Analysis of Nonlinear Muskingum Flood Routing

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Abstract: The three-parameter nonlinear Muskingum method for flood routing is analyzed. Analytical solutions for simplifying cases and approximate integral solutions for general cases are derived. Its accuracy depends mainly on the parameter k. Unlike the linear case, the weighting factor is much less significant. A comparison with the linear case using four sets of inflow-outflow data shows that the nonlinear method is less accurate than its linear counterpart. Also, the accuracy varies from one nonlinear version to another.

Introduction

Flood routing is required for proper management and design of many environmental and water resources projects. The most accurate theoretical approach to flood routing is the system of the St. Venant hydrodynamic equations. This is a nonlinear partial differential hyperbolic system that cannot in general be solved analytically. Numerical techniques such as finite differences or finite elements along with digital computers must be utilized for solution of the complete St. Venant system (8,12,17). However, many less complicated methods have been developed for flood routing problems and have been found satisfactory in many practical applications. One of the most frequently used methods is the Muskingum method, which was suggested by the U.S. Corps of Engineers for the study of the Muskingum River basin in Ohio (9). This is based on a spatially lumped, water mass balance equation along with an empirical storage-discharge relation. Mathematically, it is expressed as a first order differential equation which, depending on the form of the storage-discharge relation, can be linear or nonlinear. For the completeness of the problem, initial conditions must always be given.

There exists an extensive literature on the Muskingum method, the bulk of which is devoted to its linear version. By comparison, very limited research has been done on the nonlinear version. Gill (5) has concluded that the nonlinear method is superior to the linear Muskingum method, whereas Singh (16) reasoned that more research was needed. The purpose of this study is to analyze the effectiveness and accuracy of the nonlinear Muskingum method. To this end, the study is subdivided into four parts: (1) Analytical solutions; (2) approximate integral solutions; (3) nonlinear parameters; and (4) comparison between linear and nonlinear methods. First, analytical solutions are provided as far as possible. Since these solutions are based on certain simplifying assump-

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tions, their practical value may however be limited. Second, an improvement is given for the approximate integral solutions suggested by Singh (16). The technique is to first integrate and then solve the governing differential equation instead of the usual finite difference solution. The form of the integral solutions is very simple and amenable to desk computations. Third, the behavior of the various parameters involved is investigated and their relative importance to the accuracy of the solutions assessed. Finally, the results of the linear versus nonlinear Muskingum methods are compared and discussed. All of the solutions are verified with data from inflow-outflow hydrographs reported in the literature.

Muskingum Method

The Muskingum method consists of a water mass balance equation for a specified reach. The main idea is that the rate of change of the water storage within the reach must equal at any time the difference between the inflow and outflow discharges. This can be expressed in differential form as

$$\frac{dS}{dt} = I - Q; \quad S(0) = S_0; \quad I(0) = I_0; \quad Q(0) = Q_0 \qquad (1)$$

where S = the storage; I = the rate of inflow to the reach; Q = the rate of outflow from the reach; and t = the time. The subscript (0) indicates the initial condition (t = 0).

Since the rate of inflow is always given, Eq. 1 contains two unknown variables, the storage S and the outflow discharge Q. An additional equation is given by an empirical storage inflow-outflow relation, which can be in one of the following forms:

Linear (L):

$$S = k[\alpha I + (1 - \alpha)Q]; \qquad (2)$$

Nonlinear *I* (NLI):

$$S = k_1[\alpha_1 I^m + (1 - \alpha_1)Q^m]; \qquad (3)$$

Nonlinear *II* (NLII):

$$S = k_2[\alpha_2 I + (1 - \alpha_2)Q]^p; \qquad (4)$$

where k, k_1, k_2, α_1, α_2, m and p = parameters. In the linear case (Eq. 2), k represents the average reach travel time and is equal to the time difference between the centroids of the inflow-outflow hydrographs. Therefore, it can be written as $k = \Delta x/c$ where Δx = the length of the reach and c = the phase velocity of the flood wave (11). For all practical purposes, k can be assumed as a constant. The physical interpretation of the coefficients k_1 and k_2 is not clear since their dimensions are given as the product of a discharge quantity raised to a certain power $(1 - r)$ times the time dimension, i.e., $[M^3T^{-1}]^{1-r}$ $[T]$, where M, T = mass and time dimensions, respectively, and r = either m or p. The parameter α is a weighting coefficient of the inflow-outflow relative effects on the storage. For the linear case, the value of this parameter ranges between $0 \le \alpha \le 1/2$ and can be expressed as $\alpha = (1/2)[1 - (q/i_0 c\Delta x)]$ where q

= the mean discharge per unit width within the reach, and i_0 = the slope of the reach (11). For the nonlinear methods, α_1, α_2 do not have to be the same as the one of the linear method. Regarding the exponents m and p, theoretically they take the values of 0.60 or 0.67 depending on whether Manning's or Chezy's formula is used. However, it has been indicated that for natural nonrectangular channels, their value might be higher (2).

The bulk of the research on the Muskingum method has used Eqs. 1 and 2 (3,4,10,14,15). Eq. 3 does not appear to have been used perhaps due to its complexity and difficulty to estimate its parameters (7). Eq. 4 has been utilized but in a limited way (5). The linear case can be solved easily by any standard method applicable to linear systems. The nonlinear cases require numerical techniques for their solution, but analytical solutions are also feasible for special cases. In the following, emphasis will be given to the nonlinear equations but the linear equation will also be considered for purposes of comparison.

Analytical Solutions

Substitution and rearrangement of Eqs. 2–4 to Eq. 1 yields correspondingly

$$\text{(L):}\quad \frac{dQ}{dt} + \frac{1}{k(1-\alpha)}Q = \frac{1}{k(1-\alpha)}I - \frac{\alpha}{1-\alpha}\frac{dI}{dt} \quad (5)$$

$$\text{(NLI):}\quad \frac{dQ}{dt} + \frac{1}{k_1(1-\alpha_1)m}Q^{2-m} - \frac{I - k_1\alpha_1 m I^{m-1}\dfrac{dI}{dt}}{k_1(1-\alpha_1)m}Q^{1-m} = 0 \quad (6)$$

$$\text{(NLII):}\quad \frac{dD}{dt} + \frac{1}{k_2(1-\alpha_2)p}D^{2-p}$$

$$- \frac{I}{k_2(1-\alpha_2)p}D^{1-p} = 0; \quad D = \alpha_2 I + (1-\alpha_2)Q \quad (7)$$

It can be seen that Eqs. 6 and 7 are of identical form and can therefore be written as

$$\frac{dF}{dt} + AF^{2-n} + BF^{1-n} = 0 \quad (8)$$

where A and B are certain known functions. Generally, A can be taken either as a constant or as a function of time, i.e., $k_i = k_i(t)$, $i = 1$ or 2, while B as a rule is a time dependent function since it contains the rate of inflow $I(t)$. The solution of Eq. 5 will not be included in this study since it has been extensively investigated by many researchers in the past. A thorough analysis of the linear Muskingum method has been reported by Singh and McCann (13).

Solution of NLI (Eq. 6).—Let $Q = \Psi^\lambda$. Substitution in Eq. 6 and subsequent rearrangement yields

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$$\frac{d\Psi}{dt} + \frac{1}{\lambda k_1(1-\alpha_1)m}\Psi^{\lambda-\lambda m+1} - \frac{I - k_1\alpha_1 m I^{m-1}\dfrac{dI}{dt}}{\lambda k_1(1-\alpha_1)m}\Psi^{1-\lambda m} = 0 \quad (9)$$

Eq. 9 cannot be solved in its general form analytically. However, it can be reduced to a Riccati type equation if one of the following conditions is satisfied:

$$\begin{matrix} \lambda - \lambda m + 1 = 2 \\ 1 - \lambda m = 0 \end{matrix} \quad \lambda = 2; \quad m = \frac{1}{2}; \quad (10a)$$

$$\begin{matrix} \lambda - \lambda m + 1 = 0 \\ 1 - \lambda m = 2 \end{matrix} \quad \lambda = -2; \quad m = \frac{1}{2}; \quad (10b)$$

Therefore, a closed-form solution might be feasible only if the exponent m equals 0.5, which is close to the value of $m = 0.6$ obtained by Manning's formula. To reduce the computational effort, the pair $\lambda = 2$, $m = 1/2$ is selected so that Eq. 9 becomes

$$\frac{d\Psi}{dt} + \frac{1}{k_1(1-\alpha_1)}\Psi^2 = \frac{I - 0.5k_1\alpha_1 I^{-1/2}\dfrac{dI}{dt}}{k_1(1-\alpha_1)} \quad (11)$$

Again, an analytical solution is not available for an arbitrary function of the inflow hydrograph, and so a step-wise approach is applied. For a certain time step, $\Delta t = t_i - t_{i-1}$, Eq. 11 is written as

$$\frac{d\Psi}{dt} + \frac{1}{k_1(1-\alpha_1)}\Psi^2 = \frac{I_{i-1} + I_i}{2k_1(1-\alpha_1)} - \frac{\sqrt{2}}{2}\frac{\alpha_1}{1-\alpha_1}\frac{1}{(I_{i-1}+I_i)^{1/2}}\frac{(I_i - I_{i-1})}{\Delta t} = \text{constant} \quad (12)$$

The general solution of Eq. 12 is

$$\Psi = \Psi_p + \frac{1}{v(t)} \quad (13)$$

where Ψ_p = the particular solution; and $v(t)$ = a function which satisfies the following equation:

$$\frac{dv}{dt} = \frac{2\Psi_p}{k_1(1-\alpha_1)}v + \frac{1}{k_1(1-\alpha_1)} \quad (14)$$

The particular solution is easily obtained as

$$\Psi_p = -\left[\frac{I_{i-1} + I_i}{2} - \frac{k_1\alpha_1\sqrt{2}}{2\Delta t}\frac{(I_i - I_{i-1})}{(I_{i-1}+I_i)^{1/2}}\right]^{1/2} \quad (15)$$

The integrating factor for Eq. 14 is

$$\mu = \exp\int\frac{2}{k_1(1-\alpha_1)}\left[\frac{I_{i-1}+I_i}{2} - \frac{k_1\alpha_1\sqrt{2}}{2\Delta t}\frac{(I_i - I_{i-1})}{(I_{i-1}+I_i)^{1/2}}\right]dt = \exp(E_i t) \quad (16)$$

where $$E_i = \frac{1}{k_1(1-\alpha_1)}\left[2(I_{i-1}+I_i) - \frac{2\sqrt{2}\,k_1\alpha_1}{\Delta t}\frac{(I_i - I_{i-1})}{(I_{i-1}-I_i)^{1/2}}\right]^{1/2} \qquad (17)$$

Subsequently, Eq. 14 becomes

$$v \exp(E_i t)\Big|_{t_{i-1}}^{t_i} = \int_{t_{i-1}}^{t_i} \frac{1}{k_1(1-\alpha_1)} \exp(E_i t)\, dt \qquad (18)$$

Therefore, from Eq. 18

$$v_i = v_{i-1}\exp(-E_i\Delta t) + \frac{1}{k_1(1-\alpha_1)E_i}[1-\exp(-E_i\Delta t)] \qquad (19)$$

Combining Eqs. 13, 15, and 19, the rate of inflow for the time interval $t_{i-1} \le t \le t_i$ is given as

$$Q_i = \left(\frac{k_1(1-\alpha_1)}{2}E_i + \left\{v_{i-1}\exp(-E_i\Delta t) + \frac{1}{k_1(1-\alpha_1)E_i}[1-\exp(-E_i\Delta t)]\right\}^{-1}\right)^2 \qquad (20)$$

The initial value of the variable v_i is estimated from Eqs. 13 and 15 as

$$v_0 = \frac{1}{Q_0^{1/2} + I_0^{1/2}} \qquad (21)$$

At this point the justification of the negative sign in front of Eq. 15 is evident, since otherwise Eq. 21 should be written as $v_0 = (Q_0^{1/2} - I_0^{1/2})^{-1}$ and $v_0 \to \infty$ whenever $Q_0 \to I_0$.

For the trivial case where $I(t) = I_0 =$ constant, the solution for the outflow discharge can be easily obtained as

$$Q(t) = I_0\left\{1 + \left[-\frac{1}{2} + \left(\frac{1}{2} + \frac{I_0^{1/2}}{Q_0^{1/2} - I_0^{1/2}}\right)\exp\left(\frac{2I_0^{1/2}}{k_1(1-\alpha_1)}t\right)\right]^{-1}\right\}^2 \qquad (22)$$

Solution of NLII (Eq. 7).—Following the same solution procedure as with Eq. 6, the variable D of Eq. 7 for $t_{i-1} \le t \le t_i$ is given as

$$D_i = \left(-\left(\frac{I_{i-1}+I_i}{2}\right)^{1/2} + \left\{v_{i-1}\exp(-H_i\Delta t) + \frac{1}{k_2(1-\alpha_2)H_i}[1-\exp(-H_i\Delta t)]\right\}^{-1}\right)^2 \qquad (23)$$

where $$H_i = \frac{[2(I_{i-1}+I_i)]^{1/2}}{k_2(1-\alpha_2)} \qquad (24)$$

and v_i a function similar to that of Eq. 19, where H_i stands for E_i. Solving for the outflow discharge Q, it yields

$$Q_i = -\frac{\alpha_2}{1-\alpha_2} I_i + \frac{1}{1-\alpha_2}\left(-\frac{k_2(1-\alpha_2)}{2} H_i + \left\{v_{i-1}\exp(-H_i\Delta t) + \frac{1}{k_2(1-\alpha_2)H_i}[1-\exp(-H_i\Delta t)]\right\}^{-1}\right)^2 \quad (25)$$

The initial value of the function v_i is

$$v_0 = \frac{1}{[\alpha_2 I_0 + (1-\alpha_2)Q_0]^{1/2} + I_0^{1/2}} \quad (26)$$

Again, the solution for the trivial case $I(t) = I_0$ is easily derived as

$$Q(t) = -\frac{\alpha_2}{1-\alpha_2} I_0 + \frac{I_0}{1-\alpha_2}\left[1 + \left(-\frac{1}{2} + \left\{\frac{1}{2} + \frac{I_0^{1/2}}{[\alpha_2 I_0 + (1-\alpha_2)Q_0]^{1/2} - I_0^{1/2}}\right\} \exp\left[\frac{2I_0^{1/2}}{k_2(1-\alpha_2)}t\right]\right)^{-1}\right]^2 \quad (27)$$

General Solution of Three Parameter Nonlinear Muskingum Method (Eq. 8).—Eq. 8 can be written as

$$\frac{1}{n}\frac{dF^n}{dt} + AF + B = 0 \quad (28)$$

Setting $R = F^n$, after substitution and rearrangement Eq. 28 becomes

$$\frac{dR}{dt} = -nB\left(1 + \frac{A}{B}R^{1/n}\right) \quad (29)$$

The parameter B is a function of the inflow hydrograph $I(t)$. Utilizing a step-wise function for the inflow discharge, B is assumed constant within each time step, so Eq. 29 yields

$$\left(\frac{B_i}{A}\right)^n \frac{d\left[\left(\frac{A}{B_i}\right)^n R\right]}{dt} = -nB_i\left(1 + \frac{A}{B_i}R^{1/n}\right) \quad (30)$$

or $$\frac{dP_i}{dt} = nB_i\left(\frac{A}{B_i}\right)^n (1 - P_i^{1/n}) \quad (31)$$

where $P_i = -\left(\frac{A}{B_i}\right)^n R$

Integration within a specific time interval gives

$$\int_0^{P_i} \frac{dP_i}{1-P_i^{1/n}} = \int_{t_{i-1}}^{t_i} nB_i\left(\frac{A}{B_i}\right)^n dt + \int_0^{P_{i-1}} \frac{dP_{i-1}}{1-P_{i-1}^{1/n}} = nB_i\left(\frac{A}{B_i}\right)^n \Delta t + C_{i-1} \quad (32)$$

The integral

$$\int_0^P \frac{dP}{1-P^{1/n}}$$

TABLE 1.—Values of Linear Parameters k and α

Literature source (1)	k (days) (2)	α (3)	Method of parameter estimation (4)
Wilson (18)	1.500	0.250	Least squares
Linsley, et al. (7)	0.731	0.160	Direct optimization
Lawler (6)	1.839	0.296	Direct optimization
Viessman, et al. (19)	1.840	0.245	Least squares

can be recognized as Bakhmeteff's varied flow function and its value can be taken from Table 1, or can be integrated numerically.

From this analysis, it is clear that general analytical solutions for the Muskingum method are feasible only when the inflow hydrograph is represented by a step-wise function. This fact reduces the applicability of the solutions, since the required computations are too involved for hand calculations.

Approximate Solutions

Solution of Eqs. 1 and 3, (NLI).—Substitution of Eq. 3 into Eq. 1 and subsequent integration and rearrangement results in

$$k_1(1-\alpha_1)Q^m + \int_0^t Q dt = \int_0^t I dt - k_1\alpha_1 I^m + S_0 \quad (33)$$

where $S_0 = k_1[\alpha_1 I_0^m + (1-\alpha_1)Q_0^m]$ .... (34)

For a small time step Δt, Eq. 33 can be written as

$$k_1(1-\alpha_1)Q^m + \int_{t-\Delta t}^t Q dt = -\int_0^{t-\Delta t} Q dt + \int_0^t I dt - k_1\alpha_1 I^m + S_0 \quad (35)$$

Assuming a linear variation of Q within each time step, Eq. 35 can be further written as

$$k_1(1-\alpha_1)Q^m(t) + \frac{1}{2}Q(t)\Delta t = -\frac{1}{2}Q(t-\Delta t)$$

$$-\int_0^{t-\Delta t} Q(t)dt + \int_0^t I(t)dt - k_1\alpha_1 I^m(t) + S_0 \quad (36)$$

Since $Q(t-\Delta t)$ was calculated from the previous time step computations, the right-hand side of Eq. 36 is a known quantity and therefore from the same equation the value of Q at time t can be derived. In a finite difference discrete form, Eq. 36 becomes

$$k_1(1-\alpha_1)Q_i^m + \frac{1}{2}Q_i\Delta t = -\frac{1}{2}Q_{i-1}\Delta t - \left(\frac{1}{2}Q_0 + \sum_{j=1}^{i-2} Q_j + \frac{1}{2}Q_{i-1}\right)\Delta t$$

$$+ \left(\frac{1}{2}I_0 + \sum_{j=1}^{i-1} I_j + \frac{1}{2}I_i\right)\Delta t - k_1\alpha_1 I_i^m + S_0; \quad i = 1, 2, 3, \ldots \quad (37)$$

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In the present study, the roots of Eq. 37 are obtained by the regula-falsi algorithm. An additional approximation that reduces considerably the computational effort is to represent the second integral of the left-hand side of Eq. 35 as follows:

$$\int_{t-\Delta t}^{t} Qdt = Q_{i-1}\Delta t \quad \text{(38)}$$

This implies that $Q(t)$ is represented by a histogram or step function. By this way Eq. 35 can be solved in explicit form as

$$Q(t) = \left\{\frac{1}{k_1(1-\alpha_1)}\left[-Q(t-\Delta t)\Delta t - \int_0^{t-\Delta t} Q(t)dt + \int_0^t I(t)dt - k_1\alpha_1 I^m(t) + S_0\right]\right\}^{1/m} \quad \text{(39)}$$

Solution of Eqs. 1 and 4, (NLII).—Integration of Eq. 1 and substitution in Eq. 4 yields

$$k_2[\alpha_2 I + (1-\alpha_2)Q]^P = -\int_0^t Qdt + \int_0^t Idt + S_0 \quad \text{(40)}$$

where $S_0 = k_2[\alpha_2 I_0 + (1-\alpha_2)Q_0]^P$ ...... (41)

Again, under the assumptions of small time step Δt and linear variation of the outflow function Q within each time step, Eq. 40 is written as

$$k_2[\alpha_2 I(t) + (1-\alpha_2)Q(t)]^P + \frac{1}{2}Q(t)\Delta t = -\frac{1}{2}Q(t-\Delta t)\Delta t - \int_0^{t-\Delta t} Q(t)\Delta t + \int_0^t I(t)dt + S_0 \quad \text{(42)}$$

Therefore, having the values of the outflow function for the previous time steps, the present value of Q can be obtained implicitly from Eq. 42, which in discrete form reads as

$$k_2[\alpha_2 I_i + (1-\alpha_2)Q_i]^P + \frac{1}{2}Q_i\Delta t = -\frac{1}{2}Q_{i-1}\Delta t - \left(\frac{1}{2}Q_0 + \sum_{j=1}^{i-2} Q_j + \frac{1}{2}Q_{i-1}\right)\Delta t + \left(\frac{1}{2}I_0 + \sum_{j=1}^{i-1} I_j + \frac{1}{2}I_i\right)\Delta t + S_0 \quad \text{(43)}$$

An explicit form of Eq. 43 can be obtained by utilizing Eq. 38 so that after some rearrangment

$$Q(t) = \frac{1}{1-\alpha_2}\left\{\frac{1}{k_2}\left[-Q(t-\Delta t)\Delta t - \int_0^{t-\Delta t} Q(t)dt + \int_0^t I(t)dt + S_0\right]\right\}^{1/P} - \frac{\alpha_2}{1-\alpha_2}I(t) \quad \text{(44)}$$

Eqs. 39 and 44 are very convenient for a tabular form desk calculations. Solutions for the linear Muskingum method can be readily obtained from Eqs. 37 and 39 or Eqs. 43 and 44 when the exponents m or P are set equal to one, respectively.

Application Cases

For calibration, testing and comparison of the results of the nonlinear Muskingum method, four different inflow-outflow hydrographs were used. These were taken from the following sources: Wilson (18), Linsley, et al. (7), Lawler (6), and Viessman, et al. (19). Plots of these hdyrographs are given respectively in Figs. 1–4.

Estimation of Nonlinear Parameters by Equivalence with Linear Method.—The first step for employment of the Muskingum method is the determination of the parameters k and α. This can be done by calibration using existing data. For the linear case, the adjustment of the parameters can be performed effectively by various techniques (13), but the same is not true for the nonlinear case. Therefore, for the nonlinear cases as a first approximation, the parameters were derived from those of the linear case and then improved on properly. The linear parameters were estimated elsewhere (13,16) by four different techniques, i.e., least squares, method of moments, method of cumulants and direct optimization. The values of k and α that were found to describe more accurately the flood routing problem for each individual case are reported in Table 1. Since it is not feasible to estimate k_2 and α_2 in Eq. 4, for the sake of comparison for the two nonlinear methods, the estimation of k's is based on the linear data. By assuming that $\alpha_1 = \alpha_2 = \alpha$, the parameters k_1 and k_2 can be estimated from Eqs. 1–3 as follows:

$$k_1 = kI^{1-m}\left[\frac{\alpha + (1-\alpha)\dfrac{Q}{I}}{\alpha + (1-\alpha)\left(\dfrac{Q}{I}\right)^m}\right] = kI^{1-m}F_1(t) \quad (45)$$

$$k_2 = kI^{1-p}\left[\alpha + (1-\alpha)\frac{Q}{I}\right]^{1-p} = kI^{1-p}F_2(t) \quad (46)$$

As can be seen from Eqs. 45 and 46, k_1 and k_2 are functions of the dimensionless quantities $F_1(t)$ and $F_2(t)$ and of the inflow discharge raised to a certain power. It may be interesting to investigate the change of the nonlinear parameters in regard to those quantities. In Figs. 5 and 6 the parameters k_1 and k_2 as well as the quantities $F_1(t)$ and $F_2(t)$ are plotted for the sample data due to Wilson (18) for various values of the exponents m, p and the weighting coefficient α. The results for other data sets were similar and are not plotted for purposes of conserving space.

Since it was found that under similar conditions k_1 and k_2 are almost identical, only one plot of the parameter k_2 and function F_2 was included (Fig. 6). Therefore, for simplicity the conclusions will be drawn in general for the parameter k_1 and only when it is necessary the parameter k_2 will be mentioned. The parameter k_1 is strongly dependent on the ex-

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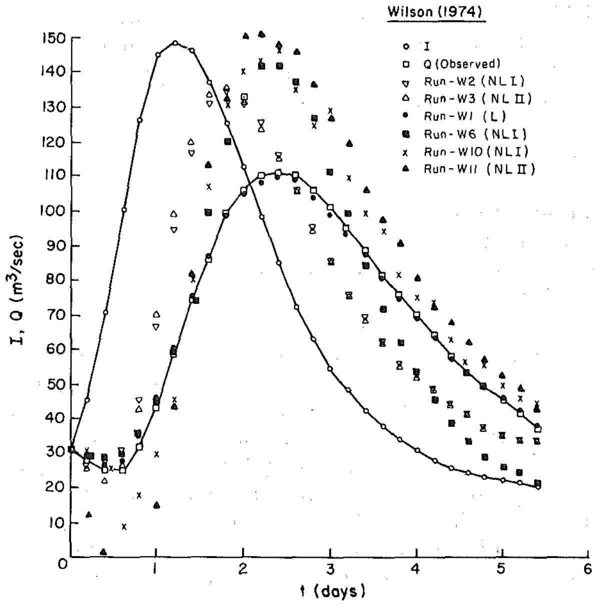

FIG. 1.—Inflow-Outflow Hydrographs for Wilson's Data

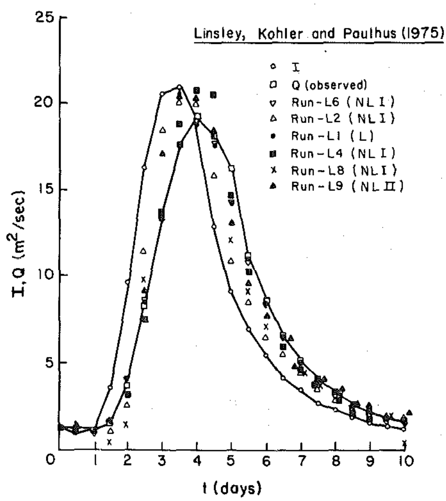

FIG. 2.—Inflow-Outflow Hydrographs for Linsley's, et al., Data

ponent m. The smaller the m is the more drastic the change of k_1 is. The maximum value k_{1max} is related to the exponent m (Eq. 45) according to the relation

$$k_{1max} = kI_{eq}^{1-m} \quad (47)$$

where I_{eq} = the value of inflow at the instance when the rates of inflow

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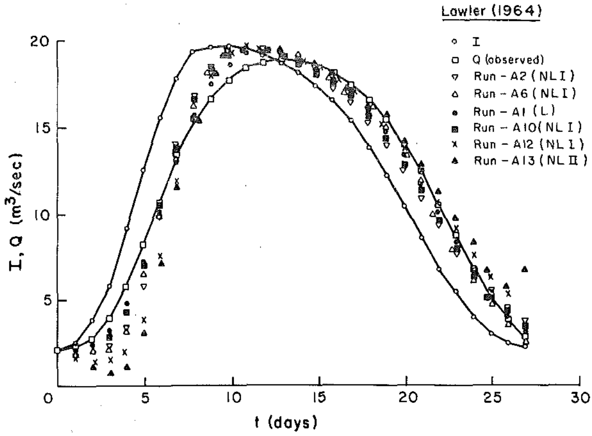

FIG. 3.—Inflow-Outflow Hydrographs for Lawler's Data

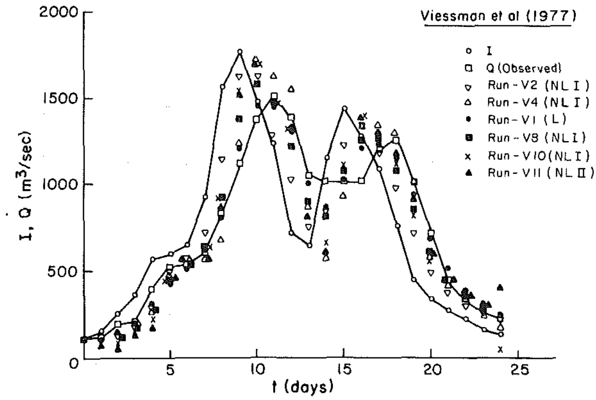

FIG. 4.—Inflow-Outflow Hydrographs for Viessman's, et al., Data

and outflow are equal. Eq. 47 is valid only when the coefficient α_1 is taken equal to the one derived for the linear method α. For higher values of the coefficient α_1, the $k_{1\max}$ is slightly higher than the previous one and the time of its occurrence is shifted to the left, which is reasonable since the infuence of the inflow hydrograph becomes stronger. The opposite is true when a value α_1, that is smaller than the linear coefficient α, is used. Generally, the change of the weighting coefficient α_1 does not produce a significant change in k_1. Eq. 47 also indicates that the deviation of the nonlinear parameter k_1 from the linear one is greater for higher rates of inflow discharge. Parameter k_1 is related also to the maximum rate of inflow as

$$k_{1\max} = k\beta I_{\max}^{\rho} \quad (48)$$

where β = a dimensional factor equal to unity; and ρ = an exponent. From the data the estimated values for ρ are given in Table 2. The av-

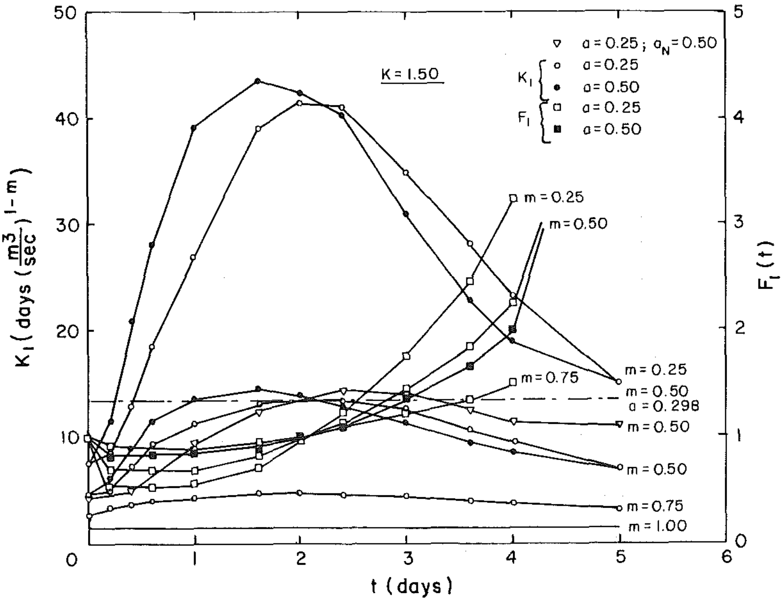

FIG. 5.—Nonlinear Parameter k_1 and Function $F_1(t)$ for Wilson's Data

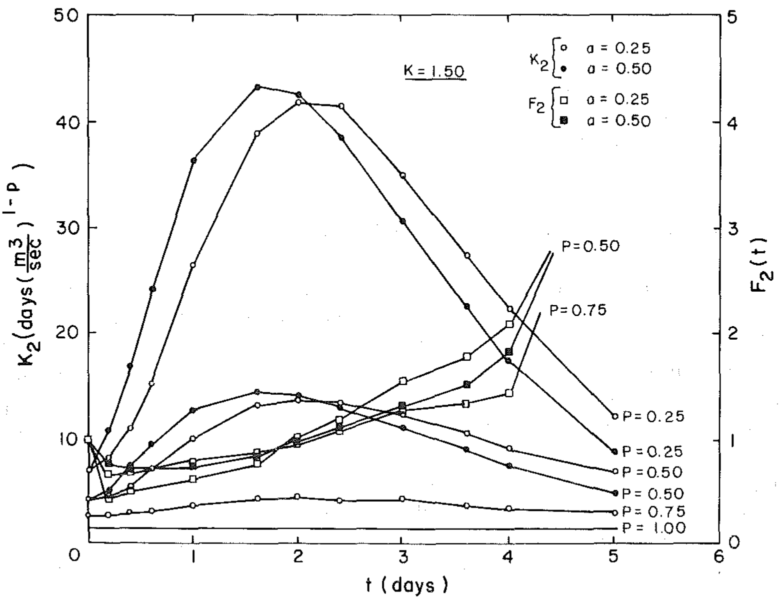

FIG. 6.—Nonlinear Parameter k_2 and Function $F_2(t)$ for Wilson's Data

TABLE 2.—Values of Exponent ρ

q / m (1)	Literature Source: Wilson (18), ρ (2)	Linsley, et al. (7), ρ (3)	Lawler (6), ρ (4)	Viessman, et al. (19), ρ (5)	ρ_{av} (6)
0.25	0.669	0.740	0.740	0.719	0.717
0.50	0.454	0.500	0.500	0.485	0.485
0.73	0.220	0.240	0.250	0.240	0.238
1.00	0	0	0	0	0

erage value ρ_{av} of the exponent ρ is estimated to be a linear relation with the exponent m as

$$\rho_{av} = 0.97(1 - m) \quad (49)$$

Combining Eqs. 48 and 49, the k_{1max} can be approximated as

$$k_{1max} = k\beta I_{max}^{0.97(1-m)} \quad (50)$$

Eq. 50 is more convenient than Eq. 47 because I_{eq} is not readily available so that it cannot be used for prediction purposes. The functions $F_1(t)$ and $F_2(t)$ are almost identical since they differ by less than 4%. Their mean value is around unity and they depend strongly on the base flow. The higher the base flow the more uniform the $F_{1,2}(t)$ functions are. This can be seen from Figs. 5 and 6 where these functions were calculated with zero base flow instead of 20 m^3/s as it really was. Functions $F_{1,2}(t)$ are quite sensitive also to the changes of the coefficient α.

Estimation of Nonlinear Parameters k_1 and α_1 by Least Squares Method.—For the nonlinear method NLI, the nonlinear parameter k_1 can be estimated independently from the linear parameters k and α by utilizing the least squares method. The results obtained by this approach for $m = 0.5$ are given in Table 3. The details of the least squares technique for the determination of the nonlinear parameters are given in Appendix I. Comparison of the values of k_1 estimated by the least squares method with those obtained from Eq. 45 shows that the former is between the mean and maximum values of the latter (Fig. 5). The nonlinear coefficient α_1 is always less than 0.5 and different from the linear one. The parameter k_2 can be estimated from Eqs. 45 and 46 for $p = m$ and $\alpha_2 = \alpha_1$ as

$$k_2 = k_1 \frac{\left[\alpha_1 + (1 - \alpha_1)\left(\dfrac{Q}{I}\right)^m\right]}{\left[\alpha_1 + (1 - \alpha_1)\left(\dfrac{Q}{I}\right)\right]^m} \quad (51)$$

Since the numerator is a truncated form of the denominator, k_2 is always less than or equal to k_1. However, they do not differ more than 4%. This is evident also by comparing the plots in Figs. 5 and 6.

Approximation of Nonlinear Parameters.—Summarizing, the parameter k_1 (or k_2) can be approximated as follows: First, by a constant value obtained by the least squares method. The approach is good only when

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TABLE 3.—Values of Nonlinear Parameters k_1 and α_1 (m = 0.5)

Literature source (1)	k_1 (days) (2)	α_1 (3)	Method of parameter estimation (4)
Wilson (18)	10.343	0.250	Least squares
Linsley, et al. (7)	2.018	0.160	Least squares
Lawler (6)	8.008	0.296	Least squares
Viessman, et al. (19)	42.364	0.245	Least squares

the estimated exponent is close to unity and the inflow hydrograph is a smooth one. Second, by assuming the function $F_{1,2}(t)$ as a constant and expressing the parameter k_1 as a function of the linear parameter k and the rate of inflow. The approximation is good only when there is a significant base flow so that $F_{1,2}(t) \cong 1.0$. And third, by utilizing Eqs. 45 or 46 where the functions $F_{1,2}(t)$ will be evaluated by the values of I and Q obtained at the immediate previous time step. This approximation essentially coincides with the linear method.

Sensitivity Analysis and Results

The performance of the nonlinear Muskingum method was investigated through a series of numerical simulations for various cases. The nonlinear parameters were estimated from the preceding discussion. Information about the input for the simulation cases is given in Table 4. In the same table information about the accuracy of the methods is provided by defining a goodness of fit criterion D as

$$D = \frac{Q_0 - Q_{comp}}{Q_0} \times 100 \quad (52)$$

where Q_0 = the observed discharge; and Q_{comp} = the simulated discharge. The results are plotted in Figs. 1–4. From these results, the following conclusions were drawn:

1. The nonlinear Muskingum method NLI is better than the nonlinear method NLII for all choices of the parameters $k_{1,2}$ and $\alpha_{1,2}$. More specifically, the latter tends to underestimate drastically the initial part of the rising stage of the outflow hydrograph. For the rest of the hydrograph, their accuracy is comparable.
2. The simulation by the nonlinear methods with constant parameters $k_{1,2}$ and $\alpha_{1,2}$ obtained by the least squares method (Appendix I) is not accurate for small values of the exponent m or p ($\cong$0.50). Indeed, the solution underestimates the outflow at the beginning and the end of the outflow hdyrograph, while it exaggerates the peak value of outflow.
3. A reasonable improvement of the previous approach can be seen when the exponent m or p is equal to 0.75. This is an indication that the assumption of constant $k_{1,2}$ for the nonlinear Muskingum method leads to inaccurate results, which are improved when the exponent approaches unity.
4. Approximation of the nonlinear parameter $k_{1,2}$ by Eqs. 45 or 46, with $F_{1,2}(t)$ taken as unity, can give good results only when function

TABLE 4.—Information of Simulation Cases

Literature source and method (1)	Solution technique (2)	Simulation run number (3)	Exponent m or p (4)	Parameter α or α_1 or α_2 (5)	Parameter k or k_1 or k_2 (6)	Fitness criterion, D (7)	Utilized Eq. number (8)
Wilson (18)							
Linear	Integrated approach	W1	0	0.250	1.500	0.91	36
NLI	Integrated approach	W2	0.50	0.298	10.343	−12.73	36
NLI	Simple integrated approach	W4	0.50	0.298	10.343	−10.91	39
NLI	Integrated approach (I)	W6	0.50	0.250	$1.5\sqrt{I}$	−28.18	36
NLI	Integrated approach (I & F_1)	W8	0.50	0.250	$1.5\sqrt{I}F_1$	0.91	36
NLI	Analytic solution	W10	0.50	0.298	10.343	−31.82	20
NLII	Integrated approach	W3	0.50	0.298	10.343	−12.73	43
NLII	Simple integrated approach	W5	0.50	0.298	10.343	−10.91	44
NLII	Integrated approach (I)	W7	0.50	0.250	$1.5\sqrt{I}$	−28.18	43
NLII	Integrated approach (I & F_2)	W9	0.50	0.250	$1.5\sqrt{I}F_2$	0.91	43
NLII	Analytic solution	W11	0.50	0.298	10.343	−36.36	25
Linsley, et al. (7)							
Linear	Integrated approach	L1	0	0.160	0.731	0.53	36
NLI	Integrated approach	L2	0.50	0.499	2.018	−2.63	36
NLI	Integrated approach (I)	L4	0.50	0.160	$0.731\sqrt{I}$	−7.89	36
NLI	Integrated approach (I & F_1)	L6	0.50	0.160	$0.731\sqrt{I}F_1$	0.53	36
NLI	Analytic solution	L8	0.50	0.499	2.018	−7.11	20
NLII	Integrated approach	L3	0.50	0.499	2.018	−2.63	43
NLII	Integrated approach (I)	L5	0.50	0.160	$0.731\sqrt{I}$	−7.89	43
NLII	Integrated approach (I & F_2)	L7	0.50	0.160	$0.731\sqrt{I}F_2$	0.53	43
NLII	Analytic solution	L9	0.50	0.499	2.018	−6.25	25
Lawler (6)							
Linear	Integrated approach	A1	0	0.296	1.839	−1.60	36
NLI	Integrated approach	A2	0.50	0.190	8.008	−3.19	36

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TABLE 4.—*Continued*

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
NLI	Simple integrated approach	A4	0.50	0.190	8.008	−3.19	39
NLI	Integrated approach (I)	A6	0.50	0.296	$1.839\sqrt{I}$	−3.19	36
NLI	Integrated approach (I & F_1)	A8	0.50	0.296	$1.839\sqrt{I}F_1$	−1.60	36
NLI	Integrated approach	A10	0.75	0.296	3.400	−3.19	36
NLI	Analytic solution	A12	0.50	0.190	8.008	−4.26	20
NLII	Integrated approach	A3	0.50	0.190	8.008	−3.19	43
NLII	Simple integrated approach	A5	0.50	0.190	8.008	−3.19	44
NLI	Integrated approach (I)	A7	0.50	0.296	$1.839\sqrt{I}$	−3.19	43
NLII	Integrated approach (I & F_2)	A9	0.50	0.296	$1.839\sqrt{I}F_2$	−1.60	43
NLII	Integrated approach	A11	0.75	0.296	3.400	−3.19	43
NLII	Analytic solution	A13	0.50	0.190	8.008	−3.72	25
Viessman, et al. (19)							
Linear	Integrated approach	V1	0	0.245	1.840	0.67	36
NLI	Integrated approach	V2	0.50	0.257	42.364	−6.66	36
NLI	Integrated approach (I)	V4	0.50	0.245	$42.364\sqrt{I}$	−14.66	36
NLI	Integrated approach (I & F_1)	V6	0.50	0.245	$42.364\sqrt{I}F_1$	0.67	36
NLI	Integrated approach	V8	0.75	0.245	9.500	−5.33	36
NLI	Analytic solution	V10	0.50	0.257	42.364	−13.33	20
NLII	Integrated approach	V3	0.50	0.257	42.364	−6.66	43
NLII	Integrated approach (I)	V5	0.50	0.245	$42.364\sqrt{I}$	−14.66	43
NLII	Integrated approach (I & F_2)	V7	0.50	0.245	$42.364\sqrt{I}F_2$	0.67	43
NLII	Integrated approach	V9	0.75	0.245	9.500	−5.33	43
NLII	Analytic solution	V11	0.50	0.257	42.364	−13.33	25

$F_{1,2}(t)$ does not deviate from unity. This is true only when the ratio Q/I is close to one, i.e., when the time lag between inflow-outflow hydrographs is small, or when the flooding discharge is small in comparison with the existing base flow.

5. Application of the nonlinear Muskingum method with parameters $k_{1,2}$ given by Eqs. 45 or 46 gives the more accurate results for all cases. These results are identical to the ones obtained by the linear Muskingum method.

6. From a numerical point of view, the integral approximations (Eqs. 36 and 42) represent the solution of the equations very accurately and for all practical purposes, their simplified versions (Eqs. 39 and 44) can be adequatley utilized.

7. Since the analytical solutions were derived from $m = p = 0.5$ and $k_{1,2}$ a constant, they have limited applicability. However, they provide a reasonable simulation of the receding part of the outflow hydrograph.

Conclusions

The conclusions drawn from this study are:

1. The nonlinear Muskingum methods (Eqs. 6 and 7) are less accurate than the linear method (Eq. 5).

2. The nonlinear Muskingum method NLI described by Eq. 6 is more accurate than the NLII given by Eq. 7.

3. The performance of the nonlinear methods depends mainly on the parameter $k_{1,2}$, while changes of the weighting factor $\alpha_{1,2}$ are not essential.

4. The parameter $k_{1,2}$ cannot in general be assumed as constant, especially for small values of the exponents m or p. The simulation with constant values of $k_{1,2}$ is very poor even if k_1 and α are obtained by the least squares method.

5. Application of the nonlinear methods with exponents equal to the theoretical values of 0.60 and 0.67 does not improve the performance of the method. Therefore, $m = 0.5$ or $p = 0.5$ can be used so that advantage can be taken of the analytical solutions.

6. The integral approximation is a very good technique that represents the solution of the equations very efficiently.

7. The analytical solutions are good only for the receding part of the outflow hydrograph.

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Appendix I.—Nonlinear Parameter Estimation

Direct determination of the nonlinear Muskingum parameters can be done for the equation

$$S = k_1[\alpha_1 I^m + (1 - \alpha_1)Q^m] \quad (53)$$

If N is the number of data sets, then by the least squares method the pair of parameters k_1 and α that minimizes the error in the function

$$G(k_1, \alpha_1) = \sum_{j=1}^{N} [S_j - k_1\alpha_1 I_j^m - k_1(1 - \alpha_1)Q_j^m] \quad (54)$$

can be obtained as follows:

$$\frac{\partial G}{\partial (k_1\alpha_1)} = 0 \rightarrow k_1\alpha_1 \sum_{j=1}^{N} I_j^{2m} + k_1(1 - \alpha_1) \sum_{j=1}^{N} (I_j Q_j) = \sum_{j=1}^{N} I_j^m S_j \quad (55)$$

$$\frac{\partial G}{\partial [k_1(1 - \alpha_1)]} = 0 \rightarrow k_1\alpha_1 \sum_{j=1}^{N} (I_j Q_j)^m + k_1(1 - \alpha_1) \sum_{j=1}^{N} Q_j^{2m} = \sum_{j=1}^{N} Q_j^m S_j \quad (56)$$

Dropping the indices for convenience, the solution of Eqs. 55–56 is given as

$$k_1\alpha_1 = \frac{\Sigma(Q^m S)\,\Sigma(IQ)^m - \Sigma(I^m S)\,\Sigma Q^{2m}}{[\Sigma(IQ)^m]^2 - \Sigma I^{2m}\,\Sigma Q^{2m}} = A \quad (57)$$

$$k_1(1 - \alpha_1) = \frac{\Sigma(I^m S)\,\Sigma(IQ)^m - \Sigma(Q^m S)\,\Sigma Q^{2m}}{[\Sigma(IQ)^m]^2 - \Sigma I^{2m}\,\Sigma Q^{2m}} = B \quad (58)$$

and subsequently

$$k_1 = A + B \quad (59)$$

$$\alpha_1 = \frac{A}{A + B} \quad (60)$$

are the values of the parameters of the nonlinear Muskingum method.

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Appendix III.—Notation

The following symbols are used in this paper:

A = known function of Muskingum parameters;
B = known function of Muskingum parameters and rate of inflow;
c = wave celerity;
D = $\alpha I + (1 - \alpha)Q$;
E_i = known function of Muskingum parameters and rate of inflow;
F = either Q or D;
$F_{1,2}$ = function of rates of inflow-outflow;
H_i = known function of Muskingum parameters and rate of inflow;
I = inflow rate;
I_{eq} = rate of inflow at time that equals rate of outflow;
i_0 = slope of channel;
k = average travel time;
$k_{1,2}$ = nonlinear parameters;
m = exponent of nonlinear Muskingum method;
n = either m or p;
P_i = unknown function of rate of outflow;
p = exponent of nonlinear Muskingum method;
Q = outflow rate;
q = rate of inflow per unit width;
R = F^n;
S = storage;
t = time;
Δt = time step;
v = dependent variable related to rate of outflow;
α = weighting coefficient;
β = dimensional factor equal to unity;
Δx = length of the reach;
λ = exponent;
μ = integrating factor;
ρ = empirical exponent;
ψ = $Q^{1/\lambda}$; and
ψ_p = particular solution.

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