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Method for constructing bijections for classical partition identities

(Rogers–Ramanujan identities/involution/labeled graph)

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ABSTRACT We sketch the construction of a bijection between the partitions of *n* with parts congruent to 1 or 4 (mod 5) and the partitions of *n* with parts differing by at least 2. This bijection is obtained by a cut-and-paste procedure that starts with a partition in one class and ends with a partition in the other class. The whole construction is a combination of a bijection discovered quite early by Schur and two bijections of our own. A basic principle concerning pairs of involutions provides the key for connecting all these bijections. It appears that our methods lead to an algorithm for constructing bijections for other identities of Rogers–Ramanujan type such as the Gordon identities.

1. Introduction

Throughout this paper we shall use the following notation. **D** will denote the class of all partitions with parts differing by at least two. **E** will denote the class of all partitions. **E1,E2,E3,E4,E5** will denote, respectively, the classes of partitions with parts congruent to 1,2,3,4,0 (mod 5). We shall think of **E** as the cartesian product

$$\mathbf{E} = \mathbf{E5} \times \mathbf{E2} \times \mathbf{E3} \times \mathbf{E1} \times \mathbf{E4}.$$

This simply corresponds to taking an arbitrary partition and separating the parts according to their congruence class (mod 5). By appending a "Z" we shall express the condition that the parts should be distinct. For instance, **EZ** denotes the class of all partitions with distinct parts all congruent to 2 (mod 5).

The sum of the parts of a partition Π will be denoted by $w(\Pi)$ and be referred to as the *weight* of Π. The weight of a k-tuple of partitions $(\Pi_1,\Pi_2,\ldots,\Pi_k)$ is simply defined as the sum of the weights of the individual components.

The object of this paper is to illustrate a general combinatorial method for constructing bijective proofs of partition identities of Rogers–Ramanujan type. To this end we sketch the construction of a weight-preserving bijection between the class **D** and the class **E1** × **E4** (a complete account will appear in ref. 1). This will be achieved by constructing a weight-preserving bijection within **EZ** × **D** × **E** that will match the subset $\varnothing \times \mathbf{D} \times \varnothing$ to the subset $\varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})$. Here the symbol $\varnothing \times \mathbf{D} \times \varnothing$ denotes the set of all triplets (EZ,D,E) of **EZ** × **D** × **E** with EZ and E the empty partitions and D an arbitrary element of **D**. The analogous interpretation should be given to the symbol $\varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})$.

We shall obtain first a weight-preserving permutation Δ of the set **EZ** × **D** × **E** onto itself with the following fundamental property:

> Each cycle of Δ contains either no element of the sets $\varnothing \times \mathbf{D} \times \varnothing$, $\varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})$ or exactly one element of each.

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This given, to produce our desired bijection between **D** and **E1** × **E4**, we interpret an element $D \in \mathbf{D}$ simply as the triplet $(\varnothing, D, \varnothing)$ and then take its images under the successive powers of Δ. The fundamental property guarantees that exactly one of these images will be of the form $(\varnothing, \varnothing, E)$ with $E \in \mathbf{E1} \times \mathbf{E4}$.

Because the cycles of a permutation are disjoint this algorithm produces the desired bijection.

The construction of such a permutation Δ gives a direct bijective proof of the corresponding Rogers–Ramanujan identity

$$\sum_{m\geq 0} \frac{q^{m^2}}{(1-q)\ldots(1-q^m)} = \prod_{m\geq 0} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}. \qquad [1.1]$$

The permutation Δ itself will be constructed as the product of two involutions β and α both acting on the space **EZ** × **D** × **E**. The involution β is a combination of a bijection initially given by Schur and a bijection of our own. Roughly speaking, Schur's bijection is closely related to the sum side of the Rogers–Ramanujan identity and our bijection is closely related to the product side.

The involutions α and β are discussed in *Section 2*. A general combinatorial principle that implies the fundamental property of Δ is presented in *Section 3*. In ref. 1 we give a step-by-step translation of our combinatorial constructs into the analytical steps that have originated them.

The problem whose solution we present here was formulated by MacMahon (2), who apparently was one of the first to have given the partition interpretation of the Rogers–Ramanujan identities. This interpretation was also given by Schur (3), who actually independently discovered the identities themselves.

There is an extensive history of the Rogers–Ramanujan identities and we need not dwell on it further here. Excellent sources for this and related matters are the books of Hardy (4) and Andrews (5, 6). Some very interesting combinatorial aspects of the problem are presented in ref. 6. In ref. 7 the reader may find a very illuminating viewpoint on how one may be led to conjecture the existence of such a bijection by purely combinatorial experimentation.

2. A description of the involutions ALPHA and BETA

As a first step in the construction of the involution BETA we consider the involution introduced by Schur in ref. 3 (see also the $d = t = 2$ case of Gordon in ref. 8). To simplify our notation we shall use the symbol **BIG** to denote the cartesian product **EZ** × **D** × **E** of triplets (EZ, D, E). A triplet (EZ, D, E) is said to be *positive* or *negative* according as EZ has an *even* or *odd* number of parts. The collection of positive and negative triplets of **BIG** will be denoted by $\mathbf{BIG}^+$ and $\mathbf{BIG}^-$, respectively, and will be referred to as the positive and negative parts of **BIG**.

Given an integer $Q \geq 1$, the pair of partitions (EZ,D) with $EZ = (2Q, 2Q-1, \ldots, Q+1)$ and $D = (2Q-1, 2Q-3, \ldots, 3, 1)$ will be referred to as a *special Schur pair of type* A and will be denoted by the symbol SSA(Q). Given an integer $P \geq 0$, the pair of partitions $EZ = (2P - 1, 2P - 2, \ldots, P)$ and $D = (2P - 1, 2P - 3, \ldots, 3, 1)$ will be referred to as a *special Schur pair of type B* and will be denoted by the symbol SSB(P). In case $P = 0$, we interpret SSB(0) as the empty partition $(\varnothing,\varnothing)$.

The subset of **BIG** consisting of the triplets of the form (SSA(Q),E) or (SSB(P),E), with E an arbitrary partition, will be denoted by the symbol **SS** × **E** and will be referred to as the *Schur special set.* The positive and negative parts of **SS** × **E** will be denoted by $\mathbf{SS}^+ \times \mathbf{E}$ and $\mathbf{SS}^- \times \mathbf{E}$, respectively.

The bijection introduced by Schur in ref. 3 induces a permutation of **BIG** onto itself which we shall denote by SBETA. This permutation has the following properties:

SBETA operates only on the first two components of a triplet (EZ, D, E). **[2.1a]**

Its square is the identity; that is, SBETA is an involution. **[2.1b]**

The fixed points of SBETA are the elements of **SS** × **E**. **[2.1c]**

SBETA bijectively interchanges the sets $\mathbf{BIG}^+ - \mathbf{SS}^+ \times \mathbf{E}$ and $\mathbf{BIG}^- - \mathbf{SS}^- \times \mathbf{E}$. **[2.1d]**

A full description of the Schur involution is presented in ref. 1 in addition to refs. 3 and 8.

Our goal is to construct a weight-preserving involution BETA of **BIG** onto itself whose fixed points are the elements of the set $\varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})$. The Schur involution does not go far enough in this respect, and we have to construct another involution GMBETA having the desired fixed points and acting on the Schur special set itself. The involution BETA can then be obtained as a combination of SBETA and GMBETA. Indeed, we shall set it equal to SBETA on **BIG** − **SS** × **E** and to GMBETA on **SS** × **E**.

To define GMBETA we need to work with the cartesian product **EZ2** × **EZ3** × **E2** × **E3** × **E1** × **E4**, which we denote by the symbol **NEWSS**. We associate a sign to each element of **NEWSS** by letting the sign of a sixtuple ($EZ2$, $EZ3$, $E2$, $E3$, $E1$, $E4$) be equal to −1 to the power the total number of parts in $EZ2$ and $EZ3$.

This, given GMBETA, is obtained as the product of:

(*i*) A weight- and sign-preserving bijection of **SS** × **E** onto **NEWSS** that we call CONVERT.

(*ii*) A weight-preserving involution we call GAMMA of **NEWSS** onto itself with fixed points the elements of the set $\varnothing \times \varnothing \times \varnothing \times \varnothing \times \mathbf{E1} \times \mathbf{E4}$. GAMMA in addition will be sign reversing outside its fixed point set.

(*iii*) A bijection of **NEWSS** back onto **SS** × **E** that we call INVERT and that is simply the inverse of CONVERT.

The action of CONVERT is best explained if we replace our triplets (EZ,D,E) by the corresponding seventuples (EZ,D,$E5$, $E2$,$E3$,$E1$,$E4$) obtained by separating the parts of E according to their congruence class (mod 5). In this setting CONVERT can be defined by giving a cut-and-paste algorithm that converts a special triplet (EZ,D,$E5$) into a pair ($EZ2$,$EZ3$), in which $EZ2$, $EZ3$ are partitions with distinct parts respectively congruent to 2 and 3 (mod 5). Once (EZ,D,$E5$) has been so converted the image by CONVERT of the seventuple (EZ,D,$E5$,$E2$, $E3$,$E1$,$E4$) is then taken to be the sixtuple ($EZ2$,$EZ3$,$E2$,$E3$, $E1$,$E4$). The conversion algorithm is given in full detail in ref. 1.

There remains to define two more involutions: ALPHA acting on **BIG** and GAMMA acting on **NEWSS**. Fortunately, they are both very simple.

Let (EZ,D,E) be a given triplet of **BIG** and suppose that EZ and E are *not both empty.* We then define **MIN** to be the smallest of the parts occurring in EZ and E. This given, the action of ALPHA on a triplet (EZ,D,E) is defined as follows:

Case 1: If $EZ = E = \varnothing$ then ALPHA $(\varnothing,D,\varnothing) = (\varnothing,D,\varnothing)$.

Case 2: If EZ or E is not empty then we set ALPHA(EZ,D,E) = (EZ',D,E'), in which:

(*i*) If **MIN** is a part of EZ, then the pair (EZ',E') is obtained from (EZ,E) by transferring **MIN** from EZ to E.

(*ii*) If **MIN** is not a part of EZ, then (EZ',E') is obtained from (EZ,E) by transferring one part equal to **MIN** from E to EZ.

The GAMMA involution has an entirely analogous definition. Indeed, given a sixtuple ($EZ2$,$EZ3$,$E2$,$E3$,$E1$,$E4$) in **NEWSS** its image by GAMMA is the sixtuple ($EZ2'$,$EZ3'$,$E2'$,$E3'$,$E1$,$E4$) obtained by appropriately moving the smallest part of ($EZ2$,$EZ3$,$E2$,$E3$) from ($EZ2$,$EZ3$) to ($E2$,$E3$) or vice versa.

Table 1 gives a global view of our maps and their properties. Of course all these maps are weight preserving.

3. A basic principle concerning pairs of involutions

We should note that what we have so far immediately yields that the partitions of n in the class **D** are equinumerous with those in the class **E1** × **E4**.

To show this we need some notation. If **S** is a set of k-tuples of partitions, let $\mathbf{S}|_n$ denote the subset of k-tuples of weight n in **S**.

This given, from Table 1 we deduce that ALPHA maps the set $\mathbf{BIG}^+ - \varnothing \times \mathbf{D} \times \varnothing|_n$ bijectively onto $\mathbf{BIG}^-|_n$ and BETA maps $\mathbf{BIG}^-|_n$ bijectively onto the set $\mathbf{BIG}^+ - \varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})|_n$. Thus, these sets have the same cardinality. Consequently the same must be true for the pair $\varnothing \times \mathbf{D} \times \varnothing|_n$ and $\varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})|_n$, or equivalently the pair $\mathbf{D}|_n$ and $\mathbf{E1} \times \mathbf{E4}|_n$. ALPHA and BETA have thus given us the *existence* of a weight-preserving bijection between **D** and **E1** × **E4**. However, the pair of involutions ALPHA and BETA actually contains the *mechanism* of our desired bijection. A basic principle is involved here that should turn out to be useful in a variety of other combinatorial situations.

Let **C** be a finite class of "signed" objects and let $\mathbf{C}^+$ and $\mathbf{C}^-$ denote the "positive" and "negative" parts of **C**. Suppose fur-

Table 1. Maps and their properties

Name	Domain	Range	Type	Fixed point set	Behavior
SBETA	**BIG**	**BIG**	Involution	**SS** × **E**	Sign reversing
CONVERT	**SS** × **E**	**NEWSS**	Bijection	–	Sign preserving
INVERT	**NEWSS**	**SS** × **E**	Bijection	–	Sign preserving
GAMMA	**NEWSS**	**NEWSS**	Involution	$\varnothing \times \varnothing \times \varnothing \times \varnothing \times \mathbf{E1} \times \mathbf{E4}$	Sign reversing
GMBETA	**SS** × **E**	**SS** × **E**	Involution	$\varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})$	Sign reversing
ALPHA	**BIG**	**BIG**	Involution	$\varnothing \times \mathbf{D} \times \varnothing$	Sign reversing
BETA	**BIG**	**BIG**	Involution	$\varnothing \times \varnothing \times (\mathbf{E1} \times \mathbf{E4})$	Sign reversing

Table 2. Matches for the case $n = 12$

D	Power of Δ	E1 × E4
12	40	6^2
11 1	2	11 1
10 2	50	1^{12}
9 3	6	9 1^3
8 4	15	4 1^8
7 5	40	6 1^6
8 3 1	29	4^3
7 4 1	21	4^2 1^4
6 4 2	5	6 4 1^2

ther that we have two involutions α and β of **C** onto itself and let F_α and F_β denote their respective fixed point sets. For the moment we shall only make the following assumption:

$$\alpha \text{ and } \beta \text{ are sign reversing outside } F_\alpha \text{ and } F_\beta, \text{ respectively.} \quad [3.1]$$

This given, our goal is a complete description of the cycle structure of the product permutation $\Delta = \alpha\beta$. This will be obtained by means of a labeled graph G with *loops* and *multiple* edges and with **C** as a vertex set, which is constructed from α and β according to the following recipe:

(*i*) Each vertex $x \in \mathbf{C}$ is labeled with a + or a − according as $x \in \mathbf{C}^+$ or $x \in \mathbf{C}^-$. **[3.2a]**

(*ii*) Between two distinct elements $x, y \in \mathbf{C}$ we put an α-labeled edge if $y = \alpha x$, a β-labeled edge if $y = \beta x$, or both types of edges if both conditions are satisfied. **[3.2b]**

(*iii*) At a fixed point x we put an α-labeled loop if $x \in F_\alpha$, a β-labeled loop if $x \in F_\beta$, or both again if $x \in F_\alpha \cap F_\beta$. **[3.2c]**

In general such a graph can have components of only the following basic types:

(*A*) Even number of nodes, both ends in F_α, **[3.3a]**

(*B*) Even number of nodes, both ends in F_β, **[3.3b]**

(*C*) Odd number of nodes, positive ends, **[3.3c]**

(*D*) Odd number of nodes, negative ends, **[3.3d]**

(*E*) No fixed points. **[3.3e]**

In ref. 1 it is shown that the permutation $\Delta = \alpha\beta$ has five types of cycles that correspond to the above five types of components of G. It is then not difficult to deduce the following fundamental fact:

THEOREM 3.4. *If α and β, in addition to assumption* 3.1, *satisfy also*

$$F_\alpha \text{ and } F_\beta \text{ are completely contained in } C^+, \quad [3.5]$$

then each cycle of the permutation $\Delta = \alpha\beta$ contains either no fixed points at all or exactly one element of F_α *and one element of* F_β.

Clearly, when $F_\alpha, F_\beta \subset \mathbf{C}^+$ the components of G of type **C** will be precisely those that give us the bijection of F_α onto F_β. Indeed, we need only match their end points. In practice, however, we are led to the following algorithm. We start with an element $x \in F_\alpha$. This automatically places us at an end point of a component of type **C**. We then take the images of x by the successive powers of Δ. In so doing we will be proceeding along this connected component. Eventually we must reach the other end, say y, which automatically will be in F_β. We then match x to y.

As the reader may have already noticed, *Theorem 3.4* does apply if we let $\mathbf{C} = \mathbf{BIG}|_n$ and take α and β to be, respectively, the restrictions of ALPHA and BETA to $\mathbf{BIG}|_n$. Thus, the fundamental property for the product bijection ALPHA × BETA does indeed hold as asserted in *Introduction*.

The actual final bijection sketched above is given in section 6 of ref. 1 as a completely debugged APL program that can be run on the IBM 5100 desk computer. In the case $n = 12$ we obtain the matches of Table 2.

At this point we should observe that almost exactly the same graphs we used here have been introduced in ref. 9 and used in refs. 9 and 10 for entirely different purposes.

In closing, we note that a generalization of *Theorem 3.4* and the techniques introduced above can be used to *derive* bijective proofs of other classical partition identities such as the theorems of Euler (5), Gordon (8), and Schur (11). This work will appear elsewhere.

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