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Decoupling Limit, Lens Spaces and Taub-NUT: D=4 Black Hole Microscopics from D=5 Black Holes

M. Cvetič [†1], H. Lü [†1] and C.N. Pope [‡2]

[†] *Department of Physics and Astronomy*
University of Pennsylvania, Philadelphia, Pennsylvania 19104

[‡] *Center for Theoretical Physics*
Texas A&M University, College Station, Texas 77843

ABSTRACT

We study the space-times of non-extremal intersecting p-brane configurations in M-theory, where one of the components in the intersection is a "NUT," *i.e.* a configuration of the Taub-NUT type. Such a Taub-NUT configuration corresponds, upon compactification to $D = 4$, to a Gross-Perry-Sorkin (GPS) monopole. We show that in the decoupling limit of the CFT/AdS correspondence, the 4-dimensional transverse space of the NUT configuration in $D = 5$ is foliated by surfaces that are cyclic lens spaces S^3/Z_N, where N is the quantised monopole charge. By contrast, in $D = 4$ the 3-dimensional transverse space of the GPS monopole is foliated by 2-spheres. This observation provides a straightforward interpretation of the microscopics of a $D = 4$ string-theory black hole, with a GPS monopole as one of its constituents, in terms of the corresponding $D = 5$ black hole with no monopole. Using the fact that the near-horizon region of the NUT solution is a lens space, we show that if the effect of the Kaluza-Klein massive modes is neglected, p-brane configurations can be obtained from flat space-time by means of a sequence of dimensional reductions and oxidations, and U-duality transformations.

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1 Introduction

Advances in the quantitative treatment of black-hole microscopics [1] represent an important "spin-off" of M-theory unification, facilitated by developments in the quantum treatment of non-pertubative objects in string theory, such as D-branes [2].

More recently, aspects of the black-hole microscopics have found an elegant reinterpretation within the framework of the CFT/AdS correspondence [3]. Namely, the microscopic interpretation of black hole entropy can be made quantitative in terms of the boundary conformal field theory, as determined by the anti-de Sitter space-time, *i.e.* the (asymptotic) geometry in the decoupling limit of certain black holes. In particular, the near-extremal black holes in $D = 5$ [4, 5] (or $D = 4$) [6, 7] have a six-dimensional (or five-dimensional) embedding [9, 10] as black strings, with or without rotation, whose geometry in the decoupling limit is BTZ$\times S^3$ (or BTZ$\times S^2$), where BTZ denotes the Bañados-Teitelboim-Zanelli three-dimensional black hole space-time, which is locally AdS$_3$ [8]. Thus its quantum states are determined asymptotically by a two-dimensional conformal field theory at the asymptotic boundary [11]. The counting of states in this CFT is then used [4, 12] to reproduce the Bekenstein-Hawking entropy. (The analysis of black holes in $D \geq 6$ involves a correspondence to CFT's in $D > 2$, and there, due to renormalisation effects [13] in strongly coupled CFT's, only a qualitative modelling of the microscopic black hole entropy is possible [14, 15].) Further study of the microscopic black hole spectrum both on the gravity AdS$_3$ side [16], as well as on the CFT$_2$ side [17, 18, 19], has been pursued.

This paper addresses a number of related topics. In section 2, we study the near-horizon geometry that is relevant in the decoupling limit for extremal and non-extremal p-branes in M-theory, in the case where one of the ingredients in the intersection is a NUT, *i.e.* a configuration of the Taub-NUT type. Such intersections become four-dimensional black holes upon dimensional reduction on T^7, with the Taub-NUT component corresponding to a magnetic charge carried by one of the Kaluza-Klein vectors.[1] If the Kaluza-Klein vector is the one coming from the reduction step from $D = 5$ to $D = 4$, this corresponds to the situation arising in the Gross-Perry-Sorkin (GPS) monopole (*i.e.* a $D = 4$ black hole where the Kaluza-Klein vector from $D = 5$ carries a magnetic charge [20]). If, on the other hand, we consider eleven-dimensional configurations with the same set of intersecting components, but with no NUT, they will now already be interpretable as black holes (as opposed to

[1] In general, the higher-dimensional configuration with the topological twist in the fibre coordinate associated with the Kaluza-Klein vector carrying the magnetic charge will be described, for brevity, as a NUT.

Taub-NUT configurations) in $D=5$, with one less charge than that in $D=4$. (This will be discussed further in section 2.)[2] The crucial observation is that in the decoupling limit, the foliating 3-surfaces in the transverse space of the NUT solution have the geometry of a cyclic lens space, with the topology S^3/Z_N, where N is the quantised NUT charge. This observation allows us to interpret, *via* the AdS/CFT correspondence, the microscopics of such $D=4$ black holes in terms of $D=5$ black holes with one less charge.

In section 3, we show that by performing a sequence of dimensional reductions and oxidations on the $U(1)$ fibres of the foliating 3-spheres of a flat 4-dimensional transverse space, supplemented by appropriate U-duality transformations, we can generate p-brane configurations in M-theory from the flat space. This provides a new way of generating BPS solutions, by starting from a flat space-time and just using symmetries of the theory as a solution-generating procedure.

2 Decoupling limit and microscopics of $D=4$ black-hole states with GPS-monopole

In four-dimensional maximal supergravity, which is the effective low-energy limit of M-theory compactified on the 7-torus, black-hole solutions form multiplets under the $E_{7(+7)}$ U-duality group. The prototype solution is specified by four charges. (The generating solution of the most general black hole, consistent with the no-hair theorem, is actually specified by five charges [21]; however the global space-time features are essentially captured by the 4-charge solution.) In the "diagonal" case where each of the charges is associated with a specific harmonic function (*i.e.* a *generating solution*), the solutions have a simple structure, and are referred to as four-charge solutions (first specified by two electric and two magnetic charges of the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector and given explicitly in [22]). The possible field strength configurations that can give rise to such simple 4-charge

[2]Note that when one speaks of a configuration of N intersecting objects, this is not the same thing as a configuration of $(N+1)$ intersecting objects in which the $(N+1)$'th charge is set to zero. The reason for this is that the individual ingredients in an intersection are distributed uniformly over the world-volumes of the other ingredients. For example, in the case where one sets the charges of all but one of the ingredients in an intersection to zero, the remaining configuration will not be a single isotropic object, but instead it will be smeared uniformly over all the spatial world-volume directions of the (now-vanished) other ingredients. Thus when we speak of an eleven-dimensional configuration "but with no NUT," we mean the configuration that one would have considered if the NUT were never introduced, as opposed to the the configuration that would result from setting the NUT charge to zero in the original intersection.

solutions are given by [23]

$$
\begin{aligned}
N=4: \qquad & \{F_{(2)ij}, F_{(2)k\ell}, F_{(2)mn}, *\mathcal{F}^p_{(2)}\}_{105+105}\ , \qquad \{F_{(2)ij}, *F_{(2)ik}, \mathcal{F}^j_{(2)}, *\mathcal{F}^k_{(2)}\}_{210}\ , \\
& \{F_{(2)ij}, F_{(2)k\ell}, *F_{(2)ik}, *F_{(2)j\ell}\}_{210}\ . \qquad (2.1)
\end{aligned}
$$

(We are using the notation of [24, 25, 23] here.) The subscripts on each bracketed set of field strengths denotes the multiplicities of the solutions (corresponding to the possible permutations of index choices on the field strengths). The Hodge duals indicate that the associated fields carry electric charges if the fields without duals carry magnetic charges, and *vice versa*.

When oxidised back to $D=6$, there are four possible near-horizon limits that can arise, namely

$$
\mathrm{AdS}_3\times S^3,\ \mathrm{AdS}_3\times(S^2\times S^1),\ (\mathrm{AdS}_2\times S^1)\times S^3,\ (\mathrm{AdS}_2\times S^1)\times(S^2\times S^1)\ . \qquad (2.2)
$$

(To be precise, the AdS_3 refers only to the local space-time, which can be globally that of the BTZ black hole, and also the S^3 can in general be squashed and/or factored by a cyclic group, in the manner described in [26] and in subsequent discussions.) If we oxidise these near-horizon solutions further, to $D=10$ or $D=11$, then the additional dimensions provide further factors of T^4 or T^5 respectively. These near-horizon geometries are related to each other by Hopf T-duality, which is a T-duality that makes use of the $U(1)$ isometry of the fibre bundle coordinate over the base space [27]. The U-duality and T-duality transformations that relate the different topologies in (2.2) leave the areas of the horizons invariant, and they may therefore be called isentropic mappings [26].

The first of the three cases in (2.1) can be viewed in $D=11$ as the intersection of three M2-branes and one NUT, or else the magnetic dual of this, namely, three M5-branes and a gravitational pp-wave [28]. (The microscopic state counting in the context of the AdS/CFT correspondence has been given in [6] for static black holes, and in [7] for rotating black holes.) The third case in (2.1) can be viewed as the intersection of two M2-branes and two M5-branes [28].

In this paper, we shall concentrate on the second case in (2.1). These configurations can be viewed as intersections of an M2-brane and an M5-brane, together with a wave and a NUT. In particular, we shall consider the case where the indices j and k take the values $\{j,k\}=\{6,7\}$, so that the solution can be oxidised back to $D=6$ to become [29] a dyonic string [30, 31, 32, 33] with a pp-wave propagating on its word-sheet, and a NUT planted in its transverse space. If the index i takes any of the values $i=2$, 3, 4, or 5, the dyonic string belongs to the R-R sector, If instead the index i takes the value $i=1$, the dyonic string belongs to the NS-NS sector [29]. In this latter case, the discussion is equally applicable to the heterotic string.

2.1 Extremal case

Although the extremal solution can be obtained by taking the extremal limit of the non-extremal solution, their respective geometries are quiet different and we shall discuss them separately. We begin with the extremal solution. The extremal 4-charge black hole for the field configuration $\{F_{(2)i6}, *F_{(2)i7}, \mathcal{F}^6_{(2)}, *\mathcal{F}^7_{(2)}\}$ is a solution of the bosonic Lagrangian

$$\kappa_4^2\, e^{-1}\, \mathcal{L} = R - \tfrac{1}{2}(\partial\vec{\phi})^2 - \tfrac{1}{4}e^{\vec{a}_{i6}\cdot\vec{\phi}}\,(F_{(2)i6})^2 - \tfrac{1}{4}e^{\vec{a}_{i7}\cdot\vec{\phi}}\,(F_{(2)i7})^2 - \tfrac{1}{4}e^{\vec{b}_6\cdot\vec{\phi}}\,(\mathcal{F}^6_{(2)})^2 - \tfrac{1}{4}e^{\vec{b}_7\cdot\vec{\phi}}\,(\mathcal{F}^7_{(2)})^2\ , \tag{2.3}$$

where the index i can be any number from 1 to 5. The dilaton vectors $\vec{c}_\alpha = \{\vec{a}_{i6}, \vec{a}_{i7}, \vec{b}_6, \vec{b}_7\}$ can be found in [24, 25]; they satisfy [34]

$$\vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta} - 1\ . \tag{2.4}$$

Four-charge black hole solutions were constructed in [22]. In this case, the solution is given by

$$\begin{aligned}
&ds_4^2 = -(H_e\, H_m\, K\, U)^{-1/2}\, dt^2 + (H_e\, H_m\, K\, U)^{1/2}\,(d\rho^2 + \rho^2\, d\Omega_2^2)\ ,\\
&A_{(1)16} = H_e^{-1}\, dt\ ,\quad F_{(2)17} = Q_m\,\Omega_{(2)},\quad \mathcal{A}^6_{(1)} = K^{-1}\, dt\ ,\quad \mathcal{F}^7_{(2)} = Q_{\rm NUT}\,\Omega_{(2)}\ ,\\
&\vec{\phi} = \tfrac{1}{2}\vec{a}_{16}\,\log H_e - \tfrac{1}{2}\vec{a}_{17}\,\log H_m + \tfrac{1}{2}\vec{b}_6\log K - \tfrac{1}{2}\vec{b}_7\log U\ ,
\end{aligned} \tag{2.5}$$

where the harmonic functions are

$$H_e = 1 + \frac{Q_e}{\rho}\ ,\qquad H_m = 1 + \frac{Q_m}{\rho}\ ,\qquad K = 1 + \frac{Q_{\rm wave}}{\rho}\ ,\qquad U = 1 + \frac{Q_{\rm NUT}}{\rho}\ . \tag{2.6}$$

The metric of the extremal 4-charge black hole (2.5) in $D = 4$ has a regular horizon at $\rho = 0$, near to which the geometry approaches $\mathrm{AdS}_2 \times S^2$. The entropy of the solution is

$$\begin{aligned}
S &\equiv \frac{\rm Area}{4\kappa_4^2}\\
&= \frac{\omega_2}{4\kappa_4^2}\,\sqrt{Q_e\, Q_m\, Q_{\rm wave}\, Q_{\rm NUT}}\ .
\end{aligned} \tag{2.7}$$

Here $\omega_2 = 4\pi$ is the volume of the unit two-sphere. (In this paper we use ω_n to denote the volume of the unit n-sphere, and $\Omega_{(n)}$ to denote its volume form. Thus $\omega_n = \int \Omega_{(n)}$. We denote the metric of the unit n-sphere by $d\Omega_n^2$.)

2.1.1 $D = 4$ and $D = 5$ black holes

First, let us consider the oxidation of the 4-charge black-hole solution (2.5) to $D = 5$. Since the internal coordinate associated with this step is z_7, which we shall denote by y, it follows that the charges Q_e, Q_m and $Q_{\rm wave}$ will now be supported by the field strengths $F_{(2)i6}$, $*F_{(3)i}$ and $\mathcal{F}^6_{(2)}$ in $D = 5$. However the charge $Q_{\rm NUT}$, being associated with the Kaluza-Klein vector $\mathcal{A}^7_{(1)}$ of the $D = 5$ to $D = 4$ reduction step, becomes instead a topological "NUT charge" in $D = 5$. The metric in (2.5) oxidises to become

$$ds_5^2 = -(H_e\, H_m\, K)^{-2/3}\, dt^2 + (H_e\, H_m\, K)^{1/3)} \Big[U\, (d\rho^2 + \rho^2\, d\Omega_2^2) + U^{-1}\, (dy + Q_{\rm NUT}\, B)^2\Big] \quad (2.8)$$

in five dimensions, where B is a 1-form on the unit 2-sphere, such that $dB = \Omega_{(2)}$.

This is an appropriate juncture at which to comment further on the observation made in footnote 2. If we set the NUT charge $Q_{\rm NUT}$ in (2.8) to zero, then we get

$$ds_5^2 = -(H_e\, H_m\, K)^{-2/3}\, dt^2 + (H_e\, H_m\, K)^{1/3)} \Big[d\rho^2 + \rho^2\, d\Omega_2^2 + dy^2\Big] \,. \quad (2.9)$$

Although this ostensibly looks like a standard 3-charge black hole in $D = 5$ it is actually a *line* of 3-charge black holes, since the remaining harmonic functions H_e, H_m and K depend only on ρ, rather than on the entire radial coordinate $R = \sqrt{\rho^2 + y^2}$ in the 4-dimensional transverse space: A standard isotropic 3-charge black hole would have harmonic functions of the form $1 + 4Q/R^2$, rather than $1 + Q/\rho$. In fact the line of black holes described by the metric (2.9) is precisely what would result from performing a normal vertical oxidation of an isotropic 3-charge black hole in $D = 4$. As we shall now show, if we instead perform the oxidation when $Q_{\rm NUT}$ is an additional non-vanishing fourth charge in a $D = 4$ black hole, with y the fibre coordinate of the $U(1)$ bundle associated with the NUT charge, we instead arrive at a configuration that *does* correspond to an isotropic 3-charge black hole in $D = 5$.

If we go to a region near the horizon, defined by the requirement that $\rho << Q_{\rm NUT}$ so that we can drop the "1" in the harmonic function $U = 1 + Q_{\rm NUT}/\rho$, and if we also define a new radial coordinate r by $\rho = r^2/4$, we find that the five-dimensional metric (2.8) can be approximated as

$$ds_5^2 = -(H_e\, H_m\, K)^{-2/3}\, dt^2 + Q_{\rm NUT}\, (H_e\, H_m\, K)^{1/3)} \Big[dr^2 + \tfrac14 r^2\, d\Omega_2^2 + \tfrac14 r^2 (\frac{dy}{Q_{\rm NUT}} + B)^2\Big] \,, \quad (2.10)$$

The metric

$$d\Omega_3^2(Q_{\rm NUT}) \equiv \tfrac14 d\Omega_2^2 + \tfrac14 (\frac{dy}{Q_{\rm NUT}} + B)^2 \quad (2.11)$$

is locally the standard metric on the unit 3-sphere. In fact it would be precisely the unit metric on S^3 if y had the period $4\pi\, Q_{\rm NUT}$. If instead y has the period 4π, then the metric describes the cyclic lens space $S^3/Z_{Q_{\rm NUT}}$, where the fibre coordinate of the $U(1)$ bundle over the S^2 base is identified[3] after a translation by a fraction $1/Q_{\rm NUT}$ of its total length in S^3. (The charges are normalised here so that the Dirac quantisation condition requires that they be integers.) In fact we are obliged to take y to have the period 4π, rather than $4\pi\, Q_{\rm NUT}$, since we require that four-dimensional black holes, for *any* integer value of the charge $Q_{\rm NUT}$, should be oxidisable to regular geometries in $D=5$. If we were instead to took the period of y to be $4\pi\, Q_{\rm NUT}$ for some value of $Q_{\rm NUT}\neq 1$, then effectively this value of $Q_{\rm NUT}$ would itself define the minimum allowed "unit" of charge, and no smaller values would be permitted. This is because one cannot have a regular geometry whose $U(1)$ fibres exceed the length that corresponds to the case of S^3 itself. Only integer fractions of the length of the fibres for S^3 give regular geometries.

Rewriting the metric (2.10) in terms of $d\Omega_3^2(Q_{\rm NUT})$, we obtain

$$ds_5^2 = -(H_e\, H_m\, K)^{-2/3}\, dt^2 + Q_{\rm NUT}\, (H_e\, H_m\, K)^{1/3})\, (dr^2 + r^2\, d\Omega_3^2(Q_{\rm NUT}))\ . \tag{2.12}$$

As a result of the coordinate transformation $\rho = r^2/4$, the functions H_e, H_m and K given in (2.6), which were harmonic in the original 3-dimensional transverse space, are now given by

$$H_e = 1 + \frac{4Q_e}{r^2}\ ,\qquad H_m = 1 + \frac{4Q_m}{r^2}\ ,\qquad K = 1 + \frac{4Q_{\rm wave}}{r^2}\ . \tag{2.13}$$

These are harmonic with respect to the 4-dimensional transverse space. Note that in this "Hopf" oxidation on the $U(1)$ fibres, unlike a standard vertical oxidation in the transverse space, the harmonic functions are still isotropic in the higher dimension. (By contrast, in the usual vertical oxidation the harmonic functions would describe smeared lines of charge in the higher dimension.) The five-dimensional solution can be recognised as having the structure of the isotropic 3-charge black hole, at least if y is identified with period $4\pi\, Q_{\rm NUT}$.

2.1.2 $D=6$ dyonic string with pp-wave and Taub-NUT charge

In order to give a microscopic interpretation for the above semi-classical Hawking entropy, we shall first oxidise the solution back to $D=6$, where it describes [29] a dyonic string,

[3] Note that the identification has no fixed points, and thus $S^3/Z_{Q_{\rm NUT}}$ is a smooth manifold, and not an orbifold. However, if one considers the flat-space metric

$$ds^2 = dr^2 + \tfrac14 r^2\, d\Omega_2^2 + \tfrac14 r^2 (\frac{dy}{Q_{\rm NUT}} + B)^2\ ,$$

and takes y to have the period 4π, then there is a fixed point at the origin when the integer $Q_{\rm NUT}$ is not equal to unity, where the foliating lens spaces shrink down to a point. This gives rise to an orbifold singularity in the manifold $R^4/Z_{Q_{\rm NUT}}$.

together with a pp-wave and a NUT. As mentioned previously, the dyonic string will be supported by charges in the NS-NS sector if the index i takes the value $i = 1$, and otherwise it will be supported by charges in the R-R sector. The associated six-dimensional Lagrangian is of the form $\kappa_6^2\, e^{-1}\,\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\phi}\,(F_{(3)})^2$. The six-dimensional metric, which is independent of the choice of i (*i.e.* it is the same whether the fields are in the R-R or the NS-NS sector), is obtained by performing a further step of dimensional oxidation of (2.8), leading to [29]:

$$\begin{aligned} ds_6^2 &= (H_e\, H_m)^{-1/2}\Big(- K^{-1}\, dt^2 + K(dx + (K^{-1} - 1)dt)^2\Big) \\ &+ (H_e\, H_m)^{1/2}\Big(U(d\rho^2 + \rho^2\, d\Omega_2^2) + U^{-1}(dy + Q_{\rm NUT}\, B)^2\Big)\ , \end{aligned} \tag{2.14}$$

where again $dB = \Omega_{(2)}$ is the volume form of the unit 2-sphere. Here, $x = z_6$ is the compactification coordinate of the S^1 reduction step from $D = 6$ to $D = 5$.

As we shall see presently, the area of the horizon (at $r = 0$) for the above metric implies that the entropy of this $D = 6$ boosted dyonic string with NUT charge is:

$$\begin{aligned} S &\equiv \frac{\rm Area}{4\kappa_6^2} \\ &= \frac{1}{4\kappa_6^2}\,((Q_e\, Q_m)^{1/4}\, {Q_{\rm wave}}^{1/2}\, L_x)\ (2{Q_{\rm NUT}}^{1/2}\,(Q_e\, Q_m)^{1/4})^3\ \omega_3/Q_{\rm NUT} \\ &= \frac{2\omega_3\, L_x}{\kappa_6^2}\,\sqrt{Q_e\, Q_m\, Q_{\rm wave}\, Q_{\rm NUT}}\ . \end{aligned} \tag{2.15}$$

Note that we have taken the period of the internal coordinate x to be L_x. The period of the internal coordinate y for the reduction step from $D = 5$ to $D = 4$ is 4π, and so the volume of the internal 2-torus is $4\pi\, L_x$. It then follows that the gravitational constants κ_4 and κ_6 in $D = 4$ and $D = 6$ satisfy the following relationship:

$$\kappa_6^2 = 4\pi\, L_x\, \kappa_4^2\ . \tag{2.16}$$

Taking into account the fact that $\omega_3 = \pi\,\omega_2/2$, it follows that the entropy (2.15) of the $D = 6$ string (with the Taub-NUT-charge) is the same as (2.7), that of the $D = 4$ black hole. This result is of course a natural consequence of the fact that entropy is preserved under dimensional reduction.

2.1.3 Near-horizon region and counting of microstates

We now turn to the near-horizon region $\rho \to 0$, which in turn corresponds to the gravity decoupling limit in the AdS/CFT correspondence:

$$\rho \ll (Q_e,\ Q_m,\ Q_{\rm NUT})\ . \tag{2.17}$$

Note that this limit does *not* impose any restriction on the value of $Q_{\rm wave}$, relative to ρ. The metric takes the following form:

$$ds_6^2 = \frac{r^2}{4\sqrt{Q_e\,Q_m}}\Big(-\frac{r^2}{r^2+4Q_{\rm wave}}\,dt^2 + (1+\frac{4Q_{\rm wave}}{r^2})\,(dx - \frac{4Q_{\rm wave}}{r^2-4Q_{\rm wave}}\,dt)^2\Big)$$
$$+4\sqrt{Q_e\,Q_m}\,Q_{\rm NUT}\,\frac{dr^2}{r^2} + 4\sqrt{Q_e\,Q_m}\,Q_{\rm NUT}\,d\Omega_3^2(Q_{\rm NUT})\ , \tag{2.18}$$

where we have, as previously, made the coordinate transformation $\rho = r^2/4$, and $d\Omega_3^2(Q_{\rm NUT})$ denotes the metric (2.11) on the unit cyclic lens space $S^3/Z_{Q_{\rm NUT}}$, where the coordinate y has the period 4π. Note that the volume of the unit-radius lens space is $\omega_3/Q_{\rm NUT}$, where ω_3 is the volume of the unit-radius three-sphere. The fact that the space associated with the line element $d\Omega_3^2(Q_{\rm NUT})$ is a lens space with the topology S^3/Z_N, where N is the quantised value of the charge $Q_{\rm NUT}$, has important implications for the microscopic interpretation. The metric (2.18) describes a direct product of two three-dimensional spaces, namely the extremal BTZ black hole [8] and the lens space $S^3/Z_{Q_{\rm NUT}}$. To see this, we first make the coordinate rescaling $r \to r/\sqrt{Q_{\rm NUT}}$, and then we dimensionally reduce the six-dimensional metric (2.18) on the lens space. Specifically, we take the lens space metric for the compactification to be scaled to $ds^2 = 4\sqrt{Q_e\,Q_m}\,Q_{\rm NUT}\,d\Omega_3^2(Q_{\rm NUT})$. With this choice for the internal metric, the (constant) breathing-mode scalar takes the value 0, and hence there is no conformal rescaling of the space-time metric. (See [38] for a detailed discussion of Kaluza-Klein reduction on spheres and other spaces.) The resulting three-dimensional space-time metric is then given by

$$ds_3^2 = -\frac{r^4}{\ell^2\,R^2}\,dt^2 + R^2\,(\frac{dx}{\ell} - \frac{2\kappa_3^2\,J}{R^2}\,dt)^2 + \ell^2\,\frac{dr^2}{r^2}\ , \tag{2.19}$$

where

$$J = M\,\ell\ ,\qquad \ell = 2(Q_e\,Q_m)^{1/4}\,{Q_{\rm NUT}}^{1/2}\ ,\qquad M = \frac{Q_{\rm wave}}{2\kappa_3^2\,\sqrt{Q_e\,Q_m}}\ ,$$
$$R^2 = r^2 + 2\kappa_3^2\,M\,\ell^2\ . \tag{2.20}$$

This is precisely the extremal BTZ black hole solution [8], *i.e.* a rotating black hole solution of three-dimensional Einstein gravity with a negative cosmological constant, described by the Lagrangian $\kappa_3^2\,e^{-1}\,\mathcal{L} = R - 2\ell^{-1}$. Note that the extremal BTZ black hole is an example of a generalised Kaigorodov metric, specialised to $D=3$ [15]. The entropy is given by

$$S = \frac{\omega_1}{4\kappa_3^2} = \frac{\pi}{\kappa_3}\,\sqrt{2\ell^2\,M}\ . \tag{2.21}$$

Since the coordinate x has to be periodic with the period $L_x = 2\pi\,\ell$, the three-dimensional gravitational constant κ_3^2, when expressed in terms of L_x and either κ_4^2 or κ_6^2, is given by:

$$\kappa_3^2 = \frac{\kappa_6^2}{8(Q_e\,Q_m)^{3/4}\,{Q_{\rm NUT}}^{1/2}\,\omega_3}$$

$$= \frac{L_x\, \kappa_4^2}{(Q_e\, Q_m)^{3/4}\, {Q_{\rm NUT}}^{1/2}\, \omega_2} \tag{2.22}$$
$$= \frac{4\pi\, \kappa_4^2}{(Q_e\, Q_m)^{1/2}\, \omega_2}\ .$$

In particular we see that when κ_3^2 is expressed in terms of κ_4^2, it is independent of $Q_{\rm NUT}$. Note that the three-dimensional gravitational constant κ_3 is related to the Newton's constant G defined in [4] by $\kappa_3^2 = 2G$.

Substuting M and ℓ from (2.20), and κ_3 from (2.23), into the extremal BTZ entropy (2.21) reproduces precisely the entropy of the four-dimensional 4-charge black hole, given in (2.7), as one would expect. Thus the microscopic counting in [4, 12], which reproduces precisely the BTZ entropy (2.21) in terms of the asymptotic two-dimensional CFT with the $SL(2,R)_L \times SL(2,R)_R$ isometry [11] (*via* the AdS/CFT correspondence), in its turn reproduces the microscopic entropy formula of the four-dimensional 4-charge black hole as well!

Here we should like to comment on the ranges of the various charges for which the above microscopic counting is valid. The discussion of the entropy of the four-dimensional 4-charge black hole splits in two parts. The first step, in which the near-horizon geometry of the $D=4$ black hole is mapped to the BTZ black hole in $D=3$, can be implemented when the gravity decoupling limit is valid. This is discussed in detail in [15], and can be specified roughly by (2.17). This is the condition for the field theory on the intersecting p-brane configuration to decouple from gravity. The second step makes use of Cardy's entropy formula for two-dimensional CFT [35], leading to the microscopic state-counting formula for the entropy of the BTZ black hole:

$$S = 2\pi \sqrt{\tfrac{1}{6} c\, N_R} + 2\pi \sqrt{\tfrac{1}{6} c\, N_L}\ . \tag{2.23}$$

Here the central charge is given by $c = 3\ell/\kappa_3^2$, and the Virasoro level numbers $N_L \equiv L_0$ and $N_R \equiv \bar{L}_0$ are related to the BTZ mass M and angular momentum J by $L_0 + \bar{L}_0 = M\,\ell$ and $L_0 - \bar{L}_0 = J$ [4]. Cardy's formula is valid only in the asymptotic limit where the growth of the numbers of states is such that $N_L + N_R >> c$. This constraint implies that we must have

$$\frac{M\,\ell}{3\ell/\kappa_3^2} = \frac{Q_{\rm wave}}{6\sqrt{Q_e\, Q_m}} >> 1\ . \tag{2.24}$$

Thus we see that in order to have a conformal-field-theoretic microscopic interpretation for the entropy, the momentum of the wave must be very large. Note, however, that if we nevertheless blindly apply Cardy's formula for the case $1 \ll N_{L,R} \le c$, *i.e.* $Q_{\rm wave} \le (Q_e, Q_m)$, we still precisely reproduce the classical results!

It is important to note that the constraint (2.24) is independent of the value of the Taub-NUT charge. A particular case corresponds to the choice $Q_{\rm NUT} = 1$, for which the six-dimensional near-horizon geometry is precisely BTZ$\times S^3$. This is exactly the same as the near-horizon geometry of the boosted dyonic string, which gives rise to the 3-charge Reissner-Nordström-type black hole in $D = 5$. (Its microscopic state counting, using the CFT/AdS correspondence, was understood for static black holes in [4], and for rotating black holes in [5].)

To summarise, we have seen that from the six-dimensional point of view the near-horizon geometries of the 3-charge $D = 5$ and 4-charge $D = 4$ black holes are given by

$$\begin{aligned} D = 5: &\qquad \text{BTZ} \times S^3 \\ D = 4: &\qquad \text{BTZ} \times S^3/Q_{\rm NUT} \ . \end{aligned} \tag{2.25}$$

Thus the oxidation to six dimensions of the near-horizon geometry of the five-dimensional black hole can be viewed as a special case of the oxidation of the four-dimensional black hole, in which $Q_{\rm NUT} = 1$. Since κ_3, when expressed in terms of κ_4, is independent of $Q_{Q_{\rm NUT}}$, the above analysis shows that the microscopics of the five-dimensional 3-charge black hole precisely reproduce those of the four-dimensional 4-charge black hole, in the case where the fourth charge comes from the reduction on the $U(1)$ fibre coordinate of the lens space $S^3/Z_{Q_{\rm NUT}}$. In other words, we have

$$S_{D=4} = \sqrt{Q_{\rm NUT}}\, S_{D=5} \ . \tag{2.26}$$

Note that the implications are not only for the (asymptotic) microscopic counting of states, but also also for the whole black hole spectrum. (The microscopic counting of the four-dimensional black hole entropy was also discussed in [36] using D-brane techniques. However, the counting was only valid for $Q_{\rm NUT} = 1$, which corresponds to, in essence, to 5-dimensional black holes.)

2.2 Non-extremal case

We now turn to the consideration of non-extremal solutions, highlighting the new features that arise here. The non-extremal 4-charge black hole solution can be found in [37]. In terms of our field configuration, it is given by

$$\begin{aligned} ds_4^2 &= -(H_e\, H_m\, K\, U)^{-1/2}\, e^{2f}\, dt^2 + (H_e\, H_m\, K\, U)^{1/2}\, (e^{-2f}\, d\rho^2 + \rho^2\, d\Omega_2^2) \ , \\ A_{(1)16} &= \coth\mu_e\, H_e^{-1}\, dt \ , \qquad A_{(1)17} = Q_m\, \Omega_{(2)} \ , \\ \mathcal{A}^6_{(1)} &= \coth\mu_{\rm wave}\, K^{-1}\, dt \ , \qquad \mathcal{A}^7_{(1)} = Q_{\rm NUT}\, \Omega_{(2)} \ , \end{aligned} \tag{2.27}$$

where the functions H_e, H_m, K, U and f are

$$\begin{aligned} H_e = 1 + \frac{k\ \sinh^2\mu_e}{\rho}\ , \qquad & H_e = 1 + \frac{k\ \sinh^2\mu_m}{\rho}\ , \\ K = 1 + \frac{k\ \sinh^2\mu_{\rm wave}}{\rho}\ , \qquad & U = 1 + \frac{k\ \sinh^2\mu_{\rm NUT}}{\rho}\ , \qquad e^{2f} = 1 - \frac{k}{\rho}\ . \end{aligned} \tag{2.28}$$

The four charges are given by

$$\begin{aligned} Q_e = \tfrac12 k\ \sinh 2\mu_e\ , \qquad & Q_m = \tfrac12 k\ \sinh 2\mu_m\ , \\ Q_{\rm wave} = \tfrac12 k\ \sinh 2\mu_{\rm wave}\ , \qquad & Q_{\rm NUT} = \tfrac12 k\ \sinh 2\mu_{\rm NUT}\ . \end{aligned} \tag{2.29}$$

The Hawking temperature and entropy are of the form

$$\begin{aligned} T &= (4\pi\, k\ \cosh\mu_e\ \cosh\mu_m\ \cosh\mu_{\rm wave}\ \cosh\mu_{\rm NUT})^{-1}\ , \\ S &= \frac{k^2\,\omega_2}{4\kappa_4^2}\ \cosh\mu_e\ \cosh\mu_m\ \cosh\mu_{\rm wave}\ \cosh\mu_{\rm NUT}\ . \end{aligned} \tag{2.30}$$

2.2.1 $D=6$ boosted dyonic string with a NUT charge

As in the extremal case, we may oxidise the 4-charge solution in $D=4$ back to $D=6$, to obtain the metric

$$\begin{aligned} ds_6^2 &= (H_e\,H_m)^{-1/2}\Big(-K^{-1}\,e^{2f}\,dt^2 + K\,(dx + \coth\mu_{\rm wave}\,(K^{-1}-1)\,dt)^2\Big) \\ &\quad + (H_e\,H_m)^{1/2}\Big(U\,(e^{-2f}\,d\rho^2 + \rho^2\,d\Omega_2^2) + U^{-1}\,(dy + Q_{\rm NUT}\,B)^2\Big)\ . \end{aligned} \tag{2.31}$$

We shall be concerned with the near-extremal regime, which is defined by taking k to be small, with μ_e, μ_m and $\mu_{\rm NUT}$ large, so that

$$Q_e \sim \tfrac14 k\,e^{2\mu_e}\ , \qquad Q_m \sim \tfrac14 k\,e^{2\mu_m}\ , \qquad Q_{\rm NUT} \sim \tfrac14 k\,e^{2\mu_{\rm NUT}}\ , \tag{2.32}$$

are all finite and non-vanishing. It follows that in the near-horizon region $\rho\to 0$, the "1" in the functions H_e, H_m and U can be dropped, and we have $k\sinh^2\mu_e \sim Q_e$, $k\sinh^2\mu_m \sim Q_m$ and $k\sinh^2\mu_{\rm NUT} \sim Q_{\rm NUT}$. Note that we do not impose any restriction on $\mu_{\rm wave}$. Making the coordinate transformation

$$\rho = \frac{r^2}{4k\sinh^2\mu_{\rm NUT}}\ , \tag{2.33}$$

the metric (2.31) becomes

$$\begin{aligned} ds_6^2 &= -\frac{r^2(r^2 - 4k^2\ \sinh^2\mu_{\rm NUT})}{\ell^2\,(r^2 + 4k^2\ \sinh^2\mu_{\rm wave}\ \sinh^2\mu_{\rm NUT})}\,dt^2 \\ &\quad + \frac{r^2 + 4k^2\ \sinh^2\mu_{\rm wave}\ \sinh^2\mu_{\rm NUT}}{\ell^2}\Big(dx - \frac{4Q_{\rm wave}\,k\ \sinh^2\mu_{\rm NUT}}{r^2 + 4k^2\ \sinh^2\mu_{\rm wave}\ \sinh^2\mu_{\rm NUT}}\,dt\Big)^2 \\ &\quad + \frac{\ell^2\,dr^2}{r^2 - 4k^2\ \sinh^2\mu_{\rm NUT}} + \ell^2\Big(\tfrac14\coth^2\mu_4(\frac{dy}{Q_{\rm NUT}} + B)^2 + \tfrac14\,d\Omega_2^2\Big)\ , \end{aligned} \tag{2.34}$$

where $\ell^2 = 4k^2 \sinh\mu_e \sinh\mu_m \sinh^2\mu_{\rm NUT}$. We see that this six-dimensional metric is the direct sum of two three-dimensional metrics. In particular, the factor

$$d\bar{s}_3^2 = \ell^2 \Big(\tfrac14 \coth^2\mu_{\rm NUT} \big(\frac{dy}{Q_{\rm NUT}} + B\big)^2 + \tfrac14 d\Omega_2^2 \Big) \tag{2.35}$$

describes a *squashed* three-dimensional cyclic lens space, where the squashing parameter is $\coth^2\mu_{\rm NUT}$ and the 3-sphere is factored by $Z_{Q_{\rm NUT}}$. (We are assuming that, as usual, y has period 4π.) In the extremal limit with non-vanishing $Q_{\rm NUT}$, which requires that $\mu_{\rm NUT} \to \infty$, the squashing parameter $\coth\mu_{\rm NUT}$ becomes 1 and the space becomes the unsquashed lens space. In the decoupling limit, which corresponds to the near-extremal region with $\mu_{\rm NUT} >> 1$, the squashing effect is very small, and the lens space is almost round. (If we set $Q_{\rm NUT}$ to zero, and hence $\mu_{\rm NUT} = 0$, the space will instead be untwisted, becoming $S^2 \times S^1$.) The volume of this squashed lens space is given by

$$V = \frac{\pi\,\omega_2\,\ell^3 \coth\mu_{\rm NUT}}{2Q_{\rm NUT}} = 2\pi\,\ell\,\omega_2\,k \sinh\mu_e \sinh\mu_m\ . \tag{2.36}$$

2.3 Microscopic counting and the BTZ black hole

Dimensionally reducing the metric (2.34) on the squashed lens space (2.35), we obtain precisely the three-dimensional BTZ black hole, given by

$$ds_3^2 = -\frac{r^2\,(r^2 - r_+^2)}{\ell^2\,R^2}\,dt^2 + R^2\big(\frac{dx}{\ell} - \frac{2\kappa_3^2\,J}{R^2}\,dt\big)^2 + \frac{\ell^2}{r^2 - r_+^2}\,dr^2\ , \tag{2.37}$$

where

$$\begin{aligned}
R^2 &= r^2 + \tfrac12(4\kappa_3^2\,M\ell^2 - r_+^2)\ ,\\
\ell^2 &= 4k^2 \sinh\mu_e \sinh\mu_m \sinh^2\mu_{\rm NUT} \sim 4\sqrt{Q_e\,Q_m}\,Q_{\rm NUT}\ ,\\
M\,\ell^2 &= \frac{k^2}{\kappa_3^2}\cosh 2\mu_{\rm wave} \sinh^2\mu_{\rm NUT} \sim \frac{Q_{\rm NUT}}{\kappa_3^2}\,k\cosh 2\mu_{\rm wave}\ ,\qquad J = M\,\ell\tanh 2\mu_{\rm wave}\ ,\\
r_+^2 &= 4\kappa_3^2\,M\ell^2\sqrt{1 - (\frac{J}{M\ell})^2} = 4k^2 \sinh^2\mu_{\rm NUT} \sim 4k\,Q_{\rm NUT}\ .
\end{aligned} \tag{2.38}$$

Thus the coordinate x has the period of $L_x = 2\pi\,\ell$. Following the same discussion as in the extremal case, we find that the three-dimensional gravitational constant κ_3 is related to that of the original four-dimensional theory by

$$\kappa_3^2 = \frac{4\pi\,\kappa_4^2}{\omega_2\,k \sinh\mu_e \sinh\mu_m} \sim \frac{4\pi\,\kappa_4^2}{(Q_e\,Q_m)^{1/2}\,\omega_2}\ . \tag{2.39}$$

The entropy of the BTZ black hole in this case takes the form:

$$S = \frac{\pi}{\kappa_3}\Big(\sqrt{\ell(M\,\ell + J)} + \sqrt{\ell(M\,\ell - J)}\Big)\ . \tag{2.40}$$

Substituting the variables in (2.38) and κ_3^2 into the above entropy formula, we obtain

$$\begin{aligned} S &= \frac{k^2\,\omega_2}{4\kappa_4^2}\,\sinh\mu_e\,\sinh\mu_m\,\cosh\mu_{\rm wave}\,\sinh\mu_{\rm NUT} \\ &\sim \frac{\omega_2}{4\kappa_4^2}\sqrt{Q_e\,Q_m\,Q_{\rm NUT}\,k\,\cosh^2 2\mu_{\rm wave}}\ , \end{aligned} \tag{2.41}$$

which is precisely the entropy (2.30) for the four-dimensional black hole in the near-extremal region where μ_e, μ_m and $\mu_{\rm NUT}$ are all much greater than 1, and k tends to zero, while keeping Q_e, Q_m and $Q_{\rm NUT}$ fixed. Note that this agreement of the entropy formulae occurs only in the near-extremal region.

It is instructive to study the limit where the counting of the string states (2.40) on the boundary of the BTZ black hole is valid. In this non-extremal case, the central charge is given by $c = 3\ell/\kappa_3^2$, and we have $L_0 + \bar L_0 = M\,\ell$ and $L_0 - \bar L_0 = J$. The state counting gives the expression $S = 2\pi\sqrt{\frac16 c\,N_R} + 2\pi\sqrt{\frac16 c\,N_L}$ for the entropy, valid when $N_R + N_L >> c$. This expression is in agreement with (2.40). The constraint on the level-numbers implies that

$$\frac{M\,\ell}{3\ell/\kappa_3^2} = \frac{\cosh 2\mu_{\rm wave}}{12\sinh\mu_e\,\sinh\mu_m} \sim \frac{Q_{\rm wave}}{6\sqrt{Q_e\,Q_m}} >> 1\ . \tag{2.42}$$

Again, we see that this constraint is independent of the NUT charge $Q_{\rm NUT}$.

3 p-branes from flat space-time

In the previous section, we made use of the fact that the fourth charge of the 4-charge black hole in $D = 4$ can be obtained from the Hopf reduction of the $D = 5$ 3-charge black hole on the fibre coordinate of the lens space S^3/Z_N, described as a $U(1)$ bundle over S^2. This is a special case of general discussion that can be given for any N-charge p-brane solution whose transverse space is four-dimensional; by a similar Hopf reduction we can obtain an $(N+1)$-charge p-brane solution in one dimension less [26]. In this section, we show that if the effect of Kaluza-Klein massive modes is neglected, p-branes configurations can be obtained from flat space-time by a sequence of dimensional reductions and oxidations, and U-duality transformations. This provides an alternative way of constructing BPS p-brane solitons, without needing to go through the process of explicitly solving the supergravity equations of motion. In other words, the non-trivial BPS soliton solutions can be obtained by acting with symmetry transformations on the trivial flat space-time solution. In this context, therefore, U dualities play the rôle of solution-generating symmetries.

3.1 $D = 4$ black holes from $D = 5$ Minkowski space-time

We begin with five-dimensional Minkowski space-time, written as

$$ds_5^2 = -dt^2 + dr^2 + r^2\, d\Omega_3^2\ , \tag{3.1}$$

where $d\Omega_3^2$ is a metric on the unit 3-sphere. Exploiting the fact that S^3 is a $U(1)$ bundle over S^2, we may write the 3-sphere metric as

$$d\Omega_3^2 = \tfrac{1}{4} d\Omega_2^2 + \tfrac{1}{4}(dz + B)^2\ , \tag{3.2}$$

where $d\Omega_2^2$ is the metric on the unit 2-sphere, and $dB = \Omega_{(2)}$, the volume form on the unit 2-sphere. The $U(1)$ fibre coordinate z has period 4π.

Adopting the standard Kaluza-Klein ansatz for the metric, we now reduce from $D = 5$ to $D = 4$ on a circle:

$$ds_5^2 = e^{-\phi_1/\sqrt{3}}\, ds_4^2 + e^{2\phi_1/\sqrt{3}}\, (dz_1 + \mathcal{A}^1_{(1)})^2\ . \tag{3.3}$$

With this ansatz, the pure Einstein-Hilbert Lagrangian $\mathcal{L}_5 = R\,{*\mathbb{1}}$ in $D = 5$ reduces to

$$\mathcal{L}_4 = R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\phi_1} \wedge d\phi_1 - \tfrac{1}{2} e^{\sqrt{3}\phi_1}\, {*\mathcal{F}^1_{(2)}} \wedge \mathcal{F}^1_{(2)}\ . \tag{3.4}$$

We may now apply the reduction (3.3) to the five-dimensional Minkowski space-time (3.1), which is, of course, a solution of the pure gravity equations in $D = 5$. Writing $d\Omega_3^2$ as in (3.2), we take the compactification coordinate z_1 to be the Hopf fibre coordinate z, and the Kaluza-Klein vector $\mathcal{A}^1_{(1)} = B$. Thus from (3.3) we obtain the four-dimensional configuration

$$\begin{aligned} ds_4^2 &= \tfrac{1}{2} r \Big[- dt^2 + dr^2 + \tfrac{1}{4} r^2\, d\Omega_2^2 \Big]\ , \\ e^{-\frac{2}{\sqrt{3}}\phi_1} &= \frac{4}{r^2}\ , \qquad \mathcal{F}^1_{(2)} = dB = \Omega_{(2)}\ . \end{aligned} \tag{3.5}$$

This is necessarily a solution of the equations following from the dimensionally-reduced Lagrangian (3.4).

We now make the coordinate transformation $r = 2\,\rho^{1/2}$, and define $H = \rho^{-1}$, in terms of which the four-dimensional solution (3.5) becomes

$$\begin{aligned} ds_4^2 &= -H^{-1/2}\, dt^2 + H^{1/2}\, (d\rho^2 + \rho^2\, d\Omega_2^2)\ , \\ e^{-\frac{2}{\sqrt{3}}\phi_1} &= H\ , \qquad \mathcal{F}^1_{(2)} = \Omega_{(2)}\ . \end{aligned} \tag{3.6}$$

We observe that H is a harmonic function in the flat three-dimensional "transverse space" with metric $d\rho^2 + \rho^2\, d\Omega_2^2$.

The solution (3.6) is superficially like the standard four-dimensional single-charge extremal black hole. The only difference is that in (3.6) the harmonic function tends to zero at infinity, while in the usual black hole solution one has $H = 1 + Q/\rho$, and the harmonic function is asymptotically constant. In fact, although the metric (3.6) has the same structure as the usual black hole in the near-horizon ($\rho \to 0$ limit), its asymptotic behaviour is quite different, and in fact it has no asymptotically Minkowskian limit. However it has been shown that, by any of a number of somewhat different procedures, one can use U-duality transformations to change the values of the constant terms in the harmonic functions in black-hole or p-brane solutions [9, 39, 41, 40]. The most convenient of these for our purposes is the one introduced in [41]. This is a universal prescription, in which one diagonally dimensionally reduces a D-dimensional p-brane on all its world-volume dimensions (including time), thereby obtaining a Euclidean instanton solution in $D-p-1$ dimensions. The dimensionally-reduced Lagrangian describing this solution has a global symmetry group that includes a number of independent $SL(2,\mathrm{I\!R})$ factors, one associated with each harmonic function. In fact there is an $SL(2,\mathrm{I\!R})/O(1,1)$ scalar coset associated with each $SL(2,\mathrm{I\!R})$ factor. By making $SL(2,\mathrm{I\!R})$ transformations on a given solution, a new one with harmonic functions that are shifted and scaled by constants can be obtained [41]. The original motivation for transforming the harmonic functions was in fact to strip off the constant terms, so that the black-hole or p-brane solution was transformed into its near-horizon limit. Here, our interest lies in the opposite direction, in that we want to transform the harmonic function H in (3.6) from the degenerate form $H = \rho^{-1}$ into the standard black-hole form where there is a constant term.

To apply the procedure of [41], we first diagonally reduce the solution (3.6) to $D = 3$, with the metric ansatz

$$ds_4^2 = e^{-\phi_2}\, ds_3^2 - e^{\phi_2}\, (dt + \mathcal{A}^2_{(1)})^2\ . \tag{3.7}$$

Thus we obtain the three-dimensional configuration

$$\begin{aligned} ds_3^2 &= d\rho^2 + \rho^2\, d\Omega_2^2\ ,\\ e^{\phi} &= H\ , \qquad e^{\varphi} = 1\ ,\\ \mathcal{F}^1_{(2)} &= \Omega\ , \qquad \mathcal{F}^2_{(2)} = 0\ , \qquad \mathcal{F}^1_{(1)2} = 0\ , \end{aligned} \tag{3.8}$$

where we have defined the dilatonic scalars ϕ and φ by

$$\phi = -\tfrac{\sqrt{3}}{2}\phi_1 - \tfrac12\phi_2\ , \qquad \varphi = -\tfrac{\sqrt{3}}{2}\phi_2 + \tfrac12\phi_1\ . \tag{3.9}$$

The dimensionally-reduced Euclidean-signature theory in $D = 3$ has the Lagrangian

$$\mathcal{L}_3 \;=\; R\,{*\mathbb{1}} - \tfrac12 {*d\phi}\wedge d\phi - \tfrac12 {*d\varphi}\wedge d\varphi - \tfrac12 e^{-2\phi}\, {*\mathcal{F}^1_{(2)}}\wedge \mathcal{F}^1_{(2)}$$

$$+\tfrac{1}{2}e^{-\phi-\sqrt{3}\varphi}\,{*\mathcal{F}^2_{(2)}}\wedge\mathcal{F}^2_{(2)}+\tfrac{1}{2}e^{-\phi+\sqrt{3}\varphi}\,{*\mathcal{F}^1_{(1)2}}\wedge\mathcal{F}^1_{(1)2}\ , \tag{3.10}$$

where $\mathcal{F}^1_{(2)}=d\mathcal{A}^1_{(1)}-d\mathcal{A}^1_{(0)2}\,\mathcal{A}^2_{(1)}$, $\mathcal{F}^2_{(2)}=d\mathcal{A}^2_{(1)}$ and $\mathcal{F}^1_{(1)2}=d\mathcal{A}^1_{(0)2}$. The unusual signs for the kinetic terms for $\mathcal{F}^2_{(2)}$ and $\mathcal{F}^1_{(1)2}$ are the consequence of having performed the dimensional reduction on the time direction. The three-dimensional configuration (3.8) is a solution of the equations of motion following from this Lagrangian. In fact, we may consistently truncate the fields φ, $\mathcal{A}^2_{(1)}$ and $\mathcal{A}^1_{(0)2}$ (which in any case vanish in our solution) in the Lagrangian (3.10). In the resulting Lagrangian we then dualise $\mathcal{A}^1_{(1)}$ to an axion χ, giving the purely scalar Lagrangian

$$\mathcal{L}_3=R\,{*\mathbb{1}}-\tfrac{1}{2}{*d\phi}\wedge d\phi+\tfrac{1}{2}e^{2\phi}\,{*d\chi}\wedge d\chi\ . \tag{3.11}$$

The unusual sign for the kinetic term for χ is the result of having performed a dualisation in a Euclidean-signatured theory. The Lagrangian (3.11) has an $SL(2,\mathbb{R})$ global symmetry, and in fact the scalars parameterise the coset $SL(2,\mathbb{R})/O(1,1)$. Defining $\tau=\chi+\mathrm{j}\,e^{-\phi}$, where j satisfies $\mathrm{j}^2=1$ and $\bar{\mathrm{j}}=-\mathrm{j}$, the $SL(2,\mathbb{R})$ transformations can be written as

$$\tau\longrightarrow\frac{a\,\tau+b}{c\,\tau+d}\ , \tag{3.12}$$

where $ad-bc=1$.

In terms of the dualised axion field χ, the form of the 3-dimensional solution will be the same as (3.8), except that now we will have $\chi=\chi_0+H^{-1}$, where χ_0 is an arbitrary constant of integration. After performing an $SL(2,\mathbb{R})$ transformation, we therefore obtain the new primed solution

$$\begin{aligned}
ds_3^2 &= d\rho^2+\rho^2\,d\Omega_2^2\ ,\\
e^{\phi'} &= H'\equiv 2c(c\,\chi_0+d)\Big(1+\frac{c\,\chi_0+d}{2c\,\rho}\Big)\ ,\qquad e^{\varphi}=1\ ,\\
\chi' &= H'^{-1}+\frac{a\,\chi_0+b}{c\,\chi_0+d}\ ,\qquad \mathcal{F}^2_{(2)}=0\ ,\qquad \mathcal{F}^1_{(1)2}=0\ .
\end{aligned} \tag{3.13}$$

(Quantities that are inert under $SL(2,\mathbb{R})$ are written without primes.)

Dualising the axion χ' back to a potential $\mathcal{A}^1_{(1)}{}'$, and oxidising back to $D=4$, we obtain the new metric $ds_4^2=-H'^{-1/2}\,dt^2+H'^{1/2}\,(d\rho^2+\rho^2\,d\Omega_2^2)$. In order to put this in the standard form, where it is asymptotic to the canonical form of the Minkowski metric, we make the constant general coordinate transformations $t\rightarrow(2c(c\,\chi_0+d))^{-1/4}\,t$ and $\rho\rightarrow(2c(c\,\chi_0+d))^{1/4}\,\rho$. The final solution is given by

$$\begin{aligned}
ds_4^2 &= -\widetilde{H}^{-1/2}\,dt^2+\widetilde{H}^{1/2}\,(d\rho^2+\rho^2\,d\Omega_2^2)\ ,\\
e^{-\frac{2}{\sqrt{3}}(\phi_1-\phi_1^0)} &= \widetilde{H}\ ,\qquad \mathcal{F}^1_{(2)}=(c\,\chi_0+d)^2\,\Omega_{(2)}\ ,
\end{aligned} \tag{3.14}$$

where the new harmonic function $\widetilde{H}$ and the dilaton modulus ϕ_1^0 are given by

$$\widetilde{H} = 1 + \frac{(c\,\chi_0 + d)^2\, e^{-\frac{\sqrt{3}}{2}\phi_1^0}}{\rho}\ , \qquad e^{\frac{\sqrt{3}}{2}\phi_1^0} = 2c\,(c\,\chi_0 + d)\ . \tag{3.15}$$

More generally, we may introduce a second modulus parameter, namely a constant ϕ_0 to supplement χ_0 in the (ϕ,χ) system. This can be done by rescaling the coordinates in (3.6), so that $t = e^{-\phi_0}\, t'$, $\rho = e^{\phi_0}\, \rho'$. Now following the same steps as before, we arrive at the solution

$$\begin{aligned} ds_4^2 &= -\widetilde{H}^{-1/2}\, dt^2 + \widetilde{H}^{1/2}\,(d\rho^2 + \rho^2\, d\Omega_2^2)\ , \\ e^{-\frac{2}{\sqrt{3}}(\phi_1 - \phi_1^0)} &= \widetilde{H}\ , \qquad \mathcal{F}^1_{(2)} = (c\,\chi_0 + d)^2\, e^{-\phi_0}\, \Omega_{(2)}\ , \end{aligned} \tag{3.16}$$

where the new harmonic function $\widetilde{H}$ and the dilaton modulus ϕ_1^0 are given by

$$e^{\frac{\sqrt{3}}{2}\phi_1^0} = 2c\,(c\,\chi_0 + d)\, e^{-\phi_0}\ , \qquad \widetilde{H} = 1 + \frac{(c\,\chi_0 + d)^2\, e^{-\phi_0}\, e^{-\frac{\sqrt{3}}{2}\phi_1^0}}{\rho}\ . \tag{3.17}$$

The magnetic charge of this four-dimensional black-hole solution is given by

$$Q = \tfrac{1}{4\pi} \int \mathcal{F}_2 = (c\,\chi_0 + d)^2\, e^{-\phi_0}\ . \tag{3.18}$$

The free parameters ϕ_0 and χ_0 enable us to set the dilaton modulus ϕ_1^0 and the magnetic charge Q to any desired values. Note that if we oxidise this four-dimensional magnetic black-hole solution to $D = 5$, we obtain the NUT solution (*i.e.* $\mathbb{R}\times$ Taub-NUT)

$$ds_5^2 = -e^{-\frac{1}{\sqrt{3}}\phi_1^0}\, dt^2 + e^{-\frac{1}{\sqrt{3}}\phi_1^0} \Big(\widetilde{H}\,(d\rho^2 + \rho^2\, d\Omega_2^2) + e^{\phi_1^0}\, \widetilde{H}^{-1}\,(dz_1 + Q\,\cos\theta\, d\phi)^2\Big)\ . \tag{3.19}$$

The near-horizon limit, after making the replacement $\rho = \frac{1}{4} r^2$, is

$$ds_5^2 = -e^{-\frac{1}{\sqrt{3}}\phi_1^0}\, dt^2 + Q\, e^{\frac{1}{2\sqrt{3}}\phi_1^0}\,(dr^2 + \tfrac{1}{4} r^2\, d\widetilde{\Omega}_3^2)\ , \tag{3.20}$$

where $d\widetilde{\Omega}_3^2 = \frac{1}{4} d\Omega_2^2 + \frac{1}{4}(dz/Q + \cos\theta\, d\phi)^2$ is the metric on a the unit-radius lens space S^3/Z_Q.

Instead of oxidising the four-dimensional black hole (3.16) directly back to $D = 5$, we can first perform a four-dimensional U-duality transformation to map the solution into one where the charge becomes electric, and is carried by a 2-form field strength coming from the 4-form of M-theory or the NS-NS 3-form of the type II string. For definiteness, let us consider the case where after the U-duality transformation, it is $F_{(2)12}$ that carries the electric charge. This solution can be viewed as the vertical dimensional reduction of a five-dimensional black hole, where the $U(1)$ isometry on the transverse space is achieved by making a continuous "stack" of black holes along the z_7 axis. Viewed from distances

r in the remaining transverse space that are large compared with the period of z_7, one may approximate the continuous stack by a periodic array of black holes. As we have seen, there is already a natural $U(1)$ isometry in the transverse space of an *isotropic* five-dimensional black hole, namely the $U(1)$ of the Hopf fibres of the foliating 3-spheres. We can dimensionally reduce the new solution on this fibre coordinate, and then generate a 2-charge black hole in $D = 4$, with charges carried by the field strengths $\{F_{(2)12}, *\mathcal{F}^7_{(2)}\}$. Again, the new harmonic function associated with the field strength $\mathcal{F}^7_{(2)}$ lacks a constant term, and so we have to repeat the steps of reducing to $D = 3$ and performing an $SL(2, \mathbb{R})$ transformation in order to introduce a constant term.

A discrete Weyl rotation can then again be used in $D = 4$, to rotate the 2-charge black hole to one that involves only the field strengths $F_{(2)12}$ and $F_{(2)34}$. This can be vertically oxidised to $D = 5$, followed by another dimensional reduction on the $U(1)$ Hopf fibre coordinate of the foliating 3-spheres. This gives a 3-charge black hole in $D = 4$, supported by the field strengths $\{F_{(2)12}, F_{(2)34}, *\mathcal{F}^7_{(2)}\}$. (Again, a constant term in the new harmonic function can be introduced by following the steps described previously.) This 3-charge configuration can be rotated by a Weyl duality transformation to $\{F_{(2)12}, F_{(2)34}, F_{(2)56}\}$. Repeating the vertical oxidation, followed by Hopf reduction, once more, we eventually arrive at a 4-charge black hole in $D = 4$. There are in total 630 4-charge configurations in maximal supergravity in $D = 4$, given by (2.1).

Note that we are *not* saying that there is a complete sequence of solution-mapping symmetries that takes us from the original 5-dimensional flat space-time to the four-dimensional 4-charge black hole. This is because there is one step in each of the processes of adding the second, third and fourth charges which is not implemented purely by a symmetry transformation. This is the step where we vertically oxidise an N-charge black hole from $D = 4$ to $D = 5$. If the four-dimensional solution is literally the given black hole, with all other four-dimensional fields vanishing, then the mathematical process of vertical oxidation gives a uniform line of five-dimensional black holes distributed along the z_7 axis. The harmonic function describing the black holes with therefore have a dependence of the form $1 + Q/R$ in $D = 5$, where R is the radial coordinate of the remaining 3-dimensional space transverse to the line of black holes. In order to proceed with the next step of Hopf reduction, we need instead to be able to consider an *isotropic* single black hole in $D = 5$, with harmonic function of the form $1+Q/r^2$ where r is the radial coordinate of the full 4-dimensional space transverse to the t coordinate. To justify this step, one first has to view the four-dimensional black hole as an approximate solution that could be thought of as the dimensional reduction of a periodic array of five-dimensional black holes. (This effectively means that one is neglecting massive Kaluza-Klein modes in $D = 4$.) Sufficiently near to one of the black holes in this array, it approximates to a genuinely isotropic five-dimensional black hole, with no periodic identification of any of the Cartesian transverse coordinates. It is this solution, with the transverse space then written in hyperspherical polar coordinates, that is then used for the next step of Hopf reduction on the $U(1)$ fibres of the foliating 3-spheres, in order to generate the next charge in $D = 4$.

The five-dimensional metric is independent of the $U(1)$ fibre coordinate, and hence massive Kaluza-Klein modes all rigorously vanish in the Hopf reduction. On the other hand, a periodic array of five-dimensional black holes does depend on the compactifying coordinate (along the periodic axis), and so Kaluza-Klein massive modes are non-zero after the dimensional reduction. Thus to say that the N-charge and $(N+1)$-charge four-dimensional black holes are equivalent under this vertical-oxidation/Hopf-reduction procedure is to ignore the discrepancies in the massive Kaluza-Klein modes in four dimensions. This neglect of massive modes is in the same spirit as in the usual discussion of the U-duality symmetry group: The Cremmer-Julia global symmetry groups in supergravities arise only when the massive Kaluza-Klein modes are set to zero. One can see this from a string-theory standpoint by noting, for example, that the 56-dimensional U-duality multiplet of four-dimensional single-charge black holes come from the vertical and diagonal dimensional reduction of M-branes, NUTs and waves in $D = 11$. Again, the vertical reduction steps involve the neglect of Kaluza-Klein massive modes, while the diagonal reduction steps are on coordinates which have genuine and exact $U(1)$ isometries. In fact a similar philosophy to the one that interprets all the four-dimensional black holes as coming from fewer fundamental objects in $D = 11$ allows us to interpret all the black holes as coming from flat space in $D = 5$. In both cases, the result follows once one ignores the massive Kaluza-Klein modes.

3.2 D-branes and NS-NS branes from $D = 11$ Minkowski space-time

The discussion of the previous section can easily be generalised to obtain ten-dimensional p-branes from $D = 11$ Minkowski space-time. Thus to begin, we consider the eleven-dimensional metric

$$ds_{11}^2 = dx^\mu\, dx_\mu + dr^2 + r^2\, d\Omega_3^2\ . \tag{3.21}$$

We again perform a dimensional reduction on the $U(1)$ fibre coordinate in S^3, giving rise to D6-brane in $D = 10$ type IIA. (In order to introduce a constant term in the harmonic function, we can diagonally dimensionally reduce on the world-volume to $D = 3$, perform an $SL(2,\mathrm{I\!R})$ transformation, and diagonally oxidise back to $D = 10$. An alternative procedure is to apply a sequence of IIA/IIB T-duality transformations and a type IIB S-duality transformation to map the solution to a wave. The harmonic function describing the wave can then be shifted and rescaled by general coordinate transformations [9, 39]. Yet another possibility is to map the solution instead to an instanton in type IIB, and perform an $SL(2,\mathrm{I\!R})$ transformation there [40].)

Having obtained the D6-brane in the $D = 10$ type IIA theory, we can then use the IIA/IIB T-duality that relates a Dp-brane to a D$(p+1)$-brane to generate all the D-branes in ten dimensions. Using the S-duality of the type IIB theory, we can generate the NS-NS string and 5-brane from the D-string and D5-brane respectively. The NS-NS string and 5-brane are T-dual to ten-dimensional waves and NUTs respectively.

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