QUANTITATIVE K-THEORY FOR BANACH ALGEBRAS AND ITS APPLICATIONS

A Dissertation

by

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ABSTRACT

This dissertation can be said to fall under the broad theme of computability of K-theory of L_p operator algebras (and perhaps more general Banach algebras).

The first part of the dissertation is about a variant of K-theory known as quantitative K-theory, which has been defined for C^*-algebras and applied in a number of situations. Our goal is to extend the theory to a larger class of Banach algebras so that it becomes applicable to L_p operator algebras and thus a tool for investigating an L_p version of the Baum-Connes conjecture. We develop the general framework for this theory, culminating in a version of the controlled Mayer-Vietoris sequence that has featured prominently in existing applications in the C^*-algebra setting.

In the second part of the dissertation, we study the L_p version of one of these applications. This application involves the notion of dynamic asymptotic dimension, which is a notion of dimension associated to group actions on spaces (and more generally to groupoids). In the C^*-algebra setting, the work of Guentner-Willett-Yu showed that when a group G acts on a compact space X with finite dynamic asymptotic dimension, the Baum-Connes conjecture with coefficients in $C(X)$ holds for the group G. We will formulate an L_p version of the Baum-Connes conjecture with coefficients and show that under the same assumption, the L_p Baum-Connes conjecture with coefficients in $C(X)$ holds for the group G. As a consequence, the K-theory of the L_p reduced crossed product of $C(X)$ by G does not depend on p if the action has finite dynamic asymptotic dimension.

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1. INTRODUCTION

Quantitative (or controlled) operator K-theory has its roots in [40], where the idea was used by Yu in his work on the Novikov conjecture, which is a conjecture in topology on the homotopy invariance of certain higher signatures. In fact, by applying a certain controlled Mayer-Vietoris sequence to the Roe C^*-algebras associated to proper metric spaces with finite asymptotic dimension, Yu was able to prove the coarse Baum-Connes conjecture for these metric spaces, from which it follows that the Novikov conjecture holds for finitely generated groups with finite asymptotic dimension and whose classifying space has the homotopy type of a finite CW-complex.

The underlying philosophy of quantitative K-theory is that the K_0 and K_1 groups of a complex Banach algebra can essentially be recovered by using "quasi-idempotent" and "quasi-invertible" elements respectively. A striking consequence of the flexibility gained by considering such elements instead of actual idempotent or invertible elements is that in place of closed (two-sided) ideals, which are often needed to apply the standard machinery in K-theory, we can sometimes use closed subalgebras that are almost ideals in an appropriate sense. This feature could already be seen in the controlled cutting and pasting technique used in [40].

In [26], Oyono-Oyono and Yu developed the theory for general filtered C^*-algebras, i.e., C^*-algebras equipped with a filtration, and they formulated a quantitative version of the Baum-Connes conjecture, proving it for a large class of groups. They also suggested that the theory can be extended to more general filtered Banach algebras. For most algebras of interest in noncommutative geometry, we can obtain a natural filtration from a length function defined on the algebra. This gives the algebra the necessary geometric structure in order to define quantitative K-theory, and we may regard these algebras as "geometric" algebras in the spirit of geometric group theory. It does seem, however, that one needs the algebras to be equipped with an appropriate matrix norm structure, such as a p-operator space structure, as one needs to incorporate norm control in the framework. This norm control is automatic in the C^*-algebra setting but not for arbitrary Banach algebras. Motivated by the successful application of quantitative K-theory in investigations of (variants of) the Baum-Connes conjecture [40, 14], our goal is to develop a framework of quantitative K-theory that can be applied to filtered L_p operator algebras, i.e., closed subalgebras of $B(L_p(X,\mu))$ for some measure space (X,μ), where $p \in [1,\infty)$. This will then give us a tool to investigate an L_p version of the Baum-Connes conjecture, which in turn gives us information about the K-theory of certain classes of L_p operator algebras.

In trying to extend techniques and results for C^*-algebras to more general Banach algebras, the algebras of bounded linear operators on L_p spaces seem to be a natural class to begin with. Moreover, L_p operator algebras have a natural p-operator space structure. From the point of view of noncommutative geometry, in particular the Baum-Connes conjecture [1], there are also reasons to study L_p operator algebras. Indeed, in Lafforgue's work on the Baum-Connes conjecture [20, 35], he considered certain generalized Schwartz spaces whose elements act on (weighted) L_p spaces. There is also the Bost conjecture, which is the Banach algebra analog of the Baum-Connes conjecture obtained by replacing the group C^*-algebra by the L_1 group convolution algebra. We refer the reader to [23] for a survey on the Baum-Connes conjecture and similar isomorphism conjectures. Also, in yet unpublished work [18], Kasparov and Yu have been investigating an L_p version of the Baum-Connes conjecture.

While we are mainly interested in L_p operator algebras, the class of algebras of bounded linear operators on subquotients (i.e., subspaces of quotients) of L_p spaces, which we will refer to as SQ_p algebras, is a more natural class to work with, as suggested by the theory of p-operator spaces. Indeed, an abstract p-operator space (as defined in [9]) can be

p-completely isometrically embedded in $B(E, F)$, where E and F are subquotients of L_p spaces [24]. Moreover, the class of L_p operator algebras is not closed under taking quotients by closed ideals when $p \neq 2$ [13] while the class of SQ_p algebras is [25], which is relevant to us when we are considering short exact sequences of algebras.

For general Banach algebras, one can still consider quantitative K-theory in a similar way as we do, provided one has an appropriate matrix norm structure on the Banach algebra (see Remark 3.1.8). However, since notation becomes more cumbersome in this generality, and the application we have in mind involves only L_p operator algebras, we develop our framework in the setting of SQ_p algebras.

In the first part of this dissertation, we develop the general framework for this theory, culminating in a version of the controlled Mayer-Vietoris sequence that has featured prominently in existing applications in the C^*-algebra setting. This part of the dissertation is based on [7].

In the second part of this dissertation, we study the L_p version of one of these applications. This application involves the notion of dynamic asymptotic dimension, which is a notion of dimension associated to group actions on spaces. This part of the dissertation is based on [8].

Notions of dimension abound in mathematics, and they give us quantitative measures of the sizes of various mathematical objects in a broad sense. In some instances, one wishes to know the exact dimension while in other instances, one just wishes to determine finiteness of the dimension. Finiteness of various dimensions has been considered in connection with central problems in the theory of C^*-algebras and in noncommutative geometry. For instance, finiteness of nuclear dimension plays a crucial role in the classification of C^*-algebras [11, 12, 36], while finiteness of asymptotic dimension or of dynamic asymptotic dimension has featured in work on the Baum-Connes conjecture [40, 14]. In these works on the Baum-Connes conjecture, finiteness of dimension allows one to apply cutting-and-pasting techniques (i.e., Mayer-Vietoris sequences) a finite number of times to compute the K-theory of a certain C^*-algebra. In this dissertation, we will consider the implication of finiteness of dynamic asymptotic dimension on an L_p version of the Baum-Connes conjecture with coefficients.

Dynamic asymptotic dimension is a property of topological dynamical systems introduced by Guentner, Willett, and Yu in [15] for discrete groups acting by homeomorphisms on locally compact Hausdorff spaces, and it can be defined as follows.

Definition 1.0.1. *[15] An action of a countable discrete group Γ on a locally compact Hausdorff space X has dynamic asymptotic dimension d if d is the smallest natural number with the following property: for any compact subset K of X and finite subset E of Γ, there are open subsets $U_0, \ldots, U_d$ of X that cover K such that for each $i \in \{0, \ldots, d\}$, the set*

$$\left\{ g \in \Gamma : \begin{array}{l} \text{there exist } x \in U_i \text{ and } g_n, \ldots, g_1 \in E \text{ such that } g = g_n \cdots g_1 \\ \text{and } g_k \cdots g_1 x \in U_i \text{ for all } k \in \{1, \ldots, n\} \end{array} \right\}$$

is finite.

One thinks of finite dynamic asymptotic dimension as a condition that allows one to break up the action into at most a certain number of parts whenever we are given a finite subset of the group, and such that on each part the action is fairly simple if we restrict our attention to the given finite subset. One can also think of it as measuring the extent to which we can decompose the dynamical system into neighborhoods of partial orbits determined by the finite subset of the group. The main motivation of the authors of [15] was the implications for K-theory of associated C^*-algebras, such as crossed products, and thus for manifold topology. They investigated some connections with controlled topology, coarse geometry, and structure of C^*-algebras (in particular nuclear dimension). They also defined the dynamic asymptotic dimension of locally compact Hausdorff étale groupoids, and showed that an action has dynamic asymptotic dimension d if and only if the corresponding transformation groupoid has dynamic asymptotic dimension d. Indeed, the groupoid point of view is useful if one wishes to decompose algebras associated with such actions in order to apply cutting-and-pasting techniques.

In [14], the same authors considered a model for the Baum-Connes assembly map for an action based on Yu's localization algebras [39] and Roe algebras. In the appendix of [14], the authors show that their model for the Baum-Connes assembly map agrees with the one stated in terms of Kasparov's KK-theory [1]. The main result in that paper is the following:

Theorem 1.0.2. *[14] Let a countable discrete group* Γ *act with finite dynamic asymptotic dimension on a compact Hausdorff space* X*. Then the Baum-Connes conjecture holds for* Γ *with coefficients in* $C(X)$.

Many interesting actions have finite dynamic asymptotic dimension. For example, it was shown in [15] that all free minimal $\mathbb{Z}$-actions on compact spaces (such as irrational rotation of the circle) have dynamic asymptotic dimension one, and that groups with finite asymptotic dimension act with finite dynamic asymptotic dimension on some compact space.

Although the aforementioned result follows from earlier work of Tu [37] on the Baum-Connes conjecture for amenable groupoids, the proof given in [14] is completely different, and in some sense more direct. In fact, the proof is very much inspired by Yu's proof of the coarse Baum-Connes conjecture for spaces with finite asymptotic dimension in [40]. The main tool in both cases is a controlled Mayer-Vietoris sequence, which is part of the framework of quantitative (or controlled) K-theory for C^*-algebras mentioned earlier. Finiteness of the appropriate notion of dimension allows one to apply the Mayer-Vietoris argument a finite number of times to arrive at the quantitative K-theory of the algebra in question. Passing to the limit in an appropriate sense, one gets the K-theory of the algebra.

Having extended the framework of quantitative K-theory to a larger class of Banach algebras so that it can be applied to algebras of bounded linear operators on L_p spaces, our goal is to consider the L_p analog of the assembly map in [14], and use our extended framework of quantitative K-theory to show that this assembly map is an isomorphism under the assumption of finite dynamic asymptotic dimension. In fact, one sees that the techniques and proofs in [14] carry over to our setting with minor adjustments, the main difference being the exposition of the base case in the Mayer-Vietoris argument. In order to state our main result, let us first recall the usual Baum-Connes conjecture with coefficients.

Let A be a C^*-algebra and let a countable discrete group Γ act on A by $*$-automorphisms. One may then form the reduced crossed product C^*-algebra $A \rtimes_\lambda \Gamma$. The usual Baum-Connes conjecture with coefficients posits that a certain homomorphism

$$\mu : K_*^\Gamma(\underline{E}\Gamma; A) \to K_*(A \rtimes_\lambda \Gamma)$$

is an isomorphism [1], where the left-hand side is the equivariant K-homology with coefficients in A of the classifying space $\underline{E}\Gamma$ for proper Γ-actions, and the right-hand side is the K-theory of the reduced crossed product C^*-algebra. We will consider a particular model for $\underline{E}\Gamma$, namely $\bigcup_{s\geq 0} P_s(\Gamma)$ equipped with the ℓ_1 metric (cf. [1, Section 2]), where $P_s(\Gamma)$ is the Rips complex of Γ at scale s, i.e., it is the simplicial complex with vertex set Γ, and where a finite subset $E \subset \Gamma$ spans a simplex if and only if $d(g,h) \leq s$ for all $g, h \in E$. Here we assume that Γ is equipped with a proper length function and d is the associated metric. One may then reformulate the Baum-Connes map as

$$\lim_{s\to\infty} K_*(C_L^*(P_s(\Gamma); A)) \stackrel{\epsilon_0}{\to} \lim_{s\to\infty} K_*(C^*(P_s(\Gamma); A)) \cong K_*(A \rtimes_\lambda \Gamma),$$

where $C_L^*(P_s(\Gamma); A)$ is Yu's localization algebra [39] with coefficients in A, $C^*(P_s(\Gamma); A)$ is the equivariant Roe algebra with coefficients in A, and ϵ_0 is (induced by) the evaluation-at-zero map. The fact that K-homology can be identified with the K-theory of the localization algebra was shown for finite-dimensional simplicial complexes in [39], and in full generality in [33]. The fact that the equivariant Roe algebra with coefficients is stably isomorphic to the reduced crossed product forms the basis for the coarse-geometric approach to the Baum-Connes conjecture with coefficients (see [34] for the case without coefficients).

Now let A be a norm-closed subalgebra of $B(L_p(Z, \mu))$ for some measure space (Z, μ) and $p \in (1, \infty)$. We refer to such algebras as L_p operator algebras. Let Γ be a countable discrete group acting on A by isometric automorphisms. Set $A\Gamma$ to be the set of finite sums of the form $\sum_{g \in \Gamma} a_g g$ with $a_g \in A$ and with the product given by

$$\left(\sum_{g \in \Gamma} a_g g\right)\left(\sum_{h \in \Gamma} b_h h\right) = \sum_{g,h \in \Gamma} a_g \alpha_g(b_h) gh,$$

where α denotes the Γ-action on A. There is a natural faithful representation of $A\Gamma$ on $\ell_p(\Gamma, L_p(Z, \mu))$ given by

$$(a\xi)(h) = \alpha_{h^{-1}}(a)\xi(h),$$
$$(g\xi)(h) = \xi(g^{-1}h)$$

for $a \in A$, $g, h \in \Gamma$, and $\xi \in \ell_p(\Gamma, L_p(Z, \mu))$. We then define the L_p reduced crossed product $A \rtimes_{\lambda,p} \Gamma$ to be the operator norm closure of $A\Gamma$ in $B(\ell_p(\Gamma, L_p(Z, \mu)))$.

We can formulate the L_p Baum-Connes conjecture with coefficients by replacing the reduced crossed product C^*-algebra by the L_p reduced crossed product, and also considering L_p versions of Roe algebras and localization algebras, so that the map in question essentially becomes

$$\lim_{s\to\infty} K_*(C_L^*(P_s(\Gamma); A)) \to K_*(A \rtimes_{\lambda,p} \Gamma),$$

and this map is posited to be an isomorphism. Here we note that one can show that the left-hand side is independent of p by Mayer-Vietoris arguments (cf. [2, 39]), and the L_p version of the equivariant Roe algebra is stably isomorphic to the L_p reduced crossed product by the same argument as in the C^*-algebra case.

Our main result may then be stated as follows.

Theorem 1.0.3. *(cf. Theorem 4.3.23) Let a countable discrete group Γ act with finite dynamic asymptotic dimension on a compact Hausdorff space X. Then the L_p Baum-Connes conjecture holds for Γ with coefficients in $C(X)$ for $p \in (1, \infty)$.*

Since the left-hand side of the map is independent of p, we have the following corollary, which gives a partial answer to [29, Problem 11.2].

Corollary 1.0.4. *Let a countable discrete group Γ act with finite dynamic asymptotic dimension on a compact Hausdorff space X. Then the K-theory of the L_p reduced crossed product $C(X) \rtimes_{\lambda,p} \Gamma$ is independent of p for $p \in (1, \infty)$.*

In the L_p setting, we note that Kasparov and Yu have some (yet unpublished) work on the L_p Baum-Connes conjecture [18]. We also remark that at the moment, there seems to be substantial difficulty in carrying over other approaches to the usual Baum-Connes conjecture, such as the Dirac-dual Dirac method, to the L_p setting. However, quantitative K-theory still works well in the L_p setting.

The question of whether the K-theory of the L_p reduced crossed product depends on p remains open in general. In the case where X is a point with the trivial Γ-action, one gets the L_p reduced group algebra. When Γ is hyperbolic or amenable, the K-theory of this algebra is known to be independent of p [18, 22]. The significance of this question is that sometimes the K-theory of these algebras may be more computable for large p so if the K-theory of these algebras is independent of p, then we get in particular a computation of the K-theory of the respective C^*-algebras.

2. PRELIMINARIES

2.1 Banach Algebras

In this section, we record some basic facts about Banach algebras that can be found, for instance, in [17]. Throughout this dissertation, we will only work with complex Banach algebras.

Definition 2.1.1. *A Banach algebra A is an algebra equipped with a submultiplicative norm, i.e., $||ab|| \leq ||a||||b||$ for all $a, b \in A$, and such that $(A, ||\cdot||)$ is a Banach space.*

A is said to be unital if there exists $1 \in A$ such that $1a = a1 = a$ for all $a \in A$. If there is no such element, then A is said to be non-unital.

Definition 2.1.2. *Let A be an algebra. A map $* : A \to A, a \mapsto a^*$, is called an involution if it satisfies*

1. $(a+b)^* = a^* + b^*$ *for all* $a, b \in A$,
2. $(\lambda a)^* = \bar{\lambda} a^*$ *for all* $\lambda \in \mathbb{C}$, $a \in A$,
3. $(ab)^* = b^* a^*$ *for all* $a, b \in A$,
4. $(a^*)^* = a$ *for all* $a \in A$.

*If A is a Banach algebra equipped with an isometric involution, i.e., $||a^*|| = ||a||$ for all $a \in A$, then A is called a Banach *-algebra.*

If the involution also satisfies $||a^*a|| = ||a||^2$ for all $a \in A$, then A is called a C^*-algebra.

Definition 2.1.3. *If A is a Banach algebra, and S is a subset of A, then the subalgebra of A generated by S consists of all linear combinations of finite products of elements in S. The closure of this subalgebra in A is the Banach subalgebra generated by S.*

Example 2.1.4.

1. *Let X be a compact Hausdorff space. Then $C(X)$, the set of continuous functions on X, is a unital Banach algebra when equipped with pointwise multiplication and the norm $||f|| := \sup_{x\in X} |f(x)|$.*

2. *If X is locally compact but not compact, then $C_0(X)$, the set of continuous functions on X vanishing at infinity, is a non-unital Banach algebra with the above multiplication and norm.*

3. *Let E be a complex Banach space. Then $B(E)$, the set of bounded linear operators on E, is a unital Banach algebra with composition as multiplication and the operator norm.*

Remark 2.1.5. *Suppose that $(A, ||\cdot||)$ is a unital Banach algebra. Then there exists another submultiplicative norm $||\cdot||'$ equivalent to $||\cdot||$ such that $||1||' = 1$. Indeed, if $||1|| \neq 1$, then $||1|| > 1$ by submultiplicativity of $||\cdot||$. Define $||a||' = ||L_a||$, where $L_a : A \to A$ is given by $L_a b = ab$. Note that L_a is a bounded linear operator on A as a consequence of submultiplicativity of $||\cdot||$, and we have $||a||' \leq ||a||$ for all $a \in A$. On the other hand, we have $||a|| = ||L_a 1|| \leq ||L_a||||1|| = ||a||'||1||$ for all $a \in A$.*

Hence, whenever A is a unital Banach algebra, we will assume that $||1|| = 1$.

Definition 2.1.6. *Suppose A is an algebra. Define $A^+ = A \times \mathbb{C}$ equipped with the operation $(a, z)(b, w) = (ab + zb + wa, zw)$ for $a, b \in A$ and $z, w \in \mathbb{C}$. Then A^+ is a unital algebra with unit $(0, 1)$. We identify A as a subalgebra in A^+ via the map $a \mapsto (a, 0)$. We call A^+ the unitization of A.*

Note that this construction makes sense even when A is already unital, but the original unit in A is not the unit in A^+.

Whenever B is a unital algebra, and $\phi : A \to B$ is an algebra homomorphism, there exists a unique algebra homomorphism $\phi^+ : A^+ \to B$ such that $\phi^+|_A = \phi$ and $\phi^+(0,1) = 1$. This homomorphism is given by $\phi^+(a,z) = \phi(a) + z1$ for all $(a,z) \in A^+$.

The unitization A^+ of A can be equipped with a submultiplicative norm extending the norm on A such that $(0,1) \in A^+$ has norm 1. One such norm is given by

$$||(a,z)||_1 = ||a|| + |z|$$

for all $(a,z) \in A^+$. If there is an isometric algebra homomorphism $\phi : A \to B$, where B is a unital normed algebra with $||1_B|| = 1$ and $1_B \notin \phi(A)$, then the homomorphism $\phi^+ : A^+ \to B$ is injective, and we can also define a submultiplicative norm $|| \cdot ||'$ on A^+ by

$$||(a,z)||' = ||\phi^+(a,z)||.$$

If A is a Banach algebra, then any submultiplicative norm on A^+ extending the norm on A such that $(0,1) \in A^+$ has norm 1 is in fact equivalent to the norm $|| \cdot ||_1$ defined above. This is a consequence of the open mapping theorem once one observes that any such norm is dominated by $|| \cdot ||_1$.

The invertible elements in a unital Banach algebra play an important role in the theory of Banach algebras, and also in K-theory.

Lemma 2.1.7. *Suppose A is a unital Banach algebra, and $a \in A$ is invertible. Suppose that $b \in A$ satisfies $||b - a|| < \frac{1}{||a^{-1}||}$. Then b is also invertible, and*

$$||b^{-1} - a^{-1}|| \leq \frac{||a^{-1}||^2||b - a||}{1 - ||a^{-1}||||b - a||}.$$

Proof. Define $y = 1 - a^{-1}b$. Then $||y|| = ||a^{-1}(a - b)|| \leq ||a^{-1}||||a - b|| < 1$. Since $||y^n|| \leq ||y||^n$ for all $n \geq 1$, the series $\sum_{n=0}^{\infty} y^n$ converges to an element $z \in A$, and we have

$$||1-z|| \leq \sum_{n=1}^{\infty} ||y||^n \leq \sum_{n=1}^{\infty} (||a^{-1}||||b-a||)^n = \frac{||a^{-1}||||b-a||}{1-||a^{-1}||||b-a||}.$$

By the definition of z, we have $z(1-y) = (1-y)z = 1$, so $a^{-1}b = 1-y$ is invertible, and $(a^{-1}b)^{-1} = z$. Since a is invertible, it follows that b is invertible with inverse $b^{-1} = za^{-1}$. We then have

$$||b^{-1} - a^{-1}|| = ||(1-z)a^{-1}|| \leq ||1-z||||a^{-1}|| \leq \frac{||a^{-1}||^2||b-a||}{1-||a^{-1}||||b-a||}.$$

□

Corollary 2.1.8. *If A is a unital Banach algebra, then the set of invertible elements in A, denoted by $GL(A)$, is open in A, and inversion is continuous.*

Definition 2.1.9. *Let A be a Banach algebra. A character of A is a nonzero algebra homomorphism $\gamma : A \to \mathbb{C}$.*

Note that if A is unital, then the requirement that γ be nonzero forces $\gamma(1) = 1$.

Proposition 2.1.10. *Let A be a Banach algebra, and let $\gamma : A \to \mathbb{C}$ be a character. Then γ is continuous, and $||\gamma|| \leq 1$.*

Proof. We may assume that A is unital and $||1|| = 1$. If not, we extend γ to a character γ^+ of A^+ by $\gamma^+(a, z) = \gamma(a) + z$ and it will suffice to show that γ^+ is continuous and $||\gamma^+|| \leq 1$.

Suppose there exists $a_0 \in A$ with $|\gamma(a_0)| > ||a_0||$. Let $\lambda = \gamma(a_0)$ so that $a_1 := \lambda^{-1}a_0$ satisfies $||a_1|| < 1$. Then $1 - a_1 = 1 - \lambda^{-1}a_0$ is invertible. Since $\lambda \neq 0$, it follows that $z := \lambda(1 - a_1) = \lambda 1 - a_0$ is invertible. But $\gamma(z) = \lambda\gamma(1) - \gamma(a_0) = \lambda - \gamma(a_0) = 0$, and this forces $1 = \gamma(1) = \gamma(z^{-1}z) = \gamma(z^{-1})\gamma(z) = 0$, which is impossible. □

Definition 2.1.11. *Let A be a unital Banach algebra. For $a \in A$, the set*

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A\}$$

is called the spectrum of a (relative to A).

Theorem 2.1.12. *Let A be a unital Banach algebra. For any $a \in A$, the set $\sigma_A(a)$ is compact and non-empty.*

Sketch of proof. Given $a \in A$, consider the map $\psi : \mathbb{C} \to A$ given by $\psi(\lambda) = \lambda 1 - a$. Then $\mathbb{C} \setminus \sigma_A(a) = \psi^{-1}(GL(A))$. The fact that $\mathbb{C} \setminus \sigma_A(a)$ is open is a consequence of the continuity of ψ and the fact that $GL(A)$ is open.

If $|\lambda| > ||a||$, then the element $x = \lambda 1 - a$ satisfies $||x - \lambda 1|| = ||a|| < |\lambda| = ||(\lambda 1)^{-1}||^{-1}$ so x is invertible, which means that $\lambda \notin \sigma_A(a)$. Hence $\sigma_A(a)$ is bounded as $\sigma_A(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq ||a||\}$.

If $\sigma_A(a) = \emptyset$, define $F : \mathbb{C} \to A$ by $F(\zeta) = (\zeta 1 - a)^{-1}$. Then F is continuous, and one shows that $\lim_{\zeta \to \infty} ||F(\zeta)|| = 0$. For each continuous linear functional $\theta \in A^*$, one shows that the map $F_\theta = \theta \circ F : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\lim_{\zeta \to \infty} F_\theta(\zeta) = 0$. By Liouville's theorem, it follows that $F_\theta \equiv 0$. Fix $\zeta \in \mathbb{C}$. The fact that $\theta(F(\zeta)) = F_\theta(\zeta) = 0$ for all $\theta \in A^*$, combined with the Hahn-Banach theorem, forces $F(\zeta) = 0$, which is impossible. □

Example 2.1.13.

1. *For $T \in M_n(\mathbb{C})$, we have $\sigma_{M_n(\mathbb{C})}(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue for } T\}$.*

2. *Let X be a compact Hausdorff space. For $f \in C(X)$, we have $\sigma_{C(X)}(f) = f(X)$.*

The notion of functional calculus is also an important one in the theory of Banach algebras. It allows one to make sense of expressions like $f(a)$, where a is an element of a Banach algebra and $f : \mathbb{C} \to \mathbb{C}$ is an appropriate function.

Fix some element a in a unital Banach algebra A. Suppose that $p : \mathbb{C} \to \mathbb{C}$ is a polynomial, i.e., $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ with $c_0, \ldots, c_n \in \mathbb{C}$. We can then define $p(a) = c_0 1 + c_1 a + \cdots + c_n a^n$. Now let U be an open subset of $\mathbb{C}$ containing $\sigma_A(a)$, and denote by $R(U)$ the set of all rational functions on U, i.e., $f \in R(U)$ if and only if $f = (\frac{p}{q})|_U$, where p and q are polynomials with $q(z) \neq 0$ for all $z \in U$. Since $\sigma_A(q(a)) = q(\sigma_A(a))$, we have that $0 \notin \sigma_A(q(a))$ so we may define $f(a) \in A$ by $f(a) = p(a)q(a)^{-1}$. If we let $R(a) = \bigcup\{R(U) : U \text{ open}, U \supseteq \sigma_A(a)\}$, then $R(a)$ is an algebra, and $f(a) \in A$ is well-defined for every $f \in R(a)$. In fact, the mapping $f \mapsto f(a)$ is a homomorphism from $R(a)$ into A, and satisfies $\sigma_A(f(a)) = f(\sigma_A(a))$.

More generally, we can also make sense of $f(a)$ for a function f that is holomorphic on a neighborhood of $\sigma_A(a)$. Given an open subset of $\mathbb{C}$, let $H(U)$ denote the algebra of all holomorphic functions on U. For $a \in A$, let $H(a)$ be the set of all functions that are holomorphic in some neighborhood of $\sigma_A(a)$. Then $H(a)$ is an algebra under pointwise operations.

Proposition 2.1.14. *[17, Proposition 3.15] Let A be a unital Banach algebra, let $a \in A$, and let U be an open neighborhood of $\sigma_A(a)$. Suppose that $\gamma_1, \ldots, \gamma_n$ are closed, piecewise smooth curves in $U \setminus \sigma_A(a)$ such that for any holomorphic function f on U and $z \in \sigma_A(a)$,*

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(w)}{w - z} dw.$$

Then for any rational function f on U,

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} f(z)(z 1_A - a)^{-1} dz.$$

We would like to define, for $f \in H(a)$, an element $f(a) \in A$ by setting

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} f(z)(z1_A - a)^{-1} dz,$$

where $\gamma_1, \ldots, \gamma_n$ are as above. Indeed, this definition does not depend on the choice of U and of the curves $\gamma_1, \ldots, \gamma_n$ [17, Lemma 3.16]. The set of mappings $H(a) \to A$, $f \mapsto f(a)$, is referred to as the holomorphic functional calculus.

Theorem 2.1.15. *(cf. [17, Theorem 3.18]) Let A be a unital Banach algebra, and let $a \in A$.*

1. *The mapping $f \mapsto f(a)$ is a homomorphism from $H(a)$ into A.*
2. *Suppose that f and f_n ($n \in \mathbb{N}$) are holomorphic functions on some open set U containing $\sigma_A(a)$ and that f_n converges uniformly to f on every compact subset of U. Then $||f_n(a) - f(a)|| \to 0$.*

Our main application of the holomorphic functional calculus is to produce idempotents.

Theorem 2.1.16. *(cf. [17, Theorem 3.29]) Let A be a unital Banach algebra. Let $a \in A$ and suppose that $\sigma_A(a) = \bigcup_{j=1}^{n} C_j$, where the sets C_j are nonempty, pairwise disjoint, and clopen in $\sigma_A(a)$. Then there exist idempotents $e_1, \ldots, e_n \in A$ (i.e., $e_j^2 = e_j$) such that*

1. *$1_A = \sum_{j=1}^{n} e_j$, $e_j \neq 0$, and $e_j e_k = 0$ for $j \neq k$;*
2. *each e_j is contained in the closed linear span of all elements of the form $(\lambda 1_A - a)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma_A(a)$.*

Proof. Since $C_1, \ldots, C_n$ are compact, there exist pairwise disjoint open subsets $V_1, \ldots, V_n$ of $\mathbb{C}$ such that $C_j \subseteq V_j$. For each j, choose an open subset W_j of $\mathbb{C}$ such that $W_j \cap \sigma_A(a) =$

C_j. Then the sets $U_j = V_j \cap W_j$ are pairwise disjoint, open, and satisfy $U_j \cap \sigma_A(a) = C_j$. Let $U = \bigcup_{j=1}^n U_j$. For each j, define a function f_j on U by $f_j = \chi_{U_j}$. One can then check that $e_j = f_j(a)$ are the desired idempotents. $\square$

2.2 *K*-Theory of Banach Algebras

In this section, we record some basic facts about the K-theory of Banach algebras, details of which can be found in [3] (or [38] when restricted to C^*-algebras).

In order to define the K_0 group of a Banach algebra A, we consider idempotents not only in A, but in $M_\infty(A) := \bigcup_{n\in\mathbb{N}} M_n(A)$, where we regard $M_n(A)$ as embedded in $M_{n+1}(A)$ via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Definition 2.2.1. *An idempotent in a Banach algebra A is an element e satisfying $e^2 = e$. Two idempotents e and f are orthogonal if $ef = fe = 0$.*

Definition 2.2.2. *Let e and f be idempotents in a Banach algebra A.*

1. *We say that e and f are similar, and write $e \sim_s f$, if there is an invertible element $z \in A^+$ such that $zez^{-1} = f$.*

2. *We say that e and f are homotopic, and write $e \sim_h f$, if there is a norm-continuous path of idempotents in A from e to f.*

Proposition 2.2.3. *[3, Proposition 4.3.2] Let e and f be idempotents in a Banach algebra A. If $||e - f|| < \frac{1}{||2e-1||}$, then $e \sim_s f$. In fact, there exists $z \in A^+$ with $||z - 1|| < \frac{||2e-1||}{||e-f||}$ and $z^{-1}ez = f$. Also, $e \sim_h f$.*

Proposition 2.2.4. *[3, Proposition 4.3.3] If $e \sim_h f$ via the path e_t, then there is a path z_t of invertibles with $z_0 = 1$ and $z_t^{-1}ez_t = e_t$ for all t. Thus $e \sim_s f$.*

In general, it is not true that $e \sim_s f$ implies $e \sim_h f$.

Proposition 2.2.5. *[3, Proposition 4.4.1] If* $e \sim_s f$*, then* $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$.

Since we will consider simultaneously all matrix algebras over A, the two equivalence relations become interchangeable (up to doubling matrix sizes).

Definition 2.2.6. *Let A be a Banach algebra. Define $V(A)$ to be the set of all homotopy classes of idempotents in $M_\infty(A)$. On $V(A)$, define addition by* $[e] + [f] = \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \right]$.

It is straightforward to check that this addition operation is well-defined and makes $V(A)$ into an abelian semigroup with identity $[0]$.

Example 2.2.7.

1. $V(\mathbb{C}) = V(M_n(\mathbb{C})) = V(K(H)) = \mathbb{N} \cup \{0\}$*, where $K(H)$ denotes the algebra of compact operators on a separable Hilbert space H.*

2. $V(B(H)) = \mathbb{N} \cup \{0, \infty\}$*, where $B(H)$ denotes the algebra of bounded linear operators on a separable Hilbert space H.*

If $\phi : A \to B$ is a homomorphism between Banach algebras, then ϕ extends to a homomorphism from $M_\infty(A)$ to $M_\infty(B)$, which induces a semigroup homomorphism $\phi_* : V(A) \to V(B)$ given by $\phi_*([e]) = [\phi(e)]$.

Definition 2.2.8. *Let A and B be Banach algebras. Two bounded homomorphisms $\phi, \psi : A \to B$ are said to be homotopic if there is a path of bounded homomorphisms $\omega_t : A \to B$ for $0 \leq t \leq 1$, continuous in t in the topology of pointwise norm-convergence, with $\omega_0 = \phi$ and $\omega_1 = \psi$.*

Equivalently, ϕ and ψ are homotopic if there exists a bounded homomorphism $\omega : A \to C([0,1], B)$ with $\pi_0 \circ \omega = \phi$ and $\pi_1 \circ \omega = \psi$, where $\pi_t : C([0,1], B) \to B$ is evaluation at t.

From the definitions, one sees that if $\phi, \psi : A \to B$ are homotopic, then $\phi(e) \sim_h \psi(e)$ for any idempotent $e \in M_\infty(A)$, so $\phi_* = \psi_* : V(A) \to V(B)$. This property is known as homotopic invariance.

One can also verify that

- if $A = A_1 \oplus A_2$, then $V(A) \cong V(A_1) \oplus V(A_2)$;
- if $A = \varinjlim A_i$, then $V(A) \cong \varinjlim V(A_i)$.

Definition 2.2.9. *For a unital Banach algebra A, define $K_0(A)$ to be the Grothendieck group of $V(A)$.*

For a non-unital Banach algebra A, define $K_0(A)$ to be $\ker(\pi_* : K_0(A^+) \to K_0(\mathbb{C}))$, where $\pi : A^+ \to \mathbb{C}$ is the homomorphism given by $\pi(a, z) = z$.

Example 2.2.10.

1. $K_0(\mathbb{C}) = K_0(M_n(\mathbb{C})) = K_0(K(H)) = \mathbb{Z}$;
2. $K_0(B(H)) = 0$.

Let A be a Banach algebra. Let $GL_n(A) = \{x \in GL_n(A^+) : x \equiv I_n \mod M_n(A)\}$. We embed $GL_n(A)$ into $GL_{n+1}(A)$ via the map $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, and let $GL_\infty(A) = \varinjlim GL_n(A)$, which can be thought of as the group of invertible infinite matrices that have diagonal elements in $1_{A^+} + A$, off-diagonal elements in A, and only finitely many entries different from 0 or 1.

Definition 2.2.11. *Let u and v be invertible elements in a unital Banach algebra A. We say that u and v are homotopic if there is a norm-continuous path of invertible elements in A from u to v.*

Definition 2.2.12. *Let A be a Banach algebra. Define $K_1(A)$ to be the set of homotopy classes of invertible elements in $GL_\infty(A)$.*

$K_1(A)$ has an abelian group structure under the operation $[u] + [v] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]$.

Example 2.2.13. *$K_1(\mathbb{C}) = 0$ since every invertible matrix with entries in $\mathbb{C}$ can be connected to the identity matrix.*

The properties that we stated for K_0 also hold for K_1, i.e.,

- If $\phi : A \to B$ is a homomorphism between Banach algebras, then it extends to a unital homomorphism $A^+ \to B^+$, thereby inducing a homomorphism $\phi_* : K_1(A) \to K_1(B)$.
- If $\phi, \psi : A \to B$ are homotopic, then $\phi_* = \psi_*$.
- $K_1(A_1 \oplus A_2) \cong K_1(A_1) \oplus K_1(A_2)$.
- $K_1(\varinjlim A_i) \cong \varinjlim K_1(A_i)$.

Definition 2.2.14. *Let A be a Banach algebra. The suspension of A, denoted by SA, is $C_0(\mathbb{R}, A)$ equipped with pointwise operations and the sup norm.*

Using suspensions, one can view K_1 groups as K_0 groups. More precisely, we have

Theorem 2.2.15. *[3, Theorem 8.2.2] There is an isomorphism $\theta_A : K_1(A) \to K_0(SA)$ such that whenever $\phi : A \to B$ is a homomorphism, we have the following commutative diagram:*

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\phi_*} & K_1(B) \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} \\ K_0(SA) & \xrightarrow{S\phi_*} & K_0(SB) \end{array}$$

Definition 2.2.16. *A sequence* $G \xrightarrow{f} H \xrightarrow{g} K$ *of groups and group homomorphisms is said to be exact if* $\mathrm{im} f = \ker g$.

Theorem 2.2.17. *If* J *is a closed two-sided ideal in* A*, then we have the following exact sequence:*

$$K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{q_*} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{q_*} K_0(A/J),$$

where $i : J \to A$ *is the inclusion,* $q : A \to A/J$ *is the quotient homomorphism, and* $\partial : K_1(A/J) \to K_0(J)$ *is defined as follows: Let* $u \in GL_n(A/J)$*, and let* $w \in GL_{2n}(A)$ *be a lift of* $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. *Then* $\partial([u]) = [wp_nw^{-1}] - [p_n] \in K_0(J)$*, where* p_n *is the matrix with* n *1's along the diagonal and 0 everywhere else.*

In fact, one can connect $K_0(A/J)$ to $K_1(J)$ to make the sequence a cyclic six-term exact sequence. This is a consequence of Bott periodicity, which we will now briefly describe.

If e is an idempotent in $M_n(A^+)$, write $f_e(z) = ze + (1 - e) \in C(S^1, GL_n(A^+))$. Such loops represent elements in $K_1(SA)$. Consider the homomorphism $\beta_A : K_0(A) \to K_1(SA)$ given by $\beta_A([e] - [p_n]) = [f_e f_{p_n}^{-1}]$, called the Bott map. If $\phi : A \to B$ is a homomorphism, then we have the following commutative diagram:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\phi_*} & K_0(B) \\ \downarrow \beta_A & & \downarrow \beta_B \\ K_1(SA) & \xrightarrow{S\phi_*} & K_1(SB) \end{array}$$

Theorem 2.2.18. *[3, Theorem 9.2.1](Bott Periodicity)* β_A *is an isomorphism.*

Define $\partial : K_0(A/J) \to K_1(J)$ to be the composition

$$K_0(A/J) \stackrel{\beta_A}{\to} K_1(S(A/J)) \to K_0(SJ) \stackrel{\theta_J^{-1}}{\to} K_1(J).$$

Theorem 2.2.19. *[3, Theorem 9.3.1] If J is a closed two-sided ideal in A, then we have the following six-term exact sequence:*

$$\begin{array}{ccccc}
K_1(J) & \xrightarrow{\quad i_* \quad} & K_1(A) & \xrightarrow{\quad q_* \quad} & K_1(A/J) \\
\partial \uparrow & & & & \downarrow \partial \\
K_0(A/J) & \xleftarrow[\quad q_* \quad]{} & K_0(A) & \xleftarrow[\quad i_* \quad]{} & K_0(J)
\end{array}$$

This six-term exact sequence is one of the standard computational tools in K-theory. Another useful computational tool is the following Mayer-Vietoris sequence.

Theorem 2.2.20. *(cf. [16, Exercise 4.10.21]) Let J_0 and J_1 be closed two-sided ideals in a Banach algebra A with $J_0 + J_1 = A$. Then we have the following six-term exact sequence:*

$$\begin{array}{ccccc}
K_1(J_0 \cap J_1) & \longrightarrow & K_1(J_0) \oplus K_1(J_1) & \longrightarrow & K_1(A) \\
\uparrow & & & & \downarrow \\
K_0(A) & \longleftarrow & K_0(J_0) \oplus K_0(J_1) & \longleftarrow & K_0(J_0 \cap J_1)
\end{array}$$

3. QUANTITATIVE K-THEORY FOR BANACH ALGEBRAS

3.1 Filtered Banach Algebras, Quasi-Idempotents, and Quasi-Invertibles

In this section, we introduce quasi-idempotents and quasi-invertibles in filtered Banach algebras, and we consider homotopy relations on these elements. These are the basic ingredients for our framework of quantitative K-theory. We will only consider complex Banach algebras.

3.1.1 Filtered Banach algebras and SQ_p algebras

Definition 3.1.1. *A filtered Banach algebra is a Banach algebra A with a family $(A_r)_{r>0}$ of closed linear subspaces such that*

- $A_r \subseteq A_{r'}$ *if* $r \leq r'$;
- $A_r A_{r'} \subseteq A_{r+r'}$ *for all* $r, r' > 0$;
- *the subalgebra* $\bigcup_{r>0} A_r$ *is dense in* A.

If A is unital with unit 1_A, we require $1_A \in A_r$ for all $r > 0$. In this case, we set $A_0 = \mathbb{C}1_A$. Elements of A_r are said to have propagation r. The family $(A_r)_{r>0}$ is called a filtration of A.

When A is a C^*-algebra, we also want $A_r^* = A_r$ for all $r > 0$.

Remark 3.1.2.

1. *When $A = \mathbb{C}$, we will usually set $A_r = \mathbb{C}$ for all $r > 0$.*
2. *If A is a filtered Banach algebra with filtration $(A_r)_{r>0}$, and J is a closed ideal in A, then A/J is a Banach algebra under the quotient norm, and has filtration $((A_r + J)/J)_{r>0}$.*

Example 3.1.3. *Let G be a countable discrete group equipped with a proper length function l, i.e., a function $l : G \to \mathbb{N}$ satisfying*

- $l(g) = 0$ *if and only if* $g = e$*;*
- $l(gh) \leq l(g) + l(h)$ *for all* $g, h \in G$*;*
- $l(g^{-1}) = l(g)$ *for all* $g \in G$*;*
- $\{g \in G : l(g) \leq r\}$ *is finite for all* $r \geq 0$.

Then

1. *The reduced group C^*-algebra, $C^*_\lambda(G)$, is a filtered C^*-algebra with a filtration given by*
$$(C^*_\lambda(G))_r = \{\sum a_g g : a_g \in \mathbb{C}, l(g) \leq r\}.$$
2. *Suppose that G acts on a C^*-algebra A by automorphisms. Then the reduced crossed product, $A \rtimes_\lambda G$, is a filtered C^*-algebra with a filtration given by*
$$(A \rtimes_\lambda G)_r = \{\sum a_g g : a_g \in A, l(g) \leq r\}.$$

Other examples of filtered C^*-algebras include finitely generated C^*-algebras and Roe algebras. One may also consider the L_p analogs of the group C^*-algebra and crossed product, and these are examples of filtered Banach algebras. In fact, in each of these examples, one can define a length function $l : A \to [0, \infty]$ on the algebra A, satisfying the following conditions:

- $l(0) = 0$ (or $l(1_A) = 0$ if A is unital);
- $l(a + b) \leq \max(l(a), l(b))$ and $l(ab) \leq l(a) + l(b)$ for all $a, b \in A$;

- $l(ca) \leq l(a)$ for any $a \in A$ and $c \in \mathbb{C}$;
- the set $\{a \in A : l(a) < \infty\}$ is dense in A, and $\{a \in A : l(a) \leq r\}$ is a closed subset of A for each $r \geq 0$.

This length function then gives rise to a natural filtration by setting

$$A_r = \{a \in A : l(a) \leq r\}$$

for each $r \geq 0$. Thus filtered Banach algebras may also be regarded as "geometric" Banach algebras in the spirit of geometric group theory.

If A is a non-unital Banach algebra, let $A^+ = \{(a, z) : a \in A, z \in \mathbb{C}\}$ with multiplication given by $(a, z)(b, w) = (ab + zb + wa, zw)$. We call A^+ the unitization of A. We will use the notation

$$\tilde{A} = \begin{cases} A & \text{if } A \text{ is unital,} \\ A^+ & \text{if } A \text{ is nonunital.} \end{cases}$$

Note that if A is a unital Banach algebra, then we can always give it an equivalent Banach algebra norm such that $||1_A|| = 1$, namely the operator norm from the left regular representation of A on itself. Thus we will always assume that $||1_A|| = 1$ when dealing with unital Banach algebras.

For our framework of quantitative K-theory, since we will consider matrices of all sizes simultaneously and we want to have norm control, we need our Banach algebras to have some matrix norm structure.

Definition 3.1.4. *[9] For $p \in [1, \infty)$, an abstract p-operator space is a Banach space X together with a family of norms $|| \cdot ||_n$ on $M_n(X)$ satisfying:*

$\mathcal{D}_\infty$: *For $u \in M_n(X)$ and $v \in M_m(X)$, we have*

$$\left\| \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right\|_{n+m} = \max(||u||_n, ||v||_m);$$

$\mathcal{M}_p$: *For $u \in M_m(X)$, $\alpha \in M_{n,m}(\mathbb{C})$, and $\beta \in M_{m,n}(\mathbb{C})$, we have*

$$||\alpha u \beta||_n \leq ||\alpha||_{B(\ell_p^m, \ell_p^n)} ||u||_m ||\beta||_{B(\ell_p^n, \ell_p^m)}.$$

In the general theory of p-operator spaces, one typically considers only $p \in (1, \infty)$ but the definition still makes sense when $p = 1$, and the properties of p-operator spaces that we will use still hold in this case. We also mention Le Merdy's result [24, Theorem 4.1] that for $p \in (1, \infty)$, an abstract p-operator space X can be p-completely isometrically embedded in $B(E, F)$ for some $E, F \in SQ_p$, where SQ_p denotes the collection of subspaces of quotients of L_p spaces. The class SQ_2 is precisely the class of Hilbert spaces while the class SQ_1 contains all Banach spaces since every Banach space is a quotient of some L_1 space.

Note that a p-operator space structure does not necessarily respect multiplicative structure, so given a filtered Banach algebra A with p-operator space structure $\{|| \cdot ||_n\}_{n \in \mathbb{N}}$, we also require $(M_n(A), || \cdot ||_n)$ to be a Banach algebra for each n. Here we note the following result of Le Merdy:

Theorem 3.1.5. *[25, Theorem 3.3] If A is a unital Banach algebra with a p-operator space structure $\{|| \cdot ||_n\}_{n \in \mathbb{N}}$ and $p \in (1, \infty)$, then the following are equivalent:*

1. *Each $(M_n(A), || \cdot ||_n)$ is a Banach algebra.*

2. *A is p-completely isometrically isomorphic to a subalgebra of $B(E)$ for some $E \in SQ_p$.*

Remark 3.1.6. *This theorem generalizes the result by Blecher, Ruan, and Sinclair in the $p = 2$ case [5, Theorem 3.1] (also see [4, Theorem 2.1]). The $p = 1$ case is omitted in Le Merdy's theorem but we note that at least (2.) $\Rightarrow$ (1.) is valid, i.e., algebras of bounded linear operators on SQ_1 spaces have a canonical 1-operator space structure such that the matrix algebras are Banach algebras. This goes back to Kwapien [19, Theorem 4.2'] (also see [24, Theorem 3.2]), and is sufficient for our purposes.*

For the rest of this dissertation, we will refer to norm-closed subalgebras of $B(E)$, where $E \in SQ_p$, as SQ_p algebras for $p \in [1, \infty)$. If $A \subset B(E)$ is a non-unital SQ_p algebra, we view A^+ as $A + \mathbb{C}I_E$ so that A^+ is a unital SQ_p algebra.

Remark 3.1.7. *If A is a filtered SQ_p algebra, then*

- $||I_n||_n = 1$ *for all* $n \in \mathbb{N}$, *where* I_n *is the identity in* $M_n(\tilde{A})$;
- $||a_{kl}||_1 \leq ||(a_{ij})||_n \leq \sum_{i,j} ||a_{ij}||_1$ *for all* $(a_{ij}) \in M_n(\tilde{A})$, *all* $k, l \in \{1, \ldots, n\}$, *and all* $n \in \mathbb{N}$;
- $M_n(A)$ *is closed in* $M_n(\tilde{A})$ *for each* n;
- *if* A *is unital and has filtration* $(A_r)_{r>0}$, *then* $M_n(A)$ *has filtration* $(M_n(A_r))_{r>0}$ *for each* n;
- *if* A *is non-unital and has filtration* $(A_r)_{r>0}$, *then* $M_n(A)$ *has filtration* $(M_n(A_r))_{r>0}$ *for each* n, *and* $M_n(A^+)$ *has filtration* $(M_n(A_r + \mathbb{C}))_{r>0}$ *for each* n.

Remark 3.1.8.

1. *Since every Banach space is an SQ_1 space, every Banach algebra may be regarded as an SQ_1 algebra. However, in particular situations, this may not necessarily be the most appropriate matrix norm structure to use.*

2. *Of the two properties $\mathcal{D}_\infty$ and $\mathcal{M}_p$ that define a p-operator space, $\mathcal{M}_p$ is more crucial for our purposes because it gives us some compatibility between the norm on $M_n(A)$ for a unital Banach algebra A and the norm on $M_n(\mathbb{C})$. In particular, the canonical map from $\mathbb{C}$ into the unitization of a non-unital Banach algebra will be completely contractive. Property $\mathcal{D}_\infty$ can be replaced by, for instance, $\left\|\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right\|_{n+m} \leq ||u||_n + ||v||_m$, which is automatic for most natural matrix norms.*

3. *For a general filtered Banach algebra A, one needs an appropriate matrix norm structure on A in order to consider quantitative K-theory. One (but perhaps not the only) possibility is if $\tilde{A}$ can be given a matrix norm structure $\{||\cdot||_n\}_{n\in\mathbb{N}}$ satisfying*
   - *$||\cdot||_1$ extends the norm on A and $||1_{\tilde{A}}||_1 = 1$,*
   - *$(M_n(\tilde{A}), ||\cdot||_n)$ is a Banach algebra for all n,*
   - *the canonical inclusion $M_n(\tilde{A}) \hookrightarrow M_{n+1}(\tilde{A})$ is isometric for each n,*
   - *norms of all signed permutation matrices are at most 1,*
   - *$||z||_{M_n(\tilde{A})} = ||z||_{M_n(\mathbb{C})}$ for all $z \in M_n(\mathbb{C})$ and some matrix norm on $\mathbb{C}$.*

Given a filtered SQ_p algebra, other Banach algebras constructed from A will be given matrix norm structures naturally induced by the matrix norm structure on A. We now consider some of these constructions.

In some situations, we may want to adjoin a unit to A even if A is already unital. In this case, A^+ is isomorphic to $A\oplus\mathbb{C}$ via the homomorphism $A^+ \ni (a, z) \mapsto (a+z, z) \in A\oplus\mathbb{C}$, and we will identify $M_n(A^+)$ isometrically with $M_n(A) \oplus_\infty M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is viewed as $B(\ell_p^n)$.

Remark 3.1.9. *Matrix algebras of the form $M_k(M_n(A)^+)$ will be viewed as subalgebras of $M_{kn}(A^+)$ by identifying $(a, z) \in M_n(A)^+$ with $(a, zI_n) \in M_n(A^+)$ for $a \in M_n(A)$ and*

$z \in \mathbb{C}$.

If A is a Banach algebra, and J is a closed ideal in A, then J is a closed ideal in $\tilde{A}$. We will view $\widetilde{A/J}$ as $\tilde{A}/J$, and we will identify $M_n(\tilde{A}/J)$ isometrically with $M_n(\tilde{A})/M_n(J)$ equipped with the quotient norm.

Given a Banach algebra A, recall that the cone of A is defined as $CA = C_0((0,1], A)$, and the suspension of A is defined as $SA = C_0((0,1), A)$. These are closed subalgebras of $C([0,1], A)$ with the supremum norm. We will sometimes denote $C([0,1], A)$ by $A[0,1]$.

Lemma 3.1.10. *If A is a filtered Banach algebra, then $C([0,1], A)$ is a filtered Banach algebra with filtration $(C([0,1], A_r))_{r>0}$. This induces filtrations on CA and SA.*

Proof. It is clear that $(C([0,1], A_r))_{r>0}$ satisfies the first two conditions in the definition so we just need to prove the density condition. Given $\varepsilon > 0$ and $f \in C([0,1], A)$, let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that whenever $s, t \in [t_{i-1}, t_i]$, we have $||f(s) - f(t)|| < \frac{\varepsilon}{6}$. For each i, let $a_i \in A_{r_i}$ be such that $||a_i - f(t_i)|| < \frac{\varepsilon}{6}$. Let $r = \max_{0 \leq i \leq k} r_i$. Define $g(s) = \frac{s-t_{i-1}}{t_i - t_{i-1}} a_i + \frac{t_i - s}{t_i - t_{i-1}} a_{i-1}$ for $s \in [t_{i-1}, t_i]$. Then $g(s) \in A_r$ for all $s \in [0,1]$ so $g \in C([0,1], A_r)$. Moreover, for each i and for all $s \in [t_{i-1}, t_i]$, we have

$$\begin{aligned}
||g(s) - f(s)|| &\leq ||g(s) - a_i|| + ||a_i - f(t_i)|| + ||f(t_i) - f(s)|| \\
&\leq ||a_i - a_{i-1}|| + ||a_i - f(t_i)|| + ||f(t_i) - f(s)|| \\
&\leq ||a_i - f(t_i)|| + ||f(t_i) - f(t_{i-1})|| + ||f(t_{i-1}) - a_{i-1}|| \\
&\qquad + ||a_i - f(t_i)|| + ||f(t_i) - f(s)|| \\
&< \varepsilon.
\end{aligned}$$

If $f \in C([0,1], A)$ and $f(0) = 0$, then we may take $a_0 = 0$ so that $g(0) = 0$. Likewise, if $f(1) = 0$, then we may take $a_k = 0$ so that $g(1) = 0$. It follows that CA and SA are filtered Banach algebras. □

We will view $M_n(\widetilde{A[0,1]})$ as a subalgebra of $M_n(\tilde{A})[0,1]$ for each n, and similarly for $M_n(\widetilde{SA})$ and $M_n(\widetilde{CA})$.

Definition 3.1.11. *[32] Let X and Y be p-operator spaces, and let $\phi : X \to Y$ be a bounded linear map. For each $n \in \mathbb{N}$, let $\phi_n : M_n(X) \to M_n(Y)$ be the induced map given by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$. We say that ϕ is p-completely bounded if $\sup_n ||\phi_n|| < \infty$. In this case, we let $||\phi||_{pcb} = \sup_n ||\phi_n||$.*

We say that ϕ is p-completely contractive if $||\phi||_{pcb} \leq 1$, and ϕ is p-completely isometric if $||\phi||_{pcb} = 1$.

The following lemma was proved for $p \in (1, \infty)$ but the proof, which we reproduce here, remains valid when $p = 1$.

Lemma 3.1.12. *[9, Lemma 4.2] Let X be a p-operator space, and let μ be a bounded linear functional on X. Then μ is p-completely bounded as a map to $\mathbb{C}$, and $||\mu||_{pcb} = ||\mu||$.*

Proof. We wish to show that $\mu_n : M_n(X) \to B(\ell_p^n)$ is bounded with norm $||\mu||$. Let $x = (x_{ij})_{i,j=1}^n \in M_n(X)$ so that $\mu_n(x) = (\mu(x_{ij}))_{i,j=1}^n$. Let $\alpha = (\alpha_i)_{i=1}^n \in \ell_p^n$ and $\beta = (\beta_j)_{j=1}^n \in \ell_q^n$, where $q \in (1, \infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\langle \beta, \mu_n(x)(\alpha) \rangle = \sum_{i,j=1}^n \beta_i \mu(x_{ij}) \alpha_j = \mu(\sum_{i,j=1}^n \beta_i x_{ij} \alpha_j).$$

Regarding α as an element in $M_{n,1}(\mathbb{C})$ and β as an element in $M_{1,n}(\mathbb{C})$, we have $||\alpha||_{B(\ell_p^1, \ell_p^n)} = ||\alpha||_p$ and $||\beta||_{B(\ell_p^n, \ell_p^1)} = ||\beta||_q$. By axiom $\mathcal{M}_p$ in the definition of p-operator spaces, we have $||\beta x \alpha||_1 \leq ||\beta||_q ||x||_n ||\alpha||_p$, so

$$||\langle \beta, \mu_n(x)(\alpha) \rangle|| \leq ||\mu|| ||\beta||_q ||x||_n ||\alpha||_p.$$

This implies that $||\mu_n(x)|| \leq ||\mu|| ||x||_n$, which in turn implies that $||\mu_n|| \leq ||\mu||$ as required. □

Since all characters on Banach algebras are contractive, we get the following

Corollary 3.1.13. *If A is a non-unital SQ_p algebra, then the canonical homomorphism $\pi : A^+ \to \mathbb{C}$ is p-completely contractive.*

Definition 3.1.14. *Let A and B be filtered SQ_p algebras with filtrations $(A_r)_{r>0}$ and $(B_r)_{r>0}$ respectively. A filtered homomorphism $\phi : A \to B$ is an algebra homomorphism such that*

- *ϕ is p-completely bounded;*
- *$\phi(A_r) \subset B_r$ for all $r > 0$.*

Recall that any bounded homomorphism $\phi : A \to B$ between Banach algebras induces a bounded homomorphism $\phi^+ : A^+ \to B^+$ given by $\phi^+(a, z) = (\phi(a), z)$. If ϕ is a filtered homomorphism, then so is ϕ^+.

We will see later that filtered homomorphisms induce homomorphisms between quantitative K-theory groups.

3.1.2 Quasi-idempotent elements and quasi-invertible elements

Elements of K-theory groups of Banach algebras are equivalence classes of idempotents or invertibles. For quantitative K-theory, we will consider quasi-idempotents and quasi-invertibles. We now define these elements, and give a list of simple results based on norm estimates that we will often use to conclude that two such elements represent the same class in some quantitative K-theory group.

Definition 3.1.15. *Let A be a filtered Banach algebra. For $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and $N \geq 1$,*

- *an element $e \in A$ is called an (ε, r, N)-idempotent if $||e^2 - e|| < \varepsilon$, $e \in A_r$, and $\max(||e||, ||1_{\tilde{A}} - e||) \leq N$;*

- *if A is unital, an element $u \in A$ is called an (ε, r, N)-invertible if $u \in A_r$, $||u|| \le N$, and there exists $v \in A_r$ with $||v|| \le N$ such that $\max(||uv - 1||, ||vu - 1||) < \varepsilon$.*

We call v an (ε, r, N)-inverse for u, and we call (u, v) an (ε, r, N)-inverse pair.

We will use the terms quasi-idempotent, quasi-invertible, quasi-inverse, and quasi-inverse pair when the precise parameters are not crucial.

Observe that if A is unital and $a \in A_r$ satisfies $||a|| \le N$ and $||a - 1|| < \varepsilon$, then $(a, 1)$ is an (ε, r, N)-inverse pair. Also, if $u \in A$ is an (ε, r, N)-invertible, then u is invertible and $||u^{-1}|| < \frac{N}{1-\varepsilon}$. Indeed, if v is an (ε, r, N)-inverse for u, then uv and vu are invertible so u is invertible and

$$||u^{-1}|| = ||v(uv)^{-1}|| < \frac{N}{1-\varepsilon}.$$

Lemma 3.1.16. *Let A be a filtered Banach algebra.*

1. *Let $e \in A$ be an idempotent, and let $N = ||e|| + 1$. If $a \in A_r$ and $||a - e|| < \frac{\varepsilon}{3N}$, then a is an (ε, r, N)-idempotent in A.*

2. *Suppose that A is unital. Let $u_0 \in A$ be invertible, and let $N = ||u_0|| + ||u_0^{-1}|| + 1$. If $u, v \in A_r$ and $\max(||u - u_0||, ||v - u_0^{-1}||) < \frac{\varepsilon}{N}$, then (u, v) is an (ε, r, N)-inverse pair in A.*

Proof.

1. $||a^2 - a|| \le ||a(a - e)|| + ||(a - e)e|| + ||e - a|| < (2N + 1)\frac{\varepsilon}{3N} \le \varepsilon$.

2. $||uv - 1|| \le ||(u - u_0)v|| + ||u_0(v - u_0^{-1})|| < \varepsilon$; similarly $||vu - 1|| < \varepsilon$.

□

Definition 3.1.17. *Let A be a filtered Banach algebra.*

- *Two (ε, r, N)-idempotents e_0 and e_1 in A are (ε', r', N')-homotopic for some $\varepsilon' \geq \varepsilon, r' \geq r$, and $N' \geq N$ if there exists a norm-continuous path $(e_t)_{t\in[0,1]}$ of (ε', r', N')-idempotents in A from e_0 to e_1. Equivalently, there exists an (ε', r', N')-idempotent e in $A[0,1]$ such that $e(0) = e_0$ and $e(1) = e_1$.*
- *If A is unital, two (ε, r, N)-invertibles u_0 and u_1 in A are (ε', r', N')-homotopic for some $\varepsilon' \geq \varepsilon, r' \geq r$, and $N' \geq N$ if there exists a norm-continuous path $(u_t)_{t\in[0,1]}$ of (ε', r', N')-invertibles in A from u_0 to u_1. Equivalently, there exists an (ε', r', N')-invertible u in $A[0,1]$ such that $u(0) = u_0$ and $u(1) = u_1$.*

We also call the paths (e_t) and (u_t) homotopies of (ε', r', N')-idempotents and (ε', r', N')-invertibles respectively, and we write $e_0 \overset{\varepsilon',r',N'}{\sim} e_1$ and $u_0 \overset{\varepsilon',r',N'}{\sim} u_1$.

Lemma 3.1.18. *Let A be a filtered Banach algebra.*

1. *If e is an (ε, r, N)-idempotent in A, and $f \in A_r$ satisfies $||f|| \leq N$ and $||e - f|| < \frac{\varepsilon - ||e^2 - e||}{2N+1}$, then e and f are (ε, r, N)-homotopic. More generally, if $||e - f|| < \delta$ for some $\delta > 0$, then e and f are $((2N + 1)\delta + \varepsilon, r, N)$-homotopic.*
2. *Suppose A is unital. If (u, v) is an (ε, r, N)-inverse pair in A, and $a \in A_r$ is such that $||a|| \leq N$ and $||u - a|| < \frac{\varepsilon - \max(||uv-1||, ||vu-1||)}{N}$, then a and u are (ε, r, N)-homotopic, with v being an (ε, r, N)-inverse of a. More generally, if $||u - a|| < \delta$ for some $\delta > 0$, then a and u are $(N\delta + \varepsilon, r, N)$-homotopic, with v being an $(N\delta + \varepsilon, r, N)$-inverse of a.*

Proof.

1. The following estimate shows that f is quasi-idempotent.

$$\begin{aligned}||f^2 - f|| &\leq ||f(f-e)|| + ||(f-e)e|| + ||e^2 - e|| + ||e - f|| \\ &< (||f|| + ||e|| + 1)||e - f|| + ||e^2 - e|| \\ &\leq (2N+1)||e - f|| + ||e^2 - e||.\end{aligned}$$

   The statement about homotopy follows from the inequalities $||(1-t)e + tf|| \leq N$ and $||((1-t)e + tf) - e|| \leq ||e - f||$ for $t \in [0,1]$.

2. $||av - 1|| \leq ||a - u||||v|| + ||uv - 1|| \leq N||a - u|| + ||uv - 1||$. Similarly, $||va - 1|| \leq N||a - u|| + ||vu - 1||$.

   The statement about homotopy follows from the inequalities $||(1-t)a + tu|| \leq N$ and $||((1-t)a + tu) - u|| \leq ||u - a||$ for $t \in [0,1]$.

□

Lemma 3.1.19. *Let A be a filtered Banach algebra. If e and f are (ε, r, N)-idempotents in A, then e and f are (ε', r, N)-homotopic, where $\varepsilon' = \varepsilon + \frac{1}{4}||e - f||^2$.*

Proof. If e and f are (ε, r, N)-idempotents, then for $t \in [0,1]$, we have

$$\begin{aligned}&((1-t)e + tf)^2 - ((1-t)e + tf) \\ &= (1-t)(e^2 - e) + (t^2 - t)e^2 + t(f^2 - f) + (t^2 - t)f^2 + (t - t^2)(ef + fe) \\ &= (1-t)(e^2 - e) + t(f^2 - f) + (t^2 - t)(e - f)^2\end{aligned}$$

so $||((1-t)e + tf)^2 - ((1-t)e + tf)|| < \varepsilon + \frac{1}{4}||e - f||^2$. □

Lemma 3.1.20. *Let A be a filtered SQ_p algebra, and suppose that $e, f \in A$ are orthogonal (ε, r, N)-idempotents (i.e., $ef = fe = 0$). Then $e + f$ is a $(2\varepsilon, r, 2N)$-idempotent. Moreover,* $\begin{pmatrix} e+f & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ *are $(\frac{5}{2}\varepsilon, r, \frac{5}{2}N)$-homotopic in $M_2(A)$.*

Proof. Since $(e+f)^2 = e^2 + f^2$, it follows that $e + f$ is a $(2\varepsilon, r, 2N)$-idempotent. For $t \in [0,1]$, let $c_t = \cos \frac{\pi t}{2}$ and $s_t = \sin \frac{\pi t}{2}$. Then

$$E_t = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix}$$

is a homotopy of $(\frac{5}{2}\varepsilon, r, \frac{5}{2}N)$-idempotents in $M_2(A)$ between $\begin{pmatrix} e+f & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. □

Let A be a unital filtered SQ_p algebra. It is straightforward to see that if (u, v) is an (ε, r, N)-inverse pair in A, then

1. $\left(\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}\right)$ is an (ε, r, N)-inverse pair in $M_2(A)$;
2. $\left(\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & vu \end{pmatrix}\right)$ is an $(\varepsilon, 2r, (1+\varepsilon))$-inverse pair in $M_2(A)$;
3. $\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\right)$ is an (ε, r, N)-inverse pair in $M_2(A)$.

If u_0 and u_1 are homotopic as (ε, r, N)-invertibles, and v_0 and v_1 are (ε, r, N)-inverses of u_0 and u_1 respectively, then v_0 and v_1 are (ε, r, N)-homotopic. Indeed, let (u_t) be a homotopy of (ε, r, N)-invertibles between u_0 and u_1, and let v_t be an (ε, r, N)-inverse for

u_t for each $t \in [0,1]$. Let $\delta = \sup_{t\in[0,1]}(||u_t v_t - 1||, ||v_t u_t - 1||)$, and let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that $||u_t - u_s|| < \frac{\varepsilon-\delta}{N}$ for all $s,t \in [t_{i-1}, t_i]$ and $i = 1, \ldots, k$. Let

$$w_t = \frac{t_i - t}{t_i - t_{i-1}} v_{t_{i-1}} + \frac{t - t_{i-1}}{t_i - t_{i-1}} v_{t_i}$$

if $t \in [t_{i-1}, t_i]$. Then $\max(||u_t w_t - 1||, ||w_t u_t - 1||) < \varepsilon$ for all $t \in [0,1]$ so (w_t) is a homotopy of (ε, r, N)-invertibles between v_0 and v_1.

Lemma 3.1.21. *Let A be a unital filtered SQ_p algebra. If (u,v) is an (ε, r, N)-inverse pair in A, then $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ and I_2 are $(\varepsilon, 2r, 2(N+\varepsilon))$-homotopic in $M_2(A)$.*

Proof. For $t \in [0,1]$, let $c_t = \cos\frac{\pi t}{2}$ and $s_t = \sin\frac{\pi t}{2}$. Define

$$\begin{aligned} U_t &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix} \\ &= c_t^2 \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} + s_t^2 \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} + c_t s_t \begin{pmatrix} 0 & u - uv \\ 1 - v & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} V_t &= \begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \\ &= c_t^2 \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} + s_t^2 \begin{pmatrix} 1 & 0 \\ 0 & vu \end{pmatrix} + c_t s_t \begin{pmatrix} 0 & vu - u \\ v - 1 & 0 \end{pmatrix}. \end{aligned}$$

For each $t \in [0,1]$, we have $U_t V_t = V_t U_t = \begin{pmatrix} uv & 0 \\ 0 & vu \end{pmatrix}$, $||U_t|| \leq 2(N+\varepsilon)$, and $||V_t|| \leq 2(N+\varepsilon)$ so (U_t, V_t) is an $(\varepsilon, 2r, 2(N+\varepsilon))$-inverse pair in $M_2(A)$ for each $t \in [0,1]$.

In particular, $(U_t)_{t\in[0,1]}$ is a homotopy of $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles between $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ and $\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$, while $(V_t)_{t\in[0,1]}$ is a homotopy of $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles between $\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & vu \end{pmatrix}$. Then $\begin{pmatrix} (1-t)uv+t & 0 \\ 0 & 1 \end{pmatrix}$ is a homotopy of $(\varepsilon, 2r, (1+\varepsilon))$-invertibles between $\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$ and I_2. □

Remark 3.1.22. *From the proof we see that if $u^{-1} \in A_r$ and $||u^{-1}|| \leq N$, then we may take $v = u^{-1}$ so that $(U_t)_{t\in[0,1]}$ is a homotopy between $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ and I_2 with $U_tV_t = V_tU_t = I_2$ for each t. Moreover, in this case, we have $||U_t|| \leq 2N$ and $||V_t|| \leq 2N$.*

Lemma 3.1.23. *Let A be a unital filtered SQ_p algebra. If $u, v \in A$ are (ε, r, N)-invertibles, then $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ and $\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ are $(2\varepsilon, r, 2N)$-homotopic in $M_2(A)$.*

Proof. Suppose that (u, u') and (v, v') are (ε, r, N)-inverse pairs. For $t \in [0,1]$, let $c_t = \cos \frac{\pi t}{2}$ and $s_t = \sin \frac{\pi t}{2}$. Define

$$\begin{aligned} U_t &= \begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix} \\ &= c_t^2 \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} + s_t^2 \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} + c_t s_t \begin{pmatrix} 0 & u-v \\ u-v & 0 \end{pmatrix}, \end{aligned}$$

$$V_t = \begin{pmatrix} c_t & -s_t \\ s_t & c_t \end{pmatrix} \begin{pmatrix} u' & 0 \\ 0 & v' \end{pmatrix} \begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix}$$
$$= c_t^2 \begin{pmatrix} u' & 0 \\ 0 & v' \end{pmatrix} + s_t^2 \begin{pmatrix} v' & 0 \\ 0 & u' \end{pmatrix} + c_t s_t \begin{pmatrix} 0 & u' - v' \\ u' - v' & 0 \end{pmatrix}.$$

Then $||U_t|| \leq 2N$ and $||V_t|| \leq 2N$. Also, $U_tV_t - I = \begin{pmatrix} uu' - 1 & 0 \\ 0 & vv' - 1 \end{pmatrix}$ and $V_tU_t - I = \begin{pmatrix} u'u - 1 & 0 \\ 0 & v'v - 1 \end{pmatrix}$ so $\max(||U_tV_t - I||, ||V_tU_t - I||) < \varepsilon$. □

It is a standard fact in K-theory for Banach algebras that if we consider matrices of all sizes simultaneously, then the homotopy relation and the similarity relation give us the same equivalence classes of idempotents [3, Section 4]. In the remainder of this section, we will examine the relationship between these two equivalence relations in our context.

Lemma 3.1.24. *Let A be a unital filtered SQ_p algebra. If $e \in A$ is an (ε, r, N)-idempotent and (u, v) is an (ε', r', N')-inverse pair in A, then uev is an $((NN')^2\varepsilon' + N'^2\varepsilon, r + 2r', NN'^2)$-idempotent. In particular, if $e \in M_n(A)$ is an (ε, r, N)-idempotent and $u \in M_n(\mathbb{C})$ is invertible with $\max(||u||, ||u^{-1}||) \leq 1$, then ueu^{-1} is an (ε, r, N)-idempotent in $M_n(A)$.*

Proof. The first statement holds since

$$||uevuev - uev|| \leq ||ue(vu - 1)ev|| + ||u(e^2 - e)v|| < (NN')^2\varepsilon' + N'^2\varepsilon.$$

The second statement follows from the first by setting $\varepsilon' = 0$ and $r' = 0$. □

Lemma 3.1.25. *Let A be a unital filtered SQ_p algebra. If $e \in A$ is an (ε, r, N)-idempotent, and (u, v) is an (ε', r', N')-inverse pair in A, then* $\mathrm{diag}(uev, 0)$ *and* $\mathrm{diag}(e, 0)$ *are homotopic as $(\varepsilon'', r'', N'')$-idempotents in $M_2(A)$, where*

$$\varepsilon'' = 4(N' + \varepsilon')^2(N^2\varepsilon' + \varepsilon),$$

$$r'' = r + 4r', \textit{and}$$

$$N'' = 4N(N' + \varepsilon')^2.$$

In particular, if $e \in M_n(A)$ is an (ε, r, N)-idempotent and $u \in M_n(\mathbb{C})$ is invertible with $\max(||u||, ||u^{-1}||) \leq 1$*, then* $\mathrm{diag}(ueu^{-1}, 0)$ *and* $\mathrm{diag}(e, 0)$ *are $(4\varepsilon, r, 4N)$-homotopic in $M_{2n}(A)$.*

Proof. Let U_t be a homotopy of $(\varepsilon', 2r', 2(N' + \varepsilon'))$-invertibles between $\mathrm{diag}(u, v)$ and I_2 given by Lemma 3.1.21, and let V_t be a homotopy of $(\varepsilon', 2r', 2(N' + \varepsilon'))$-invertibles between $\mathrm{diag}(v, u)$ and I_2 such that (U_t, V_t) is an $(\varepsilon', 2r', 2(N' + \varepsilon'))$-inverse pair in $M_2(A)$ for each $t \in [0, 1]$. Then $U_t\mathrm{diag}(e, 0)V_t$ is a homotopy of $(\varepsilon'', r'', N'')$-idempotents between $\mathrm{diag}(uev, 0)$ and $\mathrm{diag}(e, 0)$ by Lemma 3.1.24, where ε'', r'', N'' are given by the expressions in the statement. The second statement follows from the first by setting $\varepsilon' = 0$ and $r' = 0$. $\square$

Lemma 3.1.26. *Let A be a unital filtered Banach algebra. If e and f are (ε, r, N)-idempotents in A such that $||e - f|| < \frac{\varepsilon}{2N+1}$, then there exists an $(\varepsilon, n_\varepsilon r, \frac{1}{1-3\varepsilon})$-inverse pair (u, v) in A such that $||uev - f|| < \frac{(5N+2)\varepsilon}{1-3\varepsilon}$, where $n_\varepsilon \geq 1$ and $\varepsilon \mapsto n_\varepsilon$ is non-increasing.*

Proof. Let $v = ef + (1 - e)(1 - f)$. Then $v - 1 = (2e - 1)(f - e) + 2(e^2 - e)$. If $||e - f|| < \frac{\varepsilon}{2N+1}$, then $||v - 1|| < 3\varepsilon$ so v is invertible. We also have $||v|| < 1 + 3\varepsilon$ and $||v^{-1}|| \leq \frac{1}{1-||1-v||} < \frac{1}{1-3\varepsilon}$.

Now $ev = 2e^2f - e^2 - ef + e$ and $vf = 2ef^2 - ef - f^2 + f$ so

$$ev - vf = 2(e^2 - e)f + 2e(f - f^2) - (e^2 - e) + (f^2 - f)$$

and $||ev - vf|| \leq (4N + 2)\varepsilon$. Since $||v^{-1}|| < \frac{1}{1-3\varepsilon}$, we have

$$||v^{-1}ev - f|| < \frac{(4N + 2)\varepsilon}{1 - 3\varepsilon}.$$

Let m_ε be the smallest positive integer such that $\left\|\sum_{k=m_\varepsilon+1}^{\infty}(1 - v)^k\right\| < \frac{\varepsilon}{2}$, and let $u = \sum_{k=0}^{m_\varepsilon}(1 - v)^k$. Then $||u|| < \frac{1}{1-3\varepsilon}$ and

$$||uev - f|| \leq ||(u - v^{-1})ev|| + ||v^{-1}ev - f|| < N\varepsilon + \frac{(4N + 2)\varepsilon}{1 - 3\varepsilon} < \frac{(5N + 2)\varepsilon}{1 - 3\varepsilon}.$$

Also note that $v \in A_{2r}$, $u \in A_{2m_\varepsilon r}$,

$$||uv - 1|| \leq \left\|\sum_{k=m_\varepsilon+1}^{\infty}(1 - v)^k\right\| ||v|| < \varepsilon,$$

and similarly $||vu - 1|| < \varepsilon$ so (u, v) is an $(\varepsilon, 2m_\varepsilon r, \frac{1}{1-3\varepsilon})$-inverse pair in A. □

Proposition 3.1.27. *Let A be a unital filtered Banach algebra. If there is an M-Lipschitz homotopy of (ε, r, N)-idempotents in A between e and f with $\frac{1}{M} \leq \varepsilon$, then there exists an (ε', r', N')-inverse pair (u, v) in A such that $||uev - f|| < 3(\frac{9}{4})^{M(2N+1)+1}(5N+2)\varepsilon$, where $\varepsilon' = 2(\frac{9}{4})^{M(2N+1)+1}\varepsilon$, $r' = (M(2N + 1) + 1)n_\varepsilon r$ with $n_\varepsilon \geq 1$ and $\varepsilon \mapsto n_\varepsilon$ non-increasing, and $N' = (\frac{3}{2})^{M(2N+1)+1}$.*

Proof. Let (e_t) be an M-Lipschitz homotopy of (ε, r, N)-idempotents in A between e and f with $\frac{1}{M} \leq \varepsilon$. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that $\frac{1}{M(2N+1)+1} < |t_i - t_{i-1}| < \frac{1}{M(2N+1)}$. Note that $k < M(2N + 1) + 1$. By Lemma 3.1.26, there exists an $(\varepsilon, n_\varepsilon r, \frac{1}{1-3\varepsilon})$-

inverse pair (u_i, v_i) in A such that

$$||u_i e_{t_{i-1}} v_i - e_{t_i}|| < \frac{(5N+2)\varepsilon}{1-3\varepsilon}.$$

Set $(u, v) = (u_k \cdots u_1, v_1 \cdots v_k)$. Then

$$||uv - 1|| < 2(\frac{9}{4})^k \varepsilon$$

and similarly for $||vu - 1||$. Thus (u, v) is a $(2(\frac{9}{4})^k\varepsilon, kn_\varepsilon r, (\frac{3}{2})^k)$-inverse pair, and

$$||uev - f|| < 3(\frac{9}{4})^k (5N+2)\varepsilon.$$

□

If a homotopy of quasi-idempotents is not Lipschitz, the following lemma enables us to replace the homotopy with a Lipschitz homotopy by enlarging matrices, after which Proposition 3.1.27 becomes applicable. The Lipschitz constant depends only on the parameter N.

Lemma 3.1.28. *Let A be a unital filtered SQ_p algebra. If e and f are homotopic as (ε, r, N)-idempotents in A, then there exist $\alpha_N > 0$, $k \in \mathbb{N}$, and an α_N-Lipschitz homotopy of $(2\varepsilon, r, \frac{5}{2}N)$-idempotents between* $\mathrm{diag}(e, I_k, 0_k)$ *and* $\mathrm{diag}(f, I_k, 0_k)$.

Proof. Let $(e_t)_{t\in[0,1]}$ be a homotopy of (ε, r, N)-idempotents between e and f, and let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that

$$||e_{t_i} - e_{t_{i-1}}|| < \inf_{t\in[0,1]} \frac{\varepsilon - ||e_t^2 - e_t||}{2N+1}.$$

For each $t \in [0, 1]$, we have a Lipschitz homotopy of $(\varepsilon, r, \frac{5}{2}N)$-idempotents between

$\operatorname{diag}(e_t, 1-e_t)$ and $\operatorname{diag}(1,0)$ given by

$$\begin{pmatrix} e_t & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c_l & -s_l \\ s_l & c_l \end{pmatrix} \begin{pmatrix} 1-e_t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_l & s_l \\ -s_l & c_l \end{pmatrix},$$

where $c_l = \cos\frac{\pi l}{2}$ and $s_l = \sin\frac{\pi l}{2}$ for $l \in [0,1]$. Also, there is a Lipschitz (in fact linear) homotopy of (ε, r, N)-idempotents between $e_{t_{i-1}}$ and e_{t_i} for each i. Then we have the following sequence of Lipschitz homotopies of $(2\varepsilon, r, \frac{5}{2}N)$-idempotents:

$$\begin{pmatrix} e_{t_0} & & \\ & I_k & \\ & & 0_k \end{pmatrix} \overset{2\varepsilon,r,2N}{\sim} \begin{pmatrix} e_{t_0} & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 0 \end{pmatrix} \overset{\varepsilon,r,\frac{5}{2}N}{\sim} \begin{pmatrix} e_{t_0} & & & & & \\ & 1-e_{t_1} & & & & \\ & & e_{t_1} & & & \\ & & & \ddots & & \\ & & & & 1-e_{t_k} & \\ & & & & & e_{t_k} \end{pmatrix}$$

$$\overset{\varepsilon,r,N}{\sim} \begin{pmatrix} e_{t_0} & & & & & \\ & 1-e_{t_0} & & & & \\ & & e_{t_1} & & & \\ & & & \ddots & & \\ & & & & 1-e_{t_{k-1}} & \\ & & & & & e_{t_k} \end{pmatrix} \overset{\varepsilon,r,\frac{5}{2}N}{\sim} \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & e_{t_k} \end{pmatrix}$$

$$\overset{2\varepsilon,r,2N}{\sim} \begin{pmatrix} e_{t_k} & & \\ & I_k & \\ & & 0_k \end{pmatrix}.$$

□

Remark 3.1.29. *From the proofs above, we see that in Proposition 3.1.27, if A is a non-unital filtered Banach algebra, and $e, f \in A^+$ are such that $e-1 \in A$ and $f-1 \in A$, then we can find (u,v) satisfying the conclusion of Proposition 3.1.27 such that $u-1 \in A$ and $v-1 \in A$.*

3.2 Quantitative K-Theory

In this section, we define the quantitative K-theory groups for a filtered SQ_p algebra A. Then we establish some basic properties of these groups, and we examine the relation between these groups and the usual K-theory groups. Our setup is in parallel with the theory developed in [26] for filtered C^*-algebras.

3.2.1 Definitions of quantitative K-theory groups

Given a filtered SQ_p algebra A, we denote by $Idem^{\varepsilon,r,N}(A)$ the set of (ε, r, N)-idempotents in A. For each positive integer n, we set $Idem_n^{\varepsilon,r,N}(A) = Idem^{\varepsilon,r,N}(M_n(A))$. Then we have inclusions $Idem_n^{\varepsilon,r,N}(A) \hookrightarrow Idem_{n+1}^{\varepsilon,r,N}(A)$ given by $e \mapsto \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$, and we set

$$Idem_\infty^{\varepsilon,r,N}(A) = \bigcup_{n\in\mathbb{N}} Idem_n^{\varepsilon,r,N}(A).$$

Consider the equivalence relation $\sim$ on $Idem_\infty^{\varepsilon,r,N}(A)$ defined by $e \sim f$ if e and f are $(4\varepsilon, r, 4N)$-homotopic in $M_\infty(A)$. We will denote the equivalence class of $e \in Idem_\infty^{\varepsilon,r,N}(A)$ by $[e]$. We will sometimes write $[e]_{\varepsilon,r,N}$ if we wish to keep track of the parameters.

We define addition on $Idem_\infty^{\varepsilon,r,N}(A)/\sim$ by $[e] + [f] = [\mathrm{diag}(e, f)]$.

Proposition 3.2.1. *For any filtered SQ_p algebra A, $Idem_\infty^{\varepsilon,r,N}(A)/\sim$ is an abelian semi-group with identity $[0]$. If B is another filtered SQ_p algebra and $\phi : A \to B$ is a filtered homomorphism, then there is an induced homomorphism*

$$\phi_* : Idem_\infty^{\varepsilon,r,N}(A)/\sim \;\to Idem_\infty^{||\phi||_{pcb}\varepsilon,r,||\phi||_{pcb}N}(B)/\sim.$$

Proof. To show commutativity, we need to show that $[\mathrm{diag}(e, f)] = [\mathrm{diag}(f, e)]$ for $e, f \in$

$Idem_\infty^{\varepsilon,r,N}(A)$. We may assume that $e, f \in M_n(A)$ for some $n \in \mathbb{N}$. Letting

$$R_t = \begin{pmatrix} (\cos \frac{\pi t}{2})I_n & (\sin \frac{\pi t}{2})I_n \\ -(\sin \frac{\pi t}{2})I_n & (\cos \frac{\pi t}{2})I_n \end{pmatrix}$$

for $t \in [0,1]$, one sees that $R_t \mathrm{diag}(e,f) R_t^{-1}$ is a homotopy of $(2\varepsilon, r, 2N)$-idempotents between $\mathrm{diag}(e,f)$ and $\mathrm{diag}(f,e)$.

Using the same notation ϕ for the induced homomorphism $M_\infty(A) \to M_\infty(B)$, note that $\phi(e) \in Idem_\infty^{||\phi||_{pcb}\varepsilon, r, ||\phi||_{pcb}N}(B)$ whenever $e \in Idem_\infty^{\varepsilon,r,N}(A)$. Moreover, if $e \sim e'$, then $\phi(e) \sim \phi(e')$. Thus

$$\phi_* : Idem_\infty^{\varepsilon,r,N}(A)/\sim \; \to Idem_\infty^{||\phi||_{pcb}\varepsilon, r, ||\phi||_{pcb}N}(B)/\sim$$

given by $\phi_*([e]) = [\phi(e)]$ is a well-defined homomorphism of semigroups. $\square$

Definition 3.2.2. *Let A be a unital filtered SQ_p algebra. For $0 < \varepsilon < \frac{1}{20}, r > 0$ and $N \geq 1$, define $K_0^{\varepsilon,r,N}(A)$ to be the Grothendieck group of $Idem_\infty^{\varepsilon,r,N}(A)/\sim$.*

By the universal property of the Grothendieck group, when A and B are unital filtered SQ_p algebras and $\phi : A \to B$ is a filtered homomorphism, the induced homomorphism

$$\phi_* : Idem_\infty^{\varepsilon,r,N}(A)/\sim \; \to Idem_\infty^{||\phi||_{pcb}\varepsilon, r, ||\phi||_{pcb}N}(B)/\sim$$

extends to a group homomorphism

$$\phi_* : K_0^{\varepsilon,r,N}(A) \to K_0^{||\phi||_{pcb}\varepsilon, r, ||\phi||_{pcb}N}(B).$$

Moreover, if $\psi : B \to C$ is another filtered homomorphism between unital filtered SQ_p

algebras, then

$$(\psi \circ \phi)_* = \psi_* \circ \phi_* : K_0^{\varepsilon,r,N}(A) \to K_0^{||\psi||_{pcb}||\phi||_{pcb}\varepsilon,r,||\psi||_{pcb}||\phi||_{pcb}N}(C).$$

If A is a non-unital filtered SQ_p algebra, we have the usual quotient homomorphism $\pi : A^+ \to \mathbb{C}$, which is p-completely contractive by [9, Lemma 4.2] and the standard fact that all characters on Banach algebras are contractive. Thus π induces a homomorphism $\pi_* : K_0^{\varepsilon,r,N}(A^+) \to K_0^{\varepsilon,r,N}(\mathbb{C})$.

Definition 3.2.3. *Let A be a non-unital filtered SQ_p algebra. For $0 < \varepsilon < \frac{1}{20}, r > 0$ and $N \geq 1$, define*

$$K_0^{\varepsilon,r,N}(A) = \ker(\pi_* : K_0^{\varepsilon,r,N}(A^+) \to K_0^{\varepsilon,r,N}(\mathbb{C})).$$

Note that for $0 < \varepsilon \leq \varepsilon' < \frac{1}{20}, 0 < r \leq r'$, and $1 \leq N \leq N'$, we have a canonical group homomorphism

$$\iota_0^{\varepsilon,\varepsilon',r,r',N,N'} : K_0^{\varepsilon,r,N}(A) \to K_0^{\varepsilon',r',N'}(A)$$

given by $\iota_0^{\varepsilon,\varepsilon',r,r',N,N'}([e]_{\varepsilon,r,N}) = [e]_{\varepsilon',r',N'}$.

We have already observed that if A and B are both unital filtered SQ_p algebras, then a filtered homomorphism $\phi : A \to B$ induces a group homomorphism

$$\phi_* : K_0^{\varepsilon,r,N}(A) \to K_0^{||\phi||_{pcb}\varepsilon,r,||\phi||_{pcb}N}(B).$$

If A and B are both non-unital and $\phi^+ : A^+ \to B^+$ denotes the induced homomorphism between their unitizations, then we get a homomorphism

$$K_0^{\varepsilon,r,N}(A^+) \to K_0^{||\phi^+||_{pcb}\varepsilon,r,||\phi^+||_{pcb}N}(B^+),$$

which restricts to a homomorphism

$$\phi_* : K_0^{\varepsilon,r,N}(A) \to K_0^{||\phi^+||_{pcb}\varepsilon,r,||\phi^+||_{pcb}N}(B).$$

Given a unital filtered SQ_p algebra A, we denote by $GL^{\varepsilon,r,N}(A)$ the set of (ε, r, N)-invertibles in A. For each positive integer n, we set $GL_n^{\varepsilon,r,N}(A) = GL^{\varepsilon,r,N}(M_n(A))$. Then we have inclusions $GL_n^{\varepsilon,r,N}(A) \hookrightarrow GL_{n+1}^{\varepsilon,r,N}(A)$ given by $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, and we set

$$GL_\infty^{\varepsilon,r,N}(A) = \bigcup_{n\in\mathbb{N}} GL_n^{\varepsilon,r,N}(A).$$

Consider the equivalence relation $\sim$ on $GL_\infty^{\varepsilon,r,N}(A)$ given by $u \sim v$ if u and v are $(4\varepsilon, 2r, 4N)$-homotopic in $M_\infty(A)$. We denote the equivalence class of $u \in GL_\infty^{\varepsilon,r,N}(A)$ by $[u]$. We will sometimes write $[u]_{\varepsilon,r,N}$ if we wish to keep track of the parameters.

We define addition on $GL_\infty^{\varepsilon,r,N}(A)/\sim$ by $[u] + [v] = [\mathrm{diag}(u, v)]$.

Proposition 3.2.4. *For any unital filtered SQ_p algebra A, $GL_\infty^{\varepsilon,r,N}(A)/\sim$ is an abelian group. If B is another unital filtered SQ_p algebra and $\phi : A \to B$ is a unital filtered homomorphism, then there is an induced homomorphism*

$$\phi_* : GL_\infty^{\varepsilon,r,N}(A)/\sim \to GL_\infty^{||\phi||_{pcb}\varepsilon,r,||\phi||_{pcb}N}(B)/\sim.$$

Proof. By Lemma 3.1.21, if (u, v) is an (ε, r, N)-inverse pair in $M_n(A)$, then $[u] + [v] = [1]$. By Lemma 3.1.23, we have $[u] + [v] = [v] + [u]$. Hence $GL_\infty^{\varepsilon,r,N}(A)/\sim$ is an abelian group.

If $\phi : A \to B$ is a unital filtered homomorphism, and $u \in GL_n^{\varepsilon,r,N}(A)$, then $\phi(u) \in$

$GL_n^{||\phi||_{pcb}\varepsilon,r,||\phi||_{pcb}N}(B)$. Moreover, if $u \sim u'$, then $\phi(u) \sim \phi(u')$. Thus

$$\phi_* : GL_\infty^{\varepsilon,r,N}(A)/\sim \to GL_\infty^{||\phi||_{pcb}\varepsilon,r,||\phi||_{pcb}N}(B)/\sim$$

given by $\phi_*([u]) = [\phi(u)]$ is a well-defined group homomorphism. □

Definition 3.2.5. *Let A be a unital filtered SQ_p algebra. For $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and $N \geq 1$, define*

$$K_1^{\varepsilon,r,N}(A) = GL_\infty^{\varepsilon,r,N}(A)/\sim.$$

If A is non-unital, define $K_1^{\varepsilon,r,N}(A) = \ker(\pi_* : K_1^{\varepsilon,r,N}(A^+) \to K_1^{\varepsilon,r,N}(\mathbb{C}))$.

By the previous proposition, $K_1^{\varepsilon,r,N}(A)$ is an abelian group for any filtered SQ_p algebra A.

For $0 < \varepsilon \leq \varepsilon' < \frac{1}{20}$, $0 < r \leq r'$, and $1 \leq N \leq N'$, we have a canonical group homomorphism

$$\iota_1^{\varepsilon,\varepsilon',r,r',N,N'} : K_1^{\varepsilon,r,N}(A) \to K_1^{\varepsilon',r',N'}(A)$$

given by $\iota_1^{\varepsilon,\varepsilon',r,r',N,N'}([u]_{\varepsilon,r,N}) = [u]_{\varepsilon',r',N'}$.

Remark 3.2.6. *We will sometimes refer to the canonical homomorphisms*

$$\iota_*^{\varepsilon,\varepsilon',r,r',N,N'} : K_*^{\varepsilon,r,N}(A) \to K_*^{\varepsilon',r',N'}(A)$$

as relaxation of control maps, and we will also omit the superscripts, writing just ι_*, when they are clear from the context so as to reduce notational clutter.

Just as in the usual K-theory, we can give a unified treatment of the unital and non-unital cases. First, we make some observations about the behavior of quantitative K-theory with respect to direct sums of unital filtered SQ_p algebras.

Let A_1 and A_2 be unital filtered SQ_p algebras and let

$$\pi_i : A_1 \oplus A_2 \to A_i \ (i = 1, 2)$$

be the respective projection homomorphisms. If $e \in M_n(A_1 \oplus A_2)$ is an (ε, r, N)-idempotent, then $\pi_i(e)$ is an (ε, r, N)-idempotent in $M_n(A_i)$ for $i = 1, 2$. Conversely, if $e_i \in M_n(A_i)$ is an (ε, r, N)-idempotent for $i = 1, 2$, then $(e_1, e_2) \in M_n(A_1 \oplus A_2)$ is an (ε, r, N)-idempotent. It follows that we have an isomorphism

$$\pi_{1*} \oplus \pi_{2*} : K_0^{\varepsilon,r,N}(A_1 \oplus A_2) \overset{\cong}{\to} K_0^{\varepsilon,r,N}(A_1) \oplus K_0^{\varepsilon,r,N}(A_2).$$

Similarly, we see that

$$\pi_{1*} \oplus \pi_{2*} : K_1^{\varepsilon,r,N}(A_1 \oplus A_2) \to K_1^{\varepsilon,r,N}(A_1) \oplus K_1^{\varepsilon,r,N}(A_2)$$

is an isomorphism.

In the usual K-theory, when A is unital, we have

$$K_0(A) = \ker(\pi_* : K_0(A^+) \to K_0(\mathbb{C}))$$

and

$$K_1(A) = K_1(A^+) = \ker(\pi_* : K_1(A^+) \to K_1(\mathbb{C}))$$

since $K_1(\mathbb{C}) = 0$. Recall that when A is unital, we identify $M_n(A^+)$ isometrically with $M_n(A) \oplus_\infty M_n(\mathbb{C})$ via the canonical isomorphism $A^+ \to A \oplus \mathbb{C}$. In our current setting, we have a homomorphism $K_0^{\varepsilon,r,N}(A) \to \ker \pi_*^{\varepsilon,r,N}$ given by $[e] \mapsto [(e, 0)]$, and we also have a homomorphism $\ker \pi_*^{\varepsilon,r,N} \to K_0^{\varepsilon,r,N}(A)$ given by the composition

$$\ker \pi_*^{\varepsilon,r,N} \hookrightarrow K_0^{\varepsilon,r,N}(A^+) \overset{\cong}{\to} K_0^{\varepsilon,r,N}(A \oplus \mathbb{C}) \cong K_0^{\varepsilon,r,N}(A) \oplus K_0^{\varepsilon,r,N}(\mathbb{C}) \to K_0^{\varepsilon,r,N}(A).$$

The composition $K_0^{\varepsilon,r,N}(A) \to \ker \pi_*^{\varepsilon,r,N} \to K_0^{\varepsilon,r,N}(A)$ is the identity map while the composition $\ker \pi_*^{\varepsilon,r,N} \to K_0^{\varepsilon,r,N}(A) \to \ker \pi_*^{\varepsilon,r,N}$ is given by $[(e,z)]_{\varepsilon,r,N} \mapsto [(e+z,0)]_{\varepsilon,r,N}$. Let $\psi : A^+ \to A \oplus \mathbb{C}$ be the canonical isomorphism. Since $[z] = 0$ in $K_0^{\varepsilon,r,N}(\mathbb{C})$, we have

$$[\psi((e,z))] = [(e+z,z)] = [(e+z,0)] = [\psi((e+z,0))]$$

in $K_0^{\varepsilon,r,N}(A \oplus \mathbb{C})$. It follows that $[(e,z)] = [(e+z,0)]$ in $K_0^{\varepsilon,r,N}(A^+)$, and we have $K_0^{\varepsilon,r,N}(A) \cong \ker \pi_*^{\varepsilon,r,N}$.

It is not clear that $K_1^{\varepsilon,r,N}(\mathbb{C}) = 0$ for all ε, r, N, and for all choices of norms on $M_n(\mathbb{C})$ that we are considering, but by a similar argument as in the even case above, we still have $K_1^{\varepsilon,r,N}(A) \cong \ker \pi_*^{\varepsilon,r,N}$.

Thus we have

Proposition 3.2.7. *For any filtered SQ_p algebra A,*

$$K_*^{\varepsilon,r,N}(A) \cong \ker(\pi_*^{\varepsilon,r,N} : K_*^{\varepsilon,r,N}(A^+) \to K_*^{\varepsilon,r,N}(\mathbb{C})).$$

3.2.2 Some basic properties of quantitative K-theory groups

If A and B are filtered SQ_p algebras, and $\phi : A \to B$ is a filtered homomorphism, then we have the following commutative diagram:

$$\begin{array}{ccccc}
\ker \pi_{A*}^{\varepsilon,r,N} & \hookrightarrow & K_*^{\varepsilon,r,N}(A^+) & \longrightarrow & K_*^{\varepsilon,r,N}(\mathbb{C}) \\
\downarrow & & \downarrow \phi_*^+ & & \downarrow \iota_* \\
\ker \pi_{B*}^{||\phi^+||_{pcb}\varepsilon,r,||\phi^+||_{pcb}N} & \hookrightarrow & K_*^{||\phi^+||_{pcb}\varepsilon,r,||\phi^+||_{pcb}N}(B^+) & \longrightarrow & K_*^{||\phi^+||_{pcb}\varepsilon,r,||\phi^+||_{pcb}N}(\mathbb{C})
\end{array}$$

The homomorphism on the left is the restriction of ϕ_*^+. By the discussion above, we then get a homomorphism

$$\phi_* : K_*^{\varepsilon,r,N}(A) \to K_*^{||\phi^+||_{pcb}\varepsilon,r,||\phi^+||_{pcb}N}(B).$$

Moreover, if $\psi : B \to C$ is another filtered homomorphism between filtered SQ_p algebras, then $\psi_* \circ \phi_* = (\psi \circ \phi)_*$.

Definition 3.2.8. *A homotopy between two filtered homomorphisms $\phi_0 : A \to B$ and $\phi_1 : A \to B$ of filtered SQ_p algebras is a filtered homomorphism $\Phi : A \to C([0,1], B)$ such that $ev_0 \circ \Phi = \phi_0$ and $ev_1 \circ \Phi = \phi_1$, where $ev_t : C([0,1], B) \to B$ denotes the homomorphism given by evaluation at t.*

Recall that a filtered homomorphism $\Phi : A \to C([0,1], B)$ is a p-completely bounded homomorphism such that $\Phi(A_r) \subset C([0,1], B_r)$ for each $r > 0$. Given such a filtered homomorphism Φ, we may consider $\phi_t = ev_t \circ \Phi$ for $t \in [0,1]$. Then $\phi_t : A \to B$ is a filtered homomorphism, the map $t \mapsto \phi_t(a)$ is continuous for each $a \in A$, and $\sup_{t\in[0,1]} ||\phi_t||_{pcb} \leq ||\Phi||_{pcb}$.

Proposition 3.2.9. *Let $\phi_0 : A \to B$ and $\phi_1 : A \to B$ be filtered homomorphisms between filtered SQ_p algebras. Suppose that $\Phi : A \to C([0,1], B)$ is a homotopy between ϕ_0 and ϕ_1. Then*

$$\phi_{0*} = \phi_{1*} : K_*^{\varepsilon,r,N}(A) \to K_*^{||\Phi^+||_{pcb}\varepsilon,r,||\Phi^+||_{pcb}N}(B).$$

Proof. For $t \in [0,1]$, let $\phi_t = ev_t \circ \Phi$. For any (ε, r, N)-idempotent e in $M_n(A^+)$, $\phi_t^+(e)$ is a homotopy of $(||\Phi^+||_{pcb}\varepsilon, r, ||\Phi^+||_{pcb}N)$-idempotents in $M_n(B^+)$ between $\phi_0^+(e)$ and $\phi_1^+(e)$. Similarly, for any (ε, r, N)-invertible u in $M_n(A^+)$, $\phi_t^+(u)$ is a homotopy of $(||\Phi^+||_{pcb}\varepsilon, r, ||\Phi^+||_{pcb}N)$-invertibles in $M_n(B^+)$ between $\phi_0^+(u)$ and $\phi_1^+(u)$. □

Proposition 3.2.10. *If A is a filtered SQ_p algebra, then the canonical embedding $A \to M_n(A)$ induces an isomorphism*

$$K_*^{\varepsilon,r,N}(A) \cong K_*^{\varepsilon,r,N}(M_n(A))$$

for each $n \in \mathbb{N}$.

Proof. Let A be a unital filtered SQ_p algebra, and let $i : A \to M_n(A)$ be the canonical embedding $a \mapsto \mathrm{diag}(a, 0)$. Then we get the induced maps $i_k : M_k(A) \to M_k(M_n(A))$ for $k \in \mathbb{N}$. Let

$$\zeta : M_k(M_n(A)) \to M_{kn}(A)$$

be the isomorphism given by removing parentheses. Recall that we equip $M_k(M_n(A))$ with the norm induced by this isomorphism. If e is an (ε, r, N)-idempotent in $M_k(A)$, then $i_k(e)$ is an (ε, r, N)-idempotent in $M_k(M_n(A))$, and $\zeta(i_k(e))$ is an (ε, r, N)-idempotent in $M_{kn}(A)$. Now

$$\zeta(i_k(e)) = u\mathrm{diag}(e, 0)u^{-1}$$

for some permutation matrix u. By Lemma 3.1.25, $\mathrm{diag}(\zeta(i_k(e)), 0)$ and $\mathrm{diag}(e, 0)$ are $(4\varepsilon, r, 4N)$-homotopic in $M_{2kn}(A)$. Hence the composition $\zeta_* \circ i_*$ is the identity on $K_0^{\varepsilon,r,N}(A)$.

On the other hand, if f is an (ε, r, N)-idempotent in $M_k(M_n(A))$, then $\zeta(f)$ is an (ε, r, N)-idempotent in $M_{kn}(A)$, and $i_{kn}(\zeta(f))$ is an (ε, r, N)-idempotent in $M_{kn}(M_n(A))$. As above, we see that $\mathrm{diag}(i_{kn}(\zeta(f)), 0)$ and $\mathrm{diag}(f, 0)$ are $(4\varepsilon, r, 4N)$-homotopic in $M_{2kn}(M_n(A))$. Hence the composition $i_* \circ \zeta_*$ is the identity on $K_0^{\varepsilon,r,N}(M_n(A))$.

The odd case and the non-unital case are proved in a similar way. We just remark that in the non-unital case, we have

$$\zeta(i_k(e)) = u\mathrm{diag}(e, \pi(e), \ldots, \pi(e))u^{-1}$$

for some permutation matrix u, where $\pi : A^+ \to \mathbb{C}$ is the canonical quotient homomorphism. □

We also have a "standard picture" for the quantitative K_0 groups analogous to that for the usual K_0 group.

Lemma 3.2.11. *If $e, f \in M_\infty(A)$ are (ε, r, N)-idempotents and $[e]-[f] = 0$ in $K_0^{\varepsilon,r,N}(A)$, then there exists $m \in \mathbb{N}$ such that $\mathrm{diag}(e, I_m, 0_m)$ and $\mathrm{diag}(f, I_m, 0_m)$ are homotopic as $(4\varepsilon, r, 4N)$-idempotents in $M_\infty(\tilde{A})$.*

Proof. If $[e]-[f] = 0$ in $K_0^{\varepsilon,r,N}(A)$, then $[e]+[g] = [f]+[g]$ in $Idem_\infty^{\varepsilon,r,N}(\tilde{A})/\sim$ for some $g \in Idem_m^{\varepsilon,r,N}(\tilde{A})$, so $\mathrm{diag}(e, g)$ and $\mathrm{diag}(f, g)$ are $(4\varepsilon, r, 4N)$-homotopic in $M_\infty(\tilde{A})$. Now I_m-g is (ε, r, N)-idempotent in $M_m(\tilde{A})$, and $\mathrm{diag}(e, g, I_m-g) \overset{4\varepsilon,r,4N}{\sim} \mathrm{diag}(f, g, I_m-g)$ in $M_\infty(\tilde{A})$. But we have a homotopy of $(\varepsilon, r, \frac{5}{2}N)$-idempotents between $\mathrm{diag}(I_m, 0_m)$ and $\mathrm{diag}(g, I_m - g)$ given by

$$\mathrm{diag}(g, 0) + R_t\mathrm{diag}(I_m - g, 0)R_t^{-1},$$

where $R_t = \begin{pmatrix} (\cos\frac{\pi t}{2})I_m & (\sin\frac{\pi t}{2})I_m \\ -(\sin\frac{\pi t}{2})I_m & (\cos\frac{\pi t}{2})I_m \end{pmatrix}$. Hence we have a homotopy of $(4\varepsilon, r, 4N)$-idempotents between $\mathrm{diag}(e, I_m, 0_m)$ and $\mathrm{diag}(f, I_m, 0_m)$. □

Lemma 3.2.12. *If $[e]-[f] \in K_0^{\varepsilon,r,N}(A)$, where $e, f \in M_k(\tilde{A})$, then $[e]-[f] = [e']-[I_k]$ in $K_0^{\varepsilon,r,N}(A)$ for some $e' \in M_{2k}(\tilde{A})$.*

Proof. Let $e' = \mathrm{diag}(e, I_k - f)$. Then $e' \in Idem_{2k}^{\varepsilon,r,N}(\tilde{A})$. We have a homotopy of $(\varepsilon, r, \frac{5}{2}N)$-idempotents between $\mathrm{diag}(e, I_k - f, f)$ and $\mathrm{diag}(e, I_k, 0)$. If A is non-unital, and $[\pi(e)] = [\pi(f)]$ in $K_0^{\varepsilon,r,N}(\mathbb{C})$, then $[\pi(e')] = [I_k]$. Hence $[e] - [f] = [e'] - [I_k]$ in $K_0^{\varepsilon,r,N}(A)$. □

If we allow ourselves to relax control, then we can write elements in $K_0^{\varepsilon,r,N}(A)$ in the form $[e] - [I_k]$ with $\pi(e) = \mathrm{diag}(I_k, 0)$.

Lemma 3.2.13. *There exist a non-decreasing function* $\lambda : [1,\infty) \to [1,\infty)$ *and a function* $h : (0, \frac{1}{20}) \times [1,\infty) \to [1,\infty)$ *with* $h(\cdot, N)$ *non-increasing for each fixed* N *such that for any filtered* SQ_p *algebra* A*, if* $[e] - [f] \in K_0^{\varepsilon,r,N}(A)$*, where* $e, f \in M_k(\tilde{A})$*, then* $[e]-[f] = [e'']-[I_k]$ *in* $K_0^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A)$ *for some* $e'' \in M_{2k}(\tilde{A})$ *with* $\pi(e'') = \mathrm{diag}(I_k, 0_k)$.

Proof. By the previous proposition, $[e] - [f] = [e'] - [I_k]$ in $K_0^{\varepsilon,r,N}(A)$ for some $e' \in M_{2k}(\tilde{A})$. Since $[\pi(e')] = [I_k]$, up to rescaling ε and N, and up to stabilization, $\pi(e')$ and $\mathrm{diag}(I_k, 0_k)$ are homotopic as (ε, r, N)-idempotents in $M_{2k}(\mathbb{C})$. By Lemma 3.1.27 and Lemma 3.1.28, up to stabilization, there exist functions λ and h depending only on ε and N, and there exists a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse pair (u, v) in $M_{2k}(\mathbb{C})$ such that

$$||u\pi(e')v - \mathrm{diag}(I_k, 0_k)|| < \lambda_N\varepsilon.$$

Then

$$||\mathrm{diag}(u, v)\mathrm{diag}(\pi(e'), 0_{2k})\mathrm{diag}(v, u) - \mathrm{diag}(I_k, 0_{3k})|| < \lambda_N\varepsilon.$$

Let $e'' = ue'v - u\pi(e')v + \mathrm{diag}(I_k, 0_k)$. Then $\pi(e'') = \mathrm{diag}(I_k, 0_k)$ and $||e'' - ue'v|| < \lambda_N\varepsilon$. By Lemma 3.1.25 and Lemma 3.1.18, $\mathrm{diag}(e'', 0_{2k})$ is homotopic to $\mathrm{diag}(e', 0_{2k})$ as $(\lambda'_N\varepsilon, h'_{\varepsilon,N}r, \lambda'_N)$-idempotents. □

In the odd case, if we allow ourselves to relax control, then we can write elements in $K_1^{\varepsilon,r,N}(A)$ in the form $[u]$ with $\pi(u) = I_k$.

Lemma 3.2.14. *There exist a non-decreasing function $\lambda : [1,\infty) \to [1,\infty)$ and a function $h : (0, \frac{1}{20}) \times [1,\infty) \to [1,\infty)$ with $h(\cdot, N)$ non-increasing for each fixed N such that for any filtered SQ_p algebra A, if $[u] \in K_1^{\varepsilon,r,N}(A)$ with $u \in M_k(\tilde{A})$, then $[u] = [w]$ in $K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A)$ for some $w \in M_k(\tilde{A})$ with $\pi(w) = I_k$.*

Moreover, if u and v are homotopic as (ε, r, N)-invertibles in $M_k(\tilde{A})$, and $\pi(u) = \pi(v) = I_k$, then there is a homotopy of $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-invertibles w_t between u and v such that $\pi(w_t) = I_k$ for each $t \in [0,1]$.

Proof. Suppose that $\max(||uv-1||, ||vu-1||) < \varepsilon$. Let $w = \pi(u^{-1})u$. Then $||w|| \leq ||u^{-1}||||u|| < \frac{N^2}{1-\varepsilon} < \frac{20}{19}N^2$ and $\pi(w) = I_k$. Note that

$$||u^{-1}v^{-1} - 1|| \leq ||(vu)^{-1}||||1 - vu|| < \frac{\varepsilon}{1-\varepsilon} < \frac{20}{19}\varepsilon.$$

Similarly $||v^{-1}u^{-1} - 1|| < \frac{20}{19}\varepsilon$. It follows that

$$\max(||\pi(u^{-1})uv\pi(v^{-1}) - 1||, ||v\pi(v^{-1})\pi(u^{-1})u - 1||) < \left(\left(\frac{20}{19}N\right)^2 + \frac{20}{19}\right)\varepsilon.$$

Thus w is a $(((\frac{20}{19}N)^2 + \frac{20}{19})\varepsilon, r, \frac{20}{19}N^2)$-invertible in $M_k(\tilde{A})$.

Up to stabilization, we may assume that $\pi(u)$ is $(4\varepsilon, 2r, 4N)$-homotopic to I_k, so $\pi(u^{-1})$ is $(4\varepsilon, 2r, \frac{80}{19}N)$-homotopic to I_k. Hence w and u are $(((\frac{80}{19}N)^2 + 1)\varepsilon, 2r, \frac{80}{19}N^2)$-homotopic.

For the second statement, if (u_t) is a homotopy of (ε, r, N)-invertibles between u and v, then $w_t = \pi(u_t^{-1})u_t$ defines a homotopy with the desired property. □

If A is a filtered SQ_p algebra with filtration $(A_r)_{r>0}$, and $(B_k)_{k\in\mathbb{N}}$ is an increasing sequence of Banach subalgebras of A such that $\bigcup_{k\in\mathbb{N}} B_k$ is dense in A and $\bigcup_{r>0}(B_k \cap A_r)$ is dense in B_k for each k, then each B_k has filtration $(B_k \cap A_r)_{r>0}$. In this case, we have a canonical homomorphism $\varinjlim_k K_*^{\varepsilon,r,N}(B_k) \to K_*^{\varepsilon,r,N}(A)$ but we can say more.

Proposition 3.2.15. *Let A be a unital SQ_p algebra with filtration $(A_r)_{r>0}$ and let $(B_k)_{k\in\mathbb{N}}$ be an increasing sequence of unital Banach subalgebras of A such that*

- *$\bigcup_{r>0}(B_k \cap A_r)$ is dense in B_k for each $k \in \mathbb{N}$;*
- *$\bigcup_{k\in\mathbb{N}}(B_k \cap A_r)$ is dense in A_r for each $r > 0$.*

Then for each $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and $N \geq 1$, there is a homomorphism

$$K_*^{\varepsilon,r,N}(A) \to \varinjlim_k K_*^{\varepsilon,r,N+\varepsilon}(B_k)$$

such that the compositions

$$K_*^{\varepsilon,r,N}(A) \to \varinjlim_k K_*^{\varepsilon,r,N+\varepsilon}(B_k) \to K_*^{\varepsilon,r,N+\varepsilon}(A)$$

and

$$\varinjlim_k K_*^{\varepsilon,r,N}(B_k) \to K_*^{\varepsilon,r,N}(A) \to \varinjlim_k K_*^{\varepsilon,r,N+\varepsilon}(B_k)$$

are (induced by) the relaxation of control maps ι_*.

Proof. Note that $\bigcup_{k\in\mathbb{N}} B_k$ is dense in A. Let e be an (ε, r, N)-idempotent in $M_n(A)$, and let $\delta = \frac{\varepsilon-||e^2-e||}{6(N+1)}$. Since $\bigcup_{k\in\mathbb{N}}(B_k \cap A_r)$ is dense in A_r, there exist $k \in \mathbb{N}$ and $f \in M_n(B_k \cap A_r)$ such that $||e - f|| < \delta$. By Lemma 3.1.18, f is an $(\varepsilon, r, N + \varepsilon)$-idempotent. If $f' \in M_n(B_k \cap A_r)$ also satisfies $||e - f'|| < \delta$, then $||f - f'|| < 2\delta$. Set $f_t = (1 - t)f + tf'$. Then

$$\begin{aligned}
||f_t^2 - f_t|| &\leq ||f_t^2 - f_t f|| + ||f_t f - f^2|| + ||f^2 - f|| + ||f - f_t|| \\
&\leq ||f_t - f||(||f_t|| + ||f|| + 1) + ||f^2 - f|| \\
&< 2\delta(2(N + \delta) + 1) + (2N + 2)\delta + ||e^2 - e|| \\
&< \varepsilon.
\end{aligned}$$

Thus f and f' are $(\varepsilon, r, N+\varepsilon)$-homotopic. This gives us a well-defined homomorphism $K_0^{\varepsilon,r,N}(A) \to \varinjlim_k K_0^{\varepsilon,r,N+\varepsilon}(B_k)$ given by

$$[e] - [I_n] \mapsto [f] - [I_n].$$

It is straightforward to see that the compositions

$$K_0^{\varepsilon,r,N}(A) \to \varinjlim_k K_0^{\varepsilon,r,N+\varepsilon}(B_k) \to K_0^{\varepsilon,r,N+\varepsilon}(A)$$

and

$$\varinjlim_k K_0^{\varepsilon,r,N}(B_k) \to K_0^{\varepsilon,r,N}(A) \to \varinjlim_k K_0^{\varepsilon,r,N+\varepsilon}(B_k)$$

are the canonical maps ι_0.

The proof for the odd case is similar. □

Remark 3.2.16.

1. *In the preceding proposition, if each $a \in M_n(A_r)$ can be approximated arbitrarily closely by some $b \in \bigcup_{k\in\mathbb{N}} M_n(B_k \cap A_r)$ with $||b|| \leq ||a||$, then we have $K_*^{\varepsilon,r,N}(A) \cong \varinjlim_k K_*^{\varepsilon,r,N}(B_k)$.*

2. *Regarding $M_n(\mathbb{C})$ as $B(\ell_p^n)$, we may view $M_n(A)$ as $M_n(\mathbb{C}) \otimes_p A$ when A is an L_p operator algebra and $\otimes_p$ denotes the spatial L_p operator tensor product (see Remark 1.14 and Example 1.15 in [31]). Writing $\overline{M_\infty^p}$ for $\overline{\bigcup_{n\in\mathbb{N}} M_n(\mathbb{C})}$, we see that $\overline{M_\infty^p}$ is a closed subalgebra of $B(\ell_p)$. Let P_n be the projection onto the first n coordinates with respect to the standard basis in ℓ_p. When $p \in (1,\infty)$, we have $\lim_{n\to\infty} ||a - P_n a P_n|| = 0$ for any compact operator $a \in K(\ell_p)$. It follows that $\overline{M_\infty^p} = K(\ell_p)$ for $p \in (1,\infty)$. However, when $p = 1$, we can only say that $\lim_{n\to\infty} ||a - P_n a|| = 0$ for $a \in K(\ell_1)$. In fact, there is a rank one operator on ℓ_1 that is not in $\overline{M_\infty^1}$. This*

operator is given by $\xi \mapsto (\sum_j \xi_j)\delta_{n_0}$*, where* $n_0 \in \mathbb{N}$ *is fixed. We refer the reader to Proposition 1.8 and Example 1.10 in [31] for details.*

Corollary 3.2.17. *If* A *is a filtered* L_p *operator algebra, then*

$$K_*^{\varepsilon,r,N}(\overline{M_\infty^p} \otimes_p A) \cong K_*^{\varepsilon,r,N}(A)$$

for $p \in [1,\infty)$*. In particular, when* $p \in (1,\infty)$*, we have*

$$K_*^{\varepsilon,r,N}(K(\ell_p) \otimes_p A) \cong K_*^{\varepsilon,r,N}(A).$$

3.2.3 Relating quantitative K-theory to usual K-theory

If e is an (ε, r, N)-idempotent in a unital filtered Banach algebra A, then its spectrum $\sigma(e)$ is contained in $B_{\sqrt{\varepsilon}}(0) \cup B_{\sqrt{\varepsilon}}(1) \subset \mathbb{C}$, where $B_r(z)$ denotes the open ball of radius r centered at $z \in \mathbb{C}$. In particular, if $\varepsilon < \frac{1}{4}$, then the two balls are disjoint. By choosing a function κ_0 that is holomorphic on a neighborhood of $\sigma(e)$, takes value 0 on $\bar{B}_{\sqrt{\varepsilon}}(0)$, and takes value 1 on $\bar{B}_{\sqrt{\varepsilon}}(1)$, we may apply the holomorphic functional calculus to get an idempotent

$$\kappa_0(e) = \frac{1}{2\pi i}\int_\gamma \kappa_0(z)(z-e)^{-1}dz \in A,$$

where γ may be taken to be the contour

$$\{z \in \mathbb{C} : |z| = \sqrt{\varepsilon}\} \cup \{z \in \mathbb{C} : |z-1| = \sqrt{\varepsilon}\}.$$

This enables us to pass from the quantitative K_0 groups to the usual K_0 groups.

Note that if e is an idempotent, then $\kappa_0(e) = e$. Indeed, for $z \in \gamma$, we have

$$\begin{aligned}
\kappa_0(e) - e &= \frac{1}{2\pi i}\int_\gamma (\kappa_0(z) - z)(z-e)^{-1}dz \\
&= \frac{1}{2\pi i}\int_\gamma (\kappa_0(z) - z)\left(\frac{1-e}{z} + \frac{e}{z-1}\right)dz \\
&= 0.
\end{aligned}$$

When e is an (ε, r, N)-idempotent, we expect $\kappa_0(e)$ to be close to e. In fact, we can estimate $||\kappa_0(e) - e||$ and $||\kappa_0(e)||$ in terms of ε and N. We can also estimate $||\kappa_0(e) - \kappa_0(f)||$ in terms of $||e - f||$.

Proposition 3.2.18. *Let e be an (ε, r, N)-idempotent in a unital filtered Banach algebra A. Then*

$$||\kappa_0(e) - e|| < \frac{2(N+1)\varepsilon}{(1-\sqrt{\varepsilon})(1-2\sqrt{\varepsilon})}$$

and

$$||\kappa_0(e)|| < \frac{N+1}{1-2\sqrt{\varepsilon}}.$$

Proof. Let $\gamma = \gamma_0 \cup \gamma_1$, where $\gamma_j = \{z \in \mathbb{C} : |z - j| < \sqrt{\varepsilon}\}$ for $j = 0, 1$. Let $y = \frac{1}{z}(1-e) + \frac{1}{z-1}e$ for $z \in \gamma$. Then

$$\begin{aligned}
||\kappa_0(e) - e|| &= \frac{1}{2\pi}\left\|\int_\gamma (\kappa_0(z) - z)(z-e)^{-1}dz\right\| \\
&= \frac{1}{2\pi}\left\|\int_{\gamma_0} -z(z-e)^{-1}dz + \int_{\gamma_1}(1-z)(z-e)^{-1}dz\right\| \\
&\leq \frac{1}{2\pi}\left[\left\|\int_{\gamma_0} -z((z-e)^{-1} - y)dz\right\| + \left\|\int_{\gamma_1}(1-z)((z-e)^{-1} - y)dz\right\|\right] \\
&\leq \varepsilon\left[\max_{z\in\gamma_0}||(z-e)^{-1} - y|| + \max_{z\in\gamma_1}||(z-e)^{-1} - y||\right].
\end{aligned}$$

For $z \in \gamma$, we have

$$||(z-e)y-1|| = \left\|\left(\frac{1}{z-1}-\frac{1}{z}\right)(e-e^2)\right\| < \frac{1}{|z(z-1)|}\varepsilon$$
$$\leq \frac{1}{\sqrt{\varepsilon}(1-\sqrt{\varepsilon})}\varepsilon = \frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}.$$

Thus $||((z-e)y)^{-1}-1|| < \frac{\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}}{1-\frac{\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}}} = \frac{\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}}$. Also,

$$||y|| = \left\|\frac{1}{z}+\left(\frac{1}{z-1}-\frac{1}{z}\right)e\right\| \leq \frac{1}{\sqrt{\varepsilon}}+\frac{N}{\sqrt{\varepsilon}(1-\sqrt{\varepsilon})} < \frac{N+1}{\sqrt{\varepsilon}(1-\sqrt{\varepsilon})}.$$

Hence

$$\begin{aligned}||(z-e)^{-1}-y|| &\leq ||y||||((z-e)y)^{-1}-1|| \\ &< \frac{N+1}{\sqrt{\varepsilon}(1-\sqrt{\varepsilon})}\frac{\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}} \\ &= \frac{N+1}{(1-\sqrt{\varepsilon})(1-2\sqrt{\varepsilon})}\end{aligned}$$

for all $z \in \gamma$, and $||\kappa_0(e)-e|| < \frac{2(N+1)\varepsilon}{(1-\sqrt{\varepsilon})(1-2\sqrt{\varepsilon})}$.

We also get

$$\begin{aligned}||\kappa_0(e)|| &= \frac{1}{2\pi}\left\|\int_{\gamma_1}(z-e)^{-1}dz\right\| \leq \sqrt{\varepsilon}(\max_{z\in\gamma_1}||(z-e)^{-1}-y||+||y||) \\ &< \sqrt{\varepsilon}\left(\frac{N+1}{(1-\sqrt{\varepsilon})(1-2\sqrt{\varepsilon})}+\frac{N+1}{\sqrt{\varepsilon}(1-\sqrt{\varepsilon})}\right) \\ &= \frac{N+1}{1-2\sqrt{\varepsilon}}.\end{aligned}$$

□

Proposition 3.2.19. *If e and f are (ε, r, N)-idempotents in a unital filtered Banach algebra A, then*

$$||\kappa_0(e) - \kappa_0(f)|| < \frac{(N+1)^2}{\sqrt{\varepsilon}(1-2\sqrt{\varepsilon})^2}||e-f||.$$

In particular, if $||e-f|| < \frac{\varepsilon}{9(2N+3)(N+1)^2}$, then $\kappa_0(e)$ and $\kappa_0(f)$ are homotopic idempotents.

Proof. We have

$$\begin{aligned}
||\kappa_0(e) - \kappa_0(f)|| &= \left\| \frac{1}{2\pi i} \int_\gamma \kappa_0(z)[(z-e)^{-1} - (z-f)^{-1}]dz \right\| \\
&= \left\| \frac{1}{2\pi i} \int_{\gamma_1} (z-e)^{-1}(e-f)(z-f)^{-1} dz \right\| \\
&< \sqrt{\varepsilon} \max_{z \in \gamma_1} ||(z-e)^{-1}(e-f)(z-f)^{-1}|| \\
&\leq \frac{(N+1)^2}{\sqrt{\varepsilon}(1-2\sqrt{\varepsilon})^2} ||e-f||
\end{aligned}$$

. If $||e-f|| < \frac{\varepsilon}{9(2N+3)(N+1)^2}$, then since

$$||2\kappa_0(e) - 1|| < \frac{2N+3-2\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}} < 6N+7,$$

we have

$$||\kappa_0(e) - \kappa_0(f)|| < \frac{\sqrt{\varepsilon}}{9(2N+3)(1-2\sqrt{\varepsilon})^2} < \frac{1}{6N+9} < \frac{1}{||2\kappa_0(e)-1||}$$

so $\kappa_0(e)$ and $\kappa_0(f)$ are homotopic idempotents [3, Proposition 4.3.2]. □

If A is a filtered SQ_p algebra, and e is an (ε, r, N)-idempotent in $M_n(\tilde{A})$, then we may apply the holomorphic functional calculus to get an idempotent $\kappa_0(e)$ in $M_n(\tilde{A})$. This gives us a group homomorphism $K_0^{\varepsilon,r,N}(A) \to K_0(A)$ given by $[e] \mapsto [\kappa_0(e)]$. Also, since every (ε, r, N)-invertible is actually invertible, we have a homomorphism $K_1^{\varepsilon,r,N}(A) \to K_1(A)$ given by $[u]_{\varepsilon,r,N} \mapsto [u]$. We will denote this homomorphism by

κ_1. These homomorphisms allow us to represent elements in $K_0(A)$ and $K_1(A)$ in terms of quasi-idempotents and quasi-invertibles respectively.

Proposition 3.2.20.

1. *Let A be a filtered SQ_p algebra. Let f be an idempotent in $M_n(\tilde{A})$, and let $0 < \varepsilon < \frac{1}{20}$. Then there exist $r > 0$, $N \geq 1$, and $[e] \in K_0^{\varepsilon,r,N}(A)$ with $e \in Idem_n^{\varepsilon,r,N}(\tilde{A})$ such that $[\kappa_0(e)] = [f]$ in $K_0(A)$.*

2. *Let A be a filtered SQ_p algebra. Let u be an invertible element in $M_n(\tilde{A})$, and let $0 < \varepsilon < \frac{1}{20}$. Then there exist $r > 0$, $N \geq 1$, and $[v] \in K_1^{\varepsilon,r,N}(A)$ with $v \in GL_n^{\varepsilon,r,N}(\tilde{A})$ such that $[v] = [u]$ in $K_1(A)$.*

Proof.

1. Let $N = ||f||+1$. There exist $r > 0$ and $e \in M_n(\tilde{A}_r)$ such that $||e-f|| < \frac{\varepsilon}{9N(N+1)^2}$. Then e is an (ε, r, N)-idempotent in $M_n(\tilde{A})$ by Lemma 3.1.16. Moreover,

$$\begin{aligned} ||\kappa_0(e) - f|| &= ||\kappa_0(e) - \kappa_0(f)|| \\ &< \frac{(N+1)^2}{\sqrt{\varepsilon}(1-2\sqrt{\varepsilon})^2} \frac{\varepsilon}{9N(N+1)^2} \\ &< \frac{1}{2N} < \frac{1}{||2f-1||} \end{aligned}$$

so $\kappa_0(e)$ and f are homotopic as idempotents [3, Proposition 4.3.2]. When A is non-unital, by increasing N if necessary, we get $[\pi(e)] = 0$ in $K_0^{\varepsilon,r,N}(\mathbb{C})$ so that $[e] \in K_0^{\varepsilon,r,N}(A)$.

2. Let $N = ||u|| + ||u^{-1}|| + 1$. There exist $r > 0$ and $v, v' \in M_n(\tilde{A}_r)$ such that $||v - u|| < \frac{\varepsilon}{N}$ and $||v' - u^{-1}|| < \frac{\varepsilon}{N}$. Then $v \in GL_n^{\varepsilon,r,N}(\tilde{A})$ by Lemma 3.1.16. Moreover, since $||v-u|| < \frac{1}{||u^{-1}||}$, we have $[v] = [u]$ in $K_1(A)$ [38, Lemma 4.2.1].

When A is non-unital, we may assume that $\pi(u) \sim I_n$. Then, by increasing N if necessary, we get $[\pi(v)] = [I_n]$ in $K_1^{\varepsilon,r,N}(\mathbb{C})$ so that $[v] \in K_1^{\varepsilon,r,N}(A)$.

□

Proposition 3.2.21.

1. *There exists a (quadratic) polynomial ρ with positive coefficients such that for any filtered SQ_p algebra A, if $0 < \varepsilon < \frac{1}{20\rho(N)}$, and $[e]_{\varepsilon,r,N}, [f]_{\varepsilon,r,N} \in K_0^{\varepsilon,r,N}(A)$ are such that $[\kappa_0(e)] = [\kappa_0(f)]$ in $K_0(A)$, then there exist $r' \geq r$ and $N' \geq N$ such that $[e]_{\rho(N)\varepsilon,r',N'} = [f]_{\rho(N)\varepsilon,r',N'}$ in $K_0^{\rho(N)\varepsilon,r',N'}(A)$.*

2. *Let A be a filtered SQ_p algebra. Suppose that $0 < \varepsilon < \frac{1}{20}$, and $[u]_{\varepsilon,r,N}, [v]_{\varepsilon,r,N} \in K_1^{\varepsilon,r,N}(A)$ are such that $[u] = [v]$ in $K_1(A)$. Then there exist $r' \geq r$ and $N' \geq N$ such that $[u]_{\varepsilon,r',N'} = [v]_{\varepsilon,r',N'}$ in $K_1^{\varepsilon,r',N'}(A)$.*

Proof.

1. Let $(p_t)_{t\in[0,1]}$ be a homotopy of idempotents in $M_n(\tilde{A})$ between $\kappa_0(e)$ and $\kappa_0(f)$. Then $P := (p_t)$ is an idempotent in $C([0,1], M_n(\tilde{A}))$. There exist $r' \geq r$ and $E := (e_t) \in C([0,1], M_n(\tilde{A}_{r'}))$ such that $||E - P|| < \frac{\varepsilon}{4N'}$, where $N' = \max(N, ||P||+1)$. In particular, we have $||e_0 - \kappa_0(e)|| < \frac{\varepsilon}{4N'}$ and $||e_1 - \kappa_0(f)|| < \frac{\varepsilon}{4N'}$. By Lemma 3.1.16, e_t is an (ε, r', N')-idempotent in $M_n(\tilde{A})$ for each $t \in [0,1]$. Also

$$\begin{aligned}||e_0 - e|| &\leq ||e_0 - \kappa_0(e)|| + ||\kappa_0(e) - e|| \\ &< \frac{\varepsilon}{4N'} + \frac{2(N+1)\varepsilon}{(1-\sqrt{\varepsilon})(1-2\sqrt{\varepsilon})} \\ &< (6N+7)\varepsilon\end{aligned}$$

and similarly $||e_1 - f|| < (6N+7)\varepsilon$. By Lemma 3.1.19, e_0 and e are (ε', r', N')-homotopic, where $\varepsilon' = \varepsilon + \frac{1}{4}(6N+7)^2\varepsilon^2$, and similarly for e_1 and f. Hence

$[e]_{\varepsilon',r',N'} = [f]_{\varepsilon',r',N'}$.

2. Let $(u_t)_{t\in[0,1]}$ be a homotopy of invertibles in $M_n(\tilde{A})$ between u and v. We may regard $U = (u_t)$ as an invertible element in $C([0,1], M_n(\tilde{A}))$. Let $N' = \max(N, ||U|| + ||U^{-1}|| + 1)$. There exist $r' \geq r$ and $W \in C([0,1], M_n(\tilde{A}_{r'}))$ such that

$$||W - U|| < \frac{\varepsilon - \max(||uu'-1||, ||u'u-1||, ||vv'-1||, ||v'v-1||)}{N'},$$

where u' is an (ε, r, N)-inverse for u, and v' is an (ε, r, N)-inverse for v. Then W is an (ε, r', N')-invertible in $C([0,1], M_n(\tilde{A}))$ by Lemma 3.1.16, and we have a homotopy of (ε, r', N')-invertibles $u \sim W_0 \sim W_1 \sim v$ by Lemma 3.1.18.

□

3.3 Controlled Long Exact Sequence in Quantitative K-Theory

In this section, we establish a controlled long exact sequence in quantitative K-theory analogous to the one in usual K-theory. The proofs are adaptations of those in usual K-theory (cf. [3]) and also those in quantitative K-theory for filtered C^*-algebras (cf. [26]). First, we introduce some terminology, adapted from [26] and [27], that will allow us to describe the functorial properties of quantitative K-theory.

3.3.1 Controlled morphisms and controlled exact sequences

The following notion of a control pair provides a convenient way to keep track of increases in the parameters associated with quantitative K-theory groups. It has already appeared in some of the earlier results, and we now give a formal definition.

Definition 3.3.1. *A control pair is a pair* (λ, h) *such that*

- $\lambda : [1,\infty) \to [1,\infty)$ *is a non-decreasing function;*

- $h : (0, \frac{1}{20}) \times [1, \infty) \to [1, \infty)$ *is a function such that* $h(\cdot, N)$ *is non-increasing for fixed* N.

We will write λ_N *for* $\lambda(N)$, *and* $h_{\varepsilon,N}$ *for* $h(\varepsilon, N)$.

Given two control pairs (λ, h) *and* (λ', h'), *we write* $(\lambda, h) \leq (\lambda', h')$ *if* $\lambda_N \leq \lambda'_N$ *and* $h_{\varepsilon,N} \leq h'_{\varepsilon,N}$ *for all* $\varepsilon \in (0, \frac{1}{20})$ *and* $N \geq 1$.

Remark 3.3.2. *These functions will appear as coefficients attached to the parameters* ε, r, *and* N. *One may choose to use three functions, one for each of the parameters, but we have chosen to use just two to reduce notational clutter, with* λ *controlling both* ε *and* N.

Given a filtered SQ_p algebra A, we consider the families

$$\mathcal{K}_0(A) = (K_0^{\varepsilon,r,N}(A))_{0<\varepsilon<\frac{1}{20},r>0,N\geq 1},$$
$$\mathcal{K}_1(A) = (K_1^{\varepsilon,r,N}(A))_{0<\varepsilon<\frac{1}{20},r>0,N\geq 1}.$$

Definition 3.3.3. *Let* A *and* B *be filtered* SQ_p *algebras, and let* (λ, h) *be a control pair. A* (λ, h)*-controlled morphism* $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$, *where* $i, j \in \{0, 1\}$, *is a family*

$$\mathcal{F} = (F^{\varepsilon,r,N})_{0<\varepsilon<\frac{1}{20\lambda_N},r>0,N\geq 1}$$

of group homomorphisms

$$F^{\varepsilon,r,N} : K_i^{\varepsilon,r,N}(A) \to K_j^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(B)$$

such that whenever $0 < \varepsilon \leq \varepsilon' < \frac{1}{20\lambda_{N'}}$, $h_{\varepsilon,N}r \leq h_{\varepsilon',N'}r'$, *and* $N \leq N'$, *we have the* *following commutative diagram:*

$$\begin{array}{ccc}
K_i^{\varepsilon,r,N}(A) & \xrightarrow{\iota_i} & K_i^{\varepsilon',r',N'}(A) \\
{\scriptstyle F^{\varepsilon,r,N}}\downarrow & & \downarrow{\scriptstyle F^{\varepsilon',r',N'}} \\
K_j^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(B) & \xrightarrow{\iota_j} & K_j^{\lambda_{N'}\varepsilon',h_{\varepsilon',N'}r',\lambda_{N'}}(B)
\end{array}$$

We say that $\mathcal{F}$ is a controlled morphism if it is a (λ, h)-controlled morphism for some control pair (λ, h).

In some cases, the family of group homomorphisms $F^{\varepsilon,r,N}$ may only be defined for values of r within some finite interval rather than for all $r > 0$, for example the boundary homomorphism in our controlled Mayer-Vietoris sequence in section 3.4. Thus we make the following refinement of the above definition.

Definition 3.3.4. *Let A and B be filtered SQ_p algebras, let (λ, h) be a control pair, and let $R > 0$. A (λ, h)-controlled morphism $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ of order R, where $i, j \in \{0, 1\}$, is a family*

$$\mathcal{F} = (F^{\varepsilon,r,N})_{0<\varepsilon<\frac{1}{20\lambda_N},0<r\leq\frac{R}{h_{\varepsilon,N}},N\geq 1}$$

of group homomorphisms

$$F^{\varepsilon,r,N} : K_i^{\varepsilon,r,N}(A) \to K_j^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(B)$$

such that whenever $0 < \varepsilon \leq \varepsilon' < \frac{1}{20\lambda_{N'}}$, $h_{\varepsilon,N}r \leq h_{\varepsilon',N'}r' \leq R$, and $N \leq N'$, we have the same commutative diagram as above.

Remark 3.3.5. *The definitions in the rest of this section will be stated in terms of controlled morphisms but they have obvious extensions to the setting of controlled morphisms of a given order.*

Given a filtered SQ_p algebra A, we will denote by $\mathcal{I}d_{\mathcal{K}_i(A)}$ the $(1,1)$-controlled morphism given by the family $(Id_{K_i^{\varepsilon,r,N}(A)})_{0<\varepsilon<\frac{1}{20},r>0,N\geq 1}$, where $i \in \{0,1\}$.

A filtered homomorphism between filtered SQ_p algebras A and B will induce a controlled morphism between $\mathcal{K}_*(A)$ and $\mathcal{K}_*(B)$. Moreover, such a controlled morphism will induce a homomorphism in K-theory.

Proposition 3.3.6. *Let A, B be filtered SQ_p algebras. If $\mathcal{F} = (F^{\varepsilon,r,N}) : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ is a controlled morphism, then there is a unique group homomorphism $F : K_i(A) \to K_j(B)$ satisfying*

$$F([\kappa_i(x)]) = [\kappa_j(F^{\varepsilon,r,N}([x]))]$$

for all $0 < \varepsilon < \frac{1}{20}$, $r > 0$, $N \geq 1$, and $[x] \in K_i^{\varepsilon,r,N}(A)$.

Moreover, if $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_i(B)$ is induced by a filtered homomorphism $\phi : A \to B$, then $F = \phi_* : K_i(A) \to K_i(B)$.

Proof. Given $[f] \in K_0(A)$ and $0 < \varepsilon < \frac{1}{20}$, there exist $r > 0$, $N \geq 1$, and $e \in Idem_n^{\varepsilon,r,N}(\tilde{A})$ such that $[\kappa_0(e)] = [f]$. Then define

$$F([f]) = [\kappa_j(F^{\varepsilon,r,N}([e]))] \in K_j(B).$$

This is well-defined by Propositions 3.2.20 and 3.2.21. The proof in the odd case is similar, and the last statement follows from the definition of F. □

Let A, B, and C be filtered SQ_p algebras, and let $i, j, l \in \{0,1\}$. Suppose that $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ is a (λ, h)-controlled morphism, and that $\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_l(C)$ is a (λ', h')-controlled morphism. Then we denote by $\mathcal{G} \circ \mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_l(C)$ the family

$$(G^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N} \circ F^{\varepsilon,r,N})_{0<\varepsilon<\frac{1}{20\lambda'_{\lambda_N}\lambda_N},r>0,N\geq 1}.$$

Note that $\mathcal{G} \circ \mathcal{F}$ is a (λ'', h'')-controlled morphism, where $\lambda''_N = \lambda'_{\lambda_N}\lambda_N$ and $h''_{\varepsilon,N} = h'_{\lambda_N\varepsilon,\lambda_N} h_{\varepsilon,N}$.

Hereafter, given two control pairs (λ, h) and (λ', h'), we will write $(\lambda' \cdot \lambda)_N$ for $\lambda'_{\lambda_N}\lambda_N$, and $(h' \cdot h)_{\varepsilon,N}$ for $h'_{\lambda_N\varepsilon,\lambda_N} h_{\varepsilon,N}$.

Definition 3.3.7. *Let A and B be filtered SQ_p algebras. Let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ and $\mathcal{G} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$-controlled and $(\lambda^{\mathcal{G}}, h^{\mathcal{G}})$-controlled morphisms respectively. Let (λ, h) be a control pair. We write $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ if $(\lambda^{\mathcal{F}}, h^{\mathcal{F}}) \leq (\lambda, h)$, $(\lambda^{\mathcal{G}}, h^{\mathcal{G}}) \leq (\lambda, h)$, and the following diagram commutes whenever $0 < \varepsilon < \frac{1}{20\lambda_N}$, $r > 0$, and $N \geq 1$:*

$$\begin{array}{ccccc} & & K_j^{\lambda_N^{\mathcal{F}}\varepsilon, h^{\mathcal{F}}_{\varepsilon,N} r, \lambda_N^{\mathcal{F}}}(B) & & \\ & \overset{F^{\varepsilon,r,N}}{\nearrow} & & \overset{\iota_j}{\searrow} & \\ K_i^{\varepsilon,r,N}(A) & & & & K_j^{\lambda_N\varepsilon, h_{\varepsilon,N} r, \lambda_N}(B) \\ & \underset{G^{\varepsilon,r,N}}{\searrow} & & \underset{\iota_j}{\nearrow} & \\ & & K_j^{\lambda_N^{\mathcal{G}}\varepsilon, h^{\mathcal{G}}_{\varepsilon,N} r, \lambda_N^{\mathcal{G}}}(B) & & \end{array}$$

Observe that if $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ for some control pair (λ, h), then $\mathcal{F}$ and $\mathcal{G}$ induce the same homomorphism in K-theory.

Definition 3.3.8. *Let A, B, C, and D be filtered SQ_p algebras. Let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$, $\mathcal{F}' : \mathcal{K}_i(A) \to \mathcal{K}_l(C)$, $\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_m(D)$, and $\mathcal{G}' : \mathcal{K}_l(C) \to \mathcal{K}_m(D)$ be controlled morphisms, where $i, j, l, m \in \{0, 1\}$, and let (λ, h) be a control pair. We say that the diagram*

$$\begin{array}{ccc} \mathcal{K}_i(A) & \xrightarrow{\quad \mathcal{F} \quad} & \mathcal{K}_j(B) \\ {\scriptstyle \mathcal{F}'}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}} \\ \mathcal{K}_l(C) & \xrightarrow{\quad \mathcal{G}' \quad} & \mathcal{K}_m(D) \end{array}$$

is (λ, h)-commutative if $\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}' \circ \mathcal{F}'$.

Definition 3.3.9. *Let A and B be filtered SQ_p algebras. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$-controlled morphism with $(\lambda^{\mathcal{F}}, h^{\mathcal{F}}) \leq (\lambda, h)$.*

- *We say that $\mathcal{F}$ is left (resp. right) (λ, h)-invertible if there exists a controlled morphism $\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$ such that $\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_i(A)}$ (resp. $\mathcal{F} \circ \mathcal{G} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_j(B)}$). In this case, we call $\mathcal{G}$ a left (resp. right) (λ, h)-inverse for $\mathcal{F}$.*

- *We say that $\mathcal{F}$ is (λ, h)-invertible or a (λ, h)-isomorphism if there exists a controlled morphism $\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$ that is both a left (λ, h)-inverse and a right (λ, h)-inverse for $\mathcal{F}$. In this case, we call $\mathcal{G}$ a (λ, h)-inverse for $\mathcal{F}$.*

  We say that $\mathcal{F}$ is a controlled isomorphism if it is a (λ, h)-isomorphism for some control pair (λ, h).

Note that if $\mathcal{F}$ is left (λ, h)-invertible and right (λ, h)-invertible, then there exists a control pair $(\lambda', h') \geq (\lambda, h)$, depending only on (λ, h), such that $\mathcal{F}$ is (λ', h')-invertible. Also, a controlled isomorphism will induce an isomorphism in K-theory.

Definition 3.3.10. *Let A and B be filtered SQ_p algebras. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$-controlled morphism.*

- *We say that $\mathcal{F}$ is (λ, h)-injective if $(\lambda^{\mathcal{F}}, h^{\mathcal{F}}) \leq (\lambda, h)$, and for any $0 < \varepsilon < \frac{1}{20\lambda_N}$, $r > 0$, $N \geq 1$, and $x \in K_i^{\varepsilon,r,N}(A)$, if $F^{\varepsilon,r,N}(x) = 0$ in $K_j^{\lambda_N^{\mathcal{F}}\varepsilon, h^{\mathcal{F}}_{\varepsilon,N} r, \lambda_N^{\mathcal{F}}}(B)$, then $\iota_i(x) = 0$ in $K_i^{\lambda_N\varepsilon, h_{\varepsilon,N} r, \lambda_N}(A)$.*

- *We say that $\mathcal{F}$ is (λ, h)-surjective if for any $0 < \varepsilon < \frac{1}{20(\lambda^{\mathcal{F}}\cdot\lambda)_N}$, $r > 0$, $N \geq 1$, and $y \in K_j^{\varepsilon,r,N}(B)$, there exists $x \in K_i^{\lambda_N\varepsilon, h_{\varepsilon,N} r, \lambda_N}(A)$ such that $F^{\lambda_N\varepsilon, h_{\varepsilon,N} r, \lambda_N}(x) = \iota_j(y)$ in $K_j^{(\lambda^{\mathcal{F}}\cdot\lambda)_N\varepsilon, (h^{\mathcal{F}}\cdot h)_{\varepsilon,N} r, (\lambda^{\mathcal{F}}\cdot\lambda)_N}(B)$.*

It is clear from the definitions that if $\mathcal{F}$ is left (λ, h)-invertible, then $\mathcal{F}$ is (λ, h)-injective. If $\mathcal{F}$ is right (λ, h)-invertible, then there exists a control pair $(\lambda', h') \geq (\lambda, h)$, depending only on (λ, h), such that $\mathcal{F}$ is (λ', h')-surjective. On the other hand, if $\mathcal{F}$ is both (λ, h)-injective and (λ, h)-surjective, then there exists a control pair $(\lambda', h') \geq (\lambda, h)$, depending only on (λ, h), such that $\mathcal{F}$ is a (λ', h')-isomorphism.

Definition 3.3.11. *Let A, B, and C be filtered SQ_p algebras, and let (λ, h) be a control pair. Let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\lambda^{\mathcal{F}}, h^{\mathcal{F}})$-controlled morphism, and let $\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_l(C)$ be a $(\lambda^{\mathcal{G}}, h^{\mathcal{G}})$-controlled morphism, where $i, j, l \in \{0, 1\}$. Then the composition*

$$\mathcal{K}_i(A) \xrightarrow{\mathcal{F}} \mathcal{K}_j(B) \xrightarrow{\mathcal{G}} \mathcal{K}_l(C)$$

is said to be (λ, h)-exact (at $\mathcal{K}_j(B)$) if

- $\mathcal{G} \circ \mathcal{F} = 0$;
- *for any $0 < \varepsilon < \frac{1}{20 \max((\lambda^{\mathcal{F}} \cdot \lambda)_N, \lambda^{\mathcal{G}}_N)}$, $r > 0$, $N \geq 1$, and $y \in K_j^{\varepsilon,r,N}(B)$ such that $G^{\varepsilon,r,N}(y) = 0$ in $K_l^{\lambda^{\mathcal{G}}_N \varepsilon, h^{\mathcal{G}}_{\varepsilon,N} r, \lambda^{\mathcal{G}}_N}(C)$, there exists $x \in K_i^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N}(A)$ such that $F^{\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N}(x) = \iota_j(y)$ in $K_j^{(\lambda^{\mathcal{F}} \cdot \lambda)_N \varepsilon, (h^{\mathcal{F}} \cdot h)_{\varepsilon,N} r, (\lambda^{\mathcal{F}} \cdot \lambda)_N}(B)$.*

A sequence of controlled morphisms

$$\cdots \to \mathcal{K}_{i_{k-1}}(A_{k-1}) \to \mathcal{K}_{i_k}(A_k) \to \mathcal{K}_{i_{k+1}}(A_{k+1}) \to \mathcal{K}_{i_{k+2}}(A_{k+2}) \to \cdots$$

is said to be (λ, h)-exact if the composition $\mathcal{K}_{i_{k-1}}(A_{k-1}) \to \mathcal{K}_{i_k}(A_k) \to \mathcal{K}_{i_{k+1}}(A_{k+1})$ is (λ, h)-exact for every k.

Note that controlled exact sequences in quantitative K-theory induce exact sequences in K-theory.

3.3.2 Completely filtered extensions of Banach algebras

Let A be a filtered Banach algebra with filtration $(A_r)_{r>0}$, and let J be a closed ideal in A. Then A/J has filtration

$$(q(A_r))_{r>0} = ((A_r + J)/J)_{r>0},$$

where $q : A \to A/J$ is the quotient homomorphism. We will consider extensions of filtered Banach algebras in which the ideal J has the natural filtration inherited from the filtration for A, and for which we can perform controlled lifting from A/J to A.

For $r > 0$, let $J_r = J \cap A_r$. Suppose there exists $C \geq 1$ such that $\inf_{y \in J_r} ||x + y|| \leq C \inf_{y \in J} ||x + y||$ for all $r > 0$ and $x \in A_r$. For any $y \in J$ and $\varepsilon > 0$, there exist $r > 0$ and $a \in A_r$ such that $||a - y|| < \frac{\varepsilon}{C+1}$. Then there exists $z \in J_r$ such that $||a - z|| < \frac{C\varepsilon}{C+1}$, and $||y - z|| < \varepsilon$. It follows that $(J_r)_{r>0}$ is a filtration for J.

Definition 3.3.12. *Let A be a filtered SQ_p algebra with filtration $(A_r)_{r>0}$, and let J be a closed ideal of A. The extension of Banach algebras*

$$0 \to J \to A \to A/J \to 0$$

is called a C-completely filtered extension of SQ_p algebras if there exists $C \geq 1$ such that for any $n \in \mathbb{N}$, $r > 0$, and $x \in M_n(A_r)$, we have

$$\inf_{y \in M_n(J_r)} ||x + y|| \leq C \inf_{y \in M_n(J)} ||x + y||.$$

Remark 3.3.13.

1. *If A is a non-unital filtered SQ_p algebra, J is a closed ideal of A, and the extension $0 \to J \to A \to A/J \to 0$ is C-completely filtered, then the extension*

$$0 \to J \to \tilde{A} \to \tilde{A}/J \to 0$$

   is $3C$-completely filtered.

2. *If $0 \to J \to A \to A/J \to 0$ is a C-completely filtered extension of filtered SQ_p*

algebras, then the suspended extension

$$0 \to SJ \to SA \to S(A/J) \to 0$$

is $3C$-completely filtered.

A particular class of completely filtered extensions are the extensions

$$0 \to J \to A \overset{q}{\to} A/J \to 0$$

that admit p-completely contractive sections $s : A/J \to A$ such that $s(q(A_r)) \subset A_r$ for all $r > 0$.

Example 3.3.14. *If A is a filtered SQ_p algebra, then the cone CA and the suspension SA have filtrations induced by the filtration for A. The extension $0 \to SA \to CA \to A \to 0$ admits a p-completely contractive section s such that $s(q((CA)_r)) \subset (CA)_r$ for all $r > 0$.*

Lemma 3.3.15. *Extensions that admit p-completely contractive sections s with $s(q(A_r)) \subset A_r$ for all $r > 0$ are 1-completely filtered.*

Proof. Let $x \in M_n(A_r)$. Since $s(q(A_r)) \subset A_r$, there exists $z \in M_n(J_r)$ such that $s(q(x)) = x + z$. Then

$$||x + z|| = ||s(q(x))|| \leq ||q(x)|| = \inf_{y \in M_n(J)} ||x + y||$$

so $\inf_{y \in M_n(J_r)} ||x + y|| = \inf_{y \in M_n(J)} ||x + y||$. □

3.3.3 Controlled half-exactness of $\mathcal{K}_0$ and $\mathcal{K}_1$

Lemma 3.3.16. *Let A be a unital filtered SQ_p algebra. For any $C \geq 1$ and any C-completely filtered extension $0 \to J \to A \overset{q}{\to} A/J \to 0$, for any $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and*

$N \geq 1$, *if* (u, v) *is an* (ε, r, N)*-inverse pair in* $M_n(A/J)$*, then there exists an invertible* w *in* $M_{2n}(A)$ *such that*

- $w, w^{-1} \in M_{2n}(A_{3r})$*;*
- $\max(||w||, ||w^{-1}||) \leq (N + \varepsilon + 1)^3$*;*
- $\max(||q(w) - \mathrm{diag}(u, v)||, ||q(w^{-1}) - \mathrm{diag}(v, u)||) < (N + 1)\varepsilon$*;*
- w *is homotopic to* I_{2n} *via a homotopy of invertible elements in* $M_{2n}(A_{3r})$ *with norm at most* $\sqrt{2}(N + \varepsilon + 1)^3$*, and likewise for* w^{-1}.

Proof. Given an (ε, r, N)-inverse pair (u, v) in $M_n(A/J)$, there exist $U, V \in M_n(A)$ such that $q(U) = u$, $q(V) = v$, and $\max(||U||, ||V||) < N + \varepsilon$. Consider

$$w = \begin{pmatrix} I & U \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -V & I \end{pmatrix} \begin{pmatrix} I & U \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M_{2n}(A_{3r}).$$

Then $w^{-1} \in M_{2n}(A_{3r})$, and $\max(||w||, ||w^{-1}||) < (N + \varepsilon + 1)^3$.

Moreover, $q(w) - \mathrm{diag}(u, v) = \begin{pmatrix} u(1 - vu) & uv - 1 \\ 1 - vu & 0 \end{pmatrix}$ so

$$||q(w) - \mathrm{diag}(u, v)|| < (N + 1)\varepsilon,$$

and similarly $||q(w^{-1}) - \mathrm{diag}(v, u)|| < (N + 1)\varepsilon$. Finally,

$$w_t = \begin{pmatrix} I & tU \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -tV & I \end{pmatrix} \begin{pmatrix} I & tU \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos \frac{\pi t}{2} & -\sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix}$$

is a homotopy of invertible elements in $M_{2n}(A_{3r})$ between w and I_{2n}, w_t^{-1} is a homotopy of invertible elements in $M_{2n}(A_{3r})$ between w^{-1} and I_{2n}, and $\max(||w_t||, ||w_t^{-1}||) < \sqrt{2}(N+$

$\varepsilon + 1)^3$ for all $t \in [0,1]$. $\square$

Proposition 3.3.17. *For any $C \geq 1$, there exists a control pair (λ, h) such that for any C-completely filtered extension of SQ_p algebras $0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0$, we have a (λ, h)-exact sequence*

$$\mathcal{K}_0(J) \xrightarrow{j_*} \mathcal{K}_0(A) \xrightarrow{q_*} \mathcal{K}_0(A/J).$$

Proof. Clearly the composition $K_0^{\varepsilon,r,N}(J) \to K_0^{\varepsilon,r,N}(A) \to K_0^{\varepsilon,r,N}(A/J)$ is the zero map. Let $[e] - [I_n] \in K_0^{\varepsilon,r,N}(A)$ be such that $[q(e)] - [I_n] = 0$ in $K_0^{\varepsilon,r,N}(A/J)$, where e is an (ε, r, N)-idempotent in $M_n(\tilde{A})$. Up to stabilization and relaxing control, we may assume that $q(e)$ and I_n are (ε, r, N)-homotopic in $M_n(\tilde{A}/J)$. By Proposition 3.1.27 and Lemma 3.1.28, there exists a control pair (λ, h) such that, up to stabilization, $||uq(e)v - I_n|| < \lambda_N \varepsilon$ for some $(\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N)$-inverse pair (u, v) in $M_n(\tilde{A}/J)$.

By Lemma 3.3.16, there exists an invertible $w \in M_{2k}(\tilde{A}_{3h_{\varepsilon,N}r})$ with $w^{-1} \in M_{2k}(\tilde{A}_{3h_{\varepsilon,N}r})$ such that $\max(||w||, ||w^{-1}||) \leq (\lambda_N + \varepsilon + 1)^3$, $||q(w) - \mathrm{diag}(u, v)|| < (\lambda_N + 1)\varepsilon$, and $||q(w^{-1}) - \mathrm{diag}(v, u)|| < (\lambda_N + 1)\varepsilon$. Set

$$e' = w\mathrm{diag}(e, 0)w^{-1}.$$

Since $||q(e') - \mathrm{diag}(I_n, 0)|| < 3N(\lambda_N + 1)(\lambda_N + \varepsilon + 1)^3\varepsilon$, there exists $f \in M_{2n}(\tilde{J}_{(6h_{\varepsilon,N}+1)r})$ such that

$$||f - e'|| < 3CN(\lambda_N + 1)(\lambda_N + \varepsilon + 1)^3\varepsilon.$$

By further enlarging the control pair (λ, h) if necessary, and applying Lemma 3.1.25 and Lemma 3.1.18, we get $[f] - [I_n] = [e] - [I_n]$ in $K_0^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A)$. $\square$

Lemma 3.3.18. *For any $C \geq 1$, there exists a control pair (λ, h) such that for any C-completely filtered extension $0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0$ of SQ_p algebras with A unital, if $u \in GL_n^{\varepsilon,r,N}(A/J)$ is (ε, r, N)-homotopic to I_n, then there exist $k \in \mathbb{N}$ and*

$a \in GL_{(2k+2)n}^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A)$ *homotopic to* $I_{(2k+2)n}$ *such that* $q(a) = \operatorname{diag}(u, I_{(2k+1)n})$.

Proof. Let $(u_t)_{t\in[0,1]}$ be a homotopy of (ε, r, N)-invertibles with $u_0 = u$ and $u_1 = I_n$. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be such that $||u_{t_i} - u_{t_{i-1}}|| < \frac{\varepsilon}{N}$ for $i = 1, \ldots, k$. For each t, let u'_t be an (ε, r, N)-inverse for u_t with $u'_1 = I_n$. Set

$$V = \operatorname{diag}(u_{t_0}, \ldots, u_{t_k}, u'_{t_0}, \ldots, u'_{t_k}),$$
$$W = \operatorname{diag}(I_n, u'_{t_0}, \ldots, u'_{t_{k-1}}, u_{t_0}, \ldots, u_{t_k}).$$

By Lemma 3.3.16, there exists an invertible $v \in M_{(2k+2)n}(A_{3r})$ homotopic to $I_{(2k+2)n}$ such that $||v|| < (N + \varepsilon + 1)^3$ and $||q(v) - V|| < (N + 1)\varepsilon$. Since

$$W = \operatorname{diag}(P, I)\operatorname{diag}(u'_{t_0}, \ldots, u'_{t_k}, u_{t_0}, \ldots, u_{t_k})\operatorname{diag}(P^{-1}, I)$$

for some permutation matrix P, by Lemma 3.3.16 again, there exists an invertible element $w \in M_{(2k+2)n}(A_{3r})$ homotopic to $I_{(2k+2)n}$ such that $||w|| < (N + \varepsilon + 1)^3$ and $||q(w) - W|| < (N + 1)\varepsilon$. Then vw is homotopic to $I_{(2k+2)n}$ and

$$\begin{aligned} ||q(vw) - VW|| &\leq ||(q(v) - V)q(w)|| + ||V(q(w) - W)|| \\ &< (N + 1)((N + \varepsilon + 1)^3 + N)\varepsilon. \end{aligned}$$

Let $b \in M_n(A_r)$ be a lift of u. Then

$$||q(\operatorname{diag}(b, I_{(2k+1)n}) - vw)|| < ((N + 1)((N + \varepsilon + 1)^3 + N) + 2)\varepsilon$$

so there exists $d \in M_{(2k+2)n}(J_r)$ such that

$$||\operatorname{diag}(b, I_{(2k+1)n}) - vw + d|| < ((N + 1)((N + \varepsilon + 1)^3 + N) + 2)C\varepsilon.$$

Then $a = \mathrm{diag}(b, I_{(2k+1)n}) + d$ has the desired properties. $\square$

Proposition 3.3.19. *For any $C \geq 1$, there exists a control pair (λ, h) such that for any C-completely filtered extension of SQ_p algebras $0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0$, we have a (λ, h)-exact sequence*

$$\mathcal{K}_1(J) \xrightarrow{j_*} \mathcal{K}_1(A) \xrightarrow{q_*} \mathcal{K}_1(A/J).$$

Proof. It is clear that the composition is the zero map. Let $[u] \in K_1^{\varepsilon,r,N}(A)$ with $u \in M_n(\tilde{A})$ and $[q(u)] = [I] \in K_1^{\varepsilon,r,N}(A/J)$. By Lemma 3.3.18, there exist a control pair (λ, h), $k \in \mathbb{N}$, and $w \in GL_{(2k+2)n}^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(\tilde{A})$ homotopic to $I_{(2k+2)n}$ such that $q(w) = \mathrm{diag}(q(u), I_{(2k+1)n})$.

Let w' be a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse for w. Then w' is homotopic to $I_{(2k+2)n}$, and up to relaxing control, $w'\mathrm{diag}(u, I_{(2k+1)n})$ is homotopic to $\mathrm{diag}(u, I_{(2k+1)n})$ as $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-invertibles.

Since

$$\begin{aligned}
&||q(w'\mathrm{diag}(u, I_{(2k+1)n})) - I_{(2k+2)n}|| \\
\leq\ &||q(w')||||\mathrm{diag}(q(u), I_{(2k+1)n}) - q(w)|| + ||q(w'w - I_{(2k+2)n})|| \\
<\ &(\lambda_N^2 + \lambda_N)\varepsilon,
\end{aligned}$$

there exists $U \in M_{(2k+2)n}(\tilde{J}_{h_{\varepsilon,N}r})$ such that

$$||U - w'\mathrm{diag}(u, I_{(2k+1)n})|| < C(\lambda_N^2 + \lambda_N)\varepsilon.$$

Thus by further enlarging the control pair (λ, h), we get $j_*([U]) = [u]$ in $K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A)$. $\square$

3.3.4 Controlled boundary map and controlled long exact sequence

Proposition 3.3.20. *For any $C \geq 1$, there exists a control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ such that if $0 \to J \to A \stackrel{q}{\to} A/J \to 0$ is a C-completely filtered extension of SQ_p algebras, then there is a $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$-controlled morphism*

$$\mathcal{D}_1 = (\partial_1^{\varepsilon,r,N}) : \mathcal{K}_1(A/J) \to \mathcal{K}_0(J)$$

which induces the usual boundary map $\partial_1 : K_1(A/J) \to K_0(J)$ in K-theory.

Proof. Let $u \in GL_n^{\varepsilon,r,N}(\tilde{A}/J)$ and let $v \in GL_m^{\varepsilon,r,N}(\tilde{A}/J)$ be such that $\mathrm{diag}(u, v)$ is $(\varepsilon, 2r, 2(N + \varepsilon))$-homotopic to I_{n+m}. For instance, we may take $v \in GL_n^{\varepsilon,r,N}(\tilde{A}/J)$ to be an (ε, r, N)-inverse for u by Lemma 3.1.21. By Lemma 3.3.18, up to stabilization, there exist a control pair (λ, h) and a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-invertible $w \in M_{n+m}(\tilde{A})$ with $q(w) = \mathrm{diag}(u, v)$. Let w' be a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse for w, and set

$$x = w\mathrm{diag}(I_n, 0)w' \in M_{n+m}(\tilde{A}).$$

There is a control pair $(\lambda', h') \geq (\lambda, h)$ such that

- (w, w') is a $(\lambda'_N\varepsilon, h'_{\varepsilon,N}r, \lambda'_N)$-inverse pair,
- $||q(w') - \mathrm{diag}(u', v')|| < \lambda'_N\varepsilon$, where u' and v' are (ε, r, N)-inverses for u and v respectively,
- x is homotopic to $\mathrm{diag}(I_n, 0)$ as $(\lambda'_N\varepsilon, h'_{\varepsilon,N}r, \lambda'_N)$-idempotents in $M_{n+m}(\tilde{A})$, and
- $||q(x) - \mathrm{diag}(I_n, 0)|| < \lambda'_N\varepsilon$.

Then there exists $y \in M_{n+m}(\tilde{A}_{h'_{\varepsilon,N}r} \cap J)$ such that

$$||x - \mathrm{diag}(I_n, 0) - y|| < 3C\lambda'_N\varepsilon$$

so there is a control pair $(\lambda'', h'') \geq (\lambda', h')$ such that $y + \text{diag}(I_n, 0)$ is a $(\lambda''_N \varepsilon, h''_{\varepsilon,N} r, \lambda''_N)$-idempotent in $M_{n+m}(\tilde{J})$, and it is homotopic to x as $(\lambda''_N \varepsilon, h''_{\varepsilon,N} r, \lambda''_N)$-idempotents in $M_{n+m}(\tilde{A})$. We will define

$$\partial_1([u]) = [y + \text{diag}(I_n, 0)] - [\text{diag}(I_n, 0)].$$

It remains to show that there is a control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}}) \geq (\lambda'', h'')$ so that

$$\partial_1 : K_1^{\varepsilon,r,N}(A/J) \to K_0^{\lambda^{\mathcal{D}}_N \varepsilon, h^{\mathcal{D}}_{\varepsilon,N} r, \lambda^{\mathcal{D}}_N}(J)$$

is a well-defined homomorphism. We do so in several steps as follows:

1. Suppose that $y' \in M_{n+m}(\tilde{A}_{h'_{\varepsilon,N} r} \cap J)$ also satisfies

$$||x - \text{diag}(I_n, 0) - y'|| < 3C\lambda'_N \varepsilon.$$

   Then $||y - y'|| < 6C\lambda'_N \varepsilon$ so $y + \text{diag}(I_n, 0)$ and $y' + \text{diag}(I_n, 0)$ are homotopic as $(\lambda^{\mathcal{D}}_N \varepsilon, h^{\mathcal{D}}_{\varepsilon,N} r, \lambda^{\mathcal{D}}_N)$-idempotents for an appropriate control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$.

2. Suppose that $z \in M_{n+m}(\tilde{A})$ is another $(\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N)$-invertible such that $||q(z) - \text{diag}(u, v)|| < \lambda_N \varepsilon$, and $z' \in M_{n+m}(\tilde{A})$ is a $(\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N)$-inverse for z. Set $x' = z\text{diag}(I_n, 0)z'$. Then there exists $y' \in M_{2n}(\tilde{A}_{h'_{\varepsilon,N} r} \cap J)$ such that

$$||x' - \text{diag}(I_n, 0) - y'|| < 3C\lambda'_N \varepsilon.$$

   Since $||q(x) - q(x')|| < 2\lambda'_N \varepsilon$, there exists $y'' \in M_{n+m}(\tilde{A}_{h'_{\varepsilon,N} r} \cap J)$ such that $||x - x' - y''|| < 6C\lambda'_N \varepsilon$. Then

$$||x - \text{diag}(I_n, 0) - y' - y''|| < 9C\lambda'_N \varepsilon.$$

This reduces to the case in (1.).

3. Replacing u by $\operatorname{diag}(u, I_k)$ and replacing v by $\operatorname{diag}(v, I_j)$, we get $z \in M_{n+m+k+j}(\tilde{A})$ such that

$$||q(z) - \operatorname{diag}(u, I_k, v, I_j)|| < \lambda_N \varepsilon.$$

In fact, if we write w as a block matrix $\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$, where w_{11} has size $n \times n$ and w_{22} has size $m \times m$, then by (ii), we may take $z = \begin{pmatrix} w_{11} & 0 & w_{12} & 0 \\ 0 & I_k & 0 & 0 \\ w_{21} & 0 & w_{22} & 0 \\ 0 & 0 & 0 & I_j \end{pmatrix}$, and similarly for z' corresponding to w'. Set $x' = z\operatorname{diag}(I_{n+k}, 0)z'$. Then x' is homotopic to $\operatorname{diag}(w, I_{k+j})\operatorname{diag}(I_n, 0, I_k, 0)\operatorname{diag}(w', I_{k+j}) = \operatorname{diag}(x, I_k, 0)$. There exists $y' \in M_{n+m+k+j}(\tilde{A}_{h'_{\varepsilon,N}r} \cap J)$ such that

$$||x' - \operatorname{diag}(I_{n+k}, 0) - y'|| < 3C\lambda'_N \varepsilon.$$

Conjugating by permutation matrices yields $y'' \in M_{n+m+k+j}(\tilde{A}_{h'_{\varepsilon,N}r} \cap J)$ such that

$$||\operatorname{diag}(x, I_k, 0) - \operatorname{diag}(I_n, 0, I_k, 0) - y''|| < 3C\lambda'_N \varepsilon.$$

Then $||y'' - \operatorname{diag}(y, 0)|| < 6C\lambda'_N \varepsilon$. It follows that there is an appropriate control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ such that in $K_0^{\lambda^{\mathcal{D}}_N\varepsilon, h^{\mathcal{D}}_{\varepsilon,N}r, \lambda^{\mathcal{D}}_N}(J)$ we have

$$\begin{aligned}\partial_1([\mathrm{diag}(u, I_m)]) &= [y' + \mathrm{diag}(I_{n+k}, 0_{m+j})] - [\mathrm{diag}(I_{n+k}, 0_{m+j})] \\ &= [y'' + \mathrm{diag}(I_n, 0_{m+k+j})] - [\mathrm{diag}(I_n, 0_{m+k+j})] \\ &= [\mathrm{diag}(y, 0_{k+j}) + \mathrm{diag}(I_n, 0_{m+k+j})] - [\mathrm{diag}(I_n, 0_{m+k+j})] \\ &= [y + \mathrm{diag}(I_n, 0_m)] - [\mathrm{diag}(I_n, 0_m)] = \partial_1([u]).\end{aligned}$$

4. Suppose that u_0 and u_1 are homotopic as (ε, r, N)-invertibles in $M_n(\tilde{A}/J)$, and that $v_i \in GL_m^{\varepsilon,r,N}(\tilde{A}/J)$ is such that $\mathrm{diag}(u_i, v_i)$ is $(\varepsilon, 2r, 2(N+\varepsilon))$-homotopic to I_{n+m} for $i = 0, 1$. Let u_0' and v_0' be (ε, r, N)-inverses for u_0 and v_0 respectively. Then there exists a control pair $(\lambda^1, h^1) \geq (\lambda, h)$ such that as $(\lambda^1_N\varepsilon, h^1_{\varepsilon,N}r, \lambda^1_N)$-invertibles, $u_0'u_1 \sim I_n$, and $\mathrm{diag}(v_0'v_1, I_n) \sim \mathrm{diag}(I_n, v_0'v_1) \sim \mathrm{diag}(u_0'u_1, v_0'v_1) \sim I_{n+m}$. Let $w_0 \in M_{n+m}(\tilde{A})$ be such that $||q(w_0) - \mathrm{diag}(u_0, v_0)|| < \lambda_N\varepsilon$. There is a control pair $(\lambda^2, h^2) \geq (\lambda^1, h^1)$ such that, up to stabilization, there exist $a \in M_n(\tilde{A})$ and $b \in M_{n+m}(\tilde{A})$ such that

   - $||q(a) - u_0'u_1|| < \lambda^2_N\varepsilon$,
   - $||q(b) - \mathrm{diag}(v_0'v_1, I_n)|| < \lambda^2_N\varepsilon$,
   - $||q(\mathrm{diag}(w_0, I_n)\mathrm{diag}(a, b)) - \mathrm{diag}(u_1, v_1, I_m)|| < \lambda^2_N\varepsilon$, and
   - $||\mathrm{diag}(w_0a, b)\mathrm{diag}(I_n, 0)\mathrm{diag}(a'w_0', b') - \mathrm{diag}(I_n, 0) - \mathrm{diag}(y, 0)|| < \lambda^2_N\varepsilon$,

   where a', b', w_0' are quasi-inverses for a, b, w_0 respectively. By the previous cases, there is a control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ such that

$$\begin{aligned}\partial_1([u_1]) &= [\mathrm{diag}(y,0) + \mathrm{diag}(I_n,0)] - [\mathrm{diag}(I_n,0)] \\ &= [y + \mathrm{diag}(I_n,0)] - [\mathrm{diag}(I_n,0)] \\ &= \partial_1([u_0])\end{aligned}$$

in $K_0^{\lambda^{\mathcal{D}}_N\varepsilon, h^{\mathcal{D}}_{\varepsilon,N} r, \lambda^{\mathcal{D}}_N}(J)$.

To see that this controlled boundary map induces the usual boundary map in K-theory, let $u_0 \in M_n(\tilde{A}/J)$ be invertible, and let $u \in M_n(\tilde{A}/J)$ be sufficiently close to u_0 so that u is (ε, r, N)-invertible for some ε, r, and N. Similarly, let v correspond to u_0^{-1}. Up to relaxing control, we may assume that (u, v) is an (ε, r, N)-inverse pair. Let $w_0 \in M_{2n}(\tilde{A})$ be a lift of $\mathrm{diag}(u_0, u_0^{-1})$. Recall that the usual boundary map $\partial_1 : K_1(A/J) \to K_0(J)$ is defined by

$$\partial_1([u_0]) = [w_0 \mathrm{diag}(I_n, 0) w_0^{-1}] - [\mathrm{diag}(I_n, 0)].$$

Let $N = ||w_0|| + ||w_0^{-1}|| + 1$, and let $w, w' \in M_{2n}(\tilde{A}_r)$ be such that $||w - w_0|| < \frac{\varepsilon}{N}$ and $||w' - w_0^{-1}|| < \frac{\varepsilon}{N}$ so that (w, w') is an (ε, r, N)-inverse pair for some $r > 0$. Moreover, $||q(w) - \mathrm{diag}(u, v)|| < \frac{2\varepsilon}{N}$. We use this w in the definition of the controlled boundary map to obtain y such that

$$||w\mathrm{diag}(I_n, 0)w' - \mathrm{diag}(I_n, 0) - y|| < 3C\lambda'_N\varepsilon$$

and

$$\partial_1^{\varepsilon,r,N}([u]) = [y + \mathrm{diag}(I_n, 0)] - [\mathrm{diag}(I_n, 0)].$$

Now

$$
\begin{aligned}
&||y + \mathrm{diag}(I_n, 0) - w_0 \mathrm{diag}(I_n, 0) w_0^{-1}|| \\
\leq\ &||y + \mathrm{diag}(I_n, 0) - w \mathrm{diag}(I_n, 0) w'|| + ||w \mathrm{diag}(I_n, 0) w' - w_0 \mathrm{diag}(I_n, 0) w_0^{-1}||
\end{aligned}
$$

so by making ε sufficiently small, $y + \mathrm{diag}(I_n, 0)$ and $w_0 \mathrm{diag}(I_n, 0) w_0^{-1}$ will be sufficiently close in norm so that

$$[\kappa_0(y + \mathrm{diag}(I_n, 0))] = [w_0 \mathrm{diag}(I_n, 0) w_0^{-1}]$$

in $K_0(J)$. □

Remark 3.3.21. *The following can be deduced from the definition of the controlled boundary map.*

1. *If $0 \to I \to A \to A/I \to 0$ and $0 \to J \to B \to B/J \to 0$ are C-completely filtered extensions of SQ_p algebras, and $\phi : A \to B$ is a filtered homomorphism such that $\phi(I) \subset J$ (so ϕ induces a filtered homomorphism $\tilde{\phi} : A/I \to B/J$), then $\mathcal{D}_1^B \circ \tilde{\phi}_* = \phi_* \circ \mathcal{D}_1^A$,*
2. *If $0 \to J \to A \to A/J \to 0$ is a completely filtered split extension of SQ_p algebras, i.e., there is a filtered homomorphism $s : A/J \to A$ such that $q \circ s = Id_{A/J}$ and the induced homomorphism $\tilde{A}/J \to \tilde{A}$ is p-completely contractive, then $\mathcal{D}_1 = 0$.*

Proposition 3.3.22. *For any $C \geq 1$, there exists a control pair (λ, h) such that for any C-completely filtered extension of SQ_p algebras $0 \to J \to A \stackrel{q}{\to} A/J \to 0$, we have a (λ, h)-exact sequence*

$$\mathcal{K}_1(A) \to \mathcal{K}_1(A/J) \stackrel{\mathcal{D}_1}{\to} \mathcal{K}_0(J).$$

Proof. Let $[u] \in K_1^{\varepsilon,r,N}(A)$, where $u \in GL_n^{\varepsilon,r,N}(\tilde{A})$, and let v be an (ε, r, N)-inverse for u. Then $q(v)$ is an (ε, r, N)-inverse for $q(u)$, and $\mathrm{diag}(u, v)$ is a lift of $\mathrm{diag}(q(u), q(v))$. In the definition of the controlled boundary map, we may take $w = \mathrm{diag}(u, v)$ and $w' = \mathrm{diag}(v, u)$. Then $x = w\mathrm{diag}(I_n, 0)w' = \mathrm{diag}(uv, 0)$, and $||x - \mathrm{diag}(I_n, 0)|| < \varepsilon$ so we may take $y = 0$. Thus $\partial_1([q(u)]) = [\mathrm{diag}(I_n, 0)] - [\mathrm{diag}(I_n, 0)] = 0$, i.e., the composition of the two morphisms is the zero map.

Now suppose $\partial_1([u]) = [y + \mathrm{diag}(I_n, 0)] - [\mathrm{diag}(I_n, 0)] = 0$ with $u \in GL_n^{\varepsilon,r,N}(\tilde{A}/J)$. We may assume that $y+\mathrm{diag}(I_n, 0)$ and $\mathrm{diag}(I_n, 0)$ are homotopic as (ε, r, N)-idempotents in $M_{2n}(\tilde{J})$ and thus in $M_{2n}(\tilde{A})$. By Proposition 3.1.27 and Lemma 3.1.28, up to stabilization, there exist a control pair (λ'', h'') and a $(\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N)$-inverse pair (z, z') in $M_{2n}(\tilde{J})$ such that

$$||y + \mathrm{diag}(I_n, 0) - z\mathrm{diag}(I_n, 0)z'|| < \lambda''_N\varepsilon.$$

Moreover, with $x = w\mathrm{diag}(I_n, 0)w'$ as in the definition of the controlled boundary map, we have $||x - \mathrm{diag}(I_n, 0) - y|| < 3C\lambda'_N\varepsilon$ so

$$||x - z\mathrm{diag}(I_n, 0)z'|| < (3C\lambda'_N + \lambda''_N)\varepsilon.$$

Let $V = \pi(z)z'w$, where $\pi : \tilde{J} \to \mathbb{C}$ is the usual quotient homomorphism. Then $q(V) = \pi(z)\pi(z')q(w)$ so

$$||q(V) - \mathrm{diag}(u, v)|| < 2\lambda_N\varepsilon.$$

Also, $||\pi(z)\mathrm{diag}(I_n, 0)\pi(z') - \pi(w)\mathrm{diag}(I_n, 0)\pi(w')|| < (3C\lambda'_N + \lambda''_N)\varepsilon$ so

$$||\pi(z)\mathrm{diag}(I_n, 0)\pi(z') - \mathrm{diag}(\pi(uu'), 0)|| < (2\lambda_N + 3C\lambda'_N + \lambda''_N)\varepsilon.$$

Thus

$$||\pi(z)\mathrm{diag}(I_n,0)\pi(z') - \mathrm{diag}(I_n,0)|| < \lambda'''_N\varepsilon,$$

where $\lambda'''_N = 2\lambda_N + 3C\lambda'_N + \lambda''_N + 1$. It follows that, up to relaxing control, we have

$$||V\mathrm{diag}(I_n,0) - \mathrm{diag}(I_n,0)V|| < \lambda'''_N\varepsilon.$$

Let X be the upper left $n \times n$ corner of V. Then there exists a control pair (λ^1, h^1) such that X is a $(\lambda^1_N\varepsilon, h^1_{\varepsilon,N}r, \lambda^1_N)$-invertible element in $M_n(\tilde{A})$ and $[q(X)] = [u]$ in $K_1^{\lambda^1_N\varepsilon, h^1_{\varepsilon,N}r, \lambda^1_N}(A/J)$. □

Proposition 3.3.23. *For any $C \geq 1$, there exists a control pair (λ, h) such that for any C-completely filtered extension of SQ_p algebras $0 \to J \to A \stackrel{q}{\to} A/J \to 0$, we have a (λ, h)-exact sequence*

$$\mathcal{K}_1(A/J) \stackrel{\mathcal{D}_1}{\to} \mathcal{K}_0(J) \to \mathcal{K}_0(A).$$

Proof. Let $[u] \in K_1^{\varepsilon,r,N}(A/J)$ and

$$\partial_1([u]) = [y + \mathrm{diag}(I_n,0)] - [\mathrm{diag}(I_n,0)] \in K_0^{\lambda^{\mathcal{D}}_N\varepsilon, h^{\mathcal{D}}_{\varepsilon,N}r, \lambda^{\mathcal{D}}_N}(J).$$

From the definition of the controlled boundary map, we have

$$||x - \mathrm{diag}(I_n,0) - y|| < \lambda'_N\varepsilon,$$

and x is homotopic to $\mathrm{diag}(I_n,0)$ as $(\lambda'_N\varepsilon, h'_{\varepsilon,N}r, \lambda'_N)$-idempotents in $M_{2n}(\tilde{A})$. Hence $[y + \mathrm{diag}(I_n,0)] = [\mathrm{diag}(I_n,0)]$ in $K_0^{\lambda^{\mathcal{D}}_N\varepsilon, h^{\mathcal{D}}_{\varepsilon,N}r, \lambda^{\mathcal{D}}_N}(A)$, i.e., the composition of the two morphisms is the zero map.

Suppose that $[e] - [I_k] \in K_0^{\varepsilon,r,N}(J)$ with $e \in M_n(\tilde{J})$, and $[e] - [I_k] = 0$ in $K_0^{\varepsilon,r,N}(A)$. Up to stabilization and relaxing control, we may assume that e is (ε, r, N)-homotopic to

$\operatorname{diag}(I_k, 0)$ in $M_n(\tilde{A})$, and there exists a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse pair (u, v) in $M_n(\tilde{A})$ such that

$$||uev - \operatorname{diag}(I_k, 0)|| < \lambda_N\varepsilon.$$

Then $||q(uev) - \operatorname{diag}(I_k, 0)|| < \lambda_N\varepsilon$. We may assume that

$$||q(e) - \operatorname{diag}(I_k, 0)|| < \lambda_N\varepsilon$$

by relaxing control so that $[e] = [uev]$. There exists another $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse pair (w, w') in $M_n(\tilde{A})$ such that $||e - w\operatorname{diag}(I_k, 0)w'|| < \lambda_N\varepsilon$. Then

$$||\operatorname{diag}(I_k, 0) - \operatorname{diag}(q(w), q(w'))\operatorname{diag}(I_k, 0)\operatorname{diag}(q(w'), q(w))|| < 2\lambda_N\varepsilon$$

and it follows that

$$||\operatorname{diag}(I_k, 0)\operatorname{diag}(q(w), q(w')) - \operatorname{diag}(q(w), q(w'))\operatorname{diag}(I_k, 0)|| < 3\lambda_N^2\varepsilon.$$

Let V_1 be the upper left $k\times k$ corner of $q(w)$ and let V_2 be the lower right $(2n-k)\times(2n-k)$ corner of $\operatorname{diag}(q(w), q(w'))$. Then

$$||\operatorname{diag}(V_1, V_2) - \operatorname{diag}(q(w), q(w'))|| < 3\lambda_N^2\varepsilon$$

so there exists a control pair (λ', h') such that as $(\lambda'_N\varepsilon, h'_{\varepsilon,N}r, \lambda'_N)$-invertibles, $\operatorname{diag}(V_1, V_2)$ is homotopic to I_{2n}. Since

$$||\operatorname{diag}(w, w')\operatorname{diag}(I_k, 0)\operatorname{diag}(w', w) - \operatorname{diag}(I_k, 0) - (\operatorname{diag}(e, 0) - \operatorname{diag}(I_k, 0))|| < \lambda_N\varepsilon,$$

it follows from the definition of the controlled boundary map that there is an appropriate control pair such that

$$\partial_1([V_1]) = [\mathrm{diag}(e, 0)] - [\mathrm{diag}(I_k, 0)] = [e] - [I_k].$$

□

Combining the results of this section, we get the following

Theorem 3.3.24. *For any $C \geq 1$, there exists a control pair (λ, h) such that for any C-completely filtered extension of SQ_p algebras $0 \to J \to A \to A/J \to 0$, we have a (λ, h)-exact sequence*

$$\mathcal{K}_1(J) \to \mathcal{K}_1(A) \to \mathcal{K}_1(A/J) \xrightarrow{\mathcal{D}_1} \mathcal{K}_0(J) \to \mathcal{K}_0(A) \to \mathcal{K}_0(A/J).$$

Applying the theorem to the semisplit extension

$$0 \to SA \to CA \to A \to 0,$$

and recalling that $\mathcal{K}_*(CA) = 0$, we see that there is a controlled isomorphism between $\mathcal{K}_1(A)$ and $\mathcal{K}_0(SA)$.

Corollary 3.3.25. *Let $\mathcal{D}_{1S} : \mathcal{K}_1(A) \to \mathcal{K}_0(SA)$ be the controlled boundary map associated to the semisplit extension*

$$0 \to SA \to CA \to A \to 0.$$

Then there exists a control pair (λ, h) such that $\mathcal{D}_{1S}$ is (λ, h)-invertible for any filtered SQ_p algebra A. Moreover, we can choose a (λ, h)-inverse that is natural, i.e., for any filtered SQ_p algebra A, there exists a (λ, h)-controlled morphism $\mathcal{B}_A : \mathcal{K}_0(SA) \to \mathcal{K}_1(A)$

that is a (λ, h)-inverse for $\mathcal{D}_{1S}$ and such that for any filtered homomorphism $f : A \to B$ of filtered SQ_p algebras, we have $\mathcal{B}_B \circ (Sf)_* = f_* \circ \mathcal{B}_A$.

Corollary 3.3.26. *There exists a control pair (λ, h) such that if*

$$0 \to J \xrightarrow{j} A \to A/J \to 0$$

is a completely filtered split extension of SQ_p algebras (so there exists a filtered homomorphism $s : A/J \to A$ such that $q \circ s = Id_{A/J}$ and the induced homomorphism $\tilde{A}/J \to \tilde{A}$ is p-completely contractive), then we have (λ, h)-exact sequences

$$0 \to \mathcal{K}_*(J) \to \mathcal{K}_*(A) \to \mathcal{K}_*(A/J) \to 0,$$

and we have (λ, h)-isomorphisms

$$\mathcal{K}_*(J) \oplus \mathcal{K}_*(A/J) \to \mathcal{K}_*(A)$$

given by $(x, y) \mapsto j_*(x) + s_*(y)$.

Remark 3.3.27. *At this point, it seems appropriate to make a remark about Bott periodicity. Indeed, one can recover Bott periodicity in the usual K-theory from our controlled long exact sequence by considering suspensions. In [26], a proof of controlled Bott periodicity in the C^*-algebraic setting was given using the power of KK-theory. However, we do not have a proof of controlled Bott periodicity in our setting.*

3.4 A Controlled Mayer-Vietoris Sequence

In this section, we follow the approach in [27] to obtain a controlled Mayer-Vietoris sequence for filtered SQ_p algebras. Throughout this section, whenever A is a Banach subalgebra of a unital Banach algebra B, we will view A^+ as $A + \mathbb{C}1_B \subset B$.

3.4.1 Preliminary definitions

Definition 3.4.1. *Let A be a filtered SQ_p algebra with filtration $(A_r)_{r>0}$. Let $s>0$ and let (Δ_1, Δ_2) be a pair of closed linear subspaces of A_s. Then (Δ_1, Δ_2) is called a completely coercive decomposition pair of degree s for A if there exists $c>0$ such that for every $r \leq s$, any positive integer n, and any $x \in M_n(A_r)$, there exist $x_1 \in M_n(\Delta_1 \cap A_r)$ and $x_2 \in M_n(\Delta_2 \cap A_r)$ such that $\max(||x_1||, ||x_2||) \leq c||x||$ and $x = x_1 + x_2$.*

The number c is called the coercity of the pair (Δ_1, Δ_2).

Definition 3.4.2. *Let A be a filtered SQ_p algebra with filtration $(A_r)_{r>0}$. Let r and R be positive numbers, and let Δ be a closed linear subspace of A_r. We define the R-neighborhood of Δ, denoted by $\mathfrak{N}_{\Delta}^{(r,R)}$, to be $\Delta + A_R\Delta + \Delta A_R + A_R\Delta A_R$.*

We will denote $\mathfrak{N}_{\Delta}^{(r,R)} \cap A_s$ by $\mathfrak{N}_{\Delta,s}^{(r,R)}$. For $s \leq r$, we also denote the R-neighborhood of $\Delta \cap A_s$ by $\mathfrak{N}_{\Delta}^{(s,R)}$.

Definition 3.4.3. *Let A be a filtered SQ_p algebra with filtration $(A_r)_{r>0}$. Let $s>0$ and let Δ be a closed linear subspace of A_s. Then a Banach subalgebra B of A is called an s-controlled Δ-neighborhood in A if B has filtration $(B \cap A_r)_{r>0}$, and B contains $\mathfrak{N}_{\Delta}^{(s,4s)}$.*

The choice of the coefficient 4 in the preceding definition is determined by the proof of Lemma 3.4.8.

Definition 3.4.4. *Let S_1 and S_2 be subsets of a SQ_p algebra A. The pair (S_1, S_2) is said to have the complete intersection approximation property if there exists $c>0$ such that for any $\varepsilon > 0$, any positive integer n, any $x \in M_n(S_1)$ and $y \in M_n(S_2)$ with $||x-y|| < \varepsilon$, there exists $z \in M_n(S_1 \cap S_2)$ such that $\max(||z-x||, ||z-y||) < c\varepsilon$.*

The number c is called the coercity of the pair (S_1, S_2).

Definition 3.4.5. *Let A be a filtered SQ_p algebra with filtration $(A_r)_{r>0}$, let $s>0$, and let $c>0$. An (s,c)-controlled Mayer-Vietoris pair for A is a pair $(A_{\Delta_1}, A_{\Delta_2})$ of Banach* *subalgebras of A associated with a pair (Δ_1, Δ_2) of closed linear subspaces of A_s such that*

- *(Δ_1, Δ_2) is a completely coercive decomposition pair for A of degree s with coercity c;*
- *A_{Δ_i} is an s-controlled Δ_i-neighborhood in A for $i = 1, 2$;*
- *$(A_{\Delta_1,r}, A_{\Delta_2,r})$ has the complete intersection approximation property for every $r \leq s$ with coercity c.*

Remark 3.4.6. *If A is a non-unital filtered SQ_p algebra, and $(A_{\Delta_1,r}, A_{\Delta_2,r})$ has the complete intersection approximation property with coercity c with respect to A, then also $(\widetilde{A_{\Delta_1,r}}, \widetilde{A_{\Delta_2,r}})$ has the complete intersection approximation property with coercity $2c + \frac{1}{2}$ with respect to $\tilde{A}$. Indeed, if $x \in M_n(\widetilde{A_{\Delta_1,r}})$ and $y \in M_n(\widetilde{A_{\Delta_2,r}})$ are such that $||x-y|| < \varepsilon$, then $||(x - \pi(x)) - (y - \pi(y))|| < 2\varepsilon$ so there exists $z_0 \in M_n(A_{\Delta_1,r} \cap A_{\Delta_2,r})$ such that*

$$\max(||x - \pi(x) - z_0||, ||y - \pi(y) - z_0||) < 2c\varepsilon.$$

Letting $z = z_0 + \frac{1}{2}(\pi(x) + \pi(y)) \in M_n(\widetilde{A_{\Delta_1,r}} \cap \widetilde{A_{\Delta_2,r}})$, we have

$$\max(||x - z||, ||y - z||) < (2c + \frac{1}{2})\varepsilon.$$

Let A be a filtered SQ_p algebra with filtration $(A_r)_{r>0}$, let $s > 0$, and let $c > 0$. Then $C([0,1], A)$ has filtration $(C([0,1], A_r))_{r>0}$. Suppose that $(A_{\Delta_1}, A_{\Delta_2})$ is an (s, c)-controlled Mayer-Vietoris pair for A. Note that $C([0,1], \Delta_i)$ is a closed linear subspace of $C([0,1], A_s)$ for $i = 1, 2$. Also, $C([0,1], A_{\Delta_i})$ is a Banach subalgebra of $C([0,1], A)$ for $i = 1, 2$. For $n \in \mathbb{N}$, we view $M_n(C([0,1], A))$ as a subalgebra of $C([0,1], M_n(A))$ with the supremum norm.

One can show that $(A_{\Delta_1}[0,1], A_{\Delta_2}[0,1])$ is an $(s, 2c)$-controlled Mayer-Vietoris pair for $A[0,1]$. Moreover, if the decomposition $x = x_1 + x_2$ in the definition of completely coercive decomposition pair satisfies $||x(k)_i - x(l)_i|| \leq c||x(k) - x(l)||$ for all $k, l \in [0,1]$, $i = 1, 2$, and $x \in M_n(A[0,1]_r)$, then $(A_{\Delta_1}[0,1], A_{\Delta_2}[0,1])$ is an (s, c)-controlled Mayer-Vietoris pair for $A[0,1]$. Similar statements hold for the respective suspensions.

Our main goal is to show that a Mayer-Vietoris pair gives rise to a controlled Mayer-Vietoris sequence. The following example is our motivating example. It illustrates the potential applicability of the notions introduced here, and thus the existence of a controlled Mayer-Vietoris sequence, in a situation where we have a group action with finite dynamic asymptotic dimension as defined in [15]. Indeed, this idea was implemented to investigate the C^*-algebraic Baum-Connes conjecture in [14], and we discuss the L_p operator algebra version in [8].

Example 3.4.7. *Let X be a compact Hausdorff space, and let G be a discrete group acting on X by homeomorphisms. Then we get an isometric action of G on $C(X)$, where we regard $C(X)$ as an L_p operator algebra with functions in $C(X)$ acting as multiplication operators on $L_p(X, \mu)$ for some regular Borel measure μ. Consider the reduced L_p crossed product $A = C(X) \rtimes_{\lambda,p} G$, which is defined in an analogous manner to the reduced C^* crossed product by completing the skew group ring in $B(L_p(G \times X))$ with the operator norm (cf. [31]). Equip G with a proper length function l. Then A is a filtered L_p operator algebra (and thus a filtered SQ_p algebra) with $A_r = \{\sum f_g g : f_g \in C(X), l(g) \leq r\}$.*

Suppose that for any finite subset E of G, there exist open subsets U_0 and U_1 such that $X = U_0 \cup U_1$, and the set

$$\left\{ g \in G : \begin{array}{l} \text{there exist } x \in U_i \text{ and } g_n, \ldots, g_1 \in E \text{ such} \\ \text{that } g = g_n \cdots g_1 \text{ and } g_k \cdots g_1 x \in U_i \text{ for all} \\ k \in \{1, \ldots, n\} \end{array} \right\}$$

is finite for $i = 0, 1$. *Let* $E = \{g \in G : l(g) \leq 4R\}$ *for some fixed* $R > 0$, *let* V_0 *and* V_1 *be the open subsets of* X *corresponding to the finite set* E^3, *and let* $U_i = \bigcup_{g \in E} gV_i$ *for* $i = 0, 1$. *Let* A_i *be the Banach subalgebra of* A *generated by* $\{f_g g : supp(f_g) \subset U_i, g \in G\}$. *By considering*

$$\Delta_i = \{\sum f_g g : supp(f_g) \subset V_i, l(g) \leq R\},$$

one can show that (A_0, A_1) *is an* $(R, 1)$*-controlled Mayer-Vietoris pair for* A.

3.4.2 Key technical lemma

In order to build a controlled Mayer-Vietoris sequence from a Mayer-Vietoris pair, we need to be able to factor a quasi-invertible as a product of two invertibles, one from each subalgebra in the Mayer-Vietoris pair. We will prove a lemma that allows us to do so.

Let A be a unital filtered SQ_p algebra. For $x, y \in A$, set

$$X(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, Y(y) = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}.$$

Note that $X(x_1 + x_2) = X(x_1)X(x_2)$, $Y(y_1 + y_2) = Y(y_1)Y(y_2)$, $X(x)^{-1} = X(-x)$, and $Y(y)^{-1} = Y(-y)$. Also, if (u, v) is an (ε, r, N)-inverse pair in A, then

$$\left\| \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} - X(u)Y(-v)X(u) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} (uv-1)u & 1-uv \\ vu-1 & 0 \end{pmatrix} \right\|$$
$$< (N+1)\varepsilon.$$

Observe that if $u = u_1 + u_2$ and $v = v_1 + v_2$, then

$$X(u)Y(-v)X(u) = X(u_1 + u_2)Y(-v_1)X(u_1 - u_2)X(u_2 - u_1)Y(-v_2)X(u_1 + u_2).$$

Moreover, if $u_i \in A_R$, and $v_i \in \Delta_i \subset A_r$, then

$$X(u_1+u_2)Y(-v_1)X(u_1-u_2) - I \in M_2(\mathfrak{N}^{(r,R)}_{\Delta_1,r+2R}),$$

and

$$X(u_2-u_1)Y(-v_2)X(u_1+u_2) - I \in M_2(\mathfrak{N}^{(r,R)}_{\Delta_2,r+2R}).$$

If A is non-unital, and (u,v) is an (ε,r,N)-inverse pair in A^+ with $\pi(u)=\pi(v)=1$, $u-1=u_1+u_2$, and $v-1=v_1+v_2$, then letting

$$P_1 = X(u_1+1)X(u_2)Y(-v_1-1)X(u_1+1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X(u_2)$$

and

$$P_2 = X(-u_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(-u_1-1)Y(-v_2)X(u_1+1)X(u_2),$$

we have $X(u)Y(-v)X(u) = P_1P_2$. Moreover, if $u_i, v_i \in \Delta_i \subset A_r$, then $P_1 - I \in M_2(\mathfrak{N}^{(r,2r)}_{\Delta_1,4r})$ and $P_2 - I \in M_2(\mathfrak{N}^{(r,2r)}_{\Delta_2,4r})$.

Lemma 3.4.8. *There exists a polynomial Q with positive integer coefficients such that for any filtered SQ_p algebra A with filtration $(A_r)_{r>0}$, any (s,c)-controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A, and any (ε,r,N)-inverse pair (u,v) in $\tilde{A}$ such that u is homotopic to 1 with $0<r\leq s$, $u-1\in A$, and $v-1\in A$, there exist an integer $k\geq 2$, $M=M(c,N)>1$, and $z_1,z_2 \in M_k(\widetilde{A_{\Delta_i}} \cap \tilde{A}_{16r})$ such that*

- *z_i is invertible in $M_k(\widetilde{A_{\Delta_i}})$ for $i=1,2$;*
- *$\max(||z_i||, ||z_i^{-1}||) \leq M$ for $i=1,2$;*
- *$||\mathrm{diag}(u, I_{k-1}) - z_1z_2|| < Q(N)\varepsilon$;*

- *for $i = 1, 2$, there is a homotopy $(z_{i,t})_{t\in[0,1]}$ of invertible elements in $M_k(\widetilde{A_{\Delta_i}} \cap \tilde{A}_{16r})$ between I_k and z_i such that*

$$\max(||z_{i,t}||, ||z_{i,t}^{-1}||) \leq M$$

for each $t \in [0, 1]$. Moreover, $\pi_i(z_{i,t})$ and $\pi_i(z_{i,t}^{-1})$ are homotopic to I_k in $M_k(\mathbb{C})$ via homotopies of invertible elements of norm at most $\sqrt{2}$, where $\pi_i : \widetilde{A_{\Delta_i}} \to \mathbb{C}$ is the quotient homomorphism.

Proof. If $||u - 1|| < \varepsilon$, then we may simply take $z_1 = z_2 = I_2$. In the general case, we proceed as follows.

Let $(u_t)_{t\in[0,1]}$ be a homotopy of (ε, r, N)-invertibles in $\tilde{A}$ with $u_0 = u$ and $u_1 = 1$. Up to relaxing control, we may assume that $u_t - 1 \in A$ for each t. For each $t \in [0, 1]$, let u'_t be an (ε, r, N)-inverse for u_t with $u'_1 = 1$. Up to relaxing control, we may also assume that $u'_t - 1 \in A$ for each t. Let $0 = t_0 < \cdots < t_m = 1$ be such that $||u_{t_i} - u_{t_{i-1}}|| < \frac{\varepsilon-\delta}{N}$ for $i = 1, \ldots, m$, where

$$\delta = \sup_{t\in[0,1]} \{||u_t v_t - 1||, ||v_t u_t - 1||\}.$$

Set

$$V = \mathrm{diag}(u_{t_0}, \ldots, u_{t_m}, u'_{t_0}, \ldots, u'_{t_m}),$$
$$W = \mathrm{diag}(1, u'_{t_0}, \ldots, u'_{t_{m-1}}, u_{t_0}, \ldots, u_{t_{m-1}}, 1).$$

Then

$$||\mathrm{diag}(u, I_{2m+1}) - VW|| < \varepsilon.$$

In the rest of this proof, we shall write $X^\wedge$ for $\mathrm{diag}(1, X, 1) \in M_{2k+2}(\tilde{A})$ whenever

$X \in M_{2k}(\tilde{A})$.

For each $i \in \{0, \dots, m\}$, there exist $v_i, v_i' \in \Delta_1 \cap A_r$ and $w_i, w_i' \in \Delta_2 \cap A_r$ such that

$$u_{t_i} - 1 = v_i + w_i, \quad u_{t_i}' - 1 = v_i' + w_i',$$
$$\max(||v_i||, ||w_i||) \leq c||u_{t_i} - 1||, \quad \max(||v_i'||, ||w_i'||) \leq c||u_{t_i}' - 1||.$$

Set

$$x_1 = \mathrm{diag}(v_0 + 1, \dots, v_m + 1), x_2 = \mathrm{diag}(w_0, \dots, w_m),$$
$$x_1' = \mathrm{diag}(v_0' + 1, \dots, v_m' + 1), x_2' = \mathrm{diag}(w_0', \dots, w_m'),$$
$$y_1 = \mathrm{diag}(v_0 + 1, \dots, v_{m-1} + 1), y_2 = \mathrm{diag}(w_0, \dots, w_{m-1}),$$
$$y_1' = \mathrm{diag}(v_0' + 1, \dots, v_{m-1}' + 1), y_2' = \mathrm{diag}(w_0', \dots, w_{m-1}').$$

Then

$$\begin{pmatrix} x_1 + x_2 & 0 \\ 0 & x_1' + x_2' \end{pmatrix} = V$$

and

$$\begin{pmatrix} y_1' + y_2' & 0 \\ 0 & y_1 + y_2 \end{pmatrix}^{\wedge} = W.$$

Set

$$P_1(x, y) = X(x)X(y)Y(-x')X(x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}X(y)$$

and

$$P_2(x, y) = X(-y)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}X(-x)Y(-y')X(x)X(y)$$

with the convention that $(x')' = x$. We have

$$\left\| V - P_1(x_1,x_2)P_2(x_1,x_2)\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} \right\| < (N+1)\varepsilon$$

and

$$\left\| \begin{pmatrix} y_1'+y_2' & 0 \\ 0 & y_1+y_2 \end{pmatrix} - P_1(y_1',y_2')P_2(y_1',y_2')\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} \right\| < (N+1)\varepsilon$$

so

$$\left\| W - P_1(y_1',y_2')^\wedge P_2(y_1',y_2')^\wedge \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}^\wedge \right\| < (N+1)\varepsilon$$

and

$$\left\| VW - P_1(x_1,x_2)P_2(x_1,x_2)\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} P_1(y_1',y_2')^\wedge P_2(y_1',y_2')^\wedge \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}^\wedge \right\|$$
$$< 3N(N+1)\varepsilon.$$

Hence

$$\left\| \begin{pmatrix} u & 0 \\ 0 & I_{2m+1} \end{pmatrix} - P_1(x_1,x_2)P_2(x_1,x_2)\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} P_1(y_1',y_2')^\wedge P_2(y_1',y_2')^\wedge \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}^\wedge \right\|$$
$$< (3N(N+1)+1)\varepsilon.$$

We will show that

$$P_1(x_1,x_2)P_2(x_1,x_2)\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} P_1(y_1',y_2')^\wedge P_2(y_1',y_2')^\wedge \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}^\wedge$$

can be factored as a product $z_1 z_2$ with z_1 and z_2 having the required properties.

Let

$$z_1 = P_1(x_1, x_2)\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} P_1(y_1', y_2')^\wedge$$

and

$$z_2 = z_3 P_2(y_1', y_2')^\wedge \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}^\wedge,$$

where

$$z_3 = (P_1(y_1', y_2')^\wedge)^{-1}\begin{pmatrix} 0 & I_{m+1} \\ -I_{m+1} & 0 \end{pmatrix} P_2(x_1, x_2)\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} P_1(y_1', y_2')^\wedge$$

Then z_1 and z_2 are matrices in $M_{2m+2}(\widetilde{A_{\Delta_i}} \cap \tilde{A}_{16r})$ that are invertible in $M_{2m+2}(\widetilde{A_{\Delta_i}})$, and

$$P_1(x_1, x_2)P_2(x_1, x_2)\begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix} P_1(y_1', y_2')^\wedge P_2(y_1', y_2')^\wedge \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}^\wedge = z_1 z_2.$$

Let $\pi_i : \widetilde{A_{\Delta_i}} \to \mathbb{C}$ denote the quotient homomorphism. Then

$$\pi_1(P_1(x_1, x_2)) = \pi_1(P_1(y_1', y_2')^\wedge) = I_{2m+2},$$

and

$$\pi_2(P_2(x_1, x_2)) = \pi_2(P_2(y_1', y_2')^\wedge) = I_{2m+2}.$$

Thus $\pi_1(z_1) = \begin{pmatrix} 0 & -I_{m+1} \\ I_{m+1} & 0 \end{pmatrix}$ and $\pi_2(z_2) = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}^\wedge$. Similarly, $\pi_1(z_1^{-1}) = \begin{pmatrix} 0 & I_{m+1} \\ -I_{m+1} & 0 \end{pmatrix}$ and $\pi_2(z_2^{-1}) = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}^\wedge$. These are all homotopic to I_{2m+2} in $M_{2m+2}(\mathbb{C})$ via homotopies of invertible elements of norm at most $\sqrt{2}$.

Note that $||X(x)|| \leq 1+||x||$ and $||Y(y)|| \leq 1+||y||$. For $i=1,2$, since

$$\max(||x_i||, ||x_i'||, ||y_i||, ||y_i'||) \leq c(N+1)+1,$$

we have

$$\max(||z_i||, ||z_i^{-1}||) \leq (1+N)^4(1+c(N+1))^6(2+c(N+1))^6.$$

For $t \in [0,1]$, set

$$\begin{aligned} R_{m,t} &= \begin{pmatrix} (\cos\frac{\pi t}{2})I_m & -(\sin\frac{\pi t}{2})I_m \\ (\sin\frac{\pi t}{2})I_m & (\cos\frac{\pi t}{2})I_m \end{pmatrix}, \\ z_{1,t} &= P_1(tx_1, tx_2)R_{m+1,t}P_1(ty_1', ty_2')^\wedge, \\ z_{2,t} &= \left[(P_1(ty_1', ty_2')^\wedge)^{-1}R_{m+1,t}^{-1}P_2(tx_1, tx_2)R_{m+1,t}P_1(ty_1', ty_2')^\wedge\right] P_2(ty_1', ty_2')^\wedge R_{m,t}^\wedge. \end{aligned}$$

Then $(z_{i,t})_{t\in[0,1]}$ is a homotopy of invertibles in $M_{2m+2}(\widetilde{A_{\Delta_i}} \cap \tilde{A}_{16r})$ between I_{2m+2} and z_i for $i=1,2$. Moreover, for each $t \in [0,1]$, $\pi_i(z_{i,t})$ and $\pi_i(z_{i,t}^{-1})$ are homotopic to I_{2m+2} in $M_{2m+2}(\mathbb{C})$ via homotopies of invertible elements of norm at most $\sqrt{2}$. Finally, for $i=1,2$,

$$\max(||z_{i,t}||, ||z_{i,t}^{-1}||) \leq 2\sqrt{2}(1+N)^4(1+c(N+1))^6(2+c(N+1))^6.$$

□

3.4.3 Controlled Mayer-Vietoris sequence

Given a filtered SQ_p algebra A and an (s,c)-controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A, we will consider the inclusion maps $j_1 : A_{\Delta_1} \to A$, $j_2 : A_{\Delta_2} \to A$, $j_{1,2} : A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_1}$, and $j_{2,1} : A_{\Delta_1} \cap A_{\Delta_2} \to A_{\Delta_2}$.

For any $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and $N \geq 1$, it is clear that the composition

$$K_0^{\varepsilon,r,N}(A_{\Delta_1} \cap A_{\Delta_2}) \xrightarrow{(j_{1,2*},j_{2,1*})} K_0^{\varepsilon,r,N}(A_{\Delta_1}) \oplus K_0^{\varepsilon,r,N}(A_{\Delta_2}) \xrightarrow{j_{1*}-j_{2*}} K_0^{\varepsilon,r,N}(A)$$

is the zero map.

Proposition 3.4.9. *For every $c > 0$, there exists a control pair (λ, h) such that for any filtered SQ_p algebra A, any (s, c)-controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A, any $N \geq 1$, $0 < \varepsilon < \frac{1}{20\lambda_N}$, and $0 < r \leq \frac{s}{h_{\varepsilon,N}}$, if $y_1 \in K_0^{\varepsilon,r,N}(A_{\Delta_1})$ and $y_2 \in K_0^{\varepsilon,r,N}(A_{\Delta_2})$ are such that $j_{1*}(y_1) = j_{2*}(y_2)$ in $K_0^{\varepsilon,r,N}(A)$, then there exists $z \in K_0^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(A_{\Delta_1} \cap A_{\Delta_2})$ such that $j_{1,2*}(z) = y_1$ in $K_0^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(A_{\Delta_1})$ and $j_{2,1*}(z) = y_2$ in $K_0^{\lambda_N\varepsilon,h_{\varepsilon,N}r,\lambda_N}(A_{\Delta_2})$.*

Proof. Up to rescaling ε, r, and N, and up to stabilization, we may write $y_i = [e_i] - [I_k]$, where e_i is an (ε, r, N)-idempotent in $M_n(\widetilde{A_{\Delta_i}})$ for $i = 1, 2$ with $e_i - \mathrm{diag}(I_k, 0) \in M_n(A_{\Delta_i})$ and with e_1 and e_2 homotopic as (ε, r, N)-idempotents in $M_n(\tilde{A})$. By Lemma 3.1.28 and Proposition 3.1.27, and up to stabilization, there exists a control pair (λ, h), and a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse pair (u, v) in $M_n(\tilde{A})$ such that $||ue_2v - e_1|| < \lambda_N\varepsilon$. Then

$$||\mathrm{diag}(u, v)\mathrm{diag}(e_2, 0)\mathrm{diag}(v, u) - \mathrm{diag}(e_1, 0)|| < \lambda_N\varepsilon.$$

We may also assume that $u - I_n \in M_n(A)$ and $v - I_n \in M_n(A)$.

By Lemma 3.4.8, there exist $Q = Q(\lambda_N)$ and $M = M(c, \lambda_N)$, and up to stabilization, there exist invertibles w_1 and w_2 in $M_{2n}(\tilde{A}_{16h_{\varepsilon,N}r})$ such that $\max(||w_i||, ||w_i^{-1}||) \leq M$ for $i = 1, 2$, $\pi_i(w_i)$ and $\pi_i(w_i^{-1})$ are homotopic to I via homotopies of invertible elements with norm at most $\sqrt{2}$, where $\pi_i : \widetilde{A_{\Delta_i}} \to \mathbb{C}$ is the quotient homomorphism, and $||\mathrm{diag}(u, v) -$

$w_1w_2|| < Q\varepsilon$. Then we also have

$$
\begin{aligned}
&||\text{diag}(v,u) - w_2^{-1}w_1^{-1}|| \\
&= ||(\text{diag}(v,u)(w_1w_2 - \text{diag}(u,v)) + \text{diag}(vu-1, uv-1))w_2^{-1}w_1^{-1}|| \\
&\leq \lambda_N(Q+1)M^2\varepsilon.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&||w_1^{-1}\text{diag}(e_1,0)w_1 - w_2\text{diag}(e_2,0)w_2^{-1}|| \\
&= ||w_1^{-1}(\text{diag}(e_1,0) - w_1w_2\text{diag}(e_2,0)w_2^{-1}w_1^{-1})w_1|| \\
&\leq M^2||(\text{diag}(u,v) - w_1w_2)\text{diag}(e_2,0)w_2^{-1}w_1^{-1} \\
&\qquad + \text{diag}(u,v)\text{diag}(e_2,0)(\text{diag}(v,u) - w_2^{-1}w_1^{-1}) + \text{diag}(e_1 - ue_2v, 0)|| \\
&\leq M^2(QNM^2 + \lambda_N^2 N(Q+1)M^2 + \lambda_N)\varepsilon.
\end{aligned}
$$

Note that $M^2(QNM^2 + \lambda_N^2 N(Q+1)M^2 + \lambda_N)$ depends only on c and N. We will denote it by $M' = M'(c,N)$.

Thus there exists $e \in M_{2n}(\widetilde{A}_{\Delta_1,(32h_{\varepsilon,N}+1)r} \cap \widetilde{A}_{\Delta_2,(32h_{\varepsilon,N}+1)r})$ such that

$$||e - w_1^{-1}\text{diag}(e_1,0)w_1|| < \left(2c + \frac{1}{2}\right)M'\varepsilon$$

and

$$||e - w_2\text{diag}(e_2,0)w_2^{-1}|| < \left(2c + \frac{1}{2}\right)M'\varepsilon.$$

By Lemma 3.1.25 and Lemma 3.1.18, there exists a control pair (λ'', h'') such that

- $w_1^{-1}\text{diag}(e_1,0)w_1$ and $w_2\text{diag}(e_2,0)w_2^{-1}$ are in $Idem_{2n}^{\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N}(\tilde{A})$,
- $\text{diag}(w_1^{-1},0)\text{diag}(e_1,0)\text{diag}(w_1,0)$ and $\text{diag}(w_2,0)\text{diag}(e_2,0)\text{diag}(w_2^{-1},0)$ are ho-

motopic to $\mathrm{diag}(e_1, 0)$ and $\mathrm{diag}(e_2, 0)$ respectively as $(\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N)$-idempotents in $M_{4n}(\tilde{A})$,

- e is $(\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N)$-homotopic to $w_1^{-1}\mathrm{diag}(e_1, 0)w_1$ and $w_2\mathrm{diag}(e_2, 0)w_2^{-1}$,
- $\pi(e)$ is $(\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N)$-homotopic to I in $M_{2n}(\mathbb{C})$, where $\pi : \widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}} \to \mathbb{C}$ is the quotient homomorphism.

Thus we conclude that $j_{1,2*}([e]-[I_k]) = y_1$ in $K_0^{\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N}(A_{\Delta_1})$ and $j_{2,1*}([e]-[I_k]) = y_2$ in $K_0^{\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N}(A_{\Delta_2})$. □

In the odd case, we also have that for any $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and $N \geq 1$, the composition

$$K_1^{\varepsilon,r,N}(A_{\Delta_1} \cap A_{\Delta_2}) \xrightarrow{(j_{1,2*}, j_{2,1*})} K_1^{\varepsilon,r,N}(A_{\Delta_1}) \oplus K_1^{\varepsilon,r,N}(A_{\Delta_2}) \xrightarrow{j_{1*}-j_{2*}} K_1^{\varepsilon,r,N}(A)$$

is the zero map.

Proposition 3.4.10. *For every $c > 0$, there exists a control pair (λ, h) such that for any filtered SQ_p algebra A, any (s, c)-controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A, any $N \geq 1$, $0 < \varepsilon < \frac{1}{20\lambda_N}$, and $0 < r \leq \frac{s}{h_{\varepsilon,N}}$, if $y_1 \in K_1^{\varepsilon,r,N}(A_{\Delta_1})$ and $y_2 \in K_1^{\varepsilon,r,N}(A_{\Delta_2})$ are such that $j_{1*}(y_1) = j_{2*}(y_2)$ in $K_1^{\varepsilon,r,N}(A)$, then there exists $z \in K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_1} \cap A_{\Delta_2})$ such that $j_{1,2*}(z) = y_1$ in $K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_1})$ and $j_{2,1*}(z) = y_2$ in $K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_2})$.*

Proof. By relaxing control, we may write $y_i = [u_i]$, where u_i is an (ε, r, N)-invertible in $M_n(\widetilde{A_{\Delta_i}})$ for $i = 1, 2$ with $\pi_i(u_i) = I_n$. Then $[u_1] = [u_2]$ in $K_1^{\varepsilon,r,N}(A)$ so up to stabilization and relaxing control, we may assume that u_1 and u_2 are homotopic as (ε, r, N)-invertibles in $M_n(\tilde{A})$. Let $v_2 \in M_n(\widetilde{A_{\Delta_2}})$ be an (ε, r, N)-inverse for u_2 with $\pi_2(v_2) = I_n$. If $(u_{t+1})_{t\in[0,1]}$ is a homotopy of (ε, r, N)-invertibles in $M_n(\tilde{A})$ between u_1 and u_2 with $\pi(u_{t+1}) = I_n$ for all t, then $(u_{t+1}v_2)_{t\in[0,1]}$ is a homotopy of $((N^2+1)\varepsilon, 2r, N^2)$-invertibles in $M_n(\tilde{A})$ between u_1v_2 and u_2v_2. Moreover, since $||u_2v_2 - I_n|| < \varepsilon$, we have that u_2v_2 and I_n are homotopic as $(\varepsilon, 2r, 1+\varepsilon)$-invertibles in $M_n(\tilde{A})$. Hence u_1v_2 and I_n are homotopic as $((N^2+1)\varepsilon, 2r, N^2)$-invertibles in $M_n(\tilde{A})$.

By Lemma 3.4.8, there exist $Q(N)$, $M = M(c, N)$, and up to stabilization, there exist invertibles w_i in $M_n(\widetilde{A_{\Delta_i}} \cap \tilde{A}_{32r})$ such that $\max(||w_i||, ||w_i^{-1}||) \leq M$, $\pi(w_i)$ and $\pi(w_i^{-1})$ are homotopic to I_n via homotopies of invertible elements in $M_n(\mathbb{C})$ with norm at most $\sqrt{2}$, and $||u_1v_2 - w_1w_2|| < Q(N)\varepsilon$. Now

$$\begin{aligned} ||w_1^{-1}u_1 - w_2u_2|| &\leq ||w_1^{-1}(u_1v_2 - w_1w_2)u_2|| + ||w_1^{-1}u_1(v_2u_2 - 1)|| \\ &< MN(Q(N)+1)\varepsilon. \end{aligned}$$

Thus there exists $z \in M_n(\widetilde{A_{\Delta_1,33r}} \cap \widetilde{A_{\Delta_2,33r}})$ such that

$$||z - w_1^{-1}u_1|| < \left(2c + \frac{1}{2}\right) MN(Q(N)+1)\varepsilon$$

and

$$||z - w_2u_2|| < \left(2c + \frac{1}{2}\right) MN(Q(N)+1)\varepsilon.$$

It follows from Lemma 3.1.18 that there exists a control pair (λ, h) such that z is homotopic to u_i as $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-invertibles in $M_n(\widetilde{A_{\Delta_i}})$ for $i = 1, 2$, and $\pi(z)$ is homotopic to I_n as $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-invertibles in $M_n(\mathbb{C})$, where $\pi : \widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}} \to \mathbb{C}$ is the quotient homomorphism. Hence

- $[z] \in K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_1} \cap A_{\Delta_2})$,
- $j_{1,2*}([z]) = y_1$ in $K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_1})$, and
- $j_{2,1*}([z]) = y_2$ in $K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_2})$.

□

Next, we want to define a boundary map

$$\partial^{\varepsilon,r,N} : K_1^{\varepsilon,s,N}(A) \to K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}} s, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$$

for an appropriate control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ depending only on the coercity c.

Lemma 3.4.11. *For every $c > 0$, there exists a control pair (λ, h) such that for any filtered SQ_p algebra A, any (s, c)-controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A, any $N \geq 1$, $0 < \varepsilon < \frac{1}{20\lambda_N}$, and $0 < r \leq \frac{s}{h_{\varepsilon,N}}$, if $u \in GL_n^{\varepsilon,r,N}(\tilde{A})$ and $v \in GL_m^{\varepsilon,r,N}(\tilde{A})$ are such that $u - I_n \in M_n(A)$ and $v - I_n \in M_n(A)$, and $w_i \in GL_{n+m}^{\varepsilon,r,N}(\widetilde{A_{\Delta_i}})$ for $i = 1, 2$ are such that $||\mathrm{diag}(u, v) - w_1 w_2|| < \varepsilon$, then letting w_i' be an (ε, r, N)-inverse for w_i, there exists a $(\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N)$-idempotent $e \in M_{n+m}(\widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}})$ such that*

$$\max(||e - w_1'\mathrm{diag}(I_n, 0)w_1||, ||e - w_2 \mathrm{diag}(I_n, 0)w_2'||) < \lambda_N \varepsilon,$$

and $\mathrm{diag}(\pi(e), 0)$ is $(\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N)$-homotopic to $\mathrm{diag}(I_n, 0)$ in $M_{2(n+m)}(\mathbb{C})$, where $\pi : \widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}} \to \mathbb{C}$ is the quotient homomorphism.

Proof. Let u' be an (ε, r, N)-inverse for u with $u' - I_n \in M_n(A)$, and let v' be an (ε, r, N)-inverse for v with $v' - I_n \in M_n(A)$. Then

$$\begin{aligned}
&||w_1'\mathrm{diag}(I_n, 0)w_1 - w_1'\mathrm{diag}(u, v)\mathrm{diag}(I_n, 0)\mathrm{diag}(u', v')w_1|| \\
&= ||w_1'\mathrm{diag}(I_n - uu', 0)w_1|| \\
&< N^2\varepsilon,
\end{aligned}$$

Since $||\text{diag}(u,v) - w_1 w_2|| < \varepsilon$, we have

$$\begin{aligned} ||w_1'\text{diag}(u,v) - w_2|| &\leq ||w_1'(\text{diag}(u,v) - w_1 w_2)|| + ||(w_1' w_1 - 1)w_2|| \\ &< 2N\varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} &||w_1'\text{diag}(I_n,0)w_1 - w_2\text{diag}(I_n,0)w_2'|| \\ &< N^2\varepsilon + ||w_1'\text{diag}(u,v)\text{diag}(I_n,0)\text{diag}(u',v')w_1 - w_2\text{diag}(I_n,0)w_2'|| \\ &\leq N^2\varepsilon + ||(w_1'\text{diag}(u,v) - w_2)\text{diag}(I_n,0)\text{diag}(u',v')w_1|| \\ &\quad + ||w_2\text{diag}(I_n,0)(\text{diag}(u',v')w_1 - w_2')|| \\ &< N^2\varepsilon + 2N^3\varepsilon + N||\text{diag}(u',v')w_1 - w_2'|| \\ &\leq (N^2 + 2N^3)\varepsilon + N(||\text{diag}(u',v')w_1(w_2 - w_1'\text{diag}(u,v))w_2'|| \\ &\quad + ||\text{diag}(u',v')w_1(1 - w_2 w_2')|| + ||\text{diag}(u',v')(w_1 w_1' - 1)\text{diag}(u,v)w_2'|| \\ &\quad + ||\text{diag}(u'u - 1, v'v - 1)w_2'||) \\ &< (N^2 + 2N^3)\varepsilon + N(2N^4 + N^2 + N^3 + N)\varepsilon \\ &= (2N^5 + N^4 + 3N^3 + 2N^2)\varepsilon. \end{aligned}$$

There exists $e \in M_{n+m}(\widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}})$ such that

$$||e - w_1'\text{diag}(I_n,0)w_1|| < \left(2c + \frac{1}{2}\right)(2N^5 + N^4 + 3N^3 + 2N^2)\varepsilon$$

and

$$||e - w_2\text{diag}(I_n,0)w_2'||) < \left(2c + \frac{1}{2}\right)(2N^5 + N^4 + 3N^3 + 2N^2)\varepsilon.$$

By Lemma 3.1.18 and Lemma 3.1.25, there exists a control pair (λ, h) depending only on c

such that e is a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-idempotent, and $\mathrm{diag}(\pi(e), 0)$ is homotopic to $\mathrm{diag}(I_n, 0)$ as $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-idempotents in $M_{2(n+m)}(\mathbb{C})$. $\square$

Given $u \in GL_n^{\varepsilon,r,N}(\tilde{A})$ with $u - I_n \in M_n(A)$, pick $v \in GL_m^{\varepsilon,r,N}(\tilde{A})$ with $v - I_n \in M_n(A)$ such that $\mathrm{diag}(u, v)$ is homotopic to I_{n+m} as $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles. For instance, by Lemma 3.1.21, we may pick $v \in GL_n^{\varepsilon,r,N}(\tilde{A})$ to be an (ε, r, N)-inverse for u. By Lemma 3.4.8, there exists a control pair (λ, h) depending only on the coercity c such that, up to replacing v by $\mathrm{diag}(v, I_k)$ for some integer k, there exist $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-invertibles w_1 and w_2 in $M_{n+m}(\tilde{A}_{h_{\varepsilon,N}r})$ such that $||\mathrm{diag}(u, v) - w_1w_2|| < \lambda_N\varepsilon$.

By Lemma 3.4.11, there exists a control pair (λ', h') depending only on the coercity c, and a $((\lambda'\cdot\lambda)_N\varepsilon, (h'\cdot h)_{\varepsilon,N}s, (\lambda'\cdot\lambda)_N)$-idempotent $e \in M_{n+m}(\widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}})$ such that letting w_i' be a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse for w_i, we have

$$\max(||e - w_1'\mathrm{diag}(I_n, 0)w_1||, ||e - w_2\mathrm{diag}(I_n, 0)w_2'||) < (\lambda'\cdot\lambda)_N\varepsilon,$$

and $\mathrm{diag}(\pi(e), 0)$ is $((\lambda'\cdot\lambda)_N\varepsilon, (h'\cdot h)_{\varepsilon,N}r, (\lambda'\cdot\lambda)_N)$-homotopic to $\mathrm{diag}(I_n, 0)$.

We would like to define the boundary map by $\partial([u]) = [e] - [I_n]$ but in order for it to be well-defined (i.e., independent of the various choices made), we need to place $[e] - [I_n]$ in the appropriate quantitative K_0 group.

More precisely, we need to check that there is a control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ (depending only on the coercity c) such that for $N \geq 1$, $0 < \varepsilon < \frac{1}{20\lambda_N^{\mathcal{D}}}$ and $0 < r \leq \frac{s}{h_{\varepsilon,N}^{\mathcal{D}}}$, the map $\partial : K_1^{\varepsilon,r,N}(A) \to K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$ given by $[u] \mapsto [e] - [I_n]$

- does not depend on the choice of e satisfying the conclusion of Lemma 3.4.11;
- does not depend on the choice of w_1, w_2 satisfying the hypotheses of Lemma 3.4.11;
- does not change upon replacing u (resp. v) by $\mathrm{diag}(u, 1)$ (resp. $\mathrm{diag}(v, 1)$);

- does not depend on the choice of $v \in GL_m^{\varepsilon,r,N}(A)$ such that $\mathrm{diag}(u,v)$ is homotopic to I_{n+m} as $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles;

- only depends on the equivalence class of u.

We also want the compositions

$$K_1^{\varepsilon,r,N}(A_{\Delta_1}) \oplus K_1^{\varepsilon,r,N}(A_{\Delta_2}) \xrightarrow{(j_{1*}-j_{2*})} K_1^{\varepsilon,r,N}(A) \xrightarrow{\partial} K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$$

and

$$\begin{aligned} K_1^{\varepsilon,r,N}(A) \quad &\xrightarrow{\partial} \quad K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2}) \\ &\xrightarrow{(j_{1,2*}, j_{2,1*})} K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1}) \oplus K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_2}) \end{aligned}$$

to be the zero maps.

Now we will address each of these points in turn. When there is a need to relax control by increasing the parameters, we will sometimes omit precise expressions of the parameters involved with the understanding that they increase in a controlled manner.

1. If e_0 and e_1 both satisfy the conclusion of Lemma 3.4.11, then $||e_0 - e_1|| < 2\lambda_N\varepsilon$ so e_0 and e_1 are $(2\lambda_N^2\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-homotopic by Lemma 3.1.19.

2. Suppose that $w_3 \in GL_{n+m}^{\varepsilon,r,N}(\widetilde{A_{\Delta_1}})$ and $w_4 \in GL_{n+m}^{\varepsilon,r,N}(\widetilde{A_{\Delta_2}})$ have the same properties as w_1, w_2 so $||\mathrm{diag}(u,v) - w_1w_2|| < \lambda_N\varepsilon$ and $||\mathrm{diag}(u,v) - w_3w_4|| < \lambda_N\varepsilon$. Let w_i' be a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse for w_i. Then $||w_1w_2 - w_3w_4|| < 2\lambda_N\varepsilon$ and

$$\begin{aligned} &||w_3'w_1 - w_4w_2'|| \\ \leq\ &||w_3'(w_1w_2 - w_3w_4)w_2'|| + ||w_3'w_1(1 - w_2w_2')|| + ||(w_3'w_3 - 1)w_4w_2'|| \\ <\ &4\lambda_N^3\varepsilon. \end{aligned}$$

The complete intersection approximation property yields $y \in M_{n+m}(\widetilde{A_{\Delta_1, h_{\varepsilon,N}r}} \cap$

$\widetilde{A_{\Delta_2,h_{\varepsilon,N}r}}$) with $||y - w_3'w_1|| < 4\lambda_N^3(2c + \frac{1}{2})\varepsilon$ and $||y - w_4w_2'|| < 4\lambda_N^3(2c + \frac{1}{2})\varepsilon$. Thus

$$\max(||w_3y - w_1||, ||yw_2 - w_4||) < \left(4\lambda_N^4\left(2c + \frac{1}{2}\right) + \lambda_N^2\right)\varepsilon.$$

Similarly, there exists $z \in M_{n+m}(\widetilde{A_{\Delta_1,h_{\varepsilon,N}r}} \cap \widetilde{A_{\Delta_2,h_{\varepsilon,N}r}})$ such that

$$\max(||z - w_1'w_3||, ||z - w_2w_4'||) < 4\lambda_N^3\left(2c + \frac{1}{2}\right)\varepsilon$$

so

$$\max(||zw_3' - w_1'||, ||zw_4 - w_2||) < \left(4\lambda_N^4\left(2c + \frac{1}{2}\right) + \lambda_N^2\right)\varepsilon.$$

Moreover,

$$\begin{aligned}
&||yz - 1|| \\
&\leq ||(y - w_3'w_1)z|| + ||w_3'w_1(z - w_1'w_3)|| + ||w_3'(w_1w_1' - 1)w_3|| + ||w_3'w_3 - 1|| \\
&< \left(4\lambda_N^3\left(2c + \frac{1}{2}\right)\left(\lambda_N^2 + 4\lambda_N^3\left(2c + \frac{1}{2}\right)\right) + 4\lambda_N^5\left(2c + \frac{1}{2}\right) + \lambda_N^3 + \lambda_N\right)\varepsilon \\
&= \left(16\lambda_N^6\left(2c + \frac{1}{2}\right)^2 + 8\lambda_N^5\left(2c + \frac{1}{2}\right) + \lambda_N^3 + \lambda_N\right)\varepsilon,
\end{aligned}$$

and similarly for $||zy - 1||$, so (y, z) is a quasi-inverse pair.

If e is the quasi-idempotent element obtained from w_1 and w_2, then with respect to an appropriate control pair (λ'', h'') depending only on the coercity c, yez is the quasi-idempotent element obtained from w_3 and w_4, and in $K_0^{\lambda_N''\varepsilon,h_{\varepsilon,N}''r,\lambda_N''}(A_{\Delta_1} \cap A_{\Delta_2})$ we have $[yez] = [e]$.

3. Replacing u by $\mathrm{diag}(u, 1)$, and letting w_1, w_2 be such that

$$||\mathrm{diag}(u, v) - w_1w_2|| < \lambda_N\varepsilon,$$

we have $||\mathrm{diag}(u, v, I_2) - \mathrm{diag}(w_1 w_2, I_2)|| < \lambda_N \varepsilon$. Now

$$U \mathrm{diag}(u, v, I_2) U^{-1} = \mathrm{diag}(u, 1, v, 1)$$

for some permutation matrix U so

$$||\mathrm{diag}(u, 1, v, 1) - U \mathrm{diag}(w_1 w_2, I_2) U^{-1}|| < \lambda_N \varepsilon.$$

If $||e - w_1' \mathrm{diag}(I_n, 0) w_1|| < (\lambda' \cdot \lambda)_N \varepsilon$, then

$$||\mathrm{diag}(e, 1, 0) - \mathrm{diag}(w_1', I_2) \mathrm{diag}(I_n, 0_m, 1, 0) \mathrm{diag}(w_1, I_2)|| < (\lambda' \cdot \lambda)_N \varepsilon.$$

But $\mathrm{diag}(I_n, 0_m, 1, 0) = U^{-1} \mathrm{diag}(I_{n+1}, 0) U$ so

$$||\mathrm{diag}(e, 1, 0) - \mathrm{diag}(w_1', I_2) U^{-1} \mathrm{diag}(I_{n+1}, 0) U \mathrm{diag}(w_1, I_2)|| < (\lambda' \cdot \lambda)_N \varepsilon.$$

Similarly, we have

$$||\mathrm{diag}(e, 1, 0) - \mathrm{diag}(w_2, I_2) U^{-1} \mathrm{diag}(I_{n+1}, 0) U \mathrm{diag}(w_2', I_2)|| < (\lambda' \cdot \lambda)_N \varepsilon.$$

Thus $\partial([\mathrm{diag}(u, 1)]) = [\mathrm{diag}(e, 1, 0)] - [I_{n+1}] = [e] - [I_n] = \partial([u])$. Similarly, one sees that replacing v by $\mathrm{diag}(v, 1)$ does not change $\partial([u])$.

4. Suppose that $v_0 \in GL_m^{\varepsilon,r,N}(\tilde{A})$ and $v_1 \in GL_k^{\varepsilon,r,N}(\tilde{A})$ are such that $\mathrm{diag}(u, v_0)$ is homotopic to I_{n+m} as $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles, and $\mathrm{diag}(u, v_1)$ is homotopic to I_{n+k} as $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles. Assume that $m \geq k$. Then $\mathrm{diag}(u, v_1, I_{m-k})$ is homotopic to I_{n+m} as $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles. Let $(U_t)_{t \in [0,1]}$ be a homotopy of $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles between $\mathrm{diag}(u, v_0)$ and $\mathrm{diag}(u, v_1, I_{m-k})$. We may as-

sume that $\pi(U_t) = I_{n+m}$ for all t. Then we may regard $U = (U_t)$ as an $(\varepsilon, 2r, 2(N + \varepsilon))$-invertible in $C([0,1], M_{n+m}(\tilde{A}_{2r}))$ with $\pi(U) = I_{n+m}$. Moreover, U is homotopic to 1 as $(\varepsilon, 2r, 2(N+\varepsilon))$-invertibles. By Lemma 3.4.8, there exist $l \in \mathbb{N}$ and invertibles $W_1, W_2 \in C([0,1], M_{n+m+l}(\tilde{A}))$ with $||\text{diag}(U, I_l) - W_1W_2|| < \lambda_N \varepsilon$. We obtain a quasi-idempotent $E \in C([0,1], M_{n+m+l}(\tilde{A}))$ by Lemma 3.4.11. Using v_0 in the definition of ∂ yields $[E_0] - [I_n]$ while using v_1 in the definition yields $[E_1] - [I_n]$, but $[E_0] - [I_n] = [E_1] - [I_n]$.

5. Suppose that $[u_0] = [u_1]$ in $K_1^{\varepsilon,r,N}(A)$. Then up to stabilization, we may assume that u_0 and u_1 are homotopic as $(4\varepsilon, 2r, 4N)$-invertibles in $M_n(\tilde{A})$. Let $(u_t)_{t\in[0,1]}$ be such a homotopy. We may assume that $\pi(u_t) = I_n$ for all t. Then we may regard $u = (u_t)$ as a $(4\varepsilon, 2r, 4N)$-invertible in $C([0,1], M_n(\tilde{A}_{2r}))$. Let u' be a $(4\varepsilon, 2r, 4N)$-inverse for u with $\pi(u') = I_n$. Up to stabilization, there exist quasi-invertible elements w_i in $C([0,1], M_n(\widetilde{A_{\Delta_i}}))$ such that $||\text{diag}(u, u') - w_1w_2|| < \lambda_{4N}\varepsilon$. Then there exists a $(\lambda'''_N\varepsilon, h'''_{\varepsilon,N}r, \lambda'''_N)$-idempotent $e \in C([0,1], M_n(\widetilde{A_{\Delta_1,h'''_{\varepsilon,N}r}} \cap \widetilde{A_{\Delta_2,h'''_{\varepsilon,N}r}}))$ such that

$$\max(||e - w'_1\text{diag}(I_n, 0)w_1||, ||e - w_2\text{diag}(I_n, 0)w'_2||) < \lambda'''_N\varepsilon.$$

Now $(e_t)_{t\in[0,1]}$ is a homotopy of $(\lambda'''_N\varepsilon, h'''_{\varepsilon,N}r, \lambda'''_N)$-idempotents, and $\partial([u_0]) = [e_0] - [I_n] = [e_1] - [I_n] = \partial([u_1])$.

Consider the composition

$$K_1^{\varepsilon,r,N}(A_{\Delta_1}) \oplus K_1^{\varepsilon,r,N}(A_{\Delta_2}) \xrightarrow{(j_{1*}-j_{2*})} K_1^{\varepsilon,r,N}(A) \xrightarrow{\partial} K_0^{\lambda^{\mathcal{D}}_N\varepsilon, h^{\mathcal{D}}_{\varepsilon,N}r, \lambda^{\mathcal{D}}_N}(A_{\Delta_1} \cap A_{\Delta_2}).$$

Suppose that $u \in M_n(\widetilde{A_{\Delta_1}})$ and $v \in M_n(\widetilde{A_{\Delta_2}})$ are (ε, r, N)-invertibles. In the preliminary definition of $\partial([u])$, we may take $w_1 = \text{diag}(u, u')$, where u' is an (ε, r, N)-inverse for u, and $w_2 = I_{2n}$. Then $\partial([u]) = [e_0] - [I_n]$, where e_0 is a $((\lambda' \cdot \lambda)_N\varepsilon, (h' \cdot h)_{\varepsilon,N}r, (\lambda' \cdot \lambda)_N)$-

idempotent in $M_{2n}(\tilde{A})$ such that $||e_0 - \mathrm{diag}(I_n, 0)|| < (\lambda' \cdot \lambda)_N \varepsilon$. Similarly, $\partial([v]) = [e_1] - [I_n]$, where e_1 is an $((\lambda' \cdot \lambda)_N \varepsilon, (h' \cdot h)_{\varepsilon,N} r, (\lambda' \cdot \lambda)_N)$-idempotent in $M_{2n}(\tilde{A})$ such that $||e_1 - \mathrm{diag}(I_n, 0)|| < (\lambda' \cdot \lambda)_N \varepsilon$. Now $||e_0 - e_1|| < 2(\lambda' \cdot \lambda)_N \varepsilon$ so by Lemma 3.1.19, e_0 and e_1 are homotopic as $(2(\lambda' \cdot \lambda)_N^2 \varepsilon, (h' \cdot h)_{\varepsilon,N} r, (\lambda' \cdot \lambda)_N)$-idempotents. Thus there exists a control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ depending only on the coercity c such that $\partial([u] - [v]) = 0$ in $K_0^{\lambda_N^{\mathcal{D}} \varepsilon, h_{\varepsilon,N}^{\mathcal{D}} r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$.

Finally, consider the composition

$$\begin{aligned} K_1^{\varepsilon,r,N}(A) \quad &\xrightarrow{\partial} \quad K_0^{\lambda_N^{\mathcal{D}} \varepsilon, h_{\varepsilon,N}^{\mathcal{D}} r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2}) \\ &\xrightarrow{(j_{1,2*}, j_{2,1*})} K_0^{\lambda_N^{\mathcal{D}} \varepsilon, h_{\varepsilon,N}^{\mathcal{D}} r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1}) \oplus K_0^{\lambda_N^{\mathcal{D}} \varepsilon, h_{\varepsilon,N}^{\mathcal{D}} r, \lambda_N^{\mathcal{D}}}(A_{\Delta_2}). \end{aligned}$$

When $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ is sufficiently large, the preliminary definition of the boundary map yields $j_{1,2*}(\partial([u])) = 0$ in $K_0^{\lambda_N^{\mathcal{D}} \varepsilon, h_{\varepsilon,N}^{\mathcal{D}} r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1})$ and $j_{2,1*}(\partial([u])) = 0$ in $K_0^{\lambda_N^{\mathcal{D}} \varepsilon, h_{\varepsilon,N}^{\mathcal{D}} r, \lambda_N^{\mathcal{D}}}(A_{\Delta_2})$ so that the composition is the zero map.

Now we give a formal definition of the boundary map in terms of a control pair $(\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ making all the above hold.

Definition 3.4.12. *Let A be a filtered SQ_p algebra, and let $(A_{\Delta_1}, A_{\Delta_2})$ be an (s, c)-controlled Mayer-Vietoris pair for A. Let (λ, h) be the control pair from Lemma 3.4.8, and let (λ', h') be the control pair from Lemma 3.4.11. Given $[u] \in K_1^{\varepsilon,r,N}(A)$, where $N \geq 1$, $0 < \varepsilon < \frac{1}{20\lambda_N^{\mathcal{D}}}$, $0 < r \leq \frac{s}{h_{\varepsilon,N}^{\mathcal{D}}}$, and $u \in GL_n^{\varepsilon,r,N}(\tilde{A})$ with $u - I_n \in M_n(A)$,*

1. *find $v \in GL_m^{\varepsilon,r,N}(\tilde{A})$ with $v - I_n \in M_n(A)$ such that* $\mathrm{diag}(u, v)$ *is homotopic to I_{n+m} as $(\varepsilon, 2r, 2(N + \varepsilon))$-invertibles,*

2. *let w_1, w_2 be $(\lambda_N \varepsilon, h_{\varepsilon,N} r, \lambda_N)$-invertibles in $M_{n+m}(\tilde{A})$ such that*

$$||\mathrm{diag}(u, v) - w_1 w_2|| < \lambda_N \varepsilon,$$

3. *let w_i' be a $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse for w_i, and*

4. *let $e \in M_{n+m}(\widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}})$ be such that*

$$\max(||e - w_1'\mathrm{diag}(I_n, 0)w_1||, ||e - w_2\mathrm{diag}(I_n, 0)w_2'||) < (\lambda' \cdot \lambda)_N\varepsilon$$

and $\mathrm{diag}(\pi(e), 0)$ is $((\lambda' \cdot \lambda)_N\varepsilon, (h' \cdot h)_{\varepsilon,N}r, (\lambda' \cdot \lambda)_N)$-homotopic to $\mathrm{diag}(I_n, 0)$ in $M_{2(n+m)}(\mathbb{C})$.

Define $\partial : K_1^{\varepsilon,r,N}(A) \to K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$ by $[u] \mapsto [e] - [I_n]$.

Proposition 3.4.13. *For every $c > 0$, there exists a control pair (λ, h) such that for any filtered SQ_p algebra A, any (s, c)-controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A, and $N \geq 1$, $0 < \varepsilon < \frac{1}{20\lambda_N}$, and $0 < r \leq \frac{s}{h_{\varepsilon,N}}$,*

1. *if $u \in M_n(\tilde{A})$ is an (ε, r, N)-invertible such that $\partial([u]) = [e] - [I_n] = 0$ in $K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$, then $j_{1*}(y_1) - j_{2*}(y_2) = [u]$ in $K_0^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A)$ for some $y_i \in K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_i})$;*

2. *if $[e]-[I_n] \in K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$ satisfies $[e]-[I_n] = 0$ in $K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_i})$ for $i = 1, 2$, then $\partial(y) = [e] - [I_n]$ for some $y \in K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A_{\Delta_1} \cap A_{\Delta_2})$.*

Proof. For (1.), suppose that $u \in M_n(\tilde{A})$ is an (ε, r, N)-invertible such that $\partial([u]) = [e] - [I_n] = 0$ in $K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$. Up to stabilization, we may assume that e is $(4\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, 4\lambda_N^{\mathcal{D}} + 1)$-homotopic to $\mathrm{diag}(I_n, 0)$ in $M_{2n}(\widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}})$. By Proposition 3.1.27 and Lemma 3.1.28, there exists a $(\lambda_N''\varepsilon, h_{\varepsilon,N}''r, \lambda_N'')$-inverse pair (v, v') in $M_{2n}(\widetilde{A_{\Delta_1}} \cap \widetilde{A_{\Delta_2}})$ such that

$$||e - v\mathrm{diag}(I_n, 0)v'|| < \lambda_N''\varepsilon.$$

Since $\max(||e - w_1'\mathrm{diag}(I_n, 0)w_1||, ||e - w_2\mathrm{diag}(I_n, 0)w_2'||) < \lambda_N^{\mathcal{D}}\varepsilon$, we have

$$||w_1'\mathrm{diag}(I_n, 0)w_1 - v\mathrm{diag}(I_n, 0)v'|| < (\lambda_N^{\mathcal{D}} + \lambda_N'')\varepsilon,$$
$$||w_2\mathrm{diag}(I_n, 0)w_2' - v\mathrm{diag}(I_n, 0)v'|| < (\lambda_N^{\mathcal{D}} + \lambda_N'')\varepsilon.$$

By successively relaxing control if necessary, there exists a control pair (λ''', h''') depending only on the coercity c such that

- $||\mathrm{diag}(I_n, 0)w_1v - w_1v\mathrm{diag}(I_n, 0)|| < \lambda_N'''\varepsilon$;
- $||v'w_2\mathrm{diag}(I_n, 0) - \mathrm{diag}(I_n, 0)v'w_2|| < \lambda_N'''\varepsilon$;
- $\max(||w_1v - \mathrm{diag}(u_1, v_1)||, ||v'w_2 - \mathrm{diag}(u_2, v_2)||) < \lambda_N'''\varepsilon$, where u_i, v_i are quasi-invertible elements in $M_n(\widetilde{A_{\Delta_i}})$ for $i = 1, 2$;
- $||\mathrm{diag}(u, u') - \mathrm{diag}(u_1u_2, v_1v_2)|| < \lambda_N'''\varepsilon$, where u' is an (ε, r, N)-inverse for u;
- $||u - u_1u_2|| < \lambda_N'''\varepsilon$;
- u and u_1u_2 are homotopic as $(\lambda_N'''\varepsilon, h_{\varepsilon,N}'''s, \lambda_N''')$-invertibles;
- $[u] = [u_1] + [u_2]$ in $K_1^{\lambda_N'''\varepsilon, h_{\varepsilon,N}'''r, \lambda_N'''}(A)$.

For (2.), suppose that $[e] - [I_n] \in K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_1} \cap A_{\Delta_2})$ satisfies $[e] - [I_n] = 0$ in $K_0^{\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, \lambda_N^{\mathcal{D}}}(A_{\Delta_i})$ for $i = 1, 2$. Up to stabilization, we may assume that e is $(4\lambda_N^{\mathcal{D}}\varepsilon, h_{\varepsilon,N}^{\mathcal{D}}r, 4\lambda_N^{\mathcal{D}} + 1)$-homotopic to $\mathrm{diag}(I_n, 0)$ in $M_{2n}(\widetilde{A_{\Delta_i}})$ for $i = 1, 2$. By successively relaxing control if necessary, there exists a control pair $(\lambda'', h'') \geq (\lambda^{\mathcal{D}}, h^{\mathcal{D}})$ depending only on the coercity c such that

- $\max(||e - w_1'\mathrm{diag}(I_n, 0)w_1||, ||e - w_2\mathrm{diag}(I_n, 0)w_2'||) < \lambda_N''\varepsilon$, where (w_i, w_i') are $(\lambda_N''\varepsilon, h_{\varepsilon,N}''s, \lambda_N'')$-inverse pairs in $M_{2n}(\widetilde{A_{\Delta_i}})$;

- $||w_1w_2\mathrm{diag}(I_n, 0) - \mathrm{diag}(I_n, 0)w_1w_2|| < 3(\lambda''_N)^3\varepsilon$;
- $||w_1w_2 - \mathrm{diag}(u, v)|| < 3(\lambda''_N)^3\varepsilon$, where u, v are $(\lambda''_N\varepsilon, h''_{\varepsilon,N}s, \lambda''_N)$-invertibles in $M_n(A)$;
- $\partial([u]) = [e] - [I_n]$ in $K_0^{\lambda''_N\varepsilon, h''_{\varepsilon,N}r, \lambda''_N}(A_{\Delta_1} \cap A_{\Delta_2})$.

□

We summarize the results of this section as follows:

Theorem 3.4.14. *For every $c > 0$, there exists a control pair (λ, h) such that for any filtered SQ_p algebra A and any (s, c)-controlled Mayer-Vietoris pair $(A_{\Delta_1}, A_{\Delta_2})$ for A, we have the following (λ, h)-exact sequence of order s:*

$$\begin{array}{ccccc}
\mathcal{K}_1(A_{\Delta_1} \cap A_{\Delta_2}) & \xrightarrow{(j_{1,2*},\, j_{2,1*})} & \mathcal{K}_1(A_{\Delta_1}) \oplus \mathcal{K}_1(A_{\Delta_2}) & \xrightarrow{j_{1*} - j_{2*}} & \mathcal{K}_1(A) \\
 & & & & \big\downarrow \partial \\
\mathcal{K}_0(A) & \xleftarrow[j_{1*} - j_{2*}]{} & \mathcal{K}_0(A_{\Delta_1}) \oplus \mathcal{K}_0(A_{\Delta_2}) & \xleftarrow[(j_{1,2*},\, j_{2,1*})]{} & \mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_2})
\end{array}$$

In view of the observations after Remark 3.4.6, one can also obtain the corresponding controlled exact sequence for the suspensions by considering the controlled Mayer-Vietoris pair $(SA_{\Delta_1}, SA_{\Delta_2})$ for SA.

3.5 Remarks on the C^*-algebra Case

Finally, we make some remarks about the case where A is a filtered C^*-algebra. In this case, both our definition of the quantitative K-theory groups and the definition in [26] are applicable, and we shall briefly explain (without detailed proofs) that the two definitions give us essentially the same information.

In [26], the quantitative K-theory groups are defined by equivalence relations similar to ours but in terms of quasi-projections and quasi-unitaries instead of quasi-idempotents and quasi-invertibles.

Definition 3.5.1. *Let A be a filtered C^*-algebra. Fix $0 < \varepsilon < \frac{1}{20}$ and $r > 0$.*

1. *An element $p \in A$ is called an (ε, r)-projection if $p \in A_r$, $p^* = p$, and $||p^2 - p|| < \varepsilon$.*

2. *If A is unital, then an element $u \in A$ is called an (ε, r)-unitary if $u \in A_r$ and $\max(||uu^* - 1||, ||u^*u - 1||) < \varepsilon$.*

It is straightforward to see that every (ε, r)-projection is an $(\varepsilon, r, 1+\varepsilon)$-idempotent, and every (ε, r)-unitary is an $(\varepsilon, r, 1+\varepsilon)$-invertible [26, Remark 1.4]. Writing $K'^{\varepsilon,r}_*(A)$ for the quantitative K-theory groups defined in [26], we thus have canonical homomorphisms $\phi_* : K'^{\varepsilon,r}_*(A) \to K^{\varepsilon,r,N}_*(A)$ for $0 < \varepsilon < \frac{1}{20}$, $r > 0$, and $N \geq 1 + \varepsilon$.

Given an (ε, r, N)-idempotent e in A, by considering

$$p_1 = ((2e^* - 1)(2e - 1) + 1)^{1/2} e((2e^* - 1)(2e - 1) + 1)^{-1/2},$$
$$p_2 = Q((2e^* - 1)(2e - 1))eR((2e^* - 1)(2e - 1)),$$
$$p_3 = \frac{p_2 + p_2^*}{2},$$

where $Q(t)$ and $R(t)$ are polynomials such that

$$||(t+1)^{1/2} - Q(t)|| < \frac{\varepsilon}{6N^5},$$
$$||(t+1)^{-1/2} - R(t)|| < \frac{\varepsilon}{6N^5}$$

on $[0, (2N+1)^2]$, one can show the existence of a control pair (λ, h) such that $\mathrm{diag}(e, 0)$ and $\mathrm{diag}(p_3, 0)$ are homotopic as $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-idempotents. Since p_3 is a quasi-projection, we get a homomorphism $K^{\varepsilon,r,N}_0(A) \to K'^{\lambda_N\varepsilon, h_{\varepsilon,N}r}_0(A)$, which is a controlled inverse for ϕ_0.

In the odd case, we have the following analog of polar decomposition:

Lemma 3.5.2. *[27, Lemma 2.4] There exists a control pair (λ, h) such that for any unital filtered C^*-algebra A and any (ε, r, N)-invertible $x \in A$, there exists a positive $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-invertible $y \in A$ and a $(\lambda_N\varepsilon, h_{\varepsilon,N}r)$-unitary $u \in A$ such that $|||x| - y|| < \lambda_N\varepsilon$ and $||x - uy|| < \lambda_N\varepsilon$. Moreover, we can choose u and y such that*

- *there exists a real polynomial Q with $Q(1) = 1$ such that $u = xQ(x^*x)$ and $y = x^*xQ(x^*x)$;*
- *y has a positive $(\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N)$-inverse;*
- *if x is homotopic to 1 as (ε, r, N)-invertibles, then u is homotopic to 1 as $(\lambda_N\varepsilon, h_{\varepsilon,N}r)$-unitaries.*

By applying Lemma 3.1.18 and using an appropriate polynomial approximation of $t \mapsto \exp(t \log y)$, one can find a control pair $(\lambda', h') \geq (\lambda, h)$ such that $x \sim uy \sim u$ as $(\lambda'_N\varepsilon, h'_{\varepsilon,N}r, \lambda'_N)$-invertibles. This gives us a controlled inverse for ϕ_1.

In summary, we have a controlled isomorphism between $\mathcal{K}'_*(A)$ and $\mathcal{K}_*(A)$ when A is a filtered C^*-algebra.

4. DYNAMIC ASYMPTOTIC DIMENSION AND THE L_p BAUM-CONNES CONJECTURE

4.1 Dynamic Asymptotic Dimension

Dynamic asymptotic dimension is a property of topological dynamical systems introduced by Guentner, Willett, and Yu in [15] for discrete groups acting by homeomorphisms on locally compact Hausdorff spaces. In this section, we recall the definition and some facts about this notion of dimension.

Definition 4.1.1. *[15] An action of a countable discrete group Γ on a locally compact Hausdorff space X has dynamic asymptotic dimension d if d is the smallest natural number with the following property: for any compact subset K of X and finite subset E of Γ, there are open subsets $U_0, \ldots, U_d$ of X that cover K such that for each $i \in \{0, \ldots, d\}$, the set*

$$\left\{ g \in \Gamma : \begin{array}{l} \text{there exist } x \in U_i \text{ and } g_n, \ldots, g_1 \in E \text{ such that } g = g_n \cdots g_1 \\ \text{and } g_k \cdots g_1 x \in U_i \text{ for all } k \in \{1, \ldots, n\} \end{array} \right\}$$

is finite.

Many interesting actions have finite dynamic asymptotic dimension. For example, it was shown in [15] that all free minimal $\mathbb{Z}$-actions on compact spaces (such as irrational rotation of the circle) have dynamic asymptotic dimension one, and that groups with finite asymptotic dimension act with finite dynamic asymptotic dimension on some compact space.

In this dissertation, we will restrict to actions on compact Hausdorff spaces. Moreover, for the Mayer-Vietoris argument that we will use later, it will be more convenient (and perhaps more natural) to use the language of groupoids instead of group actions. Indeed, in [15], dynamic asymptotic dimension is defined for locally compact Hausdorff étale groupoids. We will only consider the transformation groupoid $\Gamma \ltimes X$ associated to the action of Γ on X. We will denote the action of Γ on X by $\Gamma \curvearrowright X$. We also assume that Γ is equipped with a proper length function $l : \Gamma \to \mathbb{N}$ and the associated right invariant metric.

Definition 4.1.2. *The transformation groupoid* $\Gamma \ltimes X$ *associated to* $\Gamma \curvearrowright X$ *is*

$$\{(gx, g, x) : g \in \Gamma, x \in X\}$$

topologized such that the projection $\Gamma \ltimes X \to \Gamma \times X$ *onto the second and third factors is a homeomorphism, and equipped with the following additional structure:*

1. *A pair* $((hy, h, y), (gx, g, x))$ *of elements in* $\Gamma \ltimes X$ *is said to be composable if* $y = gx$. *In this case, their product is defined by*

$$(hgx, h, gx)(gx, g, x) = (hgx, hg, x).$$

2. *The inverse of an element* $(gx, g, x) \in \Gamma \ltimes X$ *is*

$$(gx, g, x)^{-1} = (x, g^{-1}, gx).$$

3. *The units of* $\Gamma \ltimes X$ *are the elements of the clopen subspace*

$$G^{(0)} = \{(x, e, x) : x \in X\},$$

where e *is the identity in* Γ. *We refer to* $G^{(0)}$ *as the unit space of* $\Gamma \ltimes X$.

Definition 4.1.3. *Let* $\Gamma \ltimes X$ *be the transformation groupoid associated to* $\Gamma \curvearrowright X$. *A* *subgroupoid of* $\Gamma \ltimes X$ *is a subset* $G \subset \Gamma \ltimes X$ *that is closed under composition, taking inverses, and units, i.e.,*

1. *If* (hgx, h, gx) *and* (gx, g, x) *are in* G*, then so is* (hgx, hg, x)*.*

2. *If* $(gx, g, x) \in G$*, then* $(gx, g, x)^{-1} \in G$*.*

3. *If* $(gx, g, x) \in G$*, then* $(x, e, x) \in G$ *and* $(gx, e, gx) \in G$*, where* e *is the identity in* Γ*.*

Such a subgroupoid is equipped with the subspace topology from $\Gamma \ltimes X$*.*

Note that if S is an open subset of $\Gamma \ltimes X$, then S generates an open subgroupoid of $\Gamma \ltimes X$ (cf. [15, Lemma 5.2]).

Definition 4.1.4. *Let* G *be an open subgroupoid of* $\Gamma \ltimes X$ *and let* $r \geq 0$*. The extension of* G *by* r*, denoted by* G^{+r}*, is the open subgroupoid of* $\Gamma \ltimes X$ *generated by*

$$G \cup \{(gx, g, x) : x \in G^{(0)}, l(g) \leq r\}.$$

Roughly speaking, one can think of G^{+r} as the subgroupoid generated by the "r-neighborhood" of G.

Lemma 4.1.5. *Let* G *be an open subgroupoid of* $\Gamma \ltimes X$*, and let* $r_1, r_2 \geq 0$*. Then* $(G^{+r_1})^{+r_2} \subset G^{+(r_1+r_2)}$*.*

Proof. It suffices to show that

$$\{(gx, g, x) : x \in (G^{+r_1})^{(0)}, l(g) \leq r_2\} \subset G^{+(r_1+r_2)}.$$

Pick such an element (gx, g, x). There exists $h \in \Gamma$ with $l(h) \leq r_1$ and $hx \in G^{(0)}$. Thus (gx, gh^{-1}, hx) and (hx, h, x) are in $G^{+(r_1+r_2)}$ so $(gx, g, x) = (gx, gh^{-1}, hx)(hx, h, x) \in G^{+(r_1+r_2)}$. □

Definition 4.1.6. *[15] An action $\Gamma \curvearrowright X$ has dynamic asymptotic dimension d if d is the smallest natural number with the following property: for every open relatively compact subset K of the transformation groupoid $\Gamma \ltimes X$, there are open subsets $U_0, \ldots, U_d$ of X covering*

$$\{x \in X : (gx, g, x) \in K \text{ or } (x, g, g^{-1}x) \in K \text{ for some } g \in \Gamma\}$$

such that for each i, the subgroupoid of $\Gamma \ltimes X$ generated by

$$\{(gx, g, x) \in K : x, gx \in U_i\}$$

is relatively compact.

One can check that this definition is equivalent to the earlier one [15, Lemma 5.4].

The main consequence of having finite dynamic asymptotic dimension that we will use is the following.

Lemma 4.1.7. *[14, Lemma 5.3] Suppose that $\Gamma \curvearrowright X$ has dynamic asymptotic dimension d. Then for any $r \geq 0$, there is an open cover $\{U_0, \ldots, U_d\}$ of X such that for each $i \in \{0, \ldots, d\}$, if G_i is the subgroupoid of $\Gamma \ltimes X$ generated by*

$$\left\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r} h \cdot U_i, l(g) \leq r\right\},$$

then G_i^{+r} is relatively compact.

We also note that when $1 \leq d < \infty$ and $r \geq 0$, if we set $W_0 = \bigcup_{i=0}^{d-1} U_i$ and $W_1 = U_d$ with $\{U_0, \ldots, U_d\}$ as in the lemma, and G_i is the subgroupoid generated by

$$\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r} h \cdot W_i, l(g) \leq r\},$$

then G_i^{+r} is also relatively compact.

In [14], the authors considered a model for the Baum-Connes assembly map for an action based on Yu's localization algebras [39] and Roe algebras. In the appendix of [14], the authors show that their model for the Baum-Connes assembly map agrees with the one stated in terms of Kasparov's KK-theory [1]. The main result in that paper is the following:

Theorem 4.1.8. *[14] Let a countable discrete group Γ act with finite dynamic asymptotic dimension on a compact Hausdorff space X. Then the Baum-Connes conjecture holds for Γ with coefficients in $C(X)$.*

As an application of our framework of quantitative K-theory, we will consider the L_p analog of the assembly map in [14], and show that it is an isomorphism under the assumption of finite dynamic asymptotic dimension. In fact, our proof is modeled after the proof in the C^*-algebraic setting in [14] with only minor adjustments.

4.2 An L_p Assembly Map

In this section, we will define an assembly map in terms of L_p versions of localization algebras and Roe algebras, where $p \in (1, \infty)$. In the case $p = 2$, we recover (a model of) the Baum-Connes assembly map for Γ with coefficients in $C(X)$ considered in [14].

Definition 4.2.1. *Let $s \geq 0$. The Rips complex of Γ at scale s, denoted $P_s(\Gamma)$, is the simplicial complex with vertex set Γ, and where a finite subset $E \subset \Gamma$ spans a simplex if and only if $d(g,h) \leq s$ for all $g, h \in E$.*

Points in $P_s(\Gamma)$ can be written as formal linear combinations $\sum_{g\in\Gamma} t_g g$, where $t_g \in [0,1]$ for each g and $\sum_{g\in\Gamma} t_g = 1$. We equip $P_s(\Gamma)$ with the ℓ_1 metric, i.e.,

$$d(\sum_{g\in\Gamma} t_g g, \sum_{g\in\Gamma} s_g g) = \sum_{g\in\Gamma} |t_g - s_g|.$$

The barycentric coordinates on $P_s(\Gamma)$ are the continuous functions

$$t_g : P_s(\Gamma) \to [0,1]$$

uniquely determined by the condition $z = \sum_{g\in\Gamma} t_g(z)g$ for all $z \in P_s(\Gamma)$.

By assumption of properness of the length function on Γ, one sees that $P_s(\Gamma)$ is finite-dimensional and locally compact. Also, the right translation action of Γ on itself extends to a right action of Γ on $P_s(\Gamma)$ by isometric simplicial automorphisms.

In the usual setting of the Baum-Connes conjecture, one considers Hilbert spaces and C^*-algebras encoding the large scale geometry of Γ and the topology of $P_s(\Gamma)$. We will replace these Hilbert spaces by L_p spaces, thereby obtaining L_p operator algebras instead of C^*-algebras. First, we recall some facts about L_p tensor products. Details can be found in [10, Chapter 7].

For $p \in [1,\infty)$, there is a tensor product of L_p spaces such that we have a canonical isometric isomorphism $L_p(X,\mu)\otimes L_p(Y,\nu) \cong L_p(X\times Y, \mu\times\nu)$, which identifies, for every $\xi \in L_p(X,\mu)$ and $\eta \in L_p(Y,\nu)$, the element $\xi\otimes\eta$ with the function $(x,y) \mapsto \xi(x)\eta(y)$ on $X \times Y$. Moreover, we have the following properties:

- Under the identification above, the linear span of all $\xi\otimes\eta$ is dense in $L_p(X\times Y, \mu\times\nu)$.
- $||\xi\otimes\eta||_p = ||\xi||_p||\eta||_p$ for all $\xi \in L_p(X,\mu)$ and $\eta \in L_p(Y,\nu)$.
- The tensor product is commutative and associative.
- If $a \in B(L_p(X_1,\mu_1), L_p(X_2,\mu_2))$ and $b \in B(L_p(Y_1,\nu_1), L_p(Y_2,\nu_2))$, then there exists a unique

$$c \in B(L_p(X_1\times Y_1, \mu_1\times\nu_1), L_p(X_2\times Y_2, \mu_2\times\nu_2))$$

such that under the identification above, $c(\xi \otimes \eta) = a(\xi)b(\eta)$ for all $\xi \in L_p(X_1, \mu_1)$ and $\eta \in L_p(Y_1, \nu_1)$. We will denote this operator by $a \otimes b$. Moreover,

$$||a \otimes b|| = ||a||||b||.$$

- The tensor product of operators is associative, bilinear, and satisfies

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2.$$

Definition 4.2.2. *For $s \geq 0$, define*

$$Z_s = \left\{ \sum_{g \in \Gamma} t_g g \in P_s(\Gamma) : t_g \in \mathbb{Q} \textit{ for all } g \in \Gamma \right\}.$$

Note that Z_s is a Γ-invariant, countable, dense subset of $P_s(\Gamma)$.

Define

$$E_s = \ell_p(Z_s) \otimes \ell_p(X) \otimes \ell_p \otimes \ell_p(\Gamma),$$

and equip E_s with the isometric Γ-action given by

$$u_g \cdot (\delta_z \otimes \delta_x \otimes \eta \otimes \delta_h) = \delta_{zg^{-1}} \otimes \delta_{gx} \otimes \eta \otimes \delta_{gh}$$

for $z \in Z_s$, $x \in X$, $\eta \in \ell_p$, and $g, h \in \Gamma$.

Note that we have a canonical isometric isomorphism

$$E_s = \ell_p(Z_s \times X, \ell_p \otimes \ell_p(\Gamma)).$$

Also note that if $s_0 \leq s$, then $P_{s_0}(\Gamma)$ identifies equivariantly and isometrically with a subcomplex of $P_s(\Gamma)$, and $Z_{s_0} \subset Z_s$. Hence we have a canonical equivariant isometric inclusion $E_{s_0} \subset E_s$.

We will write $\mathcal{K}_\Gamma$ for the algebra of compact operators on $\ell_p \otimes \ell_p(\Gamma) \cong \ell_p(\mathbb{N} \times \Gamma)$ equipped with the Γ-action induced by the tensor product of the trivial action on ℓ_p and the left regular representation on $\ell_p(\Gamma)$. We also equip the algebra $C(X) \otimes \mathcal{K}_\Gamma$ with the diagonal action of Γ. Note that the natural faithful representation of $C(X) \otimes \mathcal{K}_\Gamma$ on $\ell_p(X) \otimes \ell_p \otimes \ell_p(\Gamma)$ is covariant for the representation defined by tensoring the natural action on $\ell_p(X)$, the trivial representation on ℓ_p, and the regular representation on $\ell_p(\Gamma)$.

Now we can define the L_p operator algebras that will feature in our assembly map.

Definition 4.2.3. *Let T be a bounded linear operator on E_s, which we may think of as a $(Z_s \times Z_s)$-indexed matrix $T = (T_{y,z})$ with*

$$T_{y,z} \in B(\ell_p(X) \otimes \ell_p \otimes \ell_p(\Gamma))$$

for each $y, z \in Z_s$.

1. *T is Γ-invariant if $u_g T u_g^{-1} = T$ for all $g \in \Gamma$.*

2. *The Rips-propagation of T is $\sup\{d_{P_s(\Gamma)}(y, z) : T_{y,z} \neq 0\}$.*

3. *The Γ-propagation of T, denoted by $prop_\Gamma(T)$, is*

$$\sup\{d_\Gamma(g, h) : T_{y,z} \neq 0 \textit{ for some } y, z \in Z_s \textit{ with } t_g(y) \neq 0 \textit{ and } t_h(z) \neq 0\}.$$

4. *T is X-locally compact if $T_{y,z} \in C(X) \otimes \mathcal{K}_\Gamma$ for all $y, z \in Z_s$, and if for any compact subset $F \subset P_s(\Gamma)$, the set $\{(y, z) \in F \times F : T_{y,z} \neq 0\}$ is finite.*

Definition 4.2.4. *Let $\mathbb{C}[\Gamma \curvearrowright X; s]$ denote the algebra of all Γ-invariant, X-locally compact operators on E_s with finite Γ-propagation.*

Let $C^{*,p}(\Gamma \curvearrowright X; s)$ denote the closure of $\mathbb{C}[\Gamma \curvearrowright X; s]$ with respect to the operator norm on E_s. We will call $C^{*,p}(\Gamma \curvearrowright X; s)$ the (equivariant) L_p Roe algebra of $\Gamma \curvearrowright X$ at scale s.

We will always regard the algebras above as concretely represented on E_s, and we will often think of elements of $C^{*,p}(\Gamma \curvearrowright X; s)$ as matrices $(T_{y,z})_{y,z\in Z_s}$ with entries being continuous equivariant functions $T_{y,z} : X \to \mathcal{K}_\Gamma$ having additional properties.

Definition 4.2.5. *Let $\mathbb{C}_L[\Gamma \curvearrowright X; s]$ denote the algebra of all bounded, uniformly continuous functions $a : [0, \infty) \to \mathbb{C}[\Gamma \curvearrowright X; s]$ such that the Γ-propagation of $a(t)$ is uniformly finite as t varies, and such that the Rips-propagation of $a(t)$ tends to zero as $t \to \infty$.*

Let $C_L^{*,p}(\Gamma \curvearrowright X; s)$ denote the completion of $\mathbb{C}_L[\Gamma \curvearrowright X; s]$ with respect to the norm

$$||a|| := \sup_t ||a(t)||_{C^{*,p}(\Gamma\curvearrowright X;s)}.$$

We will call $C_L^{*,p}(\Gamma \curvearrowright X; s)$ the L_p localization algebra of $\Gamma \curvearrowright X$ at scale s.

We will regard these algebras as concretely represented on $L_p[0, \infty) \otimes E_s$. Elements of $C_L^{*,p}(\Gamma \curvearrowright X; s)$ can be regarded as bounded, uniformly continuous functions $a : [0, \infty) \to C^{*,p}(\Gamma \curvearrowright X; s)$ having additional properties.

Now consider the evaluation-at-zero homomorphism

$$\epsilon_0 : C_L^{*,p}(\Gamma \curvearrowright X; s) \to C^{*,p}(\Gamma \curvearrowright X; s),$$

which induces a homomorphism

$$\epsilon_0 : K_*(C_L^{*,p}(\Gamma \curvearrowright X; s)) \to K_*(C^{*,p}(\Gamma \curvearrowright X; s)).$$

If $s_0 \leq s$, then the equivariant isometric inclusion $E_{s_0} \subset E_s$ allows us to regard

$\mathbb{C}[\Gamma \curvearrowright X; s_0]$ as a subalgebra of $\mathbb{C}[\Gamma \curvearrowright X; s]$. We then regard $C^{*,p}(\Gamma \curvearrowright X; s_0)$ (resp. $C_L^{*,p}(\Gamma \curvearrowright X; s_0)$) as a subalgebra of $C^{*,p}(\Gamma \curvearrowright X; s)$ (resp. $C_L^{*,p}(\Gamma \curvearrowright X; s)$). Thus there are directed systems of inclusions of L_p operator algebras $(C^{*,p}(\Gamma \curvearrowright X; s))_{s \geq 0}$ and $(C_L^{*,p}(\Gamma \curvearrowright X; s))_{s \geq 0}$, and the evaluation-at-zero maps above are compatible with these inclusions.

Definition 4.2.6. *The L_p assembly map for $\Gamma \curvearrowright X$ is the direct limit*

$$\epsilon_0 : \lim_{s \to \infty} K_*(C_L^{*,p}(\Gamma \curvearrowright X; s)) \to \lim_{s \to \infty} K_*(C^{*,p}(\Gamma \curvearrowright X; s)).$$

For most of the rest of this dissertation, we will work with the kernel of this L_p assembly map.

Definition 4.2.7. *Let $C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$ be the subalgebra of $C_L^{*,p}(\Gamma \curvearrowright X; s)$ consisting of functions a such that $a(0) = 0$. We will call $C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$ the L_p obstruction algebra of $\Gamma \curvearrowright X$ at scale s.*

Lemma 4.2.8. *The L_p assembly map for $\Gamma \curvearrowright X$ is an isomorphism if and only if*

$$\lim_{s \to \infty} K_*(C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)) = 0.$$

Proof. We have a short exact sequence

$$0 \to C_{L,0}^{*,p}(\Gamma \curvearrowright X; s) \to C_L^{*,p}(\Gamma \curvearrowright X; s) \to C^{*,p}(\Gamma \curvearrowright X; s) \to 0,$$

which induces the usual six-term exact sequence in K-theory. The lemma then follows from continuity of K-theory under direct limits, and the preservation of exact sequences under direct limits of abelian groups. □

Our goal will be to show that if $\Gamma \curvearrowright X$ has finite dynamic asymptotic dimension, then $\lim_{s\to\infty} K_*(C^{*,p}_{L,0}(\Gamma \curvearrowright X; s)) = 0$, and thus the L_p assembly map is an isomorphism.

As indicated in the previous section, we will be working with the transformation groupoid $\Gamma \ltimes X$.

Definition 4.2.9. *Let $s \geq 0$, and let $P_s(\Gamma)$ be the associated Rips complex of Γ. The support of $z = \sum_{g\in\Gamma} t_g(z)g \in P_s(\Gamma)$ is the finite set*

$$supp(z) = \{g \in \Gamma : t_g(z) \neq 0\}.$$

The support of $T = (T_{y,z})_{y,z\in Z_s} \in C^{*,p}(\Gamma \curvearrowright X; s)$ is

$$supp(T) = \left\{ (gx, gh^{-1}, hx) \in \Gamma \ltimes X : \begin{array}{l} \text{there exist } y, z \in P_s(\Gamma) \text{ with } T_{y,z}(x) \neq 0, \\ g \in supp(y), \text{ and } h \in supp(z) \end{array} \right\}.$$

With this definition, one sees that

$$prop_\Gamma(T) = \sup\{l(gh^{-1}) : (gx, gh^{-1}, hx) \in supp(T) \text{ for some } x \in X\}.$$

Given two subsets $A, B \subset \Gamma \ltimes X$, we write AB for

$$\{ab : a \in A, b \in B, (a, b) \text{ is composable}\}.$$

With this notation, the following lemma says that supports of operators in $C^{*,p}(\Gamma \curvearrowright X; s)$ behave as expected under composition of operators.

Lemma 4.2.10. *Let $S, T \in C^{*,p}(\Gamma \curvearrowright X; s)$. Then $supp(ST) \subset supp(S)supp(T)$.*

Proof. Suppose that $(gx, gh^{-1}, hx) \in supp(ST)$. Then there are $y, z \in P_s(\Gamma)$ such that $(ST)_{y,z}(x) \neq 0$, $g \in supp(y)$, and $h \in supp(z)$. Thus there is $w \in P_s(\Gamma)$ such that $S_{y,w}(x) \neq 0$ and $T_{w,z}(x) \neq 0$. If $k \in supp(w)$, then $(gx, gk^{-1}, kx) \in supp(S)$ and $(kx, kh^{-1}, hx) \in supp(T)$, so $(gx, gh^{-1}, hx) = (gx, gk^{-1}, kx)(kx, kh^{-1}, hx) \in supp(S)supp(T)$. □

Subgroupoids of $\Gamma \ltimes X$ give rise to subalgebras of the Roe algebra, localization algebra, and obstruction algebra that we defined earlier.

Lemma 4.2.11. *Let G be an open subgroupoid of $\Gamma \ltimes X$. Define $\mathbb{C}[G; s]$ to be the subspace of $\mathbb{C}[\Gamma \curvearrowright X; s]$ consisting of all operators T with support contained in a compact subset of G. Then $\mathbb{C}[G; s]$ is a subalgebra of $\mathbb{C}[\Gamma \curvearrowright X; s]$.*

Proof. Given Lemma 4.2.10, it suffices to show that if A and B are two relatively compact subsets of G, then so is AB. To see this, first suppose that A and B are compact. Then any net in AB has a convergent subnet since nets in A and nets in B have this property, and so AB is compact. Now if A and B are relatively compact, then since $AB \subset \bar{A}\bar{B}$ and $\bar{A}\bar{B}$ is compact, it follows that AB is relatively compact. □

Definition 4.2.12. *Let G be an open subgroupoid of $\Gamma \ltimes X$. Let $\mathbb{C}_L[G; s]$ denote the subalgebra of $\mathbb{C}_L[\Gamma \curvearrowright X; s]$ consisting of functions a such that $\bigcup_{t\in[0,\infty)} supp(a(t))$ has compact closure in G.*

Let $\mathbb{C}_{L,0}[G; s]$ denote the ideal of $\mathbb{C}_L[G; s]$ consisting of functions a such that $a(0) = 0$.

Let $C^{*,p}(G; s)$, $C_L^{*,p}(G; s)$, and $C_{L,0}^{*,p}(G; s)$ denote the respective closures of $\mathbb{C}[G; s]$, $\mathbb{C}_L[G; s]$, and $\mathbb{C}_{L,0}[G; s]$ in $C^{*,p}(\Gamma \curvearrowright X; s)$, $C_L^{*,p}(\Gamma \curvearrowright X; s)$, and $C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$.

Since we will be working mostly with the obstruction algebras, we introduce the following shorthand notation for these algebras. We also need to construct filtrations on these algebras so as to apply quantitative K-theory to them later.

Definition 4.2.13. *Let G be an open subgroupoid of $\Gamma \ltimes X$, and let $s \geq 0$. Set $A^s(G)$ to* *be* $C^{*,p}_{L,0}(G;s)$*. For* $r \geq 0$*, define*

$$A^s(G)_r = \{a \in \mathbb{C}_{L,0}[G;s] : prop_\Gamma(a(t)) \leq r \textit{ for all } t\},$$

which is a linear subspace of $A^s(G)$*.*

When $G = \Gamma \ltimes X$*, we will simply write* A^s *and* A^s_r*.*

Lemma 4.2.14. *Let* G *be an open subgroupoid of* $\Gamma \ltimes X$*, and let* $s \geq 0$*. Then the family* $(A^s(G))_{r\geq 0}$ *of subspaces of* $A^s(G)$ *satisfies:*

1. *if* $r_1 \leq r_2$*, then* $A^s(G)_{r_1} \subset A^s(G)_{r_2}$*;*
2. $A^s(G)_{r_1} A^s(G)_{r_2} \subset A^s(G)_{r_1+r_2}$ *for all* $r_1, r_2 \geq 0$*;*
3. $\bigcup_{r\geq 0} A^s(G)_r$ *is dense in* $A^s(G)$*.*

Proof. Note that $a \in A^s(G)_r$ if and only if

- $a \in \mathbb{C}_{L,0}[G;s]$, and
- $l(g) \leq r$ whenever $(gx, g, x) \in supp(a(t))$ for some $t \geq 0$.

Properties (1) and (3) follow immediately.

For (2), if $a \in A^s(G)_{r_1}$, $b \in A^s(G)_{r_2}$, and $(gx, g, x) \in supp(a(t)b(t))$ for some t, then by Lemma 4.2.10, $(gx, g, x) = (gx, gh^{-1}, hx)(hx, h, x)$ for some $(gx, gh^{-1}, hx) \in supp(a(t))$ and $(hx, h, x) \in supp(b(t))$. Thus $l(g) \leq l(gh^{-1}) + l(h) \leq r_1 + r_2$ so $ab \in A^s(G)_{r_1+r_2}$. □

Lemma 4.2.15. *Let* G *be an open subgroupoid of* $\Gamma \ltimes X$ *and let* $r, s \geq 0$*. Then*

$$A^s(G) \cdot A^s_r \cup A^s_r \cdot A^s(G) \subset A^s(G^{+r}).$$

Proof. Follows from Lemma 4.2.10. □

4.3 L_p Baum-Connes Conjecture with Coefficients

In this section, we shall prove the following theorem.

Theorem 4.3.1. *Let a countable discrete group* Γ *act with finite dynamic asymptotic dimension on a compact Hausdorff space* X*. Then the* L_p *Baum-Connes conjecture holds for* Γ *with coefficients in* $C(X)$ *for* $p \in (1, \infty)$.

First, let us formulate the L_p Baum-Connes conjecture with coefficients. We define the L_p reduced crossed product as follows:

Let A be a closed subalgebra of $B(L_p(Z, \mu))$ for some measure space (Z, μ) and $p \in (1, \infty)$. Let Γ be a countable discrete group acting on A by isometric automorphisms. Set $A\Gamma$ to be the set of finite sums of the form $\sum_{g \in \Gamma} a_g g$ with $a_g \in A$ and with the product given by

$$\left(\sum_{g \in \Gamma} a_g g\right)\left(\sum_{h \in \Gamma} b_h h\right) = \sum_{g,h \in \Gamma} a_g \alpha_g(b_h) gh,$$

where α denotes the Γ-action on A. There is a natural faithful representation of $A\Gamma$ on $\ell_p(\Gamma, L_p(Z, \mu))$ given by

$$(a\xi)(h) = \alpha_{h^{-1}}(a)\xi(h),$$
$$(g\xi)(h) = \xi(g^{-1}h)$$

for $a \in A$, $g, h \in \Gamma$, and $\xi \in \ell_p(\Gamma, L_p(Z, \mu))$. We then define the L_p reduced crossed product $A \rtimes_{\lambda,p} \Gamma$ to be the operator norm closure of $A\Gamma$ in $B(\ell_p(\Gamma, L_p(Z, \mu)))$.

We formulate the L_p Baum-Connes conjecture with coefficients by considering the L_p reduced crossed product instead of the reduced crossed product C^*-algebra on the right-hand side of the assembly map.

Conjecture 4.3.2 (**L_p** Baum-Connes conjecture for **Γ** with coefficients in **A**). *The homomorphism*

$$\mu_p : K_*^\Gamma(\underline{E}\Gamma; A) \to K_*(A \rtimes_{\lambda,p} \Gamma)$$

is an isomorphism, where the left-hand side is the equivariant K-homology with coefficients in A of the classifying space $\underline{E}\Gamma$ for proper Γ-actions, and the right-hand side is the K-theory of the L_p reduced crossed product.

We will use a particular model for $\underline{E}\Gamma$, namely $\bigcup_{s\geq 0} P_s(\Gamma)$ equipped with the ℓ_1 metric (cf. [1, Section 2]), where $P_s(\Gamma)$ is the Rips complex of Γ at scale s. The Baum-Connes assembly map can then be thought of as a map

$$\mu_p : \lim_{s\to\infty} K_*^\Gamma(P_s(\Gamma); A) \to K_*(A \rtimes_{\lambda,p} \Gamma).$$

Just as in the C^*-algebra setting (see for example [14, Theorem A.3]), and with essentially the same proof, we have a commutative diagram

$$\begin{array}{ccc} K_*^\Gamma(P_s(\Gamma); A) & \xrightarrow{\quad\mu_p\quad} & K_*(A \rtimes_{\lambda,p} \Gamma) \\ \downarrow & & \downarrow \\ K_*(C_L^{*,p}(P_s(\Gamma); A)) & \xrightarrow{\quad\epsilon_0\quad} & K_*(C^{*,p}(P_s(\Gamma); A)) \end{array}$$

where the vertical maps are isomorphisms, which allows us to identify the L_p Baum-Connes assembly map μ_p with the evaluation-at-zero map

$$\epsilon_0 : \lim_{s\to\infty} K_*(C_L^{*,p}(P_s(\Gamma); A)) \to \lim_{s\to\infty} K_*(C^{*,p}(P_s(\Gamma); A)).$$

In the case where $A = C(X)$, this is the assembly map that we defined in Section 4.2.

Thus proving the L_p Baum-Connes conjecture for Γ with coefficients in $C(X)$ amounts to proving that the evaluation-at-zero map ϵ_0 induces an isomorphism, which is equivalent to proving that

$$\lim_{s\to\infty} K_*(C^{*,p}_{L,0}(\Gamma \curvearrowright X; s)) = 0$$

by Lemma 4.2.8. The proof that we present is modeled after the proof in the C^*-algebraic setting in [14], and the idea is as follows:

We want to show that for any $s_0 \geq 0$ and any $x \in K_*(C^{*,p}_{L,0}(\Gamma \curvearrowright X; s_0))$, there exists $s \geq s_0$ such that the map

$$K_*(C^{*,p}_{L,0}(\Gamma \curvearrowright X; s_0)) \to K_*(C^{*,p}_{L,0}(\Gamma \curvearrowright X; s))$$

induced by inclusion sends x to 0. By Proposition 3.2.20, we may pass over to the quantitative setting, so it suffices to show that the corresponding homomorphism between the quantitative K-theory groups is zero. Via an induction argument using a controlled Mayer-Vietoris sequence, it comes down to showing that if G is an open subgroupoid of $\Gamma \ltimes X$ such that $G \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\}$ for some $s \geq 0$, then $K_*(C^{*,p}_{L,0}(G; s)) = 0$. For that, roughly speaking, the point is that the assumption on G essentially makes the associated Rips complex contractible so homotopy invariance will give us the result.

4.3.1 Base case

Recall that we use the shorthand $A^s(G)$ for $C^{*,p}_{L,0}(G; s)$. As mentioned above, the base case of our induction argument involves proving the following.

Proposition 4.3.3. *Let G be an open subgroupoid of $\Gamma \ltimes X$ such that*

$$G \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\}$$

for some $s \geq 0$. Then $K_*(A^s(G)) = 0$.

Before getting into the proof of the proposition, we need to fix some terminology that is standard in the C^*-algebraic setting but perhaps less so when Hilbert spaces are replaced by other Banach spaces.

Definition 4.3.4. *Let E be a complex Banach space. We say that $T \in B(E)$ is a partial isometry if $||T|| \leq 1$ and there exists $S \in B(E)$ such that $||S|| \leq 1$, $TST = T$, and $STS = S$. We call such an S a generalized inverse of T.*

Remark 4.3.5.

1. *In [30, Section 6], Phillips considers spatial partial isometries on L_p spaces. Such spatial partial isometries are partial isometries in the sense of the preceding definition but the converse is not true.*

2. *If (Z, μ) is a σ-finite measure space, $p \in [1, \infty) \setminus \{2\}$, and $T \in B(L_p(Z, \mu))$ is an isometric (but not necessarily surjective) linear map, then it follows from Lamperti's theorem [21] that T is a partial isometry in the sense above, and one can find a generalized inverse S such that $ST = I$ (also see [30, Section 6]). Hereafter, we will denote such an S by $T^\dagger$.*

Definition 4.3.6. *If $A \subset B(L_p(\mu))$ is an L_p operator algebra, then we say that $b \in B(L_p(\mu))$ is a multiplier of A if $bA \subset A$ and $Ab \subset A$. We say that $b \in B(L_p(\mu))$ is an isometric multiplier of A if b is an isometry and both b and $b^\dagger$ are multipliers of A. The set of all multipliers of A is denoted by $M(A)$.*

Fixing G as above and $s \geq 0$, we make the following definitions:

- $P_s(G) = \{(z, x) \in P_s(\Gamma) \times X : (gx, g, x) \in G \text{ for all } g \in supp(z)\}$.

- $Z_G = (Z_s \times X) \cap P_s(G)$.

- $E_G = \ell_p(Z_G, \ell_p \otimes \ell_p(\Gamma)) = \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)$.

Note that E_G is a subspace of E_s. Moreover, the faithful representation of $C^{*,p}(G;s)$ on E_s restricts to a faithful representation on E_G. Thus we will consider $C^{*,p}(G;s)$ as faithfully represented on E_G, and $A^s(G) := C^{*,p}_{L,0}(G;s)$ as faithfully represented on $L_p([0,\infty), E_G)$.

Also, if $(z,x) \in P_s(G)$ and $supp(z) = \{g_1, \ldots, g_n\}$, then $\{e, g_1, \ldots, g_n\}$ also spans a simplex Δ in $P_s(\Gamma)$ such that $\Delta \times \{x\}$ is contained in $P_s(G)$. Hence the family of functions

$$H_r : P_s(G) \to P_s(G), (z,x) \mapsto ((1-r)z + re, x) \qquad (0 \leq r \leq 1)$$

defines a homotopy between the identity map on $P_s(G)$ and the projection onto the subset $\{(z,x) \in P_s(G) : z = e\}$, which we may identify with the unit space $G^{(0)}$.

In the definition of $\mathbb{C}[\Gamma \curvearrowright X; s]$, we may use $\mathcal{K}_\Gamma^\infty$, the algebra of compact operators on $(\bigoplus_{n=0}^\infty \ell_p \otimes \ell_p(\Gamma))_p \cong (\bigoplus_{n=0}^\infty \ell_p)_p \otimes \ell_p(\Gamma)$, thereby obtaining another L_p Roe algebra $C^{*,p}(\Gamma \curvearrowright X; \mathcal{K}_\Gamma^\infty; s)$. Moreover, fixing an isometric isomorphism $\phi : \ell_p \xrightarrow{\cong} (\bigoplus_{n=0}^\infty \ell_p)_p$ gives an isomorphism

$$C^{*,p}(\Gamma \curvearrowright X; s) \cong C^{*,p}(\Gamma \curvearrowright X; \mathcal{K}_\Gamma^\infty; s).$$

We also have the corresponding statements for the L_p localization algebras and obstruction algebras defined earlier.

For each n, define an isometry $u_{n,0} : \ell_p \to (\bigoplus_{n=0}^\infty \ell_p)_p$ by inclusion as the nth summand, and define $u_{n,0}^\dagger : (\bigoplus_{n=0}^\infty \ell_p)_p \to \ell_p$ by projection onto the nth summand. Then $u_{n,0}^\dagger u_{n,0} = I_{\ell_p}$ for all n, and $u_{n,0}^\dagger u_{m,0} = 0$ when $n \neq m$. Define $u_n : L_p([0,\infty), E_G) \to L_p([0,\infty), E_G^\infty)$ to be the operator induced by tensoring $u_{n,0}$ with the identity on the other factors, where $E_G^\infty = (\bigoplus_{n=0}^\infty E_G)_p$, and define $u_n^\dagger$ similarly using $u_{n,0}^\dagger$. Then $u_n^\dagger u_n = I$ for all n, and $u_n^\dagger u_m = 0$ when $n \neq m$.

Given $a \in B(\ell_p)$, consider $a^\infty = a \oplus a \oplus \cdots \in B((\bigoplus_{n=0}^\infty \ell_p)_p)$. Then $\mu(a) = a^\infty$ is an isometric homomorphism $B(\ell_p) \to B((\bigoplus_{n=0}^\infty \ell_p)_p)$. We may also consider the isometry $v \in B((\bigoplus_{n=0}^\infty \ell_p)_p)$ given by the right shift taking the nth summand onto the $(n+1)$st summand. Denote by $v^\dagger \in B((\bigoplus_{n=0}^\infty \ell_p)_p)$ the left shift. Then $v\mu(a)v^\dagger = 0 \oplus a \oplus a \oplus \cdots$ for all $a \in B(\ell_p)$. With $u_{0,0}$ as above, $u_{0,0}au_{0,0}^\dagger$ is given by $a \oplus 0 \oplus 0 \oplus \cdots$ for all $a \in B(\ell_p)$. Now $a \mapsto \mu^{+1}(a) := v\mu(a)v^\dagger$ and $a \mapsto \mu^0(a) := u_{0,0}au_{0,0}^\dagger$ are bounded homomorphisms $B(\ell_p) \to B((\bigoplus_{n=0}^\infty \ell_p)_p)$.

Now given $a \in B(L_p([0,\infty), E_G))$, consider the bounded linear operator $\mu(a) = a^\infty$ on $L_p([0,\infty), E_G^\infty)$. Proceeding similarly as above, we get bounded homomorphisms

$$\mu, \mu^{+1}, \mu^0 : B(L_p([0,\infty), E_G)) \to B(L_p([0,\infty), E_G^\infty)).$$

Moreover, each of them maps $A^s(G)$ into $A^s(G; \mathcal{K}_\Gamma^\infty)$, and $M(A^s(G))$ into $M(A^s(G; \mathcal{K}_\Gamma^\infty))$.

Lemma 4.3.7. *Let A be a unital Banach algebra, and let I be an ideal in A. If $u \in A$ is invertible, then $Ad_u(a) := uau^{-1}$ induces the identity on $K_*(I)$.*

Proof. Consider the double of A along I, i.e., $D = \{(a,b) \in A \oplus A : a - b \in I\}$. Then I is an ideal in D via the first coordinate inclusion ι, and $D/\iota(I) \cong A$ via the second coordinate projection. The diagonal inclusion $\delta : A \to D$ given by $a \mapsto (a,a)$ gives a splitting of the quotient homomorphism so $\iota_* : K_*(I) \to K_*(D)$ is injective.

Now $w = (u,u)$ is invertible in D, and $(a,b) \mapsto w(a,b)w^{-1}$ induces the identity in $K_*(D)$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} I & \xrightarrow{\quad \iota \quad} & D \\ {\scriptstyle a \mapsto uau^{-1}} \downarrow & & \downarrow {\scriptstyle (a,b) \mapsto w(a,b)w^{-1}} \\ I & \xrightarrow{\quad \iota \quad} & D \end{array}$$

It follows that $a \mapsto uau^{-1}$ induces the identity map on $K_*(I)$. $\square$

The following lemma is an L_p version of a fairly standard result in the K-theory of C^*-algebras (cf. [16, Lemma 4.6.2]).

Lemma 4.3.8. *Let $\alpha : A \to C$ be a bounded homomorphism of L_p operator algebras with $C \subset B(L_p(\mu))$, and let $v \in B(L_p(\mu))$ be an isometric multiplier of C. Then the map $a \mapsto v\alpha(a)v^\dagger$ is a bounded homomorphism from A to C, and induces the same map as α in K-theory.*

More generally, if $v \in B(L_p(\mu))$ is a partial isometry and a multiplier of C, w is a generalized inverse of v that is also a multiplier of C, and $\alpha(a)wv = \alpha(a) = wv\alpha(a)$ for all $a \in A$, then the map $Ad_{v,w}(\alpha(a)) := v\alpha(a)w$ is a bounded homomorphism from A to C, and induces the same map as α in K-theory.

Proof. We prove the more general statement. Consider the top-left corner inclusion $j : C \to M_2(C)$, which induces an isomorphism j_* on $K_*(C)$. Let $u = \begin{pmatrix} v & 1 - vw \\ 1 - wv & w \end{pmatrix}$, which is invertible in $M_2(M(C))$ with inverse $\begin{pmatrix} w & 1 - wv \\ 1 - vw & v \end{pmatrix}$. By the previous lemma, Ad_u induces the identity on $K_*(C)$. Since $\alpha(a)wv = \alpha(a) = wv\alpha(a)$ for all $a \in A$, we have $j \circ Ad_{v,w} \circ \alpha(a) = Ad_u \circ j \circ \alpha$ from which it follows that $(Ad_{v,w} \circ \alpha)_* = \alpha_*$. $\square$

Lemma 4.3.9. $K_*(M(A^s(G))) = 0$.

Proof. Note that $\mu, \mu^{+1} : M(A^s(G)) \to M(A^s(G; \mathcal{K}_\Gamma^\infty))$ induce the same map in K-theory by the previous lemma. Moreover, since $\mu^0(a)\mu^{+1}(a) = \mu^{+1}(a)\mu^0(a) = 0$ for all $a \in M(A^s(G))$ and $\mu = \mu^0 + \mu^{+1}$, the induced maps in K-theory satisfy $\mu_* = \mu^0_* + \mu^{+1}_* = \mu^0_* + \mu_*$. Hence $\mu^0_* = 0$. But μ^0 induces an isomorphism in K-theory so $K_*(M(A^s(G))) = 0$. $\square$

For each $z \in Z_s$ such that $(z,x) \in P_s(G)$ for some $x \in X$, let E_z be a copy of ℓ_p so that we have an isometric isomorphism $\ell_p \cong (\bigoplus_{z \in Z} E_z)_p$, and let $w_z : \ell_p \otimes \ell_p(\Gamma) \to \ell_p \otimes \ell_p(\Gamma)$ be an isometry with range $E_z \otimes \ell_p(\Gamma)$. For each $r \in \mathbb{Q} \cap [0,1]$, define

$$w(r) : \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma) \to \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)$$

by $\delta_{z,x} \otimes \eta \mapsto \delta_{(1-r)z+re,x} \otimes w_z \eta$, which is a well-defined isometry.

For $t \geq 0$ and $n \in \mathbb{N} \cup \{\infty\}$, define an isometry

$$v_n(t) : \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma) \to \ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)$$

in the following way:

1. for $t < n$, $v_n(t) = w(0)$ (in particular, $v_\infty(t) = w(0)$ for all t);

2. for $m \in [n, 2n) \cap \mathbb{N}$, $t \in [m, m+1)$, $(z,x) \in Z_G$ with $z \neq e$, and $\eta \in \ell_p \otimes \ell_p(\Gamma)$,

$$v_n(t)(\delta_{z,x} \otimes \eta) = \left(\left| \cos\left(\frac{\pi}{2}(t-m)\right) \right| w\left(\frac{m-n}{n}\right) + \left| \sin\left(\frac{\pi}{2}(t-m)\right) \right| w\left(\frac{m+1-n}{n}\right) \right) (\delta_{z,x} \otimes \eta);$$

3. for $t \in [n, 2n)$, $(e,x) \in Z_G$, and $\eta \in \ell_p \otimes \ell_p(\Gamma)$,

$$v_n(t)(\delta_{e,x} \otimes \eta) = \delta_{e,x} \otimes w_e \eta;$$

4. for $t \geq 2n$, $v_n(t) = w(1)$.

One can check that the map $[0,\infty) \to B(\ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)), t \mapsto v_n(t)$, is norm continuous for each n. Now define an isometry

$$v_n : L_p([0,\infty), E_G) \to L_p([0,\infty), E_G)$$

for each n by $(v_n\xi)(t) = v_n(t)(\xi(t))$ for $\xi \in L_p([0,\infty), E_G)$.

Let $a \in \mathbb{C}_{L,0}[G;s]$ and let $T = a(t)$ for some fixed $t \in [0,\infty)$. The matrix entries $(v_n(t)Tv_n(t)^\dagger)_{y,z}(x)$ of $v_n(t)Tv_n(t)^\dagger$ will be linear combinations of at most four terms of the form $w_z^\dagger T_{H_{r_1}(y),H_{r_2}(z)}(x)w_y^\dagger$, where $r_1, r_2 \in \mathbb{Q} \cap [0,1]$ and $|r_1 - r_2| < \frac{1}{m}$ whenever $t > \frac{2}{m}$. It follows that the Rips-propagation of $v_n(t)a(t)v_n(t)^\dagger$ is at most $prop_{Rips}a(t) + \min(1, \frac{2}{|t-1|})$, so $v_n a v_n^\dagger \in \mathbb{C}_{L,0}[G;s]$.

Also, the operators $S_t := v_{n+1}(t)v_n(t)^\dagger$ on $\ell_p(Z_G) \otimes \ell_p \otimes \ell_p(\Gamma)$ have matrix entries $(S_t)_{y,z}$ that act as constant functions $X \to B(\ell_p \otimes \ell_p(\Gamma))$, their Rips-propagation tends to zero as $t \to \infty$, and they have Γ-propagation at most s for all t. Hence $v_{n+1}v_n^\dagger$ is a multiplier of $A^s(G)$ for all n.

Lemma 4.3.10. *Let A be a unital Banach algebra, and let I be an ideal in A. Define the double of A along I to be $D = \{(a,b) \in A \oplus A : a - b \in I\}$. Assume that A has trivial K-theory. Then the inclusion $\iota : I \to D$ given by $a \mapsto (a,0)$ induces an isomorphism in K-theory, and the diagonal inclusion $\delta : I \to D$ given by $a \mapsto (a,a)$ induces the zero map on K-theory.*

Proof. Note that $\iota(I)$ is an ideal in D, and $D/\iota(I)$ is isomorphic to A via the second coordinate projection. Since $K_*(A) = 0$, it follows from the six-term exact sequence that ι induces an isomorphism in K-theory.

On the other hand, δ factors through the diagonal inclusion $A \to D, a \mapsto (a,a)$, so δ induces the zero map on K-theory since $K_*(A) = 0$. □

We shall apply the lemma in the case where $A = M(A^s(G))$ and $I = A^s(G)$ to prove the next proposition.

Proposition 4.3.11. *Let $v_n : L_p([0,\infty), E_G) \to L_p([0,\infty), E_G)$ be as defined above. Then the maps $a \mapsto v_0 a v_0^\dagger$ and $a \mapsto v_\infty a v_\infty^\dagger$ induce the same map $K_*(A^s(G)) \to K_*(A^s(G))$.*

Proof. Given $a \in A^s(G)$, define

$$\alpha(a) = (\bigoplus_{n=0}^{\infty} v_n a v_n^\dagger, \bigoplus_{n=0}^{\infty} v_\infty a v_\infty^\dagger) \in A^s(G; \mathcal{K}_\Gamma^\infty) \oplus A^s(G; \mathcal{K}_\Gamma^\infty).$$

Also define

$$\beta(a) = (\bigoplus_{n=0}^{\infty} v_{n+1} a v_{n+1}^\dagger, \bigoplus_{n=0}^{\infty} v_\infty a v_\infty^\dagger) \in A^s(G; \mathcal{K}_\Gamma^\infty) \oplus A^s(G; \mathcal{K}_\Gamma^\infty).$$

Let D be the double of $M(A^s(G; \mathcal{K}_\Gamma^\infty))$ along $A^s(G; \mathcal{K}_\Gamma^\infty)$, and let

$$C = \{(c, d) \in D : d = \bigoplus_{n=0}^{\infty} v_\infty a v_\infty^\dagger \text{ for some } a \in A^s(G)\},$$

which is a closed subalgebra of D. Moreover, α and β are bounded homomorphisms with image in C. Consider $w = (w_1, w_2)$, where $w_1 = \bigoplus_{n=0}^{\infty} v_{n+1} v_n^\dagger$ and $w_2 = \bigoplus_{n=0}^{\infty} v_\infty v_\infty^\dagger$. Note that $w_1 \in M(A^s(G; \mathcal{K}_\Gamma^\infty))$. We claim that w is a multiplier of C. Indeed, if $(c, d) \in C$, then $w_2 d = d w_2 = d$ so it suffices to show that $c w_1 - d$ and $w_1 c - d$ are in $A^s(G; \mathcal{K}_\Gamma^\infty)$. We will only consider $w_1 c - d$ since the other case is similar. Now $w_1 c - d = w_1(c - d) + (w_1 d - d)$ so it suffices to show that $w_1 d - d \in A^s(G; \mathcal{K}_\Gamma^\infty)$. But $w_1 d - d = (w_1 - w_2) d = \bigoplus_{n=0}^{\infty} (v_{n+1} v_n^\dagger - v_\infty v_\infty^\dagger) v_\infty a v_\infty^\dagger \in A^s(G; \mathcal{K}_\Gamma^\infty)$ since $v_n(t) = v_\infty(t)$ for each fixed t and all $n > t$. Similarly, $w^\dagger = (w_1^\dagger, w_2^\dagger)$ is a multiplier of C. Now $\beta(a) = w \alpha(a) w^\dagger$ for all $a \in A$. Moreover, $\alpha(a) w^\dagger w = \alpha(a) = w^\dagger w \alpha(a)$ for all $a \in A$ so α and β induce the same map $K_*(A^s(G)) \to K_*(C)$, and thus the same map $K_*(A^s(G)) \to K_*(D)$ upon composing with the map induced by the inclusion of C into D.

Let u be the isometric multiplier of $A^s(G;\mathcal{K}_\Gamma^\infty)$ induced by the right shift. Then (u,u) is a multipler of D, and conjugating $\beta(a)$ by (u,u) gives

$$\gamma(a) = \left(0 \oplus \bigoplus_{n=1}^{\infty} v_n a v_n^\dagger, 0 \oplus \bigoplus_{n=1}^{\infty} v_\infty a v_\infty^\dagger\right).$$

Thus β and γ induce the same map $K_*(A) \to K_*(D)$. On the other hand, the homomorphism $\delta : A^s(G) \to D$ given by

$$a \mapsto (v_\infty a v_\infty^\dagger \oplus 0 \oplus 0 \oplus \cdots, v_\infty a v_\infty^\dagger \oplus 0 \oplus 0 \oplus \cdots)$$

induces the zero map on K-theory by the previous lemma. Also, $\gamma(a)\delta(a) = \delta(a)\gamma(a) = 0$. Hence

$$\alpha_* = \beta_* = \gamma_* = \gamma_* + \delta_* = (\gamma + \delta)_* : K_*(A^s(G)) \to K_*(D).$$

Let $\psi_0, \psi_\infty : A^s(G) \to D$ be the homomorphisms defined by

$$\psi_0(a) = (v_0 a v_0^\dagger \oplus 0 \oplus 0 \oplus \cdots, 0),$$
$$\psi_\infty(a) = (v_\infty a v_\infty^\dagger \oplus 0 \oplus 0 \oplus \cdots, 0).$$

Also define $\zeta : A^s(G) \to D$ by

$$\zeta(a) = \left(0 \oplus \bigoplus_{n=1}^{\infty} v_n a v_n^\dagger, \bigoplus_{n=0}^{\infty} v_\infty a v_\infty^\dagger\right).$$

Note that $\zeta(a)\psi_0(a) = \psi_0(a)\zeta(a) = \zeta(a)\psi_\infty(a) = \psi_\infty(a)\zeta(a) = 0$ for all $a \in A^s(G)$. Also, $\psi_0 + \zeta = \alpha$ and $\psi_\infty + \zeta = \gamma + \delta$. Hence

$$(\psi_0)_* + \zeta_* = \alpha_* = (\gamma + \delta)_* = (\psi_\infty)_* + \zeta_*,$$

so ψ_0 and ψ_∞ induce the same maps on K-theory.

Finally, if $\iota : A^s(G) \to D$ is the inclusion into the first factor (where D is now regarded as the double of $M(A^s(G))$ along $A^s(G)$), then $\psi_i(a)$ is given by the composition

$$a \mapsto v_i a v_i^\dagger \overset{\iota}{\mapsto} (v_i a v_i^\dagger, 0) \mapsto (v_i a v_i^\dagger \oplus 0 \oplus 0 \oplus \cdots, 0),$$

and the last two maps induce isomorphisms in K-theory, so $a \mapsto v_0 a v_0^\dagger$ and $a \mapsto v_\infty a v_\infty^\dagger$ induce the same map in K-theory. □

Now it remains to show that $a \mapsto v_\infty a v_\infty^\dagger$ induces the identity map on $K_*(A^s(G))$ while $a \mapsto v_0 a v_0^\dagger$ induces the zero map on $K_*(A^s(G))$. This will complete the proof of Proposition 4.3.3.

Lemma 4.3.12. *The map $\phi_\infty : K_*(A^s(G)) \to K_*(A^s(G))$ induced by conjugation by v_∞ is the identity map.*

Proof. Since $v_\infty(t) = w(0)$ for all t, and $w(0)$ is an isometric multiplier of $A^s(G)$, ϕ_∞ induces the identity map in K-theory. □

Lemma 4.3.13. *The map $\phi_0 : K_*(A^s(G)) \to K_*(A^s(G))$ induced by conjugation by v_0 is the zero map.*

Proof. Let $G^{(0)}$ be the unit space of G, which is an open subgroupoid of $\Gamma \ltimes X$. We may then consider $A^s(G^{(0)})$. In fact, ϕ_0 factors through $K_*(A^s(G^{(0)}))$, i.e., we have a commutative diagram

$$\begin{array}{ccc} K_*(A^s(G)) & \xrightarrow{\phi_0} & K_*(A^s(G)) \\ & \searrow \quad \nearrow & \\ & K_*(A^s(G^{(0)})) & \end{array}$$

so it suffices to show that $K_*(A^s(G^{(0)})) = 0$.

For each $n \in \mathbb{N}$ and $a \in A^s(G^{(0)})$, define

$$a^{(n)}(t) = \begin{cases} a(t-n) & \text{for } t \geq n \\ 0 & \text{for } t < n \end{cases}.$$

Note that $a^{(n)} \in A^s(G^{(0)})$. Now define $\alpha : A^s(G^{(0)}) \to A^s(G^{(0)}; \mathcal{K}_\Gamma^\infty)$ by $a \mapsto \bigoplus_{n=0}^\infty a^{(n)}$. We also have the "top corner inclusion" $\iota : A^s(G^{(0)}) \to A^s(G^{(0)}; \mathcal{K}_\Gamma^\infty)$ given by $a \mapsto a \oplus 0 \oplus 0 \oplus \cdots$. Using uniform continuity of elements in $A^s(G^{(0)})$ together with arguments similar to those above, we see that $\alpha_* + \iota_* = \alpha_*$ so $\iota_* = 0$. But ι induces an isomorphism in K-theory so it follows that $K_*(A^s(G^{(0)})) = 0$. □

4.3.2 Inductive step

Given two open subgroupoids of $\Gamma \ltimes X$, we will consider associated subalgebras of $A^s := C_{L,0}^{*,p}(\Gamma \curvearrowright X; s)$. However, the filtrations we put on these subalgebras are not the induced filtrations from A^s.

Definition 4.3.14. *Fix $s_0 \geq 0$. Let G_0 and G_1 be open subgroupoids of $\Gamma \ltimes X$, and let $s \geq s_0$. For $r \geq 0$, define*

$$A_r = A^{s_0}(G_0^{+r})_r + A^{s_0}(G_1^{+r})_r + A^s(G_0^{+r} \cap G_1^{+r}),$$

and

$$A = \overline{\bigcup_{r \geq 0} A_r},$$

taking closure in the norm of A^s.

Also define

$$I_r = A^{s_0}(G_0^{+r})_r + A^s(G_0^{+r} \cap G_1^{+r}), \quad J_r = A^{s_0}(G_1^{+r})_r + A^s(G_0^{+r} \cap G_1^{+r}),$$

and

$$I = \overline{\bigcup_{r\geq 0} I_r}, \quad J = \overline{\bigcup_{r\geq 0} J_r}.$$

Lemma 4.3.15. *With notation as above, $(A_r)_{r\geq 0}$ is a filtration for A. Moreover, I and J are ideals in A.*

Proof. It is clear that $A_{r_0} \subset A_r$ if $r_0 \leq r$, and that $\bigcup_{r\geq 0} A_r$ is dense in A. By Lemmas 4.2.10 and 4.2.15, it follows that for $r_1, r_2 \geq 0$,

$$A^{s_0}(G_i^{+r_1})_{r_1} \cdot A^{s_0}(G_i^{+r_2})_{r_2} \subset A^{s_0}(G_i^{+(r_1+r_2)})_{r_1+r_2}$$

for $i = 0, 1$, while

$$A^{s_0}(G_0^{+r_1})_{r_1} \cdot A^{s_0}(G_1^{+r_2})_{r_2} \subset A^{s_0}(G_0^{+(r_1+r_2)})_{r_1+r_2} \cap A^{s_0}(G_1^{+(r_1+r_2)})_{r_1+r_2}.$$

Also,

$$A^s(G_0^{+r_1} \cap G_1^{+r_1}) \cdot A^s(G_0^{+r_2} \cap G_1^{+r_2}) \subset A^s(G_0^{+(r_1+r_2)} \cap G_1^{+(r_1+r_2)})$$

and

$$\begin{aligned} A^{s_0}(G_i^{+r_1})_{r_1} \cdot A^s(G_0^{+r_2} \cap G_1^{+r_2}) &\subset A^s((G_0^{+r_2} \cap G_1^{+r_2})^{+r_1}) \\ &\subset A^s((G_0^{+r_2})^{+r_1} \cap (G_1^{+r_2})^{+r_1}) \\ &\subset A^s(G_0^{+(r_1+r_2)} \cap G_1^{+(r_1+r_2)}). \end{aligned}$$

Hence $(A_r)_{r\geq 0}$ is a filtration for A, while I and J are ideals in A. □

Now we need to check that the ideals in Definition 4.3.14 satisfy the hypotheses for our controlled Mayer-Vietoris sequence. To do so, we shall make use of partitions of unity and associated multiplication operators.

Definition 4.3.16. *Let K be a compact subset of X, let $\{U_0, \ldots, U_d\}$ be a finite open cover of K, and let $\{\phi_0, \ldots, \phi_d\}$ be a subordinate partition of unity. Let $s \geq 0$. For $i \in \{0, \ldots, d\}$, let M_i be the multiplication operator on E_s associated to the function*

$$Z_s \times X \to [0,1], (z,x) \mapsto \sum_{g \in \Gamma} t_g(z)\phi_i(gx).$$

Lemma 4.3.17. *With notation as above, the operators M_i have the following properties:*

1. $||M_i|| \leq 1$.
2. *If $T \in C^{*,p}(\Gamma \curvearrowright X; s)$ satisfies*

$$\{x \in X : (gx, g, x) \in supp(T) \text{ for some } g \in \Gamma\} \subset K,$$

   then $T = T(M_0 + \cdots + M_d)$.
3. *For any $T \in C^{*,p}(\Gamma \curvearrowright X; s)$ with Γ-propagation at most r, and $i \in \{0, \ldots, d\}$, we have*

$$supp(TM_i) \subset \left\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq s} h \cdot U_i, l(g) \leq r\right\} \cap supp(T).$$

Proof. Each M_i is a multiplication operator associated to a function taking values in $[0,1]$ so it follows that $||M_i|| \leq 1$.

For $i \in \{0, \ldots, d\}$, $T \in C^{*,p}(\Gamma \curvearrowright X; s)$, $y, z \in P_s(\Gamma)$, and $x \in X$, we have

$$(TM_i)_{y,z}(x) = T_{y,z}(x) \cdot \sum_{h \in \Gamma} t_h(z)\phi_i(hx).$$

Hence

$$(T(M_0 + \cdots + M_d))_{y,z}(x) = T_{y,z}(x) \cdot \sum_{h \in \Gamma} t_h(z)(\phi_0(hx) + \cdots + \phi_d(hx)).$$

Suppose that $\{x \in X : (gx, g, x) \in supp(T) \text{ for some } g \in \Gamma\} \subset K$. If $T_{y,z}(x) \neq 0$, then $(gx, gh^{-1}, hx) \in supp(T)$ for all $g \in supp(y)$ and $h \in supp(z)$. In particular, $hx \in K$ for all $h \in supp(z)$, so

$$\sum_{h \in \Gamma} t_h(z)(\phi_0(hx) + \cdots + \phi_d(hx)) = \sum_{h \in \Gamma} t_h(z) = 1,$$

and this proves (2).

Suppose that $(gx, gk^{-1}, kx) \in supp(TM_i)$, where $T \in C^{*,p}(\Gamma \curvearrowright X; s)$ has Γ-propagation at most r. Then there exist $y, z \in P_s(\Gamma)$ with $g \in supp(y)$, $k \in supp(z)$, and $(TM_i)_{y,z}(x) \neq 0$. In particular, $T_{y,z}(x) \neq 0$, so $(gx, gk^{-1}, kx) \in supp(T)$ and $l(gk^{-1}) \leq r$. We also have $\sum_{h \in \Gamma} t_h(z)\phi_i(hx) \neq 0$, so there exists $h \in supp(z)$ with $\phi_i(hx) \neq 0$, and thus $hx \in U_i$. Since $h, k \in supp(z)$, and $z \in P_s(\Gamma)$, we have $l(kh^{-1}) \leq s$. Now $kx = (kh^{-1})hx \in kh^{-1} \cdot U_i$, and this proves (3). □

In Section 3.4, we have shown the existence of a controlled Mayer-Vietoris sequence under certain hypotheses. Here, we shall state the hypotheses in a slightly less general manner (by omitting certain parameters) that suffices for our application. On the other hand, we also give ourselves a bit more flexibility in terms of propagation control. One can check that the proofs carry over after adjusting the propagation parameter.

Definition 4.3.18. *Let A be a filtered L_p operator algebra with filtration $(A_r)_{r \geq 0}$. A controlled Mayer-Vietoris pair for A is a pair $(A_{\Delta_1}, A_{\Delta_2})$ of Banach subalgebras of A associated with a pair (Δ_1, Δ_2) of closed linear subspaces of A_s satisfying the following conditions:*

- *There exists* $\rho : [0,\infty) \to [0,\infty)$ *with* $\rho(r) \geq r$ *such that for any* $r \geq 0$, *any positive integer* n, *and any* $x \in M_n(A_r)$, *there exist* $x_1 \in M_n(\Delta_1 \cap A_{\rho(r)})$ *and* $x_2 \in M_n(\Delta_2 \cap A_{\rho(r)})$ *such that* $x = x_1 + x_2$ *and* $\max(||x_1||, ||x_2||) \leq ||x||$;

- A_{Δ_i} *has filtration* $(A_{\Delta_i} \cap A_r)_{r \geq 0}$, *and* A_{Δ_i} *contains* $\Delta_i + A_{4s}\Delta_i + \Delta_i A_{4s} + A_{4s}\Delta_i A_{4s}$;

- *For any* $r \geq 0$, *any* $\varepsilon > 0$, *any positive integer* n, *any* $x \in M_n(A_{\Delta_1,r})$ *and* $y \in M_n(A_{\Delta_2,r})$ *with* $||x - y|| < \varepsilon$, *there exists* $z \in M_n(A_{\Delta_1,\rho(r)} \cap A_{\Delta_2,\rho(r)})$ *such that* $\max(||z - x||, ||z - y||) < \varepsilon$, *where* ρ *is as above.*

Remark 4.3.19. *If* A_{Δ_i} *is a closed ideal in* A, *and we let* $\Delta_i = A_{\Delta_i}$, *then the second part of the second condition above is automatically satisfied.*

Theorem 4.3.20. *There exists a control pair* (λ, h) *such that for any filtered* L_p *algebra* A *and any controlled Mayer-Vietoris pair* $(A_{\Delta_1}, A_{\Delta_2})$ *for* A, *we have the following* (λ, h)-*exact sequences:*

$$\begin{array}{ccccc} \mathcal{K}_1(A_{\Delta_1} \cap A_{\Delta_2}) & \xrightarrow{(j_{1,2*}, j_{2,1*})} & \mathcal{K}_1(A_{\Delta_1}) \oplus \mathcal{K}_1(A_{\Delta_2}) & \xrightarrow{j_{1*} - j_{2*}} & \mathcal{K}_1(A) \\ & & & & \downarrow \partial \\ \mathcal{K}_0(A) & \xleftarrow[j_{1*} - j_{2*}]{} & \mathcal{K}_0(A_{\Delta_1}) \oplus \mathcal{K}_0(A_{\Delta_2}) & \xleftarrow[(j_{1,2*}, j_{2,1*})]{} & \mathcal{K}_0(A_{\Delta_1} \cap A_{\Delta_2}) \end{array}$$

$$\begin{array}{ccccc} \mathcal{K}_1(SA_{\Delta_1} \cap SA_{\Delta_2}) & \xrightarrow{(j_{1,2*}, j_{2,1*})} & \mathcal{K}_1(SA_{\Delta_1}) \oplus \mathcal{K}_1(SA_{\Delta_2}) & \xrightarrow{j_{1*} - j_{2*}} & \mathcal{K}_1(SA) \\ & & & & \downarrow \partial \\ \mathcal{K}_0(SA) & \xleftarrow[j_{1*} - j_{2*}]{} & \mathcal{K}_0(SA_{\Delta_1}) \oplus \mathcal{K}_0(SA_{\Delta_2}) & \xleftarrow[(j_{1,2*}, j_{2,1*})]{} & \mathcal{K}_0(SA_{\Delta_1} \cap SA_{\Delta_2}) \end{array}$$

where $j_{1,2}$, $j_{2,1}$, j_1, *and* j_2 *are the respective inclusion maps, and* $SA = C_0((0,1), A)$ *denotes the suspension of* A.

Lemma 4.3.21. *The pair (I, J) in Definition 4.3.14 is a controlled Mayer-Vietoris pair for A.*

Proof. Let U_i be the unit space of $G_i^{+r_0}$ for $i = 0, 1$. If $a \in A_{r_0}$, then

$$K := \overline{\{x \in X : (gx, g, x) \in supp(a(t)) \text{ for some } t \in [0, \infty), g \in \Gamma\}}$$

is a compact subset of $U_0 \cup U_1$. Let M_0, M_1 be the multiplication operators defined with respect to the compact set K, the open cover $\{U_0, U_1\}$, and some choice of subordinate partition of unity $\{\phi_0, \phi_1\}$. By the previous lemma, we have $a(t)(M_0 + M_1) = a(t)$ for all t. Moreover, $||a(t)M_i|| \leq ||a(t)||$ for $i = 0, 1$. It remains to show that $t \mapsto a(t)M_0$ is in I_r and $t \mapsto a(t)M_1$ is in J_r for some $r \geq r_0$ (that may depend on s_0 but not on s). We will focus on the case of M_0 since the other case is similar.

Write $a = b_0 + b_1 + c$ with $b_i \in A^{s_0}(G_i^{+r_0})_{r_0}$ and $c \in A^s(G_0^{+r_0} \cap G_1^{+r_0})$. By the previous lemma, we have $supp(b_0(t)M_0) \subset supp(b_0(t))$ and also $supp(c(t)M_0) \subset supp(c(t))$ so $t \mapsto b_0(t)M_0$ is in $A^{s_0}(G_0^{+r_0})_{r_0} \subset I_{r_0}$ and $t \mapsto c(t)M_0$ is in $A^s(G_0^{+r_0} \cap G_1^{+r_0}) \subset I_{r_0}$.

Now assume that $(gx, gh^{-1}, hx) \in supp(b_1(t)M_0)$ for some t. Then there exist $y, z \in P_s(\Gamma)$ such that $g \in supp(y), h \in supp(z)$, and $(b_1(t)M_0)_{y,z}(x) \neq 0$. In particular, $(b_1(t))_{y,z}(x) \neq 0$ so $y, z \in P_{s_0}(\Gamma)$ and $l(gh^{-1}) \leq r_0$. Also, $\sum_{k\in\Gamma} t_k(z)\phi_0(kx) \neq 0$ so there exists $k \in supp(z)$ such that $\phi_0(kx) \neq 0$, and thus kx is in U_0, the unit space of $G_0^{+r_0}$. Hence

$$\begin{aligned}(gx, gh^{-1}, hx) &= (gx, gk^{-1}, kx)(kx, kh^{-1}, hx) \\ &\in (G_0^{+r_0})^{+r_0} \cdot (G_0^{+r_0})^{+s_0} \subset G_0^{+(2r_0+s_0)}.\end{aligned}$$

Hence $t \mapsto b_1(t)M_0$ is in $A^{s_0}(G_0^{+(2r_0+s_0)})_{2r_0+s_0} \subset I_{2r_0+s_0}$, and so $t \mapsto a(t)M_0$ is in $I_{2r_0+s_0}$.

Next, suppose that $a_0 \in I_{r_0}$ and $a_1 \in J_{r_0}$ such that $||a_0 - a_1|| < \varepsilon$. Again, let U_i be the unit space of $G_i^{+r_0}$ for $i = 0, 1$. Consider

$$K_i := \overline{\{x \in X : (gx, g, x) \in supp(a_i(t)) \text{ for some } t \in [0, \infty), g \in \Gamma\}}$$

for $i = 0, 1$, and let $K = K_1 \cup K_2$, a compact subset of $U_0 \cup U_1$. Let M_0, M_1 be the multiplication operators defined with respect to the compact set K, the open cover $\{U_0, U_1\}$, and some choice of subordinate partition of unity $\{\phi_0, \phi_1\}$. Define $b(t) = a_0(t)M_1 + a_1(t)M_0$. Then $b \in I_{2r_0+s_0} \cap J_{2r_0+s_0}$, and since $a_i(t) = a_i(t)(M_0 + M_1)$ by the choice of K, we have $||a_i(t) - b(t)|| \le ||a_0(t) - a_1(t)|| < \varepsilon$ for $i = 0, 1$. □

Proposition 4.3.22. *Let λ and P be as in Theorem 4.3.20 and Proposition 3.2.21 respectively. Let $\Gamma \curvearrowright X$ be an action. Let $r_0, s_0 \ge 0$, $d \in \mathbb{N}$, and $N \ge 1$. Then there is $r \ge \max(r_0, s_0)$ that depends only on the action, r_0, s_0, d, and N, and there exists $0 < \varepsilon < \frac{1}{20}$ that depends only on N with the following property:*
Let G be an open subgroupoid of $\Gamma \ltimes X$ such that there are open subsets $U_0, \ldots, U_d$ of X with the following properties:

1. *the unit space $G^{(0)}$ equals $\bigcup_{i=0}^{d} U_i$;*

2. *for each $i \in \{0, \ldots, d\}$, if G_i is the subgroupoid of $\Gamma \ltimes X$ generated by*

$$\left\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \le r} h \cdot U_i, l(g) \le r\right\},$$

   then the expansion G_i^{+r} has compact closure.

Then for any $s \ge \max(r_0, s_0)$ with

$$\bigcup_{i=0}^{d} G_i^{+r} \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \le s\},$$

there exists $N' \geq N$ such that the inclusion map

$$K_*^{\varepsilon, r_0, N}(A^{s_0}(G)) \to K_*^{\frac{1}{20}, s, N'}(A^s(G^{+r}))$$

is the zero map.

Proof. First consider the case $d = 0$. By assumption, the subgroupoid G' of G generated by $\{(gx, g, x) \in \Gamma \ltimes X : x, gx \in G^{(0)}, l(g) \leq r_0\}$ has compact closure. Then

$$G \cap \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq r_0\} = G' \cap \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq r_0\}$$

so $A^{s_0}(G)_{r_0} = A^{s_0}(G')_{r_0}$, and therefore the natural map

$$K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G')) \to K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G))$$

is the identity map. On the other hand, $A^s(G')_s = A^s(G')$ for $s \geq \max(r_0, s_0)$. By Proposition 4.3.3, $K_*(A^s(G')) = 0$ so for any $x \in K_*^{\frac{1}{20P_N}, s, N}(A^s(G'))$, there exists $N' \geq N$ such that $x = 0$ in $K_*^{\frac{1}{20}, s, N'}(A^s(G'))$. We have a commutative diagram

$$\begin{array}{ccc} K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G)) & \longrightarrow & K_*^{\frac{1}{20}, s, N'}(A^s(G)) \\ =\uparrow & & \uparrow \\ K_*^{\frac{1}{20P_N}, r_0, N}(A^{s_0}(G')) & \longrightarrow & K_*^{\frac{1}{20}, s, N'}(A^s(G')) \end{array}$$

which gives us the conclusion in the case $d = 0$.

Now suppose the result holds for some $d \in \mathbb{N}$. We want to prove the statement given $r_0, s_0, d+1$, and N. We first consider the odd case. By Theorem 4.3.20, we have a control pair (λ, h) and a (λ, h)-controlled exact Mayer-Vietoris sequence. In particular, for every

$0 < \varepsilon < \frac{1}{20\lambda_N}$, $r \geq 0$, and $N \geq 1$, we have a well-defined controlled boundary map

$$\partial : K_1^{\varepsilon,r,N}(A) \to K_0^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(I \cap J)$$

such that if $x \in K_1^{\varepsilon,r,N}(A)$ and $\partial(x) = 0$, then there exist $y \in K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(I)$ and $z \in K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(J)$ with $x = y + z$ in $K_1^{\lambda_N\varepsilon, h_{\varepsilon,N}r, \lambda_N}(A)$. In the rest of the proof, we will write h_N for $h_{\frac{1}{20},N}$.

By the induction hypothesis, there exists $r_1 \geq \max(h_N r_0, s_0)$ such that the result holds with respect to d. We will show that $r = r_1 + h_N r_0 + r_0 + s_0$ works. Let G be an open subgroupoid of $\Gamma \ltimes X$, and let $U_0, \ldots, U_{d+1}$ be open subsets of X be as in the assumptions for the $d+1$ case. Set $W_0 = \bigcup_{i=0}^{d} U_i$ and $W_1 = U_{d+1}$ so that $G^{(0)} = W_0 \cup W_1$. For $i = 0, 1$, let $\mathcal{G}_i$ be the open subgroupoid of $\Gamma \ltimes X$ generated by

$$\left\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq s_0} h \cdot W_i, l(g) \leq r_0\right\}.$$

We claim that

$$A^{s_0}(G)_{r_0} \subset A^{s_0}(\mathcal{G}_0)_{r_0} + A^{s_0}(\mathcal{G}_1)_{r_0}. \tag{4.1}$$

Indeed, let $a \in A^{s_0}(G)_{r_0}$ so that

$$K := \overline{\{x \in X : (gx, g, x) \in supp(a(t)) \text{ for some } t \in [0, \infty)\}}$$

is a compact subset of $G^{(0)}$. Let M_0 and M_1 be the multiplication operators defined with respect to the compact set K, the open cover $\{W_0, W_1\}$, and some choice of subordinate partition of unity. Then $a(t) = a(t)(M_0 + M_1)$ for all t. Moreover,

$$supp(a(t)M_i) \subset \left\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq s_0} h \cdot W_i, l(g) \leq r_0\right\}$$

so $t \mapsto a(t)M_i$ is in $A^{s_0}(\mathcal{G}_i)_{r_0}$, thereby proving the claim.

Now, let $s \geq \max(r_0, s_0)$ with $\bigcup_{i=0}^{d+1} G_i^{+r} \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\}$. In particular, we have $s \geq r$. For $i \in \{0, \dots, d\}$, let

$$V_i = \bigcup_{l(h) \leq h_N r_0 + s_0} h \cdot U_i,$$

which is open in X. By the definition of $\mathcal{G}_0$, we see that the unit space of $\mathcal{G}_0^{+h_N r_0}$ is $\bigcup_{i=0}^{d} V_i$. Moreover, if H_i is the subgroupoid of $\Gamma \ltimes X$ generated by

$$\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r_1} h \cdot V_i, l(g) \leq r_1\},$$

then since $r \geq r_1 + h_N r_0 + s_0$, the assumptions on $U_0, \dots, U_{d+1}$ imply that each $H_i^{+r_1}$ has compact closure contained in $\{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\}$. By the induction hypothesis applied to $\mathcal{G}_0^{+h_N r_0}$, there exist $0 < \varepsilon_0 < \frac{1}{20}$ and $N_0 \geq N$ such that the map

$$K_1^{\varepsilon_0, s, N}(A^s(\mathcal{G}_0^{+h_N r_0})) \to K_1^{\frac{1}{20}, s, N_0}(A^s(\mathcal{G}_0^{+(h_N r_0 + r_1)})) \tag{4.2}$$

is the zero map.

Let $\varepsilon_1 = \frac{\varepsilon_0}{\lambda_N P_N^2}$ and let $x \in K_1^{\varepsilon_1, r_0, N}(A^{s_0}(G))$. Let A, I, and J be defined with respect to $\mathcal{G}_0$ and $\mathcal{G}_1$ as in Definition 4.3.14. We may regard x as an element in $K_1^{\varepsilon_1, r_0, N}(A)$ since $A^{s_0}(G)_{r_0} \subset A^{s_0}(\mathcal{G}_0)_{r_0} + A^{s_0}(\mathcal{G}_1)_{r_0} \subset A_{r_0}$. Then $\partial(x) \in K_0^{\lambda_N \varepsilon_1, h_N r_0, \lambda_N}(I \cap J)$. But $I \cap J \cap A_{h_N r_0}$ can be identified with $A^s(\mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0})$ so we may regard $\partial(x)$ as an element in $K_0^{\lambda_N \varepsilon_1, h_N r_0, \lambda_N}(A^s(\mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0}))$.

Since $r \geq \max(r_0, h_N r_0, s_0)$, if we let $\mathcal{G}_{1'}^{+r}$ be the subgroupoid generated by

$$\left\{(gx, g, x) \in \Gamma \ltimes X : x \in \bigcup_{l(h) \leq r} h \cdot W_1, l(g) \leq r\right\},$$

then we have

$$\mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0} \subset \mathcal{G}_1^{+h_N r_0} \subset \mathcal{G}_{1'}^{+r} \subset \{(gx, g, x) \in \Gamma \ltimes X : l(g) \leq s\}, \tag{4.3}$$

the final inclusion following from the assumptions on s and W_1. By Proposition 4.3.3, we have

$$K_*(A^s(\mathcal{G}_0^{+h_N r_0} \cap \mathcal{G}_1^{+h_N r_0})) = 0.$$

Thus there exists $N_1 \geq \lambda_N$ such that $\partial(x) = 0$ in $K_0^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(I \cap J)$. Now by controlled exactness, there exist $y \in K_1^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(I)$ and $z \in K_1^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(J)$ with $x = y + z$ in $K_1^{\lambda_N P_N \varepsilon_1, h_N r_0, N_1}(A)$.

Since $I_{h_N r_0} \subset A^s(\mathcal{G}_0^{+h_N r_0})_s$ and $J_{h_N r_0} \subset A^s(\mathcal{G}_1^{+h_N r_0})_s$ by the assumption on s and the observation in (4.3), we may regard y and z as elements in $K_1^{\lambda_N P_N \varepsilon_1, s, N_1}(A^s(\mathcal{G}_0^{+h_N r_0}))$ and $K_1^{\lambda_N P_N \varepsilon_1, s, N_1}(A^s(\mathcal{G}_1^{+h_N r_0}))$ respectively. Then there exists $N_2 \geq N_1$ such that $y = 0$ in $K_1^{\frac{1}{20}, s, N_2}(A^s(\mathcal{G}_0^{+(h_N r_0 + r_1)}))$ by (4.2). Since $K_*(A^s(\mathcal{G}_1^{+h_N r_0})) = 0$ by Proposition 4.3.3 and (4.3), there exists $N_3 \geq N_2$ such that $z = 0$ in $K_1^{\varepsilon_0, s, N_3}(A^s(\mathcal{G}_1^{+h_N r_0}))$.

Finally, $\mathcal{G}_0$ and $\mathcal{G}_1$ are contained in $G^{+(r_0+s_0)}$ so each $\mathcal{G}_i^{+(h_N r_0 + r_1)}$ is contained in G^{+r}. In particular, y and z are both 0 in $K_1^{\frac{1}{20}, s, N_3}(A^s(G^{+r}))$ so $x = 0$ in $K_1^{\frac{1}{20}, s, N_3}(A^s(G^{+r}))$. This concludes the proof for the odd case.

One can check that everything from (4.1) onwards holds after taking suspensions throughout and using the controlled boundary map

$$\partial : K_1^{\varepsilon, r, N}(SA) \to K_0^{\lambda_N \varepsilon, h_{\varepsilon, N} r, \lambda_N}(SI \cap SJ).$$

Thus we also get the result for the even case. □

4.3.3 Proof of main theorem

Theorem 4.3.23. *Suppose that $\Gamma \curvearrowright X$ has finite dynamic asymptotic dimension. Then*

$$\lim_{s \to \infty} K_*(C^{*,p}_{L,0}(\Gamma \curvearrowright X; s)) = 0.$$

Thus the L_p assembly map in Definition 4.2.6 is an isomorphism. In other words, the L_p Baum-Connes conjecture holds for Γ with coefficients in $C(X)$

Proof. As above, we use the shorthand A^s for $C^{*,p}_{L,0}(\Gamma \curvearrowright X; s)$. We need to show that for any $s_0 \geq 0$ and any $x \in K_*(A^{s_0})$, there is $s \geq s_0$ such that the map $K_*(A^{s_0}) \to K_*(A^s)$ induced by inclusion sends x to 0.

Consider the commutative diagram

$$\begin{array}{ccc} K_*^{\varepsilon,r,N}(A^{s_0}) & \longrightarrow & K_*^{\frac{1}{20},s,N'}(A^s) \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa} \\ K_*(A^{s_0}) & \longrightarrow & K_*(A^s) \end{array}$$

where the horizontal maps are induced by inclusion, and the vertical maps are the respective maps passing from quantitative K-theory to K-theory. By Proposition 3.2.20, for any $0 < \varepsilon < \frac{1}{20}$, there exist $r \geq 0$ and $N \geq 1$ such that x is in the image of $\kappa : K_*^{\varepsilon,r,N}(A^{s_0}) \to K_*(A^{s_0})$. In particular, we apply this to the ε given by Proposition 4.3.22. Then Proposition 4.3.22 and Lemma 4.1.7 imply that there exist $s \geq \max\{s_0, r\}$ and $N' \geq N$ such that the top horizontal map is zero. Hence the bottom horizontal map sends x to 0. □

Since the left-hand side of the assembly map can be shown to be independent of p, we have the following consequence, which gives a partial answer to [29, Problem 11.2].

Corollary 4.3.24. *Suppose that $\Gamma \curvearrowright X$ has finite dynamic asymptotic dimension. Then* *the K-theory of the L_p reduced crossed product $C(X) \rtimes_{\lambda,p} \Gamma$ does not depend on p for $p \in (1, \infty)$.*

Finally, we remark that the question of whether the K-theory of L_p reduced crossed products and L_p reduced group algebras depend on p remains open in general but there has been some recent work in this direction [18, 22].

5. SUMMARY AND CONCLUSIONS

In this dissertation, we have seen that the idea of quantitative K-theory can be transferred to the setting of L_p operator algebras (or Banach algebras with an appropriate matrix norm structure), and we have applied it to investigate the K-theory of L_p reduced crossed products. Here we give a broad outline of some future prospects.

5.1 Further Study

5.1.1 L_p operator algebras and their K-theory

It follows from my work that the K-theory of the L_p reduced crossed product does not depend on $p \in (1, \infty)$ under the assumption of finite dynamic asymptotic dimension. It will be interesting to find other conditions under which the K-theory of the L_p reduced crossed product does not depend on p, or to consider Morita equivalence of these algebras in the sense of [28]. The same questions apply to the L_p reduced group algebras. There are some recent results in this direction [18, 22] but these questions remain open in general. The importance of these questions, say in the case of the group algebras, is that sometimes it is easier to compute the K-theory of the L_p reduced group algebra for large p (cf. [18]) so if the K-theory of these algebras is independent of p, then we will get a computation of the K-theory of the reduced group C^*-algebra in particular. Perhaps related to these questions is the interaction between various analytic/geometric properties of groups and the structure or K-theory of the L_p reduced group algebras.

Also, one of the questions posed by Chris Phillips in [29] is the following: What is the K-theory of the L_p group algebra of a finitely generated free group, and more generally, is there a Pimsner-Voiculescu exact sequence for L_p reduced crossed products by finitely generated free groups? He also asks whether there are non-trivial idempotents in the L_p group algebra of free groups. One may also ask about the L_p Baum-Connes conjecture with coefficients for groups acting on trees (that is, whether it is equivalent to the L_p Baum-Connes conjecture with coefficients holding for all vertex stabilizers just like in the C^*-algebra setting).

5.1.2 Other applications of quantitative K-theory

Quantitative K-theory makes sense for any C^*-algebra with some geometric structure given by a length function. This kind of geometric structure is reminiscent of what one considers in geometric group theory. Indeed, any length function on a group will naturally give rise to a length function on the group C^*-algebra (and the crossed product C^*-algebra if the group acts on another C^*-algebra). Finitely generated C^*-algebras can also be given a natural length function, and some work has been done in extending notions from geometric group theory to the setting of finitely generated algebras [6]. It seems plausible that quantitative K-theory can be used to give an "algorithm" for computing the K-theory of finitely generated C^*-algebras (under some geometric assumption), and it will be interesting to see whether there are connections between the geometric properties introduced in [6] and K-theory of C^*-algebras.

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