DAMTP-2006-52 MIFP-06-16
hep-th/0606186

Partition Functions, the Bekenstein Bound and Temperature Inversion in Anti-de Sitter Space and its Conformal Boundary

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ABSTRACT

We reformulate the Bekenstein bound as the requirement of positivity of the Helmholtz free energy at the minimum value of the function $L = E - S/(2\pi R)$, where R is some measure of the size of the system. The minimum of L occurs at the temperature $T = 1/(2\pi R)$. In the case of n-dimensional anti-de Sitter spacetime, the rather poorly defined size R acquires a precise definition in terms of the AdS radius l, with $R = l/(n-2)$. We previously found that the Bekenstein bound holds for all known black holes in AdS. However, in this paper we show that the Bekenstein bound is not generally valid for free quantum fields in AdS, even if one includes the Casimir energy. Some other aspects of thermodynamics in anti-de Sitter spacetime are briefly touched upon.

[‡] Research supported in part by DOE grant DE-FG03-95ER40917 and NSF grant INTO3-24081.

Contents

1 Introduction

In order to extend to the concept of thermodynamic entropy gravitating systems, one must include an extra contribution

$$S_{\rm grav} = \frac{1}{4G} A \,, \tag{1.1}$$

where G is Newton's constant, and A is the sum of the areas of any event horizons within the system. The generalised second law of thermodynamics asserts that the new total entropy cannot decrease. Some years ago Bekenstein conjectured [1] that in order for the generalised second law to hold, the matter contribution to the entropy should satisfy what has come to be called the Bekenstein bound,

$$S \le 2\pi R E \,, \tag{1.2}$$

where S is the entropy, E the energy, and R the "size" of a bounded material system capable of being lowered into a black hole.

The necessity of (1.2) for the validity of the generalised second law was soon called into question [2]. Nevertheless, Bekenstein's proposed bound has attracted widespread attention over the succeeding years, not least because of the remarkable feature that it does not depend upon Newton's constant, and hence it can be construed as a very general property of all forms of matter, even in the absence of gravity.

Another remarkable feature of Bekenstein's suggested bound, which has not been noticed hitherto, is the existence of a simple, universal, criterion for the validity of the bound. It follows solely from the laws of classical thermodynamics. Let

$$L \equiv E - \frac{S}{2\pi\, R} = F + \Big(T - \frac{1}{2\pi\, R}\Big)\, S \,, \tag{1.3}$$

where $F = E - TS$ is the free energy, so that the Bekenstein bound (1.2) is equivalent to the statement that $L \ge 0$. Since $S = -\partial F/\partial T$, it follows that L is extremised if

$$\begin{aligned} 0 = \frac{\partial L}{\partial T} &= \frac{\partial F}{\partial T} + S + \Big(T - \frac{1}{2\pi\, R}\Big)\, \frac{\partial S}{\partial T} \\ &= \Big(T - \frac{1}{2\pi\, R}\Big)\, \frac{\partial S}{\partial T} \,, \end{aligned} \tag{1.4}$$

and hence at the temperature

$$T = T_L \equiv \frac{1}{2\pi\, R} \,. \tag{1.5}$$

The second derivative of L is given by

$$\frac{\partial^2 L}{\partial T^2} = \frac{\partial S}{\partial T} + (T - T_L) \frac{\partial^2 S}{\partial T^2} \,. \tag{1.6}$$

Thus, provided that the specific heat $T\,\partial S/\partial T$ is positive, it follows that there is a unique extremum, and that it is a minimum. From (1.3), the corresponding minimum value of L is

$$L_{\rm min} = F(T_L)\,, \tag{1.7}$$

i.e. the free energy evaluated at the temperature T_L given by (1.5). Thus the Bekenstein bound holds for all temperatures if $F(T_L)$ is non-negative, but it is violated, for some range of temperatures around $T = T_L$, if $F(T_L)$ is negative.

Suppose that the bound is *marginally* satisfied, i.e. the minimum value of L is zero. Then, at that value, the free energy F vanishes. At this point, the free energy of the system under consideration equals that of the vacuum, and so the system can undergo a phase transition to the vacuum. Clearly, this phenomenon is universal; an example, to be discussed below, being the AdS black hole and its associated Hawking-Page phase transition.

Later in this paper, we shall use the argument above as a criterion for testing the Bekenstein bound.

Since the original statement of the Bekenstein conjecture is rather vague, a number of questions have to be addressed before its correctness can be checked. These include

- What is meant by the entropy S, and what is the ensemble being used? Is it the microcanonical ensemble, and the Boltzmann entropy $S_B = \log N(E)$, where $N(E)$ is the number of states having energy less than or equal to E?[1] Alternatively, is it the Gibbs entropy $S_G = -{\rm Tr}\rho \log \rho$, where ρ is the normalised density matrix of some ensemble, such as the grand canonical ensemble? (Note that these definitions of entropy are not necessarily equivalent. See, for example, [3].)

- What is meant by the energy E? Does it contain a contribution from the zero-point energies of the fields, i.e. the Casimir energy? Does it contain a contribution from the box or cavity walls containing the matter? Should the stress tensor of the walls of the box therefore satisfy any restrictions, such as the dominant energy condition?

- How is the radius R defined? What boundary conditions are to be imposed on the fields at the boundary of the system? Are these boundary conditions, which typically entail divergences, consistent with some renormalisation scheme ?

[1]We prefer to use the cumulative definition of $N(E)$, which is better behaved, rather than the often-adopted number of states of energy precisely E, which jumps rather erratically with E, especially at small E where, as is well known, the Bekenstein bound is most vulnerable.

- Should the total number of fields or species be limited so as to avoid the so-called "species problem?" This problem arises if one uses the microcanonical ensemble, owing to the fact that passing to N identical replicas of the original system, the Boltzmann entropy increases as

$$S_B \rightarrow S_B + \log N \,, \tag{1.8}$$

and so in principle the left-hand side of (1.2) could be made arbitrarily large by taking N sufficiently large. In fact the species problem does not arise for the Gibbs entropy of the canonical or grand canonical ensembles. If Z is the partition function for a system with one species of particle, then Z^N is the partition function for the same system with N species. Since

$$\begin{aligned} E &= -\frac{\partial \log Z}{\partial \beta} \longrightarrow NE \,, \\ S_G &= (1 - \beta \frac{\partial}{\partial \beta}) \log Z \longrightarrow NS_G \,, \end{aligned} \tag{1.9}$$

with $\beta = T^{-1}$ being the inverse temperature, the right-hand side and left-hand side of (1.2) scale in an identical fashion[2]. The correctness of the scalings (1.9) is easily checked for a radiation gas of non-interacting particles.

- Is it reasonable at all to use thermodynamic concepts at the very low temperatures at which the Bekenstein bound is most vulnerable? Usually one argues that if $T < \tau^{-1}$, where τ is a typical relaxation time, which in a finite size cavity cannot be less than a light crossing time, thermalisation is not possible. This seems to be an argument in favour of using the microcanonical ensemble, where one is essentially just counting states, à la Boltzmann.

Many of the difficulties raised above may be avoided if one avoids the technically-demanding and possibly ill-defined, situation of a quantum field theory in a sharply localised spatially-bounded region in flat Minkowski spacetime.[3] For example, one could consider instead a static spacetime with closed spatial sections, such as the Einstein Static Universe $\mathrm{ESU}_n \equiv \mathbb{R} \times S^{n-1}$, for which there is no spatial boundary, and hence no need for boundary

[2]Note that the often-repeated statement that the micro-canonical and canonical ensembles give the same answers at large energies, or equivalently high temperatures, is not strictly speaking correct in the case of entropy, since the equivalence ignores sub-leading quantities such as $\log N$

[3]It is notorious that the sharp confinement of quantum fields leads to many additional divergences, beyond those one encounters in scattering calculations in infinite space.

conditions. Alternatively, one could consider a spacetime with spatially non-compact sections, such as those of anti-de Sitter spacetime AdS_n, for which the gravitational redshift effect is sufficient to confine a finite-energy system at non-zero temperature [4, 5]. In this case the boundary conditions are uncontroversial [6, 7], and this makes the calculation of zero-point energies straightforward [8,9]. In this respect, as in so many others, it seems that anti-de Sitter spacetime provides the theorist's perfect adiathermic box, even to the extent that it permits an infinite uniformly-rotating platform, and moreover it confines gravitons. Of course the desire to eliminate the need for boundary conditions was one of the principal reasons for Einstein's introduction of the cosmological constant, and his adoption of ESU_4 as a cosmological model. Dowker [10] has made a similar point about having no boundary. He examined the case of squashed 3-spheres, and found that for sufficiently large squashing the zero-point contribution for fermions can be made arbitrarily negative. Thus, rather remarkably, there exist temperatures for which the negative zero-point energy can overwhelm the positive thermal energy, possibly leading to a divergence of the ratio S/E, but also rendering its sign negative, in gross contradiction to the Bekenstein conjecture (1.2).

The Einstein Static Universe and anti-de Sitter spacetime are of course related, in that $\mathrm{ESU}_{n-1} = \partial\mathrm{AdS}_n$, where ∂ denotes the conformal boundary. Motivated by the AdS/CFT correspondence, there has been some recent discussion [11–15] of the validity of Bekenstein's bound for free conformal field theories on $\mathbb{R} \times S^{n-2}$. Surprisingly, there has been little work or progress on the same problem in the bulk AdS_n spacetime. In a recent paper [16], evidence was presented for the general validity of an AdS Bekenstein bound of the form

$$S < \frac{2\pi E l}{n-2}\,. \tag{1.10}$$

for all known asymptotically anti-de Sitter rotating charged black holes, where l is the radius of curvature of AdS_n. It was pointed out that the AdS Bekenstein bound is a consequence of the much deeper and more fundamental conjectured Cosmic Censorship Bound for the area A of an apparent horizon,

$$E \geq \frac{(n-2)A}{16\pi G\, l}\left[l\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{-\frac{1}{n-2}} + \frac{1}{l}\left(\frac{A}{\mathcal{A}_{n-2}}\right)^{\frac{1}{n-2}}\right]. \tag{1.11}$$

Interestingly however, the bound (1.10) still does not contain Newton's constant, and so it, like (1.2), may be construed as a statement about quantum field theory, or perhaps string theory, in a fixed background, namely AdS_n. Note that by contrast with (1.2), there is no ambiguity in (1.10) about the length scale entering the bound. It is thus a well-defined question to ask whether (1.10) is always satisfied for quantum fields, conformal or not, in

AdS_n. The main purpose of the paper is to investigate this question, at the level of free fields.

In section 2, we briefly describe the relationships between one-particle and many-particle partition functions, both for bosons and fermions, and we develop some formulae allowing us to calculate thermodynamic quantities using zeta functions. In section 3, we calculate the energy and entropy for free fields, particularly those falling into supergravity multiplets, in anti-de Sitter backgrounds in four, five and seven dimensions. We find cases where the Bekenstein bound is violated, even if the contribution of the Casimir energy is included. Section 4 contains a brief discussion, inspired by recent work on large-N Yang-Mills theory, of the novel statistics that arise when the fields are given by infinite-dimensional matrices [17–19]. We find that the Bekenstein bound can be violated in this case too. The novel statistics give rise to a maximum Hagedorn-type temperature, which we calculate. In section 5, we discuss partition functions for conformally-invariant fields on $\mathbb{R}\times S^{n-2}$, or, equivalently, AdS_n. All correlation functions exhibit periodicity both in imaginary time, as a consequence of the non-vanishing temperature, and in real time because of the periodicity of AdS_n. Such doubly-periodic functions, provided they have an appropriate analytic structure (e.g. they are meromorphic), may be expressed in terms of elliptic functions. If n is odd, the free Green functions have only poles, and in consequence the partition functions have modular properties under $SL(2,\mathbb{Z})$ transformations of the temperature. If n is even, the Green functions have branch points even for conformally-invariant fields, and the above arguments fail. At the end of the paper, there is an appendix describing how one can invert the process of constructing multi-particle partition functions from single-particle partition functions, by making use of the Möbius function and a fermionic generalisation.

2 Canonical and Grand Canonical Partition Functions

Suppose that the modes of a free quantum field in a stationary, axisymmetric spacetime background $\mathcal{M}$ are discrete, and have energies E and angular momentum projections j. One may define a one-particle partition function $Y(\beta,\Omega)$ depending upon the temperature $T=\beta^{-1}$ and chemical potential Ω for the angular momentum by

$$Y(\beta,\Omega)=\sum_{E,j}e^{-(\beta E+\alpha j)}, \tag{2.1}$$

where

$$\alpha=-\beta\Omega. \tag{2.2}$$

One may rewrite (2.1) as

$$Y(\beta,\Omega) = \mathrm{Tr}_{\mathcal{H}_1} e^{-\beta(\hat{H}-\Omega\hat{j})}, \tag{2.3}$$

where $\hat{H} - \Omega\hat{j}$ is the quantum mechanical operator corresponding to the rigidly-rotating Killing field

$$\mathbf{K} = \frac{\partial}{\partial t} - \Omega\frac{\partial}{\partial\phi}. \tag{2.4}$$

One may regard (2.3) as an analytic continuation of a character of the representation of $e^{-it(\hat{H}-\Omega\hat{j})}$ in the one-particle or "first quantised" Hilbert space $\mathcal{H}_1$ at an imaginary time $t = -i\beta$. In computations it is frequently more convenient to introduce $x^2 = e^{-\beta}$ and $y^2 = e^{-\alpha}$ and re-write (2.1) as

$$Y(\beta,\Omega) = \sum_{E,j} x^{2E} y^{2j}\,, \tag{2.5}$$

Note that x is always less than one but y may be less than or greater than one, depending upon the sign of Ω. The multi-particle partition function is given by

$$Z(\beta,\Omega) = \mathrm{Tr}_{\mathcal{H}} e^{-\beta(\hat{H}-\Omega\hat{j})}\,, \tag{2.6}$$

where the trace is taken over the full "second quantised" Hilbert space of multi-particle states. For a system of bosons, the multi-particle partition function $Z_B(\beta,\Omega)$ is given by

$$Z_B(\beta,\Omega) = \prod_{E,j} \frac{1}{(1 - e^{-\beta(E-\Omega j)})}\,. \tag{2.7}$$

The thermodynamic potential $\Phi(\beta,\Omega)$ is given by by

$$-\beta\Phi(\beta,\Omega) = \log Z_B(\beta,\Omega) = \sum_n \frac{1}{n} Y(n\beta,\Omega)\,, \tag{2.8}$$

where the summation arises from expanding the logarithm in a Taylor series.

For a system of fermions the multi-particle partition function $Z_F(\beta,\Omega)$ is given by

$$Z_F(\beta,\Omega) = \prod_{E,j} \left(1 + e^{-\beta(E-\Omega j)}\right). \tag{2.9}$$

The thermodynamic potential $\Phi(\beta,\Omega)$ is given by

$$-\beta\Phi(\beta,\Omega) = \log Z_F(\beta,\Omega) = -\sum_n \frac{(-1)^n}{n} Y(n\beta,\Omega)\,. \tag{2.10}$$

It is clear that all of the information about the thermodynamics of a free quantum field theory in a background spacetime is given by the one-particle partition function. (See appendix A for a discussion of how, conversely, the one-particle partition function can be recovered from the multi-particle partition function.)

2.1 The Hamiltonian zeta function

Even though all the relevant spectral information is encoded into $Y(\beta)$), it is often more convenient to encode it into a zeta function. Thus let the Hamiltonian zeta function be defined by

$$\zeta_H(s) = \sum_n d_n E_n^{-s} = \mathrm{Tr}_{\mathcal{H}_1} H^{-s}\,, \tag{2.11}$$

where H is the Hamiltonian acting on the one-particle Hilbert space $\mathcal{H}_1$, E_n are the energies and d_n the degeneracies of the states of the system. Note that the sum (2.11) is convergent provided that $\mathfrak{Re}(s)$ is strictly greater than the spatial dimension N. The function $\zeta_H(s)$ can be analytically continued to a meromorphic function in the entire complex plane, with poles only at $s = 1, 2, \ldots, N$.

The Hamiltonian zeta function $\zeta_H(s)$ may be obtained from the one-particle partition function $Y(\beta)$ by a Mellin transform

$$\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty \beta^{s-1} Y(\beta) d\beta\,. \tag{2.12}$$

Formally

$$\sum_n d_n E_n = \zeta_H(-1)\,, \tag{2.13}$$

and so a convenient definition of the Casimir energy E_c is

$$E_c = \tfrac{1}{2}(-1)^F \zeta_H(-1)\,, \tag{2.14}$$

where F is 0 for bosons and 1 for fermions.

Using the identity

$$e^{-\beta} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \beta^{-s}\,\Gamma(s) ds\,, \tag{2.15}$$

where $c > 0$, one finds that the "blind" grand canonical partition function (i.e. with $\Omega = 0$) for bosons may be expressed as

$$\log Z_B(\beta) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \beta^{-s}\,\Gamma(s)\zeta(s+1)\zeta_H(s) ds\,, \tag{2.16}$$

where $\zeta(s)$ is the Riemann zeta function, while for fermions

$$\log Z_F(\beta) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \beta^{-s}\,\Gamma(s)(1-2^{-s})\zeta(s+1)\zeta_H(s) ds\,. \tag{2.17}$$

In these formulae one must take γ to be greater than the spatial dimension N, so that the order of integration and summations may be freely interchanged.

From $E = -\partial \log Z/\partial\beta$, it follows that for bosons

$$E_B = \sum_n \frac{d_n E_n}{e^{\beta E_n} - 1} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \beta^{-s}\,\Gamma(s)\zeta(s)\zeta_H(s-1)ds\,, \tag{2.18}$$

while for fermions

$$E_F = \sum_n \frac{d_n E_n}{e^{\beta E_n} + 1} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \beta^{-s}\,\Gamma(s)(1-2^{1-s})\zeta(s)\zeta_H(s-1)ds\,. \tag{2.19}$$

In deriving these one uses $s\Gamma(s) = \Gamma(s+1)$, and then changes variable according to $s \to s-1$. This implies that γ must now be taken to be greater than $N+1$.

3 Entropy and Energy in Bulk AdS Spacetimes

Fields in anti-de Sitter spacetime are taken to satisfy reflecting boundary conditions at infinity. An n-dimensional AdS spacetime, satisfying $R_{\mu\nu} = -(n-1)l^{-2}g_{\mu\nu}$, thus provides a perfect realisation of a closed spherical adiathermic box of radius l, which can be used in order to study the thermodynamics of isolated closed systems. In particular, it allows one to give meaning to the otherwise somewhat ill-defined notion of a thermodynamic system of radius l that is called for in the formulation of the Bekenstein bound, which asserts that the energy of such a system is bounded by

$$E \geq \frac{(n-2)S}{2\pi l}\,, \tag{3.1}$$

where S is the entropy.

In this section, we calculate the partition functions for free fields in anti-de Sitter spacetimes, and use these to study the Bekenstein bound in the idealised AdS_n "laboratory." Before doing this, we begin with a discussion of the high-temperature limit.

3.1 High-temperature expansions and Tolman redshifting

At high temperature, free quantum fields in AdS_n behave like a radiation gas whose local temperature redshifts according to Tolman's well-known formula [20]

$$T_{\rm local}\sqrt{-g_{00}} = \frac{1}{\beta}\,, \tag{3.2}$$

where $T = \beta^{-1}$ is the global temperature as measured by an observer at rest at the origin. When allowance is made for the redshifting, the total energy is given by

$$E = \sigma\,\beta^{-n}\int (-g_{00})^{(n-1)/2}\sqrt{\det(g_{ij})}\,d^{n-1}x = \sigma\,\beta^{-n}\,V_{\rm eff}\,, \tag{3.3}$$

where

$$V_{\rm eff} \equiv \int (-g_{00})^{(n-1)/2}\, \sqrt{\det(g_{ij})}\, d^{n-1}x\,, \tag{3.4}$$

and g_{ij} denotes the spatial $(n-1)$-metric (i.e. the AdS$_n$ metric $g_{\mu\nu}$ with its indices restricted to the spatial directions). As pointed out by Hawking and Page [4], the effective volume $V_{\rm eff}$ is finite, and in fact given by $V_{\rm eff} = \frac{1}{2}\mathcal{A}_{n-1}$, where $\mathcal{A}_m$ is the volume of the unit m-sphere. The quantity σ is the generalisation of the Stefan-Boltzmann constant, and is given by

$$\begin{aligned} \text{Bosons}: &\qquad \sigma = (2\pi)^{1-n}\, \mathcal{A}_{n-2}\, \zeta(n)\, (n-1)!\,, \\ \text{Fermions}: &\qquad \sigma = (2\pi)^{1-n}\, \mathcal{A}_{n-2}\, (1-2^{-n+1})\, \zeta(n)\, (n-1)!\,. \end{aligned} \tag{3.5}$$

At high temperature, the entropy S and free energy F are related to the total energy E by

$$S \longrightarrow \frac{n}{n-1}\, \frac{E}{T}\,, \qquad F \longrightarrow \frac{1}{n-1}\, E\,. \tag{3.6}$$

Thus

$$\log Z \longrightarrow \frac{\sigma\, V_{\rm eff}}{(n-1)\, \beta^{n-1}}\,. \tag{3.7}$$

This same result may be derived microscopically either by considering the point-split non-zero-temperature Green function [5], or, more economically, by noting that at high temperature $Y(\beta) \sim Y_0/\beta^{n-1}$, and $\log Z \sim \zeta(n-1)\, Y_0/\beta^{n-1}$ for bosons, or $\log Z \sim (1-2^{1-n})\zeta(n-1)\, Y_0/\beta^{n-1}$ for fermions. The coefficient Y_0 determines the density of states at high energy in a cavity of effective volume $V_{\rm eff}$, and is well known to be universal, independent of the shape of the cavity. One readily checks that this agrees with the radiation-gas approximation.

3.2 Entropy bound in AdS$_4$

3.2.1 AdS$_4$ Partition functions in the canonical ensemble

Massless fields in AdS$_4$ are characterised by certain unitary irreducible representations $D(E_0, s)$ of $SO(2,3)$, where E_0 is the lowest energy and s is the spin. In general, for spin $s \ge \frac{1}{2}$, the massless representations correspond to taking $E_0 = s+1$. For massless conformally-invariant scalars there are two representations, namely $D(1,0)$ and $D(2,0)$. If we normalise the scale by taking $l=1$, then the energies $E_{n,j}$ and degeneracies $d_{n,j}$ are given by

$$E_{n,j} = n + j + 1\,, \qquad d_{n,j} = 2j+1\,, \qquad \text{where} \qquad n \ge 0\,, \qquad j \ge s\,, \tag{3.8}$$

with n and j increasing in integer steps.

From these expressions, and taking the angular velocity to be zero for now, one finds that the single-particle partition functions $Y_{(E_0,s)}(\beta)$ for free fields in the $D(E_0,s)$ representation are as follows:

$$
\begin{aligned}
Y_{(1,0)} &= \frac{e^{2\beta}}{(e^\beta-1)^3}\,, \qquad Y_{(2,0)} = \frac{e^{\beta}}{(e^\beta-1)^3}\,,\\
Y_{(s+1,s)} &= \frac{e^{(1-s)\beta}\,[(2s+1)e^\beta+1-2s]}{(e^\beta-1)^3}\,, \qquad s=\tfrac12,1,\tfrac32,2,\ldots
\end{aligned}
\tag{3.9}
$$

We can also consider the singleton "Di" and "Rac" representations $D(1,\frac12)$ and $D(\frac12,0)$ respectively, for which one finds the single-particle partition functions

$$
Y_{(1,\frac12)}(\beta) = \frac{2}{(e^\beta-1)^2}\,, \qquad Y_{(\frac12,0)}(\beta) = \frac{e^{\frac12\beta}\,(e^\beta+1)}{(e^\beta-1)^2}\,. \tag{3.10}
$$

The multi-particle partition functions are then calculated using the expressions (A.1) and (A.2) given in appendix A. One then has the expressions

$$
E(\beta) = -\frac{\partial}{\partial\beta}\log Z\,, \qquad S(\beta) = \log Z - \beta\,\frac{\partial}{\partial\beta}\log Z \tag{3.11}
$$

for the free energy and the entropy of the system.

The Bekenstein bound (3.1), applied to the case of AdS_4 with $l=1$, asserts that

$$
L(\beta) \equiv E(\beta) - \frac{S(\beta)}{\pi} \ge 0\,. \tag{3.12}
$$

As we showed in the introduction, and can be seen also from (3.11), $L(\beta)$ attains its minimum value when $\beta=\pi$, implying

$$
L_{\rm min} = L(\pi) = -\frac{1}{\pi}\,\log Z\,, \tag{3.13}
$$

and so it is here, at temperature $T=1/\beta=1/\pi$ that one obtains the most stringent test of the validity of the Bekenstein bound.

Whilst the Bekenstein bound is clearly satisfied in the high-temperature regime, where $E\sim T^4$ and $S\sim T^3$, it is not so easy to check the bound analytically at $T=1/\pi$. However, it is straightforward to perform the summations in (A.1) and (A.2) numerically to the required degree of accuracy. Some explicit results for fields in the $D(E_0,s)$ representations are as follows:

$$
\begin{aligned}
L_{\rm min}(1,0) &= -0.0160124\,, \qquad L_{\rm min}(2,0) = -0.00067922\,,\\
L_{\rm min}(2,1) &= -0.0020083\,, \qquad L_{\rm min}(3,2) = -0.000142841\,,\\
L_{\rm min}(\tfrac32,\tfrac12) &= -0.00650369\,, \qquad L_{\rm min}(\tfrac52,\tfrac32) = -0.000552029\,.
\end{aligned}
\tag{3.14}
$$

Additionally, for the Di and Rac singletons we find

$$L_{\rm min}(1,\tfrac{1}{2}) = -0.029472\,, \qquad L_{\rm min}(\tfrac{1}{2},0) = -0.0834542\,. \tag{3.15}$$

As can be seen, they are all negative, which would be in contradiction to the Bekenstein bound. A representative plot of $L(\beta)$ for a scalar in the $D(2,0)$ representation is given in Figure 1 below.

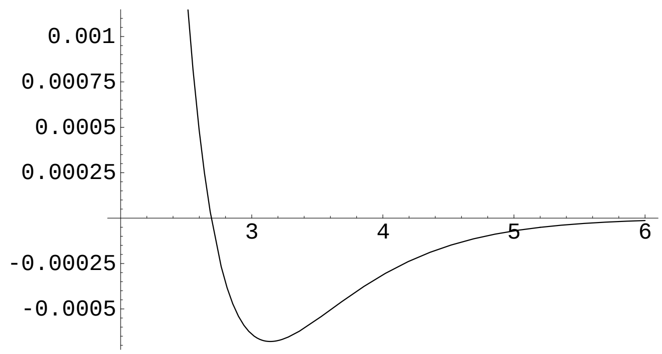

Figure 1: *A plot of $L = E - S/\pi$ for a scalar field in the $D(2,0)$ representation of $SO(3,2)$ in AdS_4, as a function of $\beta = 1/T$. The function attains its minimum at $\beta = 1/T = \pi$. It is positive at high temperature, but it is negative at sufficiently low temperature.*

The situation changes somewhat if we include the Casimir energies in the calculations. Evaluating these using zeta-function regularisation, as in (2.14), one finds the additional contributions

$$\begin{aligned} E_c(1,0) &= \tfrac{1}{480}\,, \qquad E_c(2,0) = \tfrac{1}{480}\,, \\ E_c(2,1) &= \tfrac{11}{240}\,, \qquad E_c(3,2) = \tfrac{401}{240}\,, \\ E_c(\tfrac{3}{2},\tfrac{1}{2}) &= \tfrac{17}{1920}\,, \qquad E_c(\tfrac{5}{2},\tfrac{3}{2}) = -\tfrac{863}{1920}\,. \end{aligned} \tag{3.16}$$

For the singletons, we have

$$E_c(1,\tfrac{1}{2}) = 0\,, \qquad E_c(\tfrac{1}{2},0) = 0\,. \tag{3.17}$$

With these included, the results for $f_{\rm min}$ in (3.14) are replaced by $\tilde f_{\rm min}$, given by

$$\begin{aligned} \tilde L_{\rm min}(1,0) &= -0.0139291\,, \qquad \tilde L_{\rm min}(2,0) = +0.00140411\,, \\ \tilde L_{\rm min}(2,1) &= +0.043825\,, \qquad \tilde L_{\rm min}(3,2) = +1.67069\,, \\ \tilde L_{\rm min}(\tfrac{3}{2},\tfrac{1}{2}) &= +0.00235048\,, \qquad L_{\rm min}(\tfrac{5}{2},\tfrac{3}{2}) = -0.450031\,, \end{aligned} \tag{3.18}$$

with the minima for the Di and Rac singletons unchanged from (3.15). Although in some cases the Casimir energy reverses the sign, in others the violation of the Bekenstein bound persists.

3.2.2 Entropy bounds in the microcanonical ensemble

One can also study the Bekenstein bound in the microcanonical ensemble, where the energy is held fixed. To do this, one takes an inverse Laplace transform of the expression

$$Y(\beta) = \int_0^\infty \rho(E)\, e^{-\beta E} dE \tag{3.19}$$

for the single-particle partition function, in order to obtain the expression for the density of states $\rho(E)$. From the expressions in (3.9), we therefore obtain

$$\begin{aligned} \rho_{(1,0)}(E) &= \tfrac{1}{2}\sum_{n\geq 1} \delta(E-n)\, n(n+1)\,, \\ \rho_{(2,0)}(E) &= \tfrac{1}{2}\sum_{n\geq 2} \delta(E-n)\, n(n-1)\,, \\ \rho_{(s+1,s)}(E) &= \sum_{n\geq s} \delta(E-n)\, (n^2-s^2)\,, \qquad s = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\,. \end{aligned} \tag{3.20}$$

For the Di and Rac singletons we find from (3.10)

$$\rho_{(1,\frac{1}{2})}(E) = \sum_{n\geq 1} \delta(E-n)\, n\,, \qquad \rho_{(\frac{1}{2},0)}(E) = \sum_{n\geq \frac{1}{2}} \delta(E-n)\, n\,. \tag{3.21}$$

From these expressions, one can integrate to obtain the total number of states $N(E)$ with energies less than or equal to E. Thus one has

$$\begin{aligned} N_{(1,0)}(E) &= \tfrac{1}{2}\sum_{n=1}^{E} n(n+1) = \tfrac{1}{6}E(E+1)(E+2)\,, \\ N_{(2,0)}(E) &= \tfrac{1}{2}\sum_{n=2}^{E} n(n-1) = \tfrac{1}{6}E(E^2-1)\,, \\ N_{(s+1,s)}(E) &= \sum_{n=s}^{E}(n^2-s^2) = \tfrac{1}{6}(E-s)(E+1-s)(2E+4s+1)\,, \quad s = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\,, \end{aligned} \tag{3.22}$$

and for the singletons

$$N_{(1,\frac{1}{2})}(E) = \sum_{n=1}^{E} n = \tfrac{1}{2}E(E+1)\,, \qquad N_{(\frac{1}{2},0)}(E) = \sum_{n=\frac{1}{2}}^{E} n = \tfrac{1}{2}(E+\tfrac{1}{2})^2\,. \tag{3.23}$$

At large E, the entropy in the microcanonical ensemble, which is defined as $S = \log N(E)$, is of the form

$$S \sim 3 \log E \tag{3.24}$$

for all the fields in (3.22). Thus the ratio $S/(\pi E)$ at large E approaches $3/(\pi E)\,\log E$, which tends to zero, implying that the Bekenstein bound is satisfied in this regime. Similarly, for the singletons, we have $S \sim 2\log E$, and again the Bekenstein bound is satisfied at large E.

3.3 Partition functions and entropy bounds in AdS$_5$

With the angular velocities set to zero, the resulting "blind" single-particle partition functions for the cases of (j_1, j_2) and $(j_1, 0)$ representations of the little group $SO(4)$ may be found in [21]. These are:

$$\begin{aligned}(j_1, j_2): \qquad & Y = \frac{s^{j_1+j_2+2}}{(1-s)^4}\,[(2j_1+1)(2j_2+1) - 4s j_1 j_2]\,,\\ (j_1, 0): \qquad & Y = \frac{s^{j_1+1}}{(1-s)^3}\,[2j_1 + 1 - s(2j_1-1)]\,,\end{aligned} \tag{3.25}$$

where $s \equiv e^{-\beta}$. The first line is for massless representations, and the second is for doubletons. By taking the inverse Laplace transform, we find the corresponding densities of states:

$$\begin{aligned}(j_1, j_2): \qquad & \rho(E) = \tfrac{1}{6}\sum_{m\ge j_1+j_2+2} \delta(E-m)\,(m-j_1-j_2-1)(m-j_1-j_2)\times\\ & \qquad\qquad [m(1+2j_1+2j_2) + 1 + j_1 + j_2 - 2j_1^2 - 2j_2^2 + 8j_1 j_2]\,,\\ (j_1, 0): \qquad & \rho(E) = \sum_{m\ge j_1+1} \delta(E-m)(m^2 - j_1^2)\,.\end{aligned} \tag{3.26}$$

The above calculation allows us to read off the energies E and degeneracies d for a "standard" massless field[4] in the (j_1, j_2) representation of the $SO(4)$ little group:

$$\begin{aligned}E_k &= 2 + j_1 + j_2 + k\,, \qquad\qquad k = 0, 1, 2, \ldots\\ d_k &= \tfrac{1}{6}(k+1)(k+2)[(1+2j_1+2j_2)k + 3(2j_1+1)(2j_2+1)]\,.\end{aligned} \tag{3.27}$$

For massless fields whose E_0 value exceeds the minimum $2 + j_1 + j_2$, one just takes $E_k = E_0 + k$, with d_k again given by (3.27).

The fields in the $N = 8$ massless supergravity multiplet are characterised by their $SO(4)$ little-group representations (j_1, j_2), their lowest energies E_0, and their $SU(4)$ gauge-group representations. These are given in Table 1 below:

[4] That is, a massless field with $E_0 = 2 + j_1 + j_2$.

Field	$SO(4)$ Rep.	E_0	$SU(4)$ Rep.
Scalar	$(0,0)$	2	$20'$
	$(0,0)$	3	$10+\overline{10}$
	$(0,0)$	4	$1+1$
Vector	$(\frac{1}{2},\frac{1}{2})$	3	15
A/sym. tensor	$(1,0)+(0,1)$	3	$6+6$
Spin 2	$(1,1)$	4	1
Spin $\frac{1}{2}$	$(\frac{1}{2},0)+(0,\frac{1}{2})$	$\frac{5}{2}$	$20+\overline{20}$
	$(\frac{1}{2},0)+(0,\frac{1}{2})$	$\frac{7}{2}$	$4+\overline{4}$
Spin $\frac{3}{2}$	$(1,\frac{1}{2})+(\frac{1}{2},1)$	$\frac{7}{2}$	$4+\overline{4}$

Table 1: Lowest energies E_0 for the $N=8$ massless supermultiplet

We see that all fields except the 10, $\overline{10}$, 1, 1 of scalars, and the 4 and $\overline{4}$ of spin $\frac{1}{2}$ fields, have the minimum value $E_0 = 2 + j_1 + j_2$ for their lowest-energy. From the expressions in (3.27), we can read off the energies and degeneracies for the fields in the massless supermultiplet, and then use these to calculate the Casimir energy for each field. The results are given in Table 2 below:

Field	$SU(4)$ Rep.	E	Degeneracy	Casimir Energy
Scalar	$20'$	$k+2$	$\frac{1}{6}(k+1)(k+2)(k+3)$	0
	$10+\overline{10}$	$k+3$	$\frac{1}{6}(k+1)(k+2)(k+3)$	$-\frac{1}{480}$
	$1+1$	$k+4$	$\frac{1}{6}(k+1)(k+2)(k+3)$	$\frac{3}{80}$
Vector	15	$k+3$	$\frac{1}{2}(k+1)(k+2)(k+4)$	$-\frac{11}{240}$
A/sym. tensor	$6+6$	$k+3$	$\frac{1}{2}(k+1)(k+2)(k+3)$	$-\frac{1}{160}$
Spin 2	1	$k+4$	$\frac{1}{6}(k+1)(k+2)(5k+27)$	$-\frac{553}{240}$
Spin $\frac{1}{2}$	$20+\overline{20}$	$k+\frac{5}{2}$	$\frac{1}{3}(k+1)(k+2)(k+3)$	$-\frac{17}{3840}$
	$4+\overline{4}$	$k+\frac{7}{2}$	$\frac{1}{3}(k+1)(k+2)(k+3)$	$\frac{29}{3840}$
Spin $\frac{3}{2}$	$4+\overline{4}$	$k+\frac{7}{2}$	$\frac{1}{3}(k+1)(k+2)(2k+9)$	$\frac{141}{320}$

Table 2: Energies and degeneracies for each field in the $N=8$ massless supermultiplet. In each case $k=0,1,2,3\ldots$. The Casimir energy, calculated using zeta-function regularisation, is given *per field*.

If we now add up the contributions to the Casimir energy from each field, we obtain the total

$$E_c = \tfrac{3}{8} \tag{3.28}$$

for the entire massless supermultiplet. This can be compared with the Casimir energy $\frac{3}{16}$ for the $N = 4$ super-Yang-Mills multiplet in the boundary conformal field theory.

The total single-particle partition functions for the bosonic and fermionic fields of the $N = 8$ supermultiplet are given by

$$\begin{aligned} Y_{\rm boson}(\beta) &= \frac{4(e^{5\beta}-1)(e^{\beta}+e^{-\beta}+6)}{(e^{\beta}-1)^4}\,, \\ Y_{\rm fermion}(\beta) &= \frac{16(e^{5\beta}-1)(e^{\frac12\beta}+e^{-\frac12\beta})}{(e^{\beta}-1)^4}\,. \end{aligned} \tag{3.29}$$

The total single-particle partition function for the entire $N = 8$ supermultiplet is then given by

$$Y_{\rm tot}(\beta) = \frac{4(5-e^{-\beta})}{(e^{\frac12\beta}-1)^4}\,. \tag{3.30}$$

Calculating the total free energies and entropies for the bosonic and fermionic sectors from their respective multi-particle partition functions, and then summing these to get the total free energy $E_{\rm tot}(\beta)$ and entropy $S_{\rm tot}(\beta)$ for the $N = 8$ supermultiplet, we can examine the Bekenstein bound. This asserts that

$$L(\beta) \equiv E_{\rm tot}(\beta) - \frac{3S_{\rm tot}(\beta)}{2\pi} \ge 0\,. \tag{3.31}$$

It is easily seen that the lowest value for $L(\beta)$ will occur at $\beta = 2\pi/3$, and for this value we find

$$L_{\rm min} = -0.838186\,. \tag{3.32}$$

Including the Casimir contribution $E_c = 3/8$ is insufficient to outweigh this, and so there is a range of temperatures corresponding to approximately to

$$1.84884 < \beta < 2.744356 \tag{3.33}$$

within which the Bekenstein bound is violated.

3.4 Partition functions and entropy bounds in AdS_7

First, we need to determine the partition functions for massless fields in AdS_7. We use equation (3.25) in [21] for this, with the simplifying specialisation to the "blind" case where the x_i factors parameterising the chemical potentials for angular momenta are all set to 1.

In the dominant highest-weight labelling (ℓ_1, ℓ_2, ℓ_3) of $SO(6)$ little-group representations, the partition function, as defined in [21], is given by

$$D^{(6)}_{[\ell_1+4;\ell_1,\ell_2,\ell_3]}(s,1) = s^{\ell_1+4}\,[\chi^{(6)}_{(\ell_1,\ell_2,\ell_3)} - s\,\chi^{(6)}_{(\ell_1-1,\ell_2,\ell_3)}]\,P^{(6)}(s,1)\,, \tag{3.34}$$

where

$$\begin{aligned} \chi^{(6)}_{(\ell_1,\ell_2,\ell_3)} &= \tfrac{1}{12}(1+\ell_1-\ell_2)(1+\ell_2-\ell_3)(1+\ell_2+\ell_3)(2+\ell_1-\ell_3)\times \\ &\qquad \times(2+\ell_1+\ell_3)(3+\ell_1+\ell_2)\,, \\ P^{(6)}(s,1) &= \frac{1}{(1-s)^6}\,, \end{aligned} \tag{3.35}$$

and $s \equiv e^{-\beta}$.

The relation between dominant highest-weight labels and Dynkin labels for representations of $SO(2n)$ is that with $\underline{\ell} = \sum \ell_i \underline{e_i}$ in the orthonormal basis $\underline{e_i}$ for $\mathbb{R}^n$, one has the simple roots

$$\begin{aligned} \underline{\alpha}_i &= \underline{e}_i - \underline{e}_{i+1}\,, \qquad 1 \le i \le n-1\,, \\ \underline{\alpha}_n &= \underline{e}_{n-1} + \underline{e}_n\,. \end{aligned} \tag{3.36}$$

One then gets the Dynkin labels as the dot products of $\underline{\ell}$ with the simple root vectors $\underline{\alpha}_i$. For the case of interest to us here, i.e. $SO(6)$, we shall use the Dynkin labelling for $SU(4)$, which means one orders the labels as

$$(\underline{\alpha}_2\cdot\underline{\ell}, \underline{\alpha}_1\cdot\underline{\ell}, \underline{\alpha}_3\cdot\underline{\ell})\,. \tag{3.37}$$

In other words, the $SU(4)$ Dynkin label for a highest-weight representation (ℓ_1, ℓ_2, ℓ_3) is written as

$$(\ell_2-\ell_3, \ell_1-\ell_2, \ell_2+\ell_3)\,. \tag{3.38}$$

From [22], the AdS_7 massless supergravity multiplet is as given in Table 3 below. We list the fields, their $SU(4)$ little-group Dynkin labels, their corresponding highest-weight labels, their E_0 values and their $SO(5)$ R-symmetry representations

Field	$SU(4)$ Dynkin	$SU(4)$ HW	E_0	$SO(5)$ rep.
Scalar	$(0,0,0)$	$(0,0,0)$	4	14
Spin $\frac{1}{2}$	$(1,0,0)$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$\frac{9}{2}$	16
Vector	$(0,1,0)$	$(1,0,0)$	5	10
3-form	$(2,0,0)$	$(1,1,-1)$	5	5
Spin $\frac{3}{2}$	$(1,1,0)$	$(\frac{3}{2},\frac{1}{2},-\frac{1}{2})$	$\frac{11}{2}$	4
Spin 2	$(0,2,0)$	$(2,0,0)$	6	1

Table 3: The fields of the $D = 7$ massless supergravity multiplet, with their $SU(4)$ little-group representations in Dynkin and highest-weight labelling, their E_0 values, and their $SO(5)$ R-symmetry representations.

Reading off the relevant partition functions from (3.34), we can then take the inverse Laplace transforms to get the energies E_k and degeneracies d_k for the fields in the supergravity multiplet. Then we evaluate the Casimir energies for each field. The results are given in Table 4 below

Field	Partition function	Degeneracies d_k	k range	Casimir energy
Scalar	$\frac{s^4}{(1-s)^6}$	$\frac{1}{120}(k-3)(k-2)(k-1)k(k+1)$	$k \geq 4$	$\frac{31}{120960}$
Spin $\frac{1}{2}$	$\frac{4s^{9/2}}{(1-s)^6}$	$\frac{1}{960}(2k-7)(2k-5)(2k-3)(2k-1)(2k+1)$	$k \geq \frac{9}{2}$	$-\frac{1021}{322560}$
Vector	$\frac{(6-s)s^5}{(1-s)^6}$	$\frac{1}{24}(k-4)(k-3)(k-2)(k-1)(k+1)$	$k \geq 5$	$-\frac{39}{896}$
3-form	$\frac{10s^5}{(1-s)^6}$	$\frac{1}{12}(k-4)(k-3)(k-2)(k-1)k$	$k \geq 5$	$-\frac{95}{6048}$
Spin $\frac{3}{2}$	$\frac{4(5-s)s^{11/2}}{(1-s)^6}$	$\frac{1}{480}(2k-9)(2k-7)(2k-5)(2k-3)(4k+3)$	$k \geq \frac{11}{2}$	$\frac{15293}{17920}$
Spin 2	$\frac{2(10-3s)s^6}{(1-s)^6}$	$\frac{1}{60}(k-5)(k-4)(k-3)(k-2)(7k+8)$	$k \geq 6$	$-\frac{4143}{1120}$

Table 4: The partition functions, degeneracies and Casimir energies for each field of the types listed. In all cases the energies are given by $E_k = k$, with the starting values of k as given in the table.

It can be seen from Table 3 and Table 4 that in this seven-dimensional case, the lowest energies E_0 for each type of field are always the minimum allowed in each case, and so there is no need to shift upwards by integers. Taking the sum of the individual Casimir energies, weighted by the numbers for each field, we find that the total Casimir energy for the massless supergravity multiplet is given by

$$E_c = -\frac{325}{384}\,. \tag{3.39}$$

This can be compared with the total Casimir energy for the $(2,0)$ antisymmetric tensor multiplet in the boundary CFT_6 field theory, which has [23]

$$E_c = -\frac{25}{384}\,. \tag{3.40}$$

The Casimir energy for the bulk theory is larger in magnitude by a factor of 13.

3.5 Rotating quantum fields in AdS_4

A new feature, which enters when considering rigidly-rotating quantum fields, is the appearance of a velocity of light surface (VLS), on which the rotation speed equals the speed of light. If the system extends beyond the VLS, one expects that the partition functions will exhibit singularities.

To see this, consider AdS_4, for which the metric is

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.41}$$

where we take $\Lambda = -3$ for simplicity. If $\mathbf{K}$ is given by (2.4) then $\mathbf{K}$ is everywhere timelike, as long as $\Omega^2 < 1$, and so there is no VLS. However if $\Omega^2 > 1$ then there is a VLS. Thus one expects a singularity in the partition function as $\Omega^2 \to 1$ from below. In fact it should be harder and harder to rotate the system as Ω tends to this limiting value.

To see this reflected in the partition function, we begin by defining

$$x = e^{-\frac{1}{2}\beta}\,, \qquad y = e^{-\frac{1}{2}\beta\Omega}\,. \tag{3.42}$$

The expected singularities should therefore arise as $xy \to 1$ or $x/y \to 1$. To check this idea one must calculate the one-particle partition functions for the representations of $SO(3,2)$. They have been given by Flato and Fronsdal [24]:

$$Y(\beta,\Omega) = \sum_E \sum_j \sum_{j_3} x^{2E} y^{2j_3} = \sum_E \sum_j n(E,j)\chi_j(y), \tag{3.43}$$

where the $SU(2)$ rotor partition function $\chi_j(y)$ is given by

$$\chi_j(y) = \frac{(y^{2j+1} - y^{-2j-1})}{y - y^{-1}}. \tag{3.44}$$

For scalars in the $D(1,0)$ representation

$$Y_{D(1,0)} = x^{-2} Y_{D(2,0)}, \tag{3.45}$$

whilst for scalars in the $D(2,0)$ representation

$$Y = \frac{x}{(x-x^{-1})(xy-(xy)^{-1})(yx^{-1}-xy^{-1})}\,. \tag{3.46}$$

The massless representations $\mathrm{D(s+1,s)}$ have

$$Y = \frac{1}{(x-x^{-1})(y-y^{-1})}\Big[\frac{(xy)^{2s}}{xy-(xy)^{-1}} - \frac{(x/y)^{2s}}{(x/y)-(y/x)}\Big]\,. \tag{3.47}$$

The singletons are given by the "Di" representation $D(1, \frac{1}{2})$

$$Y(\beta, \Omega) = \frac{(y + y^{-1})}{(xy - 1/xy)(x/y - y/x)}, \tag{3.48}$$

and the "Rac" representation $\mathrm{D}(\frac{1}{2}, 0)$

$$Y(\beta, \Omega) = \frac{(x + x^{-1})}{(xy - 1/xy)(x/y - y/x)}. \tag{3.49}$$

The expected singularities are clearly visible in these expressions.

It is straightforward to see from numerical studies that the Bekenstein bound is more and more strongly violated, at low temperatures, as the angular velocity Ω approaches unity. Consider, for example, a massless scalar field in the $D(2, 0)$ representation, for which the single-particle partition function is given by (3.46). If we let $\Omega = 1 - \epsilon$, then we find

$$Y(\beta, 1 - \epsilon) = \frac{e^{\beta}\, \epsilon^{-1}}{\beta\, (e^{\beta} - 1)^2\, (e^{\beta} + 1)} + \frac{e^{\beta}(e^{2\beta} + 1)}{2(e^{\beta} - 1)^3\, (e^{\beta} + 1)^2} + \mathcal{O}(\epsilon)\,. \tag{3.50}$$

From this we find that $L(\beta) \equiv E(\beta) - S(\beta)/\pi$ as always attains its minimum value at $\beta = \pi$, and this is given by

$$L_{\min} = -\frac{0.000198216}{\epsilon} - 0.000312661 + \mathcal{O}(\epsilon)\,. \tag{3.51}$$

Thus by taking ϵ sufficiently small, we can achieve an arbitrarily large violation of the Bekenstein bound. The situation for massless fields of other spins is similar.

4 Matrix Valued Fields

So far, we have been calculating $Z(\beta)$ using the standard rules for free bosons and fermions. The result is a violation of Bekenstein's bound and certainly no Hagedorn temperature.

However, we must take into account the fact that on $S^3 \times \mathbb{R}$ the fields can be matrix-valued [5] and thus one must consider single and multi-trace partition functions. For very large, strictly infinite matrices, and for a free theory this has been solved in [17–19]

The answers for bosons (in the "blind" case) are

$$Y_{\text{Single Trace}}(\beta) = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(1 - Y(n\beta)) \tag{4.1}$$

where $\phi(n)$ is the number of integers no larger than n which are relatively prime to it and

$$\log Z_{\text{Multi Trace}}(\beta) = -\sum_{n=1}^{\infty} \log(1 - Y(n\beta)) \tag{4.2}$$

[5] For $\mathcal{N} = 4$ Yang-Mills theory, for example, they all transform according to to the adjoint of $SU(N)$.

or

$$Z_{\text{Multi Trace}}(\beta) = \prod_{n=1}^{\infty} \frac{1}{1 - Y(n\beta)}\,. \tag{4.3}$$

This is to be contrasted with

$$Z_B(\beta) = \prod_{n=1}^{\infty} e^{\frac{1}{n}Y(n\beta)}\,. \tag{4.4}$$

There are similar results for fermions and extensions to include chemical potentials. Note that using Möbius inversion one may pass freely between $Z_{\text{Multi Trace}}(\beta)$ and $Y(\beta)$.

For a quantum field theory,there can be no Hagedorn-like behaviour, but using infinite dimensional matrix valued fields this can happen Indeed, one now finds that there is a Hagedorn temperature $T_H = 1/\beta_H$, at which the partition function $Z(\beta)$ blows up which is located at $\beta = \beta_H$ where $Y(\beta_H) = 1$ In the context of the AdS/CFT correspondence, this Hagedorn transition has been associated with the Hawking-Page transition.

It is interesting to note that the Bekenstein bound can still be violated if we make use of the multi-trace partition function rather than the standard ones we discussed in previous sections. For example, let us consider the case of the multi-trace partition function for scalars in $\mathbb{R} \times S^3$, for which $Y(b)$ is given in (5.2). From (4.2) and the standard expressions $E = -\partial/\partial\beta \log Z$, $S = (1 - \beta\partial/\partial\beta)\log Z$ for the energy and entropy, we find that the function $f(\beta) = E(\beta) - S(\beta)/\pi$ has a minimum at $\beta = \pi$, and $f(\pi)$ is negative; we have the Bekenstein-violating result that

$$E(\pi) - \frac{S(\pi)}{\pi} \approx -0.0174463\,. \tag{4.5}$$

Inclusion of the Casimir energy contribution $E_c = 1/240$ is insufficient to turn (4.5) positive. Note that the Hagedorn temperature is given by $\beta_H \approx 1.25606$ in this example.

5 Partition Functions and Temperature Inversion

There has been considerable interest in the behaviour under temperature inversion of thermal correlation functions in conformally-invariant quantum field theories. (See, for example, [26] for a recent discussion, with references to earlier work.) Thermal correlators are always periodic or antiperiodic in imaginary time, and for conformally-invariant fields on $\mathbb{R} \times S^{n-2}$, or on AdS_n, they are typically also periodic or antiperiodic in real time. Thus one expects elliptic functions and modular behaviour to arise. This has been seen to happen, for free fields at least, on $\mathbb{R} \times S^{n-2}$ [25,26] and on AdS_4 [5]. In what follows, we shall discuss the behaviour of the energies under temperature inversion in certain $\mathbb{R} \times S^{n-2}$ and AdS_n

examples. The results for $\mathbb{R} \times S^{n-2}$ have been discussed previously in the literature, but we believe that our results for AdS$_4$ are new.

5.1 Partition functions for $\mathbb{R} \times S^3$

Here we calculate the free-field partition functions for the scalar, vector and spinor fields that constitute the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory on the boundary of AdS$_5$. Our focus will be on the relations between the high-temperature and low-temperature limits of the free energies, which are obtained from the multi-particle partition functions via

$$E(\beta) = -\frac{\partial \log Z}{\partial \beta}\,. \tag{5.1}$$

First, we consider the case where the angular momentum vanishes.

The energies E_n and degeneracies d_n for the relevant massless fields are given by

$$\begin{aligned}
&\text{Scalar}: && E_n = n+1\,, && d_n = (n+1)^2\,,\\
&\text{Vector}: && E_n = n+2\,, && d_n = 2(n+2)^2 - 2\,,\\
&\text{Spinor}: && E_n = n+\tfrac{3}{2}\,, && d_n = 2(n+\tfrac{3}{2})^2 - \tfrac{1}{2}\,,
\end{aligned} \tag{5.2}$$

where in each case, n ranges over the non-negative integers. From these, we obtain the single-particle partition functions $Y(\beta) = \sum_{n\ge 0} d_n\, e^{-\beta E_n}$, given by

$$\begin{aligned}
&\text{Scalar}: && Y(\beta) = \frac{e^\beta\,(e^\beta+1)}{(e^\beta-1)^3}\,,\\
&\text{Vector}: && Y(\beta) = \frac{2(3e^\beta-1)}{(e^\beta-1)^3}\,,\\
&\text{Spinor}: && Y(\beta) = \frac{4e^{\frac{3}{2}\beta}}{(e^\beta-1)^3}\,.
\end{aligned} \tag{5.3}$$

The multi-particle partition functions can be calculated from these using (A.1) and (A.2). Alternatively, and completely equivalently, they can be expressed directly via

$$\begin{aligned}
&\text{Boson}: && \log Z_B = -\sum_{n\ge 0} d_n\, \log(1-e^{-\beta E_n})\,,\\
&\text{Fermion}: && \log Z_F = \sum_{n\ge 0} d_n\, \log(1+e^{-\beta E_n})\,,
\end{aligned} \tag{5.4}$$

whence we obtain the free energies

$$\begin{aligned}
&\text{Boson}: && E(\beta) = \sum_{n\ge 0} \frac{d_n E_n}{e^{\beta E_n}-1}\,,\\
&\text{Fermion}: && E(\beta) = \sum_{n\ge 0} \frac{d_n E_n}{e^{\beta E_n}+1}\,.
\end{aligned} \tag{5.5}$$

5.1.1 Temperature-inversion relations

The behaviour under temperature inversion, i.e. under $\beta \to \beta^{-1}$, may be determined from the behaviour of the integrands in (2.18) and (2.19) under $s \to -s$. This is particularly easy to investigate in cases where $\zeta_H(s)$ is known explicitly. An alternative procedure, as discussed in [25], is to make use of the Ramanujan formulae [27]

$$\mu^p \sum_{n\geq 1} \frac{n^{2p-1}}{e^{2\mu n}-1} - (-\tilde{\mu})^t \sum_{n\geq 1} \frac{n^{2p-1}}{e^{2\tilde{\mu} n}-1} = [\mu^p - (-\tilde{\mu})^p]\, \frac{B_{2p}}{4p}\,, \tag{5.6}$$

$$\mu^p \sum_{n\geq 0} \frac{(2n+1)^{2p-1}}{e^{(2n+1)\mu}+1} - (-\tilde{\mu})^p \sum_{n\geq 1} \frac{(2n+1)^{2p-1}}{e^{(2n+1)\tilde{\mu}}+1} = [\mu^p - (-\tilde{\mu})^p]\, (2^{2p-1}-1)\, \frac{B_{2p}}{4p}\,, \tag{5.7}$$

for the bosonic and fermionic sums, respectively, where

$$\mu\, \tilde{\mu} = \pi^2\,, \tag{5.8}$$

p is a positive integer, and B_n denotes the n'th Bernoulli number. We take $\mu = \frac{1}{2}\beta$.

5.1.2 Scalar field

For the scalar field, one obtains

$$E(\beta) = \Big(\frac{2\pi}{\beta}\Big)^4\, E\Big(\frac{4\pi^2}{\beta}\Big) + \tfrac{1}{8}\Big[1 - \Big(\frac{2\pi}{\beta}\Big)^4\Big]\, B_4\,. \tag{5.9}$$

From (2.14), we see that the Casimir energy is

$$E_c = \tfrac{1}{2}\zeta(-3) = -\tfrac{1}{8}B_4 = \tfrac{1}{240}\,, \tag{5.10}$$

where $\zeta(s) = \sum_{n\geq 1} n^{-s}$ is the Riemann zeta function. It follows that the total energy $E_{\rm tot}(\beta)$ of the scalar field, defined by

$$E_{\rm tot}(\beta) \equiv E(\beta) + E_c\,, \tag{5.11}$$

satisfies the inversion relation

$$E_{\rm tot}(\beta) = \Big(\frac{2\pi}{\beta}\Big)^4\, E_{\rm tot}\Big(\frac{4\pi^2}{\beta}\Big)\,. \tag{5.12}$$

Recalling that we have set the AdS radius to $l = 1$ in our calculations, it can be seen that (5.12) implies a temperature inversion symmetry under $T \longrightarrow l^2/(4\pi^2 T)$.

5.1.3 Vector field

The total energy of the scalar field in $\mathbb{R} \times S^3$ obeys the elegant inversion formula (5.12). It turns out that for fields of non-zero spin, the analogous inversion formulae are not so elegant, but they still take rather simple forms. It is is helpful first to define functions

$$f_p(\beta) \equiv \sum_{n\geq 1} \frac{n^{2p-1}}{e^{n\beta}-1}\,. \tag{5.13}$$

Using (5.6), these functions satisfy the inversion relations

$$f_p(\beta) = (-1)^p \left(\frac{2\pi}{\beta}\right)^{2t} f_t\Big(\frac{4\pi^2}{\beta}\Big) + \Big[1-(-1)^p\Big(\frac{2\pi}{\beta}\Big)^{2p}\Big]\,\frac{B_{2p}}{4p}\,. \tag{5.14}$$

From (5.2) and (5.5), it then follows that the free energy for the vector field is given by

$$E(\beta) = 2[f_2(\beta) - f_1(\beta)]\,. \tag{5.15}$$

The Casimir energy calculated using (2.14) is

$$E_c = \zeta(-3) - \zeta(-1) = -\tfrac{1}{4}B_4 + \tfrac{1}{2}B_2 = \tfrac{1}{120} + \tfrac{1}{12} = \tfrac{11}{120}\,, \tag{5.16}$$

and hence we see from (5.14) that the total energy $E_{\rm tot}(\beta) = E_c + E(\beta)$ for the vector field on $\mathbb{R} \times S^3$ satisfies the inversion relation

$$E_{\rm tot}(\beta) = 2\Big(\frac{2\pi}{\beta}\Big)^4 \Big[f_2\Big(\frac{4\pi^2}{\beta}\Big) + \tfrac{1}{240}\Big] + 2\Big(\frac{2\pi}{\beta}\Big)^2 \Big[f_1\Big(\frac{4\pi^2}{\beta}\Big) + \tfrac{1}{24}\Big]\,. \tag{5.17}$$

5.1.4 Spinor field

Again, one finds that the total energy for a spin-$\frac{1}{2}$ Weyl fermion in $\mathbb{R} \times S^3$ does not satisfy as simple an inversion relation as the scalar field in (5.12), but a more complicated relation that is similar to the vector-field relation (5.17). It is helpful first to define

$$g_p(\beta) \equiv \sum_{n\geq 0} \frac{(2n+1)^{2p-1}}{e^{(n+\frac{1}{2})\beta}+1}\,, \tag{5.18}$$

Using (5.7), these functions satisfy the inversion relations

$$g_p(\beta) = (-1)^p \left(\frac{2\pi}{\beta}\right)^{2p} g_p\Big(\frac{4\pi^2}{\beta}\Big) + \Big[1-(-1)^p\Big(\frac{2\pi}{\beta}\Big)^{2t}\Big](2^{2p-1}-1)\,\frac{B_{2t}}{4p}\,. \tag{5.19}$$

It follows from (5.2) and (5.5) that the free energy for a spinor field is given by

$$E(\beta) = \tfrac{1}{4}[g_2(\beta) - g_1(\beta)]\,. \tag{5.20}$$

From (5.2) and (2.14) it follows that the Casimir energy for the spinor field is given by

$$E_c = -\zeta(-3,\tfrac{1}{2}) + \tfrac{1}{4}\zeta(-1,\tfrac{1}{2}) = \tfrac{7}{960} + \tfrac{1}{96} = \tfrac{17}{960}\,, \tag{5.21}$$

and hence using (5.19) we find that the total energy $E_{\rm tot}(\beta) = E_c + E(\beta)$ obeys the inversion relation

$$E_{\rm tot}(\beta) = \tfrac14\Big(\frac{2\pi}{\beta}\Big)^4 \Big[g_2\Big(\frac{4\pi^2}{\beta}\Big) - \tfrac{7}{240}\Big] + \tfrac14\Big(\frac{2\pi}{\beta}\Big)^2 \Big[g_1\Big(\frac{4\pi^2}{\beta}\Big) - \tfrac{1}{24}\Big]\,. \qquad (5.22)$$

5.2 Partition functions for $\mathbb{R}\times S^5$

In this six-dimensional boundary case, the relevant fields fill out a $(2,0)$ supermultiplet, comprising one vector, four spinors, and five self-dual tensor multiplets. Since the analysis is very similar to that in the $\mathbb{R}\times S^3$ boundary theory that we discussed in the previous section, where we shall be rather brief, and just focus on the results.

The energies and degeneracies for the three fields are given by

$$\begin{aligned}
&\text{Scalar}: \quad && E_n = n+2\,, \quad && d_n = \tfrac{1}{12}(n+2)^4 - \tfrac{1}{12}(n+2)^2\,,\\
&\text{Tensor}: \quad && E_n = n+3\,, \quad && d_n = \tfrac14(n+3)^4 - \tfrac54(n+3)^2 + 1\,,\\
&\text{Spinor}: \quad && E_n = n+\tfrac52\,, \quad && d_n = \tfrac13(n+\tfrac52)^4 - \tfrac56(n+\tfrac52)^2 + \tfrac{3}{16}\,,
\end{aligned} \qquad (5.23)$$

where $n\ge 0$, implying that the single-particle partition functions are

$$\begin{aligned}
&\text{Scalar}: \quad && Y(\beta) = \frac{e^{2\beta}(e^\beta+1)}{(e^\beta-1)^5}\,,\\
&\text{Tensor}: \quad && Y(\beta) = \frac{10e^{2\beta} - 5e^\beta + 1}{(e^\beta-1)^5}\,,\\
&\text{Spinor}: \quad && Y(\beta) = \frac{8e^{\frac52\beta}}{(e^\beta-1)^5}\,.
\end{aligned} \qquad (5.24)$$

In terms of the functions $f_p(\beta)$ and $g_p(\beta)$ defined in (5.13) and (5.18), the free energies turn out to be given by

$$\begin{aligned}
&\text{Scalar}: \quad && E(\beta) = \tfrac{1}{12}[f_3(\beta) - f_2(\beta)]\,,\\
&\text{Tensor}: \quad && E(\beta) = \tfrac14[f_3(\beta) - 5f_2(\beta) + 4f_1(\beta)]\,,\\
&\text{Spinor}: \quad && E(\beta) = \tfrac{1}{96}[g_3(\beta) - 10g_2(\beta) + 9g_1(\beta)]\,.
\end{aligned} \qquad (5.25)$$

The Casimir energies, calculated from (2.14) and (5.23), are given by

$$\begin{aligned}
&\text{Scalar}: \quad && E_c = \tfrac{1}{24}\zeta(-5) - \tfrac{1}{24}\zeta(-3) = -\tfrac{31}{60480}\,,\\
&\text{Tensor}: \quad && E_c = \tfrac18\zeta(-5) - \tfrac58\zeta(-3) + \tfrac12\zeta(-1) = -\tfrac{191}{4032}\,,\\
&\text{Spinor}: \quad && E_c = -\tfrac16\zeta(-5,\tfrac12) + \tfrac{5}{12}\zeta(-3,\tfrac12) - \tfrac{3}{32}\zeta(-1,\tfrac12) = -\tfrac{367}{48384}\,.
\end{aligned} \qquad (5.26)$$

As we saw in the previous four-dimensional case, here again the Casimir energies coincide with the (negatives of the) β-independent terms coming from the right-hand sides of the temperature-inversion relations (5.14) and (5.19), once these are assembled into the combinations occurring in (5.25). This implies that the total energies $E_{\rm tot}(\beta) = E_c + E(\beta)$ satisfy the following temperature-inversion relations:

$$\begin{aligned}
\text{Scalar}: \quad & E_{\rm tot}(\beta) = -\tfrac{1}{12}\Big(\frac{2\pi}{\beta}\Big)^6\Big[f_3\Big(\frac{4\pi^2}{\beta}\Big) - \tfrac{1}{504}\Big] - \tfrac{1}{12}\Big(\frac{2\pi}{\beta}\Big)^4\Big[f_2\Big(\frac{4\pi^2}{\beta}\Big) + \tfrac{1}{240}\Big]\,, \\
\text{Tensor}: \quad & E_{\rm tot}(\beta) = -\tfrac{1}{4}\Big(\frac{2\pi}{\beta}\Big)^6\Big[f_3\Big(\frac{4\pi^2}{\beta}\Big) - \tfrac{1}{504}\Big] - \tfrac{5}{4}\Big(\frac{2\pi}{\beta}\Big)^4\Big[f_2\Big(\frac{4\pi^2}{\beta}\Big) + \tfrac{1}{240}\Big] \\
& \qquad -\Big(\frac{2\pi}{\beta}\Big)^2\Big[f_1\Big(\frac{4\pi^2}{\beta}\Big) - \tfrac{1}{24}\Big]\,, \\
\text{Spinor}: \quad & E_{\rm tot}(\beta) = -\tfrac{1}{96}\Big(\frac{2\pi}{\beta}\Big)^6\Big[g_3\Big(\frac{4\pi^2}{\beta}\Big) - \tfrac{31}{504}\Big] - \tfrac{5}{48}\Big(\frac{2\pi}{\beta}\Big)^4\Big[g_2\Big(\frac{4\pi^2}{\beta}\Big) + \tfrac{7}{24}\Big] \\
& \qquad +\tfrac{3}{32}\Big(\frac{2\pi}{\beta}\Big)^2\Big[g_1\Big(\frac{4\pi^2}{\beta}\Big) + \tfrac{3}{8}\Big]\,. \qquad (5.27)
\end{aligned}$$

5.3 Temperature-inversion formulae in AdS$_4$

The energies E_n and degeneracies d_n for the modes in AdS$_4$ can conveniently be read off from (3.20) and (3.21). Thus we have

$$\begin{aligned}
D(1,0): \quad & E_n = n\,, \quad d_n = \tfrac{1}{2}n(n+1)\,, \quad n \ge 1\,, \\
D(2,0): \quad & E_n = n\,, \quad d_n = \tfrac{1}{2}n(n-1)\,, \quad n \ge 2\,, \\
D(s+1,s): \quad & E_n = n\,, \quad d_n = n^2 - s^2\,, \quad n \ge s\,, \\
D(1,\tfrac{1}{2}): \quad & E_n = n\,, \quad d_n = n\,, \quad n \ge 1\,, \\
D(\tfrac{1}{2},0): \quad & E_n = n\,, \quad d_n = n\,, \quad n \ge \tfrac{1}{2}\,, \qquad (5.28)
\end{aligned}$$

where in each case n increases in integer step.

The free energies can be calculated using the same formulae (5.5) that we used when studying the case of $\mathbb{R} \times S^3$. Thus we shall have

$$\begin{aligned}
E_{(1,0)}(\beta) &= \tfrac{1}{2}\sum_{n\ge1}\frac{n^2(n+1)}{e^{n\beta}-1}\,, \qquad E_{(2,0)}(\beta) = \tfrac{1}{2}\sum_{n\ge1}\frac{n^2(n-1)}{e^{n\beta}-1}\,, \\
E_{(s+1,s)}(\beta) &= \sum_{n\ge s}\frac{n(n^2-s^2)}{e^{n\beta}-1}\,, \qquad s\in\mathbb{Z} \\
E_{(s+1,s)}(\beta) &= \sum_{n\ge s}\frac{n(n^2-s^2)}{e^{n\beta}+1}\,, \qquad s\in\mathbb{Z}+\tfrac{1}{2} \\
E_{(1,\frac{1}{2})}(\beta) &= \sum_{n\ge1}\frac{n^2}{e^{n\beta}+1}\,, \qquad E_{(\frac{1}{2},0)}(\beta) = \sum_{n\ge\frac{1}{2}}\frac{n^2}{e^{n\beta}-1}\,. \qquad (5.29)
\end{aligned}$$

Let us consider scalars first. It is evident from the Ramanujan formula (5.6) that, since p must be an integer there, we cannot obtain a temperature-inversion formula for the $D(1,0)$

or $D(2,0)$ scalars separately. However, if we add up the two, we get

$$E_{\rm scal}(\beta) = \sum_{n\ge 1} \frac{n^3}{e^{n\beta}-1}\,, \tag{5.30}$$

which does fall into the category covered by (5.6). In fact, from (5.13) and (5.14), and noting from (3.16) that the total scalar Casimir energy is $\frac{1}{240}$, we shall have for the total scalar energy $E_{\rm tot}(\beta) \equiv E_{\rm scal}(\beta) + E_c$ the same total energy, and temperature-inversion formula

$$E_{\rm tot}(\beta) = \Big(\frac{2\pi}{\beta}\Big)^4\, E_{\rm tot}(\frac{4\pi^2}{\beta})\,, \tag{5.31}$$

as we obtained for scalars in $\mathbb{R}\times S^3$.

6 Conclusions

In this paper, motivated by our earlier work on black holes and the cosmic censorship bound in AdS [16], we have investigated a precise formulation of the Bekenstein bound four quantum fields in AdS$_n$. We have given it a new formulation in terms of the function $L \equiv E - (n-2)S/(2\pi l)$, where l is the AdS radius. If the specific heat is positive, the function L has a unique minimum at the temperature $T = T_L \equiv (n-2)/(2\pi l)$, at which $L = F(T_L)$, where F is the Helmholtz free energy. Thus the Bekenstein bound is satisfied if and only if $F(T_L) \ge 0$. Interestingly, the marginal case corresponds to the free energy having the same value as that of the vacuum, as it does in the case of the Hawking-Page phase transition. Although we found previously that the Bekenstein bound was satisfied for all known black holes, we are able to exhibit violations of the bound for free quantum fields of various spins in AdS, including in particular those which come from supermultiplets. We do this by calculating the bulk entropies and energies in AdS$_4$, AdS$_5$ and AdS$_7$. We have also examined rotating quantum fields in AdS$_4$, where we find the expected divergence in the partition function as the rotation rate tends to its maximum value. Violations of the Bekenstein bound can be arbitrarily large as this limit is approached.

A summary of the status of the conjectured Bekenstein bound is as follows. As has been observed previously, it is trivially invalid if one uses the Boltzmann definition of entropy in the microcanonical ensemble, because of the species problem. Furthermore, inclusion of the Casimir energy cannot always rescue it, since sometimes the Casimir energy is negative. In this paper, we have resolved a previously difficulty with the bound, which is the precise definition of the radius R, and the problem of dealing with the boundary conditions on the surface of the box. These problems, which plagued previous discussions, were avoided in a simple and natural way by working in anti-de Sitter spacetime. We considered the Gibbs definition of entropy in the canonical and grand canonical ensembles, and showed that we could obtain violations of the Bekenstein bound in this well-defined situation. Thus it appears that there is no definition of entropy for which a rigorous Bekenstein bound holds.

It has recently been realised that the statistics of matrix-valued fields in Yang-Mills theory are non-conventional, and we have calculated some entropies and energies for bulk fields in AdS using these non-conventional statistics. Again, we find that the Bekenstein bound may be violated.

A topic of some interest in AdS_n and in $\mathbb{R} \times S^{n-2}$ is the issue of a possible symmetry of thermodynamic quantities under temperature inversion [25, 26]. We have investigated this in $\mathbb{R} \times S^3$, $\mathbb{R} \times S^5$ and AdS_4. The results for $\mathbb{R} \times S^{n-2}$ have appeared previously, but our results for AdS_4 are new.

String theory has a T-duality symmetry which implies that amplitudes transform covariantly under $R \longrightarrow \alpha'/R$, where α' is the inverse string tension and R is the radius of a Kaluza-Klein circle. Various authors [28–30] have consequently speculated that there should be a temperature inversion symmetry in string theory, with $T \longrightarrow \alpha'/T$. It should be noted, however, that the temperature inversion we consider, where $T \longrightarrow l^2/(4\pi^2 T)$, is distinct from the type of temperature inversion envisaged in string theory.

Finally, in the appendix, we give what we believe to be a novel way of obtaining the one-particle partition function from the many-particle partition function, using a version of the Möbius inversion formula.

Acknowledgement

We should like to thank Francis Dolan for helpful conversations and for sharing with us some of his results prior to publication. C.N.P. is grateful to the DAMTP Relativity and Gravitation group, and the Centre for Theoretical Cosmology at the CMS in Cambridge, for hospitality during the course of this work.

A Multi-Particle to Single-Particle Partition Functions

As is well known, and we discussed in section 2, one constructs the multi-particle partition function $\widetilde{Y}(\beta) = \log Z$ for a system of non-interacting particles whose single-particle partition function is $Y(\beta)$ according to the formulae

$$\text{Bosons}: \qquad \widetilde{Y}(\beta) = \sum_{n\ge 1} \frac{1}{n}\, Y(n\beta)\,, \tag{A.1}$$

$$\text{Fermions}: \qquad \widetilde{Y}(\beta) = -\sum_{n\ge 1} \frac{(-1)^n}{n}\, Y(n\beta)\,. \tag{A.2}$$

What appears to be less well known is that one can invert this construction, and express the single-particle partition functions in terms of the multi-particle partition functions.

For the bosonic case, the inversion can be performed by using the Möbius function $\mu(n)$, which is defined for positive integers n as follows. If n is not a square-free integer then $\mu(n) = 0$, whilst $\mu(n) = (-1)^m$ if n is a product of m distinct primes, with $\mu(1) = 1$. It is a standard result that if

$$g(x) \equiv \sum_{n\ge 1} f(nx)\,, \tag{A.3}$$

then

$$f(x) = \sum_{n\ge 1} \mu(n)\, g(nx)\,, \tag{A.4}$$

where $f(x)$ is an arbitrary function restricted only by the requirement that the sums converge. From this, if follows that for bosons we may invert (A.1) to express the single-particle partition function $Y(\beta)$ in terms of the multi-particle partition function $\widetilde{Y}(\beta)$ as

$$Y(\beta) = \sum_{n\ge 1} \frac{\mu(n)}{n}\, \widetilde{Y}(n\beta)\,. \tag{A.5}$$

In the fermionic case, we may again seek an inversion of the form

$$Y(\beta) = \sum_{n\ge 1} f_n\, \widetilde{Y}(n\beta)\,. \tag{A.6}$$

We find that the coefficients f_n in this expansion are given as follows. We first express n as

$$n = 2^s \prod_{i=1}^m p_i^{c_i}\,, \tag{A.7}$$

where p_i denotes the prime factors in n that are ≥ 3. The f_n are then given by $f_1 = 1$ and

$$f_n = \begin{cases} \frac{(-1)^m}{n}\, 2^{s-1} & \text{if } s\ge 1 \text{ and } c_i = 1 \text{ for all } i\,,\\ \frac{(-1)^m}{n} & \text{if } s = 0 \text{ and } c_i = 1 \text{ for all } i\,,\\ \frac{1}{2} & \text{if } s\ge 1 \text{ and } c_i = 0 \text{ for all } i\,,\\ 0 & \text{otherwise}\,. \end{cases} \tag{A.8}$$

To prove this, we define

$$G(x) = \sum_{n\ge 1} \frac{1}{n}\, F(nx)\,, \qquad H(x) = \sum_{n\ge 1} \frac{(-1)^{n+1}}{n}\, F(nx)\,. \tag{A.9}$$

By splitting the summation in the latter into the terms where n is even and n is odd, it follows that

$$H(x) = G(x) - G(2x)\,, \tag{A.10}$$

which can be iterated to give

$$G(x) = \sum_{p\ge 0} H(2^p\, x) \tag{A.11}$$

since $G(x)$ is assumed to go to zero as x goes to infinity. Using the standard result that (A.3) implies (A.4), we therefore have

$$\begin{aligned} F(x) &= \sum_{n\ge 1, p\ge 0} \frac{\mu(n)}{n}\, H(2^p n\, x) \\ &= \sum_{n\ge 1, p\ge 0} \frac{\mu(2n)}{2n}\, H(2^{p+1} n\, x) \; + \sum_{n\,\mathrm{odd}\,\ge 1, p\ge 0} \frac{\mu(n)}{n}\, H(2^p n\, x)\,, \end{aligned} \tag{A.12}$$

where the two terms in the second line were obtained by splitting the original sum over n into the cases where n is even and odd respectively. Since $\mu(2n) = -\mu(n)$ if n is odd, whilst $\mu(2n) = 0$ if n is even, it follows that

$$\begin{aligned} F(x) &= \sum_{n\,\mathrm{odd}\,\ge 1, p\ge 0} \frac{\mu(n)}{n} [H(2^p n\, x) - \tfrac{1}{2} H(2^{p+1} n\, x)] \\ &= \sum_{n\,\mathrm{odd}\,\ge 1} \frac{\mu(n)}{n}\, H(nx) \; + \sum_{n\,\mathrm{odd}\,\ge 1, p\ge 1} \frac{\mu(n)}{(2^p n)}\, 2^{p-1}\, H(2^p nx)\,. \end{aligned} \tag{A.13}$$

Comparing with (A.2) and (A.6), we see that (A.8) is indeed established.

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