Qubit feedback and control with kicked quantum nondemolition measurements: A quantum Bayesian analysis

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The informational approach to continuous quantum measurement is derived from POVM formalism for a mesoscopic scattering detector measuring a charge qubit. Quantum Bayesian equations for the qubit density matrix are derived, and cast into the form of a stochastic conformal map. Measurement statistics are derived for kicked quantum nondemolition measurements, combined with conditional unitary operations. These results are applied to derive a feedback protocol to produce an arbitrary pure state after a weak measurement, as well as to investigate how an initially mixed state becomes purified with and without feedback.

I. INTRODUCTION

Quantum measurement is usually taught in textbooks as an instantaneous process. However, in nature, all processes take a finite time. Physical projective measurement (or wave-function collapse) must therefore happen over some time period, and is often a sequence of *weak measurements*, run for a sufficiently long time. Weak quantum measurements are characterized by an intrinsic uncertainty about the state of the measured system. In the parlance of detector physics, this is equivalent to the statement that the signal cannot be confidently distinguished from the noise without a sufficiently long integration time. At some intermediate time, the eigenstates of the measurement operator can only be assigned a value with some confidence, determined with the probability of a given realization of the detector output (similarly to classical Bayesian inference).

In solid state systems, the typically weak coupling constant between the quantum system and measuring apparatus imply that weak measurements with long measurement times are the norm. While this is often frustrating to experimentalists who wish to preform projective measurement for quantum computation purposes, we view this situation as an opportunity to discover and implement ideas in quantum measurement that are qualitatively different from standard projective measurement. Although many results in this paper are abstract and apply to many different physical systems and detectors, the results will be discussed in terms of solid state physics. Quantum detection in mesoscopic structures began with the "controlled dephasing" experiments of Ref. 1 and related theoretical works,[2] which has continued to be an active area of research.[3] The particular mesoscopic structure we shall focus on presently is a quantum point contact (QPC) detector measuring a double quantum dot charge qubit (DD), a system that has been extensively investigated, both theoretically[4] and experimentally.[5]

A generic problem that arises if one is interested in

making a projective measurement, made out of many weak measurements, is that the dynamics from the Hamiltonian evolution combines in a nontrivial way with the measurement dynamics. The way around this problem is with quantum nondemolition (QND) measurements.[6] The QND scheme employed in this paper is that of kicked QND measurements[7,8,9,10,11,12] on a qubit. This idea was discussed for the QPC in Ref. 9, and was inspired by a similar rotating QND scheme of Ref. 13. By turning the detector on and off in a time scale much faster than the Rabi oscillation period, the detector gives a little information about the quantum state. The qubit is then allowed to make a full Rabi oscillation before the next weak measurement, so the measured observable is static in time from the perspective of the measurement device, effectively turning off Hamiltonian evolution. In this fashion, the information contained in the qubit is teased out over many measurements, or detector kicks. Kicked QND measurements have many advantages that recommend them as a technique of choice both for theoretical treatment, as well as for experimental implementation:

- Theoretically, the kicked measurements may be described with a non-unitary quantum map. The map is discrete in the time index, but the measurements are weak, not projective. The quantum map formalism allows for a technically simple treatment of the combined Hamiltonian and measurement-induced dynamics in a global manner, in contrast to continuous measurement analysis using conditional differential Langevin equations.

- The kicking mechanism is convenient, in that it allows the measurement strengths to be fully tunable by adding more kicks, as well as the easy inclusion of unitary operations by simply waiting a fraction of a Rabi period.

- Kicked QND measurements may be implemented in experiments with several advantages. The kicks

may be accomplished with a pulse generator on a QPC measuring a DD, and the waiting time between kicks gives external decision circuitry the needed time to process the data in order to do real-time feedback. Also, the pump variation introduced in Ref. 11 removes the uninteresting background signal of the measurement, and just gives the bare output signal as either positive or negative pumped current.

The purpose of this paper is two-fold. The first topic is formal: To start with the well-known POVM approach to generalized measurements, and derive the quantum Bayesian formalism from it, starting with a scattering detector. The detector physics allows a natural translation of the abstract POVM formalism into physical processes, and the quantum Bayesian formalism is recovered in the weak coupling limit. After discussing kicked-QND measurements, we show how both kicked measurements and unitary operations may be recast in terms of conformal maps, and demonstrate a close parallel with the mathematics of the special theory of relativity.

The second topic is physical: the formal results are applied to make predictions using conditional operations with real-time feedback: (1) We derive an algorithm to deterministically produce an arbitrary pure state after a (random) weak measurement using feedback. (2) We investigate the purification process under measurement and generalize Jacobs' qubit feedback protocol to speed up purification with feedback.[14]

The paper is organized as follows. In Sec. II, we derive the quantum Bayesian formalism from POVMs applied to a mesoscopic scattering detector in the weak coupling limit. Kicked QND measurements are reviewed in Sec. III, in the context of the quantum Bayesian formalism. In Sec. IV we introduce a stereographic projection representation, and rewrite the measurement dynamics as a stochastic conformal mapping. A close analogy to special relativity is also discussed. Sec. V combines kicked measurements with unitary operations, and calculates measurement statistics. Sec. VI introduces conditional phase shifts in order to deterministically produce the same quantum state after a measurement. In Sec. VII, we investigate the purification process of any initially mixed density matrix under kicked measurement. Sec. VIII contains our conclusions.

II. DERIVATION OF THE QUANTUM BAYESIAN FORMALISM FROM POVM

The formalism used in this paper is called the quantum Bayesian approach[15] because it may be considered as a generalization of classical Bayesian inference. An analogous approach to quantum measurement that is better known in the quantum information community has been given the unfortunate name of positive operator-valued measure (POVM) formalism.[16,17] In this section,

the quantum Bayesian formalism for a solid state system is derived from POVMs.

Consider a bipartite system composed of A and B, where the states of B are expressed in the orthonormal basis $|Q\rangle_B$, and the states of A are expressed in the orthonormal basis $|j\rangle_A$. A unitary transformation that entangles the states in A with the states in B is given by

$$|\psi\rangle_A|0\rangle_B \to \sum_Q \mathbf{M}_Q|\psi\rangle_A|Q\rangle_B, \qquad (1)$$

where we consider an initial state $|\psi\rangle_A$ in A, described by a density operator ρ_A, an initial state $|0\rangle_B$ in B, and have introduced the measurement operators $\mathbf{M}_Q$ that are indexed by the states in B, and operate in A. The normalization of the states gives the completeness relation, $\sum_Q \mathbf{M}_Q^\dagger \mathbf{M}_Q = \mathbf{1}$. Now make a projective measurement on B alone, and find the result Q. Any measurement of this kind may be described as a POVM in A. The probability of finding the result Q, called $P(Q)$, is given by

$$P(Q) = \mathrm{Tr}(\rho_A \mathbf{M}_Q^\dagger \mathbf{M}_Q), \qquad (2)$$

while the outcome of this measurement prepares a new density operator of A, conditioned on the outcome Q, and is given by

$$\rho_A' = \frac{\mathbf{M}_Q \, \rho_A \, \mathbf{M}_Q^\dagger}{\mathrm{Tr}(\rho_A \mathbf{M}_Q^\dagger \mathbf{M}_Q)}. \qquad (3)$$

This defines a mapping $\rho_A' = \$(\rho_A)$ from density operators to density operators, known as a super-operator, which is not unitary in general.

While the above results are standard generalizations of projective measurement on A, the abstract formulation obscures how to practically apply the POVM to a specific physical system. We now consider such a system in the solid-state: the quantum point contact (QPC), measuring the state of a double quantum dot (DD), in order to see how the POVM translates into physical language. The Coulomb interaction between the DD and QPC alters the transport properties of the QPC, and can thus be used to detect which quantum dot the DD electron occupies. The QPC is described with the help of a scattering matrix S_j that depends on the physical state of the DD. Following Averin and Sukhorukov,[18] the unitary evolution of the total state during the scattering process is comprised of the state of an individual electron (system B) incident from the left electrode, $|in\rangle_B$, and the state of the DD (system A), $\alpha|1\rangle_A + \beta|2\rangle_A$, evolving as

$$|in\rangle_B(\alpha|1\rangle_A + \beta|2\rangle_A) \to \qquad (4)$$
$$\alpha(r_1|L\rangle_B + t_1|R\rangle_B)|1\rangle_A + \beta(r_2|L\rangle_B + t_2|R\rangle_B)|2\rangle_A,$$

where $|L, R\rangle_B$ represent scattering states of the QPC that have either been reflected or transmitted, and the transmission and reflection amplitudes t_j and r_j are elements

of the scattering matrix S_j:

$$S_j = \begin{pmatrix} r_j & \bar{t}_j \\ t_j & \bar{r}_j \end{pmatrix}. \qquad (5)$$

The state of the "logical" qubit of the DD, is now entangled with the "ancilla" qubit of the left/right position of the QPC electron, and this comprises the bipartite system. Using the evolved state (4), we can now read off the measurement operators, $\mathbf{M}_Q$, of this unitary operation in the $(|0\rangle_A, |1\rangle_A)$ basis,

$$\mathbf{M}_L = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \qquad \mathbf{M}_R = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \qquad (6)$$

and easily verify that $\mathbf{M}_L^\dagger \mathbf{M}_L + \mathbf{M}_R^\dagger \mathbf{M}_R = \mathbf{1}$ from probability conservation. Counting the electron in the collector of the QPC gives a random outcome, $Q = 1$ if the electron is counted, or $Q = 0$ if the electron is not counted, and makes a projective measurement on the B part of the Hilbert space. Equation (2) gives the probability of counting the electron (or not),

$$P(1) = \rho_{11} T_1 + \rho_{22} T_2, \qquad P(0) = \rho_{11} R_1 + \rho_{22} R_2, \qquad (7)$$

where ρ_{ij} are the elements of the DD density matrix in the $|1, 2\rangle$ basis, $T_j = |t_j|^2$, $R_j = |r_j|^2$, and $T_j + R_j = 1$. The density matrix of the DD qubit may be updated, given the outcome of the measurement with Eq. (3). If $Q = 1$, so an electron is counted, then

$$\begin{aligned} \rho'_{11} &= T_1 \rho_{11}/P(1), \quad \rho'_{22} = 1 - \rho'_{11}, \\ \rho'_{12} &= (\rho'_{21})^* = t_1 t_2^* \, \rho_{12}/P(1) \\ &= \rho_{12} \, e^{i\xi} \sqrt{\rho'_{11}\rho'_{22}/\rho_{11}\rho_{22}}, \end{aligned} \qquad (8)$$

where $\xi = \mathrm{Arg}(t_1 t_2^*)$; while if $Q = 0$, so an electron is not counted, or equivalently, a hole is counted, then

$$\begin{aligned} \rho'_{11} &= R_1 \rho_{11}/P(0), \quad \rho'_{22} = 1 - \rho'_{11}, \\ \rho'_{12} &= (\rho'_{21})^* = r_1 r_2^* \, \rho_{12}/P(0) \\ &= \rho_{12} \, e^{i\chi} \sqrt{\rho'_{11}\rho'_{22}/\rho_{11}\rho_{22}}, \end{aligned} \qquad (9)$$

where $\chi = \mathrm{Arg}(r_1 r_2^*)$.

The results (8,9) have a natural interpretation as a quantum Bayes formula: The diagonal density matrix elements are interpreted as classical probabilities, and are updated according to the classical Bayes formula, while the off-diagonal elements have a more exotic rule.[15] Note that if the initial DD qubit state is pure, it remains pure after the measurement. This is because while the entanglement enlarged the effective Hilbert space which would lead to decoherence if the entangled information went undetected, the measurement of the QPC electron collapses the 2-particle state back down to a different pure DD state.

It is instructive to contrast the POVM procedure with the well known "decoherence" approach to quantum measurement in this most simple case. The decoherence approach corresponds to explicitly averaging the elements of the density matrix over all possible outcomes of the detector. In this case, the two possible outcomes of the measurement (8,9) are used to obtain

$$\langle \rho'_{11} \rangle = P(0)\rho'_{11}(0) + P(1)\rho'_{11}(1) = \rho_{11} \qquad (10)$$

for the diagonal elements ($\rho_{22} = 1 - \rho_{11}$), and

$$\langle \rho'_{12} \rangle = P(0)\rho'_{12}(0) + P(1)\rho'_{12}(1) = (t_1 t_2^* + r_1 r_2^*)\rho_{12} \qquad (11)$$

for the off-diagonal elements ($\rho_{21} = \rho_{12}^*$), in agreement with Averin and Sukhorukov.[18,19] The new off-diagonal matrix elements are reduced because $|t_1 t_2^* + r_1 r_2^*| \le 1$, resulting in effective decoherence, while the diagonal matrix elements are preserved. The predictive advantage of the quantum Bayesian approach comes from not averaging over the measurement results, but rather conditioning the quantum density matrix on the result obtained in a particular physical realization.

The above POVM analysis is not difficult to extend to M "ancilla" qubits, or QPC electrons. The basis $|Q\rangle$ is now spanned by M qubits, each being projected to either 0 or 1. Rather than find the probability of obtaining a given sequence of 0s and 1s in the output, it happens that it is sufficient to find the probability of just obtaining the total charge $n = \sum_{i=1}^M Q_i$, given M attempts, where $Q_i = (0, 1)$. In other words, sequence does not matter, only the total number of counted electrons. This mapping is illustrated in Fig. 1, where the ancilla outcome 1 or 0, is mapped respectively into either counting an electron, or not counting an electron in the current stream. The generalization of the quantum Bayesian rules (7-9) for M ancilla qubits is done by replacing the success probabilities T_j, and the failure probabilities $R_j = 1 - T_j$ by the probability $P(m, M|j)$ to measure m electrons in M attempts, under the condition that the qubit is in state $|j\rangle$,

$$P(m, M|j) = \binom{M}{m} T_j^m (1 - T_j)^{M-m}, \qquad (12)$$

which is the binomial distribution.

While this is sufficient for the diagonal matrix elements, the off-diagonal matrix elements get a different phase shift after each electron is measured. The total phase shift is $\Phi(m, M) = M\chi + m(\xi - \chi)$, which is deterministic if $\xi = \chi$. The symmetric quantum point contact has the property that $\xi = \chi = 0$,[2,4] so this simplification will be made in the rest of the paper. For many electrons M, the current is determined by m/M. The physically relevant weak-coupling (weakly responding) limit corresponds to $(T_1 - T_2)/(T_1 + T_2) \ll 1$. Appealing to the central limit theorem, we treat the detector shot noise in the Gaussian approximation with little lost information.[20] The QPC detector is efficient, in the sense that no information about the DD qubit is lost in the noisy current output.[4,15] This is analogous to saying in the logical language that no ancilla qubit was left unprojected, and that the unitary operations did not hide any qubit information in the phase of the ancilla qubits that is destroyed after projection in the left/right basis.
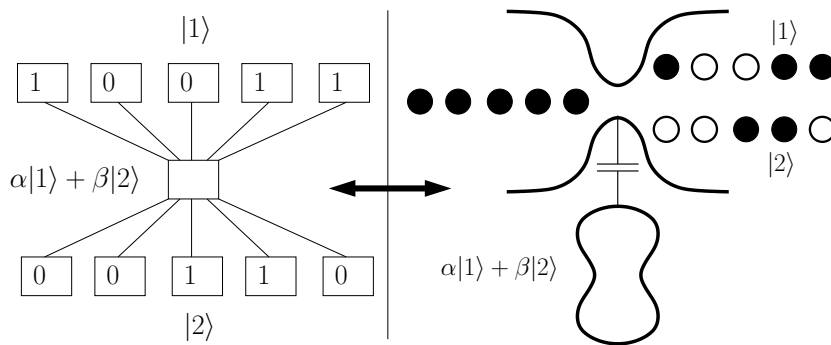
FIG. 1: Weak entangling of a logical qubit with many ancilla qubits, followed by projective measurement on the ancilla qubits can be mapped onto the measurement of a quantum double dot by the transport electrons of a quantum point contact. Two given measurement realizations are shown, depending on whether the logical qubit state is either $|1\rangle$ or $|0\rangle$. The random ancilla result 1 is mapped onto measuring an electron in the current collector (denoted with a filled circle), while the random ancilla result 0 is mapped onto not measuring an electron in the current collector, or equivalently, measuring a hole (denoted with an empty circle). The fact that the states $|1\rangle$ and $|2\rangle$ of the logical qubit alters the probability of projecting the ancillas to 0 or 1, allows a weak POVM measurement on the logical qubit.

One manifestation of an efficient detector in the dephasing approach, is that the measurement rate coincides with the measurement-induced dephasing rate.[4,15] Ref. 18 related the measurement rate to one of the Rényi entropies, or the statistical overlap $O_{1,2} = \sum_m [P(m,M|1)P(m,M|2)]^{1/2}$, and showed that for the symmetric QPC it coincides with the reduction of the off-diagonal matrix elements. It is one feature of the quantum Bayesian approach that this particular measure comes out in a natural way. To see this, we first note a general property of the density matrix, $|\rho_{12}| \leq \sqrt{\rho_{11}\rho_{22}}$, that simply comes from the density matrix eigenvalues being bounded between zero and one. Next, we consider an initially pure state ($|\rho_{12}| = \sqrt{\rho_{11}\rho_{22}}$) and notice that the elements of the density matrix after a measurement, ρ', obey the relation

$$\left|\frac{\rho'_{12}}{\rho_{12}}\right| \leq \sqrt{\frac{\rho'_{11}\rho'_{22}}{\rho_{11}\rho_{22}}}. \tag{13}$$

The above relation is valid for every given measurement outcome, so it is also valid after averaging over the distribution of results, $\langle \mathcal{O}(m)\rangle = \sum_m P(m,M)\mathcal{O}(m)$, where $\mathcal{O}$ is any observable, and $P(m,M) = \rho_{11}P(m,M|1) + \rho_{22}P(m,M|2)$. Using the classical Bayes rule for the diagonal elements, taking $\mathcal{O} = |\rho'_{12}/\rho_{12}|$, and the fact that $|\langle\mathcal{O}\rangle| \leq \langle|\mathcal{O}|\rangle$, we obtain the generalized efficiency relation

$$\left|\left\langle\frac{\rho'_{12}}{\rho_{12}}\right\rangle\right| \leq \sum_m \sqrt{P(m,M|1)P(m,M|2)}. \tag{14}$$

Notice this relation is quite general, as no particular update rule for the off-diagonal matrix element has been invoked. Therefore, a detector reaching the upper bound (14) can naturally be called ideal, or 100% efficient. (For the appropriate definition of efficiency for an asymmetric detector see, e.g., Refs. 15,21.)

Thus far, we have focused only on the dynamics of the measurement process, and have neglected the Hamiltonian evolution of the DD qubit. This evolution rotates the quantum state, and continually changes the effective measurement basis, which typically ruins the desired continuous measurement. The way to get around this bothersome detail is with QND measurements, the subject of the next section.

III. KICKED QND MEASUREMENTS

The unifying theme behind all QND schemes is to couple the measurement apparatus to the qubit with an operator that is an approximate constant of motion of the measured quantum system.[6] In this way, the detector only measures the state in the desired fixed basis, and the internal quantum dynamics that would otherwise spoil the desired measurement is circumvented. The specific scheme we employ in this paper is that of kicked QND measurements, introduced by V. Braginsky et al.[7] and K. Thorne et al.[8] for the harmonic oscillator. In Ref. 9, this idea is introduced for two-state systems by making an analogy to a cat playing with a string that moves in a circle. In the kicked QND mode, the cat sits in one spot waiting for the string to come to it, and only then bats at it. The motion in a circle comes from the simple Hamiltonian evolution of a two-state system. If $H = \epsilon\sigma_z/2 + \Delta\sigma_x/2$ is the qubit Hamiltonian, where Δ is the tunnel coupling energy, and ϵ is the energy asymmetry, then unitary evolution for a time t is given by

$$\begin{aligned}\mathbf{U} &= \exp[-it(\epsilon\sigma_z + \Delta\sigma_x)/2] = \mathbf{1}\cos(Et/2) \\ &- i\sigma_z(\epsilon/E)\sin(Et/2) - i\sigma_x(\Delta/E)\sin(Et/2),\end{aligned} \tag{15}$$

where $E = \sqrt{\epsilon^2 + \Delta^2}$, and $\hbar = 1$ throughout the paper. From the perspective of the qubit, the measurement ap-
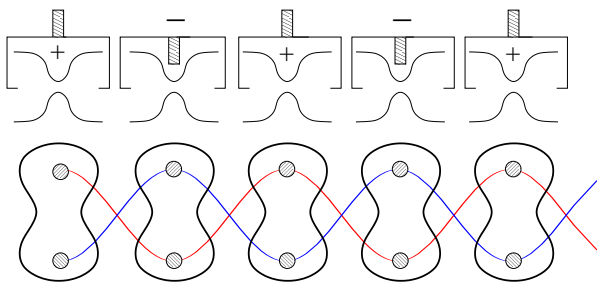
FIG. 2: (color online). (After Ref. 11). Visualization of the kicked QND measurement scheme. A voltage pulse is applied to the quantum point contact on a time scale $\tau_V \ll \tau_q$, followed by a quiet period of zero voltage bias, lasting for a Rabi oscillation period τ_q, followed by another pulse, and so on. The up/down variation is depicted, where the kicks come every half period, and the sign of the voltage pulse alternates with every kick. In this scheme, qubit read-out is by simply measuring the sign of the current, and corresponds to an elementary quantum pump.

paratus only measures at approximately discrete points in time. In this reduced problem, by choosing the waiting time between kicks to be $\tau_q = 2\pi/E$ (or some integer multiple n thereof), the unitary evolution (15) becomes $\mathbf{U} \to (-\mathbf{1})^n$. The operator we want to measure is then static in time, and is thus a QND measurement. (The evolution is also simple if n is a half-integer, especially if $\epsilon = 0$). However, from the point of view of the detector, the on/off pulse lasts much longer than any detector time scale, so many electrons pass through the QPC. If τ_0 is the time scale of the QPC electron correlation, τ_V the time scale of the pulse duration, and τ_q is the Rabi oscillation period, then the considered time scale ordering is $\tau_0 \ll \tau_V \ll \tau_q$.[22]

A pump variation on kicked QND measurements was given by Büttiker and the authors in Ref. 11, where instead of giving the same kick every Rabi oscillation, the experimentalist gives a sequence of voltage kicks to the QPC with a pulse generator, alternating in sign, every *half* oscillation period (see Fig. 2). In this scenario, we have shown that if $\epsilon = 0$, qubit readout is accomplished by pumping current: the kicks provide one AC current source, and the dynamics of the qubit provides another (intrinsically quantum mechanical) AC current source, that nevertheless causes a net DC current flow in the QPC.[23,24] There are two limiting cases the system is driven into: Either the qubit oscillations are of the same phase with the pump oscillations, pumping positive current, or the qubit oscillations are of opposite phase with the qubit oscillations, pumping negative current.

To characterize the result of each measurement kick, the parameters of the measurement process with an ideal QPC detector are specified by the currents, I_1 and I_2, produced by the detector when the qubit is in state $|1\rangle$ or $|2\rangle$, and the detector shot noise power $S_I = eI(1 - T)$ (where T is the transparency).[26] The typical integration time needed to distinguish the qubit

signal from the background noise is the measurement time $T_M = 4S_I/(I_1 - I_2)^2$. Shifted, dimensionless variables may be introduced by defining the current origin at $I_0 = (I_1 + I_2)/2$, and scaling the current per pulse as $I - I_0 = x(I_1 - I_2)/2$, so $I_{1,2}$ are mapped onto $x = \pm 1$. The weak static coupling (per pulse) between QPC and DD implies that the kick duration τ_V is less than the measurement time T_M. We take x to be normally distributed with variance $D = T_M/\tau_V$. The typical number of kicks needed to distinguish the two states is D, where we assume $D \gg 1$.

The measurement result $\mathcal{I}$ after N kicks is

$$\mathcal{I} = \frac{1}{N} \sum_{n=1}^{N} x_n, \qquad (16)$$

and we seek the conditional probability distribution $P(\mathcal{I}, N|\rho)$ of measuring the dimensionless current $\mathcal{I}$, starting with a given density operator ρ prepared before the first kick. The functions $P(\mathcal{I}, N|j)$ are defined as classical probability distributions of the current with mean $\mathcal{I} = \pm 1$ (if $j = \pm 1$) and variance $\sigma^2 = D/N$, the Gaussian equivalent of (12),

$$P(\mathcal{I}, N|1) \equiv \frac{1}{\sqrt{2\pi D/N}} \exp\left[-\frac{(\mathcal{I} - 1)^2}{2D/N} \right],$$

$$P(\mathcal{I}, N|2) \equiv \frac{1}{\sqrt{2\pi D/N}} \exp\left[-\frac{(\mathcal{I} + 1)^2}{2D/N} \right], \quad (17)$$

and the notation $P_j(x_n)$ is adopted for the $j = 1, 2$ distributions of the nth kick. The probability density of measuring the result x_n after one kick is determined by the state of the qubit just before the measurement, and is given by the analog of (7),

$$P(x_n) = \rho_{11}^{(n)} P_1(x_n) + \rho_{22}^{(n)} P_2(x_n). \qquad (18)$$

The density matrix of the qubit is updated based on the information obtained from the measurement that just occurred. This is done with the quantum Bayesian update rules,[15] that defines a non-unitary quantum map directly analogous to Eqs. (8,9),

$$\rho_{11}^{(n+1)} = \frac{\rho_{11}^{(n)} P_1(x_n)}{\rho_{11}^{(n)} P_1(x_n) + \rho_{22}^{(n)} P_2(x_n)},$$

$$\rho_{12}^{(n+1)} = \rho_{12}^{(n)} \sqrt{\rho_{11}^{(n+1)} \rho_{22}^{(n+1)} / \rho_{11}^{(n)} \rho_{22}^{(n)}},$$

$$\rho_{22}^{(n+1)} = 1 - \rho_{11}^{(n+1)}, \quad \rho_{12}^{(n+1)} = (\rho_{21}^{(n+1)})^*. \quad (19)$$

This quantum map is a probabilistic, non-unitary relative of the unitary maps studied in kicked quantum chaos.[27,28] The advantage of QND measurement in the Bayesian approach is seen by using Eqs. (19) to express the conditional probability density $P(x_n)$ in terms of the result of the preceding kick x_{n-1}. It follows from (18,19) that

$$P(x_n) = \frac{\rho_{11}^{(n-1)} P_1(x_{n-1}) P_1(x_n) + \rho_{22}^{(n-1)} P_2(x_{n-1}) P_2(x_n)}{P(x_{n-1})}.$$

$$(20)$$

This recursive relation helps in the calculation of the (unconditional) probability distribution $P(\mathcal{I}, N|\rho)$ of finding current $\mathcal{I}$, starting with the density matrix ρ, after N kicks, given by

$$P(\mathcal{I}, N|\rho) = \int \prod_{n=1}^{N} dx_n P(x_n)\, \delta\left(\mathcal{I} - \sum_{i=1}^{N} x_i/N\right). \quad (21)$$

Each application of (20) generates a denominator that cancels the probability density immediately preceding it in (21). Making N iterations of (20) gives

$$P(\mathcal{I}, N|\rho) = \int \prod_{n=1}^{N} dx_n\, [\rho_{11}P_1(x_1)\ldots P_1(x_N) + \rho_{22}P_2(x_1)\ldots P_2(x_N)]\,\delta(\mathcal{I} - \sum_{i=1}^{N} x_i/N) = \rho_{11}P(\mathcal{I}, N|1) + \rho_{22}P(\mathcal{I}, N|2), \quad (22)$$

where ρ_{11}, ρ_{22} are the diagonal matrix elements of the original density matrix, and the N Gaussians compose to form one Gaussian with a variance N times smaller. As N is increased, the two qubit states can be distinguished with greater statistical confidence, and eventually the distributions limit to delta-functions, giving either $\mathcal{I} = 1$ with probability ρ_{11}, or $\mathcal{I} = -1$ with probability ρ_{22}. A one-sigma confidence is obtained when $N = D$, as previously stated.

It is worthwhile to point out several features of the above QND measurement. First, N weak measurements simply compose to make an N-times stronger measurement. Second, the QND measurement output only involves the diagonal density matrix elements. It is for this reason that the output of a quantum nondemolition measurement is equivalent to noisy classical measurement, where the detected "classical probabilities" are given by the diagonal density matrix elements in the preferred measurement basis. In spite of the classical nature of the detector output, the qubit state prepared after the N measurements can be deduced from the outcome of the random variable $\mathcal{I}$. To characterize the post-measurement density matrix, we note another recursion relation from (19),

$$\begin{aligned} \rho_{11}^{(n+1)}/\rho_{22}^{(n+1)} &= [\rho_{11}^{(n)}/\rho_{22}^{(n)}][P_1(x_n)/P_2(x_n)] \\ &= [\rho_{11}^{(n)}/\rho_{22}^{(n)}]\exp(2x_n/D). \end{aligned} \quad (23)$$

This result may be composed N times, and the update of the off-diagonal matrix element (19) follows from (23). Using the definition $\mathcal{I} = (1/N)\sum_{n=1}^{N} x_i$, the measurement of a given random current $\mathcal{I}$ prepares a new density matrix of the DD,

$$\rho' = \frac{1}{\rho_{11}\, e^{\gamma} + \rho_{22}\, e^{-\gamma}} \begin{pmatrix} \rho_{11}\, e^{\gamma} & \rho_{12} \\ \rho_{12}^{*} & \rho_{22}\, e^{-\gamma} \end{pmatrix}, \quad (24)$$

where $\gamma = \mathcal{I}N/D$ is the rescaled measurement result, named the *rapidity* of the measurement, for reasons given in the Sec. IV. The conditional quantum dynamics of Eq. (24) is illustrated in Fig. 3, for all pure states, and the density matrix is parameterized as $\rho = (\mathbf{1} + \sum_i X_i\sigma_i)/2$,
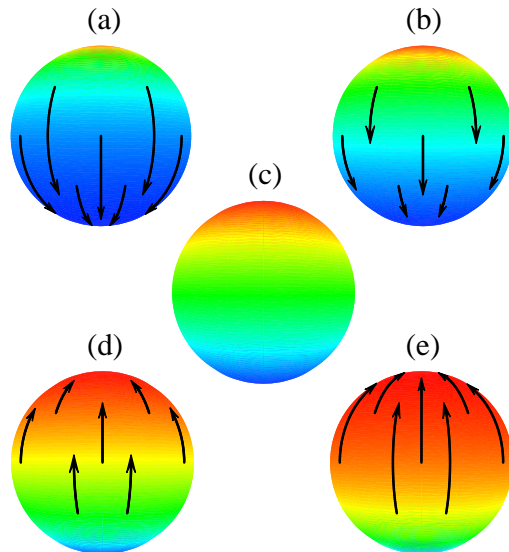

FIG. 3: (color online). (After Ref. 11). The conditional evolution of all initial pure states, represented on the Bloch sphere, under continuous QND measurement by an efficient detector. From (a-e), the *rapidity* of the measurement is $\gamma = N\mathcal{I}/D = (-1, -.5, 0, .5, 1)$ respectively. As the detector obtains more information about the quantum state, we can with greater statistical certainty distinguish the post-measurement quantum state, so the Bloch sphere is more and more red ($|1\rangle$) or blue ($|2\rangle$), depending on the value of the rapidity measured. The conditional evolution of several representative states is also indicated with black arrows. The view-point is parallel to the equator of the Bloch sphere, so $|1\rangle \to$ North pole, and $|2\rangle \to$ South pole.

so (X, Y, Z) give coordinates on the Bloch sphere. The X and Y behavior follows from Z, which is in turn conditioned on the detector output $\mathcal{I}$, so the sphere is colored according to the conditional evolution of Z,

$$Z' = \frac{\sinh\gamma + Z\cosh\gamma}{\cosh\gamma + Z\sinh\gamma}. \quad (25)$$

If the rapidity γ is positive, then states are "attracted" toward the North pole, while if the rapidity γ is nega-

tive, then states are "attracted" toward the South pole. As the rapidity grows increasingly positive or negative, we become more confident which state the qubit has continuously collapsed to, but this also depends on the initial state. The conditional evolution of several representative states is indicated with black arrows.

While the whole point of the kicked QND proposal was to effectively turn off the qubit unitary evolution while the continuous measurement is taking place, a much more interesting situation arises when continuous (non-unitary) measurements are combined with controlled unitary rotations. The kicked measurement set-up provides a simple way of generating a single-qubit rotation: waiting. Rather than spacing the pulses by a full Rabi oscillation as previously described, we choose to wait some fraction r of a Rabi oscillation, $t_{\text{wait}} = r\tau_q$, that defines a phase shift $\phi = 2\pi r$. A single-qubit unitary operation (expressed in the z-eigenbasis),

$$\mathbf{U} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}, \qquad (26)$$

may be executed by choosing r, such that $a = \cos(\phi/2) - i(\epsilon/E)\sin(\phi/2)$, and $b = -i(\Delta/E)\sin(\phi/2)$, by using the Hamiltonian evolution of the qubit, Eq. (15). Varying r, any point may be reached on a circle on the Bloch sphere, which is fixed by ϵ and Δ. In order to reach any pure state (up to an overall phase) by Hamiltonian evolution starting with a pure state, the qubit asymmetry ϵ should also be varied between the kicks with gate voltages.

IV. POVM MEASUREMENT AS A STOCHASTIC CONFORMAL MAP

Before considering specific examples of weak measurement, combined with unitary operations, we first reformulate the above set of weak measurements (18,19), and unitary operations (26) in an important special case: where the initial state is pure, the detector is efficient (as considered in this paper), so the post-measurement state is also pure. The initial arbitrary DD state is defined as $|\psi\rangle = \alpha|1\rangle + \beta|2\rangle$, with density matrix elements $\rho_{11} = |\alpha|^2$, $\rho_{12} = \alpha\beta^*$, $\rho_{21} = \rho_{12}^*$, $\rho_{22} = |\beta|^2 = 1 - \rho_{11}$. Represented as coordinates on the Bloch sphere, (X, Y, Z), both measurements and unitary operations leave the state on the surface, $X^2 + Y^2 + Z^2 = 1$.

Now make a stereographic projection of the Bloch sphere onto the complex plane, with the complex variable ζ, defined as

$$\zeta = \rho_{12}/\rho_{22} = (X + iY)/(1 - Z) = \alpha/\beta. \qquad (27)$$

The South pole of the Bloch sphere is identified as the origin of the ζ plane, while the North pole is identified as ∞ on the ζ plane. Translating the unitary operation (26) on the qubit into an operation on the complex variable ζ, we find the conformal mapping,

$$\zeta' = \frac{a\zeta - b^*}{b\zeta + a^*}, \qquad (28)$$

known as a Möbius transformation. The group algebra of unitary rotations maps onto the group algebra of conformal Möbius transformations. Reference 29 points out that this property can be used to demonstrate operator product identities for one qubit.

Translating the non-unitary Bayesian update equations for the density matrix (19), as an operation on the complex coordinate ζ, we find the following *stochastic conformal mapping:*

$$\zeta' = \zeta\sqrt{P_1(x)/P_2(x)} = \zeta\exp(x/D). \qquad (29)$$

This conformal mapping is simply a random scale transformation with two fixed points: one at 0 (the South pole, state $|2\rangle$) and the other at ∞ (the North pole, state $|1\rangle$). The random variable x is chosen from the probability distribution (18), which translates to

$$P(x) = \frac{\zeta P_1(x) + (\zeta^*)^{-1} P_2(x)}{\zeta + (\zeta^*)^{-1}}. \qquad (30)$$

Thus, any sequence of weak measurements, combined with unitary operations can be translated into repeated conformal mapping. We note that after an arbitrary sequence of weak measurements and unitary operations, the definition $\zeta = \alpha/\beta$, together with the normalization of the state, $|\alpha|^2 + |\beta|^2 = 1$, immediately allows the wavefunction to be read off (up to an overall phase). The inclusion of asymmetric measurements, where the phase shift in Eq. (9) is kept for a broader class of scattering matrices may also be easily included. This phase shift has the effect of twisting the Bloch sphere proportionally to the value of current measured, which is equivalent to including phases in the scale factor,

$$\zeta' = \zeta\exp(i\theta_1)\exp[x(1 + i\theta_2)/D], \qquad (31)$$

where θ_1, θ_2 correspond to the continuous limit of the total acquired phase shift $\Phi(m, M) = M\chi + m(\xi - \chi)$, described in Sec. II. The mapping (31) is obviously still conformal.

If we momentarily let x be a deterministic variable, then the set of Möbius mappings form a group, and the set of deterministic scale transformations form a different group. A natural question that arises is what (if any) group is described by the composition of (28), with (29)? Amusingly, the answer is provided by the special theory of relativity. Consider a relativistic four-vector (X, Y, Z, T). As is well known, any element of the Lorentz group may be produced by making a spatial rotation, followed by a boost in the (say) Z direction, followed by another spatial rotation. The boost from (X, Y, Z, T) to (X', Y', Z', T') in the Z direction may be described as a hyperbolic rotation

$$\begin{pmatrix} T' \\ Z' \end{pmatrix} = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}\begin{pmatrix} T \\ Z \end{pmatrix}, \quad X' = X, \quad Y' = Y,$$

$$\gamma = (1/2)\log\left(\frac{1 + v}{1 - v}\right), \qquad (32)$$

where the rapidity γ is introduced in terms of the velocity parameter v, the physical velocity measured in units of the speed of light, $c = 1$.

To connect this to spinor formalism, we follow the discussion in Penrose and Rindler,[30] and define a Hermitian coordinate operator,

$$\mathbf{C} = \begin{pmatrix} T+Z & X+iY \\ X-iY & T-Z \end{pmatrix}. \tag{33}$$

Translating the boost (32) into an operation on the coordinate operator $\mathbf{C}$ yields

$$\mathbf{C}' = \mathbf{A}\mathbf{C}\mathbf{A}^\dagger, \qquad \mathbf{A} = \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix}. \tag{34}$$

By fixing $T = 1$, the celestial sphere $X^2 + Y^2 + Z^2 = 1$ is defined. The celestial sphere is then stereographically projected, defining the complex variable $\zeta = (X + iY)/(1 - Z)$. In the complex plane, the boost is simply a scale transformation,

$$\zeta' = \zeta \exp\gamma. \tag{35}$$

To extend the analysis to N boosts with rapidities γ_i, the mapping simply composes the N scalings, to produce a boost with rapidity $\gamma = \sum_{i=1}^{N} \gamma_i$. This conformal mapping is identical with (29), the analogous quantum measurement composition, if $\gamma = N\mathcal{I}/D$, the quantum measurement parameter, is identified with the rapidity of the boost. Furthermore, any spatial rotation of the sphere may be interpreted as a unitary operation on the Bloch sphere, which projects to the Möbius mapping (up to an overall phase). Thus, the group described by the composition of (28), with (29) is the (restricted) Lorentz group.

The difference with the relativity analogy comes when we recall that $\gamma = \sum_i x_i/D$ is a random variable. The distribution of this random variable (30), explicitly depends on the "space-time" coordinates ζ, and thus breaks the Lorentz invariance by introducing a preferred reference frame, the Z-axis, with $Z = \pm 1$ as the attracting fixed points. From the quantum measurement point of view, this is a consequence of choosing to measure along the Z-axis. Therefore, for pure states, the mapping (29, 30) may be viewed as a stochastic Lorentz semi-group.

V. COMBINED WEAK MEASUREMENTS AND UNITARY OPERATIONS

After having separately described weak measurement and unitary operations, we now combine them. Consider an experiment, where N_1 kicks are made, followed by a single qubit unitary operation $\mathbf{U}$ (produced by inserting a dislocation into the pulse sequence), followed by N_2 kicks. The measurement results $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined as

$$\mathcal{I}_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} x_i, \qquad \mathcal{I}_2 = \frac{1}{N_2} \sum_{i=N_1+1}^{N_1+N_2} x_i. \tag{36}$$

We seek the normalized probability distribution $P(\mathcal{I}_1, N_1; \mathcal{I}_2, N_2)$ of finding current $\mathcal{I}_1$ after N_1 kicks, and $\mathcal{I}_2$ after N_2 subsequent kicks. This distribution may also be interpreted as joint counting statistics.

The analysis from Sec. III indicates that after the first N_1 kicks, the measured current $\mathcal{I}_1$ will occur with a probability given by (22), and prepares a post-measurement density matrix ρ', given by (24), described with the rapidity of the measurement $\gamma = \mathcal{I}_1 N_1/D$. The subsequent unitary operation $\mathbf{U}$ (26), (characterized by a phase ϕ) rotates the post-measurement density matrix,

$$\rho^{\text{new}} = \mathbf{U}\,\rho'\,\mathbf{U}^\dagger. \tag{37}$$

The following set of N_2 kicks start with the density matrix (37), and continue to measure in the z-basis as before. Equation (22) may be applied again with the modified initial density matrix (37) to deduce the (unconditional) probability distribution of finding result $\mathcal{I}_1$ after N_1 kicks, and result $\mathcal{I}_2$ after N_2 kicks,

$$P(\mathcal{I}_1, N_1; \mathcal{I}_2, N_2) = P(\mathcal{I}_1, N_1|\rho) \times P(\mathcal{I}_2, N_2|\rho^{\text{new}}), \tag{38}$$

where $P(\mathcal{I}_i, N_i|\rho)$ is given in (22), and the new density matrix elements are given in terms of the phase ϕ and rapidity γ as (we let $\epsilon = 0$ for simplicity),

$$\rho_{11}^{\text{new}} = \frac{\cos^2(\phi/2)\rho_{11}e^\gamma + \sin^2(\phi/2)\rho_{22}e^{-\gamma} - 2\sin(\phi/2)\cos(\phi/2)\,\text{Im}\rho_{12}}{\rho_{11}e^\gamma + \rho_{22}e^{-\gamma}}, \qquad \rho_{22}^{\text{new}} = 1 - \rho_{11}^{\text{new}},$$

$$\rho_{12}^{\text{new}} = \frac{\text{Re}\rho_{12} + (i/2)\sin\phi\,(\rho_{11}e^\gamma - \rho_{22}e^{-\gamma}) + i\cos\phi\,\text{Im}\rho_{12}}{\rho_{11}e^\gamma + \rho_{22}e^{-\gamma}}, \qquad \rho_{21}^{\text{new}} = (\rho_{12}^{\text{new}})^*, \tag{39}$$

and the natural Hamiltonian dynamics performs the unitary operation (15,26). One interesting feature of the result (38,39) is that the outcome of the first N_1 measurements, $\mathcal{I}_1$, appears in the expression involving the variables of the second set of kicks. This immediately implies that the distribution does not factorize,

$P(\mathcal{I}_1, N_1; \mathcal{I}_2, N_2) \neq P_1(\mathcal{I}_1, N_1) P_2(\mathcal{I}_2, N_2)$. The effect comes from the first set of measurements preparing a given density matrix of the DD, which affects the results of the next set of measurements. From the distribution (38), the average current in each interval, as well as the correlation between the two may be calculated:

$$
\begin{aligned}
\langle \mathcal{I}_1 \rangle &= \rho_{11} - \rho_{22}, \\
\langle \mathcal{I}_2 \rangle &= (\rho_{11} - \rho_{22}) \cos\phi - 2 \exp(-N_1/2D) \sin\phi \, \mathrm{Im}\, \rho_{12}, \\
\langle \mathcal{I}_1 \mathcal{I}_2 \rangle &= \cos\phi. \quad (40)
\end{aligned}
$$

Also, as N_1, N_2 are taken to infinity in (38,39), the distribution $P_{\mathrm{PM}}(\mathcal{I}_1, \mathcal{I}_2)$ from making simple projective measurements on the DD is recovered,

$$
\begin{aligned}
P_{\mathrm{PM}} &= \rho_{11} \cos^2(\phi/2) \, \delta(\mathcal{I}_1 - 1)\delta(\mathcal{I}_2 - 1) \\
&+ \rho_{11} \sin^2(\phi/2) \, \delta(\mathcal{I}_1 - 1)\delta(\mathcal{I}_2 + 1) \\
&+ \rho_{22} \sin^2(\phi/2) \, \delta(\mathcal{I}_1 + 1)\delta(\mathcal{I}_2 - 1) \\
&+ \rho_{22} \cos^2(\phi/2) \, \delta(\mathcal{I}_1 + 1)\delta(\mathcal{I}_2 + 1). \quad (41)
\end{aligned}
$$

It is now straightforward to generalize the result (38,39) to any number of $m-1$ dislocations in the pulse sequence, each of which has a phase shift of ϕ_k (and now ϵ is arbitrary),

$$
P(\{\mathcal{I}_j, N_j\}) = \prod_{k=1}^{m} \left[\rho_{11}^{(k)} P(\mathcal{I}_k, N_k | 1) + \rho_{22}^{(k)} P(\mathcal{I}_k, N_k | 2) \right],
$$
(42)

and each density matrix $\rho^{(k+1)}$ is defined in terms of the density matrix $\rho^{(k)}$ after the previous dislocation,

$$
\rho^{(k+1)} = \mathbf{U}_k \frac{1}{\mathcal{D}_k} \begin{pmatrix} \rho_{11}^{(k)} e^{\gamma_k} & \rho_{12}^{(k)} \\ \left[\rho_{12}^{(k)}\right]^* & \rho_{22}^{(k)} e^{-\gamma_k} \end{pmatrix} \mathbf{U}_k^\dagger, \quad (43)
$$

where $\mathcal{D}_k = \rho_{11}^{(k)} e^{\gamma_k} + \rho_{22}^{(k)} e^{-\gamma_k}$, the rapidities are $\gamma_k = \mathcal{I}_k N_k / D$, the initial density matrix before the first kick is $\rho^{(1)}$, and the matrix $\mathbf{U}_k$ has elements

$$
\begin{aligned}
(\mathbf{U}_k)_{11} &= \cos(\phi_k/2) - i(\epsilon/E) \sin(\phi_k/2), \\
(\mathbf{U}_k)_{12} &= -i(\Delta/E) \sin(\phi_k/2), \\
(\mathbf{U}_k)_{21} &= -i(\Delta/E) \sin(\phi_k/2), \\
(\mathbf{U}_k)_{22} &= \cos(\phi_k/2) + i(\epsilon/E) \sin(\phi_k/2). \quad (44)
\end{aligned}
$$

Reference 11 applied these results to violate a Bell inequality in time.

VI. CONDITIONAL PHASE SHIFTS AND FEEDBACK PROTOCOLS

In the preceding Section, the phase shift ϕ was chosen beforehand, independently of the result $\mathcal{I}_1$. We can now use the information gained in the first N_1 measurements, and make a *conditional* phase shift, pending the outcome of the random variable $\mathcal{I}_1$. This is essentially a feedback protocol that the experimentalist can choose to

execute, defined by a function $\phi(\mathcal{I}_1)$, so a different phase shift is assigned to every possible random outcome of the continuous measured current.

Kicked QND measurement provides a realistic mechanism for implementing general qubit feedback protocols. The reason for this is two-fold: First, as seen in the previous section, any combination of weak measurements and unitary operations may be accomplished with a sequence of voltage pulses to the detector. Second, the feedback circuitry must take the result obtained from the measurement, execute logical operations, and command the experimental apparatus to do something it otherwise would not have done (like make a given phase shift). The intrinsic waiting time between the kicks provides the needed time delay for all of the above to take place.

We now explicitly find the feedback protocol $\phi(\mathcal{I}_1)$ to take a given pure state to any desired pure state after a weak measurement. As the simplest case, consider any pure state on the $Z - Y$ great circle of the Bloch sphere ($X = 0, Y^2 + Z^2 = 1$), and a symmetric qubit, $\epsilon = 0$. Both Hamiltonian evolution, and weak measurements (according to (24)) do not take these states out of the $Z - Y$ great circle. Therefore, knowing the outcome $\mathcal{I}_1$, a conditional phase shift may be applied to deterministically prepare *any* quantum state on the $Z - Y$ great circle of the Bloch sphere. For definiteness, we choose to shift to the state $|1\rangle$. This choice has the advantage that after the $N_2 \gg D$ measurements, the current will be $\mathcal{I}_2 = 1$ deterministically. The parametrization $(Y, Z) = (-\sin\theta, \cos\theta)$ of the initial state is chosen, so that if no measurement is made, the shift to the North pole may be done with a phase shift $\phi = \theta$. The result (39) is applied by setting $\rho_{11}^{\mathrm{new}} = 1$, and solving for ϕ as a function of the rapidity γ, to find

$$
\tan(\phi/2) = \tan(\theta/2) \, \exp(-\gamma). \quad (45)
$$

This answer interpolates between two extreme strategies:[31] (1) If no measurement is made, just make the desired phase shift, $\phi = \theta$. (2) If a projective measurement is made, either do nothing if $\mathcal{I}_1 = 1$, or flip the state by applying the phase shift $\phi = \pi$ if $\mathcal{I}_1 = -1$. The asymptotic limits in the later case may be obtained by expanding the inverse tangent to obtain

$$
\phi/2 \approx \begin{cases} \tan(\theta/2) \exp(-\gamma) & \text{if } \gamma \gg 1, \text{and } \theta \neq \pm\pi, \\ \frac{\pi}{2} \, \mathrm{sign}(\theta) - \frac{\exp(\gamma)}{\tan(\theta/2)} & \text{if } \gamma \ll -1, \text{and } \theta \neq 0. \end{cases}
$$
(46)

The above real-time feedback proposal is experimentally promising in the kicked scheme. However, it demands fast time resolution and feedback circuitry. An experimentally simpler proposal to verify the above protocol is to make many realizations of weak measurement, phase shift, weak measurement, where the phase shift is chosen randomly in each realization. After the run is finished, the data record may be reviewed, and all instances of phase shifts where condition (45) is approximately satisfied are post-selected. In this data subset,

the prediction is that the following set of $N_2 \gg D$ kicks will deterministically find $\mathcal{I}_2 = 1$.

VII. PURIFICATION OF INITIALLY MIXED DENSITY MATRICES BY WEAK MEASUREMENT

Under repeated weak QND measurements, eventually all states collapse to either $|1\rangle$ or $|2\rangle$, including mixed initial states. However, the states $|1\rangle$ and $|2\rangle$ are both pure, and therefore if the initial state is mixed, a purification occurs during the measurement process.[32] This phenomenon is especially counter-intuitive from the point of view of the dephasing approach to quantum measurement. Jacobs has shown that the average purification in a given time can be increased by the use of continuous feedback.[14] Jacobs' protocol is somewhat counterintuitive for the qubit: always use Hamiltonian evolution to rotate the state to $Z = 0$, i.e. perpendicular to the measurement axis. The purpose of this section is (1) To show how this idea can be easily implemented for our set-up, (2) To demonstrate that the "equatorial plane" protocol (i.e. $Z = 0$) is also optimal for kicked QND measurements which have a continuous output of tunable measurement strength, and (3) To generalize Jacobs' no-feedback purification solution to any initial density matrix.

It is well known that any unitary operation preserves the purity (or entropy) of the state. It is interesting to note from (19), that during weak measurement there is a different preserved physical quantity, that we name the *murity*. For the qubit, the purity $\mathcal{P}$ and the murity M are defined as

$$\mathcal{P} = X^2 + Y^2 + Z^2, \quad M = (X^2 + Y^2)/(1 - Z^2). \quad (47)$$

If the purity $\mathcal{P} = 1$, then the murity $M = 1$, reflecting the statement made in Sec. II that if the initial state is pure, the post-measurement state is also pure. We note that $\mathcal{P}$ may be expressed in terms of M and Z by $\mathcal{P} = M(1 - Z^2) + Z^2$. After one kick, the change in the purity, $\Delta\mathcal{P}$, (or purification) is given by

$$\Delta\mathcal{P} = \mathcal{P}' - \mathcal{P} = (1 - M)[(Z')^2 - Z^2], \quad (48)$$

where we have used the fact that murity does not change during measurement. Application of the quantum Bayesian update rules (19) yields $Z' = [\rho_{11}P_1(x) - \rho_{22}P_2(x)]/(\rho_{11}P_1(x) + \rho_{22}P_2(x))$ (also given in (25)), where x is the measurement result, so the purification is

$$\Delta\mathcal{P} = (1 - \mathcal{P})\left(1 - \frac{1}{[\cosh(x/D) + Z\sinh(x/D)]^2}\right). \quad (49)$$

Several observations are in order: First, if $\mathcal{P} = 1$, the purification $\Delta\mathcal{P}$ is automatically 0, while the first (deterministic) factor is maximal if $\mathcal{P} = 0$. Second, if $x = 0$, the second (random) factor is zero, so there is no purification, which corresponds to no gained information. Finally, the first factor is between $[0, 1]$, while the second factor could be negative or positive, implying that either purification or further mixing is possible in a given run.

The *average* purification is given by averaging (49) over the distribution of x, given in Eq. (18), to yield

$$\langle\Delta\mathcal{P}\rangle = (1 - \mathcal{P})[1 - f(D, Z)], \quad (50)$$

where

$$f(D, Z) = e^{-\frac{1}{2D}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi D}} \frac{\exp(-x^2/2D)}{\cosh(x/D) + Z\sinh(x/D)}. \quad (51)$$

It is straightforward to check that $0 \leq f \leq 1$, so there is nonnegative average purification for all density matrices.[33] Changing variables to $\gamma = x/D$, it is also straightforward to check asymptotic limits. Taking $D \to 0$, the projective limit, $f(0, Z) = 0$ is recovered, so $\langle\Delta\mathcal{P}\rangle = 1 - \mathcal{P}$, implying that the final state is pure with unit probability. The opposite limit, $D \to \infty$, corresponds to an vanishingly weak measurement, so $f(\infty, Z) = 1$, or $\langle\delta\mathcal{P}\rangle = 0$, giving no purification.

The results (50,51) allow us to find the optimum average purification strategy for one kick. The best strategy on average is to rotate the density matrix to where the purification (50) is maximum. These point(s) may be found by maximizing $\langle\Delta\mathcal{P}\rangle$ on the Bloch ball, under the constraint that D and $\mathcal{P}$ are fixed. The $\mathcal{P}$ constraint simply reflects the fact that unitary operations do not alter the purity. This problem is equivalent to minimizing f with respect to Z, by solving $df/dZ = 0$, which leads to the equation

$$-\int_{-\infty}^{\infty} d\gamma \frac{\exp(-D\gamma^2/2)\sinh\gamma}{(\cosh\gamma + Z\sinh\gamma)^2} = 0. \quad (52)$$

The solution is immediate because at the point $Z = 0$, the integrand is odd, so the integral is zero. The fact that f is minimized at $Z = 0$ is seen by noting that the integrand of $d^2f/dZ^2(Z = 0)$ is nonnegative. Therefore, the average purification is maximized by applying a phase shift after the first measurement that rotates the qubit to the equatorial plane of the Bloch ball before the measurement. This result is in agreement with Jacobs, who considered purification from a two-outcome POVM of variable strength (and also the stochastic Schrödinger equation limit).[14] Our approach is from the complimentary perspective of a continuous outcome measurement of variable strength.

Results (50,51) have a simpler form in the large D limit, for very weak measurements. The average purification $\langle\Delta\mathcal{P}\rangle$ and the noise in the purification, $\langle(\Delta\mathcal{P})^2\rangle$, are given to leading order in D^{-1} as

$$\langle\Delta\mathcal{P}\rangle = (1 - \mathcal{P})(1 - Z^2)/D,$$
$$\langle(\Delta\mathcal{P})^2\rangle = 4(1 - \mathcal{P})^2 Z^2/D. \quad (53)$$

In order to compare purification with and without feedback, we first consider Jacobs' feedback protocol $Z = 0$ at

every time step. Equation (53) implies that for this feedback protocol the purification noise vanishes. Therefore, the dynamical purification is described by the deterministic rate equation $d\mathcal{P}/dN = (1 - \mathcal{P})/D$. Solving the equation with initial condition $\mathcal{P}_0$ yields[14]

$$\langle \mathcal{P}_N \rangle = 1 + (\mathcal{P}_0 - 1)\exp(-N/D), \qquad (54)$$

showing an exponential approach to a pure state, with

rate D^{-1}.

In the no feedback case, the purity after N kicks may be found from the murity relation $\mathcal{P}' = M[1 - (Z')^2] + (Z')^2$, where Z' is given in (25). Averaging this relation over the distribution (22) yields the average purity after N kicks,[34]

$$\langle \mathcal{P}_N \rangle = \sqrt{\frac{D}{2\pi N}} \exp\left(-\frac{N}{2D}\right) \int_{-\infty}^{\infty} d\gamma \exp\left(-\frac{D\gamma^2}{2N}\right) \frac{M(\cosh\gamma + Z\sinh\gamma)^2 - (M-1)(\sinh\gamma + Z\cosh\gamma)^2}{\cosh\gamma + Z\sinh\gamma}. \qquad (55)$$

After some manipulation, the above integral expression may be simplified to

$$\begin{aligned} \langle \mathcal{P}_N \rangle &= 1 - (M-1)(Z^2-1)\sqrt{\frac{D}{2\pi N}} \exp(-N/2D) \\ &\times \int_{-\infty}^{\infty} d\gamma \frac{\exp(-D\gamma^2/2N)}{\cosh\gamma + Z\sinh\gamma}. \end{aligned} \qquad (56)$$

For large N/D, the dominant dependence comes from the term outside the integral, and the N/D dependence inside the integrand may be neglected. In this limit, the purity may be approximated as

$$\langle \mathcal{P}_N \rangle \approx 1 - \pi(1-M)\sqrt{1-Z^2}\sqrt{D/(2\pi N)}\exp(-N/2D), \qquad (57)$$

yielding an approach to purity with rate $(2D)^{-1}$, half as fast as the feedback case, in agreement with Jacobs. The result (56) generalizes Jacobs' no-feedback purification result to arbitrary initial states. Before concluding, we point out that Wiseman and Ralph have recently shown that the advantage of feedback for purification depends on how the question is formulated.[35] If instead of asking about the average purification for a fixed time, we ask about the average time taken to reach a given purity, then the no feedback case is actually better.

VIII. CONCLUSIONS

The quantum Bayesian approach to the problem of quantum measurement has been derived from POVM formalism, applied to a mesoscopic scattering detector. By considering an elementary scattering event, measurement operators associated with the successful or failed detection of the electron in the current collector can be identified. We recover the quantum Bayesian formalism in the continuous current approximation.

Kicked QND measurements have been analyzed within the quantum Bayesian formalism. We derive a quantum map representation that, while discrete in the time index,

describes a sequence of weak measurements. Unitary operations (easily implemented by waiting a fraction of a Rabi period), together with kicked measurements, can be represented as a sequence of conformal mappings, where the unitary maps are deterministic, and the kicked measurement maps are stochastic. A close analogy exists between these quantum maps and the Lorentz transformations of special relativity.

We have calculated the measurement statistics associated with combined weak measurements, and unitary operations. These results are applied to find the feedback protocol that deterministically takes a given pure state to any other desired pure state after a weak measurement, using conditional phase shifts.

Next, we have investigated the process of purification of mixed density matrices under kicked QND measurement. The concept of "murity" (the physical quantity that is preserved under measurement) has been introduced, and applied to calculate the change in the state's purity associated with a measurement. Purification with and without feedback has also been investigated.

We stress that kicked QND measurements provides an experimentally viable way of implementing ideas in quantum feedback: Any combination of weak measurements and unitary operations can be accomplished by applying a sequence of voltage pulses to the detector, and the intrinsic quiet time between kicks allows the necessary processing time for feedback to occur.

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[34] This purification result may be expressed more generally as $\langle \mathcal{P}_N \rangle = \int d\mathcal{I}\{[\rho_{11}P_1(\mathcal{I}) - \rho_{22}P_2(\mathcal{I})]^2 + 4|\rho_{12}|^2 P_1(\mathcal{I})P_2(\mathcal{I})\}/[\rho_{11}P_1(\mathcal{I}) - \rho_{22}P_2(\mathcal{I})]$.

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