

A New Construction of Einstein-Sasaki Metrics in $D \geq 7$

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ABSTRACT

We construct explicit Einstein-Kähler metrics in all even dimensions $D = 2n + 4 \geq 6$, in terms of a $2n$ -dimensional Einstein-Kähler base metric. These are cohomogeneity 2 metrics which have the new feature of including a NUT-type parameter, in addition to mass and rotation parameters. Using a canonical construction, these metrics all yield Einstein-Sasaki metrics in dimensions $D = 2n + 5 \geq 7$. As is commonly the case in this type of construction, for suitable choices of the free parameters the Einstein-Sasaki metrics can extend smoothly onto complete and non-singular manifolds, even though the underlying Einstein-Kähler metric has conical singularities. We discuss some explicit examples in the case of seven-dimensional Einstein-Sasaki spaces. These new spaces can provide supersymmetric backgrounds in M-theory, which play a rôle in the AdS₄/CFT₃ correspondence.

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1 Introduction

There have recently been many developments in the construction of explicit Einstein-Sasaki metrics on complete and non-singular manifolds. An Einstein-Sasaki space is an odd-dimensional Einstein space that admits two Killing spinors, and thus it can be viewed as a generalisation of the round sphere metric in the same dimension. Einstein-Sasaki spaces are therefore important in string theory or M-theory, since they can provide examples of supersymmetric embeddings of AdS spacetimes. Most of the attention has concentrated on the case of five-dimensional Einstein-Sasaki spaces, since the generalisations of the type IIB background $\text{AdS}_5 \times S^5$ are of great importance in the $\text{AdS}_5/\text{CFT}_4$ correspondence [1]. Another important case is that of seven-dimensional Einstein-Sasaki spaces, since these provide supersymmetric backgrounds in M-theory that generalise $\text{AdS}_4 \times S^7$.

For quite some time, the only explicitly-known examples of five-dimensional Einstein-Sasaki spaces were S^5 , which is the homogeneous space $SO(5)/SO(4)$, and $T^{1,1}$, which is the homogeneous space $(SU(2) \times SU(2))/U(1)$, as well as quotients thereof.¹ A countably infinite class of inhomogeneous examples $Y^{p,q}$ was recently obtained in [4], where p and q are coprime positive integers. These were soon generalised to arbitrary higher odd dimensions in [5], with some further generalisations in [6, 7]. A much larger class of Einstein-Sasaki spaces in five dimensions was then constructed in [8]; they are denoted by $L^{p,q,r}$, where p , q and r are coprime positive integers with $0 < p \leq q$ and $0 < r < p + q$. The previous examples $Y^{p,q}$ arise as the special cases $L^{p-q,p+q,p}$. Generalisations to new Einstein-Sasaki spaces $L^{p,q,r_1,\dots,r_{n-1}}$ spaces in $D = 2n + 1$ dimensions were also given in [8, 9].

According to the $\text{AdS}_5/\text{CFT}_4$ correspondence, five-dimensional Einstein-Sasaki spaces are associated with four-dimensional $\mathcal{N} = 1$ superconformal field theories [10]. These can be described in terms of “quiver” gauge theories, which has been extensively discussed, for example, in [11–16]. It has also been conjectured that the seven-dimensional Einstein-Sasaki spaces are associated with three-dimensional $\mathcal{N} = 2$ superconformal field theories [10, 17], although much less is known about these.

There is a one-to-one correspondence between Einstein-Sasaki metrics in dimension $D = 2n + 1$ and Einstein-Kähler metrics in dimension $D = 2n$ (see, for example, [18] for a recent discussion of this). Specifically, if $d\bar{s}^2$ is a $(2n)$ -dimensional Einstein-Kähler metric satisfying $\bar{R}_{ij} = (2n + 2) \lambda \bar{g}_{ij}$, then the metric

$$ds^2 = (d\tau + A)^2 + d\bar{s}^2 \tag{1}$$

¹General proofs of the existence of inhomogeneous Einstein-Sasaki spaces were given in [2, 3].

on the $U(1)$ bundle over $d\bar{s}^2$ is a $(2n + 1)$ -dimensional Einstein-Sasaki metric with $R_{ab} = 2n \lambda g_{ab}$, where $dA = 2\sqrt{\lambda} J$ and J is the Kähler form on $d\bar{s}^2$. Thus, the local construction of $(2n + 1)$ -dimensional Einstein-Sasaki metrics is equivalent to the local construction of $(2n)$ -dimensional Einstein-Kähler metrics. However, a subtle point first emphasised in the physics literature in the work of [4] is that the criteria for being able to extend the local $(2n)$ -dimensional Einstein-Kähler metric onto a complete and non-singular manifold are much stricter than those for extending the $(2n + 1)$ -dimensional Einstein-Sasaki metric onto a complete and non-singular manifold. To put it another way, it is clearly the case that if $d\bar{s}^2$ extends onto a complete and non-singular manifold $\overline{\mathcal{M}}$, then so will ds^2 , provided only that $\overline{\mathcal{M}}$ is Hodge and that the period of τ is chosen appropriately.² However, it can be the case that the Einstein-Sasaki metric ds^2 extends onto a complete and non-singular manifold even though the base-space metric $d\bar{s}^2$ itself has no such extension. This feature played a crucial role in the explicit construction of non-singular Einstein-Sasaki spaces in [4–9].

In this paper, we present a construction of Einstein-Sasaki metrics in odd dimensions $D \geq 7$ that provides new examples over and above those that have been found in [5–9]. Our procedure involves first constructing new classes of local Einstein-Kähler metrics in all even dimensions $D \geq 6$, and then using (1) to generate the associated local Einstein-Sasaki metrics. Having obtained the local metrics, we then investigate the conditions under which they can be extended onto complete and non-singular manifolds. We find that, except in rather trivial cases, the Einstein-Kähler metrics do not admit such smooth extensions whereas the Einstein-Sasaki metrics do. Since a general discussion of the circumstances under which complete and non-singular spaces arise is quite involved, we restrict ourselves to presenting some examples which suffice to establish the basic principle.

The organisation of the paper is as follows. We begin in section 2 by presenting the local construction of the Einstein-Kähler metrics in dimension $D = 6$ and their lifting, *via* (1), to Einstein-Sasaki metrics in $D = 7$. In section 3, we analyse the global structure of these six-dimensional and seven-dimensional metrics. We show that the Einstein-Kähler metrics themselves generally do not extend smoothly onto complete and non-singular manifolds, except in the special limits of either $\mathbb{C}\mathbb{P}^3$ or $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^1$. However, we find that the seven-dimensional Einstein-Sasaki metrics do extend onto complete non-singular manifolds, provided that the various parameters in the metrics are chosen appropriately. We give the general criteria for such smooth extensions and we present some explicit examples that

²A Kähler manifold is Hodge if the integrals of J over all 2-cycles are rationally related, thus allowing a choice of period for τ that removes all conical singularities.

establish the principle that this construction yields a non-empty set of new examples. In section 4, we generalise the local construction to give new Einstein-Kähler metrics in all even dimensions $D \geq 6$, and thus new Einstein-Sasaki metrics in all odd dimensions $D \geq 7$. We do not analyse the details of the global structures in these cases but expect the results to be similar to those of the $D = 7$ Einstein-Sasaki metrics. The paper ends with conclusions in section 5.

2 Local construction in $D = 6$ and $D = 7$

As discussed in the introduction, our strategy is to first construct local expressions for Einstein-Kähler metrics in six dimensions and then lift these, using (1), to give local expressions for Einstein-Sasaki metrics in seven dimensions. The global analysis, as well as the extension to higher dimensions, will be given in subsequent sections.

2.1 $D = 6$ Einstein-Kähler metric

We begin by making the following ansatz for six-dimensional metrics:

$$ds_6^2 = \frac{(x-y)dx^2}{X} + \frac{(x-y)dy^2}{Y} + \frac{X}{x-y} \left(d\chi - \frac{y}{\beta} \sigma_3\right)^2 + \frac{Y}{x-y} \left(d\chi - \frac{x}{\beta} \sigma_3\right)^2 + \frac{xy}{\beta} (\sigma_1^2 + \sigma_2^2), \quad (2)$$

where X is a function of x and Y is a function of y . σ_1, σ_2 and σ_3 are the standard $SU(2)$ left-invariant 1-forms, satisfying $d\sigma_i = -\frac{1}{2}\epsilon_{ijk} \sigma_j \wedge \sigma_k$. We parameterise these in terms of Euler angles (θ, ϕ, ψ) in the usual way:

$$\sigma_1 + i\sigma_2 = e^{-i\psi} (d\theta + i \sin \theta d\phi), \quad \sigma_3 = d\psi + \cos \theta d\phi. \quad (3)$$

A straightforward calculation shows that (2) is an Einstein metric, satisfying $R_{\mu\nu} = 5g^2 g_{\mu\nu}$, if the functions X and Y are taken to be

$$X = -\frac{2\mu}{x} - \beta x - \alpha x^2 - \frac{5}{2}g^2 x^3, \quad Y = \frac{2\nu}{y} + \beta y + \alpha y^2 + \frac{5}{2}g^2 y^3, \quad (4)$$

where α, β, μ and ν are arbitrary constants. We also find that it is Kähler, with the Kähler form given by

$$J = dx \wedge \left(d\chi - \frac{y}{\beta} \sigma_3\right) + dy \wedge \left(d\chi - \frac{x}{\beta} \sigma_3\right) + \frac{xy}{\beta} \sigma_1 \wedge \sigma_2. \quad (5)$$

We can write this locally as $J = \frac{1}{2}dB$, where the 1-form B is given by

$$B = 2(x+y)d\chi - \frac{2xy}{\beta} \sigma_3. \quad (6)$$

The Kählerity of the metric is easily verified by checking that $J_\mu{}^\nu J_\nu{}^\rho = -\delta_\mu^\rho$, and that J is covariantly constant.

Although the metric is ostensibly parameterised by the four constants α, β, μ and ν , there is a scaling symmetry under which

$$(x, y, \alpha, \beta, \mu, \nu) \longrightarrow (\lambda x, \lambda y, \lambda \alpha, \lambda^2 \beta, \lambda^4 \mu, \lambda^4 \nu). \quad (7)$$

This can be used, for example, to set the parameter β to be either 1 or -1 . An alternative choice would be to use the scaling symmetry to fix the value of μ .

For vanishing ν , the metrics (2) lie within a subset of the Einstein-Kähler metrics considered in [8, 9], which are the base metrics of the BPS limit of the Euclideanised Kerr-de Sitter black holes found in [19, 20]. The six-dimensional Einstein-Kähler metrics and their associated seven-dimensional Einstein-Sasaki metrics discussed in [8, 9] are generally of cohomogeneity 3 and have three non-trivial parameters. These parameters can be taken to be α_1, α_2 and α_3 , associated with the three independent rotation parameters of the original seven-dimensional Euclideanised black holes. With this choice, the ‘‘mass’’ parameter μ is then trivial and rescalable. The overlap with the metrics we are discussing in this section occurs if we set $\nu = 0$ in (4) and if we set $\alpha_2 = \alpha_3$ in the six-dimensional metrics in [8, 9], thus reducing the cohomogeneity from 3 to 2. It is worth emphasizing that the generalisation away from the common overlap that we are presently discussing, for nonvanishing ν , is distinct from the generalisation where α_1, α_2 and α_3 are unequal in [8, 9], since the metrics (2) are still of cohomogeneity 2 when $\nu \neq 0$. The ν parameter can be thought of as a NUT-type parameter which complements the mass parameter μ .

The curvature invariant $R_{abcd} R^{abcd}$ for the metric (2) is given by

$$\begin{aligned} R_{abcd} R^{abcd} &= \frac{96\mu^2(2x-y)(2x^2-2xy+y^2)}{x^6(x-y)^5} + \frac{96\nu^2(x-2y)(x^2-2xy+2y^2)}{y^6(x-y)^5} \\ &+ \frac{96(\mu y - \nu x)^2}{(x-y)^6} + 150g^4. \end{aligned} \quad (8)$$

From this, we see that there are curvature singularities when $x = y$ or when x or y vanishes.

2.2 Einstein-Sasaki metrics in $D = 7$

Having obtained the six-dimensional Einstein-Kähler metrics (2), we now substitute into (1) in order to obtain the associated seven-dimensional Einstein-Sasaki metrics. This yields

$$ds_7^2 = (d\tau + \sqrt{\frac{5}{8}}g A)^2 + ds_6^2, \quad (9)$$

where A is given by (6). The metric is Einstein-Sasaki, satisfying

$$R_{\mu\nu} = \frac{15}{4}g^2 g_{\mu\nu}. \quad (10)$$

Without loss of generality, we set $g^2 = \frac{8}{5}$ so that the Einstein-Sasaki metric has the same Ricci tensor $R_{\mu\nu} = 6g_{\mu\nu}$ as that of a unit 7-sphere. Then the metric takes the local form

$$\begin{aligned} ds_7^2 = & \left(d\tau + 2(x+y)d\chi - \frac{2xy}{\beta}\sigma_3 \right)^2 + \frac{(x-y)dx^2}{X} + \frac{(x-y)dy^2}{Y} \\ & + \frac{X}{x-y} \left(d\chi - \frac{y}{\beta}\sigma_3 \right)^2 + \frac{Y}{x-y} \left(d\chi - \frac{x}{\beta}\sigma_3 \right)^2 + \frac{xy}{\beta}(\sigma_1^2 + \sigma_2^2), \end{aligned} \quad (11)$$

where

$$X = -\frac{2\mu}{x} - \beta x - \alpha x^2 - 4x^3, \quad Y = \frac{2\nu}{y} + \beta y + \alpha y^2 + 4y^3. \quad (12)$$

As in our discussion of the six-dimensional Einstein-Kähler base in section 2.1, the seven-dimensional Einstein-Sasaki metrics we have obtained here reduce, upon setting $\nu = 0$, to the subset of the metrics obtained in [8, 9] corresponding to setting $\alpha_2 = \alpha_3$. When ν is non-zero, the metrics we have constructed here are new.

3 Global Analysis

Our principal goal in this section is to study the global structure of the seven-dimensional Einstein-Sasaki metrics obtained in (11) and to establish the conditions on the parameters in order to have metrics that extend smoothly onto complete and non-singular compact 7-manifolds. Before doing this, we first examine the global structure of the six-dimensional Einstein-Kähler base metrics themselves and show that, except in “trivial” limiting cases, namely \mathbb{CP}^3 or $\mathbb{CP}^2 \times \mathbb{CP}^1$, they will necessarily have conical singularities.

3.1 $D = 6$ Einstein-Kähler metric

If a non-singular compact Einstein-Kähler space existed, it would be defined by having x and y run between adjacent roots (x_1, x_2) and (y_1, y_2) of $X = 0$ and $Y = 0$ respectively. Owing to the scaling symmetry (7), we can set $y_1 = 1$ without loss of generality. One can then parameterise the metric by the three constants y_2, x_1 and x_2 . The parameters α, β, μ and ν can be expressed in terms of these roots, by using the equations following from $X(x_i) = 0$ and $Y(y_i) = 0$, namely

$$\begin{aligned} 4\mu + 2\beta x_1^2 + 2\alpha x_1^3 - 8x_1^4 &= 0, & 4\mu + 2\beta x_2^2 + 2\alpha x_2^3 - 8x_2^4 &= 0, \\ 4\nu + 2\beta y_1^2 + 2\alpha y_1^3 - 8y_1^4 &= 0, & 4\nu + 2\beta y_2^2 + 2\alpha y_2^3 - 8y_2^4 &= 0. \end{aligned} \quad (13)$$

As can be seen from (8), the metric has power-law singularities at $x = y$, $x = 0$ and $y = 0$. Such singularities can be avoided, while also having Euclidean signature, for the following cases:

$$\begin{aligned} \text{Case 1 : } & \quad x_1 > y_1, \quad x_1 > y_2, \quad x_2 > y_1, \quad x_2 > y_2, \quad X > 0, \quad Y > 0, \quad \beta > 0, \\ \text{Case 2 : } & \quad x_1 \text{ and } x_2 < 0, \quad y_1 \text{ and } y_2 > 0, \quad X < 0, \quad Y < 0, \quad \beta < 0. \end{aligned} \quad (14)$$

With either of these choices, the coordinates x and y range between endpoints in two non-overlapping intervals, thus ensuring that none of $x - y$, x or y vanishes.

While the above constraints ensure that the solution has no power-law singularities, there can still be δ -function conical singularities at surfaces where the metric degenerates. Specifically, these degeneracies occur at $x = x_1$ and x_2 , $y = y_1$ and y_2 , and $\theta = 0$ and π . At each degeneracy, there is an associated Killing vector K whose norm $K^2 = K^\mu K_\mu$ goes to zero. These are given by

$$\begin{aligned} x = x_1 : \quad K_1 &= \frac{2x_1}{2\beta + 3\alpha x_1 + 16x_1^2} \left(\frac{\partial}{\partial\chi} + \frac{\beta}{x_1} \frac{\partial}{\partial\psi} \right), \\ x = x_2 : \quad K_2 &= \frac{2x_2}{2\beta + 3\alpha x_2 + 16x_2^2} \left(\frac{\partial}{\partial\chi} + \frac{\beta}{x_2} \frac{\partial}{\partial\psi} \right), \\ y = y_1 : \quad K_3 &= \frac{2y_1}{2\beta + 3\alpha y_1 + 16y_1^2} \left(\frac{\partial}{\partial\chi} + \frac{\beta}{y_1} \frac{\partial}{\partial\psi} \right), \\ y = y_2 : \quad K_4 &= \frac{2y_2}{2\beta + 3\alpha y_2 + 16y_2^2} \left(\frac{\partial}{\partial\chi} + \frac{\beta}{y_2} \frac{\partial}{\partial\psi} \right), \\ \theta = 0 : \quad K_5 &= \frac{\partial}{\partial\psi} - \frac{\partial}{\partial\phi}, \\ \theta = \pi : \quad K_6 &= \frac{\partial}{\partial\psi} + \frac{\partial}{\partial\phi}. \end{aligned} \quad (15)$$

In each case, we have normalised the Killing vector so that the associated ‘‘Euclidean surface gravity,’’ defined by

$$\kappa^2 = \frac{g^{\mu\nu} (\partial_\mu K^2)(\partial_\nu K^2)}{4K^2}, \quad (16)$$

in the limit that the degenerate surface is reached, is equal to unity. As discussed in [8, 9], this means that the translation generated by the Killing vector should have period 2π if a conical singularity is to be avoided on the degenerate surface.

The six Killing vectors (15) all lie in the three-dimensional vector space spanned by $\partial/\partial\phi$, $\partial/\partial\psi$ and $\partial/\partial\chi$. Following the arguments given in [8, 9], this implies that they should be linearly dependent with integer coefficients. In particular, any three among the four Killing vectors K_1 , K_2 , K_3 and K_4 should be linearly dependent. For example

$$n_1 K_1 + n_2 K_2 + n_3 K_3 = 0, \quad (17)$$

for co-prime integers n_i . Such conditions can easily be satisfied, for example, by choosing the parameters so that the roots x_i and y_i are rational. However, there are further restrictions that must be taken into account. For example, K_2 and K_3 can both vanish simultaneously, where $x = x_2$ and $y = y_1$. This implies that the Killing vector $K = n_2 K_2 + n_3 K_3$ also vanishes there. It generates translations with the period $2\pi \gcd(n_2, n_3)$. Thus, we have $n_1 = \gcd(n_2, n_3)$. Analogously, we have $n_2 = \gcd(n_1, n_3)$. This leads to the conditions

$$\begin{aligned} K_1 \pm K_2 &= n_1^\pm K_3, & K_1 \pm K_2 &= n_2^\pm K_4, \\ K_3 \pm K_4 &= m_1^\pm K_1, & K_3 \pm K_4 &= m_2^\pm K_2. \end{aligned} \quad (18)$$

These conditions can only be satisfied in certain special cases. Firstly, if μ and ν both vanish then the metric becomes the standard Fubini-Study metric on $\mathbb{C}\mathbb{P}^3$. Secondly, there is a particular case, in which either μ or ν vanishes, that corresponds to the metric on $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^1$. The associated seven-dimensional Einstein-Sasaki spaces are S^7 and $M^{1,1,1}$, respectively. With the exception of these special cases, the Einstein-Kähler metric (2) is singular in the sense that it cannot be extended onto a smooth manifold without conical singularities.

3.2 $D = 7$ Einstein-Sasaki metric

We shall now demonstrate that, even though the six-dimensional Einstein-Kähler base space is singular, we can nevertheless obtain non-singular seven-dimensional Einstein-Sasaki spaces from the local metrics (11). The conditions for avoiding power-law curvature singularities are the same as those that we discussed previously for the base metrics. Namely, we must ensure that the x and y range in non-overlapping intervals such that $x - y$, x and y never vanish. The locations of the degenerate surfaces are also the same. However, the Killing vectors that vanish on these surfaces are now given by

$$\begin{aligned} x = x_1 : & \quad K_1 = c_1 \left[\frac{\partial}{\partial \tau} - \frac{1}{2x_1} \left(\frac{\partial}{\partial \chi} + \frac{\beta}{x_1} \frac{\partial}{\partial \psi} \right) \right], \\ x = x_2 : & \quad K_2 = c_2 \left[\frac{\partial}{\partial \tau} - \frac{1}{2x_2} \left(\frac{\partial}{\partial \chi} + \frac{\beta}{x_2} \frac{\partial}{\partial \psi} \right) \right], \\ y = y_1 : & \quad K_3 = \tilde{c}_1 \left[\frac{\partial}{\partial \tau} - \frac{1}{2y_1} \left(\frac{\partial}{\partial \chi} + \frac{\beta}{y_1} \frac{\partial}{\partial \psi} \right) \right], \\ y = y_2 : & \quad K_4 = \tilde{c}_2 \left[\frac{\partial}{\partial \tau} - \frac{1}{2y_2} \left(\frac{\partial}{\partial \chi} + \frac{\beta}{y_2} \frac{\partial}{\partial \psi} \right) \right], \\ \theta = 0 : & \quad K_5 = \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \phi}, \\ \theta = \pi : & \quad K_6 = \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \phi}, \end{aligned} \quad (19)$$

where

$$c_i = \frac{4x_i^2}{2 + 3\alpha x_i + 16x_i^2}, \quad \tilde{c}_i = \frac{4y_i^2}{2 + 3\alpha y_i + 16y_i^2}. \quad (20)$$

Again, we have normalised the Killing vectors so that each has unit Euclidean surface gravity, implying that they must each generate translations with period 2π at the corresponding degenerate surface. Following the earlier discussion, it is clear that they must satisfy

$$b_1 K_1 + b_2 K_2 + b_3 K_3 + b_4 K_4 + b_5 (K_5 + K_6) = 0, \quad (21)$$

for appropriate constant coefficients b_i . It is straightforward to verify from (19) that these constants satisfy

$$b_1 + b_2 + b_3 + b_4 + 2b_5 = 0. \quad (22)$$

To avoid conical singularities, these constants must all be rationally related. Without loss of generality, we can then scale them so that they are integers.

We can solve (21) by considering the two conditions

$$n_1 K_1 + n_2 K_2 + n_3 K_3 + n_4 K_4 = 0, \quad m_1 K_1 + m_2 K_2 + m_3 K_3 + m_{56} (K_5 + K_6) = 0, \quad (23)$$

where n_i and m_i are two sets of co-prime integers. From (19), these equations imply that

$$n_1 + n_2 + n_3 + n_4 = 0, \quad m_1 + m_2 + m_3 + m_{56} = 0. \quad (24)$$

Once these equations are satisfied, the constants b_i can be given by

$$\begin{aligned} b_1 &= k_1 n_1 + k_2 m_2, & b_2 &= k_1 n_2 + k_2 m_2, & b_3 &= k_1 n_3 + k_2 m_3, \\ b_4 &= k_1 n_4, & b_5 &= k_2 m_{56}, \end{aligned} \quad (25)$$

for arbitrary integers k_1 and k_2 . Note that in a space of Euclidean signature, if n Killing vectors K_i simultaneously vanish at a certain degenerate surface then any linear combination of these Killing vectors, $K = m_i K_i$, also vanishes. In particular, if m_i are integers and K_i all generate translations with period 2π , then the period for K is $2\pi \gcd(m_1, \dots, m_n)$. In our example, any combination of three Killing vectors from the three sets (K_1, K_2) , (K_3, K_4) and (K_5, K_6) will vanish on the corresponding surface where $x = x_1$ or $x = x_2$ and simultaneously $y = y_1$ or $y = y_2$ and also $\theta = 0$ or $\theta = \pi$. This implies the following constraints on the integers, in order to avoid conical singularities:

$$\begin{aligned} \gcd(k_1 n_1 + k_2 m_2, k_1 n_3 + k_2 m_3, k_2 m_{56}) &= \gcd(k_1 n_2 + k_2 m_2, k_1 n_4, k_2 m_{56}), \\ \gcd(k_1 n_1 + k_2 m_2, k_1 n_4, k_2 m_{56}) &= \gcd(k_1 n_2 + k_2 m_2, k_1 n_3 + k_2 m_3, k_2 m_{56}), \end{aligned} \quad (26)$$

for all integers k_1 and k_2 .

If the parameters in the metric are chosen so that the conditions stated above are satisfied, then the metric will extend smoothly onto a complete and non-singular compact 7-manifold.

The analysis to determine when the above regularity conditions are satisfied is quite involved. Rather than presenting a complete analysis, we shall just give explicit examples which show that non-trivial solutions do, in fact, exist. The condition (23) implies that

$$\begin{aligned}
\frac{(2\beta + x_1(3\alpha + 16x_1))(x_2 - y_1)(y_1 - y_2)}{(x_1 - x_2)(2\beta + y_1(3\alpha + 16y_1))(x_1 - y_2)} &= \frac{n_1}{n_3} \equiv r_1, \\
-\frac{(2\beta + x_2(3\alpha + 16x_2))(x_1 - y_1)(y_1 - y_2)}{(x_1 - x_2)(2\beta + y_1(3\alpha + 16y_1))(x_2 - y_2)} &= \frac{n_2}{n_3} \equiv r_2, \\
\frac{(2\beta + x_1(3\alpha + 16x_1))(x_2 - y_1)y_1}{x_1(x_1 - x_2)(2\beta + y_1(3\alpha + 16y_1))} &= \frac{m_1}{m_3} \equiv r_3, \\
-\frac{(2\beta + x_2(3\alpha + 16x_2))(x_1 - y_1)y_1}{(x_1 - x_2)x_2(2\beta + y_1(3\alpha + 16y_1))} &= \frac{m_2}{m_3} \equiv r_4, \tag{27}
\end{aligned}$$

where r_1, r_2, r_3 and r_4 are rational numbers. These equations place severe constraints on the existence of solutions. Recalling that the scaling symmetry (7) allows us to set $y_1 = 1$ without loss of generality, we see that the right-hand sides of the four equations (27) have only three independent variables. Thus, for any given set of rational numbers (r_1, r_2, r_3, r_4) there are, in general, no solutions. We can eliminate the quantities x_1, x_2 and y_2 from (27), which gives rise to a 26th-order polynomial P in (r_1, r_2, r_3, r_4) involving 1866 non-factorisable terms; we shall not present this here.

A strategy for finding a solution is to start by selecting two rational numbers (r_1, r_2) that satisfy the conditions in (26) for $k_1 = 1$ and $k_2 = 0$. Next, we substitute these into the polynomial P and look for rational solutions for (r_3, r_4) . If such solutions exist, we can then check if the set (r_1, r_2, r_3, r_4) satisfies the conditions in (26) for all integers k_1 and k_2 . If it does, then we can use (27) to determine x_1, x_2 and y_2 (since we have set $y_1 = 1$). We can then check if either of the sets of inequalities in (14) is satisfied. If this is the case, then we have obtained an Einstein-Sasaki metric that extends smoothly onto a complete and non-singular compact 7-manifold. Of course, for many of the starting choices for r_1 and r_2 the procedure will fail, since not all of the regularity conditions will be satisfied. With the aid of a computer, one can repeat the procedure for different choices of r_1 and r_2 until one finds a solution which satisfies all of the constraints.³

³One might think that a simpler search strategy would be to start by choosing rational roots x_1, x_2 and y_2 that satisfy either of the sets of inequalities in (14), and then obtain the (necessarily) rational numbers r_i via (27). However, the results obtained for (r_1, r_2, r_3, r_4) will typically not satisfy the conditions (26), and

We will explicitly present two regular solutions which we have obtained by following the above procedure. The first solution satisfies the conditions for Case 1 in (14), whilst the second satisfies the conditions in Case 2. The first example is given by

$$x_1 = \frac{13}{8}, \quad x_2 = \frac{59}{24}, \quad y_1 = 1, \quad y_2 = \frac{4}{3}, \quad (28)$$

which corresponds to

$$\begin{aligned} (r_1, r_2, r_3, r_4) &= \left(-\frac{5}{2}, -\frac{1}{2}, \frac{35}{26}, \frac{81}{118}\right), \\ (\alpha, \beta, \mu, \nu) &= \left(-\frac{686}{39}, \frac{2326}{117}, -\frac{45253}{18432}, -\frac{368}{117}\right). \end{aligned} \quad (29)$$

The six Killing vectors (19) that vanish on the degenerate surfaces satisfy the linear relations:

$$\begin{aligned} 5K_1 + K_2 - 2K_3 - 4K_4 &= 0, \\ 2065K_1 + 1053K_2 + 1534K_3 - 2326(K_5 + K_6) &= 0. \end{aligned} \quad (30)$$

The condition (26) applied to this case becomes

$$\begin{aligned} \gcd(5k_1 + 2065k_2, -2k_1 + 1534k_2, -2326k_2) &= \gcd(k_1 + 1053k_2, -4k_1, -2326k_2), \\ \gcd(5k_1 + 2065k_2, -4k_1, -2326k_2) &= \gcd(k_1 + 1053k_2, -2k_1 + 1534k_2, -2326k_2), \end{aligned} \quad (31)$$

which is satisfied for all integers k_1 and k_2 .

The corresponding functions X and Y are

$$\begin{aligned} X &= \frac{(8x - 13)(59 - 24x)(2496x^2 - 784x - 767)}{59904x}, \\ Y &= \frac{4(y - 1)(4 - 3y)(92 + 161y - 67y^2)}{117y}. \end{aligned} \quad (32)$$

It can be seen from Figure 1 that this example satisfies all of the inequalities specified in Case 1 of (14).

The second example, which instead satisfies the inequalities listed in Case 2 in (14), is given by

$$x_1 = -\frac{2}{9}, \quad x_2 = -2, \quad y_1 = 1, \quad y_2 = \frac{1}{4}. \quad (33)$$

The parameters α, β, μ and ν and the rational numbers r_i are given by

$$\begin{aligned} (r_1, r_2, r_3, r_4) &= \left(-\frac{9}{2}, -\frac{1}{2}, -\frac{51}{4}, -\frac{3}{4}\right), \\ (\alpha, \beta, \mu, \nu) &= \left(\frac{35}{9}, -\frac{25}{3}, \frac{2}{9}, \frac{2}{9}\right). \end{aligned} \quad (34)$$

in practice one finds that the search for a valid solution using this approach takes much longer.

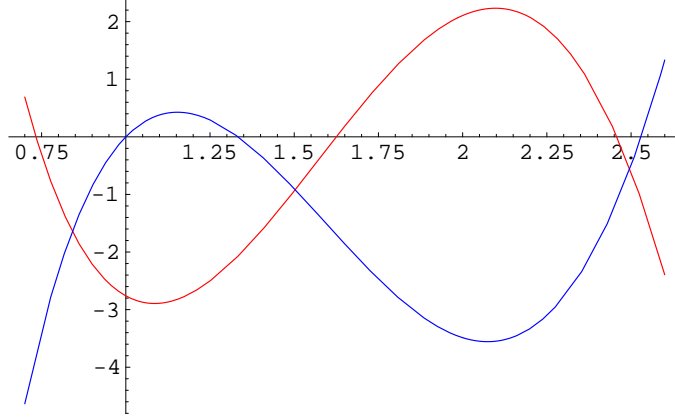


Figure 1: *The functions $X(x)$ (in red) and $Y(y)$ (in blue), showing the non-overlapping closed intervals in which they are greater than or equal to zero. The horizontal axis is x and y , respectively, for the two functions. ($X(x)$ is heading to $-\infty$ at large positive x , whilst $Y(y)$ is heading to $+\infty$ at large positive y .)*

The six Killing vectors (19) that vanish on the degenerate surfaces satisfy the linear relations

$$\begin{aligned} 9K_1 + K_2 - 2K_3 - 8K_4 &= 0, \\ 51K_1 + 3K_2 - 4K_3 - 25(K_5 + K_6) &= 0. \end{aligned} \quad (35)$$

To avoid conical singularities, it is necessary to satisfy the condition (26), which, applied to this example, is given by

$$\begin{aligned} \gcd(9k_1 + 51k_2, -2k_1 - 4k_2, -25k_2) &= \gcd(k_1 + 3k_2, -8k_1, -25k_2), \\ \gcd(9k_1 + 51k_2, -8k_1, -25k_2) &= \gcd(k_1 + 3k_2, -2k_1 - 4k_2, -25k_2). \end{aligned} \quad (36)$$

It is straightforward to verify that this condition is satisfied for all integers k_1 and k_2 .

The functions $X(x)$ and $Y(y)$ in this example are given by

$$\begin{aligned} X &= -\frac{2}{9x}(1-x)(1-4x)(2+x)(2+9x), \\ Y &= \frac{2}{9y}(1-y)(1-4y)(2+y)(2+9y). \end{aligned} \quad (37)$$

As can be seen from Figure 2, these functions are negative in the non-overlapping ranges $x_2 \leq x \leq x_1$ and $y_2 \leq y \leq y_1$ of their respective arguments, thus satisfying the conditions of Case 2.

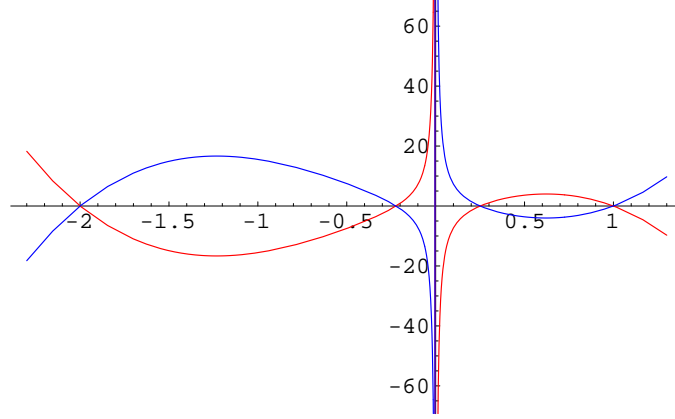


Figure 2: The functions $X(x)$ (in red) and $Y(y)$ (in blue), showing the non-overlapping closed intervals in which they are less than or equal to zero. The horizontal axis is respectively x and y for the two functions. ($X(x)$ is heading to $-\infty$ at large positive x and to $+\infty$ at large negative x , whilst $Y(y)$ is heading to $+\infty$ at large positive y and to $-\infty$ at large negative y .)

4 Generalisation to Arbitrary Dimension

We may generalise the construction of six-dimensional Einstein-Kähler metrics given in section 2.1 to arbitrary even dimensions $D \geq 6$. To do this, it is useful first to derive a set of conditions for having a $(2n)$ -dimensional Einstein-Kähler metric ds^2 that is normalised, for convenience, to satisfy $R_{ij} = 2(n+1)g_{ij}$. If the Kähler form is $J = \frac{1}{2}dB$, then we may construct the $(2n+1)$ -dimensional Einstein-Sasaki metric using (1), and hence the $(2n+2)$ -dimensional Ricci-flat Kähler metric $d\tilde{s}^2$ on the Calabi-Yau cone over this. With the normalisations we are using, the metric on the Calabi-Yau cone will be given by

$$d\tilde{s}^2 = dr^2 + r^2(d\tau + B)^2 + r^2 ds^2. \quad (38)$$

It is easily verified that this has Kähler form

$$\tilde{J} = r dr \wedge (d\tau + B) + r^2 J. \quad (39)$$

We may also take the canonical holomorphic $(n+1)$ -form $\tilde{\Omega}$ to be given by

$$\tilde{\Omega} = e^{i(n+1)\tau} r^n [dr + i r(d\tau + B)] \wedge \Omega, \quad (40)$$

where n is the canonical holomorphic n -form on ds^2 . It can now easily be verified that the conditions $d\tilde{J} = 0$ and $d\tilde{\Omega} = 0$, which ensure that $d\tilde{s}^2$ is Ricci-flat and Kähler, imply that

$$dJ = 0, \quad d\Omega = i(n+1)B \wedge \Omega. \quad (41)$$

These, then, are the conditions for the original $2n$ -dimensional metric ds^2 to be Einstein-Kähler, satisfying $R_{ij} = 2(n+1)g_{ij}$.

Equipped with these equations, we now make the following ansatz for a $(2n+4)$ -dimensional Einstein-Kähler metric $d\hat{s}^2$:

$$d\hat{s}^2 = \frac{x-y}{X} dx^2 + \frac{x-y}{Y} dy^2 + \frac{X}{x-y} (d\chi - y\sigma)^2 + \frac{Y}{x-y} (d\chi - x\sigma)^2 + xy ds^2, \quad (42)$$

where X is a function of x , Y is a function of y ,

$$\sigma = d\psi + B, \quad (43)$$

and ds^2 is an Einstein-Kähler metric satisfying $R_{ij} = 2(n+1)g_{ij}$, with Kähler form $J = \frac{1}{2}dB$. If we define

$$\hat{B} = 2(x+y)d\chi - 2xy\sigma, \quad (44)$$

then it is easily seen that

$$\hat{J} \equiv \frac{1}{2}d\hat{B} = dx \wedge (d\chi - y\sigma) + dy \wedge (d\chi - x\sigma) - xy J \quad (45)$$

defines an almost-complex structure with respect to the metric (42). Its manifest closure is one of the two conditions for $d\hat{s}^2$ to be Einstein-Kähler, with Kähler form given by \hat{J} .

Next, we define the $(n+2)$ -form

$$\hat{\Omega} = e^{i(\frac{1}{2}(n+2)\alpha\chi + \gamma\psi)} (xy)^{n/2} \epsilon_1 \wedge \epsilon_2 \wedge \Omega, \quad (46)$$

where Ω is the holomorphic n -form on ds^2 (satisfying the second equation in (41)), and

$$\begin{aligned} \epsilon_1 &= \left(\frac{X}{x-y}\right)^{-1/2} dx - i \left(\frac{X}{x-y}\right)^{1/2} (d\chi - y\sigma), \\ \epsilon_2 &= \left(\frac{Y}{x-y}\right)^{-1/2} dy - i \left(\frac{Y}{x-y}\right)^{1/2} (d\chi - x\sigma). \end{aligned} \quad (47)$$

It can be seen that $\hat{\Omega}$ is holomorphic with respect to the almost-complex structure defined by \hat{J} in equation (45). The remaining condition given in (41) for a metric to be Einstein-Kähler, which in the present case becomes

$$d\hat{\Omega} = i(n+3)\hat{B} \wedge \hat{\Omega}, \quad (48)$$

is satisfied if $\gamma = 2n+2$ and the functions X and Y satisfy

$$\begin{aligned} xX' + nX + 4(n+3)x^3 + (n+2)\alpha x^2 + 4(n+1)x &= 0, \\ yY' + nY - 4(n+3)y^3 - (n+2)\alpha y^2 - 4(n+1)y &= 0, \end{aligned} \quad (49)$$

where a prime indicates a derivative with respect to the argument of the functions $X(x)$ and $Y(y)$ respectively. Thus, the metric $d\hat{s}^2$ given in (42) is Einstein-Kähler, satisfying $\hat{R}_{ab} = 2(n+3)\hat{g}_{ab}$, if ds^2 is Einstein-Kähler with $R_{ij} = 2(n+1)g_{ij}$ and the functions X and Y are given by

$$X = -4x^3 - \alpha x^2 - 4x - \frac{\mu}{x^n}, \quad Y = 4y^3 + \alpha y^2 + 4y + \frac{\nu}{y^n}. \quad (50)$$

The quantities μ , ν and α are arbitrary constants, the Kähler form is given by (45) and the holomorphic $(n+2)$ -form is given by (46). The Einstein-Kähler metrics that we have obtained in this construction are generalisations of the six-dimensional Einstein-Kähler metrics in section 2.1.⁴ In that case, the starting point was the canonically-normalised two-dimensional Einstein-Kähler metric $\frac{1}{4}(\sigma_1^2 + \sigma_2^2)$ on $\mathbb{C}\mathbb{P}^1 = S^2$, while in the generalisation (42) we instead began with an arbitrary $(2n)$ -dimensional Einstein-Kähler base metric ds^2 .

Having obtained the $(2n+4)$ -dimensional Einstein-Kähler metrics (42), we can of course immediately write down the associated $(2n+5)$ -dimensional Einstein-Sasaki metrics by using the construction given in (1). With the normalisations we are using here, these will be given by

$$\begin{aligned} d\tilde{s}^2 = & [d\tau - 2(x+y)d\chi + \frac{2xy}{\beta}\sigma]^2 + \frac{x-y}{X}dx^2 + \frac{x-y}{Y}dy^2 \\ & + \frac{X}{x-y}(d\chi - \frac{y}{\beta}\sigma)^2 + \frac{Y}{x-y}(d\chi - \frac{x}{\beta}\sigma)^2 + \frac{xy}{\beta}ds^2, \end{aligned} \quad (51)$$

where σ is given by (43),

$$X = -4x^3 - \alpha x^2 - 4\beta x - \frac{\mu}{x^n}, \quad Y = 4y^3 + \alpha y^2 + 4\beta y + \frac{\nu}{y^n}, \quad (52)$$

and ds^2 is an Einstein-Kähler metric satisfying $R_{ij} = 2(n+1)g_{ij}$ and with Kähler form given locally by $J = \frac{1}{2}dB$. The Einstein-Sasaki metric $d\tilde{s}^2$ satisfies $\tilde{R}_{ab} = (2n+4)\tilde{g}_{ab}$. For convenience, we have included the “trivial” β parameter mentioned in footnote 3; it can be set to ± 1 by using the previously-mentioned scaling symmetry.

It can easily be seen that if we set the parameter ν in (50) to zero, the metrics reduce to special cases of those discussed previously in [8, 9], namely where all except one of the “rotation parameters” α_i in those papers are set equal. The inclusion of the “NUT” parameter ν yields new metrics that lie outside those previously discussed in the literature.

⁴Note that we could, as in section 2.1, introduce an additional “trivial” parameter β , which would multiply the linear terms in X and Y in (50) and divide the $xyds^2$ term and the σ 1-forms in (42). One can then set $\beta = \pm 1$ using the scaling symmetry analogous to the one discussed in the six-dimensional case in section 2.1.

It is worth remarking that the construction we have discussed here can also be applied in the case when $n = 0$, for which (42) is a four-dimensional Einstein-Kähler metric, with no $xy ds^2$ term. In this special case, the extra “NUT” parameter ν is trivial, and can be absorbed by performing a constant shift transformation with $x \rightarrow x + c$, $y \rightarrow y + c$. In fact, the construction when $n = 0$ merely reproduces the metrics discussed in [4] which, in turn, are equivalent to the five-dimensional Einstein-Sasaki metrics in [8, 9] when the “rotation parameters” are set equal. Thus, it is only for Einstein-Kähler metrics in $D \geq 6$ and the associated Einstein-Sasaki metrics in $D \geq 7$ that the new parameter ν is non-trivial.

Using these local expressions for Einstein-Sasaki metrics, one can again perform an analysis to find choices for the free parameters such that the metrics will extend smoothly onto complete and non-singular compact manifolds. The analysis will be similar to the one we described in detail in section 3 for the seven-dimensional case, and we shall not discuss it further here.

5 Conclusions

We have constructed new explicit Einstein-Kähler metrics in all even dimensions $D = 2n + 4 \geq 6$, in terms of a $(2n)$ -dimensional Einstein-Kähler base metric. The metrics have cohomogeneity 2 (if one chooses a homogeneous Einstein-Kähler metric such as $\mathbb{C}\mathbb{P}^n$ for the base) and have the new feature of including a NUT-type parameter, along with mass and rotation parameters. In $D \geq 8$, this construction can be iterated to yield Einstein-Kähler metrics of cohomogeneity greater than 2.

Using a canonical construction, these metrics all yield Einstein-Sasaki metrics of the form (1) in odd dimensions $D = 2n + 5 \geq 7$. For the case $D = 7$, we showed in detail that, for suitable choices of the free parameters, the Einstein-Sasaki metrics can extend smoothly onto complete and non-singular compact manifolds even though the underlying Einstein-Kähler 6-metrics have conical singularities. These new metrics generalise certain previously-known countably infinite classes of Einstein-Kähler and Einstein-Sasaki metrics (i.e. a subset of those obtained in [8, 9], which arose as supersymmetric limits of the Kerr-de Sitter metrics). Although we have only explicitly presented two examples, it is natural to conjecture that this construction provides a countably infinite class of new non-singular Einstein-Sasaki spaces.

These spaces can be Wick-rotated to yield supersymmetric Kerr-Taub-NUT-de Sitter metrics. In a forthcoming paper, it will be shown how these solutions arise in a supersym-

metric limit of more general Kerr-Taub-NUT-de Sitter black holes [21].

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