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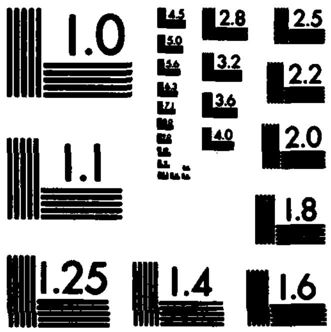
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A FREE-BOUNDARY PROBLEM ARISING FROM A GALVANIZING PROCESS

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UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

A free-boundary problem which arises from a galvanizing process is studied. The physical problem is that of an infinite cylinder $\Omega' \times R$ withdrawn from a fluid bath. Formally, this is a gravity-driven unidirectional viscous fluid flow on the exterior of the cylinder $\Omega' \times R$. The existence of a unique classical solution is shown under certain conditions on Ω', and asymptotic rerults for the thickness of the coat are obtained for large and small withdrawal speeds. If Ω' is a convex set, then the region bounded by the free surface of the fluid is shown to be convex, using level curve techniques. Finally, level curve techniques are used to bound the curvature of the free boundary in terms of that of the fixed boundary.

AMS (MOS) Subject Classification: 35R35, 35J20

Key Words: Free-boundary problem, galvanizing, variational inequality, level curves

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Coating processes are important in many industrial applications, and are currently receiving much attention from applied mathematicians. Recently, a mathematical model for continuous hot-dip galvanizing has been proposed by Tuck, Bentwich, and van der Hoek. The physical process described is that of a steel wire or sheet pulled vertically from a bath of molten zinc. The coating of zinc which adheres to the steel gradually solidifies. In the present paper, this galvanizing problem is analyzed using modern variational methods. The results presented here, especially the asymptotic dependence of the thickness of the coat, will help in an evaluation of the Tuck-Bentwich-van der Hoek model.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A FREE-BOUNDARY PROBLEM ARISING FROM A GALVANIZING PROCESS

Thomas I. Vogel

§1. Introduction. Many industrial processes involve applying a thin coat of liquid to some material. To coat an infinitely long cylinder, a common method is to pull it out of a liquid bath so that the gravity vector points parallel to the generators of the cylinder. This is typically used for galvanizing, where the cylinder (not necessarily circular) is a wire or sheet of steel, and the liquid is molten zinc. As the cylinder moves up, it carries with it a coat of liquid, which gradually solidifies. Over a substantial length of the cylinder, the flow of the liquid is steady and straight down, with the outer boundary of the region of flow a free surface. The driving forces are gravity and viscosity.

Tuck, Bentwich, and van der Hoek [7] (hereafter referred to as TBH) have recently given a formulation of this problem. Let Ω' R^2 be the cross section of the cylinder, and let Σ be its boundary. Let Ω R^2 be the cross section of the region of flow plus the cylinder, and let Γ be the boundary of Ω (which is free). The region of flow is exterior to the given Ω'. Then they show that under certain assumptions, the upward velocity field $w(x,y)$ must satisfy

$$\begin{aligned} \Delta w &= g/\nu && \text{in } \Omega - \overline{\Omega}' \\ w &= W_B && \text{on } \Sigma \\ w &= W_B/2 && \text{on } \Gamma \\ \frac{\partial w}{\partial n} &= 0 && \text{on } \Gamma \,. \end{aligned}$$

Here g is the downward acceleration due to gravity, ν is the kinematic viscosity of the liquid, and W_B is the withdrawal speed of the cylinder. It is important to keep in

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condition that the above system has a solution. The fact that we impose both Dirichlet and Neumann boundary conditions on Γ will prevent a solution w from existing for a general Ω. This situation is typical of free boundary problems, where the fact that the boundary conditions are overdetermined is compensated by the freeness of the boundary.

The model of TBH neglects surface tension and assumes that the net rate of transport

$$Q = \iint_{\Omega-\bar{\Omega}'} w(x,y)dx\, dy$$

is maximized. More precisely, they show that if $w(x,y;U)$ satisfies

$$\Delta w = g/\nu \quad \text{in} \quad U - \bar{\Omega}'$$

$$w = W_B \quad \text{on} \quad \Sigma$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial U,$$

then if $Q(U) = \iint_{U-\bar{\Omega}'} w(x,y;U)dx\, dy$ is maximized over all admissible U, the maximum will be obtained when $w(U) \equiv \frac{W_B}{2}$ on $\partial\Omega$.

In this paper, we will work with the normalization:

$$u = \nu(w-\frac{W_B}{2})/g \quad \text{and}$$

$$c = \frac{\nu\, W_B}{2g},$$

so that the equations become

$$\text{(1.1)} \qquad \begin{aligned} \Delta u &= 1 && \text{in } \Omega - \bar{\Omega}' \\ u &= c && \text{on } \Sigma \\ u &= 0 && \text{on } \Gamma \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma \end{aligned}$$

Notice that the last condition is equivalent to $|\nabla u| = 0$ on Γ, since Γ is a level surface of u. If the dependence on c is to be emphasized, we will write Γ_c and u_c.

In Section 2, we show the existence of a classical solution of (1.1) in n dimensions if Ω' is starshaped with respect to a ball. This section consists mainly of arguments in TBH placed on firmer theoretical grounds. In Section 3, we obtain asymptotic results in two dimensions for c large and c small, and also some useful comparison results. In particular, as c tends to infinity, the free surfaces Γ_c tend to circles of radius $2\sqrt{\frac{c}{\log 2c}}$. In Section 4, we will prove the convexity of the set $\{u > t\}$ for $c > t > 0$ if Ω' is convex, and in Section 5, we show that for $n = 2$, if Ω' is convex, then each point of the ridge of Ω is closer to Σ than to Γ.

§2. Existence and Regularity

Let $\Omega' \subset \mathbb{R}^n$ be a bounded open set which is starshaped with respect to all points contained in some ball, and let c be a positive constant. Suppose that $\Sigma = \partial\Omega'$ is sufficiently smooth. Then we will prove in this section that there exists a set Ω containing Ω' with $\Gamma = \partial\Omega$ analytic, and a non-negative function u which satisfies equation (1.1). This will be done variationally by using the functional

$$J_R(v) = \int_{B_R} |\nabla v|^2 + 2v. \tag{2.1}$$

where B_R is a ball containing $\bar{\Omega}'$ of some sufficiently large radius R centered in Ω'. J_R will be minimized over the set $K_{c,R} = \{v \in L^1(\mathbb{R}^n),\ \nabla v \in L^2(\mathbb{R}^n),$ $v = c$ in Ω', $v = 0$ in $\mathbb{R}^n - B_R$, $v \geq 0$ everywhere$\}$

Theorem 2.1 If Ω' is a bounded set in $\mathbb{R}^n$ and Σ $(=\partial\Omega')$ is in $C^{2+\alpha}$, then there exists a unique $u \in K_{c,R}$ such that

$$J_R(u) = \inf_{v \in K_{c,R}} J_R(v) .$$

Moreover, $u \in W^{2,p}(B_R-\bar{\Omega}') \cap W^{2,\infty}_{loc}(B_R-\bar{\Omega}')$ for all $p < \infty$, where $W^{2,p}(B_R-\bar{\Omega}') = \{v \in L^p(B_R-\bar{\Omega}'),\ \nabla v \in L^2(B_R-\bar{\Omega}')\}$. As a consequence, u is C^1 in $B_R - \bar{\Omega}'$ (see Gilbarg and Trudinger [5]). Moreover, u is analytic in $\Omega - \bar{\Omega}'$ and $\Delta u = 1$ there, where $\Omega = \{u > 0\}$.

Proof: This follows from standard theorems (Friedman [2], Sections 1.3 and 1.4).

Note: The relation between the above variational formulation and equations (1.1) can now be demonstrated. Let ζ be contained in $C_0^\infty(\Omega-\bar{\Omega}')$. For small enough ϵ, $v = u + \epsilon\zeta \in K_{c,R}$. Since

$$J_R(u+\epsilon\zeta) \geq J_R(u),$$

it follows that

$$\int_{\Omega-\bar{\Omega}'} (\nabla u\cdot\nabla\zeta+\zeta) = 0.$$

Integrating by parts,

$$\int_{\Omega-\bar{\Omega}'} (\Delta u-1)\zeta = 0,$$

hence $\Delta u = 1$ in $\Omega-\bar{\Omega}'$.

The free boundary conditions now follow assuming $\bar{\Omega} \cap \partial B_R = \emptyset$, for on $\Gamma = \partial\{u>0\}$, we have $|\nabla u| = 0$ by the continuity of ∇u in $B_R - \bar{\Omega}'$. In Section 3, we will see that for R_o sufficiently large, $\Omega \cap \partial B_{R_o} = \emptyset$. It is clear that for any $R_1, R_2 > R_o$, the minimizers u_{R_1} and u_{R_2} will be identical. We will assume from now on that R is larger than this R_o, thus eliminating the dependence of u on R.

Corollary 2.2: The minimizer obtained in Theorem 2.1 satisfies $0 \leq u \leq c$.

Proof: One easily checks that

$$J_R(u\wedge c) \leq J_R(u) ,$$

where $u \wedge c = \min(u(X),c)$. Moreover, $u \wedge c \in K_{c,R}$, so that the uniqueness part of Theorem 2.1 applies.

Definition: A region U is almost star-like with respect to a point $P \in U$ if the characteristic function χ_U is non-increasing along rays from P. The difference between an almost star-like region and a star-like region is that an almost star-like region may contain a portion of a ray through P in its boundary.

Lemma 2.3: If $\partial\Omega'$ is $C^{2+\alpha}$ and Ω' is almost star-like with respect to the origin, then $\frac{\partial u}{\partial r} < 0$ in $\Omega - \bar{\Omega}'$ and Ω is almost star-like with respect to the origin. (Here $r = \sqrt{x_1^2+\cdots+x_n^2}$)

Proof: This is proven in TBH [7] for $n = 2$ by showing that $r\frac{\partial u}{\partial r}$ is subharmonic in $\Omega - \bar{\Omega}'$ with non-positive boundary values. The proof is the same in n dimensions. The almost star-likeness of Ω follows, since u and hence χ_Ω is non-increasing along rays.

Theorem 2.4: If Ω' is starlike with respect to each point contained in a ball B_ε, then Γ is analytic, and u satisfies:

$$\Delta u = 1 \quad \text{in } \Omega - \bar{\Omega}'$$

$$u = c \quad \text{on } \Sigma$$

$$u = 0 \quad \text{on } \Gamma$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma$$

Σ need not be $C^{2+\alpha}$, although it is clearly Lipschitz continuous.

Proof: First we assume that $\partial\Omega'$ is in $C^{2+\alpha}$. Since Γ is almost starlike with respect to each point in B_ε, it is therefore Lipschitz continuous. This is enough to apply a theorem of Caffarelli (Friedman [2], p. 162) to show that Γ is C^1 and hence analytic. Once we have the smoothness of Γ, the boundary conditions on u will necessarily be satisfied.

If Σ is not $C^{2+\alpha}$, then we may approximate Ω' by an increasing nested series of sets Ω'_i with $\partial\Omega'_i$ smooth and Ω'_i starlike with respect to each point in B_ε. To see this, let $f(\theta)$ be the radius of Σ at angle θ. This continuous function can be approximated from below by smooth functions $f_i(\theta)$ whose graphs will be the $\partial\Omega'_i$'s. Let u_i be the minimizer corresponding to Ω'_i. We shall see (Theorem 3.1) that $\{u_i(X)\}$ is a bounded, increasing sequence for each X. Let $u(X) = \lim_{i\to\infty} u_i(X)$. Then $u(X)$ clearly is equal to c on Ω' and zero outside of B_R.

We may bound $J_R(u_i)$ uniformly by considering a radial function v which is equal to c on a ball B_ρ with $\Omega \subset B_\rho \subset B_R$, 0 on B_R, and C^∞ on R^n. Then each $J_R(u_i)$ is less than $J_R(v)$. Thus $\int_{B_R} |\nabla u_i|^2$ is uniformly bounded, so that there is a weakly convergent subsequence to ∇u. We therefore have

$$J_R(u) \leq \liminf_{i\to\infty} J_R(u_i)$$

But

$$J_R(u) \geq J_R(u_i)$$

for each i, so that

$$J_R(u) = \lim_{i\to\infty} J_R(u_i)$$

I claim that u is minimal over the class of functions equal to c on Ω' and 0 outside of B_R. For, if this is not true, then for some v with $J_R(v) < J_R(u)$, we must have $J_R(v) < J_R(u_i)$ for large enough i. Since $\Omega_i' \subset \Omega'$, this is a contradiction.

The free boundaries Γ_i increase out to Γ, the boundary of $\{u > 0\}$. Since each Γ_i is starshaped with respect to B_ε, so is Γ. But then we apply the same argument as before to say that Γ is analytic and u satisfies the correct boundary conditions on Γ.

<u>Theorem 2.5</u> (Uniqueness) If Ω' is starlike, then there exists at most one solution (u,Γ) to (1.1).

<u>Proof</u>: Suppose there are two solutions, (u,Γ) and (u^*, Γ^*), to (1.1). We assume that Ω' is starlike with respect to the origin. Define

$$u_r = u(rX)$$

$$\Omega_r = \frac{1}{r}\Omega$$

$$\Omega_r' = \frac{1}{r}\Omega'$$

$$\Gamma_r = \frac{1}{r}\Gamma$$

$$\Sigma_r = \frac{1}{r}\Sigma$$

Since Ω' is starshaped, $\Omega' \subset \Omega_r'$ for $r < 1$. Let $s = \sup\ \{r |\ (\Omega^*)' \subset \Omega_r'\}$. We may assume without loss of generality that $s < 1$, for we could exchange the role of u and u^* if this were not so. Then Γ_s and Γ^* are tangent at some point Y.

The boundary of $\Omega^* \subset (\Omega_s - \bar{\Omega}'_s)$ consists of $\Gamma^* \cap (\Omega_s - \bar{\Omega}'_s)$ and Σ_s. On both of these surfaces, $u_s > u^*$, so that $u_s > u^*$ everywhere in $\Omega^* \cap (\Omega_s - \bar{\Omega}'_s)$. However, at Y we have $\frac{\partial u_s}{\partial n} = \frac{\partial u^*}{\partial n} = 0$, so that $u_s = u^*$ on $\Omega^* \cap (\Omega_s - \bar{\Omega}'_s)$ by the strong maximum principle. Hence $s = 1$ and we have the desired uniqueness.

Combining Theorems 2.4 and 2.5, we see that there exists a unique solution to (1.1) if Ω' is starlike with respect to all the points in some B_ε. If Ω' is starlike with respect to only one point, then there is at most one solution to (1.1).

§3. Asymptotic Results

The following comparison theorem is basic to our work, and is needed in the proof of Theorem 2.4.

Theorem 3.1: Let Ω' and Ω_1' satisfy the hypotheses of Theorem 2.1 with $\Omega_1' \subset \Omega'$, and let $c_1 < c$. Let u and u_1 minimize J_R with boundary values c on Σ and c_1 on Σ_1 respectively. Then $u > u_1$ everywhere.

Proof: We have

$$J_R(u \wedge u_1) + J_R(u \vee u_1) = J_R(u) + J_R(u_1)$$

by a simple computation (here $u \wedge u_1 = \min(u,u_1)$, and $u \vee u_1 = \max(u,u_1)$). But $u \wedge u_1 \in (K_1)_{c_1,R}$ and $u \vee u_1 \in K_{c,R}$, so the uniqueness result of Theorem 2.1 applies.

We may use the same technique of proof to give an interesting characterization of u.

Theorem 3.2: Let u, Ω', Ω_1' be as above, and let v_A minimize

$$J_A(v) = \int_A |\nabla v|^2 + 2v,$$

where $A \subset B_R$, over the set $\{v \in L^1(A),\ \nabla v \in L^2(A),\ v = c_1 \text{ on } \Sigma_1,\ v = 0 \text{ on } \partial A\}$ (here we are not requiring v to be non-negative). Then $u > v_A$ everywhere.

Proof: This is proven by the same argument as in the proof of Theorem 3.1.

Corollary 3.3: We may therefore characterize $u(X)$ when Σ is $C^{2+\alpha}$ and Ω' is starshaped with respect to a ball as:

$$u(X) = \sup_{\substack{A \supset \Omega' \\ \partial A \text{ smooth}}} v_A(X),$$

where $v_A(X)$ solves the Dirichlet problem

$$\begin{aligned} v_A &= c && \text{on } \Sigma \\ v_A &= 0 && \text{on } \partial A \\ \Delta v_A &= 1 && \text{in } A - \bar{\Omega}' \end{aligned}$$

$v_A(X)$ will not in general be non-negative.

We now deal exclusively with the case $n = 2$.

To determine the asymptotic behavior of Γ as $c \to \infty$ and $c \to 0$, it is necessary to look at radial solutions. That is, given ρ, the radius of the circular fixed boundary, and $c > 0$, we seek a $v_{\rho,c}(r)$ to solve:

$$
(3.1) \qquad \begin{aligned}
v''_{\rho,c}(r) + \frac{1}{r} v'_{\rho,c}(r) &= 1 \\
v_{\rho,c}(\rho) &= c \\
v_{\rho,c}(\gamma) &= 0 \\
v'_{\rho,c}(\gamma) &= 0,
\end{aligned}
$$

where γ, to be determined, is the radius of the free boundary. One can calculate that γ is given by the implicit relation:

$$
(3.2) \qquad c = \frac{\rho^2 - \gamma^2}{4} + \frac{\gamma^2}{2} \log\left(\frac{\gamma}{\rho}\right),
$$

where $c > 0$, $\rho > 0$, $\gamma > \rho$, and log is $\log_e$.

__Lemma 3.4__: For $R_o = R_o(c,\Omega')$ sufficiently large, $\Omega \cap \partial B_{R_o} = \phi$, where c is fixed and B_{R_o} is centered in Ω'.

__Proof__: Since Ω is contained in some ball B_ρ, the result will follow if it is proven for symmetric solutions. But γ in equation (3.2) will be bounded if ρ and c are bounded, since the highest order term in γ on the right hand side, $\gamma^2 \log \gamma$, must be bounded.

This lemma was already used extensively in Section 2.

From the radial solutions we may investigate the behavior as $c \to \infty$ for a larger class of Ω' 's.

__Theorem 3.5__: Let Ω' be a bounded set containing a ball $B_\varepsilon(0)$ around the origin. Then both $d(\Gamma_c,0) = \inf_{x \in \Gamma_c} |x|$ and $d_1(\Gamma_c,0) = \sup_{x \in \Gamma_c} |x|$ are equal to

$$
2\sqrt{\frac{c}{\log 2c}} + o\left(\sqrt{\frac{c}{\log 2c}}\right)
$$

as $c \to \infty$. (Here Γ is subscripted to emphasize the dependence on c). Less formally, Γ_c is asymptotic to a circle of radius $2\sqrt{\frac{c}{\log 2c}}$ as c tends to infinity.

Proof: From Theorem 3.1, we know that Γ_c is contained in the annulus centered at the origin with inner radius $\gamma(\varepsilon,c)$ and outer radius $\gamma(\rho,c)$, where B_ρ contains Ω'. Therefore, we must show that if $\gamma(\rho,c)$ satisfies (3.2), then as $c \to \infty$,

$$\gamma = 2\sqrt{\frac{c}{\log 2c}} + o\left(\sqrt{\frac{c}{\log 2c}}\right) ,$$

where dependence on ρ will only appear in the second term.

First, it is clear from (3.1) that γ cannot stay bounded as c tends to infinity, and that $\frac{\partial\gamma}{\partial c} > 0$. Dividing by c we obtain:

$$1 = \frac{\rho^2}{4c} - \frac{\gamma^2}{4c} + \frac{\gamma^2\log\gamma}{2c} - \frac{\gamma^2\log\rho}{2c} .$$

For the largest order term on the right hand side, we must have

$$\lim_{c\to\infty} \frac{\gamma^2\log\gamma}{2c} = 1,$$

and the other terms must go to zero.

If we write

$$\gamma = 2f(c)\sqrt{\frac{c}{\log 2c}} , \quad \text{then}$$

$$\text{(3.3)} \qquad \lim_{c\to\infty} 2f^2(c)\left[\frac{\log 2 + \frac{1}{2}\log c - \frac{1}{2}\log(\log 2c) + \log f(c)}{\log 2 + \log c}\right] = 1$$

We can observe from the above expression that $f(c)$ is bounded. The largest order term on the left of (3.3) is

$$f^2(c)\,\frac{\log c}{\log c + \log 2} ,$$

which must approach 1 as $c \to \infty$. The other terms in (3.2) will go to zero. We then conclude that

$$\lim_{c\to\infty} \frac{\gamma(\rho,c)}{2\sqrt{\frac{c}{\log 2c}}} = 1$$

so that $\gamma(\rho,c) = 2\sqrt{\frac{c}{\log 2c}} + o\left(\sqrt{\frac{c}{\log 2c}}\right)$, as desired.

We now examine the thickness of the coat $\Omega - \bar{\Omega}'$ as c tends to zero if $\Sigma \in C^{2+\alpha}$. Fix a point P on Σ, and let ρ and ρ_1 be two radii so that a ball of radius ρ is contained in Ω' and tangent to Σ at P, and a ball of radius ρ_1 is exterior to Ω' and tangent to Σ at P. One choice for ρ is $1/\kappa(\Sigma)$ where $\kappa(\Sigma)$ is the maximum curvature of Σ. If Ω' is convex, then ρ_1 can be chosen to be infinity. From Theorem 3.1, we have

$$d(\Gamma,P) > \gamma(\rho,c) - \rho$$

For an upper bound on $d(\Gamma,P)$, we must look at interior radially symmetric solutions. That is, for the fixed radius ρ_1, we seek a function $v_{\rho_1,c}$ solving equations (3.1) for a value $\gamma_1 < \rho_1$. The same calculation as before yields that γ_1 solves the implicit relation (3.2). Here, now, we seek a root γ_1 less than ρ_1. We conclude:

$$(3.4) \qquad \rho_1 - \gamma_1(\rho_1,c) > d(\Gamma,P) > \gamma(\rho,c) - \rho$$

A straightforward calculation using (3.2) yields that:

$$\lim_{c\to 0} \frac{c}{(\rho-\gamma)^2} = \lim_{c\to 0} \frac{c}{(\rho_1-\gamma_1)^2} = \tfrac{1}{2},$$

for $\rho_1 \neq +\infty$. If $\rho_1 = +\infty$, then the upper bound for the thickness of the coat is $\sqrt{2c}$.

To sharpen the asymptotics in (3.4), we must analyze $\rho - \gamma$ and $\rho_1 - \gamma_1$ more closely for small c. One can calculate that

$$(3.5) \qquad \lim_{c\to 0} \frac{1}{\rho-\gamma}\left(\frac{c}{(\rho-\gamma)^2} - \tfrac{1}{2}\right) = \frac{-1}{6\rho},$$

by substituting (3.2) in for c, and taking the limit as γ approaches ρ.

Now, letting $\rho - \gamma = \sqrt{2c}\, f(c)$, where $\lim_{c\to 0} f(c) = -1$, and substituting into (3.5), one obtains:

$$\lim_{c\to 0} \frac{1+f(c)}{\sqrt{c}} = \frac{1}{3\rho\sqrt{2}},$$

after some manipulation. Therefore,

$$\rho - \gamma = -\sqrt{2c} + \frac{c}{3\rho} + o(c).$$

Similiarly, for the interior radially symetric solution,

$$\rho_1 - \gamma_1 = \sqrt{2c} + \frac{c}{3\rho_1} + o(c) .$$

We have proven the following theorem:

<u>Theorem 3.6</u>: Let Ω' be a set with $C^{2+\alpha}$ boundary Σ satisfying the hypothesis of Theorem 2.4. Let P be a point of Σ, and let ρ and ρ_1 be the radii of disks tangent to Σ at P which are interior and exterior to Ω' respectively. Then $\sqrt{2c} - \frac{c}{3\rho} + o(c) < d(P,\Gamma) < \sqrt{2c} + \frac{c}{3\rho_1} + o(c)$. If Ω' is convex, then the right hand side is simply $\sqrt{2c}$.

<u>Note</u>: This is similiar to a result obtained by Friedman and Phillips [3] for an interior free boundary problem for a more general equation.

<u>Remark</u>: If Σ has an angle at P, then the free boundaries for the scaled functions $u_c = \frac{1}{c} u(\sqrt{c}\, X)$ will approach the free boundary corresponding to a wedge as fixed boundary. Thus, to investigate the asymptotic thickness for small c when Σ has corners, one must look at wedge solutions (see TBH[7] for some numerical results).

§4. Convexity

In this section we investigate what results if Ω' is assumed to be convex.

Theorem 4.1: If Ω' is convex, then the sets $\{u > r\}$, $r > 0$ are convex, including $\Omega = \{u > 0\}$. (This is true in n dimensions)

Proof: (This approach was suggested by Daniel Phillips.) Assume first that $\partial\Omega'$ is smooth. From Caffarelli and Spruck [1], we know that if u_p satisfies the free boundary problem:

$$\Delta u_p = u_p^p \quad \text{in} \quad \Omega_p - \bar{\Omega}'$$
$$u_p = 1 \quad \text{on} \quad \Sigma$$
$$u_p = 0 \quad \text{on} \quad \Gamma_p$$
$$|\nabla u_p| = 0 \quad \text{on} \quad \Gamma_p \, ,$$

then Ω_p and all the sets $\{u_p > r\}$, $r > 0$ are convex. We deal with the particular u_p which minimizes

$$J_p(v) = \int_{B_R - \bar{\Omega}'} \frac{|\nabla v|^2}{2} + \frac{1}{p+1} v^{p+1} \, .$$

These functions have been studied by Phillips [6]. It is not difficult to show that the functions u_p are uniformly bounded in $W^{1,2}(B_R)$ as p tends to zero. Therefore a subsequence converges weakly in $W^{1,2}$ to some function u, which must be the unique minimizer to our original functional (2.1). Using the Rellich lemma, $u_p(x) \to u(x)$ pointwise almost everywhere in B_R, by going to another subsequence. (This is a standard technique : see Friedman [2]). Let $A \subset B_R$ be the set on which u_p converges pointwise to u. If the level sets of u are not convex, then there are three colinear points X, Y, Z in B_R with $u(Y) < \min(u(X), u(Z))$, and Y between X and Z. Since $\mu(A) = \mu(B_R)$, where μ is Lebesque measure, we may assume that X, Y, and Z are contained in A. But this, combined with the pointwise convergence of $\{u_p\}$ contradicts the convexity of the level sets of u_p. I now present an independent proof of the convexity of the free surface in 2 dimensions which is more elementary.

Lemma 4.2: Let $(x(s), y(s))$, $0 \le s \le \ell$ be a parametrization of the free boundary curve Γ. Suppose at $X_o = (x(s_o), y(s_o))$, $x(s)$ has a local extremum. Then there is a level curve $\{u_y = 0\}$ extending into Ω from X_o.

Proof: Let $\epsilon_k \downarrow 0$ be a decreasing sequence such that $\{u = \epsilon_k\}$ is a C^∞ curve. We have that $\{u = \epsilon_k\}$ will contain a point X_k near X_o with a locally extreme x value for small enough ϵ_k, with $\lim_{k\to\infty} X_k = X_o$. Since $\{u=\epsilon_k\}$ has a vertical tangent at X_k, $u_y(X_k) = 0$. But u_y is harmonic in $\Omega - \bar{\Omega}'$, so that the properties of its level curves are well known. In particular, the set $u_y = 0$ must consist of piecewise analytic curves with a finite number of branch points. Therefore, some analytic curve along which $u_y = 0$ must start at X_o and extend into Ω.

Alternate proof of convexity of Ω for $n = 2$: Suppose that Ω is not convex. Assume first that $\partial\Omega'$ is smooth. We can then rotate the coordinate system so that the x coordinate has a local extremum on Γ for at least four points X_1, X_2, X_3, and X_4. At each point X_i there is a level curve γ_i on which $u_y = 0$ extending into Ω. Since u_y is identically zero on Γ and u_y is harmonic, it follows that no γ_i can both start and end on Γ, nor can any two γ_i's meet at a branch point or a point of Σ. Since γ_i cannot terminate in the region $\Omega - \bar{\Omega}'$, it follows that these curves must terminate at four distinct points $Y_i \in \Sigma$. However, the normal derivative of u is non-zero on Σ, so that u_y can equal zero only at the two points of Σ where the normal is horizontal, since Σ is smooth. This contradicts the fact that the Y_i are distinct. If Σ is not smooth, we can approximate from within by smooth sets Ω_k'.

Note: The method of the alternate proof generalizes to elliptic operators with constant coefficients

$$a^{ij}u_{ij} + b^i u_i + cu = f(u) \ , \ i,j = 1,2,$$

with $c < 0$ and $f'(u) > 0$, and with the same boundary conditions on Σ and Γ as before.

§5. The Ridge of Ω

In this section we prove that each point of the ridge (defined later) of Ω must be closer to Ω' than Γ if Ω' is convex. We first need another level curve lemma.

Lemma 5.1 Let $\gamma_b \subset \Omega - \bar{\Omega}'$ be a smooth curve along which $\psi(r,\theta) \equiv \frac{r^2}{2} - ru_r$ is a constant b. Then u_θ is strictly monotone along γ_b. Specifically, if γ_b is traversed so that $\{\psi > b\}$ lies to the right and $\{\psi < b\}$ lies to the left, then u_θ is strictly increasing. This does not depend on where the origin for polar coordinates is placed.

Proof: The functions ψ and u_θ are harmonic conjugates, so that this follows from a well-known result. See Friedman and Vogel [4] for a proof.

Lemma 5.2: At every point $P \in \Gamma$, there initiates at least one level curve of ψ. If P is a local extremum of ψ restricted to Γ, (we will write this as $\psi|_\Gamma$) then there are at least two level curves of ψ initiating at P and going into Ω.

Proof: Since u_θ and ψ are harmonic conjugates, the normal derivative of ψ at P equals the tangential derivative of u_θ at P which is zero (since $u_\theta = 0$ on Γ). By the boundary point lemma, ψ cannot attain a local extremum on Γ, hence every point of Γ is the start of a level curve of ψ.

To prove the second assertion of the lemma, assume that $\psi(P)$ is a strict local minimum of $\psi|_\Gamma$. Since $\psi(P)$ cannot be a local minimum of ψ in any $B_r(P) \cap \Omega$, it follows that there is a region $Q = \{\psi < \psi(P)\}$ which contains P in its closure. But ∂Q contains no points of Γ except for P in some neighborhood of P, we conclude that there are at least two curves $\{\psi = \psi(P)\}$ initiating at P and going into Ω, as desired.

Now, suppose that the origin 0 for polar coordinates is placed outside of Ω'. Then I introduce the following notation.

$$\Sigma_1 = \{X \in \Sigma \mid u_r(X) > 0\}$$

$$\Sigma_2 = \{X \in \Sigma \mid u_r(X) < 0\},$$

$$\Sigma^+ = \{X \in \Sigma \mid u_\theta(X) > 0\}$$

$$\Sigma^- = \{X \in \Sigma \mid u_\theta(X) < 0\}.$$

In addition, Σ_1^+, Σ_1^-, etc., are intersections of the appropriate above sets.

Lemma 5.3: There is precisely one point on Σ_1 at which $u_\theta = 0$, and this is the closest point of Σ to 0.

Proof: Suppose $Y \in \Sigma$ satisfies $u_\theta(Y) = 0$, $u_r(Y) > 0$. Then the tangent ℓ to Σ at Y is perpendicular to the line $0Y$, and Ω' lies to one side of ℓ. Since $u_r(Y) > 0$, Ω' lies on the far side of ℓ from 0. It is clear then, that Y is the unique closest point of Σ to 0.

Hence we know that Σ_1 is divided into two segments, Σ_1^+ and Σ_1^-, and a point Σ_1^0 where $u_\theta = 0$.

Definition: The ridge R of Ω is the set of all points $X_o \in \Omega$ such that $d(X) \equiv \text{dist}(X,\partial\Omega)$ is not in $C^{1,1}(V)$ for any neighborhood V of X.

Let $R_o = \{X_o \in \Omega \mid d(X_o) = |X_o - Y| = |X_o - Z|$ for two distinct $Y, Z \in \Gamma\}$, and $R_1 = \{X_o \in \Omega \mid$ there exists precisely one point $Y \in \Gamma$ with $d(X_o) = |X_o - Y|$ and X_o is the center of the osculating circle at $Y\}$. Then $R = R_o \cup R_1$ and, since Ω is convex, $R = \overline{R}_o$ (Friedman [2] Chapter 2, Section 7).

Theorem 5.4 If $X_o \in R_o$ and Ω' is convex, then $\text{dist}(X_o,\Gamma) > \text{dist}(X_o,\Sigma)$. In consequence, if $X_o \in R$, then $\text{dist}(X_o,\Gamma) > \text{dist}(X_o,\Sigma)$.

Proof: Suppose this is not the case, and let X_o be the polar origin. Let P_1 and P_2 be points of Γ such that $d(X_o) = |X_o - P_1| = |X_o - P_2| = t$. Since $u_r \equiv 0$ on Γ, $\psi|_\Gamma$ has a local minimum at P_1 and P_2. Therefore, from Lemma 5.2, there are level curves γ_1^+, γ_1^- starting at P_1, and γ_2^+, γ_2^- starting at P_2, on which $\psi \equiv t^2/2$. As γ_1^+ and γ_2^+ are traversed in the direction away from Γ, u_θ increases, and as γ_1^- and γ_2^- are traversed in this direction, u_θ decreases. Since by assumption, the distance from X_o to each point of Σ is greater than t, all of the level curves $\gamma_1^\pm$, $\gamma_2^\pm$ must terminate on Σ_1. Indeed, γ_1^+ and γ_2^+ must end on Σ_1^+, and γ_1^- and γ_2^- must end on Σ_1^-.

Let $U \subset \Omega - \bar{\Omega}'$ be the open set whose boundary consists of γ_1^+, the segment of Σ_1^+ between the endpoints of γ_1^+ and γ_2^+, γ_2^+, and Γ from P_1 to P_2. Here the condition that $U \subset \Omega - \bar{\Omega}'$ forces the direction that Γ is traversed from P_1 to P_2 for $\partial\Omega$. Let $U^+ = \{X \in U \mid \psi(X) > t^2/2\}$ and $U^- = \{X \in U \mid \psi(X) < t^2/2\}$. Then neither U^+ nor U^- is empty, and either $\gamma_1^+ \subset \partial U^+$ and $\gamma_2^+ \subset \partial U^-$ or vice versa. For if this were not the case, then Lemma 5.1 would be violated, since γ_1 and γ_2 both have a region where $\psi > t^2/2$ lying to their right as they are traversed from Γ to Σ.

We are now led to a contradiction, since there must then be a curve $\gamma^* \subset U$ on which $\psi \equiv t^2/2$ which goes from Σ^+ to Γ to separate U^+ and U^-. Then u_θ will be increasing along γ^* from Σ^+ to Γ, violating the free boundary condition.

As a corollary, we get a rough bound on the curvature of Γ.

<u>Corollary 5.5</u>: Assume that Ω' is convex, and Σ is $C^{2+\alpha}$, and let $\kappa(\Gamma_c)$ be the maximum of the curvature of Γ_c, and $\kappa(\Sigma)$ be the maximum of the curvature of Σ. Then

$$\frac{2}{\gamma(c,1/_{\kappa(\Sigma)}) - 1/_{\kappa(\Sigma)}} > \kappa(\Gamma_c)$$

<u>Proof</u>: At each point X of Σ we may place a circle of radius $1/\kappa(\Sigma)$ contained in Ω' and tangent to Σ at X. From the proof of Theorem 3.6, we know that at each point $X \in \Sigma$ the distance from X to Γ is at least $\gamma(c,1/_{\kappa(\Sigma)}) - 1/_{\kappa(\Sigma)}$. Now consider the ball $B_{1/\kappa(\Gamma)}$ of radius $1/\kappa(\Gamma)$ osculating at the point of greatest curvature of Γ. From Theorem 5.4, $B_{1/\kappa(\Gamma)}$ must contain a point of Σ, hence

$$\frac{2}{\kappa(\Gamma)} > \gamma(c,1/\kappa(\Sigma)) - 1/\kappa(\Sigma),$$

yielding the desired result.

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A free-boundary problem which arises from a galvanizing process is studied. The physical problem is that of an infinite cylinder $\Omega' \times R$ withdrawn from a fluid bath. Formally, this is a gravity-driven unidirectional viscous fluid flow on the exterior of the cylinder $\Omega' \times R$. The existence of a unique classical solution is shown under certain conditions on Ω', and asymptotic results for the thickness of the coat are obtained for large and small withdrawal speeds. If Ω' is a convex set, then the region

ABSTRACT (Continued)

bounded by the free surface of the fluid is shown to be convex, using level curve techniques. Finally, level curve techniques are used to bound the curvature of the free boundary in terms of that of the fixed boundary.

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