &= 110000 \\ 3h &= 126000 \\ h &= 42000 \end{aligned}$$

Thus, the husband earns \$42,000 annually.

To check if our answer is reasonable, we must determine how much money the wife earns annually.

$$\begin{aligned} w &= 2h - 16000 \\ &= 2(42000) - 16000 \\ &= 84000 - 16000 \\ &= 68000 \end{aligned}$$

So, the wife earns \$68,000. If the wife earns \$68,000 and the husband earns \$42,000, the total is indeed \$110,000.

■

N *Often students struggle with word problems because there are multiple pieces of information that must be translated and combined to form a single equation in one variable. The authors recommend breaking the information down into smaller pieces first, as seen in the previous example, rather than always attempting to use all the information at once.*

EXERCISES

Skills Practice (Answers)

For Exercises 122 - 135, translate the information into an equation and then solve for the variable.

122. Nine more than x is equal to fifty-two.
123. Ten less than m is negative fourteen.
124. The difference of n and $\frac{1}{6}$ is $\frac{1}{2}$.
125. The sum of $-4n$ and $5n$ is -82.
126. 187 is the product of -17 and m.
127. -184 is the product of 23 and p.
128. u divided by 7 is equal to -49.
129. j divided by -20 is equal to -80.
130. The quotient of c and -19 is 38.
131. The quotient of k and 22 is -66.
132. Five-sixths of y is fifteen.
133. Three-tenths of x is fifteen.
134. The sum of two-fifths and f is one-half.
135. The difference of p and one-sixth is two-thirds.

For Exercises 136 - 150, translate into an equation and solve.

136. Eva's daughter is 15 years younger than her son. Eva's son is 22 years old. How old is her daughter?
137. For a family birthday dinner, Celeste bought a turkey that weighed five pounds less than the one she bought for Thanksgiving. The birthday turkey weighed 16 pounds. How much did the Thanksgiving turkey weigh?
138. Ron's paycheck this week was \$17.43 less than his paycheck last week. His paycheck this week was \$103.76. How much was Ron's paycheck last week?
139. Melissa's math book costs \$22.85 less than her art book. Her math book costs \$93.75. How much does her art book cost?
140. Mollie paid \$36.25 for 5 movie tickets. What was the price of each ticket?

141. Serena paid \$12.96 for a pack of 12 pairs of sport socks. What was the price of one pair of sport socks?

142. Nancy used 14 yards of fabric to make flags for one-third of the drill team. How much fabric would Nancy need to make flags for the whole team?

143. John's SUV gets 18 miles per gallon (mpg). This is half as many mpg as his wife's hybrid car. How many miles per gallon does the hybrid car get?

144. One-fourth of the hard candies in a bag are red. If there are 23 red hard candies, how many hard candies are in the bag?

145. Zachary has 25 country music CDs, which is one-fifth of his CD collection. How many CDs does Zachary have?

146. Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong has 162 paperbacks. How many hardback books are there?

147. Philip pays \$1620 in rent every month. This amount is \$120 more than twice what his brother Paul pays every month for rent. How much does Paul pay each month for rent?

148. Travis bought a pair of boots on sale for \$25 off the original price. He paid \$60 for the boots. What was the original price of the boots?

149. Alicia bought a package of eight peaches for \$3.20. Find the cost of each peach.

150. Kenji paid \$2279 for a new living room set, including \$129 tax. What was the price of the living room set?

Graphing Inequalities on the Number Line and Interval Notation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

The solution of an inequality, such as $x > 3$, is any real number that makes the inequality true. We can show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we use an 'open' circle at 3. The graph of $x > 3$ is shown in **Figure A.2.1**.

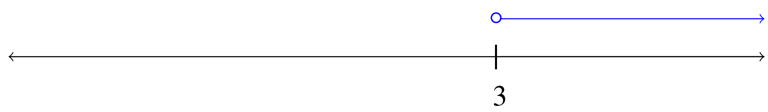


Figure A.2.1: The graphical representation of $x > 3$.

The graph of the inequality $x \geq 3$ is very much like the graph of $x > 3$, but now we need to show that 3 is a solution, too. We indicate this by using a 'closed' dot at $x = 3$, as shown in **Figure A.2.2** .

Figure A.2.2: The graphical representation of $x \geq 3$.

Notice that an 'open' circle shows that an endpoint of the inequality is not included, while a 'closed' dot shows that an endpoint is included.

▪ **Example 48** Graph on the number line:

a. $x \leq 1$
b. $x < 5$
c. $x > -1$

Solution:

a. $x \leq 1$ is read "all numbers less than or equal to 1." We shade in all the numbers on the number line in **Figure A.2.3** to the left of 1, and we use a 'closed' dot at $x = 1$ to show that $x = 1$ is included as a solution.

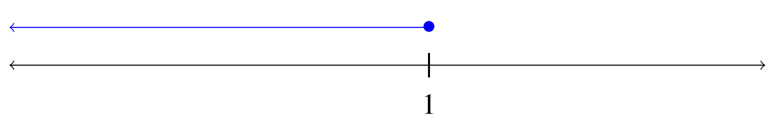


Figure A.2.3: The graphical representation of $x \leq 1$.

b. $x < 5$ is read "all numbers less than 5." We shade in all the numbers on the number line in **Figure A.2.4** to the left of 5, and we use an 'open' circle at $x = 5$ to show that $x = 5$ is not included as a solution.

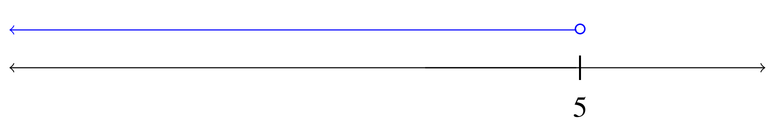


Figure A.2.4: The graphical representation of $x < 5$.

c. $x > -1$ is read "all numbers greater than -1." We shade in all the numbers on the number line in **Figure A.2.5** to the right of -1, and we use an 'open' circle at $x = -1$ to show that $x = -1$ is not included as a solution.

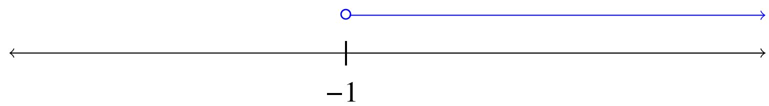


Figure A.2.5: The graphical representation of $x > -1$.

∎

We can also represent inequalities using *interval notation.* In interval notation, we start with the left endpoint of the interval and end with the right endpoint of the interval; all numbers in between the endpoints are included. If the endpoint is included, we use a square bracket, "[" or "]," but if the endpoint is not included, we use a parenthesis, "(" or ")."

As we saw above, the inequality $x > 3$ means all numbers greater than 3. The left endpoint is not included, and there is no upper limit to the solution of this inequality. So, in interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "infinity," and does not have an exact value. We think of infinity as 'bigger' than all real numbers and it indicates that the interval continues to the right forever. **Figure A.2.6** shows both the number line and the interval notation corresponding to $x > 3$.

Figure A.2.6: The graphical representation of $x > 3$, along with its interval notation.

The inequality $x \leq 1$ means all numbers less than or equal to 1. There is no lower limit to the solution of this inequality, but the right endpoint is included. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity," and does not have an exact value. We think of negative infinity as 'smaller' than all real numbers and it indicates that the interval continues to the left forever. **Figure A.2.7** shows both the number line and interval notation corresponding to $x \leq 1$.

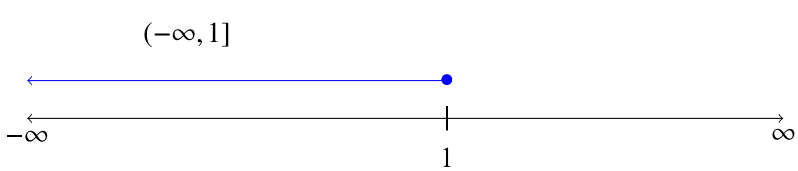
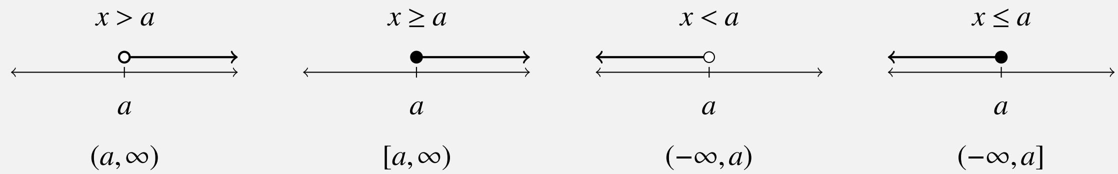


Figure A.2.7: The graphical representation of $x \leq 1$, along with its interval notation.

Equivalent Inequalities, Number Lines, and Interval Notation



▪ **Example 49** Graph the given inequality on the number line and write the equivalent interval notation.

a. $x \geq -3$
b. $x < 2.5$

Solution:

a. Shade the number line in **Figure A.2.8** to the right of -3 and include $x = -3$, using a 'closed' dot.

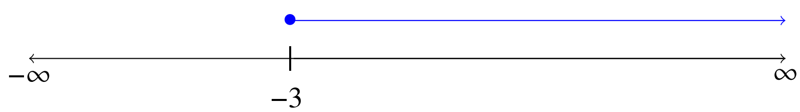


Figure A.2.8: The graphical representation of $x \geq -3$.

As the left endpoint is included, and there is no upper limit to the solution of this inequality, the corresponding interval notation is $[-3, \infty)$.

b. Shade the number line in **Figure A.2.9** the left of 2.5 and show $x = 2.5$ is not included, using an 'open' circle at $x = 2.5$.

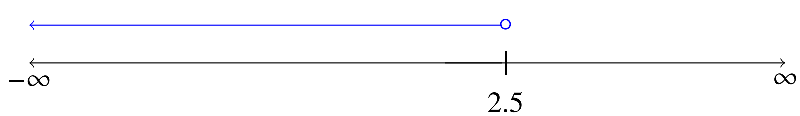


Figure A.2.9: The graphical representation of $x < 2.5$.

As there is no lower limit to the solution of this inequality, and the right endpoint is not included, the corresponding interval notation is $(-\infty, 2.5)$.

▪

Solving Inequalities using the Addition and Subtraction Properties of Inequality

The Addition and Subtraction Properties of Equality state that if two quantities are equal, when we add or subtract the same amount from both quantities, the results will be equal.

Properties of Equality for Addition and Subtraction

For any real numbers a, b, and c,

- **Addition Property**: If $a = b$, then $a + c = b + c$.
- **Subtraction Property**: If $a = b$, then $a - c = b - c$.

Similar properties hold true for inequalities.

Properties of Inequality for Addtion and Subtraction

For any real numbers a, b, and c,

- **Addition Property**: If $a < b$, then $a + c < b + c$ and if $a > b$, then $a + c > b + c$.
- **Subtraction Property**: If $a < b$, then $a - c < b - c$ and if $a > b$, then $a - c > b - c$.

We use these properties to solve inequalities, taking the same steps we used to solve equations. Solving the inequality $x + 5 > 9$, the steps would be:

$$\begin{aligned} x + 5 &> 9 \\ x + 5 - 5 &> 9 - 5 \\ x &> 4 \end{aligned}$$

Any number greater than 4 is a solution to the inequality $x + 5 > 9$. Using interval notation, we write the solution as $(4, \infty)$.

Example 50 Solve the inequality $n - \frac{1}{2} \leq \frac{5}{8}$ for n, and write the solution using the equivalent interval notation.

Solution:
To isolate the variable, n, we add $\frac{1}{2}$ to both sides of the inequality and simplify.

$$\begin{aligned} n - \frac{1}{2} &\leq \frac{5}{8} \\ n - \frac{1}{2} + \frac{1}{2} &\leq \frac{5}{8} + \frac{1}{2} \\ n &\leq \frac{9}{8} \end{aligned}$$

Using interval notation, the solution to the inequality, $n - \frac{1}{2} \leq \frac{5}{8}$, is $\left(-\infty, \frac{9}{8}\right]$.

Solving Inequalities using the Multiplication and Division Properties of Inequality

The Multiplication and Division Properties of Equality state that if two quantities are equal, when we multiply or divide both quantities by the same amount, the results will also be equal (provided we don't divide by 0).

Properties of Equality for Multiplication and Division

For any real numbers a, b, and c,

- **Multiplication Property**: If $a = b$, then $a \cdot c = b \cdot c$.
- **Division Property**: If $a = b$, then $\frac{a}{c} = \frac{b}{c}$, where $c \neq 0$.

Are there similar properties for inequalities? What happens to an inequality when we multiply or divide both sides by a constant?

Consider some numerical examples.

$$\begin{array}{cc} 10 < 15 & 10 < 15 \\ 10(5) \ ? \ 15(5) & \frac{10}{5} \ ? \ \frac{15}{5} \\ 50 \ ? \ 75 & 2 \ ? \ 3 \\ 50 < 75 & 2 < 3 \end{array}$$

The inequality signs stayed the same.

Does the inequality stay the same when we multiply or divide by a negative number?

$$\begin{array}{cc} 10 < 15 & 10 < 15 \\ 10(-5) \ ? \ 15(-5) & \frac{10}{-5} \ ? \ \frac{15}{-5} \\ -50 \ ? \ -75 & -2 \ ? \ -3 \\ -50 > -75 & -2 > -3 \end{array}$$

The inequality signs reversed their direction.

From our investigation we conclude that when we multiply or divide an inequality by a positive number, the inequality sign stays the same, but when we multiply or divide an inequality by a negative number, the inequality sign reverses direction.

Properties of Inequality for Multiplication and Division

For any real numbers a, b, and c, with $\boldsymbol{c > 0}$,

- **Multiplication Property**: If $a < b$, then $a \cdot c < b \cdot c$ and if $a > b$, then $a \cdot c > b \cdot c$.
- **Division Property**: If $a < b$, then $\frac{a}{c} < \frac{b}{c}$ and if $a > b$, then $\frac{a}{c} > \frac{b}{c}$.

For any real numbers a, b, and c, with $\boldsymbol{c < 0}$,

- **Multiplication Property**: If $a < b$, then $a \cdot c > b \cdot c$ and if $a > b$, then $a \cdot c < b \cdot c$.
- **Division Property**: If $a < b$, then $\frac{a}{c} > \frac{b}{c}$ and if $a > b$, then $\frac{a}{c} < \frac{b}{c}$.

For easy reference, in general when we **multiply** or **divide** an inequality by:

- A **positive** real number, the inequality stays the **same**.
- A **negative** real number, the inequality **reverses direction**.

N *While the Properties of Inequality for Multiplication and Division are written in terms of strict inequalities ($<$ and $>$), these properties hold true for non-strict inequalities ($\leq$ and $\geq$), as well.*

Example 51 Solve the inequality $7y < 42$ for y, and write the solution using the equivalent interval notation.

Solution:
To isolate the variable, y, we divide both sides of the inequality by 7. Given that $7 > 0$, the inequality stays the same.

$$7y < 42$$
$$\frac{7y}{7} < \frac{42}{7}$$
$$y < 6$$

Using interval notation, the solution to the inequality, $7y < 42$, is $(-\infty, 6)$.

■

Example 52 Solve the inequality $-10a \geq 50$ for a, and write the solution using the equivalent interval notation.

Solution:
To isolate the variable, a, we divide both sides of the inequality by -10. As $-10 < 0$, the inequality reverses.

$$-10a \geq 50$$
$$\downarrow$$
$$\frac{-10a}{-10} \leq \frac{50}{-10}$$
$$a \leq -5$$

Using interval notation, $(-\infty, -5]$ is the solution to the inequality, $-10a \geq 50$.

■

Sometimes when solving an inequality, the variable ends up on the right-hand side of the inequality symbol. We can rewrite the inequality, to get the variable on the left-hand side of the inequality symbol, by writing the inequality as though you were reading the given inequality right to left. For instance:

$$a \leq x \text{ has the same meaning as } x \geq a.$$

▪ **Example 53** Solve the inequality $-20 < \frac{4}{5}u$ for u, and write the solution using the equivalent interval notation.

Solution:
To isolate the variable, u, we multiply both sides of the inequality by $\frac{5}{4}$. Due to the fact that $\frac{5}{4} > 0$, the inequality stays the same.

$$\begin{aligned} -20 &< \frac{4}{5}u \\ \frac{5}{4}(-20) &< \frac{5}{4}\left(\frac{4}{5}u\right) \\ -25 &< u \end{aligned}$$

To aid us in writing the equivalent interval notation, we will rewrite the inequality so that the variable is on the left-hand side.

$$u > -25$$

Using interval notation, $(-25, \infty)$ is the solution to the inequality, $-20 < \frac{4}{5}u$.

▪

▪ **Example 54** Solve the inequality $\frac{t}{-2} \geq 8$ for t, and write the solution using the equivalent interval notation.

Solution:
To isolate the variable, t, we multiply both sides of the inequality by -2. Seeing as $-2 < 0$, the inequality reverses.

$$\begin{aligned} \frac{t}{-2} &\geq 8 \\ &\downarrow \\ -2\left(\frac{t}{-2}\right) &\leq -2(8) \\ t &\leq -16 \end{aligned}$$

Using interval notation, the solution to the inequality, $\frac{t}{-2} \geq 8$, is $(-\infty, -16]$.

▪

Unlike the examples seen thus far, most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but we generally collect variable terms on the left side and must pay close attention to the direction of the inequality during multiplication or division.

▪ **Example 55** Solve the inequality $4m \le 9m + 17$ for m, and write the solution using the equivalent interval notation.

Solution:
We can subtract $9m$ from both sides to collect the variable terms on the left side.

$$\begin{aligned} 4m &\le 9m + 17 \\ 4m - 9m &\le 9m - 9m + 17 \\ -5m &\le 17 \end{aligned}$$

To isolate the variable, m, we divide both sides of the inequality by -5. Because we are dividing by a negative number, we reverse the inequality.

$$\begin{aligned} \frac{-5m}{-5} &\ge \frac{17}{-5} \\ m &\ge -\frac{17}{5} \end{aligned}$$

Using interval notation, $\left[-\frac{17}{5}, \infty\right)$ is the solution to the inequality, $4m \le 9m + 17$.

▪

▪ **Example 56** Solve the inequality $8p + 3(p - 12) > 7p - 28$ for p, and write the solution using the equivalent interval notation.

Solution:
To begin we simplify each side of the inequality as much as possible.

$$\begin{aligned} 8p + 3(p - 12) &> 7p - 28 \\ 8p + 3p - 36 &> 7p - 28 \\ 11p - 36 &> 7p - 28 \end{aligned}$$

We can subtract $7p$ from both sides to collect the variable terms on the left side. Then we can add 36 to both sides to collect the constant terms on the right.

$$\begin{aligned} 11p - 36 - 7p &> 7p - 28 - 7p \\ 4p - 36 &> -28 \\ 4p - 36 + 36 &> -28 + 36 \\ 4p &> 8 \end{aligned}$$

Next, to isolate the variable, we divide both sides of the inequality by 4. The inequality stays the same as we are dividing by a positive number.

$$\begin{aligned} \frac{4p}{4} &> \frac{8}{4} \\ p &> 2 \end{aligned}$$

Using interval notation, the solution to the inequality, $8p + 3(p - 12) > 7p - 28$, is $(2, \infty)$.

▪

Just like some equations are identities and some are contradictions, some inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

■ **Example 57** Solve the inequality $8x-2(5-x) < 4(x+9)+6x$ for x, and write the solution using the equivalent interval notation.

Solution:
Again, we will start by simplifying each side of the inequality as much as possible.

$$\begin{aligned} 8x-2(5-x) &< 4(x+9)+6x \\ 8x-10+2x &< 4x+36+6x \\ 10x-10 &< 10x+36 \end{aligned}$$

To collect the variable terms on the left side, we subtract $10x$ from both sides of the inequality.

$$\begin{aligned} 10x-10-10x &< 10x+36-10x \\ -10 &< 36 \end{aligned}$$

After simplifying, we notice the variable terms subtract to $0x = 0$ leaving only a constant term on both sides. The resulting inequality is a true statement, meaning the inequality is an identity and true for all real numbers.

Using interval notation, a solution of all real numbers is $(-\infty, \infty)$.

■

■ **Example 58** Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$ for a, and write the solution the equivalent using interval notation.

Solution:
We will first clear the fractions by multiplying both sides of the inequality by the least common denominator of all the fractions, 24. As the LCD is positive, the inequality stays the same. Then we will simplify each side of the inequality.

$$\begin{aligned} \frac{1}{3}a - \frac{1}{8}a &> \frac{5}{24}a + \frac{3}{4} \\ 24\left(\frac{1}{3}a - \frac{1}{8}a\right) &> 24\left(\frac{5}{24}a + \frac{3}{4}\right) \\ 8a-3a &> 5a+18 \\ 5a &> 5a+18 \end{aligned}$$

To collect the variable terms on the left side, we subtract $5a$ from both sides of the inequality.

$$\begin{aligned} 5a-5a &> 5a-5a+18 \\ 0 &> 18 \end{aligned}$$

After simplifying, we notice the variable terms subtract to $0x = 0$ leaving only a constant term on both sides. The resulting inequality is a false statement. Thus, there is no solution to the inequality.

■

EXERCISES

Skills Practice (Answers)

For Exercises 151 - 154, graph the inequality on the number line and write the equivalent interval notation.

151. **a.** $x < -2$
 b. $x \geq -3.5$
 c. $x \leq \frac{2}{3}$

152. **a.** $x > 3$
 b. $x \leq -0.5$
 c. $x \geq \frac{1}{3}$

153. **a.** $x \geq -4$
 b. $x < 2.5$
 c. $x > -\frac{3}{2}$

154. **a.** $x \leq 5$
 b. $x \geq -1.5$
 c. $x < -\frac{7}{3}$

For Exercises 155 - 172, solve the inequality, and write the solution using the equivalent interval notation.

155. $6y < 48$
156. $9s \geq 81$
157. $-8v \leq 96$
158. $-7d > 105$
159. $\frac{a}{-3} \leq 9$
160. $\frac{b}{-10} \geq 30$
161. $7s < -28$
162. $\frac{3}{5}x \leq -45$
163. $4y \geq 9y - 408$
164. $5u \leq 8u - 21$
165. $13q < 7q - 29$
166. $9p > 14p - 18$
167. $12x + 3(x+7) > 10x - 24$
168. $9y + 5(y+3) < 4y - 35$
169. $4k - (k-2) \geq 7k - 26$
170. $8m - 2(14-m) \geq 7(m-4) + 3m$
171. $6n - 12(3-n) \leq 9(n-4) + 9n$
172. $9u + 5(2u-5) \geq 12(u-1) + 7u$

A.3 Introduction to Algebraic Expressions

Describing Polynomials

We have learned that a *term* is a constant or the product of a constant and one or more variables. In this text, we will focus on terms with only one variable. When a term is of the form ax^m, where a is a constant and m is a whole number, it is called a **monomial**. Some examples of monomials are $8, -2x^2, 0.4y^3$, and $11z^7$.

Definition

A **monomial** is a term of the form ax^m, where a is a constant and m is a whole number. ▪

A monomial, or two or more monomials combined by addition or subtraction, is a **polynomial**. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly *one* term, a **binomial** has exactly *two* terms, and a **trinomial** has exactly *three* terms. There are no special names for polynomials with more than three terms.

Definition

A **polynomial** is a monomial, or two or more monomials combined by addition or subtraction.

- A polynomial with exactly one term is called a **monomial.**
- A polynomial with exactly two terms is called a **binomial.**
- A polynomial with exactly three terms is called a **trinomial.**

▪

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$
Monomial	14	$8y^2$	$-5x$
Binomial	$a+7$	$4x^3-5x$	y^2-16
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$

Remember that every monomial, binomial, and trinomial is also a polynomial.

▪ **Example 1** Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

a. $4y^2-8y-6$
b. $2x^5-5x^3-9x^2+3x+4$
c. $13-5m^3$
d. q

Solution:

a. $4y^2-8y-6$ has 3 terms, thus $4y^2-8y-6$ is a trinomial.

b. $2x^5 - 5x^3 - 9x^2 + 3x + 4$ has 5 terms, therefore it has no special name and we call $2x^5 - 5x^3 - 9x^2 + 3x + 4$ a polynomial.

c. $13 - 5m^3$ has 2 terms, so $13 - 5m^3$ is a binomial.

d. q has 1 term and is called a monomial.

In addition to the number of terms, it is common to consider the powers of the variable that appear in the polynomial. The **degree of a polynomial** and the degree of each of its terms are determined by the powers of the variable.

Definition

- The **degree of a term** of a polynomial is the whole number exponent of the variable.
- The **degree of a constant** term of a polynomial is 0.
- The **degree of a polynomial** is the highest degree of all its terms.

Let's break down several polynomials to see the relationship between the degree of each term in the polynomial and the degree of the polynomial itself.

Monomial	$\mathbf{14}$	$\mathbf{8y^2}$	$\mathbf{-13a}$
Degree of Each Term	0	2	1
Degree of Polynomial	0	2	1

Binomial	$\mathbf{a+7}$	$\mathbf{-5b+4b^2}$	$\mathbf{3n^3-9n}$
Degree of Each Term	1 0	1 2	3 1
Degree of Polynomial	1	2	3

Trinomial	$\mathbf{x^2-7x+12}$	$\mathbf{6a+1+9a^2}$	$\mathbf{z^4+3z^2-1}$
Degree of Each Term	2 1 0	1 0 2	4 2 0
Degree of Polynomial	2	2	4

Polynomial	$\mathbf{1+b}$	$\mathbf{4y^2-7y+2}$	$\mathbf{7x^4+x^3-9x+1}$
Degree of Each Term	0 1	2 1 0	4 3 1 0
Degree of Polynomial	1	2	4

A polynomial is in **standard form** when the terms of a polynomial are written in descending order of degrees. The authors recommend getting in the habit of writing the term with the highest degree first.

■ **Example 2** Find the degree of the given polynomial.

a. $10y$
b. $4x^3 - 10x + 5$
c. -15

Solution:

a. The exponent of y is one, because $y = y^1$. So, the degree of $10y$ is 1.

b. The degree of the first term is 3, the second is 1, and the last is 0. The highest degree of all the terms is 3, so the degree of the trinomial, $4x^3 - 10x + 5$, is 3.

c. The degree of a constant term is 0, so the degree of the monomial, -15, is 0.

■

Adding and Subtracting Polynomials

We have learned how to simplify expressions by combining like terms. Remember, like terms must have all the same variables raised to all the same powers. Considering monomials are terms, if the monomials are like terms, adding and subtracting monomials is the same as combining like terms; we just combine them by adding or subtracting the coefficients.

■ **Example 3** Compute $25y^2 + 15y^2$.

Solution:

$$\begin{aligned} 25y^2 + 15y^2 &= (25 + 15)y^2 \\ &= 40y^2 \end{aligned}$$

■

■ **Example 4** Compute $16p - (-7p)$.

Solution:

$$\begin{aligned} 16p - (-7p) &= 16p + 7p \\ &= (16 + 7)p \\ &= 23p \end{aligned}$$

■

Remember, like terms must have all the same variables raised to all the same powers.

■ **Example 5** Simplify $c^2+7d^2-6c^2$.

Solution:

$$\begin{aligned} c^2+7d^2-6c^2 &= c^2-6c^2+7d^2 \\ &= (1-6)c^2+7d^2 \\ &= -5c^2+7d^2 \end{aligned}$$

■

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials.

■ **Example 6** Compute $(5y^2-3y+15)+(3y^2-4y-11)$.

Solution:
We begin by identifying the like terms and reordering them. Then we can combine like terms.

$$\begin{aligned} \left(5y^2-3y+15\right)+\left(3y^2-4y-11\right) &= 5y^2+3y^2+(-3y-4y)+15-11 \\ &= 8y^2-7y+4 \end{aligned}$$

■

■ **Example 7** Compute $(9w^2-7w+5)-(2w^2-4)$.

Solution:
We first distribute, so that we subtract each term of the second binomial. Then we identify like terms, reorder them, and simplify.

$$\begin{aligned} (9w^2-7w+5)-(2w^2-4) &= 9w^2-7w+5-2w^2-(-4) \\ &= 9w^2-7w+5-2w^2+4 \\ &= 9w^2-2w^2-7w+5+4 \\ &= 7w^2-7w+9 \end{aligned}$$

■

Evaluating a Polynomial for a Given Value

We have already learned how to evaluate expressions. Considering polynomials are expressions, we will follow the same procedure to evaluate a polynomial. We will substitute the given value for the variable and then simplify, using order of operations.

Example 8 Evaluate $5x^2 - 8x + 4$ at the given value of x.

a. $x = 4$
b. $x = -2$
c. $x = 0$

Solution:

a. We substitute 4 for x and simplify.

$$\begin{aligned} 5(4)^2 - 8(4) + 4 &= 5 \cdot 16 - 8(4) + 4 \\ &= 80 - 32 + 4 \\ &= 52 \end{aligned}$$

b. We substitute -2 for x and simplify.

$$\begin{aligned} 5(-2)^2 - 8(-2) + 4 &= 5 \cdot 4 - 8(-2) + 4 \\ &= 20 + 16 + 4 \\ &= 40 \end{aligned}$$

c. We substitute 0 for x and simplify.

$$\begin{aligned} 5(0)^2 - 8(0) + 4 &= 5 \cdot 0 - 8(0) + 4 \\ &= 0 - 0 + 4 \\ &= 4 \end{aligned}$$

■

Example 9 The polynomial $-16t^2 + 250$ gives the height, in feet, of a ball t seconds after it is dropped from a 250 foot tall building. Determine the height of the ball after 2 seconds.

Solution:
As the given polynomial represents the height, in feet, of the ball after t seconds, we will substitute 2 for t.

$$\begin{aligned} -16t^2 + 250 &= -16(2)^2 + 250 \\ &= -16 \cdot 4 + 250 \\ &= -64 + 250 \\ &= 186 \end{aligned}$$

After 2 seconds the height of the ball is 186 feet.

■

EXERCISES

Skills Practice (Answers)

For Exercises 1 - 4, determine whether the polynomial is a monomial, binomial, trinomial, or other polynomial.

1. $81b^5 - 24b^3 + 1$
2. $4y + 17$
3. -73
4. $y^3 - 8y^2 + 2y - 16$

For Exercises 5 - 9, determine the degree of the polynomial.

5. $5x + 2$
6. -24
7. $4m^3 + m^4 + 1 + 6m^2 + 4m$
8. $9y^3 + 2y - 10y^2 - 6$
9. $18 + 9a + a^2$

For Exercises 10 - 15, add or subtract the monomials.

10. $7x^2 + 5x^2$
11. $28x - (-12x)$
12. $-12w + 18w$
13. $-y - 5y$
14. $5u^2 + 4v^2 - 6u^2$
15. $14x - 3y - 13x$

For Exercises 16 - 19, add or subtract the polynomials.

16. $(4m^2 - 6m - 3) - (2m^2 + m - 7)$
17. $(12s^2 - 15s) - (s - 9)$
18. $(y^2 + 9y + 4) + (-2y^2 - 5y - 1)$
19. $(p^2 - 6p - 18) + (2p^2 + 11)$

For Exercises 20 - 21, evaluate the polynomial for the given values.

20. Evaluate $8y^2 - 3y + 2$ when:
    a. $y = 5$
    b. $y = -2$
    c. $y = 0$
21. Evaluate $16 - 36x^2$ when:
    a. $x = -1$
    b. $x = 0$
    c. $x = 2$

22. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a price of p dollars each is given by the polynomial $-4p^2 + 420p$. Determine the revenue received when the price of a speaker is 60 dollars.

23. The cost to rent a rug cleaner for d days is given by the polynomial $5.50d + 25$. Determine the cost to rent the cleaner for 6 days.

Simplifying Expressions with Exponents

Recall that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 is equivalent to $2 \cdot 2 \cdot 2 \cdot 2$.

Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

■ **Example 10** Simplify:

a. 4^3

b. 7^1

c. $\left(\frac{5}{6}\right)^2$

d. $(0.63)^2$

Solution:

a. We multiply three factors of 4.

$$\begin{aligned} 4^3 &= 4 \cdot 4 \cdot 4 \\ &= 64 \end{aligned}$$

b. $7^1 = 7$

c. We multiply two factors of $\frac{5}{6}$.

$$\begin{aligned} \left(\frac{5}{6}\right)^2 &= \left(\frac{5}{6}\right)\left(\frac{5}{6}\right) \\ &= \frac{25}{36} \end{aligned}$$

d. We multiply two factors of 0.63.

$$\begin{aligned} (0.63)^2 &= (0.63)(0.63) \\ &= 0.3969 \end{aligned}$$

■

■ **Example 11** Simplify each expression.

a. $(-5)^4$

b. -5^4

Solution:

a. We multiply four factors of -5.

$$\begin{aligned} (-5)^4 &= (-5)(-5)(-5)(-5) \\ &= 625 \end{aligned}$$

b. We begin by multiplying four factors of 5, and then we find the opposite of the result.

$$\begin{aligned} -5^4 &= -(5 \cdot 5 \cdot 5 \cdot 5) \\ &= -625 \end{aligned}$$

N *Notice the similarities and differences in part **a** and part **b**. Why are the answers different? As we follow the order of operations in part **a**, the parentheses tell us to raise the* (-5) *to the* 4^{th} *power. In part **b** we raise just the 5 to the* 4^{th} *power and then take the opposite.*

▪

We have seen that when we combine like terms by adding and subtracting, we need to have the same base with the same exponent. However, when we multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

We will derive the properties of exponents by looking for patterns in several examples.

Let's consider simplifying the expression $x^2 \cdot x^3$.

To simplify, we can write the exponents as repeated multiplication, as we did in the previous two examples.

$$\underbrace{\underbrace{x \cdot x}_{\text{2 factors of } x} \cdot \underbrace{x \cdot x \cdot x}_{\text{3 factors of } x}}_{\text{5 factors of } x}$$

So, we have

$$x^5$$

Notice in the result that the base, x, stayed the same and the exponent, 5, is the sum of the exponents, 2 and 3.

$$x^2 \cdot x^3 = x^5 = x^{2+3}$$

This leads to the Product Property for Exponents.

Product Property for Exponents

If a is any real number, and m and n are positive integers, then

$$a^m \cdot a^n = a^{m+n}$$

We can consider an example with numbers to show this property holds.

$$\begin{aligned} 2^2 \cdot 2^3 &\stackrel{?}{=} 2^{2+3} \\ 4 \cdot 8 &\stackrel{?}{=} 2^5 \\ 32 &= 32 \checkmark \end{aligned}$$

Example 12 Simplify $y^5 \cdot y^6$.

Solution:
As the bases are the same, we can use the Product Property for Exponents, $a^m \cdot a^n = a^{m+n}$.

$$\begin{aligned} y^5 \cdot y^6 &= y^{5+6} \\ &= y^{11} \end{aligned}$$

Example 13 Simplify each expression.

a. $a^7 \cdot a$
b. $x^{27} \cdot x^{13}$

Solution:

a. First, we rewrite a as a^1, and then simplify using the Product Property for Exponents.

$$\begin{aligned} a^7 \cdot a &= a^7 \cdot a^1 \\ &= a^{7+1} \\ &= a^8 \end{aligned}$$

b. As the bases are the same, we can use the Product Property for Exponents.

$$\begin{aligned} x^{27} \cdot x^{13} &= x^{27+13} \\ &= x^{40} \end{aligned}$$

Example 14 Simplify $d^4 \cdot d^5 \cdot d^2$.

Solution:
As all the bases are the same, we can use the Product Property for Exponents.

$$\begin{aligned} d^4 \cdot d^5 \cdot d^2 &= \left(d^4 d^5\right) d^2 \\ &= d^{4+5} \cdot d^2 \\ &= d^9 d^2 \\ &= d^{9+2} \\ &= d^{11} \end{aligned}$$

Rather than working left to right through the multiplication, we could have, from the beginning, used the Product Property for Exponents to get $d^4 \cdot d^5 \cdot d^2 = d^{4+5+2} = d^{11}$. As long as all the bases are the same, the Product Property for Exponents allows you to keep the same base and add all the exponents.

Now let's look at an exponential expression that contains a power raised to a power. Let's start with $(x^2)^3$. Again, we can replace the exponents with repeated multiplication.

$$\underbrace{\begin{array}{ccccc} (x^2) & \cdot & (x^2) & \cdot & (x^2) \\ \underbrace{x \cdot x}_{\text{2 factors of } x} & \cdot & \underbrace{x \cdot x}_{\text{2 factors of } x} & \cdot & \underbrace{x \cdot x}_{\text{2 factors of } x} \end{array}}_{\text{6 factors of } x}$$

So, we have

$$x^6$$

Notice in the result that the base, x, stayed the same and the exponent, 6, is the product of the exponents, 2 and 3.

$$(x^2)^3 = x^6 = x^{2 \cdot 3}$$

This leads to the Power Property for Exponents.

Power Property for Exponents

If a is any real number, and m and n are positive integers, then

$$(a^m)^n = a^{m \cdot n}$$

We can consider an example with numbers to show this property holds.

$$\begin{aligned} (3^2)^3 &\stackrel{?}{=} 3^{2 \cdot 3} \\ (9)^3 &\stackrel{?}{=} 3^6 \\ 729 &= 729 \checkmark \end{aligned}$$

▪ **Example 15** Simplify each expression.

a. $(y^5)^9$
b. $(4^4)^7$

Solution:

a. We use the Power Property for Exponents, $(a^m)^n = a^{m \cdot n}$.

$$\begin{aligned} (y^5)^9 &= y^{5 \cdot 9} \\ &= y^{45} \end{aligned}$$

b. Again, we use the Power Property for Exponents.

$$\begin{aligned} (4^4)^7 &= 4^{4 \cdot 7} \\ &= 4^{28} \end{aligned}$$

▪

We will now look at an expression containing a product that is raised to a power. Let's start by simplifying $(2x)^3$. Again, we replace the exponent with repeated multiplication.

$$(2x)(2x)(2x)$$

Grouping the like factors together, we have

$$2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x = 2^3 \cdot x^3$$

Notice that each factor was raised to the power of 3. Hence, the exponent applies to each of the factors.

$$(2x)^3 = 2^3 \cdot x^3$$

This leads to the Product to a Power Property for Exponents.

Product to a Power Property for Exponents

If a and b are any real numbers and m is a positive integer, then

$$(ab)^m = a^m b^m$$

We can consider an example with numbers to show this property holds.

$$\begin{aligned} (2 \cdot 3)^2 &\stackrel{?}{=} 2^2 \cdot 3^2 \\ 6^2 &\stackrel{?}{=} 4 \cdot 9 \\ 36 &= 36 \checkmark \end{aligned}$$

■ **Example 16** Simplify each expression.

a. $(-9d)^2$
b. $(3mn)^3$

Solution:

a. We use the Product to a Power Property for Exponents, $(ab)^m = a^m b^m$.

$$\begin{aligned} (-9d)^2 &= (-9)^2 d^2 \\ &= 81d^2 \end{aligned}$$

b. Again, we use the Product to a Power Property for Exponents.

$$\begin{aligned} (3mn)^3 &= ((3m) \cdot n)^3 \\ &= (3m)^3 \cdot n^3 \\ &= (3)^3 m^3 n^3 \\ &= 27m^3 n^3 \end{aligned}$$

No matter how many factors are inside the parentheses, the exponent is applied to each factor.

■

While these three properties for multiplying expressions with exponents were stated with positive exponents, all exponent properties hold true for any *real* number exponents, m and n, as long as the base and the exponent are not both 0.

Sometimes when simplifying exponential expressions we must apply multiple properties.

Example 17 Simplify each expression.

a. $(y^3)^6(y^5)^4$
b. $(-6x^4y^5)^2$

Solution:

a. We are given the product of two exponential expressions. We can simplify each factor, using the Power Property for Exponents.

$$\begin{aligned}(y^3)^6(y^5)^4 &= \left(y^{3\cdot 6}\right)\left(y^{5\cdot 4}\right)\\ &= y^{18}\cdot y^{20}\end{aligned}$$

Now we can use the Product Property for Exponents to complete our simplification.

$$\begin{aligned}&= y^{18+20}\\ &= y^{38}\end{aligned}$$

b. As nothing inside of the parentheses can be combined, we apply the Product to a Power Property for Exponents, to distribute the power to each factor.

$$(-6x^4y^5)^2 = (-6)^2\left(x^4\right)^2\left(y^5\right)^2$$

The resulting expression is similar to the given expression in part **a**, so we will apply the Power Property for Exponents to simplify.

$$\begin{aligned}&= (-6)^2\left(x^8\right)\left(y^{10}\right)\\ &= 36x^8y^{10}\end{aligned}$$

■

Example 18 Simplify $(5m)^2(3m^3)$.

Solution:
Based on order of operations we know to simplify exponents before multiplying factors. Thus, we begin by using the Product to a Power Property for Exponents on the first factor.

$$\begin{aligned}(5m)^2(3m^3) &= 5^2m^2\cdot 3m^3\\ &= 25m^2\cdot 3m^3\end{aligned}$$

By the Commutative Property of Multiplication, we can reorder the factors and multiply like factors. We use the Product Property for Exponents to simplify the variable factors.

$$\begin{aligned} &= 25 \cdot 3 \cdot m^2 \cdot m^3 \\ &= 75m^5 \end{aligned}$$

■

In the next example, we will multiply two monomials. As seen in the previous examples, we can use the properties of exponents to multiply monomials.

■ **Example 19** Compute $\left(-\frac{5}{6}x^3\right)(12x)$.

Solution:
By the Commutative Property of Multiplication, we can reorder the factors and multiply like factors. We use the Product Property for Exponents to simplify the variable factors.

$$\begin{aligned} \left(-\frac{5}{6}x^3\right)(12x) &= -\frac{5}{6} \cdot 12 \cdot x^3 \cdot x \\ &= -10x^4 \end{aligned}$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 24 - 30, simplify the expression involving an exponent.

24. 3^5
25. $(0.2)^4$
26. $\left(\frac{2}{9}\right)^2$
27. 14^1
28. $(-6)^4$
29. -6^4
30. $-\left(\frac{1}{4}\right)^4$

For Exercises 31 - 34, simplify the expression, using the Product Property for Exponents.

31. $x^4 \cdot x^2$
32. $m^x \cdot m^3$
33. $a^4 \cdot a^3 \cdot a^9$
34. $w \cdot w^2 \cdot w^3$

For Exercises 35 - 36, simplify the expression, using the Power Property for Exponents.

35. **a.** $\left(m^4\right)^2$
    b. $\left(10^3\right)^6$
36. **a.** $\left(y^3\right)^x$
    b. $(5^x)^y$

For Exercises 37 - 38, simplify the expression using the Product to a Power Property for Exponents.

37. **a.** $(6a)^2$
    b. $(3xy)^2$
38. **a.** $(-4m)^3$
    b. $(5ab)^3$

For Exercises 39 - 40, simplify the expression.

39. **a.** $(5a)^2\,(2a)^3$
    b. $\left(\frac{1}{2}y^2\right)^3\left(\frac{2}{3}y\right)^2$
40. **a.** $(3x)^2\,(5x)$
    b. $\left(5t^2\right)^3(3t)^2$

For Exercises 41 - 44, multiply the monomials.

41. $\left(6y^7\right)\left(-3y^4\right)$

42. $\left(\dfrac{1}{5}f^8\right)\left(20f^3\right)$

43. $\left(-10x^5\right)\left(-3x^3\right)$

44. $\left(\dfrac{1}{4}d^5\right)\left(36d^2\right)$

Multiplying Polynomials

We have used the Distributive Property to simplify expressions like $2(x-3)$. We multiply both terms in parentheses, x and 3, by 2, to get $2x-6$. In our new terminology, we can say we were multiplying a binomial, $x-3$, by a monomial, 2.

Let's review this process with a few examples before moving on to multiplying larger polynomials.

■ **Example 20** Compute $4(x+3)$.

Solution:
To illustrate the Distributive Property of Multiplication across Addition, we will draw arrows, as shown below.

$$4(x+3)$$

$$\begin{aligned} 4x+3 &= 4\cdot x+4\cdot 3 \\ &= 4x+12 \end{aligned}$$

■

■ **Example 21** Compute $y(y-2)$.

Solution:
To illustrate the Distributive Property of Multiplication across Subtraction, we will draw arrows, as shown below.

$$y(y-2)$$

$$\begin{aligned} y(y-2) &= y\cdot y-y\cdot 2 \\ &= y^2-2y \end{aligned}$$

■

■ **Example 22** Compute $7x(2x+y)$.

Solution:
To illustrate the Distributive Property of Multiplication across Addition, we will draw arrows, as shown below.

$$7x(2x+y)$$

$$\begin{aligned} 7x(2x+y) &= 7x\cdot 2x+7x\cdot y \\ &= 14x^2+7xy \end{aligned}$$

■

Example 23 Compute $-2y(4y^2+3y-5)$.

Solution:
To illustrate the Distributive Property, we will draw arrows, as shown below.

$$-2y(4y^2+3y-5)$$

$$\begin{aligned} -2y(4y^2+3y-5) &= (-2y)\cdot 4y^2 \cdot +(-2y)\cdot 3y - (-2y)\cdot 5 \\ &= -8y^3 - 6y^2 + 10y \end{aligned}$$

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial by a *binomial*. We will start by using the Distributive Property. Let's consider what would happen if we replaced the monomial, y, in $y(y-2)$ with the binomial, $y+7$.

We will compare the two multiplications side-by-side below.

$$\begin{aligned} y(y-2) &= y\cdot y - y\cdot 2 \\ &= y^2 - 2y \end{aligned}$$

$$\begin{aligned} (y+7)(y-2) &= (y+7)\cdot y - (y+7)\cdot 2 \\ &= y(y+7) - 2(y+7) \\ &= y^2 + 7y - 2y - 14 \\ &= y^2 + 5y - 14 \end{aligned}$$

Notice that before combining like terms in the product of the two binomials, $(y+7)(y-2)$, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial producing four multiplications.

Example 24 Compute $(y+8)(y+5)$.

Solution:
We will begin by distributing the first binomial to the two terms of the second binomial and then simply.

$$\begin{aligned} (y+8)(y+5) &= (y+8)y + (y+8)(5) \\ &= y(y+8) + 5(y+8) \\ &= y^2 + 8y + 5y + 40 \\ &= y^2 + 13y + 40 \end{aligned}$$

■ **Example 25** Compute $(4y+3)(2y-5)$.

Solution:
Instead of distributing the first binomial to the two terms of the second binomial, we will distribute the second binomial to the two terms of the first binomial. As multiplication is commutative, we will have the same result.

$$\begin{aligned}(4y+3)(2y-5) &= 4y(2y-5)+3(2y-5)\\ &= 8y^2-20y+6y-15\\ &= 8y^2-14y-15\end{aligned}$$

■

Remember that when you multiply a binomial by a binomial you get four terms before simplifying. Sometimes you can combine like terms to get a trinomial, but sometimes, there are no like terms to combine. Let's consider $(x-2)(x-y)$.

$$\begin{aligned}(x-2)(x-y) &= x(x-y)-2(x-y)\\ &= x^2-xy-2x+2y\end{aligned}$$

Notice in the result that x^2 is the product of x and x, the *first* terms in $(x-2)$ and $(x-y)$.

$$(x{-}2)(x{-}y)$$

First

The next term in the result, $-xy$, is the product of x and $-y$, the two *outer* terms of the two binomials.

$$(x{-}2)(x{-}y)$$

Outer

The third term in the result, $-2x$, is the product of -2 and x, the two *inner* terms of the two binomials.

$$(x{-}2)(x{-}y)$$

Inner

And the last term in the result, $+2y$, came from multiplying the *last* two terms, -2 and $-y$, of the two binomials.

$$(x{-}2)(x{-}y)$$

Last

We abbreviate this method for multiplying two binomials, "**F**irst, **O**uter, **I**nner, **L**ast," as FOIL.

$$(x-2)(x-y)$$
$$\underset{F}{x^2}-\underset{O}{xy}-\underset{I}{2x}+\underset{L}{2y}$$

Let's compare the methods of distributing and FOIL using the product of $(x+3)$ and $(x+7)$.

Distributive Property	**FOIL**
$(x+3)(x+7)$	$(x+3)(x+7)$
$=x(x+7)+3(x+7)$	
$=x^2+7x+3x+21$	$=\underset{F}{x^2}+\underset{O}{7x}+\underset{I}{3x}+\underset{L}{21}$
$=x^2+10x+21$	$=x^2+10x+21$

Notice how the terms in the third line of the Distributive Property method fit the FOIL method. Now we will look at an example where we use the FOIL method to multiply two binomials.

Example 26 Multiply $(x+5)(x+9)$, using the FOIL method.

Solution:

We multiply the *First* terms.	$(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{\underline{\quad}}+\underset{I}{\underline{\quad}}+\underset{L}{\underline{\quad}}$
We multiply the *Outer* terms.	$(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{9x}+\underset{I}{\underline{\quad}}+\underset{L}{\underline{\quad}}$
We multiply the *Inner* terms.	$(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{9x}+\underset{I}{5x}+\underset{L}{\underline{\quad}}$
We multiply the *Last* terms.	$(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{9x}+\underset{I}{5x}+\underset{L}{45}$

As a final step, we combine like terms to get the resulting product, $x^2+14x+45$.

When you multiply binomials using the FOIL method, drawing the arrows will help your brain focus on the pattern and make it easier to apply.

▪ **Example 27** Multiply $(y-7)(y+4)$, using the FOIL method.

Solution:

We multiply the *First* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{\underline{\quad}} + \underset{I}{\underline{\quad}} + \underset{L}{\underline{\quad}}$
We multiply the *Outer* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} + \underset{I}{\underline{\quad}} + \underset{L}{\underline{\quad}}$
We multiply the *Inner* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} - \underset{I}{7y} + \underset{L}{\underline{\quad}}$
We multiply the *Last* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} - \underset{I}{7y} - \underset{L}{28}$

As a final step, we combine like terms to get the resulting product, $y^2 - 3y - 28$.

▪

▪ **Example 28** Compute $(4x+3)(2x-5)$.

Solution:

$$(4x+3)(2x-5) = (4x+3)(2x-5)$$

Multiply the *First* terms, $4x \cdot 2x$.	$\underset{F}{8x^2} + \underset{O}{\underline{\quad}} + \underset{I}{\underline{\quad}} + \underset{L}{\underline{\quad}}$
Multiply the *Outer* terms, $4x \cdot (-5)$.	$\underset{F}{8x^2} - \underset{O}{20x} + \underset{I}{\underline{\quad}} + \underset{L}{\underline{\quad}}$
Multiply the *Inner* terms, $3 \cdot 2x$.	$\underset{F}{8x^2} - \underset{O}{20x} + \underset{I}{6x} + \underset{L}{\underline{\quad}}$
Multiply the *Last* terms, $3 \cdot (-5)$.	$\underset{F}{8x^2} - \underset{O}{20x} + \underset{I}{6x} - \underset{L}{15}$

As a final step, we combine like terms to get the resulting product, $8x^2 - 14x - 15$.

▪

The final products in the last few examples were trinomials, because we could combine the two middle terms. However, as stated previously, we are not always able to combine the two middle terms.

■ **Example 29** Compute $(n^2+4)(n-1)$.

Solution:
We will multiply using the FOIL method.

$$(n^2+4)(n-1) = (n^2+4)(n-1)$$

$$= n^3 - n^2 + 4n - 4$$

As there are no like terms to combine, the resulting product is $n^3 - n^2 + 4n - 4$.

■

We have now used two methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. Remember, FOIL only works when multiplying two binomials. When multiplying polynomials by polynomials, (other than monomials or binomials) it is necessary to use the Distributive Property.

■ **Example 30** Compute $(b+3)(2b^2-5b+8)$.

Solution:
As we have a binomial times a trinomial, we must use the Distributive Property to multiply. We begin by distributing each term of the binomial to the trinomial. Then we combine like terms to simplify.

$$\begin{aligned}(b+3)(2b^2-5b+8) &= b(2b^2-5b+8)+3(2b^2-5b+8)\\ &= 2b^3-5b^2+8b+6b^2-15b+24\\ &= 2b^3+b^2-7b+24\end{aligned}$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 45 - 56, compute the product.

45. $-3(a+7)$
46. $2(x-7)$
47. $q(q+5)$
48. $-x(x-10)$
49. $12x(x-5)$
50. $-4p(2p+7)$
51. $5q^3(q^2-2q+6)$
52. $-4z^2(3z^2+12z-1)$
53. $(2m-9)m$
54. $(8j-1)j$
55. $s(s^2-6s)$
56. $-5m(m^2+3m-18)$

For Exercises 57 - 64, multiply the binomials and simplify.

57. $(w+5)(w+7)$
58. $(q+4)(q-8)$
59. $(y-6)(y-2)$
60. $(w-4)(w+7)$
61. $(7m+1)(m+3)$
62. $(2t-9)(10t+1)$
63. $(y^2-7)(y^2-4)$
64. $(x^2+8)(x^2-5)$

For Exercises 65 - 66, multiply using the Distributive Property.

65. $(x+5)(x^2+4x+3)$
66. $(p-4)(p^2-6p+9)$

Observing Special Products

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and using the methods previously discussed, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$. This means to multiply $(x+9)$ by itself.

$$(x+9)^2 = (x+9)(x+9)$$

Then, using FOIL, we have

$$\begin{aligned} &= x^2 + 9x + 9x + 81 \\ &= x^2 + 18x + 81 \end{aligned}$$

Now let's consider $(y-7)^2$.

$$\begin{aligned} (y-7)^2 &= (y-7)(y-7) \\ &= y^2 - 7y - 7y + 49 \\ &= y^2 - 14y + 49 \end{aligned}$$

Last we square $2x+3$.

$$\begin{aligned} (2x+3)^2 &= (2x+3)(2x+3) \\ &= 4x^2 + 6x + 6x + 9 \\ &= 4x^2 + 12x + 9 \end{aligned}$$

Let's look at the results and try to find a pattern.

In each situation, we squared a binomial and the result was a trinomial.

$$(a \pm b)^2 = \underline{\qquad} \pm \underline{\qquad} + \underline{\qquad}$$

Now look at the ***first term*** in each result.

$$\begin{array}{ccc} (x+9)^2 & (y-7)^2 & (2x+3)^2 \\ (x+9)(x+9) & (y-7)(y-7) & (2x+3)(2x+3) \\ x^2 + 9x + 9x + 81 & y^2 - 7y - 7y + 49 & 4x^2 + 6x + 6x + 9 \\ x^2 + 18x + 81 & y^2 - 14y + 49 & 4x^2 + 12x + 9 \end{array}$$

The first term is the product of the first terms of each binomial. Seeing as the binomials are identical, it is just the square of the first term in the binomial.

$$(a \pm b)^2 = \boldsymbol{a^2} \pm \underline{\qquad} + \underline{\qquad}$$

Now look at the last term in each result.

$$\begin{array}{ccc} (x+9)^2 & (y-7)^2 & (2x+3)^2 \\ (x+9)(x+9) & (y-7)(y-7) & (2x+3)(2x+3) \\ x^2+9x+9x+81 & x^2-7y-7y+49 & 4x^2+6x+6x+9 \\ x^2+18x+81 & y^2-14y+49 & 4x^2+12x+9 \end{array}$$

The last term is the product of the last terms of each binomial, which is the square of the last term in the binomial.

$$(a \pm b)^2 = a^2 \pm \underline{\qquad} + \boldsymbol{b^2}$$

Finally, look at the middle term in each result.

$$\begin{array}{ccc} (x+9)^2 & (y-7)^2 & (2x+3)^2 \\ (x+9)(x+9) & (y-7)(y-7) & (2x+3)(2x+3) \\ x^2+9x+9x+81 & y^2-7y-7y+49 & 4x^2+6x+6x+9 \\ x^2+18x+81 & y^2-14y+49 & 4x^2+12x+9 \end{array}$$

Notice that it came from adding the 'outer' and the 'inner' terms, which are both the same. So the middle term is double the product of the two terms of the binomial. When squaring a sum, the resulting middle term is positive, but when squaring a difference, the resulting middle term is negative.

$$(a+b)^2 = a^2 + \boldsymbol{2ab} + b^2$$
$$(a-b)^2 = a^2 - \boldsymbol{2ab} + b^2$$

We record the general pattern for squaring a binomial below.

Binomial Squares Pattern

If a and b are real numbers, then

$$(a+b)^2 = a^2 + 2ab + b^2$$
and
$$(a-b)^2 = a^2 - 2ab + b^2$$

Now let's use our newly discovered pattern to efficiently square binomials, which are often called **perfect square trinomials**.

■ **Example 31** Simplify $(x+5)^2$.

Solution:
We are squaring a sum, so we will use the pattern: $(a+b)^2 = a^2 + 2ab + b^2$.

$$\overset{a+b}{(x+5)^2}$$

Square the first term.

$$\begin{array}{ccccc} a^2 & + & 2ab & + & b^2 \\ x^2 & + & \_\_ & + & \_\_ \end{array}$$

Square the last term.

$$\begin{array}{ccccc} a^2 & + & 2ab & + & b^2 \\ x^2 & + & \_\_ & + & 5^2 \end{array}$$

Double the product of the terms.

$$\begin{array}{ccccccccc} a^2 & + & 2 & \cdot & a & \cdot & b & + & b^2 \\ x^2 & + & 2 & \cdot & x & \cdot & 5 & + & 5^2 \end{array}$$

Simplify each term.

$$(x+5)^2 = x^2 + 10x + 25$$

■

■ **Example 32** Simplify $(y-3)^2$.

Solution:
We are squaring a difference, so we will use the pattern: $(a-b)^2 = a^2 - 2ab + b^2$.

$$\overset{a-b}{(y-3)^2}$$

Square the first term.

$$\begin{array}{ccccc} a^2 & - & 2ab & + & b^2 \\ y^2 & - & \_\_ & + & \_\_ \end{array}$$

Square the last term.

$$\begin{array}{ccccc} a^2 & - & 2ab & + & b^2 \\ y^2 & - & \_\_ & + & 3^2 \end{array}$$

Double the product of the terms.

$$\begin{array}{ccccccccc} a^2 & - & 2 & \cdot & a & \cdot & b & + & b^2 \\ y^2 & - & 2 & \cdot & y & \cdot & 3 & + & 3^2 \end{array}$$

Simplify each term.

$$(y-3)^2 = y^2 - 6y + 9$$

■

Example 33 Simplify $(4x+6)^2$.

Solution:
Using the pattern for squaring a sum, we have

$$\begin{aligned}
(\overset{a+b}{4x+6})^2 &= \overset{a^2}{(4x)^2} + \overset{2\cdot a\cdot b}{2\cdot 4x\cdot 6} + \overset{b^2}{6^2} \\
&= 16x^2+48x+36
\end{aligned}$$

Example 34 Simplify $(4u^3-1)^2$.

Solution:
Using the pattern for squaring a difference, we have

$$\begin{aligned}
(\overset{a-b}{4u^3-1})^2 &= \overset{a^2}{(4u^3)^2} - \overset{2\cdot a\cdot b}{2\cdot 4u^3\cdot 1} + \overset{b^2}{1^2} \\
&= 16u^6-8u^3+1
\end{aligned}$$

Now let's move on to another pattern which will save time when multiplying two special binomials, but before we go any further, we need to introduce some vocabulary.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name, called a **conjugate pair**.

Definition

A **conjugate pair** is two binomials of the form

$$(a-b) \text{ and } (a+b)$$

We will often use the term 'conjugates' to refer to a conjugate pair. There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x-9)(x+9)$	$(y+8)(y-8)$	$(2x-5)(2x+5)$
$x^2+9x-9x-81$	$y^2-8y+8y-64$	$4x^2+10x-10x-25$
x^2-81	y^2-64	$4x^2-25$

In each situation, the result was a binomial of the form

$$(a+b)(a-b) = \underline{\hspace{1.5cm}} - \underline{\hspace{1.5cm}}$$

Each **first term** is the product of the first terms of the binomials, and, as these terms are identical, the product is the square of the first term.

$$(a+b)(a-b) = \boldsymbol{a^2} - \underline{\hspace{1.5cm}}$$

The **last term** came from multiplying the last terms of the binomials. Again, as these terms are identical, the product is the square of the last term.

$$(a+b)(a-b) = a^2 - \boldsymbol{b^2}$$

Notice the two middle terms you get from FOIL combine to 0 in every case. The product of conjugates, $(a+b)(a-b)$, is always of the form $a^2 - b^2$, which is called a **difference of squares**.

$$\begin{array}{ccccc} & & & \textit{difference} & \\ & & & \downarrow & \\ \underbrace{(a+b)(a-b)}_{\textit{conjugates}} & = & a^2 & - & b^2 \\ & & \nwarrow & & \nearrow \\ & & & \textit{squares} & \end{array}$$

Definition

The product of conjugate pairs is called a **difference of squares**.

If a and b are real numbers,

$$(a+b)(a-b) = (a-b)(a+b) = a^2 - b^2$$

■

■ **Example 35** Simplify $(x-4)(x+4)$.

Solution:
First, recognize this is a product of conjugates. The binomials have the same first terms, the same last terms, and one binomial is a sum and the other is a difference.

$$\overset{(a-b)}{(x-4)}\overset{(a+b)}{(x+4)}$$

Begin by squaring the first term, x.

$$\begin{array}{ccc} a^2 & - & b^2 \\ x^2 & - & \_\_\_ \end{array}$$

Then, square the last term, 4.

$$\begin{array}{ccc} a^2 & - & b^2 \\ x^2 & - & 4^2 \end{array}$$

Simplify each term.

$$(x-4)(x+4) = x^2 - 16$$

■

■ **Example 36** Simplify $(3x+2)(3x-2)$.

Solution:
Again, we notice we are multiplying conjugates and the result will be the difference of squares.

$$\overset{(a+b)}{(3x+2)}\overset{(a-b)}{(3x-2)}$$

We square the first term, $3x$.

$$\begin{array}{ccc} a^2 & - & b^2 \\ (3x)^2 & - & \_\_\_ \end{array}$$

Then, we square the last term, 2.

$$\begin{array}{ccc} a^2 & - & b^2 \\ (3x)^2 & - & 2^2 \end{array}$$

Simplifying each term gives us.

$$(3x+2)(3x-2) = 9x^2 - 4$$

■

The binomials in the next example may look backwards, as the variable is in the second term. However, the two binomials are still conjugates, so the product will be a difference of squares.

Example 37 Simplify $(3+5x)(3-5x)$.

Solution:
This is the product of conjugates, $(a+b)(a-b)$, with $a=3$ and $b=5x$,

$$\begin{aligned}(3+5x)(3-5x) &= 3^2-(5x)^2 \\ &= 9-25x^2\end{aligned}$$

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. While the products are different, they look similar. It is important to recognize when it is appropriate to use each of these patterns and to notice their similarities and differences.

Comparing the Special Product Patterns

Binomial Squares
$(a+b)^2 = a^2+2ab+b^2$
$(a-b)^2 = a^2-2ab+b^2$

- Squaring a binomial
- Product is a **trinomial**
- Inner and outer terms with FOIL are **the same**
- Middle term is **double the product** of the terms

Product of Conjugates
$(a-b)(a+b) = a^2-b^2$

- Multiplying conjugates
- Product is a **binomial**
- Inner and outer terms with FOIL are **opposites**
- There is **no** middle term

EXERCISES

Skills Practice (Answers)

For Exercises 67 - 75, square the binomial, using the Binomial Squares Pattern.

67. $(w+4)^2$
68. $(q+12)^2$
69. $\left(x+\dfrac{2}{3}\right)^2$
70. $(y-6)^2$
71. $(p-13)^2$
72. $(3d+1)^2$
73. $\left(2q+\dfrac{1}{3}\right)^2$
74. $(3x^2+2)^2$
75. $(x+h)^2$

For Exercises 76 - 81, multiply the conjugate pair.

76. $(c-5)(c+5)$
77. $\left(x+\dfrac{3}{4}\right)\left(x-\dfrac{3}{4}\right)$
78. $(5k+6)(5k-6)$
79. $(11k+4)(11k-4)$
80. $(13-q)(13+q)$
81. $(4-6y)(4+6y)$

Simplifying Expressions Involving Quotients and Exponents

Previously, we developed the properties of exponents for multiplication. We summarize these properties below.

Summary of Exponent Properties for Multiplication

If a and b are any real numbers, and m and n are real numbers, as long as the base and the exponent are not both 0. Then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. We have learned to simplify fractions by dividing out common factors from the numerator and denominator using the Equivalent Fractions Property, restated below.

Equivalent Fractions Property

If a, b and c are real numbers where $b \neq 0$ and $c \neq 0$, then

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

We will develop an exponent property for division by writing the exponential notation in equivalent expanded form (repeated multiplication) and applying the Equivalent Fractions Property.

First, let's consider an expression where the exponent in the numerator is larger than the exponent in the denominator, $\dfrac{x^5}{x^2}$.

$$\begin{aligned} \frac{x^5}{x^2} &= \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} \\ &= \frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}} \\ &= x^3 \end{aligned}$$

Notice in the result that the base, x, stayed the same and the exponent, 3, is the difference of the exponents, 5 and 2.

$$\frac{x^5}{x^2} = x^3 = x^{(5-2)}$$

Now, let's consider an expression where the exponent in the numerator is smaller than the exponent in the denominator, $\frac{x^2}{x^3}$.

$$\begin{aligned}\frac{x^2}{x^3} &= \frac{x \cdot x}{x \cdot x \cdot x} \\ &= \frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x} \\ &= \frac{1}{x}\end{aligned}$$

From our definition of negative exponents, we know our result $\frac{1}{x} = x^{-1}$. Notice in the result that the base, x, stayed the same and the exponent, -1, is the difference of the exponents, 2 and 3.

$$\frac{x^2}{x^3} = \frac{1}{x} = x^{-1} = x^{(2-3)}$$

This leads us to the Quotient Property for Exponents.

Quotient Property for Exponents

If a is a nonzero real number, m and n are real numbers, then

$$\frac{a^m}{a^n} = a^{m-n}$$

■ **Example 38** Simplify each expression.

a. $\frac{x^9}{x^7}$

b. $\frac{x}{x^{-4}}$

Solution:

a. As the bases are the same, and the exponent in the numerator is greater than the exponent in the denominator $(9 > 7)$, there are more factors of x in the numerator and the resulting exponent should be positive. Using the Quotient Property for Exponents, $\frac{a^m}{a^n} = a^{m-n}$,

$$\begin{aligned}\frac{x^9}{x^7} &= x^{9-7} \\ &= x^2\end{aligned}$$

b. We can rewrite $\frac{x}{x^{-4}}$ to start, using the Property of Negative Exponents.

$$\begin{aligned}\frac{x}{x^{-4}} &= x\left(\frac{1}{x^{-4}}\right)\\ &= x\left(x^{4}\right)\end{aligned}$$

Given that $x = x^1$, using the Product Property for Exponents, we have

$$\begin{aligned}&= x^1 x^4\\ &= x^{1+4}\\ &= x^5\end{aligned}$$

Alternatively, we could just use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$, from the beginning.

$$\begin{aligned}\frac{x}{x^{-4}} &= \frac{x^1}{x^{-4}}\\ &= x^{1-(-4)}\\ &= x^{1+4}\\ &= x^5\end{aligned}$$

▪

A special case of the Quotient Property for Exponents is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From our earlier work with fractions, we know that:

$$\frac{2}{2} = 1 \qquad \frac{17}{17} = 1 \qquad \frac{-43}{-43} = 1$$

In other words, a nonzero number divided by itself is 1. So, $\frac{x}{x} = 1$, for any nonzero x.

Consider $\frac{8}{8}$, which we know is 1. We can rewrite 8 in an equivalent exponential form, 2^3

$$\frac{8}{8} = \frac{2^3}{2^3} = 1$$

Using the Quotient Property for Exponents, we then have

$$\frac{2^3}{2^3} = 2^{(3-3)} = 1$$

or

$$2^0 = 1$$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the **zero exponent**. In general, for $a \neq 0$:

$$\begin{array}{ll} \dfrac{a^m}{a^m} & \dfrac{a^m}{a^m} \\ = a^{m-m} & = \dfrac{\overbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}^{m \text{ factors}}}{\underbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}_{m \text{ factors}}} \\ = a^0 & = 1 \end{array}$$

We see $\frac{a^m}{a^m}$ simplifies to both a^0 and to 1.

Definition

If a is a nonzero number, then the **zero exponent** is $a^0 = 1$.

▪

In this text, we assume any variable that we raise to the zero power is not zero.

Now let's investigate what happens when we apply exponents to quotients.

We will consider $\left(\frac{x}{y}\right)^3$.

$$\begin{aligned} \left(\frac{x}{y}\right)^3 &= \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \\ &= \frac{x \cdot x \cdot x}{y \cdot y \cdot y} \\ &= \frac{x^3}{y^3} \end{aligned}$$

Notice in the result that the variables, x and y, remain the same and that the exponent, 3, is applied to both the numerator and the denominator.

$$\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$$

This leads us to the Quotient to a Power Property for Exponents.

Quotient to a Power Property for Exponents

If a and b are nonzero real numbers, and m is a real number, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

▪ **Example 39** Simplify each expression.

a. $\left(\frac{b}{3}\right)^4$

b. $\left(\frac{k}{j}\right)^3$

Solution:

a. Using the Quotient to a Power Property for Exponents, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, we have

$$\left(\frac{b}{3}\right)^4 = \frac{b^4}{3^4}$$
$$= \frac{b^4}{81}$$

b. Using the Quotient to a Power Property for Exponents, we have

$$\left(\frac{k}{j}\right)^3 = \frac{k^3}{j^3}$$

▪

We can simplify more complicated expressions involving quotients and exponents by using multiple properties.

▪ **Example 40** Simplify $\frac{(y^4)^2}{y^6}$.

Solution:
Following order of operations, we begin by using the Power Property for Exponents to simplify the numerator.

$$\frac{(y^4)^2}{y^6} = \frac{y^8}{y^6}$$

Now we can apply the Quotient Property for Exponents.

$$= y^{8-6}$$
$$= y^2$$

▪

■ **Example 41** Simplify $\frac{b^{12}}{(b^2)(b^{10})}$.

Solution:
First we can use the Product Property for Exponents to simplify the denominator.

$$\begin{aligned}\frac{b^{12}}{(b^2)(b^{10})} &= \frac{b^{12}}{b^{2+10}} \\ &= \frac{b^{12}}{b^{12}}\end{aligned}$$

Now we can apply the Quotient Property for Exponents.

$$\begin{aligned}&= b^{12-12} \\ &= b^0\end{aligned}$$

By the definition of a zero exponent, $b^0 = 1$, we have

$$\frac{b^{12}}{(b^2)(b^{10})} = 1$$

■

■ **Example 42** Simplify $\left(\frac{y^9}{y^4}\right)^2$.

Solution:
Following order of operations, we start inside the parentheses. As the bases of the numerator and denominator are the same, we can simplify using the Quotient Property for Exponents.

$$\begin{aligned}\left(\frac{y^9}{y^4}\right)^2 &= \left(y^{9-4}\right)^2 \\ &= \left(y^5\right)^2\end{aligned}$$

Then, using the Power Property for Exponents, this simplifies to

$$= y^{10}$$

■

Example 43 Simplify $\left(\dfrac{2m^2}{5m}\right)^4$.

Solution:
We start by simplifying inside the parentheses. Afterwards we apply the exponent, 4, to both the numerator and denominator, using the Quotient to a Power Property for Exponents, and simplify.

$$\begin{aligned}\left(\frac{2m^2}{5m}\right)^4 &= \left(\frac{2m^{2-1}}{5}\right)^4\\ &= \left(\frac{2m}{5}\right)^4\\ &= \frac{(2m)^4}{5^4}\\ &= \frac{2^4m^4}{5^4}\\ &= \frac{16m^4}{625}\end{aligned}$$

Notice, in the first step, the Quotient Property for Exponents applies to only the variable portion, not the coefficients, as the coefficients are not like bases.

While the last two examples were simplified using order of operations, we could have achieved the same results by applying the Quotient to a Power Property for Exponents before simplifying inside the parentheses. The authors will leave it to the reader to use this method for simplifying.

EXERCISES

Skills Practice (Answers)

For Exercises 82 - 95, simplify the expression. Write the final answer with only positive exponents.

82. $\dfrac{p^{21}}{p^2}$

83. $\dfrac{u^{24}}{u^{-3}}$

84. $\dfrac{t^{10}}{t^{40}}$

85. $\dfrac{x^{-1}}{x^7}$

86. **a.** -15^0

    b. $(-15)^0$

87. **a.** $(25x)^0$

    b. $25x^0$

88. $\left(\dfrac{x}{3}\right)^4$

89. $\left(\dfrac{5}{4m}\right)^2$

90. $\dfrac{n^8}{(n^6)^4}$

91. $\left(\dfrac{r^2}{r^6}\right)^3$

92. $\left(\dfrac{2j^3}{3j}\right)^4$

93. $\left(\dfrac{3m^5}{5m}\right)^3$

94. $\left(\dfrac{k^2k^8}{k^3}\right)^2$

95. $\left(\dfrac{j^2j^5}{j^4}\right)^3$

Dividing Monomials

We have now been introduced to all the properties of exponents and used them to simplify expressions. Next, we will see how to use these properties to divide a monomial, and eventually a polynomial, by a monomial.

Example 44 Compute $56x^7 \div 8x^3$.

Solution:
We will first rewrite the division as an equivalent fraction.

$$\begin{aligned} 56x^7 \div 8x^3 &= \frac{56x^7}{8x^3} \\ &= \frac{56 \cdot x^7}{8 \cdot x^3} \\ &= \frac{56}{8} \cdot \frac{x^7}{x^3} \end{aligned}$$

To simplify, we divide the numbers and apply the Quotient Property for Exponents to the variables.

$$= 7x^4$$

Once you become familiar with the process and have practiced it step-by-step several times, you may be able to divide two monomials in one step.

Example 45 Compute $\dfrac{14x^7}{21x^{11}}$, using only positive exponents.

Solution:
We simplify $\frac{14}{21}$ by dividing out a common factor of 7. Then we simplify the variables by subtracting their exponents and rewriting the result with only a positive exponent.

$$\frac{14x^7}{21x^{11}} = \frac{2}{3x^4}$$

EXERCISES

Skills Practice (Answers)

For Exercises 96 - 103, divide the monomials. Write the final answer with only positive exponents.

96. $-72u^{12} \div 12u^4$

97. $-88y^{15} \div 8y^3$

98. $\dfrac{54x^9}{-18x^6}$

99. $\dfrac{15r^4}{18r^9}$

100. $\dfrac{64q^{11}}{48q^6}$

101. $\dfrac{65a^{10}}{42a^7}$

102. $\dfrac{(-18p^4)(-6p^3)}{-36p^{12}}$

103. $\dfrac{(6b^3)(4b^5)}{(12b)(b)}$

Simplifying Rational Expressions

We have reviewed the properties of fractions and their operations. We have also introduced rational numbers, which are numbers that can be written as fractions, where the numerators and denominators are integers, and the denominator is nonzero.

Now, we will work with fractions whose numerators and denominators are polynomials, which we call **rational expressions**.

Definition

A **rational expression** is an expression of the form $\dfrac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{13}{42} \qquad \frac{7y}{8z} \qquad \frac{5x+2}{x^2-7} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{13}{42}$, is just a fraction. Similar to a nonzero constant being a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We can perform the same operations with rational expressions that we do with fractions. We will simplify, add, subtract, multiply, divide, and use them in applications.

The numerator of a rational expression may be zero, but not the denominator. If the denominator is zero, the rational expression is undefined. When we work with a numerical fraction, it is easy to avoid dividing by zero, because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator equal zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation, for example, we will know whether each algebraic solution we find is included in the solution set.

Example 46 Determine the value(s) for which the rational expression is undefined.

a. $\dfrac{9}{x}$

b. $\dfrac{4b-3}{2b+5}$

Solution:
The expression will be undefined when the expression in the denominator is zero.

a. We set the denominator, x, equal to zero, and solve for the variable.

$$x = 0$$

Therefore, $\frac{9}{x}$ is undefined at $x = 0$.

b. We set the denominator, $2b+5$, equal to zero, and solve for the variable.

$$\begin{aligned} 2b+5 &= 0 \\ 2b &= -5 \\ b &= -\frac{5}{2} \end{aligned}$$

Thus, $\frac{4b-3}{2b+5}$ is undefined at $b = -\frac{5}{2}$.

∎

When determining the value(s) for which a rational expression is undefined, we only examine the denominator, while ignoring the numerator.

A fraction is considered simplified if there are no common factors in its numerator and denominator, other than 1. Similarly, a rational expression is *simplified* if it has no common factors in its numerator and denominator, other than 1.

For example:

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{2x}{3x}$ is not simplified because x is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

If a, b, and c are real numbers where $b \neq 0$ and $c \neq 0$, then

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \qquad \text{and} \qquad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero, before simplification, are specifically disallowed; we see $b \neq 0$ and $c \neq 0$ clearly stated. Every time we write a rational expression, we should make a similar statement, disallowing values that would make the original denominator zero.

To simplify rational expressions we first write the numerator and denominator in factored form. Then, we remove the common factors, using the Equivalent Fractions Property.

■ **Example 47** Simplify $\frac{3xy}{18x^2y^2}$, where $x \neq 0$ and $y \neq 0$.

Solution:
First, we rewrite the numerator and denominator, showing the common factors.

$$\begin{aligned}\frac{3xy}{18x^2y^2} &= \frac{3 \cdot x \cdot y}{3 \cdot 6 \cdot x \cdot x \cdot y \cdot y} \\ &= \frac{1 \cdot 3xy}{6xy \cdot 3xy}\end{aligned}$$

Then, we can simplify, using the Equivalent Fractions Property.

$$\begin{aligned}&= \frac{1 \cdot \cancel{3xy}}{6xy \cdot \cancel{3xy}} \\ &= \frac{1}{6xy}, \text{ where } x \neq 0 \text{ and } y \neq 0\end{aligned}$$

■

Be very careful as you remove common factors. Factors are multiplied to make a product. We can remove a factor from a product, but we cannot remove a term from a sum.

■ **Example 48** Simplify $\frac{2(x+4)}{5(x+4)}$, where $x \neq -4$.

Solution:
Considering the numerator and denominator are already in factored form, we divide out the common factor, $x+4$, to simplify.

$$\begin{aligned}\frac{2(x+4)}{5(x+4)} &= \frac{2\cancel{(x+4)}}{5\cancel{(x+4)}} \\ &= \frac{2}{5}, \text{ where } x \neq -4\end{aligned}$$

■

■ **Example 49** Simplify $\dfrac{(x+3)(x+2)}{(x+2)(x+6)}$, where $x \neq -2$ and $x \neq -6$.

Solution:
Again, the numerator and denominator are already in factored form, so we divide out the common factor, $x+2$, from the numerator and the denominator.

$$\begin{aligned}\frac{(x+3)(x+2)}{(x+2)(x+6)} &= \frac{(x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x+6)}\\ &= \frac{x+3}{x+6}, \text{ where } x \neq -2 \text{ and } x \neq -6\end{aligned}$$

■

Usually we leave simplified rational expression in factored form, so that it is easy to check that we have removed all the common factors.

■ **Example 50** Simplify $\dfrac{3(b-2)(b-2)}{6(b+2)(b-2)}$, where $b \neq -2$ and $b \neq 2$.

Solution:
We can first write 6 in factored form.

$$\frac{3(b-2)(b-2)}{6(b+2)(b-2)} = \frac{3(b-2)(b-2)}{3 \cdot 2 \cdot (b+2)(b-2)}$$

Next, we can divide out the common factors of $(b-2)$ and 3 to simplify.

$$= \frac{\cancel{3}(b-2)\cancel{(b-2)}}{\cancel{3} \cdot 2 \cdot (b+2)\cancel{(b-2)}}$$

$$= \frac{b-2}{2(b+2)}, \text{ where } b \neq -2 \text{ and } b \neq 2.$$

■

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. Previously we have introduced opposite notation as the opposite of a is $-a$. We remember, too, that $-a = -1 \cdot a$.

With a numerical fraction, say $\frac{7}{-7}$, we know $\frac{7}{-7} = \frac{1 \cdot 7}{-1 \cdot 7} = \frac{1}{-1} = -1$.

Therefore, in general, we simplify the fraction $\dfrac{a}{-a}$, whose numerator and denominator are opposites, as follows

$$\begin{aligned}\frac{a}{-a} &= \frac{1 \cdot \cancel{a}}{-1 \cdot \cancel{a}}\\ &= \frac{1}{-1}\\ &= -1, \text{ where } a \neq 0\end{aligned}$$

So in the same way, we can simplify the fraction $\dfrac{x-3}{-(x-3)}$.

$$\begin{aligned}\frac{x-3}{-(x-3)} &= \frac{1\cdot\cancel{(x-3)}}{-1\cdot\cancel{(x-3)}}\\ &= \frac{1}{-1}\\ &= -1, \text{ where } x \neq 3\end{aligned}$$

However, the opposite of $x-3$ could be written differently:

$$-(x-3) = -x+3 = 3-x$$

This means the fraction $\dfrac{x-3}{3-x}$ also simplifies to -1, so long as $x \neq 3$.

In general, we could write the opposite of $a-b$ as $b-a$.

So, the rational expression $\dfrac{a-b}{b-a}$ simplifies to -1, when $b \neq a$.

■ **Example 51** Simplify $\dfrac{2(7-x)}{(x+7)(x-7)}$, where $x \neq -7$ and $x \neq 7$.

Solution:
We recognize that $7-x$ and $x-7$ are opposites, and simplify.

$$\begin{aligned}\frac{2(7-x)}{(x+7)(x-7)} &= \frac{2}{(x+7)}\cdot(-1)\\ &= -\frac{2}{x+7}, \text{ where } x \neq -7 \text{ and } x \neq 7\end{aligned}$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 104 - 105, determine the values for which the rational expression is undefined.

104. **a.** $\dfrac{10m}{11n}$

b. $\dfrac{6y+13}{4y-16}$

c. $\dfrac{b-8}{(b+6)}$

105. **a.** $\dfrac{4x^2y}{3y}$

b. $\dfrac{3x-2}{2x+10}$

c. $\dfrac{u-1}{(u-7)}$

For Exercises 106 - 117, simplify the rational expression.

106. $\dfrac{15xy}{3x^3y^3}$

107. $\dfrac{8m^3n}{12mn^2}$

108. $\dfrac{5(b+1)}{6(b+1)}$

109. $\dfrac{8(n-12)}{3(n-12)}$

110. $\dfrac{(y+4)(y-1)}{(y-1)(y-5)}$

111. $\dfrac{(y-3)(y+1)}{(y+3)(y-3)}$

112. $\dfrac{8b(b-4)}{2(b+5)(b-8)}$

113. $\dfrac{4d(d-6)}{2(d-6)(d+4)}$

114. $\dfrac{b-12}{12-b}$

115. $\dfrac{5-d}{d-5}$

116. $\dfrac{5(4-y)}{(y+4)(y-4)}$

117. $\dfrac{7(w-3)}{(3-w)(3+w)}$

Evaluating Rational Expressions

To evaluate a rational expression, we substitute values of the variables into the expression and simplify, just as we have for many other expressions. To let us focus on the work at hand, we will omit writing the disallowed value(s) which would make the denominator zero, but we will keep them in consideration.

■ **Example 52** Evaluate $\dfrac{2x+3}{3x-5}$ for each value of x.

a. $x = 0$
b. $x = 2$
c. $x = -3$

Solution:

a. Substitute 0 for x, and simplify.

$$\frac{2(0)+3}{3(0)-5} = \frac{3}{-5} = -\frac{3}{5}$$

b. Substitute 2 for x, and simplify.

$$\frac{2(2)+3}{3(2)-5} = \frac{4+3}{6-5} = \frac{7}{1} = 7$$

c. Substitute -3 for x, and simplify.

$$\frac{2(-3)+3}{3(-3)-5} = \frac{-6+3}{-9-5} = \frac{-3}{-14} = \frac{3}{14}$$

■

■ **Example 53** Evaluate $\dfrac{x^2+8x+7}{x^2-4}$ for each value of x.

a. $x = 0$
b. $x = 2$
c. $x = -1$

Solution:

a. Substitute 0 for x, and simplify.

$$\frac{(0)^2+8(0)+7}{(0)^2-4} = \frac{7}{-4} = -\frac{7}{4}$$

b. Substitute 2 for x, and simplify.

$$\frac{(2)^2+8(2)+7}{(2)^2-4} = \frac{4+16+7}{4-4}$$

Because $4-4$ equals 0 and we cannot divide by 0, $\frac{x^2+8x+7}{x^2-4}$ is undefined when $x = 2$. If we had determined the disallowed values for the rational expression prior to evaluating, then we would have known that $x \neq 2$, as well as $x \neq -2$. Then we could have stated that the rational expression is undefined when $x = 2$, without evaluating.

c. Substitute -1 for x, and simplify.

$$\frac{(-1)^2+8(-1)+7}{(-1)^2-4} = \frac{1-8+7}{1-4} = -\frac{-7+7}{-3} = \frac{0}{-3} = 0$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 118 - 121, evaluate the rational expression for the given values.

118. $\dfrac{4y-1}{5y-3}$

   a. $y=0$
   b. $y=2$
   c. $y=-1$

119. $\dfrac{2p+3}{p^2+1}$

   a. $p=0$
   b. $p=-1$
   c. $p=-2$

120. $\dfrac{y^2+5y+6}{y^2-1}$

   a. $y=0$
   b. $y=-1$
   c. $y=-3$

121. $\dfrac{b^2+2}{b^2-3b-4}$

   a. $b=0$
   b. $b=-2$
   c. $b=4$

Dividing a Polynomial by a Monomial

Previously, we learned how to divide a monomial by a monomial. As we continue to build up our knowledge of polynomials, the next procedure is to divide a polynomial with two by more terms by a monomial.

The method we will use to divide a polynomial by a monomial is based on the properties of fraction addition.

Let's start with an example to review fraction addition.

The sum, $\frac{y}{5}+\frac{2}{5}$, simplifies to $\frac{y+2}{5}$, as both fractions have a common denominator of 5.

Now we will apply this method in reverse to split a single fraction into separate fractions. We will state the Fraction Addition Property here, both just as we learned it and in reverse.

Fraction Addition Property

If a, b, and c are real numbers where $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \qquad \text{and} \qquad \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

We use the form on the left to add fractions, and we use the form on the right to divide a polynomial by a monomial.

For example, $\frac{y+2}{5}$ can be rewritten as $\frac{y}{5}+\frac{2}{5}$.

■ **Example 54** Simplify $\frac{7y^2+21}{7}$.

Solution:
We divide each term of the numerator by the denominator. Then, we simplify each fraction.

$$\begin{aligned}\frac{7y^2+21}{7} &= \frac{7y^2}{7}+\frac{21}{7}\\ &= y^2+3\end{aligned}$$

■

■ **Example 55** Compute $(18x^3-36x^2)\div 6x$.

Solution:
Remember that division can be represented as a fraction, so we first rewrite the division as an equivalent fraction.

$$(18x^3-36x^2)\div 6x = \frac{18x^3-36x^2}{6x}$$

Now we can divide each term of the numerator by the denominator and simplify.

$$\begin{aligned}\frac{18x^3-36x^2}{6x} &= \frac{18x^3}{6x}-\frac{36x^2}{6x}\\ &= 3x^2-6x\end{aligned}$$

■

When we divide by a negative, we must be extra careful with the signs.

■ **Example 56** Simplify $\dfrac{12d^2-16d}{-4}$.

Solution:
We divide each term of the numerator by the denominator and simplify.

$$\begin{aligned}\frac{12d^2-16d}{-4} &= \frac{12d^2}{-4}-\frac{16d}{-4}\\ &= \frac{12d^2}{-4}+\frac{-16d}{-4}\\ &= -3d^2+4d\end{aligned}$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 122 - 123, simplify the expression. Write the final answer with only positive exponents.

122. $\dfrac{8z^2+14}{4}$

123. $\dfrac{18y^2-27y}{9y}$

For Exercises 124 - 125, compute the quotient.

124. $(27b^3-33b^2)\div 36$

125. $(25x^3-55x^2)\div 5x$

For Exercises 126 - 127, simplify the expression. Write the final answer with only positive exponents.

126. $\dfrac{25y^2-15y}{-5}$

127. $\dfrac{-42b^2-18b}{-6b^2}$

Simplifying Variable Expressions with Roots

Recall from the Number Sense section, the odd root of a number can be either positive or negative. For example, we know that $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$ and $\sqrt[3]{-8} = \sqrt[3]{(-2)^3} = -2$. In general, when n is odd, $\sqrt[n]{a^n} = a$.

From the Number Sense section, we also know that $\sqrt{4} = \sqrt{2^2} = 2$ and $\sqrt{4} = \sqrt{(-2)^2} = 2$. However, $\sqrt{4} = 2$ as the radical indicates the principal square root, which is always positive or zero. To ensure the result of an even root is always the principal root, we use absolute value and say when n is even, $\sqrt[n]{a^n} = |a|$.

Simplifying Odd and Even Roots

For any integer $n \geq 2$,

- When the index, n, is odd, $\sqrt[n]{a^n} = a$.
- When the index, n, is even, $\sqrt[n]{a^n} = |a|$.

In other words, we must use the absolute value sign when we take an even root of an expression with a variable in the radical.

■ **Example 57** Simplify each expression.

a. $\sqrt{x^2}$

b. $\sqrt[3]{m^3}$

c. $\sqrt[4]{p^4}$

d. $\sqrt[5]{y^5}$

Solution:

a. Because the index, $n = 2$, is even, we use $\sqrt[n]{a^n} = |a|$. Thus, $\sqrt{x^2} = |x|$.

b. Seeing as the index, $n = 3$, is odd, we use $\sqrt[n]{a^n} = a$. Therefore, $\sqrt[3]{m^3} = m$

c. Given that the index, $n = 4$, is even, $\sqrt[4]{p^4} = |p|$

d. Considering the index, $n = 5$, is odd, $\sqrt[5]{y^5} = y$.

■

We know the Power Property for Exponents says $(a^m)^n = a^{m \cdot n}$. So, if we square b^m, the exponent will become $2m$.

$$(b^m)^2 = b^{2m}$$

Now, if we apply the square root to the result, b^{2m}, we have

$$\sqrt{b^{2m}} = \sqrt{(b^m)^2}$$

Because $\sqrt[n]{a^n} = |a|$, when n is even

$$\sqrt{(b^m)^2} = \left|b^m\right|$$

Thus, $\sqrt{b^{2m}} = |b^m|$.

We will apply this concept in the next example.

Example 58 Simplify each expression.

a. $\sqrt{x^6}$

b. $\sqrt{y^{16}}$

Solution:

a. Considering $x^6 = (x^3)^2$, we rewrite the given expression as

$$\sqrt{x^6} = \sqrt{(x^3)^2}$$

Because the index, $n = 2$, is even, we use $\sqrt[n]{a^n} = |a|$ with $a = x^3$ to simplify.

$$\sqrt{x^6} = \sqrt{(x^3)^2} = \left|x^3\right|$$

b. Due to the fact that $y^{16} = (y^8)^2$, we can rewrite the given expression as

$$\sqrt{y^{16}} = \sqrt{(y^8)^2}$$

The index $n = 2$ is even, so we use $\sqrt[n]{a^n} = |a|$ with $a = y^8$ to simplify.

$$\sqrt{y^{16}} = \sqrt{(y^8)^2} = \left|y^8\right| = y^8$$

In this case, the absolute value sign is not needed, as y^8 is non-negative for any value of y.

■ **Example 59** Simplify each expression.

a. $\sqrt[3]{y^{18}}$

b. $\sqrt[4]{z^8}$

Solution:

a. We know $y^{18} = (y^6)^3$ and the index, $n = 3$, is odd. Thus,

$$\begin{aligned}\sqrt[3]{y^{18}} &= \sqrt[3]{(y^6)^3} \\ &= y^6\end{aligned}$$

b. The index, $n = 4$, is even, and $z^8 = (z^2)^4$. So,

$$\begin{aligned}\sqrt[4]{z^8} &= \sqrt[4]{(z^2)^4} \\ &= \left|z^2\right|\end{aligned}$$

However, as z^2 is positive, we do not need the absolute value sign and so $\sqrt[4]{z^8} = z^2$.

■

In the next examples, we now have a coefficient in front of the variable.

■ **Example 60** Simplify each expression.

a. $\sqrt{16n^4}$

b. $-\sqrt{81c^2}$

Solution:

a. As $16n^4 = (4n^2)^2$, $\sqrt{16n^4} = \sqrt{(4n^2)^2} = \left|4n^2\right| = 4n^2$.

b. Given that $81c^2 = (9c)^2$, $-\sqrt{81c^2} = -\sqrt{(9c)^2} = -|9c| = -1 \cdot 9 \cdot |c| = -9|c|$.

■

EXERCISES

Skills Practice (Answers)

For Exercises 128 - 139, simplify the expression.

128. **a.** $\sqrt[3]{a^3}$

 b. $\sqrt[9]{b^9}$

129. **a.** $\sqrt[4]{y^4}$

 b. $\sqrt[7]{m^7}$

130. **a.** $\sqrt{x^6}$

 b. $\sqrt{y^{16}}$

131. **a.** $\sqrt{x^{24}}$

 b. $\sqrt{y^{22}}$

132. **a.** $\sqrt[5]{a^{10}}$

 b. $\sqrt[3]{b^{27}}$

133. **a.** $\sqrt[4]{m^8}$

 b. $\sqrt[5]{n^{20}}$

134. **a.** $\sqrt{49x^2}$

 b. $-\sqrt{81x^{18}}$

135. **a.** $\sqrt{100y^2}$

 b. $-\sqrt{100m^{32}}$

136. **a.** $\sqrt[3]{27x^{36}}$

 b. $-\sqrt[3]{125x^3}$

137. **a.** $\sqrt[3]{-8c^9}$

 b. $\sqrt[3]{125d^{15}}$

138. **a.** $\sqrt[3]{216a^6}$

 b. $\sqrt[4]{16b^{20}}$

139. **a.** $\sqrt[7]{128r^{14}}$

 b. $\sqrt[4]{81s^{24}}$

Simplifying Expressions with Rational Exponents

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

Suppose we want to find a number p such that $(8^p)^3 = 8$. We will use the Power Property of Exponents to find the value of p, as shown below.

$$\begin{aligned}(8^p)^3 &= 8\\ 8^{3p} &= 8\end{aligned}$$

Because we now have two exponential expressions where the bases are the same, the exponents must be equal. Therefore,

$$\begin{aligned}8^{3p} &= 8^1\\ 3p &= 1\\ p &= \frac{1}{3}\end{aligned}$$

So we just showed $\left(8^{\frac{1}{3}}\right)^3 = 8$, but we also know $\left(\sqrt[3]{8}\right)^3 = 8$. Thus, it must be true that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This leads us to the Rational Exponent $\frac{1}{n}$ Property.

Rational Exponent $\frac{1}{n}$ Property

If a is any real number and n is any positive integer greater than 1, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Notice, the denominator of the rational exponent, n, is the index of the radical.

There will be times when it will be easier to use rational exponents and times when it will be easier to use radicals. In the first few examples, we will practice converting expressions between these two notations.

■ **Example 61** Write each expression with a rational exponent.

a. $\sqrt{5y}$

b. $\sqrt[3]{4x}$

c. $3\sqrt[4]{5z}$

Solution:
We will write each radical in the form $a^{\frac{1}{n}}$.

a. We are given the square root of $5y$. While no index is written, we know $n = 2$. Thus the denominator of the rational exponent will be 2. The radicand is $5y$, so $a = 5y$. Thus,

$$\sqrt{5y} = (5y)^{\frac{1}{2}}$$

b. We have the cube root of $4x$. The index, n, is 3 and so the denominator of the rational exponent will be 3. As the radicand is $4x$, $a = 4x$ and we have

$$\sqrt[3]{4x} = (4x)^{\frac{1}{3}}$$

c. We are given the product of 3 and the fourth root of $5z$. The 3 is outside the radical and is not included inside the rational exponent, $\frac{1}{n}$. The index of the fourth root is 4, so the denominator of the rational exponent is 4. We know $a = 5z$, because the radicand is $5z$. Therefore, the product of 3 and the fourth root of $5z$ can be rewritten as

$$3\sqrt[4]{5z} = 3(5z)^{\frac{1}{4}}$$

■

■ **Example 62** Write each expression as a radical, and then simplify the expression.

a. $(-16)^{\frac{1}{4}}$

b. $-16^{\frac{1}{4}}$

c. $(16)^{-\frac{1}{4}}$

Solution:

a. We begin by noting the denominator of the rational exponent is 4, so we rewrite the expression as a fourth root.

$$(-16)^{\frac{1}{4}} = \sqrt[4]{-16}$$

The expression cannot be simplified as there is no real solution, because you cannot take an even root of a negative number.

b. We are given the product of -1 and $16^{\frac{1}{4}}$. The exponential factor has a rational exponent with a denominator of 4, so we rewrite this factor as the fourth root of 16 and the product becomes

$$-16^{\frac{1}{4}} = -1 \cdot 16^{\frac{1}{4}} = -\sqrt[4]{16}$$

To simplify the even root, we can rewrite 16 as 2^4.

$$\begin{aligned} -\sqrt[4]{16} &= -\sqrt[4]{2^4} \\ &= -2 \end{aligned}$$

c. The given expression includes a negative exponent, $-\frac{1}{4}$, so we begin by rewriting the expression in an equivalent form with only a positive exponent.

$$(16)^{-\frac{1}{4}} = \frac{1}{(16)^{\frac{1}{4}}}$$

Again, the denominator of the rational exponent is 4, and so we will express the rational exponent as the fourth root.

$$= \frac{1}{\sqrt[4]{16}}$$

Because $16 = 2^4$, we simplify the radical as follows.

$$= \frac{1}{\sqrt[4]{2^4}}$$

$$= \frac{1}{2}$$

▪

Thus far we have considered examples where the numerator of the rational exponent was always 1, but it is possible for the numerator to be an integer greater than one, called m. In these cases we would write the expression in the form $a^{\frac{m}{n}}$. We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property for Exponents tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \qquad \text{and} \qquad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the Rational Exponent Property.

Rational Exponent Property

If a is any real number, m is any positive integer, and n is any positive integer greater than 1,

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \qquad \text{and} \qquad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

N *The Rational Exponent $\frac{1}{n}$ Property is a special case of the Rational Exponent Property, where $m = 1$.*

While there are two ways to rewrite a rational exponent in radical form, one of the forms may be easier if simplifying an expression. When simplifying we usually take the root first, so we keep the numbers in the radicand smaller, before raising it to the power indicated.

Example 63 Write each expression with a rational exponent.

a. $\sqrt{y^3}$

b. $\left(\sqrt[3]{2x}\right)^4$

Solution:

We will write each radical expression in the form $a^{\frac{m}{n}}$.

a. We have the square root of y^3. A square root has an index of 2, and the radicand is y^3 which has an exponent of 3. Thus, $m = 3$ and $n = 2$.

$$\sqrt{y^3} = y^{\frac{3}{2}}$$

b. We have the cube root of $2x$, which is then raised to the fourth power. A cube root has an index of 3, and the exponent is 4. Thus, $m = 4$ and $n = 3$.

$$\left(\sqrt[3]{2x}\right)^4 = (2x)^{\frac{4}{3}}$$

We will now apply the properties of exponents we have already discussed to exponential expressions involving rational exponents.

Example 64 Simplify each expression.

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b. $\left(z^9\right)^{\frac{2}{3}}$

c. $\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution:

a. The Product Property for Exponents tells us that when we multiply exponential expressions with the same base, we add the exponents.

$$\begin{aligned} x^{\frac{1}{2}} \cdot x^{\frac{5}{6}} &= x^{\frac{1}{2}+\frac{5}{6}} \\ &= x^{\frac{3}{6}+\frac{5}{6}} \\ &= x^{\frac{8}{6}} \\ &= x^{\frac{4}{3}} \end{aligned}$$

b. The Power Property for Exponents tells us that when we raise a power to a power, we multiply the exponents.

$$\begin{aligned} \left(z^9\right)^{\frac{2}{3}} &= z^{9\cdot\frac{2}{3}} \\ &= z^{\frac{18}{3}} \\ &= z^6 \end{aligned}$$

c. The Quotient Property for Exponents tells us that when we divide exponential expressions with the same base, we subtract the exponents.

$$\begin{aligned}\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}} &= x^{\frac{1}{3}-\frac{5}{3}}\\ &= x^{-\frac{4}{3}}\\ &= \frac{1}{x^{\frac{4}{3}}}\end{aligned}$$

■

EXERCISES

Skills Practice (Answers)

For Exercises 140 - 141, write the expression as an equivalent radical expression.

140. **a.** $x^{\frac{1}{2}}$ **b.** $y^{\frac{1}{3}}$ **c.** $z^{\frac{1}{4}}$

141. **a.** $u^{\frac{1}{5}}$ **b.** $v^{\frac{1}{9}}$ **c.** $w^{\frac{1}{20}}$

For Exercises 142 - 145, write the expression with a rational exponent.

142. **a.** $\sqrt[7]{x}$ **b.** $\sqrt[9]{y}$ **c.** $\sqrt[5]{f}$

143. **a.** $\sqrt[8]{r}$ **b.** $\sqrt[10]{s}$ **c.** $\sqrt[4]{t}$

144. **a.** $\sqrt[4]{5x}$ **b.** $\sqrt[8]{9y}$ **c.** $7\sqrt[5]{3z}$

145. **a.** $\sqrt[3]{25a}$ **b.** $\sqrt{3b}$ **c.** $\sqrt[8]{40c}$

For Exercises 146 - 147, write the expression with a rational exponent.

146. **a.** $\sqrt[4]{r^7}$ **b.** $\left(\sqrt[5]{2pq}\right)^3$ **c.** $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

147. **a.** $\sqrt[5]{u^2}$ **b.** $\left(\sqrt[3]{6x}\right)^5$ **c.** $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$

For Exercises 148 - 149, simplify the expression.

148. **a.** $32^{\frac{2}{5}}$ **b.** $27^{-\frac{2}{3}}$ **c.** $(-25)^{\frac{1}{2}}$

149. **a.** $-64^{\frac{3}{2}}$ **b.** $-64^{-\frac{3}{2}}$ **c.** $(-64)^{\frac{3}{2}}$

For Exercises 150 - 151, simplify the expression. Assume all variables are positive. Write the final answer with only positive exponents.

150. **a.** $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$

b. $\left(p^{12}\right)^{\frac{3}{4}}$

c. $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

151. **a.** $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$

b. $\left(x^{12}\right)^{\frac{2}{3}}$

c. $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$

d. $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$

e. $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$

f. $\dfrac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

g. $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

Rationalizing a Two-Term Denominator

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove square root radicals from the denominators of fractions, we multiply by the form of 1 that will eliminate the radical, using a process called **rationalizing the denominator**.

For a denominator containing the sum or difference of two terms, where at least one term is a square root radical, we multiply the numerator and denominator by the conjugate of the denominator, using the Product of Conjugates pattern, to eliminate the radical from the denominator. As previously mentioned, the conjugate is found by changing the sign between the two terms. For example, if the denominator is $a+\sqrt{c}$, then the conjugate is $a-\sqrt{c}$, but if the denominator is $a-\sqrt{c}$, then the conjugate is $a+\sqrt{c}$. Below we demonstrate the Product of Conjugates pattern when square roots are involved.

$$\begin{aligned} \overset{(a-b)\,(a+b)}{(2-\sqrt{5})(2+\sqrt{5})} &= \overset{a^2-b^2}{2^2-(\sqrt{5})^2} \\ &= 4-5 \\ &= -1 \end{aligned}$$

Notice when we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

▪ **Example 65** Simplify $\dfrac{4}{4+\sqrt{2}}$, by rationalizing the denominator.

Solution:
We multiply the numerator and denominator by the conjugate of the denominator, $4-\sqrt{2}$.

$$\begin{aligned} \frac{4}{4+\sqrt{2}} &= \left(\frac{4}{4+\sqrt{2}}\right)\left(\frac{4-\sqrt{2}}{4-\sqrt{2}}\right) \\ &= \frac{4(4-\sqrt{2})}{(4+\sqrt{2})(4-\sqrt{2})} \\ &= \frac{4(4-\sqrt{2})}{4^2-(\sqrt{2})^2} \\ &= \frac{4(4-\sqrt{2})}{16-2} \\ &= \frac{4(4-\sqrt{2})}{14} \end{aligned}$$

To fully simplify, we divide out common factors from the numerator and denominator.

$$\begin{aligned} \frac{4(4-\sqrt{2})}{14} &= \frac{\cancel{2}\cdot 2\cdot(4-\sqrt{2})}{\cancel{2}\cdot 7} \\ &= \frac{2(4-\sqrt{2})}{7} \end{aligned}$$

We leave the numerator in factored form to make it easier to look for common factors after we have simplified the denominator.

▪

▪ **Example 66** Simplify $\dfrac{5}{2-\sqrt{3}}$, by rationalizing the denominator.

Solution:
We multiply the numerator and denominator by the conjugate of the denominator, $2+\sqrt{3}$, and simplify.

$$\begin{aligned}
\frac{5}{2-\sqrt{3}} &= \left(\frac{5}{2-\sqrt{3}}\right)\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)\\
&= \frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}\\
&= \frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}\\
&= \frac{5(2+\sqrt{3})}{4-3}\\
&= \frac{5(2+\sqrt{3})}{1}\\
&= 5(2+\sqrt{3})
\end{aligned}$$

▪

▪ **Example 67** Simplify $\dfrac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$, by rationalizing the denominator.

Solution:
We multiply the numerator and denominator by the conjugate of the denominator and simplify.

$$\begin{aligned}
\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}} &= \left(\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}\right)\left(\frac{\sqrt{u}+\sqrt{6}}{\sqrt{u}+\sqrt{6}}\right)\\
&= \frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}\\
&= \frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}\\
&= \frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}
\end{aligned}$$

▪

■ **Example 68** Simplify $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$, by rationalizing the denominator.

Solution:
We multiply the numerator and denominator by the conjugate of the denominator and simplify.

$$
\begin{aligned}
\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}} &= \left(\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}\right)\left(\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}+\sqrt{7}}\right) \\
&= \frac{\left(\sqrt{x}+\sqrt{7}\right)\left(\sqrt{x}+\sqrt{7}\right)}{\left(\sqrt{x}-\sqrt{7}\right)\left(\sqrt{x}+\sqrt{7}\right)} \\
&= \frac{\left(\sqrt{x}+\sqrt{7}\right)\left(\sqrt{x}+\sqrt{7}\right)}{\left(\sqrt{x}\right)^2-\left(\sqrt{7}\right)^2} \\
&= \frac{\left(\sqrt{x}+\sqrt{7}\right)^2}{x-7}
\end{aligned}
$$

We do not square the numerator. In factored form, we can see that there are no common factors to remove from the numerator and denominator.

■

EXERCISES

Skills Practice (Answers)

For Exercises 152 - 159, simplify the expression by rationalizing the denominator.

152. $\dfrac{3}{3+\sqrt{11}}$

153. $\dfrac{8}{1-\sqrt{5}}$

154. $\dfrac{5}{5+\sqrt{6}}$

155. $\dfrac{6}{3-\sqrt{7}}$

156. $\dfrac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

157. $\dfrac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

158. $\dfrac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

159. $\dfrac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

A.4 Factoring

Greatest Common Factor

Recall in the Number Sense section, we discuss rewriting a number in its prime factorization. Similarly, we will start with a product involving algebraic expressions and then break them down into their factors. Splitting a product into its factors is called **factoring**.

$$\underbrace{8\cdot 7}_{factors} \overset{\xrightarrow{\text{multiply}}}{\underset{\xleftarrow{\text{factor}}}{=}} \underbrace{56}_{product} \qquad \underbrace{2x(x+3)}_{factors} \overset{\xrightarrow{\text{multiply}}}{\underset{\xleftarrow{\text{factor}}}{=}} \underbrace{2x^2+6x}_{product}$$

We will factor expressions and determine the **greatest common factor** of two or more expressions.

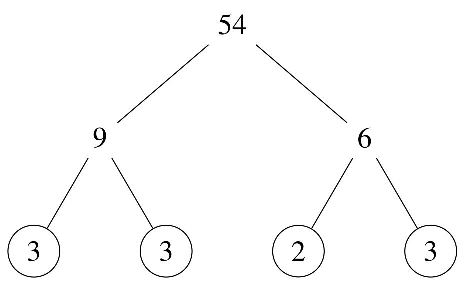
Definition

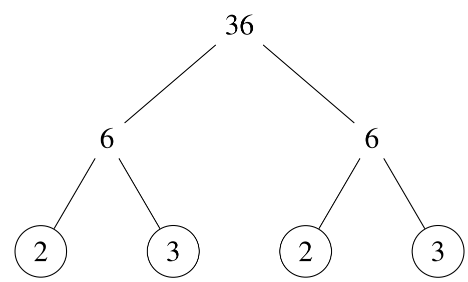
The **greatest common factor (GCF)** of two or more expressions is the largest expression that is a factor of all of these expressions. ■

We will first consider the GCF of two numbers.

■ **Example 1** Determine the GCF of 54 and 36.

Solution:
First, we factor each number, 54 and 36, into their prime factorizations.





We will write and align the factors, highlighting the common factors.

$$\begin{aligned} 54 &= 2\cdot \quad 3\cdot 3\cdot 3 \\ 36 &= 2\cdot 2\cdot 3\cdot 3 \end{aligned}$$

We multiply the shared factors to find the GCF.

$$\text{GCF} = 2\cdot 3\cdot 3 = 18$$

$\Longrightarrow$ The GCF of 54 and 36 is 18.

Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18.

$$54 = 18 \cdot 3$$
$$36 = 18 \cdot 2$$

■

Similar to the example above, we have a process for determining the GCF of expressions.

The Steps for Determining the GCF of Expressions

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

2. Write and align the factors of all expressions. Highlight the common factors.

3. Multiply the shared factors to find the GCF.

In the next two examples, we will see when variables are found in the greatest common factor.

■ **Example 2** Determine the greatest common factor of $27x^3$ and $18x^4$.

Solution:
First, we factor each coefficient into primes and write all variables with exponents in expanded form. Then, we align the factors of all expressions and highlight the common factors.

$$\begin{aligned} 27x^3 &= \boxed{3} \cdot \boxed{3} \cdot 3 \cdot \boxed{x} \cdot \boxed{x} \cdot \boxed{x} \\ 18x^4 &= 2 \cdot \boxed{3} \cdot \boxed{3} \cdot \boxed{x} \cdot \boxed{x} \cdot \boxed{x} \cdot x \end{aligned}$$

To determine the GCF, we multiply the shared factors.

$$\text{GCF} = 3 \cdot 3 \cdot x \cdot x \cdot x = 9x^3$$

$\Longrightarrow$ The GCF of $27x^3$ and $18x^4$ is $9x^3$.

■

■ **Example 3** Determine the GCF of $21x^3$, $9x^2$, and $15x$.

Solution:
Again, we factor each coefficient into primes and write all variables with exponents in expanded form. Aligning the factors of all expressions and highlighting common factors gives us

$$\begin{aligned} 21x^3 &= \boxed{3} \cdot 7 \cdot \boxed{x} \cdot x \cdot x \\ 9x^2 &= \boxed{3} \cdot 3 \cdot \boxed{x} \cdot x \\ 15x &= \boxed{3} \cdot 5 \cdot \boxed{x} \end{aligned}$$

Multiply the shared factors to determine the GCF.

$$\text{GCF} = 3 \cdot x = 3x$$

$\implies$ The GCF of $21x^3$, $9x^2$, and $15x$ is $3x$.

■

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as $2 \cdot 6$ or $3 \cdot 4$), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms.

Remember, we multiply a polynomial by a monomial as follows:

$$\begin{array}{ll} 2(x+7) & \text{factors} \\ 2 \cdot x + 2 \cdot 7 & \\ 2x + 14 & \text{product} \end{array}$$

Now we will start with a product, like $2x + 14$, and end with its factors, $2(x+7)$. To do this we apply the Distributive Property "in reverse."

Factoring the Greatest Common Factor from a Polynomial

1. Find the GCF of all the terms of the polynomial.

2. Rewrite each term as a product, with the GCF as one of two factors.

3. Use the "reverse" Distributive Property to factor the expression.

4. Check, by multiplying the factors.

■ **Example 4** Factor $4x + 12$.

Solution:
We find the GCF of all the terms of the polynomial.

$$\begin{aligned} 4x &= 2 \cdot 2 \cdot x \\ 12 &= 2 \cdot 2 \cdot 3 \end{aligned}$$

$\implies \text{GCF} = 2 \cdot 2 = 4$

Then, we rewrite each term as a product, with the GCF as one of the two factors.

$$4x + 12 = 4 \cdot x + 4 \cdot 3$$

Finally, we use the "reverse" Distributive Property to factor the expression.

$$4x + 12 = 4(x+3)$$

We can check the factorization, by multiplying the factors.

$$4(x+3) = 4 \cdot x + 4 \cdot 3 = 4x + 12 \checkmark$$

■

We use 'factor' as both a noun and a verb.

$$\begin{aligned} Noun: &\quad 7 \textit{ is a factor of } 14. \\ Verb: &\quad \textit{factor } 3 \textit{ from } 3a+3. \end{aligned}$$

■ **Example 5** Factor $5a+5$.

Solution:
We determine the GCF of $5a$ and 5.

$$\begin{aligned} 5a &= 5 \cdot a \\ 5 &= 5 \end{aligned}$$

$\Longrightarrow$ GCF = 5

Rewriting each term as a product, including the GCF, we have

$$5a + 5 = 5 \cdot a + 5 \cdot 1$$

Now we can use the Distributive Property "in reverse" to factor the expression.

$$5a + 5 = 5(a+1)$$

We check, by multiplying the factors to get the original polynomial.

$$5(a+1) = 5 \cdot a + 5 \cdot 1 = 5a + 5 \checkmark$$

■

▪ **Example 6** Factor $5x^3 - 25x^2$.

Solution:
Again, we start by determining the GCF of both terms.

$$\begin{aligned} 5x^3 &= 5 \cdot x \cdot x \cdot x \\ 25x^4 &= 5 \cdot 5 \cdot x \cdot x \end{aligned}$$

$\Longrightarrow$ GCF $= 5 \cdot x \cdot x = 5x^2$

We rewrite each term and factor the expression.

$$\begin{aligned} 5x^3 - 25x^2 &= 5x^2 \cdot x - 5x^2 \cdot 5 \\ &= 5x^2(x-5) \end{aligned}$$

Check.

$$5x^2(x-5) = 5x^2 \cdot x - 5x^2 \cdot 5 = 5x^3 - 25x^2 \checkmark$$

▪

Now we will factor the greatest common factor from a trinomial.

▪ **Example 7** Factor $4y^2 + 24y + 28$.

Solution:
We start by finding the GCF of all three terms.

$$\begin{aligned} 4y^2 &= 2 \cdot 2 \cdot y \cdot y \\ 24y &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot y \\ 28 &= 2 \cdot 2 \cdot 7 \end{aligned}$$

$\Longrightarrow$ GCF $= 2 \cdot 2 = 4$

Using the GCF, we rewrite each term as a product.

$$4y^2 + 24y + 28 = 4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$$

Factoring the expression gives us

$$4y^2 + 24y + 28 = 4(y^2 + 6y + 7)$$

As before, we can check by multiplying.

$$4(y^2 + 6y + 7) = 4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7 = 4y^2 + 24y + 28 \checkmark$$

▪

■ **Example 8** Factor $21x^3 - 9x^2 + 15x$.

Solution:
In a previous example we found the GCF of $21x^3$, $9x^2$, and $15x$ to be $3x$. So, we can rewrite each term of the trinomial with the GCF, $3x$, as one of its two factors and then factor.

$$\begin{aligned} 21x^3 - 9x^2 + 15x &= 3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5 \\ &= 3x(7x^2 - 3x + 5) \end{aligned}$$

Check.

$$3x(7x^2 - 3x + 5) = 3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5 = 21x^3 - 9x^2 + 15x \checkmark$$

■

When determining the GCF of a polynomial with a negative leading coefficient, it is a good habit to also make the GCF negative.

■ **Example 9** Factor $-8y - 24$.

Solution:
Ignoring the signs of the terms, we first find the GCF of $8y$ and 24 is $2 \cdot 2 \cdot 2 = 8$.

$$\begin{aligned} 8y &= \boxed{2} \cdot \boxed{2} \cdot \boxed{2} \cdot \quad y \\ 24 &= \boxed{2} \cdot \boxed{2} \cdot \boxed{2} \cdot 3 \end{aligned}$$

Because the polynomial, $-8y - 24$, has a negative leading coefficient, we use -8 as the GCF.

$$\begin{aligned} -8y - 24 &= -8 \cdot y + (-8) \cdot 3 \\ &= -8(y + 3) \end{aligned}$$

Check.

$$-8(y + 3) = -8 \cdot y + (-8) \cdot 3 = -8y - 24 \checkmark$$

■

Example 10 Factor $-6a^2+36a$.

Solution:
The leading coefficient is negative, so the GCF will be negative.

$$\begin{aligned} 6a^2 &= \boxed{2}\cdot \quad \boxed{3}\cdot \quad \boxed{a}\cdot a \\ 36a &= \boxed{2}\cdot 2\cdot \boxed{3}\cdot 3\cdot \boxed{a} \end{aligned}$$

Because of the negative leading coefficient, the GCF $= -2\cdot 3\cdot a = -6a$.

$$\begin{aligned} -6a^2+36a &= -6a\cdot a+(-6a)\cdot(-6) \\ &= -6a(a-6) \end{aligned}$$

We leave it to the reader to check.

■

Example 11 Factor $5q(q+7)-6(q+7)$.

Solution:
We factor each term into its prime factorization, to determine the GCF.

$$\begin{aligned} 5q(q+7) &= \quad 5\cdot q\cdot \boxed{(q+7)} \\ 6(q+7) &= 2\cdot 3\cdot \quad \boxed{(q+7)} \end{aligned}$$

$\Longrightarrow$ The GCF is the binomial $q+7$.

Factoring the expression, we have

$$5q(q+7)-6(q+7) = (q+7)(5q-6)$$

We leave it to the reader to check.

■

N *While the expression in the last example may not appear to be a polynomial, if we distribute and combine like terms, the result can be written as a polynomial in standard form.*

EXERCISES

Skills Practice (Answers)

For Exercises 1 - 5, determine the greatest common factor for each group of expressions.

1. 8, 18
2. 72, 162
3. $21b^2$, $14b$
4. $30x^2$, $18x^3$
5. $35x^3$, $10x^4$, $5x^5$

For Exercises 6 - 10, factor the polynomial, using the greatest common factor.

6. $8m-8$
7. $6m+9$
8. $8p^2+4p+2$
9. $10q^2+14q+20$
10. $5x^3-15x^2+20x$

Factor by Grouping

When there is no common factor of *all* the terms of an expression, the GCF of all the terms is 1. In this case, instead of using the GCF of all terms, we look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the expression into two parts, with two terms in each part. Then we look for the GCF in each part. If the expression can be factored, we will find a common factor emerges from both parts.

N *Not all expressions can be factored. Just like some numbers are prime, some expressions are prime.*

Factoring by Grouping

1. Check for common factors of *all* terms. When the GCF of all terms is 1, reorder the terms by grouping terms with common factors (CF).

2. Rewrite each group of terms as a product, using the common factors.

3. Factor out the common factor from the expression.

4. Check, by multiplying the factors.

Example 12 Factor $xy+3y+2x+6$.

Solution:
Because the GCF of all four terms is 1, we need to group terms with common factors. Notice the first two terms have a common factor of y and the last two terms have a common factor of 2. So, we separate the first two terms from the last two terms.

$$\underbrace{[xy \ + \ 3y]}_{CF=y} + \underbrace{[2x \ + \ 6]}_{CF=2}$$

Then, we rewrite each group of terms as a product, using the common factors.

$$xy+3y+2x+6 = y(x+3)+2(x+3)$$

Next, we factor out the common factor from the expression.

$$\begin{aligned} xy+3y+2x+6 &= y(x+3)+2(x+3) \\ &= (x+3)(y+2) \end{aligned}$$

Check, by multiplying the factors.

$$(x+3)(y+2) = xy+2x+3y+6 = xy+3y+2x+6\checkmark$$

Example 13 Factor $x^2+3x-2x-6$.

Solution:
Considering the GCF of all four terms is 1, we need to group terms with common factors. Notice the first two terms have a common factor of x and the last two terms have a common factor of -2. So, we separate the first two terms from the last two terms.

$$\underbrace{\left[x^2 \quad + \quad 3x\right]}_{CF=x} + \underbrace{\left[-2x \quad - \quad 6\right]}_{CF=-2}$$

We rewrite each group using the common factors. Then, we factor out the common factor from the expression.

$$\begin{aligned} x^2+3x-2x-6 &= [x(x+3)]+[-2(x+3)] \\ &= x(x+3)+(-2)(x+3) \\ &= (x+3)(x-2) \end{aligned}$$

We leave it to the reader to check.

■

N *When factoring out a common factor which is negative, be careful with the signs.*

EXERCISES

Skills Practice (Answers)

For Exercises 11 - 15, factor the polynomial, by grouping.

11. $b^2 + 5b - 4b - 20$
12. $m^2 + 6m - 12m - 72$
13. $p^2 + 4p + 9p + 36$
14. $x^2 - 5x - 3x + 15$
15. $x^3 + x^2 + x + 1$

Factoring Quadratic Trinomials with Leading Coefficient 1

We have already learned how to multiply binomials using FOIL. Now we will need to "undo" this multiplication by starting with the product and ending with the factors.

Let's look at an example of multiplying binomials to refresh our memory.

$$\begin{array}{ll} (x+2)(x+3) & \text{factors} \\ F \quad O \quad I \quad L & \\ x^2+3x+2x+6 & \\ x^2+5x+6 & \text{product} \end{array}$$

To factor the trinomial, x^2+5x+6, we will need to think about where each of the terms in the trinomial came from.

The *first term* of the trinomial, x^2, came from multiplying the first term in each binomial. Thus, each binomial must start with an x.

$$x^2+5x+6 = (x \qquad)(x \qquad)$$

The *last term* in the trinomial, 6, came from multiplying the last term in each binomial. So, the last terms must multiply to 6. The factors of 6 could be 1 and 6, 2 and 3, -1 and -6, or -2 and -3. Due to the fact that we have more than one factor pair for the last term, we cannot just write down the last term in each binomial.

Instead, we consider the *middle term* of the trinomial, $5x$. The middle term came from adding the product of the outer terms and the product of the inner terms of the binomials, so the coefficient of the middle term of the trinomial must be the sum of the constant terms (last terms) of the binomials.

We now know the last terms of each binomial must have a product of 6 and will need to sum to 5. We will test all possibilities and summarize the results in **Table A.18** below. The table will be very helpful when we work with numbers that can be factored in many different ways.

Factors of 6	Sum of Factors
1, 6	$1+6=7$
2, 3	$2+3=5$
$-1, -6$	$-1-6=-7$
$-2, -3$	$-2-3=-5$

Table A.18: The factor pairs of 6 and their corresponding sums.

We see that 2 and 3 is the factor pair that multiplies to 6 and adds to 5. Therefore, x^2+5x+6 will factor as

$$\underbrace{x^2+5x+6}_{\text{product}} = \underbrace{(x+2)(x+3)}_{\text{factors}}$$

We can check this factorization by multiplying and confirming the product matches the original expression.

The process just described is summarized below for factoring any trinomial where the leading coefficient is 1.

Factoring Trinomials of the Form $x^2 + bx + c$

1. Write the factors as two binomials with first terms x. $(x \quad)(x \quad)$
2. Find two numbers m and n such that their product is c and they sum to b. $m \cdot n = c$, $m + n = b$
3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
4. Check by multiplying the factors.

▪ **Example 14** Factor $x^2 + 7x + 12$.

Solution:
The trinomial, $x^2 + 7x + 12$, is of the form $x^2 + bx + c$. So, we write the factors as two binomials with first terms x.

$$x^2 + 7x + 12 = (x \qquad)(x \qquad)$$

Next, we find two numbers that multiply to $c = 12$ and add to $b = 7$.

Factors of 12	Sum of Factors
1, 12	$1 + 12 = 13$
2, 6	$2 + 6 = 8$
3, 4	$3 + 4 = 7$ ✓
−1, −12	$-1 - 12 = -13$
−2, −6	$-2 - 6 = -8$
−3, −4	$-3 - 4 = -7$

Table A.19: The factor pairs of 12 and their corresponding sums.

As the factor pair 3 and 4 multiply to 12 and add to 7, we use 3 and 4 as the last terms of the binomials. So,

$$x^2 + 7x + 12 = (x+3)(x+4)$$

We can check by multiplying the factors.

$$\begin{aligned}(x+3)(x+4) &= x^2 + 4x + 3x + 12 \\ &= x^2 + 7x + 12 \checkmark\end{aligned}$$

▪

■ **Example 15** Factor $u^2 + 11u + 24$.

Solution:
Again, the trinomial, $u^2 + 11u + 24$, is in standard form and has a leading coefficient of 1, but the variable is u. Thus, the factors are two binomials, each with a first term of u.

$$u^2 + 11u + 24 = (u \qquad)(u \qquad)$$

Now we need to find two numbers that multiply to 24 and add to 11.

Factors of 24	Sum of Factors
1, 24	$1 + 24 = 25$
2, 12	$2 + 12 = 14$
3, 8	$3 + 8 = 11$ ✓
4, 6	$4 + 6 = 10$
−1, −24	$-1 - 24 = -25$
−2, −12	$-2 - 12 = -14$
−3, −8	$-3 - 8 = -11$
−4, −6	$-4 - 6 = -10$

Table A.20: The factor pairs of 12 and their corresponding sums.

As the factor pair 3 and 8 satisfy the required product and sum, we use them as the last terms of the binomials.

$$u^2 + 11u + 24 = (u + 3)(u + 8)$$

Check.

$$\begin{aligned}(u+3)(u+8) &= u^2 + 8u + 3u + 24 \\ &= u^2 + 11u + 24 \checkmark\end{aligned}$$

■

In the examples thus far, all terms in the trinomial were positive, and both factors of the factor pair were positive.

Let's now look at a trinomial where only the middle term is negative.

■ **Example 16** Factor $t^2 - 11t + 28$.

Solution:
We write the factors as two binomials with first terms t to produce the first term of the trinomial, t^2.

$$t^2 - 11t + 28 = (t \qquad)(t \qquad)$$

Next, we find two numbers that multiply to $c = 28$ and add to $b = -11$.

Factors of 28	Sum of Factors
1, 28	$1+28 = 29$
2, 14	$2+14 = 16$
4, 7	$4+7 = 11$
$-1, -28$	$-1-28 = -29$
$-2, -14$	$-2-14 = -16$
$-4, -7$	$-4-7 = -11$ ✓

Table A.21: The factor pairs of 28 and their corresponding sums.

As shown in **Table A.21**, the factor pair -4 and -7 multiplies to 28 and adds to -11, so

$$t^2 - 11t + 28 = (t-4)(t-7)$$

Check.

$$\begin{aligned}(t-4)(t-7) &= t^2 - 7t - 4t + 28 \\ &= t^2 - 11t + 28 \checkmark\end{aligned}$$

■

In the last example, when the middle term was the only negative term, the factor pair, m and n, were both negative.

Next, we consider an example where only the last term of the trinomial is negative.

■ **Example 17** Factor $z^2 + 4z - 5$.

Solution:
With the first term of the trinomial being z^2, the factors will be two binomials with first terms z.

$$z^2 + 4z - 5 = (z \qquad)(z \qquad)$$

To get a negative last term in the trinomial, we must multiply one positive number and one negative number, such that the numbers multiply to $c = -5$ and add to $b = 4$.

Factors of -5	Sum of Factors
$1, -5$	$1+(-5) = -4$
$-1, 5$	$-1+5 = 4$ ✓

Table A.22: The factor pairs of -5 and their corresponding sums.

From **Table A.22**, we see that the factor pair -1 and 5 will be used as the last terms of the binomials. Therefore,

$$z^2 + 4z - 5 = (z-1)(z+5)$$

Check.

$$\begin{aligned}(z-1)(z+5) &= z^2+5z-1z-5\\ &= z^2+4z-5\ \checkmark\end{aligned}$$

■

In the last example, when the last term was the only negative term, the factor pair had one positive factor and one negative factor.

Last, let's consider an example where both the middle and last terms of the trinomial are negative.

■ **Example 18** Factor z^2-4z-5.

Solution:
Again, the factors will be two binomials with first terms z.

$$z^2-4z-5 = (z\qquad)(z\qquad)$$

This time, we need two numbers that multiply to $c=-5$ and that add to $b=-4$.

Factors of -5	Sum of Factors
$1,-5$	$1+(-5)=-4\ \checkmark$
$-1,5$	$-1+5=4$

Table A.23: The factor pairs of -5 and their corresponding sums.

As the factor pair 1 and -5 satisfy the required product and sum, we use them as the last terms of the binomials.

$$z^2-4z-5=(z+1)(z-5)$$

We leave it to the reader to check the factorization.

■

In the last example, when both the middle and last terms were negative, the factor pair had one positive factor and one negative factor.

N *Notice that the factors of z^2-4z-5 are very similar to the factors of z^2+4z-5. It is important to make sure you choose the factor pair that results in the correct sign of the middle term.*

In general, when factoring a trinomial of the form $x^2 + bx + c = (x+m)(x+n)$,

- If b and c are both positive, then both m and n are positive numbers.
- If b is negative and c is positive, then both m and n are negative numbers.
- If b is any number and c is negative, then one factor, m or n, is a negative number and the other is a positive number.

▪ **Example 19** Factor $q^2 - 2q - 15$.

Solution:
We know the factors will be two binomials with first terms q.

$$q^2 - 2q - 15 = (q \qquad)(q \qquad)$$

We also know that we need two numbers that multiply to $c = -15$ and that add to $b = -2$. Seeing as c is negative, the factor pair will have one positive number and one negative number.

Factors of -15	Sum of Factors
$1, -15$	$1 + (-15) = -14$
$-1, 15$	$-1 + 15 = 14$
$3, -5$	$3 + (-5) = -2$ ✓
$-3, 5$	$-3 + 5 = 2$

Table A.24: The factor pairs of -15 and their corresponding sums.

From **Table A.24**, we see the factorization will be

$$q^2 - 2q - 15 = (q+3)(q-5)$$

We leave it to the reader to check the factorization.

▪

▪ **Example 20** Factor $y^2 - 6y + 15$.

Solution:
We know the factors will be two binomials with first terms y.

$$y^2 - 6y + 15 = (y \qquad)(y \qquad)$$

We also know we need two numbers that multiply to $c = 15$ and that add to $b = -6$. Because b is negative and c is positive, both factors in our factor pair will be negative.

Negative Factors of 15	Sum of Factors
$-1, -15$	$-1 + (-15) = -16$
$-3, -5$	$-3 + (-5) = -8$

Table A.25: The negative factor pairs of 15 and their corresponding sums.

As shown in **Table A.25**, none of the factors add to -6; therefore, the polynomial is prime and cannot be factored under the integers.

N *If we had not considered the signs of b and c and instead looked at all factor pairs of 15, we would have had the two additional factor pairs of* 1 *and* 15 *and* 3 *and* 5*. Neither of these factor pairs add to* -6*, thus confirming the trinomial is prime and cannot be factored under the integers.*

■

■ **Example 21** Factor $2x + x^2 - 48$.

Solution:
First, we put the terms of the trinomial in decreasing degree order to have a trinomial of the form $x^2 + bx + c$.

$$x^2 + 2x - 48$$

Then, we know the factors will be two binomials with first terms x.

$$x^2 + 2x - 48 = (x \qquad)(x \qquad)$$

As c is negative, we know we need two numbers with opposite signs that multiply to $c = -48$, and that add to $b = 2$.

Factors of -48	Sum of Factors
$-1, 48$	$-1 + 48 = 47$
$-2, 24$	$-2 + 24 = 22$
$-3, 16$	$-3 + 16 = 13$
$-4, 12$	$-4 + 12 = 8$
$-6, 8$	$-6 + 8 = 2$ ✓
$1, -48$	$1 - 48 = -47$
$2, -24$	$2 - 24 = -22$
$3, -16$	$3 - 16 = -13$
$4, -12$	$4 - 12 = -8$
$6, -8$	$6 - 8 = -2$

Table A.26: The opposite factor pairs of -48 and their corresponding sums.

As shown in **Table A.26**, we can use -6 and 8 as the last terms of the binomials.

$$2x + x^2 - 48 = x^2 + 2x - 48 = (x - 6)(x + 8)$$

We leave it to the reader to check the factorization.

■

EXERCISES

Skills Practice (Answers)

For Exercises 16 - 39, factor the trinomial of the form $x^2 + bx + c$, if possible.

16. $x^2 + 4x + 3$
17. $y^2 + 8y + 7$
18. $m^2 + 12m + 11$
19. $a^2 + 9a + 20$
20. $m^2 + 7m + 12$
21. $p^2 + 11p + 30$
22. $x^2 - 8x + 12$
23. $q^2 - 13q + 36$
24. $y^2 - 18y + 45$
25. $m^2 - 13m + 30$
26. $x^2 - 8x + 7$
27. $y^2 - 5y + 6$
28. $p^2 + 5p - 6$
29. $n^2 + 6n - 7$
30. $y^2 - 6y - 7$
31. $v^2 - 2v - 3$
32. $x^2 - x - 12$
33. $r^2 - 2r - 8$
34. $x^2 + x + 5$
35. $x^2 - 3x - 9$
36. $8 - 6x + x^2$
37. $-11 - 10x + x^2$
38. $w^2 + 4w - 32$
39. $k^2 + 34k + 120$

Factor Quadratic Trinomials with Leading Coefficient Other than 1

Let's summarize where we are with factoring polynomials. So far, we used three methods of factoring: factoring out the GCF, factoring by grouping, and factoring a trinomial of the form x^2+bx+c, by "undoing" FOIL. More methods will follow as we continue in this section.

As we learn more methods of factoring, it will help to organize the factoring methods into a strategy that can guide us to use the correct method.

Choose a Strategy for Factoring Polynomials Completely

1. Is there a greatest common factor?
   - If yes, then factor it out and continue to Step 2 for the non-GCF factor.
   - If no, then go to Step 2.

2. Do we have a binomial, trinomial, or polynomial with more than three terms?
   - If we have a binomial, then right now we have no method to factor it. We will discuss this later in this section.
   - If we have a trinomial of the form x^2+bx+c, then factor as $(x+m)(x+n)$, where $b=m+n$ and $c=m\cdot n$.
   - If we have a polynomial with more than three terms, then we use the grouping method to factor.

Once the factorization is complete, we check our work by multiplying.

A polynomial is factored completely if, other than monomials, all of its factors are prime.

Now that we have organized what we have covered so far, we are ready to factor trinomials whose leading coefficient is not 1. In other words, we will now factor trinomials of the form ax^2+bx+c.

Remember to always check for a GCF first. Sometimes after we factor out the GCF, the leading coefficient of the trinomial factor becomes 1, and we can factor the trinomial by the methods previously discussed.

■ **Example 22** Factor $2n^2-8n-42$ completely.

Solution:
First, we notice the trinomial has a GCF of 2. Factoring the GCF out, we have

$$2n^2-8n-42=2(n^2-4n-21)$$

Inside the parentheses, we are left with a trinomial whose leading coefficient is 1, so we "undo" FOIL.

$$2(n^2-4n-21)=2(n\qquad)(n\qquad)$$

Factors of -21	Sum of Factors
$-1, 21$	$-1+21=20$
$-3, 7$	$-3+7=4$
$1, -21$	$1+(-21)=-20$
$3, -7$	$3+(-7)=-4$ ✓

Table A.27: The factor pairs for -21 and their corresponding sums.

We use the factor pair 3 and -7 as the last terms of the binomials and have

$$2n^2-8n-42=2(n+3)(n-7)$$

Check.

$$\begin{aligned}2(n+3)(n-7)&=2(n^2-7n+3n-21)\\&=2(n^2-4n-21)\\&=2n^2-8n-42\ \checkmark\end{aligned}$$

■

■ **Example 23** Factor $4y^2-36y+56$ completely.

Solution:
First, we notice the trinomial has a GCF of 4. Factoring the GCF out, we have

$$4y^2-36y+56=4(y^2-9y+14)$$

Inside the parentheses, we are left with a trinomial whose coefficient is 1, so we “undo” FOIL.

$$4y^2-36y+56=4(y\qquad)(y\qquad)$$

Negative Factors of 14	Sum of Factors
-1, -14	$-1+(-14)=-15$
-2, -7	$-2+(-7)=-9$ ✓

Table A.28: The negative factor pairs of 14 and their corresponding sums.

We use the factor pair -2 and -7 as the last terms of the binomials and have

$$4y^2-36y+56=4(y^2-9y+14)=4(y-2)(y-7)$$

Check.

$$\begin{aligned}4(y-2)(y-7)&=4(y^2-7y-2y+14)\\&=4(y^2-9y+14)\\&=4y^2-36y+56\ \checkmark\end{aligned}$$

■

So far when we have factored a trinomial of the form ax^2+bx+c, we have been able to factor out a as the GCF. We were then left with a trinomial of the form x^2+bx+c, which we developed a method for factoring previously. When the leading coefficient is not 1 and there is no GCF, there are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2+5x+2$.

From our earlier work, we expect this will factor into two binomials.

$$3x^2+5x+2 = (\qquad)(\qquad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only integer factors of $3x^2$ are $3x$ and $1x$.

$$\overset{3x,\,1x}{3x^2}+5x+2 = (3x\qquad)(x\qquad)$$

We know the last terms of the binomials will multiply to 2. Due to the fact that this trinomial has all positive terms, we only need to consider positive factors of the last term. In case you don't remember this fact, we will consider all the factors of the last term. The factors of 2 are 1 and 2 or -1 and -2, but we have four cases to consider as it will make a difference which binomial term gets which number, as the first terms of the binomial are not both x.

$$\overset{3x,\,1x}{3x^2}+5x+\overset{1,2}{2} = (3x+1)(x+2)$$

$$\overset{3x,\,1x}{3x^2}+5x+\overset{2,1}{2} = (3x+2)(x+1)$$

$$\overset{3x,\,1x}{3x^2}+5x+\overset{-1,-2}{2} = (3x-1)(x-2)$$

$$\overset{3x,\,1x}{3x^2}+5x+\overset{-2,-1}{2} = (3x-2)(x-1)$$

To decide which factorization is correct, if any, we multiply the outer terms and inner terms and determine the sum of these products.

	$(3x+1)(x+2)$	$(3x+2)(x+1)$	$(3x-1)(x-2)$	$(3x-2)(x-1)$
Outer:	$(3x)(2)=6x$	$(3x)(1)=3x$	$(3x)(-2)=-6x$	$(3x)(-1)=-3x$
Inner:	$(1)(x)=x$	$(2)(x)=2x$	$(-1)(x)=-x$	$(-2)(x)=-2x$
Sum:	$7x$	$5x$ ✓	$-7x$	$-5x$

Seeing as the middle term of the trinomial is $5x$, the second case above is the correct factorization. Our result of the factoring is

$$3x^2+5x+2 = (3x+2)(x+1)$$

We can FOIL to check.

$$\begin{aligned}(3x+2)(x+1) &= 3x^2+3x+2x+2\\ &= 3x^2+5x+2 \checkmark\end{aligned}$$

Using Trial and Error to Factor Trinomials of the Form ax^2+bx+c

1. Write the trinomial in descending order of degrees.
2. Determine all the factor pairs of the first term.
3. Determine all the factor pairs of the third term.
4. For all the possible combinations of these factors, compute the outer and inner products. Then compute the sum of the outer and inner products for each combination until the sum matches the middle term of the given trinomial. In other words, test all the possible combinations of the factors until the correct product is found.

Example 24 Factor $3y^2+22y+7$ completely, using the Trial and Error method.

Solution:
The trinomial is already written in descending order, and there is no GCF.

The only factors of $3y^2$ are $3y$ and $1y$. With only one factor pair, we know the factors of the trinomial will be two binomials with first terms $3y$ and y.

$$3y^2+22y+7 = (3y\qquad)(y\qquad)$$

The only factors of 7 are 1 and 7 or -1 and -7.

$3y^2+22y+7$			
Possible Factors	**Outer Product**	**Inner Product**	**Sum**
$(3y+7)(y+1)$	$(3y)(1)=3y$	$7y$	$3y+7y=10y$
$(3y+1)(y+7)$	$(3y)(7)=21y$	y	$21y+y=22y \checkmark$
$(3y-7)(y-1)$	$(3y)(-1)=-3y$	$-7y$	$-3y+(-7y)=-10y$
$(3y-1)(y-7)$	$(3y)(-7)=-21y$	$-y$	$-21y+(-y)=-22y$

Table A.29: The possible factors of $3y^2+22y+7$, with their corresponding outer product, inner product, and sum of the outer and inner product.

N *We only really need to consider the first two rows of* ***Table A.29****, as all terms of $3y^2+22y+7$ are positive and so both factors of 7 should be positive.*

Given that the middle term of the trinomial is $22y$, the second case in **Table A.29** is the correct factorization. Our result of the factoring is

$$3y^2+22y+7 = (3y+1)(y+7)$$

■

Example 25 Factor $6b^2 + 13b - 5$ completely, using the Trial and Error method.

Solution:
The trinomial is already in descending order with no GCF.

Here, the possible factors of the first term, $6b^2$, are b and $6b$ or $2b$ and $3b$.

As the last term, -5, is negative, its factors must have opposite signs. The possible factors are 1 and -5 or -1 and 5. We consider all the combinations of factors in **Table A.30** below.

$6b^2 + 13b - 5$	
Possible Factors	**Product**
$(b-1)(6b+5)$	$6b^2 - b - 5$
$(b-5)(6b+1)$	$6b^2 - 29b - 5$
$(b+1)(6b-5)$	$6b^2 + b - 5$
$(b+5)(6b-1)$	$6b^2 + 29b - 5$
$(2b-1)(3b+5)$	$6b^2 + 7b - 5$
$(2b-5)(3b+1)$	$6b^2 - 13b - 5$
$(2b+1)(3b-5)$	$6b^2 - 7b - 5$
$(2b+5)(3b-1)$	$6b^2 + 13b - 5$ ✓

Table A.30: The possible factors of $6b^2 + 13b - 5$ with their corresponding products.

The correct factors are those whose product is the original trinomial. Thus,

$$6b^2 + 13b - 5 = (2b+5)(3b-1)$$

■

Another way to factor trinomials of the form $ax^2 + bx + c$ is called the 'ac' method. (The 'ac' method is sometimes called the grouping method.) The 'ac' method is actually an extension of the methods used in the examples to factor trinomials with a leading coefficient of one. This method is very structured (that is, step-by-step), and it always works.

Using the 'ac' Method to Factor Trinomials of the Form ax^2+bx+c

1. Write the trinomial in descending order of degrees.

2. Compute the product of a and c.

3. List all factor pairs of the product ac.

4. Determine which factor pair, m and n, adds to b.

5. Split the middle term of the trinomial, bx, using m and n.

$$\begin{aligned} ax^2+bx+c \\ ax^2+mx+nx+c \end{aligned}$$

6. Then factor $ax^2+mx+nx+c$ by grouping.

Example 26 Factor $6x^2+7x+2$ completely, using the 'ac' method.

Solution:
The trinomial is already written in descending order, and there is no GCF.

We know $a=6$ and $c=2$, so the product $ac=(6)(2)=12$.
Because the middle term of the trinomial, $7x$, is positive, we know that both factors of 12 must be positive.

$$\begin{aligned} 1,12 \\ 2,6 \\ 3,4 \end{aligned}$$

Now we need to determine which factor pair has a sum of 7, as $b=7$.

Factors of 12	Sum of Factors
$1,12$	$1+12=13$
$2,6$	$2+6=8$
$3,4$	$3+4=7$ ✓

Table A.31: The positive factors of 12 and their corresponding sums.

We will split the middle term of the trinomial, $7x$, using $m=3$ and $n=4$.

$$6x^2+7x+2=6x^2+3x+4x+2$$

Last, we factor $6x^2+3x+4x+2$ by grouping.

$$\begin{aligned} 6x^2+7x+2 &= \underbrace{6x^2+3x}\underbrace{+4x+2} \\ &= 3x(2x+1)+2(2x+1) \\ &= (2x+1)(3x+2) \end{aligned}$$

■

Example 27 Factor $2x^2+6x+5$ completely, using the 'ac' method.

Solution:
The trinomial is already in descending order, and there is no GCF.

We know $a = 2$ and $c = 5$, so the product $ac = 10$.
Because the middle term of the trinomial, $6x$, is positive, we know that both factors of 10 must be positive.

$$\begin{gathered} 1,10 \\ 2,5 \end{gathered}$$

Now we need to determine which factor pair add to $b = 6$.

Factors of 10	**Sum of Factors**
$1,10$	$1+10=11$
$2,5$	$2+5=7$

Table A.32: The positive factors of 10 and their corresponding sums.

As shown in **Table A.32**, there are no factors that both multiply to 10 and add to 6. Therefore, the polynomial is prime and cannot be factored under the integers.

■

In the next examples, we will factor each trinomial completely without being told which method to use. While one method may be illustrated, we can factor using any method discussed and end with the same result. If the trinomial is prime, it is prime no matter the method used for factoring.

Example 28 Factor $10y^2-55y+70$ completely.

Solution:
We begin by looking for a greatest common factor. The GCF of the three terms is 5. We factor out the GCF, being careful to keep the factor of 5 all the way through the solution.

$$10y^2-55y+70 = 5(2y^2-11y+14)$$

Now we focus on factoring the resulting trinomial inside the parentheses. We will use the 'ac' method to factor $2y^2-11y+14$.

We know $a = 2$ and $c = 14$, so the product $ac = 28$. Because the middle term of the trinomial, $-11y$, is negative, both factors of 28 must be negative.

$$\begin{gathered} -1,-28 \\ -2,-14 \\ -4,-7 \end{gathered}$$

We next determine which factor pair adds to $b = -11$.

Negative Factors of 28	Sum of Factors
$-1, -28$	$-1-28 = -29$
$-2, -14$	$-2-14 = -16$
$-4, -7$	$-4-7 = -11$ ✓

Table A.33: The negative factors of 28 and their corresponding sums.

We will split the middle term of the trinomial, $-11y$, using $m = -4$ and $n = -7$ and factor by grouping.

$$\begin{aligned} 2y^2 - 11y + 14 &= 2y^2 - 4y - 7y + 14 \\ &= \underbrace{2y^2 - 4y}\underbrace{-7y + 14} \\ &= 2y(y-2) + (-7)(y-2) \\ &= (2y-7)(y-2) \end{aligned}$$

So the final factorization of $10y^2 - 55y + 70 = 5(2y-7)(y-2)$.

■

■ **Example 29** Factor $4u^3 + 16u^2 - 20u$ completely.

Solution:
First, we notice the trinomial has a GCF of $4u$. Factoring out the GCF, we have

$$4u^3 + 16u^2 - 20u = 4u(u^2 + 4u - 5)$$

Inside the parentheses, we are left with a trinomial whose coefficient is 1, so we "undo" FOIL.

$$4u^3 + 16u^2 - 20u = 4u(u \qquad)(u \qquad)$$

Because the last term of $u^2 + 4u - 5$ is negative, the factors of -5 have opposite signs.

Factors of -5	Sum of Factors
-1, 5	$-1+5 = 4$ ✓
1, -5	$1+(-5) = -4$

Table A.34: The factor pairs of -5 and their corresponding sums.

We use the factor pair -1 and 5 as the last terms of the binomials and have

$$4u^3 + 16u^2 - 20u = 4u(u^2 + 4u - 5) = 4u(u-1)(u+5)$$

■

■ **Example 30** Factor $10y^4 + 55y^3 + 60y^2$ completely.

Solution:
We notice the greatest common factor of the trinomial is $5y^2$, and we factor it out first.

$$10y^4 + 55y^3 + 60y^2 = 5y^2(2y^2 + 11y + 12)$$

Now we focus on factoring the resulting trinomial inside the parentheses. Because all terms of $2y^2 + 11y + 12$ are positive, the factors of 12 will both be positive (1 and 12 or 2 and 6 or 3 and 4). We will use trial and error to consider all the combinations of factors.

$2y^2 + 11y + 12$	
Possible Factors	**Product**
$(y+1)(2y+12)$	$2y^2 + 13y + 12$
$(y+12)(2y+1)$	$2y^2 + 25y + 12$
$(y+2)(2y+6)$	$2y^2 + 10y + 12$
$(y+6)(2y+2)$	$2y^2 + 14y + 12$
$(y+3)(2y+4)$	$2y^2 + 10y + 12$
$(y+4)(2y+3)$	$2y^2 + 11y + 12$ ✓

Table A.35: The possible factors of $2y^2 + 11y + 12$ and their corresponding products.

The correct factors, $(y+4)$ and $(2y+3)$, are those whose product is $2y^2 + 11y + 12$.

Thus, the final factorization of $10y^4 + 55y^3 + 60y^2 = 5y^2(y+4)(2y+3)$

■

EXERCISES

Skills Practice (Answers)

For Exercises 40 - 49, factor the trinomial completely, if possible.

40. $5x^2 + 35x + 30$
41. $12s^2 + 24s + 12$
42. $q^3 - 5q^2 - 24q$
43. $3m^3 - 21m^2 + 30m$
44. $5x^4 + 10x^3 - 75x^2$
45. $4w^2 - 5w + 1$
46. $10y^2 - 53y - 11$
47. $3x^2 + 5x + 4$
48. $4p^2 + 17p - 15$
49. $2x^2 - x + 7$

Factoring Differences of Squares

Thus far we have factored polynomials with a greatest common factor and trinomials with or without a leading coefficient of 1. Now we will turn our attention to factoring a special type of binomial, differences of squares.

Recall the definition of the difference of squares from the Introduction to Algebraic Expressions section.

Definition

If a and b are real numbers,

$$(a-b)(a+b) = a^2 - b^2$$

The product is called a **difference of squares**.

When you multiply conjugate binomials, the middle terms of the product add to 0, and all you have left is a binomial, which is the differences of squares. For example,

$$(3x-4)(3x+4) = 9x^2 - 16$$

To factor a difference of squares, we will use the product pattern “in reverse”. The resulting factorization will be a product of conjugates.

Factoring Differences of Squares

1. Verify the given expression is a binomial, with a subtraction sign separating the two terms.
2. Identify each term of the difference is a perfect square.
3. Write each term as a square.
4. Factor the binomial as the product of conjugates.

$$(a)^2 - (b)^2 = (a-b)(a+b)$$

Remember, "difference" refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

It is important to remember that sums of squares do not factor into a product of binomials. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime.

Don't forget that 1 is a perfect square.

▪ **Example 31** Factor $x^2 - 4$ completely.

Solution:
The binomial has no GCF, but it is a difference of two perfect squares. Thus, we write the terms of the binomial as squares,

$$x^2 - 4 = (x)^2 - (2)^2$$

and then write the product of conjugates.

$$= (x-2)(x+2)$$

To check, we multiply the factors.

$$(x-2)(x+2) = x^2 - 4 \checkmark$$

▪

▪ **Example 32** Factor $64y^2 - 1$ completely.

Solution:
The binomial has no GCF, but it is a difference of two perfect squares. Again, we write the terms of the binomial as squares,

$$64y^2 - 1 = (8y)^2 - (1)^2$$

and then write the product of conjugates.

$$= (8y-1)(8y+1)$$

▪

▪ **Example 33** Factor $121x^2 - 49$ completely.

Solution:
The binomial has no GCF, but it is a difference of two perfect squares. So, we write the terms of the binomial as squares,

$$121x^2 - 49 = (11x)^2 - (7)^2$$

and then write the product of conjugates.

$$= (11x-7)(11x+7)$$

▪

The binomial in the next example may look "backwards," but it's still the difference of squares.

▪ **Example 34** Factor $100 - h^2$ completely.

Solution:
The binomial has no GCF, but it is a difference of two perfect squares. We write the terms of the binomial as squares,

$$100 - h^2 = (10)^2 - (h)^2$$

and then write the product of conjugates.

$$= (10 - h)(10 + h)$$

Be careful not to rewrite the original expression as $h^2 - 100$.

Factor $h^2 - 100$ *on your own and then notice how the result differs from* $(10 - h)(10 + h)$.

▪

▪ **Example 35** Factor $x^4 - 16$ completely.

Solution:
The binomial has no GCF, but it is a difference of two perfect squares, as $x^4 = \left(x^2\right)^2$. We write the terms of the binomial as squares,

$$x^4 - 16 = \left(x^2\right)^2 - (4)^2$$

and then write the product of conjugates.

$$= \left(x^2 - 4\right)\left(x^2 + 4\right)$$

Notice the first binomial, $x^2 - 4$, is also a difference of squares, so we will factor it as the product of conjugates. The last factor, $x^2 + 4$, is the sum of squares and cannot be factored under the integers.

$$\begin{aligned}(x^2 - 4)(x^2 + 4) &= \left((x)^2 - (2)^2\right)\left(x^2 + 4\right) \\ &= (x - 2)(x + 2)(x^2 + 4)\end{aligned}$$

Therefore, the final factorization of $x^4 - 16 = (x - 2)(x + 2)\left(x^2 + 4\right)$.

▪

As always, we should look for a common factor first, whenever we have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and we will not recognize the perfect squares until we factor out the GCF.

▪ **Example 36** Factor $8x^2 - 98$ completely.

Solution:
We notice there is a GCF of 2, so we factor it out first.

$$8x^2 - 98 = 2(4x^2 - 49)$$

Now we have a difference of squares as our remaining binomial. Proceeding as before,

$$\begin{aligned} 2(4x^2 - 49) &= 2\left((2x)^2 - (7)^2\right) \\ &= 2(2x-7)(2x+7). \end{aligned}$$

So, the final factorization of $8x^2 - 98 = 2(2x-7)(2x+7)$.

▪

▪ **Example 37** Factor $6x^2 + 96$ completely.

Solution:
We factor out the GCF of 6 first.

$$6x^2 + 96 = 6(x^2 + 16)$$

We have a *sum* of squares for our remaining polynomial, and *sums* of squares do not factor. Thus, we have completely factored the given binomial and have

$$6x^2 + 96 = 6(x^2 + 16)$$

▪

EXERCISES

Skills Practice (Answers)

For Exercises 50 - 64, factor the binomial completely.

50. $x^2 - 16$
51. $1 - 25x^2$
52. $169q^2 - 1$
53. $121x^2 - 144$
54. $49x^2 - 81$
55. $4 - 49x^2$
56. $121 - 25s^2$
57. $16z^4 - 1$
58. $5q^2 - 45$
59. $98r^3 - 72r$
60. $24p^2 + 54$
61. $20b^2 + 140$
62. $27q^2 - 3$
63. $4p^2 - 100$
64. $8p^2 + 2$

A.5 Solving Quadratic Equations

We have already solved linear equations of the form $ax+by=c$. Quadratic equations are equations which involve polynomials of degree 2. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \qquad 3y^2+4y=10 \qquad 64u^2-81=0 \qquad n(n+1)=42$$

The last equation doesn't appear to involve a polynomial of degree 2, but when we expand the expression on the left-hand side we have $n^2+n=42$.

Definition

An equation of the form

$$ax^2+bx+c=0$$

is called a **quadratic equation** in standard form, where a, b, and c are real numbers and $a \neq 0$. ▪

To solve quadratic equations, we need methods different than the ones we used when solving linear equations.

Solving Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property.

Zero Product Property

If the product of two quantities is zero, then it must be that at least one of the quantities is zero.

So, if $a \cdot b = 0$, then either $a=0$ or $b=0$ or both a and b equal zero.

▪ **Example 1** Solve $(x+1)(x-4)=0$ for x.

Solution:
Considering the product equals zero, we know at least one factor must equal zero.

$$(x+1)(x-4)=0$$

$$x+1=0 \quad \text{or} \quad x-4=0$$

We solve each linear equation for x.

$$\begin{aligned} x+1&=0 \\ x&=-1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x-4&=0 \\ x&=4 \end{aligned}$$

Thus, the solutions to $(x+1)(x-4)=0$ are $x=-1$ or $x=4$.

We check our solutions by substituting each, separately, into the *original* quadratic equation.

$$\begin{aligned} &\underline{x=-1} \\ (x+1)(x-4)&=0 \\ (-1+1)(-1-4)&\stackrel{?}{=}0 \\ (0)(-5)&\stackrel{?}{=}0 \\ 0&=0\,\checkmark \end{aligned} \qquad \begin{aligned} &\underline{x=4} \\ (x+1)(x-4)&=0 \\ (4+1)(4-4)&\stackrel{?}{=}0 \\ (5)(0)&\stackrel{?}{=}0 \\ 0&=0\,\checkmark \end{aligned}$$

■

■ **Example 2** Solve $(5n-2)(6n-1)=0$ for n.

Solution:
We use the Zero Product Property to set each factor equal to zero.

$$5n-2=0 \quad \text{or} \quad 6n-1=0$$

Then, we solve the linear equations.

$$\begin{aligned} 5n-2&=0 \\ 5n&=2 \\ n&=\frac{2}{5} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 6n-1&=0 \\ 6n&=1 \\ n&=\frac{1}{6} \end{aligned}$$

Therefore, the solutions to $(5n-2)(6n-1)=0$ are $n=\frac{2}{5}$ or $n=\frac{1}{6}$.

We leave it to the reader to check the solutions in the *original* quadratic equation.

■

■ **Example 3** Solve $3p(10p+7)=0$ for p.

Solution:
Again, we use the Zero Product Property to set each factor equal to zero.

$$3p=0 \quad \text{or} \quad 10p+7=0$$

Solving the linear equations gives us

$$\begin{aligned} 3p&=0 \\ p&=\frac{0}{3} \\ p&=0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 10p+7&=0 \\ 10p&=-7 \\ p&=-\frac{7}{10} \end{aligned}$$

Hence, the solutions to $3p(10p+7)=0$ are $p=0$ or $p=-\frac{7}{10}$.

■

In the examples thus far, while each solution made just one factor equal zero, the resulting product of both factors was zero for both solutions.

■ **Example 4** Solve $(y-8)^2 = 0$ for y.

Solution:
While it may appear that there is only one factor in this example, remember, however, that $(y-8)^2$ means $(y-8)(y-8)$. Thus, we rewrite the left-hand side of the equation as a product.

$$(y-8)(y-8) = 0$$

Using the Zero Product Property and setting each factor equal to zero, we have

$$y-8 = 0 \quad \text{or} \quad y-8 = 0$$

We then solve the linear equations.

$$\begin{aligned} y-8 &= 0 \\ y &= 8 \end{aligned} \qquad \text{or} \qquad \begin{aligned} y-8 &= 0 \\ y &= 8 \end{aligned}$$

The solution to $(y-8)^2 = 0$ is $y = 8$.

Here, we have a solution that repeats. When a solution repeats, we call it a **double root**.

■

Up to this point, each quadratic equation has been written as the product of two factors set equal to zero, which allowed us to use the Zero Product Property. The Zero Product Product can only be applied when the product is set equal to zero, as it says "If $a \cdot b = 0$, then $a = 0$ or $b = 0$." When we are given a product of factors set equal to a nonzero value, in order to solve we must rewrite the equation in standard form. To do so we multiply the factors, and move the nonzero value to the 'variable side.'

■ **Example 5** Solve $(x-3)(x+5) = 9$ for x.

Solution:
Seeing as the right-hand side of the equation is a nonzero value, we must rewrite the the equation in standard form. We first multiply the binomials on the left.

$$\begin{aligned} (x-3)(x+5) &= 9 \\ x^2 + 2x - 15 &= 9 \end{aligned}$$

We then subtract 9 from both sides of the equation to set the equation equal to zero.

$$x^2 + 2x - 24 = 0$$

Using techniques from the Factoring section, we rewrite the left-hand side of the equation as a product.

$$(x-4)(x+6) = 0$$

Now we can use the Zero Product Property and set each factor equal to zero.

$$x-4=0 \quad \text{or} \quad x+6=0$$

We tehn solve the linear equations.

$$\begin{aligned} x-4&=0 & & & x+6&=0 \\ x&=4 & \quad \text{or} \quad & & x&=-6 \end{aligned}$$

The solutions to $(x-3)(x+5)=9$ are $x=4$ or $x=-6$. Remember to check the solutions in the *original* quadratic equation, $(x-3)(x+5)=9$.

■

Solving Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side given in factored form. However, we saw that in order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. Now let's turn our attention to solving quadratic equations which are not originally written as the product of factors set equal to a constant.

Solving a Quadratic Equation by Factoring

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
2. Factor the quadratic expression on the left.
3. Use the Zero Product Property.
4. Solve the linear equations.

Check your solution(s) by substituting them, separately, into the original equation.

■ **Example 6** Solve $x^2+2x-8=0$ for x.

Solution:
As the equation is already in standard form, we factor the quadratic expression on the left-hand side.

$$\begin{aligned} x^2+2x-8&=0 \\ (x+4)(x-2)&=0 \end{aligned}$$

Now we set each linear factor equal to zero and solve for x.

$$\begin{aligned} x+4&=0 & \quad \text{or} \quad & & x-2&=0 \\ x&=-4 & \quad \text{or} \quad & & x&=2 \end{aligned}$$

Thus, the solutions to $x^2+2x-8=0$ are $x=-4$ or $x=2$.

We substitute each solution, separately, into the original equation, $x^2+2x-8=0$, to verify our solutions.

$$\begin{aligned} &\underline{x=-4} \\ x^2+2x-8&=0 \\ (-4)^2+2(-4)-8&\stackrel{?}{=}0 \\ 16+(-8)-8&\stackrel{?}{=}0 \\ 0&=0\checkmark \end{aligned} \qquad \begin{aligned} &\underline{x=2} \\ x^2+2x-8&=0 \\ 2^2+2(2)-8&\stackrel{?}{=}0 \\ 4+4-8&\stackrel{?}{=}0 \\ 0&=0\checkmark \end{aligned}$$

▪

▪ **Example 7** Solve $2y^2=13y+45$ for y.

Solution:
First, we must write the quadratic equation in standard form.

$$\begin{aligned} 2y^2&=13y+45 \\ 2y^2-13y-45&=0 \end{aligned}$$

Then we factor the quadratic expression on the left-hand side, and we use the Zero Product Property to solve for y.

$$(2y+5)(y-9)=0$$

$$\begin{aligned} 2y+5&=0 &\quad \text{or} \quad && y-9&=0 \\ 2y&=-5 \\ y&=-\frac{5}{2} &\quad \text{or} \quad && y&=9 \end{aligned}$$

The solutions to $2y^2=13y+45$ are $y=-\dfrac{5}{2}$ or $y=9$.

We leave it to the reader to check the solutions.

▪

▪ **Example 8** Solve $5x^2-13x=7x$ for x.

Solution:
Again we write the quadratic equation in standard form.

$$\begin{aligned} 5x^2-13x&=7x \\ 5x^2-20x&=0 \end{aligned}$$

Factoring the left-hand side of the equation, we have

$$5x(x-4)=0$$

Now, using the Zero Product Propert, we solve for x.

$$\begin{array}{rcr} 5x = 0 & \text{or} & x-4=0 \\ x = \dfrac{0}{5} & & \\ x = 0 & \text{or} & x = 4 \end{array}$$

As a result, $x = 0$ or $x = 4$ is a solution to $5x^2 - 13x = 7x$.

■

■ **Example 9** Solve $144q^2 = 25$ for q.

Solution:
First, we write the quadratic equation in standard form.

$$\begin{aligned} 144q^2 &= 25 \\ 144q^2 - 25 &= 0 \end{aligned}$$

Notice the quadratic expression on the left-hand side is a difference of squares, so we rewrite it as a product of conjugates. Then, we can use the Zero Product Property and solve for q.

$$(12q-5)(12q+5) = 0$$

$$\begin{array}{rcr} 12q-5=0 & \text{or} & 12q+5=0 \\ 12q = 5 & & 12q = -5 \\ q = \dfrac{5}{12} & \text{or} & q = -\dfrac{5}{12} \end{array}$$

Consequently, the solutions to $144q^2 = 25$ are $q = \dfrac{5}{12}$ or $q = -\dfrac{5}{12}$.

■

■ **Example 10** Solve $(3x-8)(x-1) = 3x$ for x.

Solution:
As the product of factors is not set equal to zero, we begin by multiplying the binomials.

$$\begin{aligned} (3x-8)(x-1) &= 3x \\ 3x^2 - 11x + 8 &= 3x \end{aligned}$$

Next, we write the quadratic equation in standard form.

$$\begin{aligned} 3x^2 - 11x + 8 &= 3x \\ 3x^2 - 14x + 8 &= 0 \end{aligned}$$

Factoring, applying the Zero Product Property, and solving for x gives us

$$\begin{array}{rcr} & (3x-2)(x-4)=0 & \\ 3x-2=0 & \text{or} & x-4=0 \\ 3x=2 & & \\ x=\dfrac{2}{3} & \text{or} & x=4 \end{array}$$

Hence, $x=\dfrac{2}{3}$ or $x=4$ is a solution to $(3x-8)(x-1)=3x$. ∎

The Zero Product Property also applies to the product of three or more factors. If a product is zero, at least one of the factors must be zero.

Example 11 Solve $4x^2=16x+84$ for x.

Solution:
We write the quadratic equation in standard form.

$$\begin{aligned} 4x^2 &= 16x+84 \\ 4x^2-16x-84 &= 0 \end{aligned}$$

When factoring the quadratic expression completely, we notice a GCF of 4, which is factored out first.

$$\begin{aligned} 4(x^2-4x-21) &= 0 \\ 4(x-7)(x+3) &= 0 \end{aligned}$$

Here, when we use the Zero Product Property to set each factor equal to zero, we get three equations instead of two.

$$4=0 \quad \text{or} \quad x-7=0 \quad \text{or} \quad x+3=0$$

Notice the first equation, $4=0$, is a contradiction and, thus, produces no solution. We proceed to the remaining two linear equations to determine the solutions to $4x^2=16x+84$.

$$\begin{array}{rcrcr} 4=0 & \text{or} & x-7=0 & \text{or} & x+3=0 \\ 4\neq 0 & \text{or} & x=7 & \text{or} & x=-3 \\ & & x=7 & \text{or} & x=-3 \end{array}$$

Therefore, the solutions to $4x^2=16x+84$ are $x=7$ or $x=-3$. ∎

We can solve some equations of degree more than two by using the Zero Product Property, similar to how we solved quadratic equations.

Example 12 Solve $9m^3 + 100m = 60m^2$ for m.

Solution:
First, we rewrite the equation so that one side of the equation is zero.

$$\begin{aligned} 9m^3 + 100m &= 60m^2 \\ 9m^3 - 60m^2 + 100m &= 0 \end{aligned}$$

We factor the expression on the left-hand side, completely.

$$\begin{aligned} m(9m^2 - 60m + 100) &= 0 \\ m(3m - 10)(3m - 10) &= 0 \end{aligned}$$

To solve, we apply the Zero Product Property.

$$\begin{array}{ccccc} m = 0 & \text{or} & 3m - 10 = 0 & \text{or} & 3m - 10 = 0 \\ & & 3m = 10 & & 3m = 10 \\ m = 0 & \text{or} & m = \dfrac{10}{3} & \text{or} & m = \dfrac{10}{3} \end{array}$$

We have two solutions to $9m^3 + 100m = 60m^2$, $m = 0$ or $m = \dfrac{10}{3}$. Notice, $m = \dfrac{10}{3}$ is a double root.

■

EXERCISES

Skills Practice (Answers)

For Exercises 1 - 17, solve the given equation.

1. $(x-3)(x+7)=0$
2. $(5b+1)(6b+1)=0$
3. $(3y+5)^2=0$
4. $(3a-10)(2a-7)=0$
5. $2x(6x-3)=0$
6. $(2x-1)^2=0$
7. $x^2+7x+12=0$
8. $y^2-8y+15=0$
9. $4b^2+7b=-3$
10. $5a^2-26a=24$
11. $4m^2=17m-15$
12. $n^3=-5n+6n^2$
13. $(y-3)(y+2)=4y$
14. $(x+6)(x-3)=-8$
15. $16p^3=18p^2+9p$
16. $(x+6)(x-2)=9$
17. $(y+9)(y+7)=80$

Completing the Square of a Binomial Expression

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

We rewrote the equation in standard form. Then, we factored the difference of squares and used the Zero Product Property to solve.

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ x-3 = 0 \qquad &\text{or} \qquad x-3=0 \\ x = 3 \qquad &\text{or} \qquad x = -3 \end{aligned}$$

We can easily use factoring to find the solution of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In these cases, we would get two solutions, $x = 4$ or $x = -4$ and $x = 5$ or $x = -5$, respectively.

Considering 7 is not a perfect square, we cannot solve the equation $x^2 = 7$ by factoring $x^2 - 7 = 0$, using integers.

Previously we learned that because 169 is the square of 13, we can also say that 13 is a *square root* of 169. Similarly, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots (one positive and one negative). We earlier defined the square root of a number as follows:

If $n^2 = m$, then n is a square root of m.

The square root definition tells us the solutions to the equation $x^2 = k$ are the two square roots of k. This leads us to the **Square Root Property**.

Theorem A.1 Square Root Property

If $x^2 = k$, then $x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}$.

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$, which we read as "x equals positive or negative the square root of k."

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

$$\begin{aligned} x^2 &= 9 \\ x &= \pm\sqrt{9} \\ x &= \pm 3 \end{aligned}$$

So, $x = 3$ or $x = -3$ is a solution to $x^2 = 9$.

Let's use the Square Root Property to solve the equation $x^2 = 7$.

$$\begin{aligned} x^2 &= 7 \\ x &= \pm\sqrt{7} \end{aligned}$$

As we cannot simplify $\sqrt{7}$, we leave the answers as radicals, and have a solution to $x^2 = 7$ of $x = \sqrt{7}$ or $x = -\sqrt{7}$.

We are able to use the Square Root Property to solve the equation $(y-7)^2 = 12$, because the left-hand side is a perfect square.

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y &= 7 \pm \sqrt{12} \\ y = 7+\sqrt{12} \quad &\text{or} \quad y = 7-\sqrt{12} \end{aligned}$$

We can also solve an equation in which the left-hand side is a perfect square trinomial, but we have to first rewrite the left-hand side in order to use the Square Root Property.

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \\ x-5 &= \pm\sqrt{18} \\ x &= 5 \pm \sqrt{18} \\ x = 5+\sqrt{18} \quad &\text{or} \quad x = 5-\sqrt{18} \end{aligned}$$

If the 'variable side' is not part of a perfect square, we can use algebra to make a perfect square.

In the Introduction to Algebraic Expressions section, we discussed the patterns for squaring a binomial. We restate these general patterns here for reference.

Binomial Squares Pattern

If m and n are real numbers, then

$$\underbrace{(m+n)^2}_{(\text{binomial})^2} = \underbrace{m^2}_{(\text{first term})^2} + \underbrace{2mn}_{2\cdot(\text{product of terms})} + \underbrace{n^2}_{(\text{second term})^2}$$

$$\underbrace{(m-n)^2}_{(\text{binomial})^2} = \underbrace{m^2}_{(\text{first term})^2} - \underbrace{2mn}_{2\cdot(\text{product of terms})} + \underbrace{n^2}_{(\text{second term})^2}$$

Let's consider the expression $x^2 + 6x$.

Noting there is a plus sign between the two terms, we will use the $(m+n)^2$ pattern, $m^2 + 2mn + n^2 = (m+n)^2$.

$$\begin{array}{ccccc} m^2 & + & 2mn & + & n^2 \\ x^2 & + & 6x & + & \_\_\_ \end{array}$$

Our goal is to determine the last term of this trinomial that will make it a perfect square trinomial, which requires the value of n. To compute n we need to solve the equation $2mn = 6x$. Notice that the first term of $x^2 + 6x$ is x^2; this tells us that $m = x$. Thus,

$$\begin{aligned} 2mn &= 6x \\ 2 \cdot (x) \cdot n &= 6x \\ 2nx &= 6x \end{aligned}$$

We are looking for the value of n such that $2n = 6$. Therefore, $n = 3$, which is $\frac{1}{2}(6)$.

$$\overset{m^2}{x^2} \quad \overset{+}{+} \quad \overset{2mn}{2 \cdot x \cdot 3} \quad \overset{+}{+} \quad \overset{n^2}{(3)^2}$$

Now to complete the perfect square trinomial, we will simplify the last two terms.

$$\overset{m^2}{x^2} \quad \overset{+}{+} \quad \overset{2mn}{6x} \quad \overset{+}{+} \quad \overset{n^2}{9}$$

Once we have the expression written as a perfect square trinomial we can factor the trinomial.

$$\overset{(m+n)^2}{(x+3)^2}$$

So we found that adding 9 to $x^2 + 6x$ 'completes the square,' and we write it as $(x+3)^2$.

Completing the Square of $x^2 + bx$

1. Identify b, the coefficient of x.
2. Determine $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.
4. Factor the perfect square trinomial, writing it as a binomial squared.

▪ **Example 13** Complete the square of $x^2 - 26x$ to make a perfect square trinomial. Then, write the result as a binomial squared.

Solution:
The coefficient of x, b, is -26.

$$\overset{x^2 - bx}{x^2 - 26x}$$

We determine $\left(\frac{1}{2}b\right)^2 = \left(\frac{1}{2} \cdot (-26)\right)^2 = (-13)^2 = 169$.

As a result, we add 169 to the binomial to complete the square.

$$x^2 - 26x + 169$$

We factor the perfect square trinomial, $x^2 - 26x + 169$, writing it as a binomial squared.

$$(x-13)^2$$

■

■ **Example 14** Complete the square of $y^2 - 9y$ to make a perfect square trinomial. Then, write the result as a binomial squared.

Solution:
The coefficient of y, b, is -9.

$$\overset{x^2-bx}{y^2-9y}$$

We determine $\left(\frac{1}{2}b\right)^2 = \left(\frac{1}{2}\cdot(-9)\right)^2 = \left(-\frac{9}{2}\right)^2 = \frac{81}{4}$.

As a result, we add $\frac{81}{4}$ to the binomial to complete the square.

$$y^2 - 9y + \frac{81}{4}$$

We factor the perfect square trinomial, $y^2 - 9y + \frac{81}{4}$, writing it as a binomial squared.

$$\left(y - \frac{9}{2}\right)^2$$

■

■ **Example 15** Complete the square of $n^2 + \frac{1}{2}n$ to make a perfect square trinomial. Then, write the result as a binomial squared.

Solution:
The coefficient of n, b, is $\frac{1}{2}$.

$$\overset{x^2+bx}{n^2+\frac{1}{2}n}$$

We determine $\left(\frac{1}{2}b\right)^2 = \left(\frac{1}{2}\cdot\frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$.

As a result, we add $\frac{1}{16}$ to the binomial to complete the square.

$$n^2 + \frac{1}{2}n + \frac{1}{16}$$

We factor the perfect square trinomial, $n^2 + \frac{1}{2}n + \frac{1}{16}$, writing it as a binomial squared.

$$\left(n + \frac{1}{4}\right)^2$$

■

Solving Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

When solving equations, we must always perform the same operation to both sides of the equation. When we solve a quadratic equation by completing the square, we add a term to one side of the equation to make a perfect square trinomial, so we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x - 40 = 0$, and we do not realize the left-hand side factors, then we can complete the square on the left to solve the equation. To complete the square, we begin by isolating the variable terms on the left-hand side.

$$\begin{aligned} x^2 + 6x - 40 &= 0 \\ x^2 + 6x &= 40 \\ x^2 + 6x + \underline{\quad} &= 40 + \underline{\quad} \end{aligned}$$

Next, we determine $\left(\frac{1}{2}b\right)^2 = \left(\frac{1}{2} \cdot 6\right)^2 = 9$ and add it to both sides of the equation to complete the square on the left.

$$x^2 + 6x + 9 = 40 + 9$$

Rewriting as a binomial squared, we have

$$(x+3)^2 = 49$$

Now the equation can be solved using the Square Root Property. Solving for x, we have

$$x + 3 = \pm 7$$

$$x + 3 = 7 \quad \text{or} \quad x + 3 = -7$$

$$x = 4 \quad \text{or} \quad x = -10$$

So, the solutions to $x^2 + 6x - 40 = 0$ are $x = 4$ or $x = -10$.

Completing the square is a way to transform an equation so that we may apply the Square Root Property.

From the beginning, we could have solved this equation by factoring the trinomial, as follows.

$$x^2+6x-40=0$$

$$(x-4)(x+10)=0$$

$$x-4=0 \quad \text{or} \quad x+10=0$$

$$x=4 \quad \text{or} \quad x=-10$$

Again, the solutions to $x^2+6x-40=0$ are $x=4$ or $x=-10$.

Notice, we get the same solutions no matter which method we use, if done correctly.

Solving a Quadratic Equation of the Form $x^2+bx+c=0$ by Completing the Square

1. Isolate the variable terms on the left-hand side and the constant terms on the right.
2. Determine $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square on the left-hand side, and add it to both sides of the equation.
3. Factor the perfect square trinomial, writing it as a binomial squared, on the left and simplify by adding the terms on the right.
4. Use the Square Root Property to solve the resulting equation.

Check the solution(s) in the original equation.

▪ **Example 16** Solve $x^2+8x=48$, by completing the square.

Solution:
The variable terms and constant terms are already separated on either side of the equals sign. So, we determine $\left(\frac{1}{2}\cdot b\right)^2=\left(\frac{1}{2}\cdot 8\right)^2=4^2=16$, the number to complete the square; we add it to both sides of the equation.

$$x^2+8x=48$$
$$x^2+8x+\_\_\_=48+\_\_\_$$
$$x^2+8x+16=48+16$$

We factor the perfect square trinomial as a binomial squared.

$$(x+4)^2=64$$

We use the Square Root Property to solve.

$$\begin{aligned} x+4 &= \pm\sqrt{64} \\ x+4 &= \pm 8 \end{aligned}$$

$$\begin{array}{ccc} x+4=8 & \text{or} & x+4=-8 \\ x=4 & \text{or} & x=-12 \end{array}$$

Consequently, the solutions to $x^2+8x=48$ are $x=4$ or $x=-12$.

As always, we can check the solutions in the *original* equation.

$$\begin{array}{ccc} \underline{x=4} & \quad & \underline{x=-12} \\ x^2+8x=48 & & x^2+8x=48 \\ (4)^2+8(4)\stackrel{?}{=}48 & & (-12)^2+8(-12)\stackrel{?}{=}48 \\ 16+32\stackrel{?}{=}48 & & 144-96\stackrel{?}{=}48 \\ 48=48\ \checkmark & & 48=48\ \checkmark \end{array}$$

■

■ **Example 17** Solve $y^2-18y=-6$, by completing the square.

Solution:
The variable terms and constant terms are already separated on either side of the equals sign. So, we take half of b, -18, and square it.

$$\left(\frac{1}{2}b\right)^2=\left(\frac{1}{2}(-18)\right)^2=81$$

We add the result, 81, to both sides.

$$\begin{aligned} y^2-18y &= -6 \\ y^2-18y+\underline{\quad} &= -6+\underline{\quad} \\ y^2-18y+81 &= -6+81 \end{aligned}$$

We now factor the perfect square trinomial, writing it as a binomial squared, and use the Square Root Property to solve for y.

$$\begin{aligned} (y-9)^2 &= 75 \\ y-9 &= \pm\sqrt{75} \\ y &= 9\pm\sqrt{75} \end{aligned}$$

$$y=9+\sqrt{75} \quad \text{or} \quad y=9-\sqrt{75}$$

Thus, the solutions to $y^2-18y=-6$ are $y=9+\sqrt{75}$ or $y=9-\sqrt{75}$.

We can check our solutions in the original equation, but another way to check them would be to use a calculator. Evaluating y^2-18y for both of the solutions in the calculator should result in -6.

■

Example 18 Solve $x^2 + 10x + 4 = 15$, by completing the square.

Solution:
We begin by isolating the variable terms on the left-hand side. To do so we subtract 4 from both sides of the equation.

$$\begin{aligned} x^2 + 10x + 4 &= 15 \\ x^2 + 10x &= 11 \end{aligned}$$

We complete the square on the left-hand side, adding the necessary constant to both sides of the equation. Then, we solve for x.

$$\begin{aligned} x^2 + 10x &= 11 \\ x^2 + 10x + \_\_\_ &= 11 + \_\_\_ \\ x^2 + 10x + \left(\frac{1}{2}\cdot(10)\right)^2 &= 11 + \left(\frac{1}{2}\cdot(10)\right)^2 \\ x^2 + 10x + 25 &= 11 + 25 \\ (x+5)^2 &= 36 \\ x + 5 &= \pm\sqrt{36} \\ x + 5 &= \pm 6 \end{aligned}$$

$$\begin{aligned} x + 5 = 6 \quad &\text{or} \quad x + 5 = -6 \\ x = 1 \quad &\text{or} \quad x = -11 \end{aligned}$$

Therefore, $x = 1$ or $x = -11$ is a solution to $x^2 + 10x + 4 = 15$.

■

Example 19 Solve $n^2 = 3n + 11$, by completing the square.

Solution:
We subtract $3n$ from both sides to isolate the variable terms on the left-hand side.

$$\begin{aligned} n^2 &= 3n + 11 \\ n^2 - 3n &= 11 \end{aligned}$$

We complete the square on the left-hand side, adding the necessary constant to both sides of the equation. Then, we solve for n.

$$\begin{aligned}
n^2 - 3n &= 11 \\
n^2 - 3n + ___ &= 11 + ___ \\
n^2 - 3n + \left(\frac{1}{2}\cdot(-3)\right)^2 &= 11 + \left(\frac{1}{2}\cdot(-3)\right)^2 \\
n^2 - 3n + \frac{9}{4} &= 11 + \frac{9}{4} \\
n^2 - 3n + \frac{9}{4} &= \frac{44}{4} + \frac{9}{4} \\
\left(n - \frac{3}{2}\right)^2 &= \frac{53}{4} \\
n - \frac{3}{2} &= \pm\sqrt{\frac{53}{4}} \\
n - \frac{3}{2} = \sqrt{\frac{53}{4}} \quad &\text{or} \quad n - \frac{3}{2} = -\sqrt{\frac{53}{4}} \\
n = \frac{3}{2} + \sqrt{\frac{53}{4}} \quad &\text{or} \quad n = \frac{3}{2} - \sqrt{\frac{53}{4}}
\end{aligned}$$

Therefore, $n = \frac{3}{2} + \sqrt{\frac{53}{4}}$ or $n = \frac{3}{2} - \sqrt{\frac{53}{4}}$ is a solution to $n^2 = 3n + 11$.

▪

Solving Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of x^2 is 1, so the left-hand side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient of x^2 can be factored from all three terms of the trinomial as the GCF, but often the coefficient of x^2 is not the GCF. In either case, divide all terms, on both sides, by the coefficient of x^2 before proceeding.

Solving a Quadratic Equation of the Form $ax^2 + bx + c = 0$ by Completing the Square

1. Divide both sides of the equation by the coefficient of x^2, a, to make the coefficient of the x^2 term 1.
2. Isolate the variable terms on the left-hand side and the constant terms on the right.
3. Determine $\left(\frac{1}{2}\cdot b\right)^2$, and add it to both sides of the equation.
4. Factor the perfect square trinomial, writing it as a binomial squared, on the left and simplify by adding the terms on the right.
5. Use the Square Root Property to solve the resulting equation.

> Check the solution(s) in the original equation.

■ **Example 20** Solve $3x^2 - 12x - 15 = 0$, by completing the square.

Solution:
The coefficient of x^2, $a = 3$, is the GCF of the expression on the left-hand side, so we factor out the GCF.

$$3(x^2 - 4x - 5) = 0$$

To isolate the trinomial with coefficient 1, we divide both sides by 3 and simplify.

$$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$$
$$x^2 - 4x - 5 = 0$$

At this point, we have a quadratic equation of the form $x^2 + bx + c = 0$. We can complete the square and solve for x, using the same technique as previous examples.

$$\begin{aligned} x^2 - 4x - 5 &= 0 \\ x^2 - 4x &= 5 \\ x^2 - 4x + \underline{\quad} &= 5 + \underline{\quad} \\ x^2 - 4x + \left(\frac{1}{2} \cdot (-4)\right)^2 &= 5 + \left(\frac{1}{2} \cdot (-4)\right)^2 \\ x^2 - 4x + 4 &= 5 + 4 \\ (x-2)^2 &= 9 \\ x - 2 &= \pm\sqrt{9} \\ x - 2 &= \pm 3 \end{aligned}$$

$$\begin{aligned} x - 2 = 3 \quad &\text{or} \quad x - 2 = -3 \\ x = 5 \quad &\text{or} \quad x = -1 \end{aligned}$$

Hence, $x = 5$ or $x = -1$ is a solution to $3x^2 - 12x - 15 = 0$.

■

■ **Example 21** Solve $2x^2 - 3x = 20$, by completing the square.

Solution:
While the coefficient of x^2, $a = 2$, is not the GCF of all terms, we still divide both sides of the equation by 2 to get the coefficient of x^2 to be 1, and simplify.

$$\frac{2x^2 - 3x}{2} = \frac{20}{2}$$
$$x^2 - \frac{3}{2}x = 10$$

Again, we have a quadratic equation of the form $x^2+bx+c=0$, and we can complete the square and solve for x, using the same technique as previous examples.

$$\begin{aligned}
x^2 - \frac{3}{2}x &= 10 \\
x^2 - \frac{3}{2}x + ___ &= 10 + ___ \\
x^2 - \frac{3}{2}x + \left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2 &= 10 + \left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2 \\
x^2 - \frac{3}{2}x + \frac{9}{16} &= 10 + \frac{9}{16} \\
x^2 - \frac{3}{2}x + \frac{9}{16} &= \frac{160}{16} + \frac{9}{16} \\
\left(x - \frac{3}{4}\right)^2 &= \frac{169}{16} \\
x - \frac{3}{4} &= \pm\sqrt{\frac{169}{16}} \\
x - \frac{3}{4} &= \pm\frac{13}{4} \\
x - \frac{3}{4} = \frac{13}{4} \quad &\text{or} \quad x - \frac{3}{4} = -\frac{13}{4} \\
x = \frac{3}{4} + \frac{13}{4} \quad & \quad\; x = \frac{3}{4} - \frac{13}{4} \\
x = 4 \quad &\text{or} \quad x = -\frac{5}{2}
\end{aligned}$$

So, the solutions to $2x^2-3x=20$ are $x=4$ or $x=-\frac{5}{2}$.

■

■ **Example 22** Solve $3x^2+2x=4$, by completing the square.

Solution:

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, $a=3$, we can then continue with solving the equation by completing the square.

$$\begin{aligned}
\frac{3x^2+2x}{3} &= \frac{4}{3} \\
x^2 + \frac{2}{3}x &= \frac{4}{3}
\end{aligned}$$

$$x^2 + \frac{2}{3}x + \_\_\_ = \frac{4}{3} + \_\_\_$$

$$x^2 + \frac{2}{3}x + \left(\frac{1}{2} \cdot \left(\frac{2}{3}\right)\right)^2 = \frac{4}{3} + \left(\frac{1}{2} \cdot \left(\frac{2}{3}\right)\right)^2$$

$$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$$

$$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{12}{9} + \frac{1}{9}$$

$$\left(x + \frac{1}{3}\right)^2 = \frac{13}{9}$$

$$x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$$

$$x + \frac{1}{3} = \sqrt{\frac{13}{9}} \quad \text{or} \quad x + \frac{1}{3} = -\sqrt{\frac{13}{9}}$$

$$x = -\frac{1}{3} + \sqrt{\frac{13}{9}} \qquad x = -\frac{1}{3} - \sqrt{\frac{13}{9}}$$

So, the solutions to $3x^2 + 2x = 4$ are $x = -\frac{1}{3} + \sqrt{\frac{13}{9}}$ or $x = -\frac{1}{3} - \sqrt{\frac{13}{9}}$.

∎

EXERCISES

Skills Practice (Answers)

For Exercises 18 - 23, complete the square to make a perfect square trinomial.
Then, write the result as a binomial squared.

18. $m^2 - 24m$
19. $x^2 - 11x$
20. $q^2 + \frac{3}{4}q$
21. $y^2 + 5y$
22. $q^2 + 6q$
23. $n^2 - \frac{2}{3}n$

For Exercises 24 - 37, solve by completing the square.

24. $u^2 + 2u = 3$
25. $z^2 + 12z = -11$
26. $x^2 - 20x = 21$
27. $a^2 - 10a = -5$
28. $u^2 - 14u + 12 = -1$
29. $z^2 + 2z - 5 = 2$
30. $w^2 = 5w - 1$
31. $y^2 - 14 = 6y$
32. $3m^2 + 30m - 27 = 6$
33. $2x^2 - 14x + 12 = 0$
34. $5x^2 + 20x = 35$
35. $2c^2 + c = 6$
36. $2p^2 + 8p = 15$
37. $3q^2 - 5q = 9$

Solving Quadratic Equations Using the Quadratic Formula

Mathematicians look for patterns when they do things over and over, in order to make their work easier. Because completing the square is tedious to some, we will derive and use a formula to find the solution of a quadratic equation.

We will go through the steps of completing the square, using the standard form of a quadratic equation, to solve a quadratic equation for x.

$$ax^2+bx+c=0 \quad \text{where } a \neq 0$$

We isolate the variable terms on the left-hand side.

$$ax^2+bx=-c$$

We make the coefficient of x^2 equal to 1, by dividing by a, and simplifying.

$$\frac{ax^2}{a}+\frac{b}{a}x=-\frac{c}{a}$$
$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Then to complete the square, we determine $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$ and add it to both sides of the equation.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

The left-hand side is now a perfect square trinomial that can be factored.

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

Now we find the common denominator of the right-hand side, and write equivalent fractions with the common denominator.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$
$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$$

By simplifying we can combine the right-hand side into one fraction.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$
$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Using the Square Root Property gives us

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Unlike with our examples thus far, we must simplify the radical to make our formula easier to remember.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Now, we add $-\frac{b}{2a}$ to both sides of the equation to isolate x.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Seeing as the terms on the right-hand side have a common denominator, we can combine the terms and rewrite the formula in the following manner:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the

Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

■

To use the Quadratic Formula, we substitute the numerical values of a, b, and c from the standard form of the quadratic equation into the Quadratic Formula. Then, we simplify the solutions to the quadratic equation. Notice the formula is an equation; make sure you include the x from the left-hand side of the equation.

Solving a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$, and identify the numerical values of a, b, and c.
2. Write the Quadratic Formula, substituting in the values of a, b, and c.
3. Simplify the right-hand side of the formula.

Check the solution(s) in the original equation.

■ **Example 23** Solve $2x^2+9x-5=0$, by using the Quadratic Formula.

Solution:
The quadratic equation is given in standard form, so we identify the numerical values of a, b, and c.

$$\begin{array}{ccccccc} ax^2 & + & bx & + & c & = & 0 \\ 2x^2 & + & 9x & - & 5 & = & 0 \end{array}$$

$$a=2, \quad b=9, \quad c=-5$$

Then, we write the quadratic formula, substituting in the numerical values of a, b, and c.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$$

Finally, we simplify the fraction, and solve for x.

$$x=\frac{-9\pm\sqrt{81-(-40)}}{4}$$

$$x=\frac{-9\pm\sqrt{121}}{4}$$

$$x=\frac{-9\pm 11}{4}$$

$$\begin{array}{lcl} x=\dfrac{-9+11}{4} & \text{or} & x=\dfrac{-9-11}{4} \\ x=\dfrac{2}{4} & & x=\dfrac{-20}{4} \\ x=\dfrac{1}{2} & \text{or} & x=-5 \end{array}$$

As a result, the solutions to $2x^2+9x-5=0$ are $x=\dfrac{1}{2}$ or $x=-5$.

Again, we can check our solutions.

$$\begin{array}{rcr} \underline{x=\dfrac{1}{2}} & & \underline{x=-5} \\ 2x^2+9x-5=0 & & 2x^2+9x-5=0 \\ 2\left(\dfrac{1}{2}\right)^2+9\cdot\left(\dfrac{1}{2}\right)-5\stackrel{?}{=}0 & & 2(-5)^2+9(-5)-5\stackrel{?}{=}0 \\ 2\cdot\left(\dfrac{1}{4}\right)+9\cdot\left(\tfrac{1}{2}\right)-5\stackrel{?}{=}0 & & 2\cdot 25-45-5\stackrel{?}{=}0 \\ \dfrac{1}{2}+\dfrac{9}{2}-5\stackrel{?}{=}0 & & 50-45-5\stackrel{?}{=}0 \\ \dfrac{10}{2}-5\stackrel{?}{=}0 & & 0=0\ \checkmark \\ 5-5\stackrel{?}{=}0 & & \\ 0=0\ \checkmark & & \end{array}$$

N *The authors recommend the reader say the formula aloud as they write it in each problem to help them memorize it. Always remember the Quadratic Formula is an EQUATION; be sure to start with "$x =$."*

When we solve quadratic equations by using the Square Root Property, we sometimes get answers that have radicals. Radicals can also occur in the solution when using the Quadratic Formula.

Example 24 Solve $2x^2 + 10x + 11 = 0$, by using the Quadratic Formula.

Solution:
The given equation is already in standard form; we identify the values of a, b, and c.

$$\begin{array}{ccccccc} ax^2 & + & bx & + & c & = & 0 \\ 2x^2 & + & 10x & + & 11 & = & 0 \end{array}$$

$$a = 2, \quad b = 10, \quad c = 11$$

Then, we write the quadratic formula, substituting in the numerical values of a, b, and c.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(10) \pm \sqrt{10^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$$

Finally, we simplify the fraction, and solve for x.

$$x = \frac{-10 \pm \sqrt{100 - 88}}{4}$$
$$x = \frac{-10 \pm \sqrt{12}}{4}$$
$$x = \frac{-10 + \sqrt{12}}{4} \quad \text{or} \quad x = \frac{-10 - \sqrt{12}}{4}$$

Thus, the solutions to $2x^2 + 10x + 11 = 0$ are $x = \dfrac{-10 + \sqrt{12}}{4}$ or $x = \dfrac{-10 - \sqrt{12}}{4}$.

When we substitute a, b, and c into the Quadratic Formula, and the radicand is negative, the quadratic equation will have imaginary or complex solutions. In this text, we only consider real-number solutions; if the radicand is negative, we say the quadratic equation has no real solutions.

■ **Example 25** Solve $3p^2+2p+9=0$, by using the Quadratic Formula.

Solution:
Again, this equation is already in standard form, so we start by identifying the values of a, b, and c.

$$\begin{array}{ccccccc} ap^2 & + & bp & + & c & = & 0 \\ 3p^2 & + & 2p & + & 9 & = & 0 \end{array}$$
$$a=3, \quad b=2, \quad c=9$$

Then, we write the quadratic formula, substituting in the numerical values of a, b, and c.

$$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$p=\frac{-(2)\pm\sqrt{2^2-4\cdot 3\cdot(9)}}{2\cdot 3}$$

Finally, we simplify the fraction, and solve for p.

$$p=\frac{-2\pm\sqrt{4-108}}{6}$$
$$p=\frac{-2\pm\sqrt{-104}}{6}$$

Given that the radicand, -104, is negative, $3p^2+2p+9=0$ has no real solutions. ■

N *When a quadratic equation, $ax^2+bx+c=0$, has no real solutions, the corresponding quadratic function, $f(x)=ax^2+bx+c$, when graphed, will lie completely above or below the x-axis.*

Remember, in order to use the Quadratic Formula, the quadratic equation must be written in standard form, $ax^2+bx+c=0$. Sometimes we will need to perform some algebra to get the equation into standard form.

■ **Example 26** Solve $x(x+6)+4=0$, by using the Quadratic Formula.

Solution:
Our first step is to get the equation in standard form. To do so, we distribute.

$$x^2+6x+4=0$$

Now that we have a quadratic equation in standard form, we can identify the values of a, b, and c.

$$\begin{array}{ccccccc} ax^2 & + & bx & + & c & = & 0 \\ x^2 & + & 6x & + & 4 & = & 0 \end{array}$$
$$a=1, \quad b=6, \quad c=4$$

Then, we write the quadratic formula, substituting in the numerical values of a, b, and c.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(6) \pm \sqrt{6^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$$

Finally, we simplify the fraction, and solve for x.

$$x = \frac{-6 \pm \sqrt{36 - 16}}{2}$$

$$x = \frac{-6 \pm \sqrt{20}}{2}$$

As the radicand, 20, is not a perfect square, we leave the radical in the solutions.

The solutions to $x(x+6)+4 = 0$ are $x = \dfrac{-6 \pm \sqrt{20}}{2}$ or $x = \dfrac{-6 \pm \sqrt{20}}{2}$. ■

When solving equations, if an equation has multiple fractions, we can 'clear the fractions' by multiplying both sides of the equation by the LCD. This gives us an equivalent equation, without fractions, to solve.

■ **Example 27** Solve $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$, by using the Quadratic Formula.

Solution:
Our first step is to 'clear the fractions.' We multiply both sides of the equation by the LCD, $2 \cdot 3 = 6$, and simplify.

$$6\left(\frac{1}{2}u^2 + \frac{2}{3}u\right) = 6\left(\frac{1}{3}\right)$$

$$3u^2 + 4u = 2$$

As the result is not a quadratic equation in standard form, we subtract 2 from both sides of the equation, and then identify the values of a, b, and c.

$$\begin{array}{ccccccc} ax^2 & + & bx & + & c & = & 0 \\ 3u^2 & + & 4u & - & 2 & = & 0 \end{array}$$

$$a = 3, \quad b = 4, \quad c = -2$$

We write the quadratic formula, substituting in the numerical values of a, b, and c.

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$u = \frac{-(4) \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$$

Finally, we simplify the fraction, and solve for u.

$$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$$

$$u = \frac{-4 \pm \sqrt{40}}{6}$$

Given that the radicand, 40, is not a perfect square, we leave the radical in the solutions.

This gives us solutions to $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$ of $u = \frac{-4+\sqrt{40}}{6}$ or $u = \frac{-4-\sqrt{40}}{6}$.

■

We know from the Zero Product Property that $(x-3)^2 = 0$ has only one solution, $x = 3$. Similarly, if $(x-3)^2 = 0$ is rewritten in standard form, the Quadratic Formula can applied. Then the radicand, when simplified, becomes 0 and leaves us with the same single solution, $x = 2$.

■ **Example 28** Solve $4x^2 - 20x = -25$, by using the Quadratic Formula.

Solution:
We add 25 to both sides of the given equation to rewrite the equation in standard form. We can next identify the values of a, b, and c.

$$\begin{array}{ccccccc} ax^2 & + & bx & + & c & = & 0 \\ 4x^2 & - & 20x & + & 25 & = & 0 \end{array}$$

$$a = 4, \quad b = -20, \quad c = 25$$

Then, we write the quadratic formula, substituting in the numerical values of a, b, and c.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$$

Finally, we simplify the fraction, and solve for x.

$$x = \frac{20 \pm \sqrt{400 - 400}}{8}$$
$$x = \frac{20 \pm \sqrt{0}}{8}$$
$$x = \frac{20}{8}$$
$$x = \frac{5}{2}$$

Therefore, the only solution to $4x^2 - 20x = -25$ is $x = \frac{5}{2}$.

■

EXERCISES

Skills Practice (Answers)

For Exercises 38 - 49, solve by using the Quadratic Formula.

38. $4m^2 + m - 3 = 0$
39. $r^2 - 8r = 33$
40. $2p^2 + 8p + 5 = 0$
41. $8x^2 - 6x + 2 = 0$
42. $(x+1)(x-3) = 0$
43. $2a^2 - 6a + 3 = 0$
44. $\frac{1}{3}n^2 + n = -\frac{1}{2}$
45. $16y^2 + 8y + 1 = 0$
46. $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$
47. $q^2 + 3q - 18 = 0$
48. $(x+2)(x+6) = 21$
49. $25d^2 - 60d + 36 = 0$

Identifying the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do we decide which one to use? Factoring is often the quickest method, so we try it first. If the equation is $ax^2 = k$ or $a(x-h)^2 = k$, we use the Square Root Property. For any other quadratic equation, it is probably best to use the Quadratic Formula. Remember, we can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods, because we completed the square in general to derive the Quadratic Formula.

Identifying the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Example 29 Identify the most appropriate method to use to solve each quadratic equation, but do not solve.

a. $5z^2 = 17$
b. $4x^2 - 12x + 9 = 0$
c. $8u^2 + 6u = 11$

Solution:

a. Considering the equation $5z^2 = 17$ is in the form $ax^2 = k$, the most appropriate method is to use the Square Root Property.

b. Given the equation $4x^2 - 12x + 9 = 0$, we recognize that the left-hand side of the equation is factorable as $(2x-3)(2x-3)$. Thus, factoring will be the most appropriate method.

c. First we put the equation $8u^2 + 6u = 11$ in standard form: $8u^2 + 6u - 11 = 0$. While our first thought may be to try factoring, thinking about all the possibilities for the Trial and Error method leads us to choose the Quadratic Formula as the most appropriate method.

■

Solving Equations in Quadratic Form

While the focus up to this point of this section has been in learning techniques for solving quadratic equations, sometimes we can use these same techniques to solve equations which are not quadratic, but are of a 'quadratic form.'

Some examples of equations in 'quadratic form' are

$$x^4 - 16 = 0 \qquad p^4 - 2p^2 + 1 = 0 \qquad r + 3\sqrt{r} = 10 \qquad (y+2)^2 - (y+2) - 30 = 0$$

We will use a method called **substitution** to solve equations of this form. In the standard $ax^2 + bx + c = 0$ form, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. We will look for this relationship in order to make a 'substitution.'

Let's consider the equation $x^4 - 4x^2 - 5 = 0$.

As the equation is already set equal to 0, we turn our attention to the expression on the left-hand side of equation. We notice the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. $\left(\text{We know } \left(x^2\right)^2 = x^4.\right)$ So we will make a 'substitution', $u = x^2$, and factor.

$$\begin{aligned} x^4 - 4x^2 - 5 &= \left(x^2\right)^2 - 4\left(x^2\right) - 5 \\ &= u^2 - 4u - 5 \end{aligned}$$

After making the substitution, we rewrite the given equation, now in terms of u, and factor the trinomial to solve.

$$\begin{gathered} x^4 - 4x^2 - 5 = 0 \\ u^2 - 4u - 5 = 0 \\ (u+1)(u-5) = 0 \end{gathered}$$

$$\begin{array}{rcl} u + 1 = 0 & \text{or} & u - 5 = 0 \\ u = -1 & \text{or} & u = 5 \end{array}$$

Notice our current solutions are in terms of u, but the original equation was in terms of x. Therefore, we must replace u with x^2 in each solution and solve for x.

$$\begin{array}{rcl} x^2 = -1 & \text{or} & x^2 = 5 \\ x = \pm\sqrt{-1} & \text{or} & x = \pm\sqrt{5} \end{array}$$

As $\sqrt{-1}$ is not defined under the real numbers, $x = \pm\sqrt{-1}$ yields no real solutions, thus, the solutions to $x^4 - 4x^2 - 5 = 0$ are $x = \sqrt{5}$ or $x = -\sqrt{5}$.

In general, we follow the process outlined above to solve equations of a quadratic form.

Solving Equations in Quadratic Form

1. Identify a substitution, u, that will put the equation in quadratic form and rewrite the equation in terms of u.
2. Solve the quadratic equation for u.
3. Use the substitution to rewrite all solutions in terms of the original variable.
4. Solve for the original variable.

Check the solution(s) in the original equation.

Example 30 Solve $6x^4 - 7x^2 + 2 = 0$ for x.

Solution:
As the equation is already set equal to 0, we notice the variable part of the middle term is x^2 and its square, $\left(x^2\right)^2 = x^4$, is the variable part of the first term. So, we let $u = x^2$ and rewrite the equation in terms of u.

$$\begin{aligned} 6x^4 - 7x^2 + 2 &= 0 \\ 6\left(x^2\right)^2 - 7x^2 + 2 &= 0 \\ 6u^2 - 7u + 2 &= 0 \end{aligned}$$

Next, we solve the quadratic equation for u, by any appropriate method. Here, we can solve by factoring and using the Zero Product Property.

$$(2u-1)(3u-2) = 0$$

$$\begin{aligned} 2u - 1 &= 0 \quad \text{or} \quad & 3u - 2 &= 0 \\ 2u &= 1 \quad \text{or} \quad & 3u &= 2 \\ u &= \frac{1}{2} \quad \text{or} \quad & u &= \frac{2}{3} \end{aligned}$$

Now we substitute the original variable back into the results, using the substitution, $u = x^2$.

$$x^2 = \frac{1}{2} \quad \text{or} \quad x^2 = \frac{2}{3}$$

Here, we will use the Square Root Property to solve for the original variable, x.

$$\begin{aligned} x^2 &= \frac{1}{2} & x^2 &= \frac{2}{3} \\ x &= \pm\sqrt{\frac{1}{2}} & x &= \pm\sqrt{\frac{2}{3}} \end{aligned}$$

So, there are four solutions to $6x^4 - 7x^2 + 2 = 0$:

$$x = \sqrt{\frac{1}{2}},\ x = -\sqrt{\frac{1}{2}},\ x = \sqrt{\frac{2}{3}},\ \text{or } x = -\sqrt{\frac{2}{3}}$$

We leave it to the reader to check each solution in the *original* equation.

Example 31 Solve $(x-2)^2+7(x-2)+12=0$ for x.

Solution:
The variable part in the middle term is a binomial, $(x-2)$, which is squared in the first term. If we let $u=(x-2)$, and substitute, our trinomial will be in ax^2+bx+c form.

$$\begin{aligned}(x-2)^2+7(x-2)+12&=0\\ u^2+7u+12&=0\end{aligned}$$

Again, we can solve for u by factoring.

$$(u+3)(u+4)=0$$

$$\begin{array}{ccc} u+3=0 & \text{or} & u+4=0\\ u=-3 & \text{or} & u=-4\end{array}$$

We replace u with $x-2$ in each solution to solve for the original variable, x.

$$\begin{array}{ccc} x-2=-3 & \text{or} & x-2=-4\\ x=-1 & \text{or} & x=-2\end{array}$$

Check:

$$\begin{aligned}&\underline{x=-1}\\ (x-2)^2+7(x-2)+12&=0\\ (-1-2)^2+7(-1-2)+12&\stackrel{?}{=}0\\ (-3)^2+7(-3)+12&\stackrel{?}{=}0\\ 9-21+12&\stackrel{?}{=}0\\ 0&=0\checkmark\end{aligned}$$

$$\begin{aligned}&\underline{x=-2}\\ (x-2)^2+7(x-2)+12&=0\\ (-2-2)^2+7(-2-2)+12&\stackrel{?}{=}0\\ (-4)^2+7(-4)+12&\stackrel{?}{=}0\\ 16-28+12&\stackrel{?}{=}0\\ 0&=0\checkmark\end{aligned}$$

Consequently, the solutions to $(x-2)^2+7(x-2)+12=0$ are $x=-1$ or $x=-2$.

■

While at times checking the solution has been left to the reader, when we square both sides of an equation we may introduce **extraneous solutions**. Extraneous solutions are solutions which are found through algebraic operations, but do not satisfy the original equation. Despite the authors not always including the check for each solution, it is important to check that the solutions found satisfy the original equation.

Example 32 Solve: $x-3\sqrt{x}+2=0$.

Solution:
The variable part in the middle term is $\sqrt{x}$, which is squared in the first term, $(\sqrt{x})^2=x$. If we let $u=\sqrt{x}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

$$\begin{aligned}x-3\sqrt{x}+2&=0\\ (\sqrt{x})^2-3\sqrt{x}+2&=0\\ u^2-3u+2&=0\end{aligned}$$

We solve for u, by factoring.

$$\begin{aligned}(u-2)(u-1) &= 0\\ u-2=0 \quad &\text{or} \quad u-1=0\\ u=2 \quad &\text{or} \quad u=1\end{aligned}$$

We replace u with $\sqrt{x}$ in each solution to solve for the original variable, x. Notice, in order to solve for x in each equation, we must square both sides.

$$\begin{array}{ccc}\sqrt{x}=2 & \text{or} & \sqrt{x}=1\\ \left(\sqrt{x}\right)^2=(2)^2 & & \left(\sqrt{x}\right)^2=(1)^2\\ x=4 & \text{or} & x=1\end{array}$$

Due to the fact that we squared both sides of an equation when solving for x, we check the solutions to ensure there are no extraneous solutions.

$$\begin{array}{rr}\underline{x=4} & \underline{x=1}\\ x-3\sqrt{x}+2=0 & x-3\sqrt{x}+2=0\\ 4-3\sqrt{4}+2\stackrel{?}{=}0 & 1-3\sqrt{1}+2\stackrel{?}{=}0\\ 4-6+2\stackrel{?}{=}0 & 1-3+2\stackrel{?}{=}0\\ 0=0\,\checkmark & 0=0\,\checkmark\end{array}$$

Both $x=4$ and $x=1$ satisfied $x-3\sqrt{x}+2=0$, so they are the solutions to $x-3\sqrt{x}+2=0$.

▪

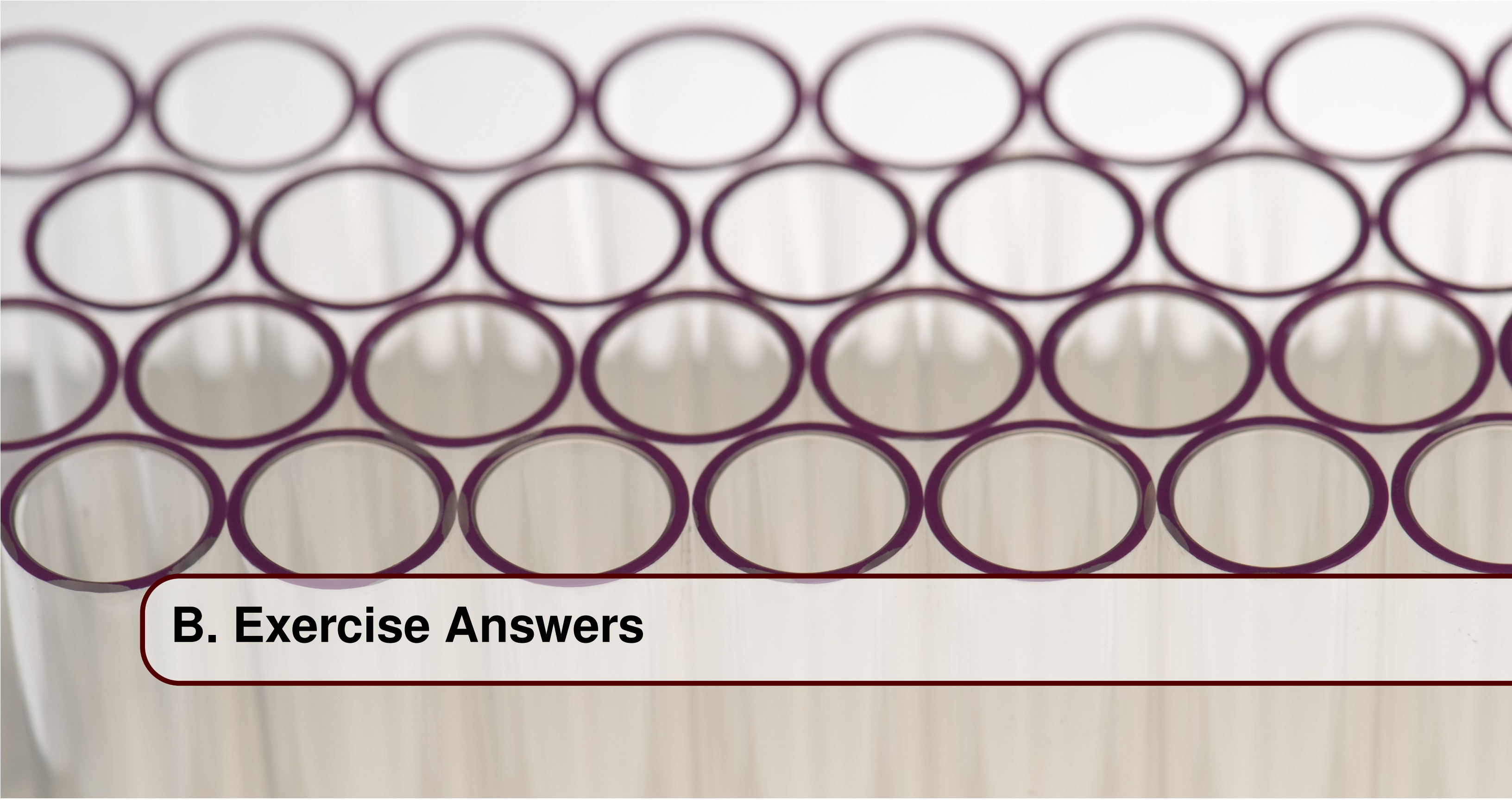
EXERCISES

Skills Practice (Answers)

For Exercises 50 - 59, solve the given equation.

50. $x^4 - 7x^2 + 12 = 0$
51. $x^4 - 9x^2 + 18 = 0$
52. $x^4 - 13x^2 - 30 = 0$
53. $x^4 + 5x^2 - 36 = 0$
54. $2x^4 - 5x^2 + 3 = 0$
55. $4x^4 - 5x^2 + 1 = 0$
56. $(x-3)^2 - 5(x-3) - 36 = 0$
57. $(3y+2)^2 + (3y+2) - 6 = 0$
58. $x - \sqrt{x} - 20 = 0$
59. $12x + 5\sqrt{x} - 3 = 0$

Exercise Answers



B. Exercise Answers

The answers to most communication exercises in this text may vary. The authors have provided a possible solution for each of these exercises.

Section 1.1 Answers

1. A is 2×2; B is 1×4; C is 2×3; D is 4×2
2. $a_{11} = 2$
3. $b_{13} = 1$
4. $c_{21} = 10$
5. d_{24} does not exist
6. b_{31} does not exist
7. $d_{42} = 7$
8. 2×2
9. Not possible
10. 1×2
11. 2×1
12. 2×2
13. Not possible
14. 3×2
15. 2×3
16. $\begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 12 \\ -8 & -2 \end{bmatrix}$

18. $\begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$

19. $\begin{bmatrix} -2 & -12 \\ 8 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 6 & 12 \\ -21 & 9 \end{bmatrix}$

21. $\begin{bmatrix} 0 & -4 \\ \frac{1}{2} & \frac{5}{2} \end{bmatrix}$

22. $\begin{bmatrix} -4 & -40 \\ 18 & 14 \end{bmatrix}$

23. $\begin{bmatrix} 10 & 28 \\ -36 & 10 \end{bmatrix}$

24. $\begin{bmatrix} 2 & -7 \\ 4 & 3 \end{bmatrix}$

25. $\begin{bmatrix} 0 & 4 \\ -32 & 20 \end{bmatrix}$

26. $\begin{bmatrix} 2 & -6 \\ -4 & 8 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$

28. $\begin{bmatrix} -5.5 & -3 \\ -3.85 & 19 \\ -0.7 & -2.25 \end{bmatrix}$

29. $\begin{bmatrix} 6.5 & -3 \\ 4.15 & 1 \\ -1.3 & 1.75 \end{bmatrix}$

30. $\begin{bmatrix} -5.5 & -3 \\ -3.85 & 19 \\ -0.7 & -2.25 \end{bmatrix}$

31. $\begin{bmatrix} -6.5 & 3 \\ -4.15 & -1 \\ 1.3 & -1.75 \end{bmatrix}$

32. $\begin{bmatrix} 50 & -300 \\ 15 & 1000 \\ -100 & -25 \end{bmatrix}$

33. $\begin{bmatrix} 36 & 0 \\ 24 & -54 \\ -1.8 & -12 \end{bmatrix}$

34. $\begin{bmatrix} 65 & -30 \\ 41.5 & 10 \\ -13 & -22.5 \end{bmatrix}$

35. $\begin{bmatrix} -3.5 & -15 \\ -3.25 & 59 \\ -4.7 & -3.25 \end{bmatrix}$

36. $\begin{bmatrix} 0.5 & 0.15 & -1 \\ -3 & 10 & -0.25 \end{bmatrix}$

37. $\begin{bmatrix} -0.6 & -0.4 & 0.03 \\ 0 & 0.9 & -0.2 \end{bmatrix}$

38. $\begin{bmatrix} -6.5 & -4.15 & 1.3 \\ 3 & -1 & -1.75 \end{bmatrix}$

39. Not possible

40. $a = 5,\ b = 6,\ c = 7,\ d = 8$

41. $a = -2,\ b = -4,\ c = 6,\ d = -18$

42. $a = 5,\ b = 7,\ c = 6,\ d = 8$

43. $a = 9,\ b = 1,\ c = -26,\ d = 2$

44. $\begin{array}{c} \\ CB \\ FM \\ VM \\ MM \end{array} \begin{array}{c} \begin{array}{ccc} \text{Fat} & \text{Carbs} & \text{Protein} \end{array} \\ \begin{bmatrix} 5 & 0 & 7 \\ 0 & 5 & 15 \\ 10 & 9 & 0 \\ 6 & 0 & 12 \end{bmatrix} \end{array}$ $\begin{array}{c} \\ \text{Fat} \\ \text{Carbs} \\ \text{Protein} \end{array} \begin{array}{c} \begin{array}{cccc} CB & FM & VM & MM \end{array} \\ \begin{bmatrix} 5 & 0 & 10 & 6 \\ 0 & 5 & 9 & 0 \\ 7 & 15 & 0 & 12 \end{bmatrix} \end{array}$

45. $\begin{array}{c} \\ JC \\ TX \\ VM \end{array} \begin{array}{c} \begin{array}{cccc} B & G & A & O \end{array} \\ \begin{bmatrix} 100 & 20 & 90 & 150 \\ 70 & 10 & 120 & 75 \\ 110 & 35 & 180 & 100 \end{bmatrix} \end{array}$ $\begin{array}{c} \\ B \\ G \\ A \\ O \end{array} \begin{array}{c} \begin{array}{ccc} JC & TX & VM \end{array} \\ \begin{bmatrix} 100 & 70 & 110 \\ 20 & 10 & 35 \\ 90 & 120 & 180 \\ 150 & 75 & 100 \end{bmatrix} \end{array}$

46. A is 3×1; B is 3×3; C is 3×2; D is 1×1

47. $c_{22} = f$

48. d_{12} does not exist

49. $a_{31} = 3$

50. $b_{23} = -5$

51. c_{13} does not exist

52. $d_{11} = 100g$

53. $\begin{bmatrix} 15 & -5 \\ 10 & 5k \end{bmatrix}$

54. Not possible

55. $\begin{bmatrix} -1 & x+1 \\ -2 & -1-k \end{bmatrix}$

56. $\begin{bmatrix} 3 & 0 \\ 3p & 3 \\ 3 & 0 \end{bmatrix}$

57. $\begin{bmatrix} 12 & 3x-2 \\ 4 & -3+2k \end{bmatrix}$

58. $\begin{bmatrix} 6y-3 & 2-3p & 1 \\ 2z & -1 & 8 \end{bmatrix}$

59. Not possible

60. Not possible

61. $\begin{bmatrix} -1 & 3 \\ 4x-6 & -4-3k \end{bmatrix}$

62. $\begin{bmatrix} 5 & 2 \\ x-1 & -1+k \end{bmatrix}$

63. $\begin{bmatrix} 3y-2 & z \\ 1-2p & -1 \\ 0 & 4 \end{bmatrix}$

64. $\begin{bmatrix} 0.25 & 0.75 \\ -0.25 & 0 \\ 0.25g & h \end{bmatrix}$

65. $\begin{bmatrix} 3y+2 & p & 3+g \\ z+3 & 2 & 4+4h \end{bmatrix}$

66. $\begin{bmatrix} 3 & -6 \\ 5p+2 & 5 \\ 5-2g & -8h \end{bmatrix}$

67. $\begin{bmatrix} 5 & 2 \\ x-1 & -1+k \end{bmatrix}$

68. $a = 15,\ b = \frac{1}{5},\ c = 5,\ d = 0$

69. $a = \frac{80}{3},\ b = -\frac{3}{4},\ c = 2,\ d = \frac{8}{7}$

70. $a = 2,\ b = 4,\ c = 3,\ d = -9$

71. $A = \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}$

72. $A = \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix}$

73. Column 1 represents the percentage of those currently subscribing to the the Tribune and whether they want to continue or switch. Column 2 represents the percentage of those currently subscribing to the Picayune and whether they want to switch or continue.

74. $\begin{bmatrix} 80 \\ 70 \end{bmatrix}$

75. $3y + 5w + 3k$

76. $\begin{bmatrix} -3x & 2 \\ -12-9y & -52 \end{bmatrix}$

77. C

78. $e_{21} = -1$

79. d_{24} does not exist

80. $b_{22} = -5w$

81. $d_{31} = -h$

82. $w = -\frac{4}{5},\ x = 0,\ y = -1$

83. $e = 10,\ f = 8,\ g = 10,\ h = 1,\ k = 9$

84. $a = -5,\ b = -5,\ c = 9,\ d = 2$

85. $X = -\frac{1}{2}A$

86. $X = -\frac{11}{3}B$

87. $X = \frac{1}{4}\left(B - \frac{1}{2}A\right)$

88. **a.** First Year $\begin{array}{c} \\ F \\ S \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 78.2 & 71.5 & 74.3 & 73.8 & 76.9 \\ 72.2 & 70.5 & 69.8 & 71.8 & 73.4 \end{bmatrix} \end{array}$

Second Year $\begin{array}{c} \\ F \\ S \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 79.4 & 73.8 & 71.9 & 75.1 & 76.9 \\ 71.6 & 72.7 & 73.1 & 72.8 & 74.9 \end{bmatrix} \end{array}$

b. $\begin{bmatrix} 1.2 & 2.3 & -2.4 & 1.3 & 0 \\ -0.6 & 2.2 & 3.3 & 1 & 1.5 \end{bmatrix}$

c. $\begin{bmatrix} 79.764 & 72.93 & 75.786 & 75.276 & 78.438 \\ 73.644 & 71.91 & 71.196 & 73.236 & 74.868 \end{bmatrix}$

89. For $(A+C)^T$ we add the corresponding entries of the two matrices and then transpose the resulting matrix, so the results of row one become the entries in column one and so on. For $A^T + C^T$ we first transpose each matrix, making row one into column one in each matrix and so on, then we add the corresponding entries of each transposed matrix. The same entries are being added, before and after the transposition.

90. When setting corresponding entries equal, in row one we have $a = b$ and $b = c$ and in row two we have $c = d$ and $d = a$. Therefore, by the properties of real numbers, $a = b = c = d$.

Section 1.2 Answers

1. Not possible
2. 2×3
3. Not possible
4. 1×2

5. 2×2
6. 1×3
7. Not possible
8. Not possible
9. $\begin{bmatrix} -23 \end{bmatrix}$
10. $\begin{bmatrix} 0 & 20 & 36 \\ 0 & 10 & 18 \\ 0 & -35 & -63 \end{bmatrix}$
11. $\begin{bmatrix} 22 \\ -1 \end{bmatrix}$
12. $\begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}$
13. $\begin{bmatrix} -1 & -6 & -2 \\ 0 & 6 & 10 \end{bmatrix}$
14. 2×2
15. Not possible
16. 3×2
17. Not possible
18. 2×3
19. 3×2
20. 2×2
21. Not possible
22. $\begin{bmatrix} 2w + 4s + 3z \\ 2t + 4sy - 15 \\ -18 + 3x \end{bmatrix}$
23. $\begin{bmatrix} 200g \\ 400sg \\ 300g \end{bmatrix}$
24. Not possible
25. $\begin{bmatrix} we + \frac{1}{4} + 6z & \frac{1}{2}w + f - 11z \\ te + \frac{1}{4}y - 30 & \frac{1}{2}t + fy + 55 \\ -9e + 6x & -\frac{9}{2} - 11x \end{bmatrix}$
26. Not possible
27. $\begin{bmatrix} 2w + 4st - 27 & 2 + 4sy & 2z - 20s + 3x \end{bmatrix}$
28. $\begin{bmatrix} 200wg + 400sg + 300zg \\ 200tg + 400syg - 1500g \\ -1800g + 300xg \end{bmatrix}$

29. $\begin{bmatrix} 400g & 800sg & 600g \\ 800sg & 1600s^2g & 1200sg \\ 600g & 1200sg & 900g \end{bmatrix}$

30. $\begin{bmatrix} 30y+5xz & 10+5x & 20+20x \\ -5z & -5 & -20 \end{bmatrix}$

31. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2}(p+3) & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}p & -\frac{1}{2} \\ \frac{g}{2} & \frac{1}{2}(gp+4h) & \frac{g}{2} \end{bmatrix}$

32. $\begin{bmatrix} 2x & -2+xk-3x \\ -2 & -k+3 \end{bmatrix}$

33. $\begin{bmatrix} 9y-z-\frac{1}{5} & 2-\frac{p}{5} & \frac{9}{5} \\ 6y+kz & \frac{9}{5}+k & 4+4k \end{bmatrix}$

34. $\begin{bmatrix} 12y & 1 & 3g-4h+8 \\ 2+3k+4z & 2 & 2g+4kh+16 \end{bmatrix}$

35. $\begin{bmatrix} 2-9y & x-3-3z \\ -5 & -x-3 \\ 2g-6 & xg-4h-12 \end{bmatrix}$

36. $\begin{bmatrix} 3y+2g-2x-7 & 9y+8h-xk+2 \\ z+4g+1 & 3z+16h+k \end{bmatrix}$

37. Not possible

38. c must equal d where $c = d =$, a positive integer

39. **a.** $e_{25} = 8h-4m-4g$
    b. $d = 0,\ g = -1,\ h = 1,\ k = 3,\ m = 9,\ p = 2$

40. **a.** $S = \begin{bmatrix} 150 \\ 0 \end{bmatrix}$
    b. $PS = \begin{bmatrix} 135 \\ 15 \end{bmatrix}$
    One week after the founding, 135 residents continue to subscribe to the Tribune and 15 residents now subscribe to the Picayune.

41. Given A is an $m\times n$ matrix and B is a $p\times q$ matrix, when we compute AB we use the entries of a row from A (of which there are n entries) and a column from B (of which there are p entries). If the inner dimensions are not equal, $n \neq p$, then we do not have matching entries to multiply.

42. If C is an $m\times n$ matrix, then when computing $C^2 = CC$ we would be multiplying an $m\times n$ matrix by an $m\times n$ matrix. Matrix multiplication requires the inner dimensions to be the same, so $n = m$. Therefore, C really has dimensions $m\times m$, which is the definition of a square matrix.

Chapter 1 Review Answers

1. **a.** A is 3×4; $a_{32} = 17$
   b. B is 2×3; $b_{23} = 0$

c. $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2. **a.** $\begin{array}{l} \\ \text{Chicken Noodle} \\ \text{Vegan Vegetable} \\ \text{Butternut Squash} \end{array} \begin{array}{c} \begin{array}{cc} \text{ER} & \text{CD} \end{array} \\ \begin{bmatrix} 24 & 18 \\ 30 & 11 \\ 10 & 2 \end{bmatrix} \end{array}$

b. $\begin{array}{l} \\ \text{price (\$)} \end{array} \begin{array}{c} \begin{array}{cccc} \text{waffle} & \text{chicken tenders} & \text{fries} & \text{soda} \end{array} \\ \begin{bmatrix} 5 & 6 & 2.50 & 1.75 \end{bmatrix} \end{array}$

3. **a.** $\begin{bmatrix} 12-b & 3 \\ 3a-2 & 3b-3c+5 \end{bmatrix}$

b. Not possible; DB is a 3×2 matrix and AC is a 2×3, matrix, so they are not the same size and cannot be subtracted.

c. $\begin{bmatrix} 7 & 4 & 12 \\ 3 & -11 & 10 \end{bmatrix}$

d. $\begin{bmatrix} -30-\frac{b}{2} & -\frac{15}{2} \\ 61 & \frac{51}{2} \end{bmatrix}$

e. Not possible; DA is a 3×2 matrix and C^T is a 3×2 matrix, so the number of columns of DA does not equal the number of rows of C^T.

f. $\begin{bmatrix} 4b+4 & ab+2b-2c \\ 2 & 3a-5b+5c \end{bmatrix}$

g. $a = 35$, $b = 8$, and $c = -23$

4. **a.** $\begin{array}{c} \\ A \\ P \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 42 & 36 & 40 \\ 28 & 20 & 26 \end{bmatrix} \end{array}$

The cost per hour for assembly and packaging doubled at each of the three plants.

b. Not possible; Not logical

5. **a.** **i.** CD is possible; The result is a 2×4 matrix.

ii. AC is possible; The result is a 2×3 matrix.

CA is not possible, as the inner dimensions of the two matrices are not equal.

Thus, matrix multiplication order matters.

iii. Not possible; The inner dimensions of the two matrices are not equal; $\left(D^T\right)^T = D$ which is a 3×4 matrix, but B is a 3×2 matrix.

b. $c_{12} = x+16$

c. $\begin{bmatrix} -g-5h & -3+7h \\ 6g+10 & 4 \\ kg-15 & 3k+21 \\ -5m & 7m \end{bmatrix}$

d. $\begin{cases} 2x+4z = 17 \\ -x+3y-5z = 20 \\ 6y+10z = -12 \end{cases}$

6. **a.** **i.** $P = \begin{array}{r} \\ \text{Fenders} \\ \text{Doors} \\ \text{Hoods} \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 280 & 236 & 266 \\ 483 & 408 & 459 \\ 238 & 200 & 226 \end{bmatrix} \end{array}$

ii. $p_{32} = 200$; The total cost of assembly and packaging of hoods at Plant 2 is \$200.

iii. Plant 2; The total costs of all parts is lower than the other plants.

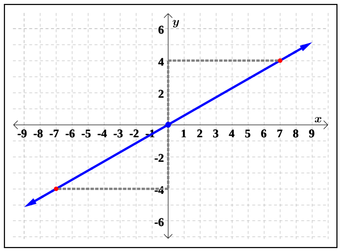
b. 2085 first generation students enrolled in the fall semester.

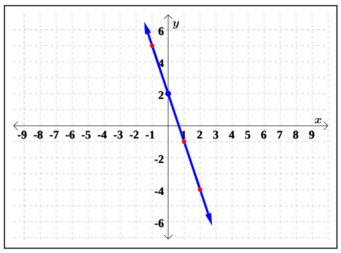
Section 2.1 Answers

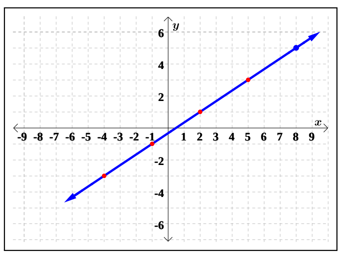
1. $m = \dfrac{5}{3}$
2. $m = \dfrac{10}{3}$
3. $m = \dfrac{8}{5} = 1.6$
4. $m = -\dfrac{1}{10} = -0.1$
5. x-intercept: $(2,0)$ and y-intercept: $(0,6)$
6. x-intercept: $\left(\dfrac{14}{5},0\right) = (2.8,0)$ and y-intercept: $(0,-14)$
7. x-intercept: $(2,0)$ and y-intercept: $(0,-3)$
8. x-intercept: $\left(\dfrac{9}{4},0\right) = (2.25,0)$ and y-intercept: $\left(0,\dfrac{9}{8}\right) = (0,1.125)$
9. Point-Slope Form: $y+1 = 3(x-3)$
   Slope-Intercept Form: $y = 3x-10$
10. Point-Slope Form: $y-1 = \dfrac{2}{3}(x+2)$
    Slope-Intercept Form: $y = \dfrac{2}{3}x+\dfrac{7}{3}$
11. Point-Slope Form: $y-8 = -2(x+5)$
    Slope-Intercept Form: $y = -2x-2$
12. Point-Slope Form: $y-4 = -\dfrac{1}{5}(x-10)$
    Slope-Intercept Form: $y = -\dfrac{1}{5}x+6$
13. Point-Slope Form: $y-10 = -2(x+3)$ or $y+6 = -2(x-5)$
    Slope-Intercept Form: $y = -2x+4$
14. Point-Slope Form: $y-3 = \dfrac{1}{2}(x-1)$ or $y-5 = \dfrac{1}{2}(x-5)$
    Slope-Intercept Form: $y = \dfrac{1}{2}x+\dfrac{5}{2}$ or $y = 0.5x+2.5$
15. Point-Slope Form: $y+15 = 2(x+6)$ or $y+7 = 2(x+2)$
    Slope-Intercept Form: $y = 2x-3$

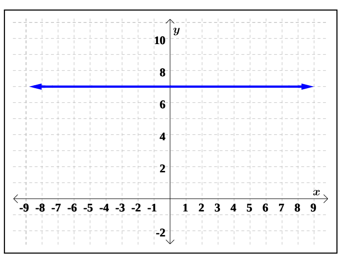
16. Point-Slope Form: $y+9=-\frac{29}{18}(x-4)$ or $y-20=-\frac{29}{18}(x+14)$

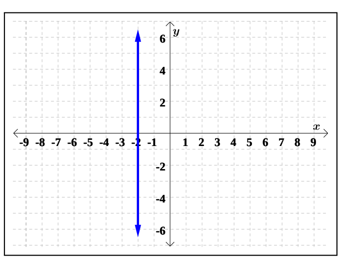
Slope-Intercept Form: $y=-\frac{29}{18}x-\frac{23}{9}$

17. 

18. 

19. 

20. 

21. 

22. **a.** y increases by 3 units
b. x increases by 4 units
c. y decreases by 3 units
d. x decreases by 4 units

23. $m=-\frac{5}{4}=-1.25$

24. $m=0$

25. $m = \frac{5}{2} = 2.5$

26. m is undefined

27. x-intercept: $\left(-\frac{1}{2}, 0\right) = (-0.5, 0)$ and y-intercept: $\left(0, -\frac{1}{4}\right) = (0, -0.25)$

28. x-intercept: $\left(\frac{3}{4}, 0\right) = (0.75, 0)$ and y-intercept: $\left(0, -\frac{3}{2}\right) = (0, -1.5)$

29. x-intercept: $(3, 0)$ and y-intercept: $\left(0, \frac{7}{2}\right) = (0, 3.5)$

30. x-intercept: $(-4, 0)$ and y-intercept: $\left(0, \frac{8}{5}\right) = (0, 1.6)$

31. Point-Slope Form: $y - 117 = 0(x - 3)$
    Slope-Intercept Form: $y = 117$

32. $x = 10$

33. Point-Slope Form: $y - 8 = 1.75(x + 4)$
    Slope-Intercept Form: $y = 1.75x + 15$

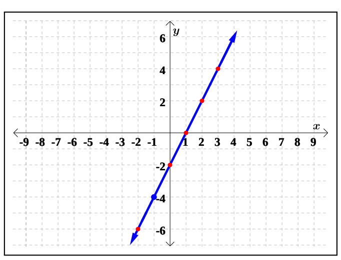
34. Point-Slope Form: $y - 0.9 = -0.1(x - 0.2)$
    Slope-Intercept Form: $y = -0.1x + 0.92$

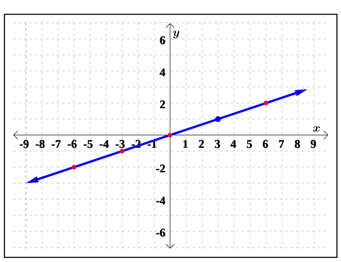
35. Point-Slope Form: $y - 7 = 0(x - 1)$
    Slope-Intercept Form: $y = 7$

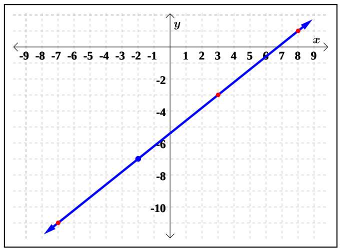
36. $x = -6$

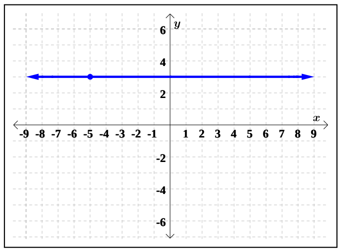
37. Point-Slope Form: $y - 6 = -2(x + 2)$ or $y - 0 = -2(x - 1)$
    Slope-Intercept Form: $y = -2x + 2$

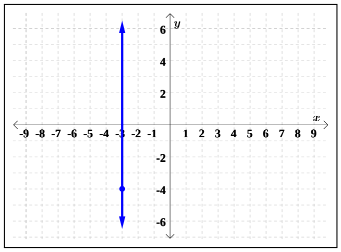
38. Point-Slope Form: $y + 1 = -\frac{3}{4}(x - 4)$ or $y - 2 = -\frac{3}{4}(x - 0)$
    Slope-Intercept Form: $y = -\frac{3}{4}x + 2$

39. 

40. 

41. 

42. 

43. 

44. **a.** y decreases by $\frac{9}{7}$ units

b. x decreases by $\frac{7}{3}$ units

c. y increases by $\frac{90}{7}$ units

d. x increases by $\frac{7}{9}$ units

45. $m = \frac{2a+5}{4-a}$, undefined slope when $a = 4$

46. slope: $-\frac{1}{2} = -0.5$

x-intercept: $(1, 0)$

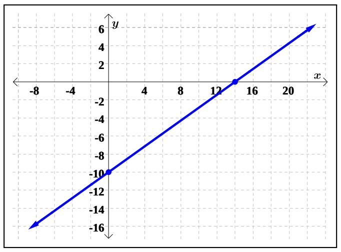
y-intercept: $\left(0, \frac{1}{2}\right) = (0, 0.5)$

47. Point-Slope Form: $y + 4 = \frac{6}{5}(x - (c-5))$

Slope-Intercept Form: $y = \frac{6}{5}x + \left(2 - \frac{6}{5}c\right)$

48. $y = \frac{7}{110}x + \frac{7}{11}$

49. x-intercept: $(14,0)$; y-intercept: $(0,-10)$



50. **a.** y decreases by $\dfrac{6}{11}$ units

    b. x decreases by 11 units

    c. y increases by $\dfrac{42}{11}$ units

    d. x increases by $\dfrac{121}{6}$ units

51. $m = \dfrac{2}{9}$

52. The y-intercept of a line is the point on the corresponding graph where the line crosses the y-axis.

53. To determine the x-intercept of a line algebraically, we substitute 0 for y and solve for x. The corresponding point $(x,0)$ is the x-intercept. To determine the y-intercept of a line algebraically, we substitute 0 for x and solve for y. The corresponding point $(0,y)$ is the y-intercept.

54. If a point is located on the y-axis, then the x-coordinate is always 0, as we did not move left or right from the origin.

55. If a point is located on the x-axis, then the y-coordinate is always 0, as we did not move up or down from the origin.

Section 2.2 Answers

1. $V(t) = -60t + 500$

2. $V(t) = -100t + 1700$

3. $V(t) = -\dfrac{425}{4}t + 850$ or $V(t) = -106.25t + 850$

4. \$1250

5. After 10 years

6. After 50 years

7. \$500

8. \$25/year

9. $C(x) = 25x + 10000$;
   $x :=$ the number of items produced
   $C(x) =$ the total cost (in dollars)

10. $R(x) = 150x$;
    $x :=$ the number of items sold
    $R(x) :=$ the total revenue (in dollars)

11. $P(x) = 17x - 800$

    $x :=$ the number of items produced and sold

    $P(x) :=$ the total profit (in dollars)

12. **a.** \$15
    b. \$40
    c. 40 items
    d. 80 items

13. **a.** \$315
    b. \$15
    c. 10 items

14. $p(x) = -\frac{3}{8}x + \frac{115}{2}$

15. $p(x) = \frac{1}{5}x + 15$

16. $V(t) = -\frac{19}{20}t + 10$

17. $V(t) = -20000t + 40000$

18. $V(t) = -300t + 3500$

19. $\$1,000,000$

20. 10 years

21. $\$575,000$

22. $\$150,000$

23. $\$85,000$/year

24. $P(x) = 4x - 15000$;

    $x :=$ the number of items produced and sold

    $P(x) :=$ the profit (in dollars)

25. $C(x) = 15x + 495$;

    $x :=$ the number of items produced

    $C(x) :=$ the total cost (in dollars)

26. $C(x) = 20x + 850$;

    $x :=$ the number of items produced

    $C(x) :=$ the total cost (in dollars)

27. $R(x) = 15x$;

    $x :=$ the number of items sold

    $R(x) :=$ the total revenue (in dollars)

28. $p(x) = -\frac{1}{2}x + 252$

29. $p(x) = -\frac{1}{100}x + 13.50$ or $p(x) = -0.01x + 13.50$

30. $p(x) = \frac{1}{100}x + 9.50$ or $p(x) = 0.01x + 9.50$

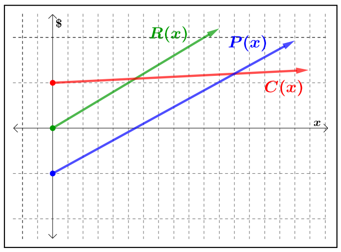
31. $p(x) = \frac{1}{32}x + 475$

32. **a.** $D(x) = p(x) = -\frac{1}{400}x + 40$

    b. $S(x) = p(x) = \frac{1}{400}x + 10$

33. \$5125

34. \$425

35. 

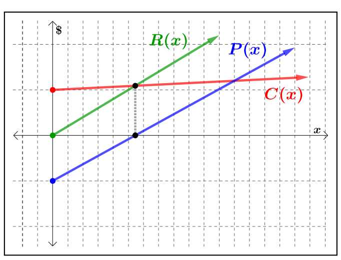
36. **a.** \$65/item
    b. $P(x) = 15x - 7200$;
    $x :=$ the number of items produced and sold
    $P(x) :=$ the profit (in dollars)

37. \$75/item

38. **a.** The demand equation is Equation B, because the slope is negative.
    b. 100 items
    c. \$150
    d. \$30

39. The rate of depreciation for an item is the absolute value of the slope of its linear depreciation model and tells the decrease in value of the item per unit of time.

40. If $V(t)$ is given only for values $0 \le t \le 20$, then for $t = 20$ the item reaches its lowest value (scrap value). For any value of t greater than 20, the item will still be worth its scrap value.

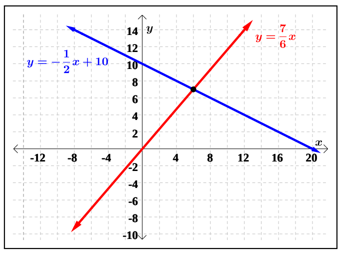
41. 

42. Supply is defined from the producers' view, who would like to make the most money per item sold. As the price per item decreases, the number of items producers are willing to supply will also decrease. Similarly, the producers are willing to increase supply as the price per item increases. These behaviors result in a positive slope for $p(x)$.

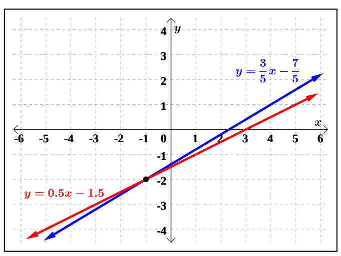
43. Demand is defined from the consumers' view, who would like to spend the least amount of money on each item they purchase. As the price per item decreases, the number of items consumers are willing to buy will increase. However, consumers will decrease the number of items they buy as the price per item increases. These behaviors result in a negative slope for $p(x)$.

Section 2.3 Answers

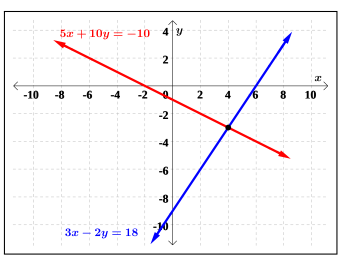
1. Not a solution
2. Not a solution
3. A solution
4. Inconsistent system, 0 solutions
5. Independent system, 1 solution
6. Dependent system, Infinitely many solutions
7. Dependent system, Infinitely many solutions
8. Inconsistent system, 0 solutions
9. Independent system, 1 solution

10. 

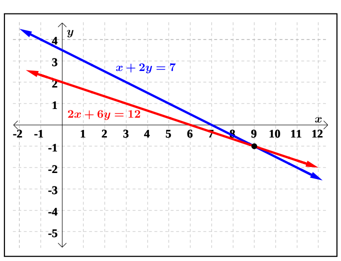
Solution: $(x, y) = (6, 7)$

11. 

Solution: $(x, y) = (-1, -2)$

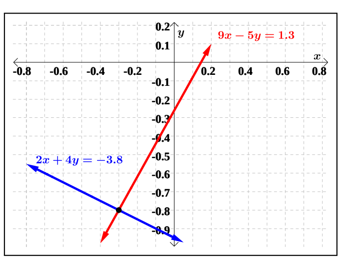
12. 

Solution: $(x, y) = (4, -3)$

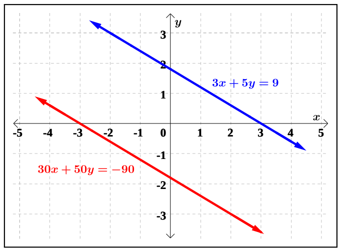
13. 

Solution: $(x, y) = (9, -1)$

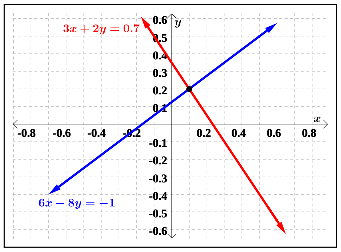
14. $(x, y) = (-3, 1)$

15. No solution
16. $(x,y) = (t, 3t+2)$ or $\left(\frac{1}{3}t - \frac{2}{3}, t\right)$, where t is any real number
17. $(x,y) = \left(-\frac{1}{50}, -\frac{33}{50}\right)$
18. $(x,y) = \left(-\frac{1}{3}, \frac{3}{2}\right)$
19. $(x,y) = \left(\frac{1}{6}, 0\right)$
20. No solution
21. $(x,y) = \left(t, -\frac{7}{6}t + \frac{1}{3}\right)$ or $\left(-\frac{6}{7}t + \frac{2}{7}, t\right)$, where t is any real number
22. Break-even point: $(2, 40)$
23. Break-even point: $(200, 40000)$
24. Break-even point: $(20, 220)$
25. Equilibrium point: $(12, 65)$
26. Equilibrium point: $(14, 38)$
27. Equilibrium point: $(30, 290)$
28. Not a solution
29. A solution
30. Not a solution
31. Independent system, 1 solution
32. Inconsistent system, 0 solutions
33. Dependent system, Infinitely many solutions
34. Inconsistent system, 0 solutions
35. Independent system, 1 solution
36. Dependent system, Infinitely many solutions
37. 

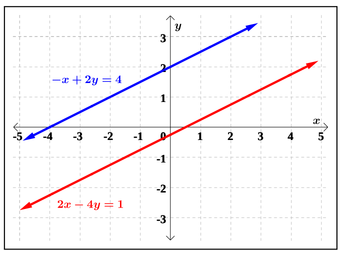
Solution: $(x,y) = (-0.3, -0.8)$

38. 

No solution

39. 

Solution: $(x, y) = \left(\dfrac{1}{10}, \dfrac{1}{5}\right)$

40. 

No solution

41. $(x, y) = (14.4, 26.4) = \left(\dfrac{72}{5}, \dfrac{132}{5}\right)$

42. $(x, y) = (-0.6, 0) = \left(-\dfrac{3}{5}, 0\right)$

43. $(x, y) = \left(t, \dfrac{12}{5}t + 11\right)$ or $\left(\dfrac{5}{12}t - \dfrac{55}{12}, t\right)$, where t is any real number

44. No solution

45. $(x, y) = (6, -6)$

46. No solution

47. $(x, y) = \left(t, \dfrac{1}{2}t - \dfrac{3}{2}\right)$ or $(3 + 2t, t)$, where t is any real number

48. $(x, y) = (-4, 2)$

49. price: \$200/guitar

    $R(x) = 200x$

50. 1250 people

    \$100,000 in total ticket sales

51. 24,000 meals

52. Market equilibrium point: $(104, 18.5)$

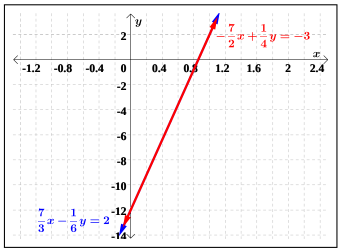
53. Market equilibrium point: $(125, 40)$

54. Market equilibrium point: $(106, 199)$

55. k is any real number, such that $k \neq 10$

56. $k = -9$

57. $y = \frac{3}{4}x + b$, where $b \neq -2$

58. 

Infinitely many solutions:

$(x, y) = (t, 14t - 12)$ or $\left(\frac{1}{14}t + \frac{6}{7}, t\right)$,

where t is any real number

59. Break-even point: $(3500, 17500)$

60. Break-even point: $(80, 16000)$

61. More than 800 items must be sold in order for the company to turn a profit.

62. Market equilibrium point: $(100, 95)$

63. The graphical method is best when both lines are easy to graph (have 'nice' intercepts) and the intersection(s) are easy to read from the graph. The substitution method is best when one of the equations is already solved for one of the variables. The addition method is best when the coefficient of one of the variables in one equation is opposite the coefficient of the same variable in the other equation.

64. The system has infinitely many solutions, because $0 = 0$ is a true statement regardless of the values of the variables.

65. The system has no solution, because $0 = 5$ is a false statement regardless of the values of the variables.

66. It costs the company \$2884, when the company produces 360.5 pounds of gluten-free flour. When 360.5 pounds of flour are sold, the company will bring in exactly \$2884 in revenue and the company will break-even. The company can truly break-even as it is possible to produce and sell 0.5 pounds of gluten-free flour.

67. If $(315, 45675)$ is the break-even point, then the point $(315, 0)$ will be on the graph of the profit function.

68. At a price of \$489, producers will supply 2600 boats to the market, and at a price of \$489, consumers will purchase all 2600 boats.

Section 2.4 Answers

1. $b :=$ the number of people who chose the basic buffet

   $d :=$ the number of people who chose the deluxe buffet

$$\begin{cases} 7.50b + 9.25d &= 227.00 \\ b + d &= 27 \end{cases}$$

2. $x :=$ the number of pounds (lbs) of \$3/lb beans

   $y :=$ the number of pounds (lbs) of \$8/lb beans

$$\begin{cases} x + y &= 50 \\ 3x + 8y &= 300 \end{cases}$$

3. $x :=$ the amount of money invested in the 3% account

   $y :=$ the amount of money invested in the 8% account

$$\begin{cases} x+y &= 10000 \\ 0.03x+0.08y &= 500 \end{cases}$$

4. $x :=$ the annual salary, in dollars, of the warehouse manager

   $y :=$ the annual salary, in dollars, of the office manager

   $z :=$ the annual salary, in dollars, of the truck driver

$$\begin{cases} x+y &= 82000 \\ y &= z+4000 \\ x+z &= 78000 \end{cases}$$

5. $p :=$ the number of pennies in the bag

   $n :=$ the number of nickels in the bag

   $d :=$ the number of dimes in the bag

$$\begin{cases} p+n+d &= 325 \\ 0.01p+0.05n+0.10d &= 19.50 \\ n &= d \end{cases}$$

6. $\left[\begin{array}{rr|r} 7 & -3 & 5 \\ 2 & 8 & 1 \end{array}\right]$

7. $\left[\begin{array}{rrr|r} 5 & 3 & -9 & 15 \\ -2 & -4 & 3 & 10 \\ 6 & -11 & 2 & 20 \end{array}\right]$

8. $\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$

9. $\left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right]$

10. $\begin{cases} 2x+5y &= 32 \\ 3x+6y &= 67 \end{cases}$

11. $\begin{cases} 3x-4y+10z &= 50 \\ x+6y-8z &= -20 \\ -7x+y+3z &= 66 \end{cases}$

12. $\begin{cases} x-4y &= 9 \\ 8y &= 24 \end{cases}$

13. $\begin{cases} 3y-5z &= 25 \\ x-4y &= 41 \\ 2x-6z &= 37 \end{cases}$

14. $\left[\begin{array}{rr|r} 1 & \frac{5}{2} & -3 \\ -3 & 7 & 8 \end{array}\right]$

15. $\left[\begin{array}{rr|r} 1 & -2 & -6 \\ 0 & 14 & 27 \end{array}\right]$

16. $\left[\begin{array}{rrr|r} 1 & 4 & -1 & 12 \\ 3 & -1 & 6 & 10 \\ 2 & 5 & -3 & 8 \end{array}\right]$

17. $\left[\begin{array}{rrr|r} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & \frac{10}{9} \end{array}\right]$

18. $\left[\begin{array}{rrr|r} 1 & 0 & -26 & 33 \\ 0 & 1 & -4 & 5 \\ 0 & 10 & -8 & 9 \end{array}\right]$

19. Not in reduced row-echelon form; Condition 2
20. In reduced row-echelon form
21. Not in reduced row-echelon form; Condition 1
22. In reduced row-echelon form
23. Not in reduced row-echelon form; Condition 3
24. Not in reduced row-echelon form; Condition 4
25. $(x, y) = (-2, 7)$
26. $(x, y, z) = (7, 8, 9)$
27. No solution
28. No solution
29. $(x, y, z) = (-3 - 9t, 20 + 4t, t)$, where t is any real number
30. $(w, x, y, z) = (4 - 3t, -6 - 6t, 2, t)$, where t is any real number
31. $x :=$ the number of non-tenure track faculty

    $y :=$ the number of tenure track faculty

    $$\begin{cases} x + y &= 130 \\ x &= y + 18 \end{cases}$$

32. $c :=$ the cost of a CD, in dollars

    $d :=$ the cost of a DVD, in dollars

    $$\begin{cases} c &= d + 5.96 \\ 5c + 2d &= 127.73 \end{cases}$$

33. $x :=$ the amount of money, in dollars, invested in the account paying 4% simple interest

    $y :=$ the amount of money, in dollars, invested in the account paying 3.125% simple interest

    $z :=$ the amount of money, in dollars, invested in the account paying 2.5% simple interest

    $$\begin{cases} x + y + z &= 80500 \\ 0.04x + 0.03125y + 0.025z &= 2670 \\ y &= 4z \end{cases}$$

34. x := the amount of your grocery bill, in dollars

y := the amount of Sarah's grocery bill, in dollars

z := the amount of Tara's grocery bill, in dollars

$$\begin{cases} x+y+z &= 82 \\ x &= \frac{1}{2}y-0.05 \\ z &= x+2.10 \end{cases}$$

35. x := the number of student tickets sold

y := the number of children tickets sold

z := the number of adult tickets sold

$$\begin{cases} x+y+z &= 1175 \\ 20x+22.50y+29z &= 28112.50 \\ z &= 2y \end{cases}$$

36. $\left[\begin{array}{rr|r} 1 & 2 & -20 \\ -3 & 1 & -2 \end{array}\right]$

37. $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 2 \end{array}\right]$

38. $\left[\begin{array}{rr|r} 1 & 4 & 6 \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{2} \end{array}\right]$

39. $\left[\begin{array}{rrr|r} 3 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 4 & 0 & 1 & 3 \\ -1 & \frac{3}{2} & 0 & \frac{5}{2} \end{array}\right]$

40. $\begin{cases} x &= -4 \\ y &= 5 \\ 0 &= 1 \end{cases}$

41. $\begin{cases} x-2y &= 12 \\ 5y+9z &= 14 \end{cases}$

42. $\begin{cases} x &= 9 \\ 2x+4y &= 24 \\ 0 &= 0 \end{cases}$

43. $\begin{cases} x+8z &= 11 \\ y-6z &= 22 \\ 0 &= 1 \end{cases}$

44. $-3R_1+R_2 \to R_2$; $\left[\begin{array}{rr|r} 1 & 4 & 5 \\ 0 & -14 & -16 \end{array}\right]$

45. $\frac{1}{4}R_2 \to R_2$; $\left[\begin{array}{rr|r} 1 & 2 & 6 \\ 0 & 1 & -3 \end{array}\right]$

46. $-5R_1+R_3 \to R_3$; $\left[\begin{array}{rrr|r} 1 & 2 & -1 & -10 \\ 0 & 5 & 10 & 30 \\ 0 & -18 & 7 & 70 \end{array}\right]$

47. $-\frac{1}{3}R_2 \to R_2$; $\left[\begin{array}{rrr|r} 1 & 2 & 7 & 8 \\ 0 & 1 & -4 & -5 \\ 0 & -1 & 4 & -9 \end{array}\right]$
48. $(x,y) = (-2,7)$
49. $(x,y,z) = (1,2,0)$
50. $(x,y,z) = (1,3,-2)$
51. No solution
52. $(x,y,z) = (5-t, 15-3t, t)$, where t is any real number
53. $(x,y) = (-4.5+1.5t, t)$, where t is any real number
54. $(x,y,z) = (-8.75-2t, -5.5-t, t)$, where t is any real number
55. $(x,y,z) = \left(\frac{1}{3}, \frac{2}{3}, 1\right)$
56. $(x,y,z) = \left(\frac{51}{13} + \frac{19}{13}t, \frac{4}{13} - \frac{11}{13}t, t\right)$, where t is any real number
57. No solution
58. 74 non-tenure track faculty and 56 tenure track faculty
59. \$147.68
60. \$25,500 in the first account, \$44,000 in the second account, and \$11,000 in the third account.
61. Your groceries total \$19.95, Sarah's groceries total \$40, and Tara's groceries total \$22.05.
62. 500 student tickets, 225 children tickets, and 450 adult tickets were sold.
63. $\frac{1}{3}R_1 \to R_1$; $4R_1 + R_2 \to R_2$; $-\frac{1}{22}R_2 \to R_2$; $6R_2 + R_1 \to R_1$
64. $R_1 \leftrightarrow R_2$; $-3R_1 + R_2 \to R_2$; $-2R_1 + R_3 \to R_3$; $-\frac{1}{6}R_2 \to R_2$; $-2R_2 + R_1 \to R_1$; $4R_2 + R_3 \to R_3$; $-\frac{1}{8}R_3 \to R_3$; $R_3 + R_2 \to R_2$; $-2R_3 + R_1 \to R_1$
65. \$750,000 is invested in Swan Peak and \$350,000 is invested in Riverside Community.
66. $x :=$ the number of lbs of Rosy Tea

    $y :=$ the number of lbs of Minty Tea

    $z :=$ the number of lbs of Sleepy Tea

    $(x,y,z) = \left(\frac{4}{3} - \frac{1}{2}t, \frac{2}{3} - \frac{1}{2}t, t\right)$, where $0 \le t \le \frac{4}{3}$
67. We would need an additional constraint of $x \ge 1$, to represent the need to use up a pound of Rosy Tea.
68. The four columns prior to the vertical bar indicate there are four variables in the corresponding system. The resulting matrix has two non-zero rows from which two equations can be written. We can easily solve the corresponding equation to the first row for the first variable and the corresponding equation to the second row for the second variable, but each of these solutions will still be in terms of the remaining two variables.

Chapter 2 Review Answers

1. **a.** Point-Slope Form: $y - \frac{4}{5} = \left(\frac{-4}{10a-35}\right)(x-7)$ or $y - 0 = \left(\frac{-4}{10a-35}\right)(x-2a)$

Slope-Intercept Form: $y = \left(\frac{-4}{10a-35}\right)x + \left(\frac{8a}{10a-35}\right)$

Standard Form: $4x + (10a-35)y = 8a$

b. $x = 4$

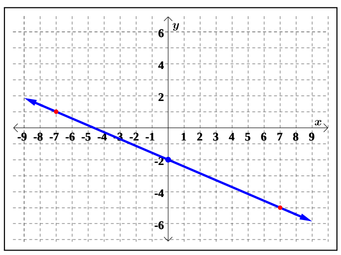
c. Point-Slope Form: $y - 0 = \frac{11}{8}(x-8)$ or $y + 11 = \frac{11}{8}(x-0)$

Slope-Intercept Form: $y = \frac{11}{8}x - 11$

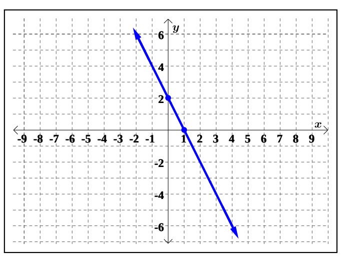
Standard Form: $-11x + 8y = -88$

d. $y = -7$

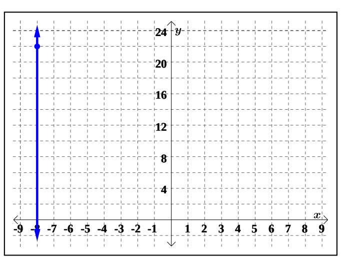
2. **a.**



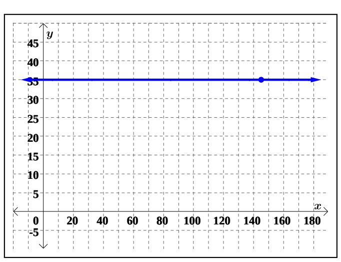
b.



c.



d.



3. **a.** x increases by $\frac{35}{3}$ units

b. y increases by 8 units

c. $m = \dfrac{\Delta y}{\Delta x} = -\dfrac{11}{6}$

4. **a.** The company will not profit when $0 \le x \le \dfrac{27}{4}$, x is an integer.

   b. $R(x)$ starts at the origin, $C(x)$ starts at the point of positive fixed costs, and $P(x)$ starts at the point of negative fixed costs. $P(x) = 0$ when $R(x) = C(x)$.

5. **a.** False

   b. False

   c. True

6. **a.** t := the number of years since purchase, V := the value of a refrigerator, in dollars
   $V(t) = -450t + 12000$
   Domain: $0 \le t \le \dfrac{80}{3}$
   Range: $0 \le V \le 12000$

   b. $C(x) = 100x + 225000$
   $R(x) = 400x$
   $P(x) = 300x - 225000$

   c. $p(x) = -\dfrac{2}{5}x + 500$; 975 computers; less than when the price was \$100.

   d. $p(x) = \dfrac{3}{4000}x + 2.25$; The price per watermelon is \$8.25.

7. **a.** In 2021, the value is \$400 and in 2030, the value is \$100.

   b. \$1400 per year

   c. The scrap value is \$14,000, which is the lowest value the car will be worth. Thus, after 10 years, the car will still be worth \$14,000.

8. **a.** Independent System, One unique solution

   b. $k = 24$

   c. $k = \dfrac{8}{5}$

9. **a.** $(x, y) = (2, 3)$

   b. $(x, y) = \left(t, \dfrac{2}{3}t - \dfrac{8}{3}\right)$, where t is any real number, or $(x, y) = \left(\dfrac{3}{2}p + 4, p\right)$, where p is any real number.

   c. No solution

10. **a.** $(x, y) = (2, 3)$

    b. $(x, y) = (4 + 1.5t, t)$, where t is any real number

    c. No solution

11. **a.** Set $R(x) = C(x)$ to find the break-even quantity. Then find the revenue at the break-even quantity, because the break-even point is (break-even quantity, break-even revenue).

    b. **i.** $P(x) = 550x - 4400$
    ii. $(8, 5160)$; It costs the company a total of \$5160 when making 8 items, and when those 8 items are sold, \$5160 in revenue is brought in by the company.

c. **i.** $(195, 20.50)$

ii. At a price of \$20.50, producers will supply 195 feeders, and at the same price consumers will buy all 195 feeders.

12. **a.** $f :=$ the amount of money, in dollars, invested in the first stock

$s :=$ the amount of money, in dollars, invested in the second stock

$t :=$ the amount of money, in dollars, invested in the third stock

$$\left.\begin{array}{c} f+s+t=100000 \\ 0.12f+0.10s+0.08t=10400 \\ s+t=f \end{array}\right\} \quad \begin{array}{rcl} f+s+t &=& 100000 \\ 0.12f+0.10s+0.08t &=& 10400 \\ -f+s+t &=& 0 \end{array}$$

b. $a :=$ the number of Package A sold

$b :=$ the number of Package B sold

$c :=$ the number of Package C sold

$$\left.\begin{array}{c} a+2b+3c=240 \\ 8a+14b+20c=1692 \\ 20c=2(8a)-32 \end{array}\right\} \quad \begin{array}{rcl} a+2b+3c &=& 240 \\ 8a+14b+20c &=& 1692 \\ -16a+20c &=& -32 \end{array}$$

13. **a.** $\left[\begin{array}{rr|r} 1 & 5 & 17 \\ -1 & -5 & 31 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} 2 & 7 & 1 & 9 \\ 1 & -5 & -9 & 8 \end{array}\right]$

c. $\left[\begin{array}{rrr|r} 4 & 8 & -9 & 0 \\ 1 & 0 & 1 & 17 \\ 18 & -11 & 0 & 2 \end{array}\right]$

14. **a.** Not in reduced row-echelon form; Condition 4

b. In reduced row-echelon form

c. In reduced row-echelon form

15. **a.** $\left[\begin{array}{rr|r} 1 & \frac{3}{4} & 4 \\ 10 & -6 & 18 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} 1 & 0 & -7 & -38 \\ 0 & 1 & 2 & 18 \\ 0 & 0 & 4 & 2 \end{array}\right]$

c. $\left[\begin{array}{rr|r} 1 & 3 & -6 \\ 0 & -36 & 84 \end{array}\right]$

16. **a.** $(x,y) = \left(-\dfrac{9}{4}, -\dfrac{37}{8}\right)$

b. $(x,y) = \left(-\dfrac{74}{7}, -\dfrac{34}{7}\right)$

17. **a.** $(x,y,z) = \left(4+\dfrac{1}{2}t, \dfrac{11}{6}+\dfrac{11}{12}t, t\right)$, where t is any real number

b. $(x,y,z) = \left(\dfrac{189}{29}, \dfrac{304}{29}, \dfrac{3188}{261}\right)$

c. No solution

18. **a.** $(x,y)=\left(\frac{4}{5},t\right)$, where t is any real number

b. No solution

c. $(x,y)=(6,-14)$

19. **a.** You invested \$50,000 in the first stock, \$20,000 in the second stock, and \$30,000 in the third stock.

b. 52 of Package A were sold, 34 of Package B were sold, and 40 of Package C were sold.

c. **i.** There are 9 solutions to the problem.

ii. Yes; $t=2\to(24,13,2)$

Section 3.1 Answers

1. **a.** x := the number of hours the student tutors per week

y := the number of hours the student grades per week

I := the student's weekly income (in dollars)

b. Objective: **Maximize** $I=30x+10y$

c. Subject to: $x+y\le 12$ (the number of working hours per week)

$2x+y\le 16$ (the number of prep hours per week)

$x\ge 0,\ y\ge 0$

2. **a.** x := the number of regular gadgets manufactured

y := the number of premium gadgets manufactured

P := the factory's daily profit (in dollars)

b. Objective: **Maximize** $P=20x+30y$

c. Subject to: $x+2y\le 12$ (the number of hours of assembly)

$2x+y\le 12$ (the number of hours of finishing)

$x+y\le 7$ (limit on the total number of gadgets per day)

$x\ge 0,\ y\ge 0$

3. **a.** x := the number of days Kit is employed each week

y := the number of days Kat is employed each week

C := the law office's weekly costs (in dollars)

b. Objective: **Minimize** $C=150x+300y$

c. Subject to: $20x+30y\ge 110$ (the number of documents to draft)

$x\ge 1,\ y\ge 1$

4. **a.** x := the number of days Professor Hamer eats the pasta meal

y := the number of days Professor Hamer eats the tofu meal

C := the total cholesterol (in mg)

b. Objective: **Minimize** $C=60x+50y$

c. Subject to: $8x+17y \geq 200$ (the number of grams of protein)

$60x+40y \geq 960$ (the number of grams of carbohydrates)

$2x+2y \geq 40$ (the number of grams of Vitamin C)

$x+y \leq 25$ (the number of lunches)

$x \geq 0,\ y \geq 0$

5. **a.** $x :=$ the amount of money, in dollars, invested in bonds

$y :=$ the amount of money, in dollars, invested in stocks

$P :=$ your profit (in dollars)

b. Objective: **Maximize** $P = 0.06x+0.08y$

c. Subject to: $x+y \leq 24000$ (the total amount of money invested)

$x \geq 2y$ (the ratio of the amounts invested in the two accounts)

$0 \leq x \leq 18000$ (limit on the amount of money invested in bonds)

$y \geq 0$

6. **a.** $x :=$ the number of objective quizzes given per semester

$y :=$ the number of recall quizzes given per semester

$G :=$ the amount of time spent grading (in minutes)

b. Objective: **Minimize** $G = x+1.5y$

c. Subject to: $x+y \geq 15$ (the number of quizzes)

$15x+30y \geq 300$ (the number of minutes students spend studying)

$7x+5y \geq 85$ (points students earn)

$x \geq 0,\ y \geq 0$

7. **a.** $x :=$ the number of Model 650s made

$y :=$ the number of Model 800s made

$P :=$ the manufacturer's profit (in dollars)

b. Objective: **Maximize** $P = 600x+500y$

c. Subject to: $x+2y \leq 70$ (the number of hours in the electrical bay)

$2x+2y \leq 90$ (the number of hours in mechanical bay)

$4x+2y \leq 160$ (the number of hours in the assembly bay)

$x \geq 0,\ y \geq 0$

8. **a.** $x :=$ the number of desktops sold per week

$y :=$ the number of laptops sold per week

$C :=$ the store's weekly marketing costs (in dollars)

b. Objective: **Minimize** $C = 75x+50y$

c. Subject to: $x+y \geq 150$ (the number of computers sold per week)

$x \geq 2y$ (the ratio of the number of desktops and laptops)

$x \geq 0,\ y \geq 0$

9. **a.** $x :=$ the number of chocolate pumpkins produced

$y :=$ the number of chocolate ghosts produced

$P :=$ the company's profit (in cents)

b. Objective: **Maximize** $P = 50x + 60y$

c. Subject to: $3x + 4y \leq 90$ (the number of minutes of manufacturing)

$x + 2y \leq 30$ (the number of minutes of packaging)

$x \geq 3y$ (the ratio of the number of pumpkins and ghosts)

$x \geq 0,\ y \geq 0$

10. **a.** $x :=$ the number of days the South College Station factory operates

$y :=$ the number of days the North Bryan factory operates

$C :=$ the order's cost (in dollars)

b. Objective: **Minimize** $C = 1500x + 2000y$

c. Subject to: $100x + 100y \geq 6000$ (the number of pairs of formal shoes)

$100x + 200y \geq 8000$ (the number of pairs of casual shoes)

$300x + 100y \geq 9000$ (the number of pairs of athletic shoes)

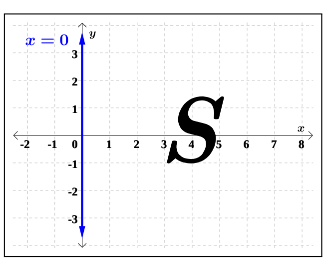
$x \geq 2y$ (the ratio of the S. College Station and the N. Bryan factories operating days)

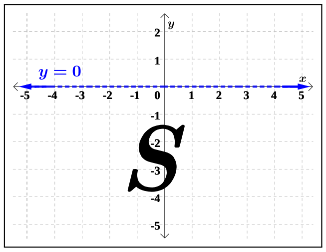
$x \geq 0,\ y \geq 5$

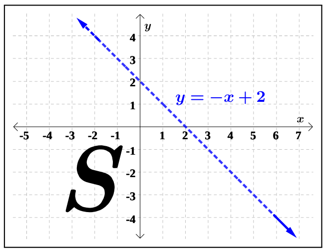
11. Real-world applications involve amounts of tangible items, so it does not make sense to have a negative amount of these items.

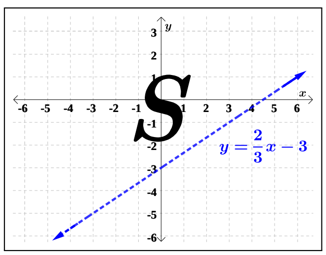
12. Often the last sentence of an application contains information about what we are looking to determine. We use this information to help us define our variables.

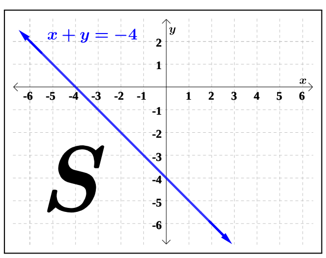
Section 3.2 Answers

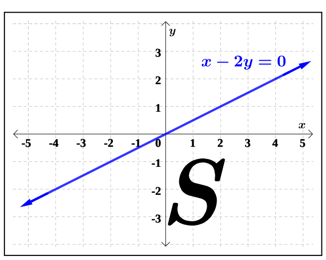
1. 

2. 

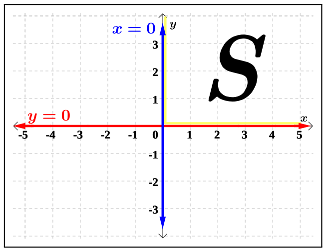
3. 

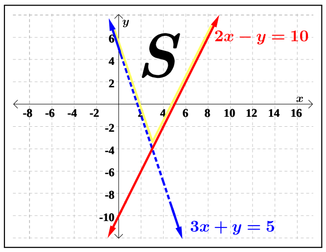
4. 

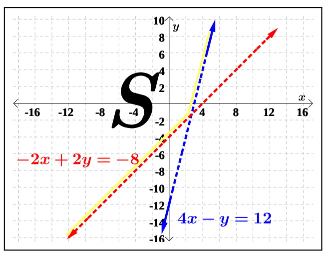
5. 

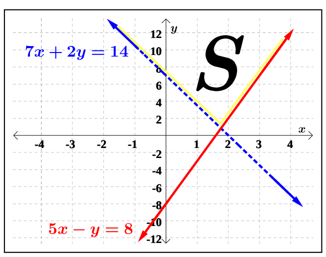
6. 

7. $x - y > 5$
8. $2x - 5y \leq 10$
9. $-x - 4y \geq 8$
10. $3x + 2y < 9$

11. 

12. 

13. 

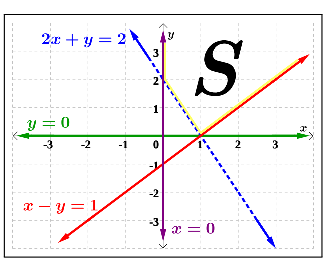
14. 

15. **S** is unbounded; $A = (-2, 0)$ and $B = \left(-\frac{2}{7}, -\frac{24}{7}\right)$

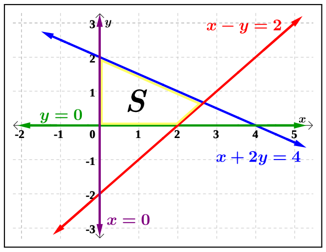
16. **S** is bounded; $A = (0, -2)$, $B = (0, 6)$, and $C = \left(\frac{160}{23}, \frac{18}{23}\right)$

17. **S** is unbounded; $A = \left(\frac{8}{3}, 0\right)$ and $B = (4, 0)$

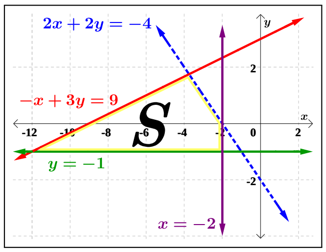
18. **S** is bounded; $A = (-8, -4)$, $B = (1, -1)$, $C = (5, -1)$, and $D = (5, -4)$

19. 

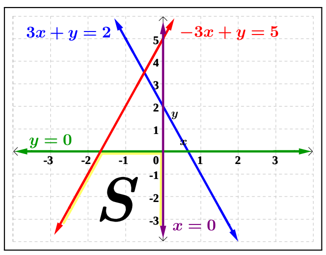
S is unbounded

20. 

S is bounded

21. 

S is bounded

22. 

S is unbounded

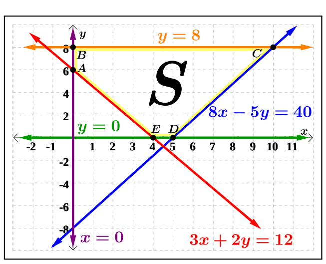
23. $\begin{cases} x & \geq & -2 \\ 4x - 3y & \leq & 4 \\ x - y & \leq & 2 \\ y & \leq & 3 \end{cases}$ corner points: $(-2,-4)$, $(-2,3)$, and $\left(\frac{13}{4},3\right)$

24. $\begin{cases} 3x + y & \geq & -3 \\ 2x + 3y & \leq & 5 \\ x & \leq & 1 \\ y & \geq & -1 \end{cases}$ corner points: $\left(-\frac{2}{3},-1\right)$, $(-2,3)$, $(1,1)$, and $(1,-1)$

25. $\begin{cases} -x + y & \geq & -3 \\ 4x + 4y & \geq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \end{cases}$ corner points: $(0,1)$, $(1,0)$, and $(3,0)$

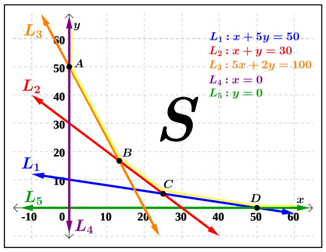
26. $\begin{cases} 2x - y & \geq & 4 \\ 2x + 3y & \geq & 12 \\ x & \geq & 5 \\ y & \geq & 0 \end{cases}$ corner points: $(5,6)$, $\left(5,\frac{2}{3}\right)$, and $(6,0)$

27. **a.**



b. **S** is bounded
c. corner points: $(0,6)$, $(0,8)$, $(10,8)$, $(5,0)$, and $(4,0)$

28. **a.**



b. **S** is unbounded

c. corner points: $(0,50)$, $\left(\frac{40}{3}, \frac{50}{3}\right)$, $(25,5)$, and $(50,0)$

29. A corner point occurs at the intersection point of two boundary lines. Finding an intersection point is equivalent to solving a system of linear equations.

30. If using the true shading method, there would be no overlap of **all** the shaded half-planes. While, if using the false shading method, there would be **no unshaded** region.

Section 3.3 Answers

1. The maximum value of P is 76 and occurs at $(x,y) = (7,2)$.

   The minimum value of P is 15 and occurs at $(x,y) = (0,5)$.

2. The maximum value of Q is 40 and occurs at $(x,y) = (4,8)$.

   The minimum value of Q is -14 and occurs at $(x,y) = (7,2)$.

3. No maximum value of P.

   The minimum value of P is 6 and occurs at $(x,y) = (6,0)$.

4. No maximum value of Q.

   The minimum value of Q is 200 and occurs at $(x,y) = (0,8)$.

5. The maximum value of P is $\frac{1370}{23}$ and occurs at $(x,y) = \left(\frac{160}{23}, \frac{18}{23}\right)$.

6. The minimum value of P is $\frac{16}{3}$ and occurs at $(x,y) = \left(\frac{8}{3}, 0\right)$.

7. The maximum value of P is 22 and occurs at $(x,y) = (2,1)$.

8. The minimum value of P is 6 and occurs at $(x,y) = (0,1)$.

9. The maximum value of P is 440 and occurs at $(x,y) = (4,8)$.

   The minimum value of P is 150 and occurs at every point on the line segment connecting $(0,5)$ and $(3,0)$.

10. The maximum value of Q is 192 and occurs at every point on the line segment connecting $(4,8)$ and $(7,2)$.

    The minimum value of Q is 60 and occurs at $(x,y) = (0,5)$.

11. No maximum value of P.

    The minimum value of P is 320 and occurs at every point on the line segment connecting $(0,8)$ and $(1,5)$.

12. No maximum value of Q.

    The minimum value of Q is 90 and occurs at every point on the line segment connecting $(1,5)$ and $(6,0)$.

13. The maximum value of P is 129 and occurs at $(x,y) = (5,4)$.

14. The minimum value of P is -37.5 and occurs at $(x,y) = \left(\frac{13}{4}, 3\right)$.

15. The maximum value of P is 2.5 and occurs at every point on the line segment connecting $(-2,3)$ and $(1,1)$.

16. The minimum value of P is 72 and occurs at $(x,y) = (6,0)$.

17. 56 servings of whipped cream leftover, but no pie crusts or sugar leftover.

18. 100 hours of labor leftover, but 0 yards of fabric and 0 units of hardware leftover.

19. The maximum value of Z is 306.

20. The minimum value of Z is 46.

21. The maximum profit is \$15 when 30 chocolate pumpkins and 0 chocolate ghosts are produced and sold.

22. The minimum weekly marketing costs are \$10000, when 100 laptops and 50 desktops are sold.

23. The maximum profit is \$26,000 when 35 Model 650s and 10 Model 800s are made and sold. There are 15 hours leftover in the electrical bay, but no time leftover in the mechanical or assembly bays.

24. The maximum annual interest you earn is \$1600, when you invest \$16,000 in bonds and \$8000 in stocks.

25. The minimum grading time is 17.5 minutes per student when 10 objective quizzes and 5 recall quizzes are given.

26. Infinitely many points give the optimal solution when two adjacent corner points give the optimal objective function value.

27. Isoprofit lines are lines corresponding to different objective function values. If the objective function is $P = ax + by$, then the only difference in the equations of the isoprofit lines is the value of P. Solving each for slope-intercept form will give the same slope $\left(m = -\frac{a}{b}\right)$, but different y-intercepts, $\left(0, \frac{P}{b}\right)$. Lines with the same slope, but different y-intercepts yield parallel lines.

Section 3.4 Answers

1. Not a standard maximization problem, because the objective is to minimize, not maximize.

2. Not a standard maximization problem, because the variables are allowed to be negative, $(x \le 0,\ y \le 0)$.

3. Standard maximization problem

4. Not a standard maximization problem, because linear inequalities cannot be written so that the variable part is $\le$ a non-negative constant, $(5x + 2 \le 6y)$.

5. $$\begin{array}{cccccc} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 35 \\ 1 & 4 & 0 & 1 & 0 & 100 \\ \hline -2.5 & -3.75 & 0 & 0 & 1 & 0 \end{array}\right]$$

6. $$\begin{array}{cccccccc} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} 3 & 0.5 & 1 & 1 & 0 & 0 & 0 & 40 \\ 0.65 & 0.35 & 0.85 & 0 & 1 & 0 & 0 & 250 \\ 30 & 20 & 45 & 0 & 0 & 1 & 0 & 3200 \\ \hline -20 & -40 & -90 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

7. $$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 6 & 7 & 0 & 1 & 0 & 0 & 8 \\ 0 & 5 & 1 & 0 & 1 & 0 & 10 \\ \hline -3 & -4 & -5 & 0 & 0 & 1 & 0 \end{array}\right]$$

8. $$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 7 & 11 & 1 & 0 & 0 & 0 & 24 \\ 2 & 4 & 0 & 1 & 0 & 0 & 6 \\ 1 & 0 & 0 & 0 & 1 & 0 & 8 \\ \hline -10 & -15 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

9. **a.** Column 1
   b. Row 1
   c. 3 (in Row 1, Column 1)

10. **a.** Column 2
    b. Row 1
    c. 21 (in Row 1, Column 2)

11. **a.** Column 3
    b. Row 2
    c. 4 (in Row 2, Column 3)

12. Basic Variables: $x = 20$, $s_2 = 20$, $P = 380$

    Non-Basic Variables: $y = 0$, $s_1 = 0$

13. Basic Variables: $y = 30$, $s_2 = \dfrac{985}{4}$, $P = 270$

    Non-Basic Variables: $x = 0$, $z = 0$, $s_1 = 0$

14. Basic Variables: $x = \dfrac{1348}{103}$, $y = \dfrac{108}{103}$, $z = \dfrac{2136}{103}$, $P = \dfrac{36936}{103}$

    Non-Basic Variables: $s_1 = 0$, $s_2 = 0$, $s_3 = 0$

15. $$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccc|c} -8 & 11 & 1 & 0 & 0 & 5 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ \hline -4 & -8 & 0 & 0 & 1 & 0 \end{array}\right]$$

16. $$\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{ccccccc|c} \frac{3}{4} & \frac{1}{3} & \frac{2}{5} & 1 & 0 & 0 & 0 & 60 \\ 1 & \frac{1}{2} & 1 & 0 & 1 & 0 & 0 & 16 \\ 4 & 0 & 5 & 0 & 0 & 1 & 0 & 10 \\ \hline -1 & -1.75 & -1.25 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

17.
$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 8 & 14 & 1 & 0 & 0 & 0 & 91 \\ 25 & 10 & 0 & 1 & 0 & 0 & 522 \\ 1 & 0 & 0 & 0 & 1 & 0 & 10 \\ \hline -75 & -52 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

18.
$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} \frac{1}{7} & \frac{3}{8} & 1 & 0 & 0 & 0 & 42 \\ -3 & 6 & 0 & 1 & 0 & 0 & 9 \\ 1 & -1 & 0 & 0 & 1 & 0 & 283 \\ \hline -17 & -15 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

19. **a.** Column 1
b. Row 1
c. 2 (in Row 1, Column 1)

20. **a.** Column 2
b. Row 2
c. $\frac{1}{8}$ (in Row 2, Column 2)

21. **a.** Column 2
b. Row 3
c. 8 (in Row 3, Column 2)

22. **a.** Basic Variables: $y = \frac{375}{4}$, $s_1 = \frac{255}{2}$, $P = 1125$
Non-Basic Variables: $x = 0$, $s_2 = 0$
b. Not the final tableau

23. **a.** Basic Variables: $x = 2480$, $y = 2000$, $s_1 = 230$, $P = 18000$
Non-Basic Variables: $z = 0$, $s_2 = 0$, $s_3 = 0$
b. The final tableau

24. **a.** Basic Variables: $y = \frac{25}{2}$, $z = \frac{143}{8}$, $s_1 = \frac{337}{4}$, $s_4 = \frac{243}{8}$, $P = \frac{2859}{8}$
Non-Basic Variables: $x = 0$, $s_2 = 0$, $s_3 = 0$
b. Not the final tableau

25. $s_1 = 350$ and $s_2 = 0$

26. $s_1 = 0$, $s_2 = 0$, and $s_3 = \frac{3620}{7}$

27. $s_1 = 0$, $s_2 = 296.5$, and $s_3 = 65$

28. The maximum value of P is $\frac{107}{3}$ and occurs at $(x, y) = \left(\frac{5}{3}, \frac{2}{3}\right)$.

29. The maximum value of P is $\frac{3100}{7}$ and occurs at $(x, y) = \left(\frac{58}{7}, \frac{68}{7}\right)$.

30. The maximum value of P is $\frac{135}{4}$ and occurs at $(x, y) = \left(\frac{45}{2}, 0\right)$.

31. The simplex method cannot be used, because it is not a standard maximization problem.

32. Objective: Maximize $P = 8x + 11y$
    Subject to: $x + 2y \leq 100$
    $4x + 3y \leq 75$
    $x \geq 0,\ 0 \leq y \leq 23$

33. The tableau is the final tableau. $P = 38,000$ at $(x, y, z) = (125, 0, 850)$, where $s_1 = 0$ and $s_2 = 0$.

34. Answers will vary.

$$\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 5 & 0 & 136 \\ 2 & 1 & 0 & 0 & 6 & 0 & 78 \\ 3 & 0 & 0 & 1 & 7 & 0 & 20 \\ \hline 4 & 0 & 0 & 0 & 8 & 1 & 15000 \end{array}\right]$$

35. Corner Point 1: $(x, y) = (0, 0)$

    Corner Point 2: $(x, y) = (0, 2)$

    Corner Point 3: $(x, y) = (6, 0)$

36. The factory should manufacture 35 Model 650s and 10 Model 800s to generate a maximum profit of $26,000. At the optimal level, there are 15 hours leftover in the electrical bay. (No time is leftover in the other two bays.)

37. Pies Galore should make and sell 38 chocolate cream pies, 0 lemon meringue pies, and 6 tart cherry pies to maximize their revenue at $690. At this level, they will have 56 servings of whipped cream left over. (No crusts or sugar will be left.)

38. You should invest $16,000 in bonds and $8000 in stocks in order to maximize your annual interest earned at $16,000.

39. The company should produce and sell 30 chocolate pumpkins and no chocolate cats or ghosts to maximize its profit at $15. At this production level, there is no time leftover in manufacturing or packing.

40. The most negative entry in the bottom row corresponds to the most positive coefficient in the objective function. Increasing the variable value paired with this coefficient, which happens when pivoting on the column, increases the objective function the most.

41. The ratio calculated from each row gives the amount by which the variable being pivoted on may increase before exhausting the constraint. The smallest is chosen, because it represents the most restrictive constraint (the constraint which will run out first).

42. Slack variables represent leftovers of a constraint, and leftovers can never be negative.

43. The number of constraints (excluding non-negativity constraints) determines the number of slack variables needed.

44. A negative entry in the last column indicates an error has occurred. With a negative entry in the last column, a variable value will be negative, which is not possible.

Chapter 3 Review Answers

1. **a.** $g :=$ the number of bottles of gentle shampoo

$r :=$ the number of bottles of regular shampoo

$P :=$ the profit, in dollars, on the sales of these bottles

Objective: **Maximize** $P = 3g + 4r$

Subject to: $14g + 12r \leq 840$ (Ounces of Water)

$4g + 6r \leq 348$ (Ounces of Surfactant)

$g \geq 0,\ r \geq 0$

b. $w :=$ the number of acres of wheat

$g :=$ the number of acres of sorghum

$P :=$ the profit, in dollars, on the sales

Objective: **Maximize** $P = 9000w + 8000g$

Subject to: $10w + 15g \leq 1500$ (Hours to Harvest)

$w + g \leq 500$ (Number of Acres Planted)

$w \geq 2g$ (Ratio of the number of acres of wheat to sorghum)

$w \geq 0,\ g \geq 0$

c. $N :=$ the number of 9-passenger vans

$T :=$ the number of 12-passenger vans

$C :=$ the total costs, in dollars

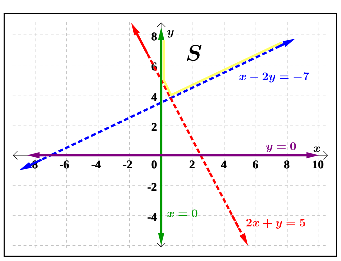
Objective: **Minimize** $C = 250N + 270T$

Subject to: $9N + 12T \geq 75$ (Number of Passengers)

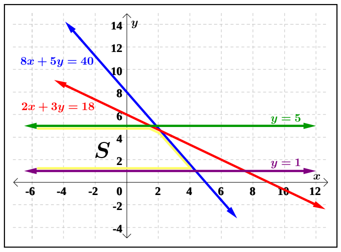
$550N + 560T \geq 3890$ (Pounds of Gear)

$N \geq 0,\ T \geq 0$

2. **a.**



b. No Solution Set

c. 

3. **a.** Find the intercepts of the boundary line for each inequality. Graph a boundary line (dashed if a strict inequality, solid if not). Label the boundary line. Choose a test point, off the boundary line. Determine if the test point makes the inequality true or false. Shade to either indicate the solution to the inequality or to indicate the points not in the solution. Repeat for each boundary line. The solution set will be determined how you shaded.

   b. $(4,6)$ is a solution to the system.

4. **a.** A corner point is the intersection of two boundary lines of a solution set, where you go from one boundary line to another as you move around the boundary of the solution set.

   b. $(0,0)$, $(0,3)$, $(1,4)$, and $(3,0)$

   c. $\left(\frac{15}{2},3\right)$ and $(12,0)$

5. **a.** **S** is bounded

   b. **S** is unbounded

   c. **S** is bounded

6. **a.** The maximum value of R is 27 and occurs at $(3,0)$.

   b. False

   c. True

7. **a.** The maximum value of P is 56 and occurs at $(7,0)$.

   b. No maximum value of $P = 8x+5y$.

   c. The minimum value of P is 7 at $(1,1)$.

   d. The minimum value of P is 20 at every point on the line segment connecting $(2,4)$ and $(5,0)$.

8. **a.** To maximize their profits at \$240, the natural shampoo maker should make 24 bottles of gentle shampoo and 42 bottles of regular shampoo.

   b. To maximize their profits at \$1,350,000, the farmer should plant 150 acres of wheat and 0 acres of sorghum.

   c. To minimize team costs at \$1830, the team should use 3 of the 9-passenger vans and 4 of the 12-passenger vans.

9. **a.** 500 oz of onions leftover and 0 oz of tomatoes leftover

10. **a.**
$$\begin{array}{c}
\begin{array}{ccccccc|c} x & y & z & s_1 & s_2 & s_3 & P & \text{constant} \end{array} \\
\left[\begin{array}{ccccccc|c}
5 & 2 & 1 & 1 & 0 & 0 & 0 & 24 \\
4 & 3 & 2 & 0 & 1 & 0 & 0 & 29 \\
6 & 1 & 5 & 0 & 0 & 1 & 0 & 26 \\
\hline
-15 & -22 & -8 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

    b. Not a standard maximization problem

$$\begin{array}{c} \\ \textbf{c.} \end{array} \begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \end{array}$$

$$\textbf{c.}\ \left[\begin{array}{cccccc|c} 2 & 1 & 1 & 0 & 0 & 0 & 5 \\ 3 & 6 & 0 & 1 & 0 & 0 & 25 \\ -3 & 4 & 0 & 0 & 1 & 0 & 0 \\ \hline -15 & -12 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

11. **a.** $P = 9$ at $(x,y) = (22,18)$ and is the optimal solution

    b. $P = 4$ at $(x,y) = (0,2)$ and is not the optimal solution

    c. $P = 380$ at $(x,y,z) = (210,102,0)$ and is not the optimal solution

12. **a.** Pivot on 4 in Row 1, Column 2

    b. Pivot on 2 in Row 2, Column 2

    c. Pivot on 9 in Row 3, Column 3

13. **a.** The maximum value of P is 210 at $(x,y) = (0,5)$.

    b. The maximum value of P is $\frac{32}{5}$ at $(x,y,z) = (8,0,0)$.

    c. The maximum value of P is 944 at $(x,y,z) = (0,0,118)$.

14. **a.** Corner points: $(0,0)$, $(0,3)$, $\left(\frac{88}{23}, \frac{135}{23}\right)$, and $(5,0)$

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$

$$\text{Initial Tableau:} \left[\begin{array}{ccccc|c} 5 & 1 & 1 & 0 & 0 & 25 \\ -3 & 4 & 0 & 1 & 0 & 12 \\ \hline -6 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \text{Corner Point: } (x,y) = (0,0)$$

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$

$$\text{Next Tableau:} \left[\begin{array}{ccccc|c} 1 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 5 \\ 0 & \frac{23}{5} & \frac{3}{5} & 1 & 0 & 27 \\ \hline 0 & -\frac{14}{5} & \frac{6}{5} & 0 & 1 & 30 \end{array}\right] \text{Corner Point: } (x,y) = (5,0)$$

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \text{constant} \end{array}$$

$$\text{Next Tableau:} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{4}{23} & -\frac{1}{23} & 0 & \frac{88}{23} \\ 0 & 1 & \frac{3}{23} & \frac{5}{23} & 0 & \frac{135}{23} \\ \hline 0 & 0 & \frac{36}{23} & \frac{14}{23} & 1 & \frac{1068}{23} \end{array}\right] \text{Optimal: Max. } P = \frac{1068}{23} \text{ at } (x,y) = \left(\frac{88}{23}, \frac{135}{23}\right)$$

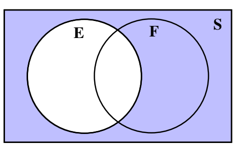
15. **a.** To maximize their profits at \$1,350,000, the farmer should plant 150 acres of wheat and 0 acres of sorghum.

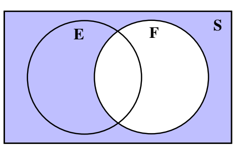
    b. The meal prep service should make and sell 100 “College Student” meals, 200 “Single on the Go” meals and 0 “Retiree” meals to maximize their profit at \$1500.

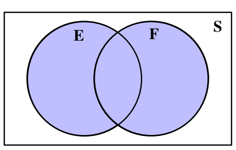
16. **a.** To maximize profit at \$104,000,000, the construction company should build and sell 1000 luxury homes, 3000 standard homes, and 0 basic homes. When doing so they will have 0 hours leftover for building, 0 hours leftover for painting, 0 hours leftover for bricking, and 1000 hours leftover for wiring.

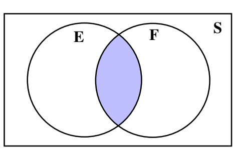
Section 4.1 Answers

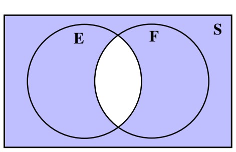
1. $S = \{black,\ red\}$
2. $S = \{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$
3. $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$
4. $S = \{red,\ blue,\ green\}$
5. $S = \{H, O, W, D, Y\}$
6. **a.** $\{a\}$, $\{e\}$, $\{i\}$, $\{o\}$, $\{u\}$
   b. 32
   c. $E = \{o, u, e\}$
7. **a.** $\{2\}$, $\{4\}$, $\{6\}$, $\{8\}$
   b. 16
   c. $E = \{4, 8\}$
8. **a.** $\{for\}$, $\{against\}$, $\{undecided\}$
   b. 8
   c. $E = \{against,\ undecided\}$
9. $E \cap F = \varnothing$, thus E and F are mutually exclusive
10. $E \cap F = \{o, e\} \neq \varnothing$, thus E and F are not mutually exclusive
11. $E \cap F = \{2, 1, 9\} \neq \varnothing$, thus E and F are not mutually exclusive

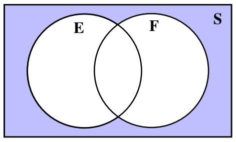
12. 

13. 

14. 

15. 

16. 

17. 

18. $A^C = \{n, q, t, w, x\}$

19. $E^C = \{m, n, p, q, w, x\}$

20. $A \cup B = \{m, p, r, t, w, x\}$

21. $B \cup D = \{t, w, x, p, q, r\}$

22. $D \cap E = \{r, t\}$

23. $B \cap E = \{t\}$

24. $S = \{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(1,3),(2,3),(3,3),(1,4),(2,4),(3,4),(1,5),(2,5),(3,5),$
$(1,6),(2,6),(3,6)\}$

25. $S = \{(red, heads),\ (red, tails),\ (blue, heads),\ (blue, tails),\ (green, heads),\ (green, tails),$
$(yellow, heads),\ (yellow, tails)\}$

26. $S = \{3, 4, 5, 6, 7\}$

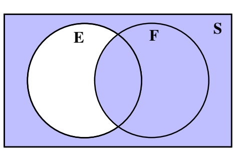
27. $S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$, where H = heads and T = tails

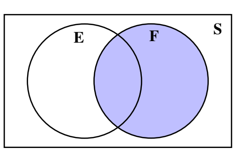
28. $S = \{(S, hearts),\ (O, hearts),\ (C, hearts),\ (I, hearts),\ (A, hearts),\ (L, hearts),\ (S, diamonds),$
$(O, diamonds),\ (C, diamonds),\ (I, diamonds),\ (A, diamonds),\ (L, diamonds),\ (S, clubs),\ (O, clubs),$
$(C, clubs),\ (I, clubs),\ (A, clubs),\ (L, clubs),\ (S, spades),\ (O, spades),\ (C, spades),$
$(I, spades),\ (A, spades),\ (L, spades)\}$

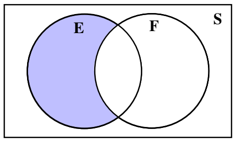
29. **a.** 10
b. 1024
c. $E = \varnothing$

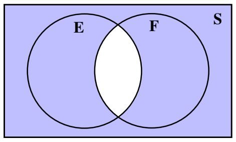
30. **a.** 4
b. 16
c. $E = \{(heart, queen),\ (heart, king),\ (spade, queen)\}$

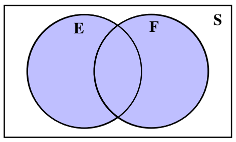
31. **a.** 6
b. 64
c. $E = \{(1,1),\ (1,3)\}$

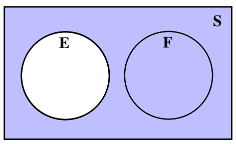
32. 

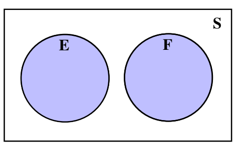
33. 

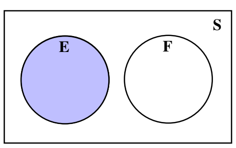
34. 

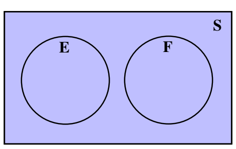
35. 

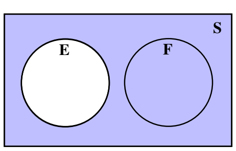
36. 

37. 

38. 

39. 

40. 

41. 

42. $A^C \cup D = \{n, q, t, w, x, p, r\}$
43. $E^C \cap B = \{w, x\}$
44. $(A \cup B)^C = \{n, q\}$
45. $B \cup D^C = \{t, w, x, m, n\}$
46. $(D \cap E)^C = \{m, n, p, q, w, x\}$
47. $(B \cap E) \cup A = \{t, m, p, r\}$
48. E^C

49. $F \cap G$

50. $H \cup F^C$

51. G^C is the event that a letter in the word *EXIT* is not drawn.

52. $H^C \cap E$ is the event that the spinner does not land on red or yellow and it does land on green.

53. $F \cup E$ is the event that a vowel is drawn or the spinner lands on green.

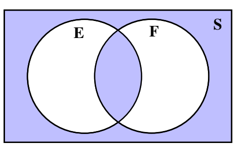
54. $S = \{(V,H,O),(V,H,E),(V,T,O),(V,T,E),(N,H,O),(N,H,E),(N,T,O),(N,T,E)\}$, where V = vowel, N = non-vowel, H = heads, T = tails, O= odd rolled, and E = even rolled

55. $\varnothing$, $\{1\}$, $\{2\}$, $\{a\}$, $\{b\}$, $\{1,2\}$, $\{1,a\}$, $\{1,b\}$, $\{2,a\}$, $\{2,b\}$, $\{a,b\}$, $\{1,2,a\}$, $\{1,2,b\}$, $\{1,a,b\}$, $\{2,a,b\}$, S

56. **a.** $S = \{(1,red), (1,blue), (1,green), (1,yellow), (2,red), (2,blue), (2,green), (2,yellow), (3,red)$ $(3,blue), (3,green), (3,yellow), (4,red), (4,blue), (4,green), (4,yellow), (5,red), (5,blue)$ $(5,green), (5,yellow)\}$

    b. $\{(1,red), (1,blue), (1,green), (1,yellow), (2,red), (2,blue), (2,green), (2,yellow), (3,red)$ $(3,blue), (3,green), (3,yellow), (4,red), (4,blue), (4,green), (4,yellow), (5,red), (5,blue)$ $(5,green), (5,yellow)\}$

    c. $\{(1,red)\}$ and $\{(1,blue)\}$; Answers will vary.

57. 

58. **a.** $E \cap G = \{(black,2), (black,4), (black,6), (black,8)\}$

    $E \cap G$ is the event that a black card is drawn and a multiple of 2 is rolled.

    b. $F^C \cup G = \{(black,2), (black,4), (black,6), (black,7), (black,8), (red,2), (red,4), (red,6)$ $(red,7), (red,8)\}$

    $F^C \cup G$ is the event that a multiple of 2 is rolled or the number rolled is not less than 6.

    c. $\left(E^C \cup F\right) \cap G^C$

59. The sample space, **S**, is the "certain" event, because it contains all outcomes of the experiment no matter what outcome occurs, so **S** is "certain" to happen.

60. Two events are mutually exclusive if they cannot both happen at the same time.

Section 4.2 Answers

1. **a.** $\frac{1}{4}$

    b. $\frac{2}{4}$

    c. $\frac{3}{4}$

2. **a.** $\frac{30}{55}$

    b. $\frac{25}{55}$

    c. $\frac{50}{55}$

    d. 1

3. **a.** $\frac{15}{66}$ **b.** $\frac{39}{66}$ **c.** $\frac{54}{66}$

4. **a.** $\frac{100}{131}$ **b.** $\frac{120}{131}$ **c.** 1

5. **a.** $\frac{105}{180}$ **b.** $\frac{63}{180}$ **c.** $\frac{32}{180}$ **d.** $\frac{121}{180}$ **e.** $\frac{120}{180}$ **f.** $\frac{122}{180}$

6. Yes; All probabilities are between 0 and 1, inclusive, and the sum of all the probabilities is 1.

7. No; The probability 'A' occurs is less than 0.

8. **a.** $\frac{1}{5}$ **b.** $\frac{3}{5}$ **c.** $\frac{4}{5}$ **d.** $\frac{4}{5}$ **e.** 0

9. **a.** $\frac{13}{52}$ **b.** $\frac{26}{52}$ **c.** $\frac{2}{52}$ **d.** $\frac{40}{52}$ **e.** $\frac{4}{52}$

10. **a.** $\frac{4}{8}$ **b.** $\frac{3}{8}$ **c.** $\frac{1}{8}$ **d.** $\frac{4}{8}$

11. **a.** $\frac{1}{8}$
b. $\frac{2}{8}$
c. $\frac{4}{8}$
d. $\frac{4}{8}$
e. 0

12. **a.** $\frac{11}{36}$
b. 0
c. $\frac{9}{36}$
d. 1
e. $\frac{6}{36}$
f. $\frac{1}{36}$
g. $\frac{21}{36}$

13. **a.** $\frac{134}{260}$
b. $\frac{47}{260}$
c. $\frac{107}{260}$
d. $\frac{110}{260}$
e. 1
f. $\frac{77}{260}$
g. $\frac{13}{260}$
h. $\frac{132}{260}$

14. No; While probabilities are between 0 and 1, inclusive, and the sum of all the probabilities is 1, all the probabilities are not equal.

15. No; The sum of all the probabilities is not 1.

16. **a.** $\frac{8}{20}$
b. $\frac{5}{20}$
c. 0
d. $\frac{4}{20}$
e. $\frac{7}{20}$
f. $\frac{8}{20}$
g. 1

17. **a.** $\frac{4}{8}$
b. $\frac{2}{8}$
c. $\frac{7}{8}$
d. $\frac{1}{8}$
e. $\frac{6}{8}$

18. **a.** $\frac{185}{460}$
b. $\frac{165}{460}$
c. $\frac{45}{460}$
d. $\frac{186}{460}$
e. $\frac{317}{460}$
f. $\frac{216}{460}$
g. $\frac{171}{460}$
h. $\frac{220}{460}$
i. $\frac{88}{460}$

19. Yes; All probabilities are between 0 and 1, inclusive, and the sum of all the probabilities is 1. The sample space is not uniform.

20. Yes; All probabilities are between 0 and 1, inclusive, and the sum of all the probabilities is 1. the sample space is uniform.

21. In a uniform distribution each outcome has the same chance of occurring.

22. The least number of outcomes an event can have is 0; thus, the smallest probability that an event occurs is $\frac{0}{n} = 0$. The largest number of outcomes an event can have is n; thus, the largest probability that an event occurs is $\frac{n}{n} = 1$. For most events the number of outcomes in the event is somewhere between these two extremes and so the corresponding probability that the event occurs is between 0 and 1.

23. An example of a theoretical probability is "Given a numbered raffle ticket is randomly selected from a box with 10,000 different numbered tickets, what is the probability the ticket numbered 01234 is selected?"

    An example of an empirical probability is "Given the survey results of 10,000 people regarding their preference in learning (remote, face-to-face, hybrid), what is the probability a randomly selected person prefers learning face-to-face?"

Section 4.3 Answers

1. $\frac{2}{6}$

2. $\frac{3}{6}$

3. **a.** $\frac{15}{20}$
b. $\frac{13}{20}$
c. $\frac{7}{20}$
d. $\frac{15}{20}$
e. 0
f. $\frac{8}{20}$
g. 1
h. $\frac{13}{20}$

4. **a.** $\frac{40}{65}$
b. $\frac{15}{65}$
c. $\frac{10}{65}$
d. $\frac{45}{65}$

e. $\frac{50}{65}$

5. **a.** 0.56
b. 0.36
c. 0.70

6. **a.** 0.74
b. 0.53
c. 0
d. 0.73

7. **a.** $\frac{33}{40}$
b. $\frac{2}{40}$
c. $\frac{16}{40}$
d. $\frac{7}{40}$

8. **a.** 1
b. 1
c. $P(F \cup E)$
d. $P\left(M \cap T^C\right)$
e. $\frac{110}{200}$
f. $\frac{40}{200}$

9. $\frac{16}{52}$

10. $\frac{5}{52}$

11. **a.** $\frac{3}{10}$
b. $\frac{9}{50}$
c. $\frac{5}{20}$
d. $\frac{73}{100}$
e. $\frac{3}{20}$
f. $\frac{6}{10}$
g. 0
h. 1

12. **a.** $\frac{90}{100}$
b. $\frac{5}{100}$
c. $\frac{55}{100}$
d. $\frac{75}{100}$

13. **a.** 0.40
b. 0.55
c. 0.21
d. 0.16
e. 0.79
f. 0.24

14. **a.** 0.57
b. 0.85
c. 0
d. 0.58
e. 1
f. 1

15. **a.** $\frac{3}{60}$
   b. $\frac{20}{60}$
   c. $\frac{4}{60}$
   d. $\frac{33}{60}$

16. **a.** $\frac{34}{260}$
   b. $\frac{183}{260}$
   c. $P(X \cap 1)$
   d. $P\left(T^C \cup 5^C\right)$
   e. $\frac{33}{260}$
   f. $\frac{109}{260}$

17. $P(s_3) = 0.33; \quad P(s_5) = 0.20$

18. **a.** $\frac{7}{16}$
   b. $\frac{14}{16}$
   c. $\frac{6}{16}$

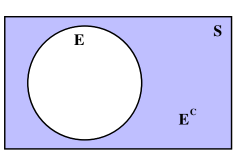
19. **a.** 0.35
   b. 0.70
   c. 0.25
   d. 0.45
   e. 0.75
   f. 0.45

20. **a.** 0.011
   b. 0.229
   c. 0.369
   d. 0.391

21. **a.** $\frac{353}{460}$
   b. 1
   c. $P\left(M \cap S^C\right)$
   d. $P((M \cup D) \cup F)$
   e. $\frac{247}{460}$
   f. $\frac{353}{460}$

22. Probability is the total number of outcomes in an event divided by the total number of outcomes possible, n. If the event is the sample space, then the event contains all possible outcomes, n outcomes. Thus, the probability of the sample space is $P(\mathbf{S}) = \frac{n}{n} = 1$.

23. Probability is the total number of outcomes in an event divided by the total number of outcomes possible, n. If the event is the impossible event, then the event contains none of the possible outcomes, 0 outcomes. Thus, the probability of the impossible event is $P(\text{Impossible Event}) = \frac{0}{n} = 0$.

24. 

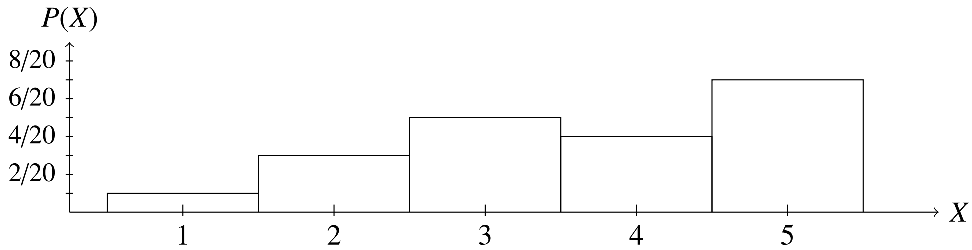
$$P(E) + P\left(E^C\right) = P(S)$$
$$P(S) = 1$$
$$P\left(E^C\right) = 1 - P(E)$$

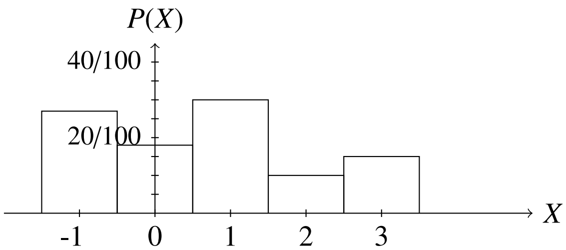
Section 4.4 Answers

1. $X = 0, 1, 2, 3, 4$
2. $X = 508, -19492$

3. $X = 48, -752, -1952$

4. $X = 0, 1, 2, 8$
   $Y = -3, -2, -1, 5$

5. 

6. 

7. $E(X) = 3.65$

8. $E(X) = 2.13$

9.

X	0	1	2
$P(X)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

10.

X	0	1
$P(X)$	$\frac{26}{30}$	$\frac{4}{30}$

11.

X	508	-19492
$P(X)$	0.88	0.12

12.

X	-48	752	1952
$P(X)$	0.875	0.09	0.035

13.

X	0	1	2	8
$P(X)$	$\frac{6}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{4}{12}$

Y	-3	-2	-1	5
$P(Y)$	$\frac{6}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{4}{12}$

14.

X	1	2	3	4
$P(X)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{1}{15}$	$\frac{8}{15}$

15. **a.** $P(X = 2) = \dfrac{4}{10}$

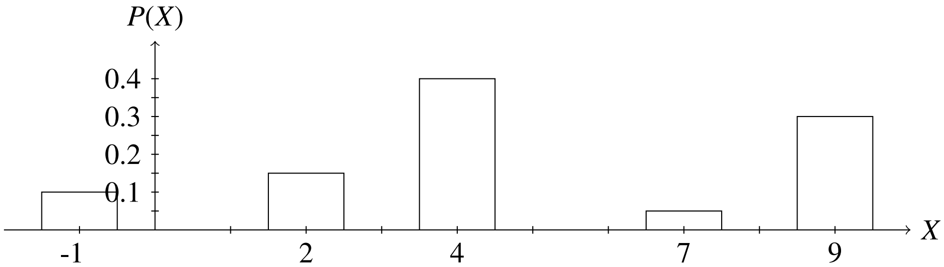
b. $\dfrac{8}{10}$

c. $\dfrac{6}{10}$

d. 0

e. $E(X) = 2.4$

16. Mathematically: 1.69 days; Realistically: $1-2$ days

17. 

18.

X	\$100	\$200	\$300
$P(X)$	0.35	0.45	0.20

19. Expect an average of $4-5$ visible chocolate chips, $E(X) \approx 4.14$

20. **a.**

X	4500	-245500
$P(X)$	0.993	0.007

b. $E(X) = \$2750$

21. **a.**

X	$-p$	$275000 - p$	$1168750 - p$
$P(X)$	0.868	0.13	0.002

p = premium paid

b. $p = \$38,087.50$

22. **a.**

X	-2	8	18
$P(X)$	$\frac{8}{16}$	$\frac{4}{16}$	$\frac{4}{16}$

b. $E(X) = \$5.50$

c. No

23. **a.**

X	0	6	8
$P(X)$	$\frac{9}{16}$	$\frac{4}{16}$	$\frac{3}{16}$

b. $E(X) = \$3.00$

c. \$3.00

24. \$5, because neither side should have a profit if the game is fair.

25. The minimum premium will have the insurance company break even. Given the goal of an insurance company is to make a profit, they must charge premiums higher than the minimum.

26. The heights of the rectangles indicate probability values, which are always numbers between 0 and 1, inclusive.

Chapter 4 Review Answers

1. **a.** The sample space of an experiment is the set of all possible outcomes, **S**.

   b. An event is a subset of the sample space of an experiment.

   c. A simple event is an event containing exactly 1 outcome. The certain event contains all possible outcomes, **S**. An impossible event is an event containing no outcomes.

2. **a.** $\mathbf{S} = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5)\}$

   b. $\mathbf{S} = \{$T, E, X, A, S$\}$

   c. Simple events: $\{heart\}$, $\{diamond\}$, $\{spade\}$, $\{club\}$

   d. An impossible event is the event "the spinner lands in a purple region." Answers may vary

   e. 32 possible events

   f. $E = \{(3,5), (5,3)\}$

3. **a.** $B \cap A$ or $A \cap B$

   b. C^C

   c. The event "a spade or a diamond is drawn."

   d. The event "a face card is not drawn, but a red card is drawn."

4. **a.** A and B are NOT mutually exclusive, because $A \cap B = \{6\} \neq \varnothing$.

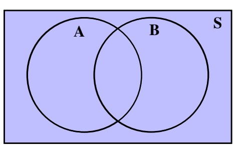
   b. B and C ARE mutually exclusive, because $B \cap C = \varnothing$.

   c. Answers will vary.

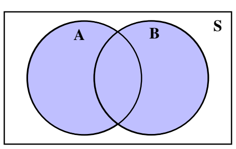
5. **a.** $\mathbf{S} = \{(1,H), (1,T), (2,H), (2,T), (3,H), (3,T), (4,H), (4,T), (5,H), (5,T)\}$, where H = heads, and T = tails

   b. $\mathbf{S} = \{(heart, black), (heart, red), (diamond, black), (diamond, red), (spade, black), (spade, red), (club, black), (club, red)\}$

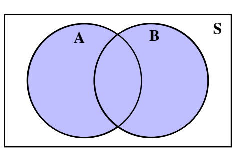
6. **a.**



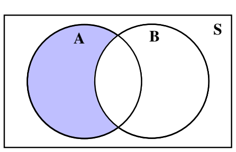
   b.



   c.



   d.



7. **a.** Not uniform; 'G' is more likely to be drawn than others – all the letters are not equally likely

   b. Yes, because each student has a unique UIN, so the sample space will contain 100 different UINs, where each is just as likely to be randomly chosen as another.

c. **i.** $P(\text{a six is rolled}) = \dfrac{1}{20}$

ii. $P(\text{an even number is rolled}) = \dfrac{10}{20}$

iii. $P(\text{a zero is rolled}) = 0$

iv. $P(\text{a number less than 7 is rolled}) = \dfrac{6}{20}$

d. **i.** $\dfrac{1}{52}$

ii. $\dfrac{4}{52}$

iii. $\dfrac{13}{52}$

iv. $\dfrac{12}{52}$

8. **a.** **i.** Theoretical

ii. Theoretical

iii. Empirical

9. **a.**

Outcome	Double	Not a Double
Probability	$\frac{6}{36}$	$\frac{30}{36}$

S = {double, not a double} is NOT uniform, because the two possible outcomes are not equally likely.

b.

Outcome	Heart	Diamond	Spade	Club
Probability	$\frac{13}{52}$	$\frac{13}{52}$	$\frac{13}{52}$	$\frac{13}{52}$

S = {heart, diamond, spade, club} is a uniform sample space, because all four possible outcomes are equally likely.

10. **a.** **i.** $\dfrac{15}{100}$

ii. $\dfrac{80}{100}$

iii. $\dfrac{94}{100}$

iv. $\dfrac{92}{100}$

v. 0

11. **a.** $P(H \cup K) = P(H) + P(K) - P(H \cap K)$

b. $P\left(M^C\right) = 1 - P(M)$

c. **i.** $P(\text{a six is rolled or the coin lands on heads}) = \frac{21}{40}$

ii. $P(\text{an even is rolled and the coin lands on tails}) = \frac{10}{40}$

iii. $P(\text{a five is not rolled}) = \frac{38}{40}$

iv. $P(\text{a number not greater than 8 is rolled, but the coin lands on tails}) = \frac{8}{40}$

d. **i.** $P(\text{the green die does not show a 4}) = \frac{30}{36}$

ii. $P(\text{the green die shows an even number and the blue die shows any number but 3}) = \frac{15}{36}$

iii. $P(\text{the green die shows a 1 and doubles are showing}) = \frac{1}{36}$

e. **i.** $\frac{16}{52}$

ii. $\frac{26}{52}$

iii. $\frac{6}{52}$

iv. $\frac{3}{52}$

v. $\frac{8}{52}$

vi. $\frac{13}{52}$

f. **i.** $P(\text{in favor of reviving the rivalry}) = \frac{50}{80}$

ii. $P(\text{a current student who opposes reviving the rivalry}) = \frac{5}{80}$

iii. $P(\text{an incoming freshman or has no opinion about reviving the rivalry}) = \frac{26}{80}$

iv. $P(\text{does not favor reviving the rivalry}) = \frac{30}{80}$

v. $P(\text{is not a current student and has no opinion about reviving the rivalry}) = \frac{19}{80}$

vi. $P(\text{is not an alumni or is in favor of reviving the rivalry}) = \frac{79}{80}$

12. **a.** 0.7
b. 0.2
c. 0.6

13. **a.**

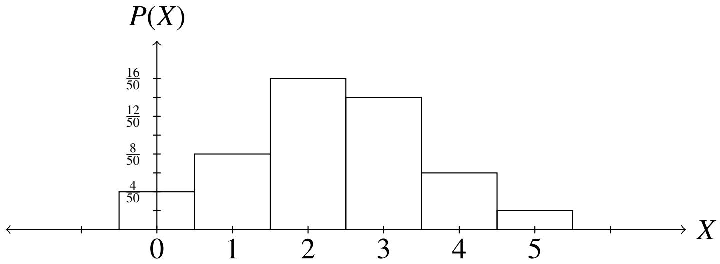
X	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

b.

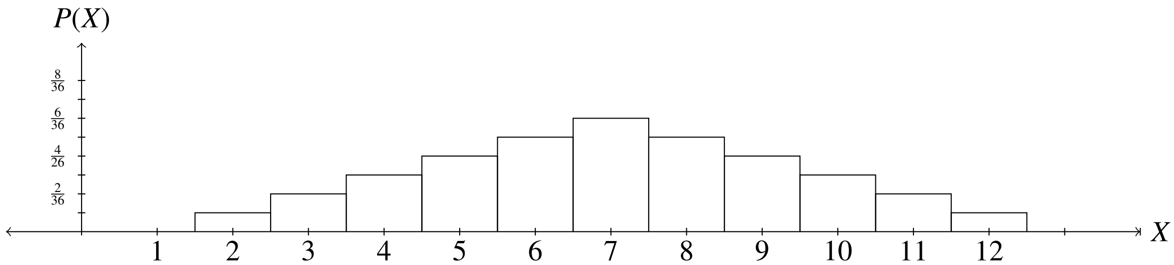
X	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

14. **a.** A histogram

b.



c.



15. **a.** The expected value is a weighted average of values; if an experiment is repeated many times, the average of the results approaches the expected value.
    b. $E(X) = 2.32$; On average, a post-op patient will ring the nurse 2 to 3 times during a 12-hr shift.
    c. $E(X) = 7$; On average, when two fair standard six-sided dice are rolled, a sum of 7 is rolled.
16. **a.** A mathematical game is fair if neither party expects to profit; each party should expect a profit of zero.
    b. \$5500; The cost of the policy is reasonable, because paying \$10,500 is not much to be given \$250,000 (paying 4.2%) and they only expect half as profit.
    c. \$0; the game is fair.

Section 5.1 Answers

1. $(-2,3]$
2. $(-\infty,1.5)$
3. $[-10,8]$
4. $[4,\infty)$
5. $(-\infty,-3]$
6. $(6,\infty)$
7. $[20,\infty)$
8. $\left(-\infty,-\dfrac{5}{9}\right)$
9. Function
10. Not a Function
11. Function
12. Function
13. $f(0) = 1$
14. $f(-3) = -4$

15. $f(2) = 4$
16. $f\left(\frac{1}{2}\right) = 1$
17. $x = -5$ or $x = 6$
18. $x = -4$ or $x = -2$
19. $x = 1$ or $x = 4$
20. $x = 5.5$
21. $h(0) = 4$; $h(2) = 1$; $h(6) = -3$
22. $h(-5) = 7$; $h(4) = 1.5$; $h(11) = 2$
23. $h(-1) = -2$; $h(-2) = -1$; $h(9) = 0$
24. $[-3, 0) \cup (1, 9]$
25. $(-\infty, 6] \cup [7.5, 11)$
26. $[-5, 2) \cup (2, \infty)$
27. $(-\infty, -1) \cup (-1, \infty)$
28. $[-3.4, 100)$
29. $(-\infty, -1) \cup (7, \infty)$
30. $(-5, 0) \cup (0, 3) \cup (3, 9]$
31. $(-\infty, 10) \cup (20, 30) \cup (30, \infty)$
32. Inputs: $\{1, 2, 3\}$
    Outputs: $\{1, 2, 4\}$
    Function
33. Inputs: $\{-1, 0, 1, 2\}$
    Outputs: $\{-3, -2, 1\}$
    Function
34. Inputs: $\{-2, 1, 2\}$
    Outputs: $\{-5, 3, 0, 4\}$
    Not a Function
35. $f(0) = 9$
36. $f(-3) = 7$
37. $f(5) = 1$
38. $f(4) = 0$
39. $f(7) = 3$
40. $f\left(-\frac{3}{2}\right) = 8$
41. $x = -8$ or $x = 4$
42. $x = -7$
43. $x = -6$ or $x = 9$

44. $x = -9$ or $x = 7$

45. $x = 3$

46. $x = 0$

47. $f(3) = 7$

48. $f(-3) = -5$

49. $f\left(\frac{3}{2}\right) = 4$

50. $f(4a) = 8a + 1$

51. $4f(a) = 8a + 4$

52. $\frac{f(a)}{4} = \frac{2a+1}{4} = \frac{1}{2}a + \frac{1}{4} = \frac{a}{2} + \frac{1}{4}$

53. $f(a-5) = 2a - 9$

54. $f(a) - f(5) = 2a - 10$

55. $f(a) - 5 = 2a - 4$

56. $f(0) = -12$

57. $f(1) = -11$

58. $f\left(\frac{1}{2}\right) = -12$

59. $f(-a) = 2a^2 + a - 12$

60. $-f(a) = -2a^2 + a + 12$

61. $\frac{f(a)}{2} = \frac{2a^2 - a - 12}{2} = a^2 - \frac{1}{2}a - 6$

62. $f(a+5) = 2a^2 + 19a + 33$

63. $f(a) + f(5) = 2a^2 - a + 21$

64. $f(a) + 5 = 2a^2 - a - 7$

65. Function
    Domain: $(-\infty, -3) \cup (-3, \infty)$
    Range: $(-\infty, \infty)$

66. Not a function

67. Function
    Domain: $(-\infty, 4) \cup (4, 7]$
    Range: $(-\infty, 3)$

68. $f(-12) = 4$

69. $f\left(\frac{3}{2}\right) = \frac{7}{4}$

70. $f(2a) = 2 - \frac{1}{3}a$

71. $6f(a) = 12 - a$

72. $f(x) + h = 2 - \frac{1}{6}x + h$

73. $f(x+h) = 2 - \frac{1}{6}x - \frac{1}{6}h$
74. $f(-4) = -67$
75. $f(1.6) = 1.32$
76. $f(a+b) = -3a^2 - 6ab - 3b^2 + 5a + 5b + 1$
77. $f(a) + f(b) = -3a^2 + 5a - 3b^2 + 5b + 2$
78. $f(x-h) = -3x^2 + 6xh - 3h^2 + 5x - 5h + 1$
79. $f(x+h) - f(x) = -6xh - 3h^2 + 5h$
80. All real values of x except for $x = -12$.
81. All real values of x less than or equal to -7, or greater than -1 and less than 4.
82. Every point on a vertical line has the same x-value. Thus, if a vertical line is drawn and crosses the graph at more than one point, then an x-value exists in the relation with more than one y-value. This means the relation is not a function.

Section 5.2 Answers

1. Polynomial
2. Polynomial
3. Not a polynomial
4. Polynomial
5. Not a polynomial
6. Polynomial
7. As $x \to -\infty$, $f(x) \to \infty$; As $x \to \infty$, $f(x) \to -\infty$
8. As $x \to -\infty$, $g(x) \to \infty$; As $x \to \infty$, $g(x) \to \infty$
9. As $x \to -\infty$, $h(x) \to -\infty$; As $x \to \infty$, $h(x) \to -\infty$
10. As $x \to -\infty$, $k(x) \to -\infty$; As $x \to \infty$, $k(x) \to \infty$
11. Cubic; $f(x) = x^3$
12. Linear; $f(x) = x$
13. Constant; $f(x) = c$
14. Quadratic; $f(x) = x^2$
15. $x = -3$ or $x = 4$
16. $x = -7$ or $x = -2$ or $x = 8$
17. $x = 5$
18. Does not exist

19. **a.** Vertex: $(1,-1)$
b. Axis of symmetry: $x=1$
c. Domain: $(-\infty,\infty)$
d. Range: $[-1,\infty)$
e. x-intercept(s): $(0,0)$, $(2,0)$
f. y-intercept: $(0,0)$
g. Maximum value: None
h. Minimum value: -1

20. **a.** Vertex: $(-2,0)$
b. Axis of symmetry: $x=-2$
c. Domain: $(-\infty,\infty)$
d. Range: $(-\infty,0]$
e. x-intercept(s): $(-2,0)$
f. y-intercept: $(0,-4)$
g. Maximum value: 0
h. Minimum value: None

21. **a.** Vertex: $(-4,4)$
b. Axis of symmetry: $x=-4$
c. Domain: $(-\infty,\infty)$
d. Range: $[4,\infty)$
e. x-intercept(s): No real x-intercepts
f. y-intercept: $(0,12)$
g. Maximum value: None
h. Minimum value: 4

22. **a.** Vertex: $(3,4)$
b. Axis of symmetry: $x=3$
c. Domain: $(-\infty,\infty)$
d. Range: $(-\infty,4]$
e. x-intercept(s): $(1,0)$, $(5,0)$
f. y-intercept: $(0,-5)$
g. Maximum value: 4
h. Minimum value: None

23. $R(x)=-2x^2+60x$

24. $R(x)=-0.05x^2+50x$

25. $R(x)=-4x^2+24x$

26. $R(x)=-\dfrac{2}{3}x^2+200x$

27. $P(x)=-5x^2+70x-120$

28. $P(x)=-3x^2+90x-375$

29. $P(x)=-4x^2+440x-4000$

30. $P(x)=-\dfrac{1}{2}x^2+40x-750$

31. Degree: 2, Leading coefficient: -3, and Constant term: 0

32. Degree: 17, Leading coefficient: $-\sqrt{3}$, and Constant term: -10^{20}

33. Degree: 2, Leading coefficient: 1, and Constant term: 2

34. Degree: 9, Leading coefficient: -5, and Constant term: 2.8

35. Degree: 12, Leading coefficient: $\dfrac{2}{11}$, and Constant term: 0

36. Degree: 0, Leading coefficient: -4, and Constant term: -4

37. As $x \to -\infty$, $f(x) \to \infty$; As $x \to \infty$, $f(x) \to \infty$

38. As $x \to -\infty$, $g(x) \to \infty$; As $x \to \infty$, $g(x) \to -\infty$

39. As $x \to -\infty$, $h(x) \to -\infty$; As $x \to \infty$, $h(x) \to -\infty$

40. As $x \to -\infty$, $k(x) \to -\infty$; As $x \to \infty$, $k(x) \to \infty$

41. $x = 3$, or $x = -\dfrac{4}{7}$

42. $x = 2$, or $x = -1$

43. $x = 0$, or $x = -\dfrac{5}{4}$, or $x = -11$

44. $x = 0$, or $x = 6$, or $x = -\dfrac{2}{3}$, or $x = \dfrac{5}{9}$

45. $x = 3$

46. $x = -\dfrac{21}{2}$, or $x = \dfrac{2}{33}$

47.
   a. Vertex: $(3, 7)$
   b. Axis of symmetry: $x = 3$
   c. Domain: $(-\infty, \infty)$
   d. Range: $(-\infty, 7]$
   e. x-intercept(s): $\left(3 - \sqrt{\dfrac{7}{2}}, 0\right)$, $\left(3 + \sqrt{\dfrac{7}{2}}, 0\right)$
   f. y-intercept: $(0, -11)$
   g. Maximum value: 7
   h. Minimum value: None

48.
   a. Vertex: $(0, -4)$
   b. Axis of symmetry: $x = 0$
   c. Domain: $(-\infty, \infty)$
   d. Range: $[-4, \infty)$
   e. x-intercept(s): $(-2, 0)$, $(2, 0)$
   f. y-intercept: $(0, -4)$
   g. Maximum value: None
   h. Minimum value: -4

49.
   a. Vertex: $\left(\dfrac{5}{6}, -\dfrac{59}{12}\right)$
   b. Axis of symmetry: $x = \dfrac{5}{6}$
   c. Domain: $(-\infty, \infty)$
   d. Range: $\left(-\infty, -\dfrac{59}{12}\right]$
   e. x-intercept(s): No real x -intercepts
   f. y-intercept: $(0, -7)$
   g. Maximum value: $-\dfrac{59}{12}$
   h. Minimum value: None

50. **a.** Vertex: $\left(1, \frac{1}{4}\right)$
b. Axis of symmetry: $x = 1$
c. Domain: $(-\infty, \infty)$
d. Range: $\left[\frac{1}{4}, \infty\right)$
e. x-intercept(s): No real x-intercepts
f. y-intercept: $\left(0, \frac{3}{4}\right)$
g. Maximum value: None
h. Minimum value: $\frac{1}{4}$

51. **a.** Vertex: $(-5, -8)$
b. Axis of symmetry: $x = -5$
c. Domain: $(-\infty, \infty)$
d. Range: $[-8, \infty)$
e. x-intercept(s): $\left(-5 - \sqrt{2}, 0\right)$, $\left(-5 + \sqrt{2}, 0\right)$
f. y-intercept: $(0, 92)$
g. Maximum value: None
h. Minimum value: -8

52. **a.** Vertex: $(0, 27)$
b. Axis of symmetry: $x = 0$
c. Domain: $(-\infty, \infty)$
d. Range: $(-\infty, 27]$
e. x-intercept(s): $(-3, 0)$, $(3, 0)$
f. y-intercept: $(0, 27)$
g. Maximum value: 27
h. Minimum value: None

53. **a.** Vertex: $\left(-\frac{5}{2}, \frac{147}{4}\right)$
b. Axis of symmetry: $x = -\frac{5}{2}$
c. Domain: $(-\infty, \infty)$
d. Range: $\left(-\infty, \frac{147}{4}\right]$
e. x-intercept(s): $(-6, 0)$, $(1, 0)$
f. y-intercept: $(0, 18)$
g. Maximum value: $\frac{147}{4}$
h. Minimum value: None

54. **a.** Vertex: $(4, -1)$
b. Axis of symmetry: $x = 4$
c. Domain: $(-\infty, \infty)$
d. Range: $(-\infty, -1]$
e. x-intercept(s): No real x-intercepts
f. y-intercept: $\left(0, -\frac{21}{5}\right)$
g. Maximum value: -1
h. Minimum value: None

55. **a.** 9 items
b. \$405
c. 7 items
d. \$125
e. 2 items and 12 items

56. **a.** 20 items
b. \$1200
c. 15 items
d. \$300
e. 5 items and 25 items

57. **a.** 65 items
b. \$16,900
c. 55 items
d. \$8100
e. 10 items and 100 items

58. **a.** 85 items
b. \$3612.50
c. 40 items
d. \$50
e. 30 items and 50 items

59. $f(x) = 100x^5 + 2x^3 - 245$ (Middle term will vary with power 1, 2, 3, or 4 and any real number coefficient.)

60. $g(x) = \sqrt{73}x^{10} + 3x^8 - 5x^7 + x + \dfrac{\pi}{e}$ (Middle three terms will vary.)

61. As $x \to -\infty$, $f(x) \to -\infty$; As $x \to \infty$, $f(x) \to -\infty$

62. As $x \to -\infty$, $f(x) \to \infty$; As $x \to \infty$, $f(x) \to \infty$

63. The degree is odd. The leading coefficient is positive

64. The degree is even. The leading coefficient is negative.

65. Zeros: $x = \dfrac{-1 - \sqrt{145}}{-8}$ or $x = \dfrac{-1 + \sqrt{145}}{-8}$
x-intercepts: $\left(\dfrac{-1 - \sqrt{145}}{-8}, 0\right)$, $\left(\dfrac{-1 + \sqrt{145}}{-8}, 0\right)$

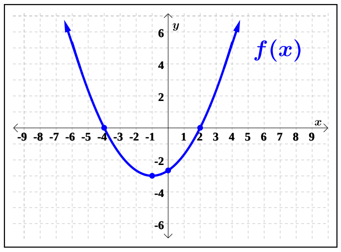
66. Zeros: $x = -3$ or $x = 0$ or $x = 3$
x-intercepts: $(-3, 0)$, $(0, 0)$, $(3, 0)$

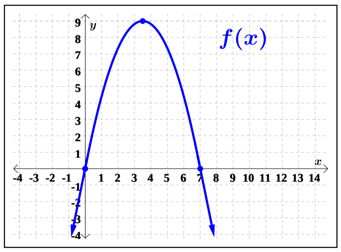
67. Zeros: $x = 0$ or $x = \sqrt[3]{10}$
x-intercepts: $(0, 0)$, $\left(\sqrt[3]{10}, 0\right)$

68. Zeros: No real zeros
x-intercepts: None

69. Zeros: $x = -8$ or $x = 0$ or $x = 2$
x-intercepts: $(-8, 0)$, $(0, 0)$, $(2, 0)$

70. Zeros: $x = -6$ or $x = \dfrac{2}{3}$
x-intercepts: $(-6, 0)$, $\left(\dfrac{2}{3}, 0\right)$

71. 

72. 

73. **a.** 9.125 feet
    b. 3.4 seconds

74. **a.** $R(x) = -6x^2 + 96x$
    b. 8000 items
    c. $384,000
    d. $48

75. **a.** $P(x) = -0.1x^2 + 25x - 400$
    b. 150 widgets
    c. 125 widgets
    d. $17.50

76. **a.** $R(x) = -\frac{1}{20}x^2 + 55x$
    b. $27.50
    c. $525

77. The domain of a function is the set of all input values that produce an output value. A polynomial function is a sum of terms, where each term is a product of a real number and an input value raised to whole number power. As any real number raised to a whole number power and then multiplied by another real number produces a real number, then a polynomial is always a sum of real numbers and always produces a real number output.

78. The real zeros and real roots of a function are the values where the function has an output value of 0. On a graph an output value of 0 corresponds to a y-value of 0. The x-intercept(s) of a function is/are the point(s) on the corresponding graph where the y-value is 0.

Section 5.3 Answers

1. Rational function
2. Not a rational function
3. Rational function
4. Rational function
5. Not a rational function

6. Rational function
7. $(-\infty,-4)\cup(-4,\infty)$
8. $(-\infty,-5)\cup(-5,-2)\cup(-2,\infty)$
9. $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
10. $(-\infty,-2)\cup(-2,4)\cup(4,\infty)$
11. $\left(-\infty,-\frac{2}{5}\right)\cup\left(-\frac{2}{5},\infty\right)$
12. $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
13. $(-\infty,\infty)$
14. $(-\infty,\infty)$
15. y-intercept: $\left(0,\frac{5}{4}\right)$

    x-intercept: $(-5,0)$
16. y-intercept: $\left(0,\frac{1}{4}\right)$

    x-intercept: None
17. y-intercept: $\left(0,\frac{3}{5}\right)$

    x-intercept: $\left(-\frac{3}{7},0\right)$
18. y-intercept: $\left(0,\frac{7}{9}\right)$

    x-intercept: $(-7,0)$
19. $\frac{2x}{x+3},\ x\neq -3$
20. $\frac{3x-18}{32+4x},\ x\neq -8$
21. $\frac{-24}{5-9x},\ x\neq \frac{5}{9}$
22. $\frac{-27x+9}{28x+98},\ x\neq -\frac{98}{28}$
23. $\frac{11}{x+1},\ x\neq -1$
24. $\frac{-5}{x-4},\ x\neq 4$
25. $\frac{4+7x}{2x-5},\ x\neq \frac{5}{2}$
26. $\frac{5x-1}{3x+11},\ x\neq -\frac{11}{3}$
27. $2x+h,\ h\neq 0$
28. $-\frac{1}{x(x+h)},\ h\neq 0,\ x\neq 0,\ x\neq -h$

29. $4x+2h,\ h\neq 0$

30. $-\dfrac{2}{x(x+h)},\ h\neq 0,\ x\neq 0$

31. $-2x-h,\ h\neq 0,\ x\neq -h$

32. $\dfrac{1}{x(x+h)},\ h\neq 0,\ x\neq 0,\ x\neq -h$

33. $(-\infty,-4)\cup(-4,-3)\cup(-3,\infty)$

34. $(-\infty,-2)\cup(-2,6)\cup(6,\infty)$

35. $(-\infty,-3)\cup(-3,-2)\cup(-2,0)\cup(0,\infty)$

36. $(-\infty,0)\cup(0,1)\cup(1,\infty)$

37. $(-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty)$

38. $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$

39. $(-\infty,-1)\cup(-1,\infty)$

40. $(-\infty,-3)\cup(-3,\infty)$

41. y-intercept: $(0,3)$
    x-intercept: None

42. y-intercept: $\left(0,-\dfrac{1}{50}\right)$
    x-intercept: None

43. y-intercept: None
    x-intercept: None

44. y-intercept: $(0,2)$
    x-intercept: $(4,0)$

45. $\dfrac{8x+9}{5x-4},\ x\neq 0,\dfrac{4}{5}$

46. $\dfrac{1}{x+3},\ x\neq -3,-2,-1$

47. $\dfrac{(x+1)(x+5)}{3(2x^2+19)}=\dfrac{x^2+6x+5}{6x^2+57},\ x\neq 0,1$

48. $\dfrac{4x(x-4)}{(x+3)^3(x-2)}=\dfrac{4x^2-16x}{x^4+7x^3+9x^2-27x-54},\ x\neq -3,2$

49. $\dfrac{(x-6)(x-5)}{(3-x)(x^2-9)}=\dfrac{x^2-11x+30}{-x^3+3x^2+9x-27},\ x\neq -3,3$

50. $-\dfrac{1}{x+5},\ x\neq -5,5,11$

51. $\dfrac{3x^2(x+6)}{x^2+6},\ x\neq 7$

52. $\dfrac{x-2}{8x(x+5)}=\dfrac{x-2}{8x^2+40x},\ x\neq 0,-5$

53. $\dfrac{7x-21}{(x+6)(x-1)},\ x\neq -6,1$

54. $\dfrac{-x-47}{(x+2)(x+7)},\ x \neq -7, -2$

55. $\dfrac{22x^2-2x-9}{(7x+1)(x+4)},\ x \neq -\dfrac{1}{7}, -4$

56. $\dfrac{-x^2+6x-6}{(9-x)(x+2)},\ x \neq -2, 9$

57. $\dfrac{3x^2-x-40}{(x-2)^2(x-8)},\ x \neq 2, 8$

58. $\dfrac{-12h}{(x+3)(x+h+3)},\ x \neq -3$

59. $6x+3h,\ h \neq 0$

60. $\dfrac{-1}{(x-7)(x+h-7)},\ h \neq 0,\ x \neq 7$

61. $-x-0.5h,\ h \neq 0$

62. $\dfrac{-21}{(7x+2)(7x+7h+2)},\ h \neq 0,\ x \neq -\dfrac{2}{7}$

63. $20x+10h+1,\ h \neq 0$

64. $\dfrac{2}{(x-5)(x+h-5)},\ h \neq 0\ x \neq 5$

65. Domain: $\left(-\infty, \dfrac{3}{2}\right) \cup \left(\dfrac{3}{2}, \infty\right)$

    Vertical Asymptote at $x = \dfrac{3}{2}$

66. Domain: $(-\infty, -4) \cup (-4, -1) \cup (-1, 3) \cup (3, \infty)$

    Hole at $\left(3, \dfrac{25}{28}\right)$

    Vertical Asymptotes at $x = -4$ and $x = -1$

67. Domain: $\left(-\infty, -\sqrt{5}\right) \cup \left(-\sqrt{5}, \sqrt{5}\right) \cup \left(\sqrt{5}, \infty\right)$

    Vertical Asymptotes at $x = -\sqrt{5}$ and $x = \sqrt{5}$

68. Domain: $(-\infty, -4) \cup \left(-4, \dfrac{4}{3}\right) \cup \left(\dfrac{4}{3}, \infty\right)$

    Hole at $\left(-4, \dfrac{11}{16}\right)$

    Vertical Asymptote at $x = \dfrac{4}{3}$

69. Domain: $(-\infty, \infty)$

70. Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$

    Hole at $\left(-2, -\dfrac{5}{8}\right)$

    Vertical Asymptotes at $x = 0$ and $x = 2$

71. $f(x) = \dfrac{(x-2)(x+1)}{(x-5)(x+5)}$

72. $g(x) = \dfrac{(x-1)(x-5)(x-2)}{(x+4)(x+1)(x-2)}$
73. $\dfrac{32x+112}{x+7}$, $x \neq -7, -3, 3$
74. $\dfrac{3\left(x^2-2x+4\right)}{(x+h+11)(x+11)}$, $h \neq 0$, $x \neq -11$
75. $\dfrac{4\left(x^2+4x+16\right)}{(2x+1)(x-7)}$, $x \neq -\dfrac{1}{2}, 7$
76. $\dfrac{-x^4+3x^3-2x^2-19x-152}{x(x-1)}$, $x \neq -2, 0, 1$
77. $-8x-4h-7$, $h \neq 0$
78. $\dfrac{-3}{(x-2)(x+h-2)}$, $h \neq 0$, $x \neq 2$
79. $16x+8h+7$, $h \neq 0$
80. $\dfrac{35}{(2x+9)(2x+2h+9)}$, $h \neq 0$, $x \neq -\dfrac{9}{2}$
81. When given a rational function, we first identify the domain restrictions and then simplify the rational function, by dividing out any common factors. Once the rational function is simplified, if a restricted domain value produces a valid output value, then the restricted domain value is the x-coordinate of a hole; otherwise the restricted domain value is the location of a vertical asymptote.
82. If we compute the intercepts of a rational function without first finding its domain, then we risk including a point on the corresponding graph where the function is actually undefined.

Section 5.4 Answers

1. $\dfrac{1}{64}$
2. $\dfrac{\sqrt{3}}{x^{\frac{5}{2}}}$
3. $23(x+1)^9$
4. $-2xy^{11}$
5. $f(x) = x^{\frac{9}{5}}$
6. $g(x) = 7(x+36)^{\frac{1}{2}}$
7. $h(x) = \dfrac{8}{(x+11)^{\frac{4}{3}}}$
8. $r(x) = (5x+2)^{\frac{3}{4}}$
9. $q(x) = -2\left(49x^7-63\right)^{\frac{1}{7}}$
10. $m(x) = \dfrac{(15x+125)^{\frac{1}{8}}}{13}$
11. $f(x) = 6\sqrt[7]{x^2}$ or $f(x) = 6\left(\sqrt[7]{x}\right)^2$

12. $g(x) = 8\sqrt[10]{(3x+4)^9}$ or $g(x) = 8\left(\sqrt[10]{3x+4}\right)^9$

13. $h(x) = \dfrac{1}{\sqrt[5]{x^2+2x+50}}$

14. $[0,\infty)$

15. $\left[-\dfrac{2}{7},\infty\right)$

16. $(-\infty,\infty)$

17. $[0,\infty)$

18. $(-\infty,\infty)$

19. $(-\infty,\infty)$

20. $\left(-\infty,\dfrac{2}{3}\right]$

21. $(-\infty,\infty)$

22. $\dfrac{\sqrt{2}}{2}$

23. $\dfrac{7\left(\sqrt{x}-6\right)}{x-36}$

24. $\dfrac{\sqrt{5}}{5}$

25. $\dfrac{-9\left(x+2\sqrt{13}\right)}{x^2-52}$

26. $\dfrac{70}{9-\sqrt{11}}$

27. $\dfrac{41-x}{\sqrt{41}+\sqrt{x}}$

28. $\dfrac{-23}{\sqrt{3}+\sqrt{26}}$

29. $\dfrac{64-x}{8-\sqrt{x}}$

30. $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

31. $\dfrac{4}{2\sqrt{x+h}+2\sqrt{x}}$ or $\dfrac{2}{\sqrt{x+h}+\sqrt{x}}$

32. $\dfrac{3}{\sqrt{3(x+h)}+\sqrt{3x}}$

33. $\left(x-y^2\right)^2 = \left(x-y^2\right)\left(x-y^2\right) = x^2-xy^2-y^2x+y^4 = x^2-2xy^2+y^4$ ✓

34. $\left(\sqrt[6]{x+y}\right)\left(\sqrt[5]{x^{-8}y^3z}\right) = (x+y)^{\frac{1}{6}}\left(x^{-8}y^3z\right)^{\frac{1}{5}} = (x+y)^{\frac{1}{6}}\left(x^{-\frac{8}{5}}y^{\frac{3}{5}}z^{\frac{1}{5}}\right) = \dfrac{(x+y)^{\frac{1}{6}}y^{\frac{3}{5}}z^{\frac{1}{5}}}{x^{\frac{8}{5}}}$ ✓

35. $25x^4 - 6x^2 + \frac{3}{2}x^{-\frac{1}{2}} = 25x^4 - 6x^2 + \frac{3}{2\sqrt{x}} = \frac{2\sqrt{x}\left(25x^4\right) - 2\sqrt{x}\left(6x^2\right) + 3}{2\sqrt{x}}$

$= \frac{50x^{\frac{1}{2}}x^4 - 12x^{\frac{1}{2}}x^2 + 3}{2x^{\frac{1}{2}}} = \frac{50x^{\frac{9}{2}} - 12x^{\frac{5}{2}} + 3}{2x^{\frac{1}{2}}}$ ✓

36. $(-\infty, 0) \cup (0, \infty)$

37. $(-2, \infty)$

38. $\left(-\infty, \frac{5}{7}\right)$

39. $\left(-\infty, \frac{1}{5}\right)$

40. $(0, \infty)$

41. $(-\infty, -1) \cup (-1, \infty)$

42. $(-\infty, 6) \cup (6, \infty)$

43. $(-\infty, -7) \cup (-7, 7) \cup (7, \infty)$

44. $\frac{2x - 126}{\sqrt{2x-5} + 11}$

45. $\frac{23}{23 - 7\sqrt{3}}$

46. $\frac{h}{\sqrt{x+h} + \sqrt{x}}$

47. $\frac{81x - 16}{3(9\sqrt{x} - 4)}$ or $\frac{81x - 16}{27\sqrt{x} - 12}$

48. $\frac{1}{\sqrt{x+h-4} + \sqrt{x-4}}$

49. $\frac{9}{3\sqrt{x+h+1} + 3\sqrt{x+1}}$ or $\frac{3}{\sqrt{x+h+1} + \sqrt{x+1}}$

50. $\frac{2}{\sqrt{2(x+h)+5} + \sqrt{2x+5}}$

51. $\frac{-3}{\sqrt{4-3(x+h)} + \sqrt{4-3x}}$

52. $(-\infty, -9) \cup (-9, \infty)$

53. $[0, 64) \cup (64, \infty)$

54. $[1, 8]$

55. $(4, \infty)$

56. $[0, 81) \cup (81, \infty)$

57. $(-\infty, 9)$

58. $\frac{2x + h}{\sqrt{(x+h)^2 - 5} + \sqrt{x^2 - 5}}$

59. $\frac{-30}{\sqrt{6(x+h)+10} + \sqrt{6x+10}}$

60. $\dfrac{1}{\sqrt{x+h-8}+\sqrt{x-8}}$

61. The numerator of $f(x)$ is a polynomial which has no domain restrictions, but the numerator of $g(x)$ is an even root function, which does have domain restrictions that must be considered.

62. To fully simplify a difference quotient, $\dfrac{f(x+h)-f(x)}{h}$, the 'h' in the denominator must be removed by dividing it out with a common factor of 'h' in the numerator. When $f(x)$ is a square root function, the only way to have this happen is to rationalize the numerator to get an 'h' outside of the root.

Section 5.5 Answers

1. -2
2. 2
3. 4
4. -2
5. -1
6. -4.75
7. -4
8. 11
9. 8
10. 12.5
11. Does Not Exist
12. Does Not Exist
13. 16.5
14. 0
15. Does Not Exist
16. Does Not Exist
17. Does Not Exist
18. 134
19. 206
20. $f(x) = |3x|$
21. $f(x) = |x+6|$
22. $f(x) = \begin{cases} -(x-5) & \text{if } x < 5 \\ x-5 & \text{if } x \geq 5 \end{cases}$
23. $g(x) = \begin{cases} -\frac{2}{7}x & \text{if } x < 0 \\ \frac{2}{7}x & \text{if } x \geq 0 \end{cases}$
24. $[-8, \infty)$
25. $(-\infty, 3) \cup (3, 7]$

26. $(-\infty,\infty)$

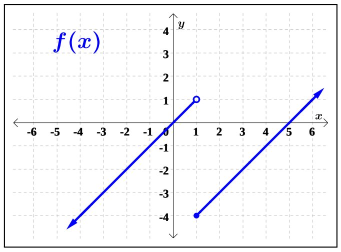
27. $(-\infty,12)\cup(12,\infty)$

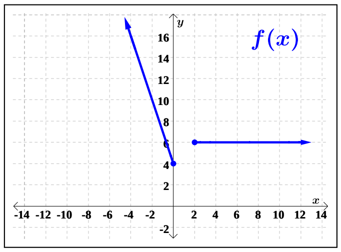
28. $(-\infty,\infty)$

29. $(-7,0]\cup(4,\infty)$

30. $(-\infty,\infty)$

31. $(0,7)\cup[8,\infty)$

32. 

33. 

34. $H(x)=\begin{cases} 12x & \text{if } 0\le x\le 10 \\ 120+8(x-10) & \text{if } x>10 \end{cases}$ or $H(x)=\begin{cases} 12x & \text{if } 0\le x\le 10 \\ 8x+40 & \text{if } x>10 \end{cases}$

35. $I(x)=\begin{cases} 475+80x & \text{if } 0\le x\le 9 \\ 1195+200(x-9) & \text{if } x>9 \end{cases}$ or $I(x)=\begin{cases} 475+80x & \text{if } 0\le x\le 9 \\ 220x-605 & \text{if } x>9 \end{cases}$

36. Does Not Exist

37. 3

38. -3

39. 6

40. -1

41. 1

42. 8

43. 5

44. 9

45. Does Not Exist

46. 0

47. 5

48. Does Not Exist

49. $\sqrt{17}$

50. Does Not Exist

51. $\sqrt[3]{-18}$

52. -5

53. -2

54. Does Not Exist

55. 89

56. -3

57. 29

58. 30.13

59. $-\dfrac{20}{3}$

60. $f(x) = |2x+4|$

61. $f(x) = |-3x+12|$

62. $f(x) = \begin{cases} 9-6x & \text{if } x \le \frac{3}{2} \\ -(9-6x) & \text{if } x > \frac{3}{2} \end{cases}$ or $f(x) = \begin{cases} 9-6x & \text{if } x \le \frac{3}{2} \\ -9+6x & \text{if } x > \frac{3}{2} \end{cases}$

63. $g(x) = \begin{cases} -100x & \text{if } x \le 0 \\ 100x & \text{if } x > 0 \end{cases}$

64. $(-\infty, 2) \cup (2, 10) \cup (10, \infty)$

65. $(-9, -7] \cup [-6, -3) \cup (-3, \infty)$

66. $(-1, 3) \cup (3, 4) \cup [6, \infty)$

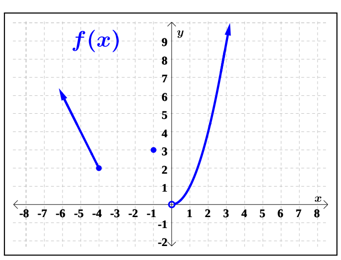
67. $(-7, -3) \cup (-3, \infty)$

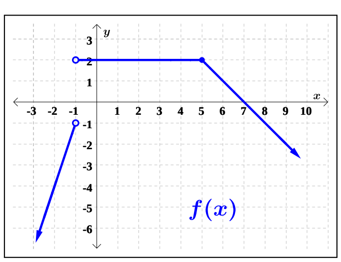
68. $[-5, -1) \cup (-1, 2) \cup (2, 7) \cup (7, 8]$

69. $(-\infty, -1) \cup (-1, \infty)$

70. $(-\infty, \infty)$

71. $(-\infty, 0) \cup (0, \infty)$

72. 

73. 

74. $C(x) = \begin{cases} 20 & \text{if } 0 \le x \le 100 \\ 20 + 0.1(x-100) & \text{if } 100 < x \le 500 \\ 20 + 0.1(400) + 0.15(x-500) & \text{if } x > 500 \end{cases}$

75. $B(t) = \begin{cases} 5 & \text{if } 0 \le t \le 100 \\ 5 + 0.2(t-100) & \text{if } 100 < t \le 1200 \\ 5 + 0.2(1100) + 0.4(t-1200) & \text{if } t > 1200 \end{cases}$

76. 2

77. 3

78. 0

79. 2

80. $[-7,-5) \cup (-5,8)$

81. $[-6,6]$

82. Does Not Exist

83. $-\frac{1}{2}$

84. Does Not Exist

85. $\sqrt{3.9801}$

86. $-\frac{200}{101}$

87. 7

88. $\sqrt{7}$

89. -2

90. $\sqrt{67}$

91. Does Not Exist

92. $(-\infty,9) \cup (10,\infty)$

93. $[-2,0) \cup \left[\sqrt{3}, \sqrt{84}\right)$

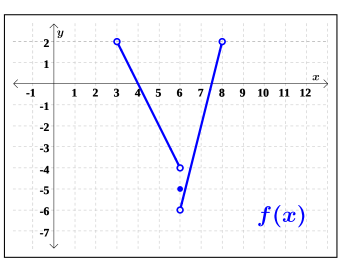
94. $f(x) = \begin{cases} -\frac{x-4}{x-4} & \text{if } x < 4 \\ \frac{x-4}{x-4} & \text{if } x > 4 \end{cases}$ or $f(x) = \begin{cases} -1 & \text{if } x < 4 \\ 1 & \text{if } x > 4 \end{cases}$

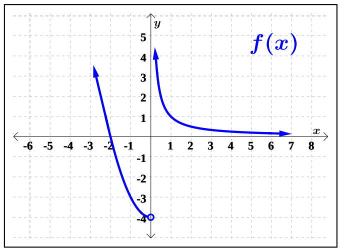
95. $(-\infty,-8) \cup (-8,-3) \cup (-3,-2) \cup (-2,\infty)$

96. $[-1,35]$

97. $(-\infty,-4) \cup (-4,5) \cup (5,6) \cup (6,7)$

98.



99. 

100. $P(t) = \begin{cases} 20t & \text{if } 0 \leq t \leq 3 \\ 20(3) + 40(t-3) & \text{if } 3 < t \leq 5 \\ 20(3) + 40(2) + 60(t-5) & \text{if } t > 5 \end{cases}$

101. We use an 'open' circle when graphing the endpoint of a strict inequality (<, >), and a 'closed' dot when graphing the endpoint of a non-strict inequality (≤, ≥).

Section 5.6 Answers

1. 4^{6x}
2. 2^{12x}
3. 5^{3x-21}
4. 3^{4x+16}
5. Exponential function
6. Not an exponential function
7. Not an exponential function
8. Exponential function
9. **a.** Domain: $(-\infty, \infty)$
   b. Range: $(0, \infty)$
   c. End Behavior: As $x \to -\infty$, $f(x) \to 0$ (H.A.: $y = 0$)
   As $x \to \infty$, $f(x) \to \infty$
   d. x-intercept: None
   e. y-intercept: $(0, 1)$
10. **a.** Domain: $(-\infty, \infty)$
   b. Range: $(0, \infty)$
   c. End Behavior: As $x \to -\infty$, $g(x) \to \infty$
   As $x \to \infty$, $g(x) \to 0$ (H.A.: $y = 0$)
   d. x-intercept: None
   e. y-intercept: $(0, 1)$

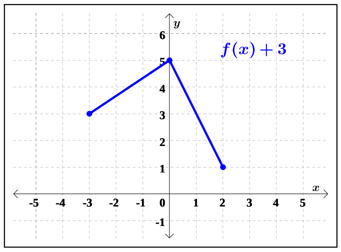
11. **a.** Domain: $(-\infty, \infty)$
    b. Range: $(0, \infty)$
    c. End Behavior: As $x \to -\infty$, $h(x) \to \infty$
    As $x \to \infty$, $h(x) \to 0$ (H.A.: $y = 0$)
    d. x-intercept: None
    e. y-intercept: $(0, 1)$
12. **a.** Domain: $(-\infty, \infty)$
    b. Range: $(0, \infty)$
    c. End Behavior: As $x \to -\infty$, $j(x) \to 0$ (H.A.: $y = 0$)
    As $x \to \infty$, $j(x) \to \infty$
    d. x-intercept: None
    e. y-intercept: $(0, 1)$
13. $(-\infty, \infty)$
14. $(-\infty, \infty)$
15. $(-\infty, \infty)$
16. $(-\infty, \infty)$
17. $x = 3$
18. $x = -1$ or $x = 1$
19. $x = \frac{5}{4}$
20. $x = \frac{3}{2}$
21. $x = \frac{12}{23}$
22. $x = -\frac{13}{7}$
23. $x = \frac{14}{5}$
24. \$57,121
25. \$3681.39
26. 7^{4x-5}
27. 2^{36x-24}
28. 11^{4x+3}
29. 6^{8x+1}
30. Exponential decay
31. Exponential growth
32. Exponential decay
33. Exponential growth

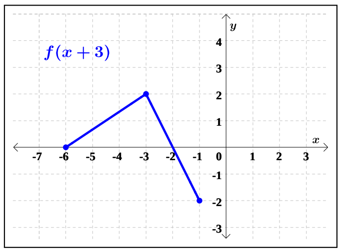
34. **a.** Domain: $(-\infty, \infty)$
    b. Range: $(0, \infty)$
    c. End Behavior: As $x \to -\infty$, $f(x) \to 0$ (H.A.: $y = 0$)
    As $x \to \infty$, $f(x) \to \infty$
    d. x-intercept: None
    e. y-intercept: $(0, 4)$
35. **a.** Domain: $(-\infty, \infty)$
    b. Range: $(-\infty, 0)$
    c. End Behavior: As $x \to -\infty$, $g(x) \to 0$ (H.A.: $y = 0$)
    As $x \to \infty$, $g(x) \to -\infty$
    d. x-intercept: None
    e. y-intercept: $\left(0, -\frac{1}{15}\right)$
36. **a.** Domain: $(-\infty, \infty)$
    b. Range: $(0, \infty)$
    c. End Behavior: As $x \to -\infty$, $h(x) \to \infty$
    As $x \to \infty$, $h(x) \to 0$ (H.A.: $y = 0$)
    d. x-intercept: None
    e. y-intercept: $(0, 6)$
37. **a.** Domain: $(-\infty, \infty)$
    b. Range: $(-\infty, 0)$
    c. End Behavior: As $x \to -\infty$, $j(x) \to -\infty$
    As $x \to \infty$, $j(x) \to 0$ (H.A.: $y = 0$)
    d. x-intercept: None
    e. y-intercept: $(0, -11)$
38. $(-\infty, 6) \cup (6, \infty)$
39. $(-\infty, 7)$
40. $[-3, \infty)$
41. $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
42. $x = -3$ or $x = 4$
43. $x = \dfrac{11}{14}$
44. $x = -2$ or $x = 6$
45. $x = -\dfrac{1}{2}$
46. $x = 5$
47. $x = 4$ or $x = 5$
48. $x = -\dfrac{1}{2}$ or $x = 2$
49. $x = -2$ or $x = 0$
50. \$23,692.78
51. \$1915.95

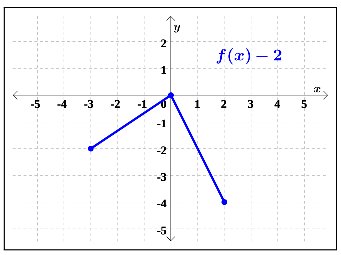
52. \$38,075.60
53. **a.** True
    b. False
54. $3^{-16x-35}$
55. Exponential decay function
56. Exponential decay function
57. **a.** Domain: $(-\infty, \infty)$
    b. Range: $(0, \infty)$
    c. End Behavior: As $x \to -\infty$, $f(x) \to \infty$
    As $x \to \infty$, $f(x) \to 0$ (H.A.: $y = 0$)
    d. x-intercept: None
    e. y-intercept: $(0, 1)$
58. $[-9, -1) \cup (-1, \infty)$
59. $(-2, 3)$
60. $x = -4$ or $x = 2$
61. $x = 0$ or $x = 1$
62. $x = -\dfrac{3}{7}$ or $x = 0$
63. **a.** \$33,491.00
    b. \$32,965.34
    c. \$32,947.06
    d. \$32,944.02
64. $f(x) = 0.125^{-4x} = \left(\dfrac{1}{8}\right)^{-4x} = \left(8^{-1}\right)^{-4x} = 8^{4x} = 8^4 \cdot 8^x$, $f(x)$ is an exponential growth function because $f(x) = a \cdot b^x$ with $a > 0$ and $b > 1$.
65. For exponential growth, over equal increments, the constant multiplicative rate of change results in multiplying the output whenever the input increases by one. For linear growth, the constant additive rate of change over equal increments results in adding to the output whenever the input is increased by one.

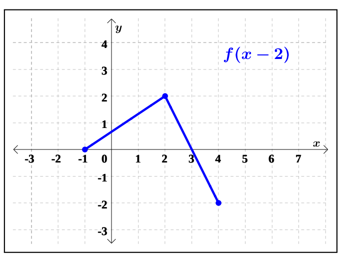
Section 5.7 Answers

1. $f(x) = x$
2. $f(x) = |x|$
3. $f(x) = \sqrt{x}$
4. $f(x) = b^{-x}$ for $b > 1$ or $f(x) = b^x$ for $0 < b < 1$

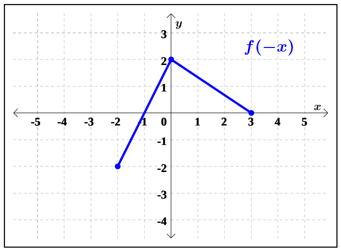
5. 

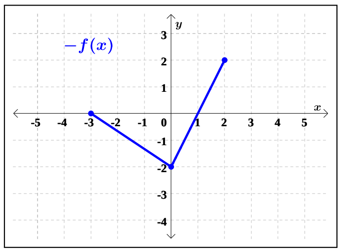
6. 

7. 

8. 

9. $f(x) = \sqrt[3]{x}$; Shift down 6 units
10. $f(x) = e^x$; Shift up 1 unit
11. $f(x) = x^3$; Shift right 7 units
12. $f(x) = |x|$; Shift left 4 units
13. $g(x) = x^2 - 8$
14. $g(x) = |x| + 20$
15. $g(x) = 3^{x-5}$
16. $g(x) = (x+8)^3$

17. 

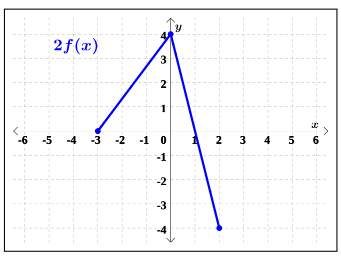
18. 

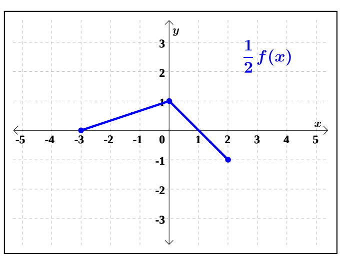
19. $f(x) = x^2$; Reflect across the x-axis

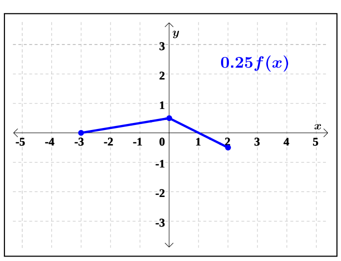
20. $f(x) = \sqrt{x}$; Reflect across the y-axis

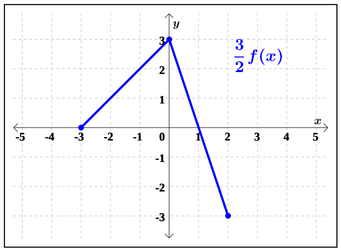
21. $g(x) = -\dfrac{1}{x^2}$

22. $g(x) = 10^{-x}$

23. 

24. 

25. 

26. 

27. $f(x) = |x|$; Vertical stretch by a factor of 6

28. $f(x) = 10^x$; Vertical stretch by a factor of $\frac{5}{3}$

29. $f(x) = x^3$; Vertical stretch by a factor of 3.8

30. $f(x) = x$; Vertical compression by a factor of 7

31. $g(x) = \frac{1}{6} \cdot \frac{1}{x} = \frac{1}{6x}$

32. $g(x) = 8e^x$

33. 13

34. -13

35. -4

36. $-\frac{1}{2}$

37. $x^3 - x^2 + x + 3$

38. $|x| - 3x - 9$

39. $-3x^3 - 9x^2 + 12x + 36$

40. $\frac{|x|}{-x^2 + 4}$

41. $\frac{1}{2}$

42. 0

43. 16

44. -21

45. 0

46. 2

47. -1

48. 1

49. 1

50. 0

51. $f(x) = g(h(x))$ where $g(x) = \sqrt[4]{x}$ and $h(x) = x - 9$

52. $f(x) = g(h(x))$ where $g(x) = \frac{3}{x}$ and $h(x) = x + 6$

53. $f(x) = g(h(x))$ where $g(x) = e^x$ and $h(x) = 7x + 10$

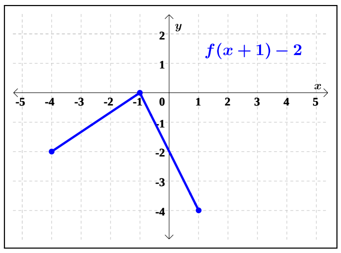
54. $f(x) = g(h(x))$ where $g(x) = \dfrac{1}{x^2}$ and $h(x) = 2x + 1$

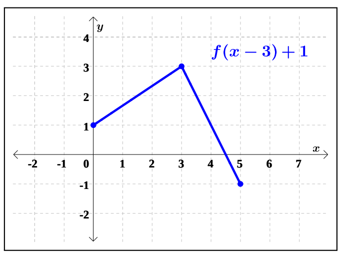
55. $f(x) = x^2$; Quadratic function

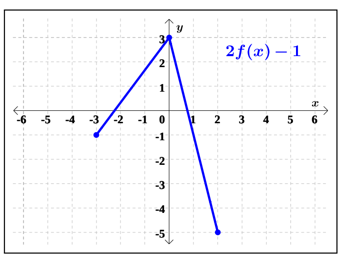
56. $f(x) = \sqrt[3]{x}$; Cube root function

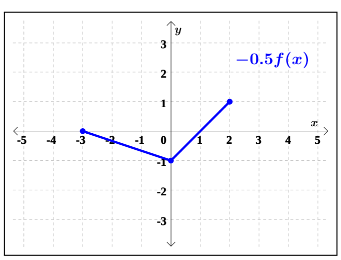
57. $f(x) = \dfrac{1}{x}$; Reciprocal function

58. $f(x) = b^x$ where $b > 1$ or $f(x) = b^{-x}$ where $0 < b < 1$; Exponential growth function

59. 

60. 

61. 

62. 

63. $f(x) = 2^x$
Reflect over the x-axis
Shift up 3 units

64. $f(x) = \sqrt{x}$
Shift left 4 units
Shift down 7 units

65. $f(x) = e^x$
Shift right 6 units
Vertical stretch by a factor of 3

66. $f(x) = x^3$
Shift right 1 unit
Vertical compression by a factor of 4

67. $g(x) = |x+3| - 8$

68. $g(x) = -5^{x-7}$

69. $g(x) = 9\sqrt[3]{x} + 1$

70. $g(x) = -\dfrac{1}{2}x^3$

71. 4

72. $\sqrt[3]{3} + 2$

73. $3\sqrt[3]{6}$

74. $\dfrac{2}{e^2}$

75. $3e^x + \sqrt[3]{6-x}$

76. $\sqrt{x+1} - e^{-x}$

77. 3

78. $\dfrac{\sqrt{x+1}}{3e^x}$

79. $5x^2 + 11x + 6$

80. $5x^2 + x + 1$

81. $125x^4 + 50x^3 + 10x^2 + x$

82. $x + 2$

83. -1

84. 1

85. 4

86. 4

87. 3

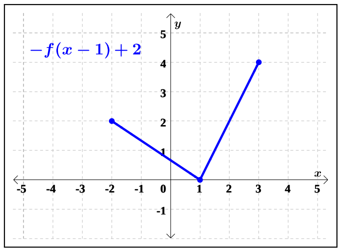
88. 0

89. $f(x) = g(h(x))$ where $g(x) = 3\sqrt{x}$ and $h(x) = 8 - x$

90. $f(x) = g(h(x))$ where $g(x) = \dfrac{x}{9}$ and $h(x) = 7x^2 + 11$

91. $f(x) = g(h(x))$ where $g(x) = e^x + 100$ and $h(x) = x^2$

92. $f(x) = g(h(x))$ where $g(x) = \dfrac{17}{4}x^2 - 11$ and $h(x) = x - 2$

93. 

94. $f(x)=\dfrac{1}{x}$
Shift right 5 units
Vertical compression by a factor of 8
Reflect over the x-axis
Shift up 6 units

95. $f(x)=10^x$
Shift left 4 units
Vertical stretch by a factor of 2
Shift down 1 unit

96. $g(x)=-6\cdot\dfrac{1}{(x+3)^2}-2=-\dfrac{6}{(x+3)^2}-2$

97. $g(x)=\dfrac{1}{12}|x-2|+1$

98. $\dfrac{56}{5}$

99. 13

100. $-\dfrac{69}{25}$

101. 0

102. $7-20e$

103. $\dfrac{6}{37}$

104. $-3x^2-2x+7$

105. $\dfrac{1-5xe^{x-2}-30e^{x-2}}{x+6}$ or $\left(\dfrac{1}{x+6}\right)-5e^{x-2}$

106. $35e^{x-2}-20xe^{x-2}$

107. $\dfrac{1}{-3x^3-16x^2+12x}$

108. $48x^2+160x-133$

109. $-27x^4-36x^3-18x^2+4x$

110. $g(f(0))=5$ and $(f\circ g)(0)=52$

111. $f(x)=g(h(x))$ where $g(x)=\dfrac{8x}{3x-32}$ and $h(x)=\sqrt{7x+6}$

112. Evaluate $A(m(t))=4$

113. Reflections across the x-axis must occur before vertical shifts.

114. $(f \cdot g)(x) = f(x) \cdot g(x)$ means we are multiplying the two functions. $(f \circ g)(x) = f(g(x))$ means we are computing a composition of the two functions.

Section 5.8 Answers

1. One-to-One
2. NOT One-to-One
3. One-to-One
4. $f(x)$ and $g(x)$ are inverse functions
5. $f(x)$ and $g(x)$ are NOT inverse functions
6. $4^2 = 16$
7. $10^{-2} = \dfrac{1}{100}$
8. $3^4 = 81$
9. $e^{\frac{1}{2}} = \sqrt{e}$
10. $\ln(1) = 0$
11. $\log_5\left(\dfrac{1}{125}\right) = -3$
12. $\log(100000) = 5$
13. $\log_2\left(\dfrac{1}{64}\right) = -6$
14. $\log_8\left(\dfrac{x^2}{y^4}\right)$
15. $\ln\left(8x^7\right)$
16. $\log_5\left(\dfrac{y^{11}}{2}\right)$
17. $\log\left(\dfrac{y^{\sqrt{2}}}{x^3}\right)$
18. $2 + \log(x) + \log(y) + \log(z)$
19. $\log_7\left(x^2 + y\right) - \log_7(y - z)$
20. $\dfrac{1}{2}\ln(x) + 4\ln(y)$
21. $1 + 2\log_{13}(x) + \log_{13}(y) - 3\log_{13}(z)$
22. **a.** Domain: $(0, \infty)$
    b. Range: $(-\infty, \infty)$
    c. End Behavior: As $x \to 0$ from the right, $f(x) \to -\infty$ (V.A.: $x = 0$)
    As $x \to \infty$, $f(x) \to \infty$
    d. x-intercept: $(1, 0)$
    e. y-intercept: None

23. **a.** Domain: $(0,\infty)$
    b. Range: $(-\infty,\infty)$
    c. End Behavior: As $x \to 0$ from the right, $g(x) \to \infty$ (V.A.: $x = 0$)
    As $x \to \infty$, $g(x) \to -\infty$
    d. x-intercept: $(1,0)$
    e. y-intercept: None
24. $(5,\infty)$
25. $(-9,\infty)$
26. $\left(\frac{7}{2},\infty\right)$
27. $\left(-\frac{8}{3},\infty\right)$
28. $x = \dfrac{\ln(7)}{\ln(5)}$
29. $x = \ln(2)$
30. $x = \dfrac{1}{2}(9+\log(3))$
31. $x = -\dfrac{1}{3}\left[\dfrac{\ln(11)}{\ln(0.4)} - 8\right]$
32. $x = 16$
33. $x = \dfrac{1005}{4}$
34. $x = -3$ or $x = 11$
35. $x = e^{\frac{1}{5}}$
36. $x = -\dfrac{1}{3}(e-2)$
37. $x = \dfrac{14}{3}$
38. 22.1 years
39. 14.415 years
40. One-to-One
41. NOT One-to-One
42. NOT One-to-One
43. One-to-One
44. $f(x)$ and $g(x)$ are inverse functions
45. $f(x)$ and $g(x)$ are NOT inverse functions

46. 3

47. -2

48. $\frac{1}{2}$

49. $\sqrt{3}$

50. $\sqrt{\pi}$

51. 16

52. $\log\left(\frac{x}{y^3 z^{\frac{1}{2}}}\right)$

53. $\ln\left(\frac{16x^{\frac{1}{3}}z^4}{x+y}\right)$

54. $\log_4\left(\frac{\left(3x^2+5\right)^{\frac{1}{7}}}{x^6}\right)$

55. $\log_3(x+5)-8\log_3(y)-\frac{1}{2}\log_3(z)$

56. $2+\frac{1}{2}\log_5(x)+\frac{3}{2}\log_5(y)$

57. $\ln(x)-\frac{2}{3}-\frac{4}{3}\ln(z)$

58. $\log_2(x)+\log_2(x+4)+\log_2(2x-3)$

59. **a.** Domain: $(0,\infty)$
    b. Range: $(-\infty,\infty)$
    c. End Behavior: As $x\to 0$ from the right, $f(x)\to-\infty$ (V.A.: $x=0$)
    As $x\to\infty$, $f(x)\to\infty$
    d. x-intercept: $(1,0)$
    e. y-intercept: None

60. **a.** Domain: $(0,\infty)$
    b. Range: $(-\infty,\infty)$
    c. End Behavior: As $x\to 0$ from the right, $g(x)\to-\infty$ (V.A.: $x=0$)
    As $x\to\infty$, $g(x)\to\infty$
    d. x-intercept: $(1,0)$
    e. y-intercept: None

61. $\left(-\infty,\frac{11}{6}\right)$

62. $(-4,\infty)$

63. $(-\infty,-7)\cup(-7,7)\cup(7,9)$

64. $\left(-\frac{10}{33},\infty\right)$

65. $(-\infty,\infty)$

66. $(-1,8)\cup(8,\infty)$

67. $x = -\frac{1}{3}\ln\left(\frac{5}{4}\right)$

68. $x = -\frac{\ln(8)}{\ln(3)}$

69. $x = \left(\frac{\ln(21)}{\ln(11)}\right)^2$

70. $x = \frac{1}{5}\left[\frac{\ln(12)}{\ln(6)} - 3\right]$

71. $x = -\frac{1}{3}(\ln(5))$

72. $x = \sqrt{\ln(14)}$ or $x = -\sqrt{\ln(14)}$

73. $x = \frac{\ln(5)}{\ln(2)}$ or $x = \frac{\ln(13)}{\ln(2)}$

74. $x = 2$ or $x = \frac{\ln(4)}{\ln(3)}$

75. $x = \frac{1}{2}\left[\frac{\ln(36)}{\ln(9)}\right]$

76. $x = \ln\left(\frac{2}{3}\right)$

77. $x = 8$

78. $x = \frac{131}{2}$

79. $x = -2$

80. $x = \frac{103}{86}$

81. $x = \frac{3 + \sqrt{21}}{6}$

82. No solution

83. 26 years

84. 10.2337%

85. NOT One-to-One

86. One-to-One

87. $f(x)$ and $g(x)$ are inverse functions

88. $\log_3\left(\frac{(9+z)\left(x^b\right)}{w^a y^c}\right)$

89. $\log_b(x-3) + \frac{7}{8}\log_b(y) - 1 - \log_b(z)$

90. **a.** Domain: $(4, \infty)$
    b. Range: $(-\infty, \infty)$
    c. End Behavior: As $x \to 4$ from the right, $f(x) \to -\infty$ (V.A.: $x = 4$)
    As $x \to \infty$, $f(x) \to \infty$
    d. x-intercept: $\left(a^{-2} + 4, 0\right)$

e. y-intercept: None

91. $[-2,3)\cup(3,4)$

92. $\left(-\infty,\frac{1}{3}[-9+\ln(2)]\right)\cup\left(\frac{1}{3}[-9+\ln(2)],0\right)$

93. $(-7,-2)\cup\left(-2,\frac{4}{3}\right]$

94. $\left(-2,\frac{1}{2}[1+\log(7)]\right)\cup\left(\frac{1}{2}[1+\log(7)],8\right)$

95. $x=\dfrac{6}{1-\ln(3)}$

96. $x=\ln(9)$

97. $x=2$

98. $x=10^{10}$

99. $x=5^{-\frac{3}{2}}-4$

100. $x=1+\sqrt{1+e}$

101. 67 years

102. 39.585 years

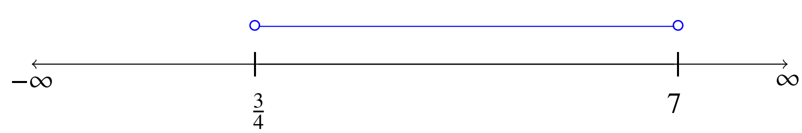
103. A function is one-to-one if it has a unique y for every x in the domain. A one-to-one function passes the Vertical Line Test and the Horizontal Line Test.

104. Every solution must be in the domain of each of the original expressions in the equation.

Chapter 5 Review Answers

1. **a.** The graph of a relation represents a function if it passes the Vertical Line Test, meaning for each x-value, there is at most one y-value.
   b. A function; each input (a state) corresponds to exactly one output (it's capital city).
   c. Not a function; an input of 4 gives two different outputs.
   d. A function; the graph passes the Vertical Line Test.

2. **a.** When a toy boat has 3 gallons of gas in its tank, it will run 8 hours.
   b. At 9 am, the tides in the Bay of Fundy are 1.76 meters high.
   c. $(3,19)$
   d. $(20,15000)$

3. **a.**


   b. $A:\ (-7,2]$, $B:\ (0,\infty)$

4. **a.** Domain: $(-\infty,-2]\cup[3,\infty)$, Range: $[-1,\infty)$
   b. Domain: $[-1,\infty)$, Range: $[0,\infty)$

5. **a.** Polynomial; Degree: 197, Leading coefficient: $-\frac{2}{3}$, Constant term: 6^{12}

   b. Not a polynomial

   c. As $x \to -\infty$, $p(x) \to -\infty$; As $x \to \infty$, $p(x) \to -\infty$

   $\swarrow \cdots \searrow$

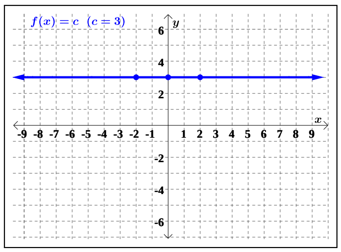
   d. As $x \to -\infty$, $q(x) \to -\infty$; As $x \to \infty$, $q(x) \to \infty$

   $\swarrow \cdots \nearrow$

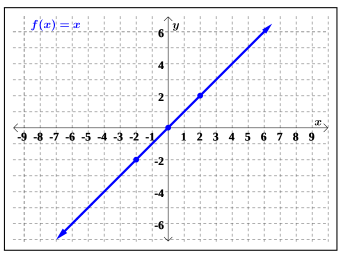
   e. x-intercepts: $(-5,0)$, $(0,0)$, $(6,0)$; y-intercept: $(0,0)$

   f. x-intercepts: $(-2,0)$, $(1,0)$, $(7,0)$; y-intercept: $(0,56)$

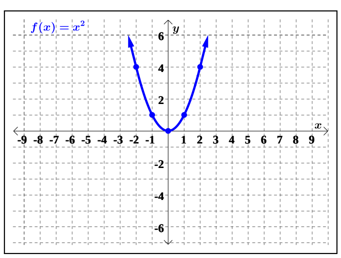
6. **a.**



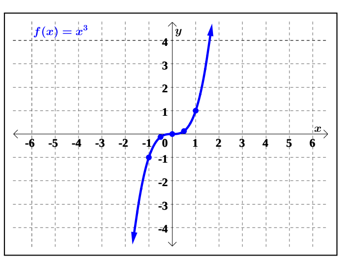
   b.



   c.



   d.



7. **a.** $(-\infty, \infty)$

   b. $(-\infty, \infty)$

8. **a.** $f(x)$ has x-intercepts at $(-2,0)$, $(3,0)$, and $(5,0)$. $f(x)$ has real roots at $x = -2$, 3, and 5. The graph of $f(x)$ touches the x-axis at $x = -2$, 3, and 5. $f(x) = 0$ when $x = -2$, 3, or 5.

   b. $x = -8$ or $x = -2$

9. **a.** Domain: $(-\infty,\infty)$; Range: $(-\infty,36]$; Vertex: $(4,36)$; y-intercept: $(0,-28)$; x-intercept: $(1,0)$ and $(7,0)$

b. Domain: $(-\infty,\infty)$; Range: $[0,\infty)$; Vertex: $(1,0)$; y-intercept: $(0,1)$; x-intercept: $(1,0)$

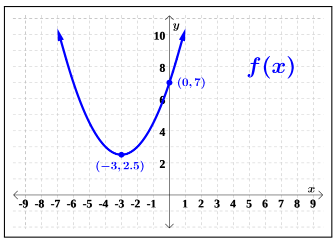
c. Domain: $(-\infty,\infty)$; Range: $\left[\frac{1}{5},\infty\right)$; Vertex: $\left(-2,\frac{1}{5}\right)=(-2,0.2)$; y-intercept: $(0,1)$; x-intercept: None

10. **a.** $g(x)$ factors; $x=\frac{7}{2}$ and $x=1$

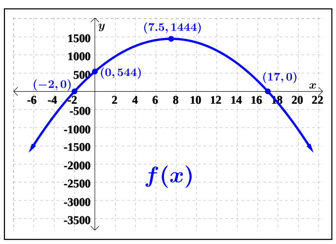
b. $f(x)$ does not factor easily; $x=\dfrac{-12+\sqrt{84}}{6}$ and $x=\dfrac{-12-\sqrt{84}}{6}$ or we say $x=\dfrac{-6\pm\sqrt{21}}{3}$

c. $h(x)$ does not factor easily; $x=\dfrac{-4+\sqrt{56}}{2}$ and $x=\dfrac{-4-\sqrt{56}}{2}$ or we say $x=-2\pm\sqrt{14}$

11. **a.**



b.



12. **a.** They should sell 155,000 items to maximize profit at \$166.25.

b. Maximum revenue is \$3721 and it occurs when 61 items are sold. To maximize profit, 60 items should be produced and sold.

c. The pumpkin landed approximately 82.8688 meters from the catapult and reaches a maximum height of 1753 meters.

13. **a.** Rational function; Domain: $(-\infty,2)\cup(2,\infty)$

b. Rational function; Domain: $(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$

c. Not a rational function

14. **a.** Domain: $(-\infty,2)\cup(2,\infty)$; y-intercept: $(0,0)$; x-intercept: $(0,0)$ and $\left(\frac{4}{3},0\right)$; V.A.: $x=2$; Hole: None

b. Domain: $(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$; y-intercept: $\left(0,\frac{7}{6}\right)$; x-intercept: None; V.A.: $x=-3$ and $x=-2$; Hole: None

c. Domain: $(-\infty,-4)\cup(-4,2)\cup(2,\infty)$; y-intercept: $(0,6)$; x-intercept: $(4,0)$; V.A.: $x=2$; Hole at $(-4,4)$

15. **a.** $\dfrac{(x+2)(x+1)}{(x+3)(x-2)},\ x\neq\pm2,\ x\neq\pm3$

b. $\dfrac{-(a-4)}{7a^2},\ a\neq 0,\ a\neq 2$

c. $\dfrac{3x^2+10x-53}{(x+7)(x-5)(x+6)},\ x\neq-7,\ x\neq-5,\ x\neq 6$

d. $\dfrac{-2(y^2+9y-2)}{y(y+2)(3y-2)(y-3)},\ y\neq-2,\ y\neq 0,\ y\neq\frac{2}{3},\ y\neq 3$

16. **a.** $\dfrac{-63}{59(59+9h)}$

b. $\dfrac{15}{(x+5)(x+h+5)}$

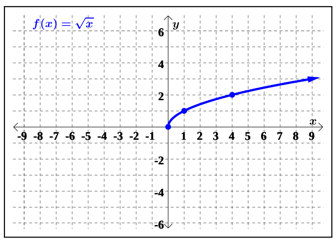
c. $\dfrac{-8x^2-8xh-2x-h}{x^2(x+h)^2}$

17. **a.** $f(x)=11\sqrt[5]{x^6}=11\left(\sqrt[5]{x}\right)^6$

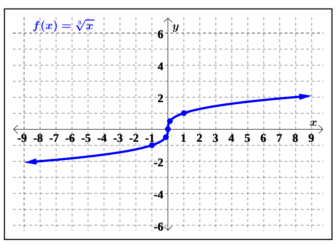
b. $f(x)=(3x-4)^{\frac{1}{6}}$

c. $f(x)=\dfrac{9}{(x^2-8)^{\frac{1}{3}}}=9\left(x^2-8\right)^{-\frac{1}{3}}$

18. **a.**



b.



19. **a.** $(-\infty,\infty)$

b. $\left(-\infty,\frac{9}{2}\right]$

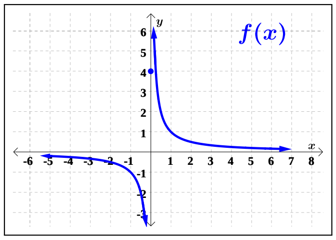
c. $\left[-\frac{4}{3},\infty\right)$

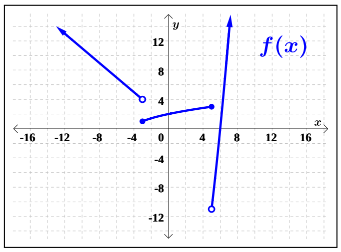
d. $\left(-\infty,-\frac{4}{3}\right)\cup\left(-\frac{4}{3},\infty\right)$

20. **a.** $\dfrac{(x-4)(\sqrt{6-x}-9)}{-x-75}$

b. $\dfrac{9}{3\sqrt{x+h}+3\sqrt{x}}$ or $\dfrac{3}{\sqrt{x+h}+\sqrt{x}}$

c. $\dfrac{2}{\sqrt{2x+2h-5}+\sqrt{2x-5}}$ or $\dfrac{2}{\sqrt{2(x+h)-5}+\sqrt{2x-5}}$

21. **a.** 

b. 

22. **a.** $(-\infty, -5] \cup (0, 4] \cup [4.5, \infty)$
b. $(-\infty, -1) \cup (-1, \infty)$
c. $(-\infty, \infty)$

23. **a.** $a :=$ the age of a person

$C(a) :=$ the amount, in dollars, of admission charged

$$C(a) = \begin{cases} 0 & \text{if} \quad 0 \le a < 3 \\ 2a & \text{if} \quad 3 \le a \le 14 \\ 29.95 & \text{if} \quad a > 14 \end{cases}$$

b. $h :=$ the number of hours a worker works per week

$P(h) :=$ the amount, in dollars, of a factory worker's pay

$$P(h) = \begin{cases} 12.50h & \text{if} \quad 0 \le h \le 40 \\ 12.50(40) + 1.5(12.50)(h-40) & \text{if} \quad 40 < h \le 60 \\ 12.50(40) + 1.5(12.50)(20) + 2(12.50)(h-60) & \text{if} \quad h > 60 \end{cases}$$

$P(71) = \$1150$

c. $t :=$ the number of tickets sold to a sporting event

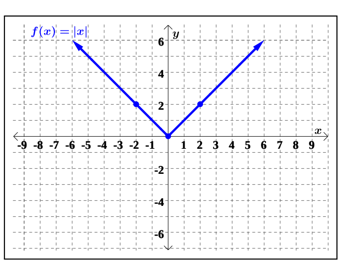
$P(t) :=$ the price, in dollars, per tickets sold

$$P(h) = \begin{cases} 120 & \text{if} \quad 0 < t < 1000 \\ 100 & \text{if} \quad 1000 \le t < 4000 \\ 75 & \text{if} \quad 4000 \le t \le 4675 \end{cases}$$

$P(900) = \$120$

$P(2300) = \$100$

$P(4500) = \$75$, the price per ticket when sold out is \$75

24. **a.** 

25. **a.** $f(x) = \begin{cases} -(x+3) & \text{if} \quad x < -3 \\ x+3 & \text{if} \quad x \geq -3 \end{cases}$ or $f(x) = \begin{cases} -x-3 & \text{if} \quad x < -3 \\ x+3 & \text{if} \quad x \geq -3 \end{cases}$

b. $f(x) = \begin{cases} -(2-x) & \text{if} \quad x > 2 \\ 2-x & \text{if} \quad x \leq 2 \end{cases}$ or $f(x) = \begin{cases} -2+x & \text{if} \quad x > 2 \\ 2-x & \text{if} \quad x \leq 2 \end{cases}$ or

$f(x) = \begin{cases} 2-x & \text{if} \quad x \leq 2 \\ -2+x & \text{if} \quad x > 2 \end{cases}$

c. $f(x) = \begin{cases} -3(8-5x) & \text{if} \quad x > \frac{8}{5} \\ 3(8-5x) & \text{if} \quad x \leq \frac{8}{5} \end{cases}$ or $f(x) = \begin{cases} 3(8-5x) & \text{if} \quad x \leq \frac{8}{5} \\ -3(8-5x) & \text{if} \quad x > \frac{8}{5} \end{cases}$ or

$f(x) = \begin{cases} 24-15x & \text{if} \quad x \leq \frac{8}{5} \\ -24+15x & \text{if} \quad x > \frac{8}{5} \end{cases}$

26. **a.** $f(x)$ is an exponential function, because the variable "x" is in the exponent.
b. $f(x)$ is a power function, because the variable "x" is the base.
c. $f(x)$ is an exponential function, because the variable "x" is in the exponent.

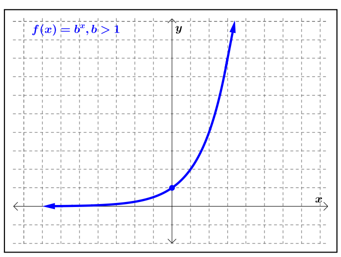
27. **a.** Exponential Growth; $5^{2c} = \left(5^2\right)^c = 25^c$ and the base is 25, which is greater than 1.

b. Exponential Decay; $\left(\frac{5}{9}\right)^x$ and the base is $\frac{5}{9}$, which is between 0 and 1.

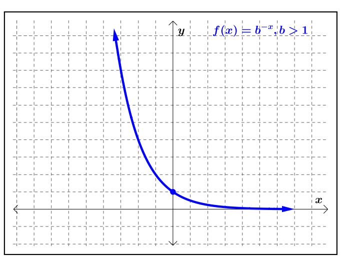
c. Exponential Growth; π^t and the base is π, which is greater than 1.

d. $a > 0$ and $k > 0$ based on the definition

28. **a.**



b.



29. **a.** $(-\infty, \infty)$
b. $\left[-\frac{1}{2}, \infty\right)$
c. $(-\infty, \infty)$

30. **a.** $\dfrac{19683y^{21}}{x^{27}}$

b. $\dfrac{8xy - 16x^{\frac{1}{2}}y^{\frac{1}{2}} + 8}{y}$

c. 1

31. **a.** $x = 2$

b. $x = \frac{1}{2}$

c. $x = -1$

32. **a.** \$27,721.78
b. \$2576.14
c. 142,616 people

33. **a.** Parent Function: $f(x) = \sqrt{x}$

Transformations:
1. Shift $f(x)$ right 1 unit: $y_1 = \sqrt{x-1}$
2. Vertically stretch y_1 by a factor of 2: $y_2 = 2\sqrt{x-1}$
3. Reflect y_2 over the x-axis: $y_3 = -2\sqrt{x-1}$
4. Shift y_3 up 3 units: $g(x) = -2\sqrt{x-1} + 3$

b. Parent Function: $f(x) = |x|$

Transformations:
1. Shift $f(x)$ left 4 units: $y_1 = |x+4|$
2. Vertically compression y_1 by a factor of 5: $y_2 = \frac{1}{5}|x+4|$
3. Shift y_2 down 2 units: $g(x) = \frac{1}{5}|x+4| - 2$

c. Parent Function: $f(x) = e^x$

Transformations:
1. Shift $f(x)$ left 8 units: $y_1 = e^{x+8}$
2. Vertically stretch y_1 by a factor of 4: $y_2 = 4e^{x+8}$
3. Shift y_2 up 7 units: $g(x) = 4e^{x+8} + 7$

34. **a.** $g(x) = -2(x-3)^2 - 1$

b. $g(x) = -\frac{1}{x+4} + 3$

c. $(g(x) = \frac{3}{4}(2^x) + \frac{5}{2}$

35. **a.** $(g - f)(x) = 6x^2 - 5x - 4$

b. $(hj)(x) = (7 - 3\sqrt[3]{2x+1})\left(\frac{x+1}{2x+5}\right)$

c. $\left(\frac{g}{j}\right)(x) = (6x^2 - x - 7)\left(\frac{2x+5}{x+1}\right)$

36. **a.** $(g \circ f)(x) = 6(4x-3)^2 + 3(4x-3) + 8$

b. $(h \circ g)(x) = 7 - 3\sqrt[3]{2(6x^2 + 3x + 8) + 1}$

c. $j(f(x)) = \frac{(4x-3)+1}{2(4x-3)+5}$

d. $j(j(x)) = \dfrac{\left(\frac{x+1}{2x+5}\right) + 1}{2\left(\frac{x+1}{2x+5}\right) + 5}$

37. **a.** $f(x)g(x)$ means the two functions are multiplied together. $f(g(x))$ means $g(x)$ is substituted into $f(x)$ for each x and is the composition of two functions.

38. **a.** $f(x)$ fails the Horizontal Line Test and therefore $f(x)$ does not have an inverse.

39. **a.** Logarithmic Function, with a base of 3 $(b > 0, b \neq 1)$
b. Logarithmic Function, with a base of $\frac{4}{5}$ $(b > 0, b \neq 1)$
c. Not a Logarithmic Function; the base is -9, which is not > 0.

40. **a.** $2^3 = 8$

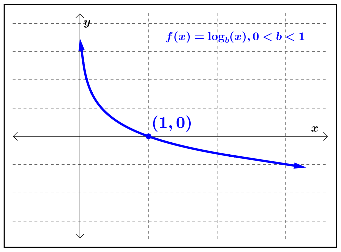
b. $e^0 = x$

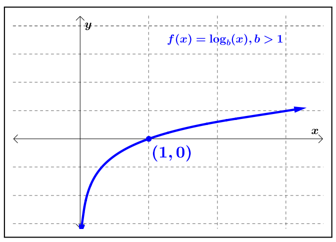
c. $10^x = 12$

d. $\log_5(125) = 3$

e. $\log_7(98) = x$

f. $\log_3(576) = x$ or $\log_3(24) = \frac{1}{2}x$

41. **a.** 

b. 

42. **a.** $(3, \infty)$

b. $(-\infty, 6)$

c. $\left(-\infty, \frac{5}{3}\right)$

d. $(-6, \infty)$

e. $(-7, -6) \cup (-6, \infty)$

43. **a.** $\log_2\left[\dfrac{(x-5)(x)}{3}\right]$ or $\log_2\left[\dfrac{x^2-5x}{3}\right]$

b. $\log\left[\dfrac{10^7(2x+1)^4}{(3x)^2(x-6)}\right]$

c. $\dfrac{1}{3}\ln(x) + \dfrac{1}{3}\ln(3x+4) - \dfrac{1}{3}\ln(6)$

d. $2\ln(x-4)$

44. **a.** $x = \dfrac{1}{2}\log(63)$

b. $t = \dfrac{\ln(10)}{0.05}$

c. $x = \dfrac{\ln(4)}{2\ln(4) - \ln(7)}$

45. **a.** $x = e^{\frac{14}{3}} - 4$

b. $x = \dfrac{9}{4}$

c. $x = 2$

d. $x = -3$

46. **a.** **i.** $r = \sqrt[3]{1.25} - 1 \approx 7.7217\%$

ii. $t = \dfrac{\ln\left(\frac{25}{12}\right)}{\ln(1.07)}$ years ≈ 10.848 years

b. **i.** $t = \dfrac{\ln(2)}{0.006}$ years ≈ 115.525 years

ii. $t = \dfrac{\ln(3)}{0.006}$ years ≈ 183.102 years

c. **i.** $t = \dfrac{\ln\left(\frac{1000}{501}\right)}{\ln(1.04)}$ years ≈ 17.622 years

ii. $t = \dfrac{\ln(3)}{\ln(1.04)}$ years ≈ 28.011 years

Section 6.1 Answers

1. \$31.88
2. \$116.67
3. \$667.19
4. \$1030
5. \$148.59
6. \$20,377.59
7. \$67.30
8. \$442.64
9. \$67.57
10. \$198,633.40
11. 2.6255%
12. 1.0756%
13. 4.5506%
14. 6.9628%
15. 3.4583%
16. 9.3627%
17. $\dfrac{50}{7}$ years
18. 4%
19. \$500
20. \$1985.88
21. \$1097.67

22. **a.** \$17,462.33

    b. \$25,414.29

23. **a.** \$304,449

    b. \$269,449

24. 19.8042 years

25. 4.50%

26. 3.8565%

27. 2.8957%

28. 2.4693%

29. **a.** \$50,592

    b. \$2592

30. \$7815

31. \$50,750

32. **a.** \$7149.53

    b. 643 months

    c. 24.3227%

33. \$873.27

34. 1.33%

35. Account B

36. Account A

37. Compounded n times per year, where n is not continuously

38. When saving money, you want to earn more interest, and so you look for the highest effective interest rate. When borrowing money, you want to pay less interest, and so you look for the lowest effective interest rate.

39. For a particular interest rate, as the number of compounding periods increases, the amount of interest increases (to a point).

Section 6.2 Answers

1. $A = \$168{,}468.40$
2. $A = \$19{,}017.74$
3. $P = \$5020.46$
4. $P = \$1545.42$
5. $PMT = \$80.31$
6. $PMT = \$292.80$
7. \$360,000
8. \$19,500

9. $144,000
10. $48,450
11. $6300
12. $16,000
13. $250,000
14. **a.** $365.53
    b. $54,473.22
15. **a.** $4811.63
    b. $300,235.33
16. **a.** $997.30
    b. $108,829.75
17. 0.04
18. 0.0025
19. 0.03768
20. 0.00525
21. 0.0003
22. 0.01875
23. $3357.28
24. $23.15
25. 5.8%
26. $2381.38
27. $488.32
28. **a.** 282 payments
    b. $55,500 in interest
29. $42,000
30. **a.** $54,010
    b. $47,601.66
    c. $37,398.34
31. **a.** $150,403.57
    b. $594.28
32.

Payment Number	Payment Amount	Payment Amount to Interest	Payment Amount to Debt	Outstanding Principal
0	–	–	–	3000
1	521.21	36.00	485.21	2514.79
2	521.21	30.18	491.03	2023.76
3	521.21	24.29	496.92	1526.84
4	521.21	18.32	502.89	1023.95
5	521.21	12.29	508.92	515.03
6	521.21	6.18	515.03	0.00

33.

Payment Number	Payment Amount	Payment Amount to Interest	Payment Amount to Debt	Outstanding Principal
0	–	–	–	850
1	228.67	25.50	203.17	646.83
2	228.67	19.40	209.27	437.56
3	228.67	13.13	215.54	222.02
4	228.67	6.66	222.01	0.01

34. 136 months
35. \$100,471.03
36. Deposits: \$50,148.72 and Interest: \$49,851.28
37. \$94,202.20
38. **a.** \$10,025.06
    b. \$139.95
39. \$21,138.05
40. \$52,014
41. **a.** \$223.11
    b. \$47.50
    c. \$177.37
    d. \$5466
42. Money is leaving your hands.
43. Interest is not paid on down payments; interest is only paid on money borrowed.
44. Refinancing means you receive a loan at a lower rate to payoff the outstanding amount on your current loan. You then pay only on your new loan, which should save you money because of the lower interest rate.
45. Because of rounding to the nearest cent throughout the amortization process, the last payment changes to account for the money difference and to 'fully' pay off the loan.

Chapter 6 Review Answers

1. **a.** Compound Interest
   b. Ordinary Annuity
   c. Simple Interest
   d. Ordinary Annuity
   e. Effective Rate
   f. Continuously Compounded Interest
2. **a.** No TVM Solver
   b. TVM Solver
   c. No TVM Solver
3. **a.** Account C
   b. Account C
4. **a.** 55.8822%
   b. 180 quarters or 45 years ago

c. $27,137.65

5. a. $569.30
   b. $707.73
   c. $89,695.93

6. a. Option C
   b. Option B
   c. If given a down payment percentage, multiply the purchase price by the percentage to get the down payment dollar amount. Then you subtract that amount from the purchase price to determine the loan amount. You would make a down payment on a large purchase to reduce your loan amount and the extra amount of money you pay in interest.

7. a. $17,137.65
   b. You will deposit a total of $425; the total interest earned is $144.30
   c. $96,460.20 less in interest
   d. $460,200.82 is needed at the beginning of your retirement; $900,000 will be pulled out; $439,799.18 is earned in interest

8. a. 155 months or $\frac{155}{12}$ years, payments of $412.56
   b. $100,695.49
   c. 29 payments
   d. You refinanced $34,218.38, the new payment is $556.15

9. a. $17,418.15
   b. $475,000

10. a. i. $367.96
       ii. $274.06
       iii. $93.90
       iv. $2662.08
    b. i. $29,250
       ii. $165,750
       iii. $869.63
       iv. $208.29
       v. $661.34
    c. i. $134,004.16
       ii. $906.20
       iii. $607.48
       iv. $298.79
       v. $45,667.20

Appendix - Number Sense Answers

1. a. Thousands
   b. Hundreds
   c. Tens
   d. Ten thousands
   e. Ones

2.
   - **a.** Ten thousands
   - **b.** Millions
   - **c.** Ones
   - **d.** Thousands
   - **e.** Ten millions
3. Five thousand, nine hundred two
4. Three hundred sixty-four thousand, five hundred ten
5. 412
6. 18,102,783
7.
   - **a.** 390
   - **b.** 2,930
8.
   - **a.** 790
   - **b.** 5,650
9.
   - **a.** 13,700
   - **b.** 391,800
10.
    - **a.** 28,200
    - **b.** 481,600
11. Divisible by 2, 3, and 6
12. Divisible by 3 and 5
13. Divisible by 2, 3, and 6
14. Divisible by 2, 5, and 10
15. $86 = 2 \cdot 43$
16. $132 = 2 \cdot 2 \cdot 3 \cdot 11$
17. $627 = 3 \cdot 11 \cdot 19$
18. $2520 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7$
19. 48
20. 220
21. 280
22. 1260
23.
    - **a.** $>$
    - **b.** $<$
    - **c.** $<$
    - **d.** $>$

24. **a.** $<$
    b. $<$
    c. $>$
    d. $<$
25. **a.** 32
    b. 0
    c. 16
26. **a.** $<$
    b. $=$
27. **a.** $>$
    b. $>$
28. **a.** -19
    b. -33
29. **a.** -160
    b. -12
30. -80
31. -15
32. -82
33. 32
34. 6
35. -2
36. -9
37. 10
38. -32
39. 65
40. 14
41. 27
42. -19
43. -63
44. -4
45. 14
46. -5
47. -16
48. $\frac{10}{18}, \frac{15}{27}, \frac{20}{36}$

49. $\frac{2}{16}, \frac{3}{24}, \frac{4}{32}$
50. $-\frac{5}{11}$
51. $-\frac{13}{6}$
52. $-\frac{7}{11}$
53. $\frac{10}{21}$
54. $-\frac{12}{7}$
55. $\frac{13}{21}$
56. $\frac{27}{40}$
57. $\frac{1}{3}$
58. $-\frac{1}{6}$
59. $-\frac{21}{44}$
60. $\frac{6}{7}$
61. 25
62. $\frac{9}{8}$
63. 1
64. $\frac{4}{9}$
65. $-\frac{4}{9}$
66. $-\frac{1}{16}$
67. -12
68. $-\frac{10}{9}$
69. $-\frac{2}{5}$
70. $\frac{11}{13}$
71. $-\frac{5}{8}$
72. $\frac{7}{17}$
73. $\frac{1}{2}$

74. $-\frac{55}{14}$

75. $\frac{1}{5}$

76. $\frac{9}{14}$

77. $\frac{3}{8}$

78. $\frac{1}{48}$

79. $\frac{37}{120}$

80. $\frac{1}{12}$

81. Two and sixty-four hundredths

82. Negative thirty-one and four tenths

83. 0.85

84. 0.76

85. **a.** \$6

    b. \$5.78

86. **a.** \$2

    b. \$1.64

87. **a.** \$84

    b. \$84.28

88. **a.** \$63

    b. \$63.48

89. 52.27

90. -40.91

91. 107.368

92. -27.50

93. 337.8914

94. -11.653

95. $-\frac{5}{2} = -2.50$

96. $\frac{469}{192}$

97. $\frac{27}{20}$

98. $\frac{17}{200}$

99. 0.85
100. -11.36
101. 0.01
102. 2.5
103. 6.25%
104. 400%
105. **a.** \$142.19

     b. \$142
106. **a.** \$67.53

     b. \$68
107. $\dfrac{1}{10^3} = \dfrac{1}{1000}$
108. $\dfrac{1}{3^4} = \dfrac{1}{81}$
109. $5^2 = 25$
110. $7^9 = 40,353,607$
111. $\left(\dfrac{10}{3}\right)^2 = \dfrac{100}{9}$
112. $\left(\dfrac{9}{4}\right)^3 = \dfrac{729}{64}$
113. $\left(\dfrac{2}{7}\right)^3 = \dfrac{8}{343}$
114. $\dfrac{32768}{3125}$
115. $(-3)^3 = -27$
116. -27
117. $9^5 = 59,049$
118. $3^4 \cdot 5^4 = 50,625$
119. $5^{12} = 244,140,625$
120. $6^8 = 1,679,616$
121. 2.6×10^{-2}
122. 1.03×10^{-6}
123. 8.75×10^{6}
124. 520
125. 0.0000413
126. 0.039
127. 6

128. 3

129. -1

130. -11

131. 3

132. -2

133. **a.** Rational number

b. Irrational number

134. **a.** Irrational number

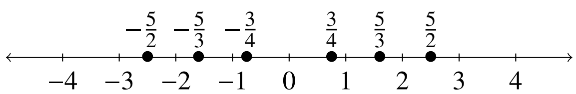
b. Rational number

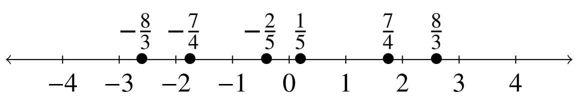
135. **a.** Real number

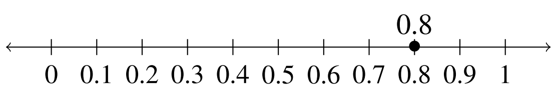
b. Not a real number

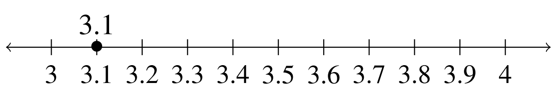
136. **a.** Not a real number

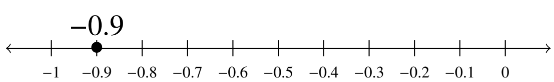
b. Real number

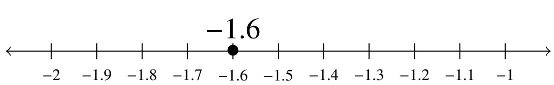
137. 

138. 

139. 

140. 

141. 

142. 

143. $\frac{7}{8}$

144. $\frac{25}{7}$

145. $\frac{11}{6}$

146. $\frac{5}{12}$

147. $\frac{23}{12}$

148. $\frac{49}{11}$

149. 17

150. 360

151. **a.** $-\frac{5}{9}$

   b. -2.1

   c. 3

   d. $\frac{9}{5}$

152. **a.** $\frac{8}{3}$

   b. 0.019

   c. -52

   d. $-\frac{5}{6}$

153. **a.** $\frac{1}{12}$

   b. $-\frac{2}{9}$

   c. $\frac{100}{13}$

154. **a.** $\frac{20}{17}$

   b. $-\frac{2}{3}$

   c. $-\frac{1}{3}$

155. 0

156. 0

157. 0

158. 0

159. Undefined/Does Not Exist

160. Undefined/Does Not Exist

161. 5400 seconds

162. 8100 seconds

163. 3.05 hours

164. $\frac{36}{7}$ weeks

Appendix - Introduction to Algebra Answers

1. the difference of sixteen and nine
2. the sum of x and eleven
3. fourteen is less than twenty-one

4. the product of six and n is equal to thirty-six
5. the product of three and nine
6. the product of two and seven
7. seventeen is less than thirty-five
8. the difference of y and 1 is greater than six
9. a is not equal to the product of one and twelve
10. the quotient of twenty-eight and four
11. the product of negative four and eight
12. thirty-six is greater than or equal to nineteen
13. Expanded Form: $5 \cdot 5 \cdot 5$

    Words: 5 to the third power or 5 cubed
14. Expanded Form: $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

    Words: 10 to the fifth power
15. Expanded Form: $8 \cdot 8 \cdot 8$

    Words: 8 to the third power or 8 cubed
16. Expanded Form: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

    Words: 2 to the eighth power
17. 43
18. 55
19. 6
20. 43
21. 58
22. 6
23. 27
24. 58
25. 76
26. 100
27. 121
28. 50
29. 50
30. 125
31. 16
32. 21
33. 9
34. 225
35. 117

36. 51
37. $15x^2$, $6x$, 2
38. $11x^2$, $8x$, 5
39. $10y^3$, y, 2
40. $9y^3$, y, 5
41. 13
42. -5
43. 1
44. x^3 and $8x^3$; 14 and 5;
    $8y$ and $8x$ do not have any like terms.
45. $3w^2$ and w^2; $6z$ and $4z$;
    $6z^2$ and 1 do not have any like terms.
46. $16b^2$ and $9b^2$; 16 and 4;
    a^2 and $9a$ do not have any like terms.
47. $25r^2$ and $4r^2$; $10s$ and $3s$;
    $10r$ and 3 do not have any like terms.
48. $19x$
49. $11y$
50. $10u+3$
51. $22c$
52. $17x^2+20x+16$
53. $7b^2+12b+6$
54. $14-9$
55. $19-8$
56. $9(7)=9\cdot 7$
57. $8(7)=8\cdot 7$
58. $\dfrac{36}{9}=36\div 9$
59. $\dfrac{42}{7}=42\div 7$
60. $8x+3x$
61. $13x+2x$
62. $\dfrac{y}{3}=y\div 3$
63. $\dfrac{x}{8}=x\div 8$
64. $8(y-9)$
65. $7(y-1)$

66. $r = c + 3$

67. $p = 2n - 7$

68. $f = 6t + 3$

69. $x = 11$

70. $y = -111$

71. $b = \frac{1}{2}$

72. $a = 121$

73. $a = 87$

74. $x = \frac{21}{5}$

75. $x = -5.03$

76. $p = \frac{16}{15}$

77. $x = 7$

78. $c = -11$

79. $p = \frac{541}{37}$

80. $z = 13$

81. $x = 0$

82. $x = 140$

83. $q = 100$

84. $y = -144$

85. $y = -6$

86. $r = 125$

87. $x = -32$

88. $u = \frac{1}{4}$

89. $p = -12$

90. $n = 88$

91. $d = 15$

92. $c = 25$

93. $x = 1$

94. $x = -\frac{72}{5} = -14.4$

95. $m = -8$

96. $x = -4$

97. $x = 6$

98. $z = 3$

99. $f = 7$

100. $a = -40$

101. $y = 20$

102. $z = 3.46$

103. $w = 80$

104. $y = 5$

105. $n = -\frac{3}{2} = -1.5$

106. $r = -2$

107. $w = -18$

108. $b = 2$

109. $z = -1$

110. $q = 3$

111. $s = 5$

112. $x = 5$

113. $d = 1$

114. $a = 69$

115. $n = 2$

116. $m = 3$

117. $u = 2$

118. $q = 22$

119. All real numbers

120. No solution

121. All real numbers

122. $x + 9 = 52; \quad x = 43$

123. $m - 10 = -14; \quad m = -4$

124. $n - \frac{1}{6} = \frac{1}{2}; \quad n = \frac{2}{3}$

125. $-4n + 5n = -82; \quad n = -82$

126. $187 = -17m; \quad -11 = m$

127. $-184 = 23p; \quad -8 = p$

128. $\frac{u}{7} = -49; \quad u = -343$

129. $\frac{j}{-20} = -80; \quad j = 1600$

130. $\frac{c}{-19} = 38; \quad c = -722$

131. $\frac{k}{22} = -66; \quad k = -1452$

132. $\frac{5}{6}y = 15; \quad y = 18$

133. $\frac{3}{10}x = 15$; $x = 50$

134. $\frac{2}{5} + f = \frac{1}{2}$; $f = \frac{1}{10}$

135. $p - \frac{1}{6} = \frac{2}{3}$; $p = \frac{5}{6}$

136. d := the age of Eva's daughter;
$d = 22 - 15$;
Eva's daughter is 7 years old.

137. t := the weight of the Thanksgiving turkey, in pounds;
$16 = t - 5$;
The Thanksgiving turkey weighed 21 pounds.

138. p := amount of Ron's paycheck last week, in dollars;
$103.76 = p - 17.43$;
Last week, Ron's paycheck was \$121.19.

139. a := the cost of Melissa's art book, in dollars;
$93.75 = a - 22.85$;
Melissa's art book costs \$116.60.

140. p := the price of one movie ticket, in dollars;
$5p = 36.25$;
The price of each movie ticket was \$7.25.

141. p := the price of one pair of sport socks, in dollars;
$12p = 12.96$;
The price of one pair of sport socks is \$1.08.

142. f := the amount of fabric to make flags for the whole team, in yards;
$14 = \frac{1}{3}f$;
Nancy needs 42 yards of fabric to make flags for the whole drill team.

143. h := the number of mpg the hybrid gets;
$18 = \frac{1}{2}h$;
John's wife's hybrid car gets 36 miles per gallon.

144. c := the total number of candies in the bag;
$23 = \frac{1}{4}c$;
92 candies are in the bag.

145. c := the number of CDs in Zachary's collection;
$25 = \frac{1}{5}c$;
Zachary has 125 CDs in his collection.

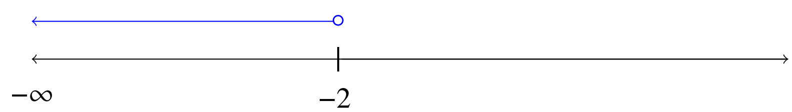
146. $h :=$ the number of hardback books;
$162 = 3h - 12$;
There are 58 hardback books.

147. $p :=$ the amount of money, in dollars, Paul pays for rent;
$1620 = 2p + 120$;
Paul pays \$750 per month in rent.

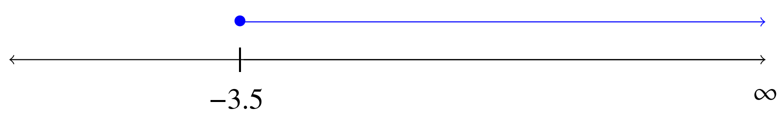
148. $p :=$ the original price of the boots, in dollars;
$60 = p - 25$;
The boots were originally priced at \$85.

149. $p :=$ the cost of each peach, in dollars;
$8p = 3.20$;
Each peach costs \$0.40.

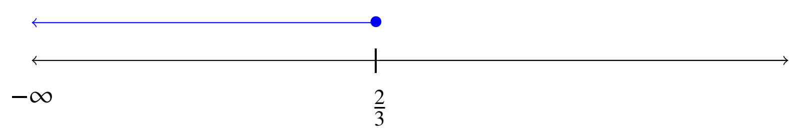
150. $p :=$ the price of the living room set, in dollars;
$2279 = p + 129$;
The price of the living room set was \$2150.

151. **a.** 

$(-\infty, -2)$

b. 

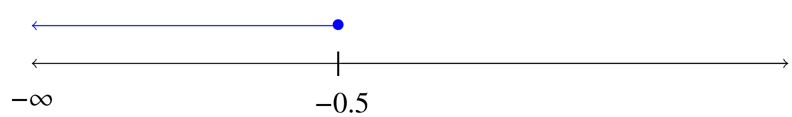
$[-3.5, \infty)$

c. 

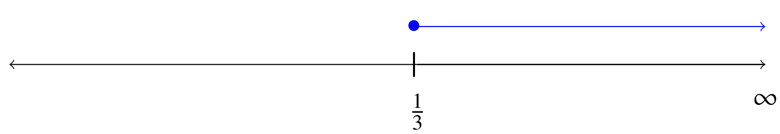
$\left(-\infty, \frac{2}{3}\right]$

152. **a.** 

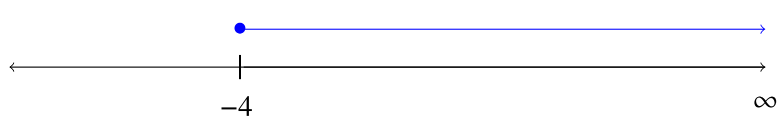
$(3, \infty)$

b. 

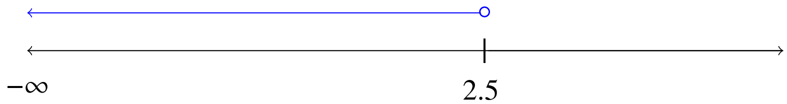
$(-\infty, -0.5]$

c. 

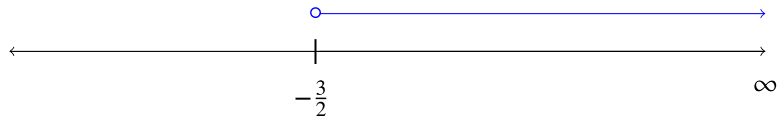
$\left[\frac{1}{3}, \infty\right)$

153. **a.** 

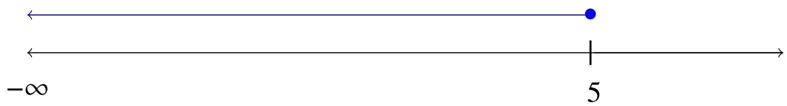
$[-4, \infty)$

b. 

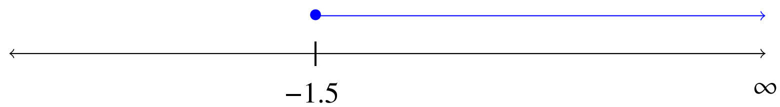
$(-\infty, 2.5)$

c. 

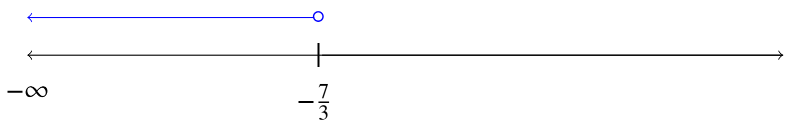
$\left(-\dfrac{3}{2}, \infty\right)$

154. **a.** 

$(-\infty, 5]$

b. 

$[-1.5, \infty)$

c. 

$\left(-\infty, -\dfrac{7}{3}\right)$

155. $(-\infty, 8)$
156. $[9, \infty)$
157. $[-12, \infty)$
158. $(-\infty, -15)$
159. $[-27, \infty)$
160. $(-\infty, -300]$
161. $(-\infty, -4)$
162. $(-\infty, -75]$
163. $\left(-\infty, \dfrac{408}{5}\right]$
164. $[7, \infty)$
165. $\left(-\infty, -\dfrac{29}{6}\right)$
166. $\left(-\infty, \dfrac{18}{5}\right)$
167. $(-9, \infty)$
168. $(-\infty, -5)$
169. $(-\infty, 7]$
170. $(-\infty, \infty)$
171. $(-\infty, \infty)$
172. No solution

Appendix - Introduction to Algebraic Expressions Answers

1. Trinomial
2. Binomial
3. Monomial
4. Polynomial
5. 1
6. 0
7. 4
8. 3
9. 2
10. $12x^2$
11. $40x$
12. $6w$
13. $-6y$
14. $-u^2+4v^2$
15. $x-3y$
16. $2m^2-7m+4$
17. $12s^2-16s+9$
18. $-y^2+4y+3$
19. $3p^2-6p-7$
20.
    a. 187
    b. 40
    c. 2
21.
    a. -20
    b. 16
    c. -128
22. $10,800
23. $58
24. 243
25. 0.0016
26. $\dfrac{4}{81}$
27. 14
28. 1296
29. -1296
30. $-\dfrac{1}{256}$

31. x^6

32. m^{x+3}

33. a^{16}

34. w^6

35. **a.** m^8

    b. 10^{18}

36. **a.** y^{3x}

    b. 5^{xy}

37. **a.** $36a^2$

    b. $9x^2y^2$

38. **a.** $-64m^3$

    b. $125a^3b^3$

39. **a.** $200a^5$

    b. $\frac{1}{18}y^8$

40. **a.** $45x^3$

    b. $1125t^8$

41. $-18y^{11}$

42. $4f^{11}$

43. $30x^8$

44. $9d^7$

45. $-3a-21$

46. $2x-14$

47. q^2+5q

48. $-x^2+10x$

49. $12x^2-60x$

50. $-8p^2-28p$

51. $5q^5-10q^4+30q^3$

52. $-12z^4-48z^3+4z^2$

53. $2m^2-9m$

54. $8j^2-j$

55. s^3-6s^2

56. $-5m^3-15m^2+90m$

57. $w^2+12w+35$

58. $q^2-4q-32$

59. $y^2-8y+12$

60. $w^2+3w-28$
61. $7m^2+22m+3$
62. $20t^2-88t-9$
63. y^4-11y^2+28
64. x^4+3x^2-40
65. $x^3+9x^2+23x+15$
66. $p^3-10p^2+33p-36$
67. $w^2+8w+16$
68. $q^2+24q+144$
69. $x^2+\frac{4}{3}x+\frac{4}{9}$
70. $y^2-12y+36$
71. $p^2-26p+169$
72. $9d^2+6d+1$
73. $4q^2+\frac{4}{3}q+\frac{1}{9}$
74. $9x^4+12x^2+4$
75. $x^2+2xh+h^2$
76. c^2-25
77. $x^2-\frac{9}{16}$
78. $25k^2-36$
79. $121k^2-16$
80. $169-q^2$
81. $16-36y^2$
82. p^{19}
83. u^{27}
84. $\frac{1}{t^{30}}$
85. $\frac{1}{x^8}$
86. **a.** -1
    b. 1
87. **a.** 1
    b. 25
88. $\frac{x^4}{81}$
89. $\frac{25}{16m^2}$

90. $\dfrac{1}{n^{16}}$

91. $\dfrac{1}{r^{12}}$

92. $\dfrac{16j^8}{81}$

93. $\dfrac{27m^{12}}{125}$

94. k^{14}

95. j^9

96. $-6u^8$

97. $-11y^{12}$

98. $-3x^3$

99. $\dfrac{5}{6r^5}$

100. $\dfrac{4q^5}{3}$

101. $\dfrac{65a^3}{42}$

102. $\dfrac{-3}{p^5}$

103. $2b^6$

104. **a.** $n = 0$

   b. $y = 4$

   c. $b = -6$

105. **a.** $y = 0$

   b. $x = -5$

   c. $u = 7$

106. $\dfrac{5}{x^2y^2}$, where $x \neq 0$, $y \neq 0$

107. $\dfrac{2m^2}{3n}$, where $m \neq 0$, $n \neq 0$

108. $\dfrac{5}{6}$, where $b \neq -1$

109. $\dfrac{8}{3}$, where $n \neq 12$

110. $\dfrac{y+4}{y-5}$, where $y \neq 1$, $y \neq 5$

111. $\dfrac{y+1}{y+3}$, $y \neq -3$, $y \neq 3$

112. $\dfrac{4b(b-4)}{(b+5)(b-8)}$, where $b \neq -5$, $b \neq 8$

113. $\dfrac{2d}{d+4}$, where $d \neq -4$, $d \neq 6$

114. -1, where $b \neq 12$

115. -1, where $d \neq 5$

116. $\dfrac{-5}{y+4}$, where $y \neq -4$, $y \neq 4$

117. $\dfrac{-7}{3+w}$, where $w \neq -3$, $w \neq 3$

118. **a.** $\dfrac{1}{3}$

 b. 1

 c. $\dfrac{5}{8}$

119. **a.** 3

 b. $\dfrac{1}{2}$

 c. $-\dfrac{1}{5}$

120. **a.** -6

 b. The expression is undefined when $y = -1$.

 c. 0

121. **a.** $-\dfrac{1}{2}$

 b. 1

 c. The expression is undefined when $b = 4$.

122. $2z^2 + \dfrac{7}{2}$

123. $2y - 3$

124. $\dfrac{3b^3}{4} - \dfrac{11b^2}{12}$

125. $5x^2 - 11x$

126. $-5y^2 + 3y$

127. $7 + \dfrac{3}{b}$

128. **a.** a

 b. b

129. **a.** $|y|$

 b. m

130. **a.** $\left|x^3\right|$

 b. y^8

131. **a.** x^{12}

 b. $\left|y^{11}\right|$

132. **a.** a^2

b. b^9

133. **a.** m^2

b. n^4

134. **a.** $7|x|$

b. $-9\left|x^9\right|$

135. **a.** $10|y|$

b. $-10m^{16}$

136. **a.** $3x^{12}$

b. $-5x$

137. **a.** $-2c^3$

b. $5d^5$

138. **a.** $6a^2$

b. $2\left|b^5\right|$

139. **a.** $2r^2$

b. $3s^6$

140. **a.** $\sqrt{x}$

b. $\sqrt[3]{y}$

c. $\sqrt[4]{z}$

141. **a.** $\sqrt[5]{u}$

b. $\sqrt[9]{v}$

c. $\sqrt[20]{w}$

142. **a.** $x^{\frac{1}{7}}$

b. $y^{\frac{1}{9}}$

c. $f^{\frac{1}{5}}$

143. **a.** $r^{\frac{1}{8}}$

b. $s^{\frac{1}{10}}$

c. $t^{\frac{1}{4}}$

144. **a.** $(5x)^{\frac{1}{4}}$

b. $(9y)^{\frac{1}{8}}$

c. $7(3z)^{\frac{1}{5}}$

145. **a.** $(25a)^{\frac{1}{3}}$
b. $(3b)^{\frac{1}{2}}$
c. $(40c)^{\frac{1}{8}}$

146. **a.** $r^{\frac{7}{4}}$
b. $(2pq)^{\frac{3}{5}}$
c. $\left(\frac{12m}{7n}\right)^{\frac{3}{4}}$

147. **a.** $u^{\frac{2}{5}}$
b. $(6x)^{\frac{5}{3}}$
c. $\left(\frac{18a}{5b}\right)^{\frac{7}{4}}$

148. **a.** 4
b. $\frac{1}{9}$
c. The expression cannot be simplified as there is no real solution.

149. **a.** -512
b. $-\frac{1}{512}$
c. The expression cannot be simplified as there is no real solution.

150. **a.** $c^{\frac{7}{8}}$
b. p^9
c. $\frac{1}{r}$

151. **a.** $y^{\frac{5}{4}}$
b. x^8
c. $\frac{1}{m}$
d. $2s^{\frac{1}{14}}$
e. $8u^{\frac{1}{4}}$
f. $r^{\frac{7}{2}}$
g. c^2

152. $\frac{3\left(3-\sqrt{11}\right)}{-2}$

153. $-2\left(1+\sqrt{5}\right)$

154. $\frac{5\left(5-\sqrt{6}\right)}{19}$

155. $3\left(3+\sqrt{7}\right)$

156. $\dfrac{\sqrt{3}\left(\sqrt{m}+\sqrt{5}\right)}{m-5}$

157. $\dfrac{\sqrt{7}\left(\sqrt{y}-\sqrt{3}\right)}{y-3}$

158. $\dfrac{\left(\sqrt{r}+\sqrt{5}\right)^2}{r-5}$

159. $\dfrac{\left(\sqrt{s}-\sqrt{6}\right)^2}{s-6}$

Appendix - Factoring Answers

1. 2
2. 18
3. $7b$
4. $6x^2$
5. $5x^3$
6. $8(m-1)$
7. $3(2m+3)$
8. $2\left(4p^2+2p+1\right)$
9. $2\left(5q^2+7q+10\right)$
10. $5x\left(x^2-3x+4\right)$
11. $(b-4)(b+5)$
12. $(m-12)(m+6)$
13. $(p+4)(p+9)$
14. $(x-5)(x-3)$
15. $\left(x^2+1\right)(x+1)$
16. $(x+1)(x+3)$
17. $(y+1)(y+7)$
18. $(m+1)(m+11)$
19. $(a+4)(a+5)$
20. $(m+3)(m+4)$
21. $(p+5)(p+6)$
22. $(x-6)(x-2)$
23. $(q-9)(q-4)$
24. $(y-15)(y-3)$
25. $(m-10)(m-3)$

26. $(x-7)(x-1)$
27. $(y-3)(y-2)$
28. $(p-1)(p+6)$
29. $(n-1)(n+7)$
30. $(y-7)(y+1)$
31. $(v-3)(v+1)$
32. $(x-4)(x+3)$
33. $(r-4)(r+2)$
34. Prime and cannot be factored under the integers.
35. Prime and cannot be factored under the integers.
36. $(-4+x)(-2+x)=(x-4)(x-2)$
37. $(-11+x)(1+x)=(x-11)(x+1)$
38. $(w-4)(w+8)$
39. $(k+4)(k+30)$
40. $5(x+1)(x+6)$
41. $12(s+1)(s+1)=12(s+1)^2$
42. $q(q-8)(q+3)$
43. $3m(m-5)(m-2)$
44. $5x^2(x-3)(x+5)$
45. $(4w-1)(w-1)$
46. $(2y-11)(5y+1)$
47. Prime and cannot be factored under the integers.
48. $(4p-3)(p+5)$
49. Prime and cannot be factored under the integers.
50. $(x-4)(x+4)$
51. $(1-5x)(1+5x)$
52. $(13q-1)(13q+1)$
53. $(11x-12)(11x+12)$
54. $(7x-9)(7x+9)$
55. $(2-7x)(2+7x)$
56. $(11-5s)(11+5s)$
57. $(2z-1)(2z+1)\left(4z^2+1\right)$
58. $5(q-3)(q+3)$
59. $2r(7r-6)(7r+6)$
60. $6\left(4p^2+9\right)$

61. $20\left(b^2+7\right)$
62. $3(3q-1)(3q+1)$
63. $4(p-5)(p+5)$
64. $2\left(4p^2+1\right)$

Appendix - Solving Quadratic Equations Answers

1. $x=-7$ or $x=3$
2. $b=-\frac{1}{5}$ or $b=-\frac{1}{6}$
3. $y=-\frac{5}{3}$
4. $a=\frac{10}{3}$ or $a=\frac{7}{2}$
5. $x=0$ or $x=\frac{1}{2}$
6. $x=\frac{1}{2}$
7. $x=-4$ or $x=-3$
8. $y=3$ or $y=5$
9. $b=-1$ or $b=-\frac{3}{4}$
10. $a=-\frac{4}{5}$ or $a=6$
11. $m=\frac{5}{4}$ or $m=3$
12. $n=0$ or $n=1$ or $n=5$
13. $y=-1$ or $y=6$
14. $x=-5$ or $x=2$
15. $p=-\frac{3}{8}$ or $p=0$ or $p=\frac{3}{2}$
16. $x=-7$ or $x=3$
17. $x=-17$ or $x=1$
18. $(m-12)^2$
19. $\left(x-\frac{11}{2}\right)^2$
20. $\left(q+\frac{3}{8}\right)^2$
21. $\left(x+\frac{5}{2}\right)^2$
22. $(q+3)^2$

23. $\left(n-\frac{1}{3}\right)^2$
24. $u=-3$ or $u=1$
25. $z=-11$ or $z=-1$
26. $x=-1$ or $x=21$
27. $a=5-\sqrt{20}$ or $a=5+\sqrt{20}$
28. $u=1$ or $u=13$
29. $z=-1-\sqrt{8}$ or $z=-1+\sqrt{8}$
30. $w=\frac{5}{2}-\sqrt{\frac{21}{4}}$ or $w=\frac{5}{2}+\sqrt{\frac{21}{4}}$
31. $y=3-\sqrt{23}$ or $y=3+\sqrt{23}$
32. $m=-11$ or $m=1$
33. $x=1$ or $x=6$
34. $x=-2-\sqrt{11}$ or $x=-2+\sqrt{11}$
35. $c=-2$ or $c=\frac{3}{2}$
36. $p=-2-\sqrt{\frac{23}{2}}$ or $p=-2+\sqrt{\frac{23}{2}}$
37. $q=\frac{5}{6}-\sqrt{\frac{133}{36}}$ or $q=\frac{5}{6}+\sqrt{\frac{133}{36}}$
38. $m=-1$ or $m=\frac{3}{4}$
39. $r=-3$ or $r=11$
40. $p=\frac{-8-\sqrt{24}}{4}$ or $p=\frac{-8+\sqrt{24}}{4}$
41. No real solutions
42. $x=-1$ or $x=3$
43. $a=\frac{6-\sqrt{12}}{4}$ or $a=\frac{6+\sqrt{12}}{4}$
44. $a=\frac{-6-\sqrt{12}}{4}$ or $a=\frac{-6+\sqrt{12}}{4}$
45. $y=-\frac{1}{4}$
46. $m=-1$ or $m=\frac{3}{4}$
47. $q=-6$ or $q=3$
48. $x=-9$ or $x=1$
49. $d=\frac{6}{5}$
50. $x=-2$ or $x=2$ or $x=-\sqrt{3}$ or $x=\sqrt{3}$
51. $x=-\sqrt{6}$ or $x=\sqrt{6}$ or $x=-\sqrt{3}$ or $x=\sqrt{3}$

52. $x = -\sqrt{15}$ or $x = \sqrt{15}$

53. $x = -2$ or $x = 2$

54. $x = -1$ or $x = 1$ or $x = -\sqrt{\frac{3}{2}}$ or $x = \sqrt{\frac{3}{2}}$

55. $x = -\frac{1}{2}$ or $x = \frac{1}{2}$ or $x = -1$ or $x = 1$

56. $x = -1$ or $x = 12$

57. $y = -\frac{5}{3}$ or $y = 0$

58. $x = 25$

59. $x = \frac{1}{9}$