Tsallis Entropy Theory for Derivation of Infiltration Equations

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Abstract. *An entropy theory is formulated for deriving infiltration equations for the potential rate (or capacity) of infiltration in unsaturated soils. The theory is comprised of five parts: (1) Tsallis entropy, (2) principle of maximum entropy (POME), (3) specification of information on the potential rate of infiltration in terms of constraints, (4) maximization of entropy in accordance with POME, and (5) derivation of the probability distribution of infiltration and its maximum entropy. The theory is illustrated with the derivation of six infiltration equations commonly used in hydrology, watershed management, and agricultural irrigation, including Horton, Kostiakov, Philip two-term, Green-Ampt, Overton, and Holtan, and the determination of the least biased probability distributions underlying these infiltration equations and the entropies thereof. The theory leads to the expression of parameters of the derived infiltration equations in terms of three measurable quantities: initial infiltration capacity (potential rate), steady infiltration rate, and soil moisture retention capacity. In this sense, these derived equations are rendered nonparametric. With parameters thus obtained, infiltration capacity rates are computed using these six infiltration equations and are compared with field experimental observations reported in the hydrologic literature as well as the capacity rates computed using parameters of these equations obtained by calibration. It is found that infiltration capacity rates computed using parameter values yielded by the entropy theory are in reasonable agreement with observed as well as calibrated infiltration capacity rates.*

Keywords. *Entropy, Infiltration, Green-Ampt equation, Holtan equation, Horton equation, Kostiakov equation, Overton equation, Philip equation, Principle of maximum entropy, Tsallis entropy.*

Infiltration is fundamental to determining the runoff hydrograph, soil moisture and groundwater recharge, irrigation efficiency, life span of pavements, and leaching of nutrients. In hydrology, irrigation engineering, and soil science, a number of infiltration equations have been developed, some of which are now commonly applied in hydrologic modeling and have been included in popular watershed hydrology models (Singh, 1989, 1995; Singh and Frevert, 2002a, 2002b, 2006; Singh and Woolhiser, 2002). Some of the commonly used equations (Singh and Yu, 1990) are: Green and Ampt (1911), Kostiakov (1932), Horton (1938), Philip two-term (Philip, 1957), Holtan (1961), and Overton (1964). These equations represent the potential or capacity rate of infiltration at a point. In this study, infiltration rate will imply capacity or potential rate, which is the maximum rate at which water enter the soil under no restriction on the supply of water. It is known that soil characteristics vary significantly from one place to another, and antecedent soil moisture, which defines the initial infiltration, also significantly varies spatially. The infiltration parameters determined using point measurements are point values, or at best reflect average values. Although large spatial variability in infiltration is recognized, little effort has been made to account for its probabilistic characteristics, except for a few watershed models, for example, BASINS (formerly Stanford Watershed Model) (Crawford and Linsley, 1966; Donigian and Imhoff, 2006).

In recent years, the entropy concept has been applied to a range of problems in hydrology, hydraulics, environmental engineering, geomorphology, and water resources engineering (see a review by Singh, 1997). Review of the literature suggests that a majority of applications have encompassed derivation of frequency distributions and estimation of their parameters (Singh, 1998), evaluation and design of monitoring networks (Harmancioglu et al., 1999), and measuring uncertainty (Klir, 2006). These applications are statistical in nature and do not invoke physical conservation laws. Other applications, such as the assessment of the reliability of water distribution systems (Awumah et al., 1991), landscape evolution (Fiorentino et al., 1993), and water resources assessment (Marayuma et al., 2005) have also been primarily statistical. On the other hand, Chiu (1987, 1988, 1989, 1991) and Barbé et al. (1991), among others, combined entropy with the laws of mass, momentum, and energy conservation and derived velocity distributions in open channels and pipes as well as solute transport models. They showed that entropy-based velocity distributions were superior to the commonly used Prandtl-von Karman and power law velocity distributions. However, this line of investigation has not been extended to other areas in hydrology and water engineering, and this essentially motivated the present study on the application of entropy to deriving infiltration equations.

Submitted for review in November 2009 as manuscript number SW 8305; approved for publication by the Soil & Water Division of ASABE in March 2010.

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Furthermore, the majority of applications of entropy in water engineering have employed the Shannon entropy. Koutsoyiannis (2005a, 2005b) was probably the first to apply the Tsallis entropy to investigate the stochastic behavior of hydrological processes. The Tsallis entropy has not been employed for describing infiltration or movement of moisture thus far. It may be interesting to explore the use of the Tsallis entropy for infiltration modeling, for it possesses a number of interesting properties and encompasses the Shannon entropy as a special case. This study, therefore, is an exploratory attempt to employ the Tsallis entropy to derive six commonly used infiltration equations.

The objective of this study, therefore, is to develop an entropy theory for deriving equations for infiltration into unsaturated soils; illustrate the theory by derivation of the well-known infiltration equations of Horton, Kostiakov, Philip, Green-Ampt, Overton, and Holtan; derive probability distributions of these equations; and test the derived forms of these equations using experimental observations reported in the literature. The theory leads to the expression of infiltration equation parameters in terms of what is easily measured or measurable and hence the physical basis of the parameters. The theory also establishes a probabilistic basis of infiltration equations and hence an estimate of uncertainty associated with each equation. The objective here is not to validate the equations nor show if one equation is better than the others.

Development of Entropy Theory

Let the infiltration capacity (or infiltrability), as a function of time t, be defined as $I(t)$. It is assumed that the soil is dry, and water is applied to the dry soil with no limitation to the supply of water. At the beginning, infiltration will be high; as time progresses, the infiltration capacity declines and may reach a steady or constant rate or approach zero. The constant rate is often called the drainage rate. This capacity of infiltration is the potential rate and will be equal to or greater than the actual rate, depending on the supply of water. Furthermore, the infiltration capacity may significantly vary from one place to another. Crawford and Linsley (1966) were probably the first to consider spatial variations in infiltration capacity. From empirical data reported in the literature (Burgy and Luthin, 1956), they found large variations in infiltration capacity, even in relatively homogeneous soils (uniform Yolo silt loam) and over small areas (12 m × 6 m; 40 ft 20 ft). Considering infiltration capacity as a random variable, they expressed the cumulative probability distribution of infiltration capacity as a function of area. Motivated by this work, it was assumed in this study that the spatially averaged infiltration capacity, $I(t)$, is a random variable and would therefore have a probability density function. It is recognized that this assumption needs to be verified or may even be tenuous, but even if it is weakly true it would not greatly mar the usefulness of the entropy theory.

The objective is to formulate the entropy theory and derive the capacity rate of infiltration as a function of time using this theory. The entropy theory for randomly varying infiltration capacity rate I can be formulated as comprising five parts: (1) Tsallis entropy, (2) principle of maximum entropy (POME), (3) specification of information on infiltration rate in terms of constraints, (4) maximization of entropy in accordance with POME, and (5) derivation of the probability distribution of infiltration rate and its entropy. Each of these parts is outlined in what follows.

Tsallis Entropy

Considering entropy as a measure of information and hence of uncertainty, Tsallis (1988) formulated what is referred to as the Tsallis entropy. The Tsallis entropy qualitatively measures the uncertainty associated with a random variable or its probability distribution in accord with several consistency measures. Consider a discrete form of infiltration capacity I that occurs with probability p_i, $i = 1, 2, ..., N$, and N is the number of values that capacity can take on. The Tsallis entropy, denoted H, can be written as:

$$H = \frac{k}{m-1}\sum_{i=1}^{N} p_i(1-p_i^{m-1}) = k\frac{1-\sum_{i=1}^{N} p_i}{m-1} \qquad (1a)$$

where k is a measure that keeps the units of H consistent and is often taken as unity, and m is any real number. Exponent m influences the variability of H with the probability. To illustrate this, a plot of H/k versus p for m = -1, -0.5, 0, 0.5, 1, and 2 is given in figure 1. For $m < 0$, the Tsallis entropy is concave, and for $m > 0$ it becomes convex. For $m = 0$, $H = k(N - 1)$ for all p_i values. For $m = 1$, it converges to the Shannon entropy. For all cases, it decreases as m increases.

If the infiltration capacity is defined as a continuous random variable with a probability density function defined as $f(I)$, then the Tsallis entropy, $H(I)$, can be expressed as:

$$H(I) = \frac{k}{m-1}\left\{1-\int_{I_L}^{I_U}[f(I)]^m\right\}dI$$

$$= \frac{k}{m-1}\int_{I_L}^{I_U} f(I)\left\{1-[f(I)]^{m-1}\right\}dI$$

$$= \frac{k}{m-1}\left\{1-\int_{I_L}^{I_U}[f(I)]^m\right\}dI \qquad (1b)$$

where I_U and I_L are, respectively, the upper and lower limits of integration for I. H describes the expected value of

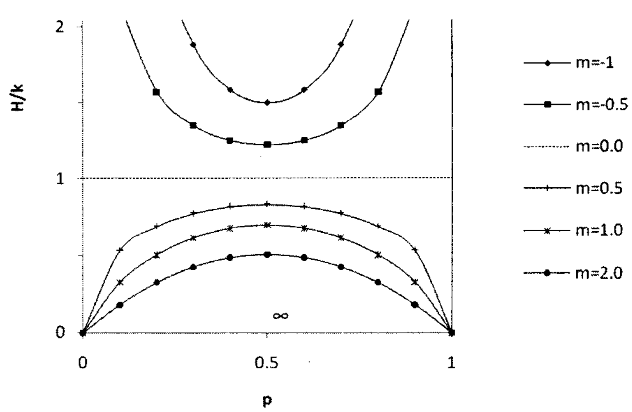

Figure 1. Plot of H/k for N = 2 for m = -1, -0.5, 0, 0.5, 1, and 2.

$1 - [f(I)]^{m-1}\}$. Considering $\{1 - [f(I)]^{m-1}\}$ as a measure of uncertainty, equation 1b defines the average uncertainty associated with $f(I)$ and in turn with I. The more uncertain I is, the more information will be needed to characterize it. In other words, information reduces uncertainty. In this sense, uncertainty and information are related to each other. Thus, the key here is to derive the least biased $f(I)$.

The Tsallis entropy is a non-extensive entropy and reduces to the Shannon entropy if exponent m in equation 1a is unity. It can also be said that for $m \to 1$, equation 1a reduces to the Boltzmann-Gibbs statistics used in statistical mechanics. H is maximum for all values of m in the case of equiprobability. H is maximum if $m > 0$ and is minimum if $m < 0$. Like the Shannon entropy, the Tsallis entropy satisfies the additivity property for independent systems. Because of these and other properties, the Tsallis entropy has received and continues to receive a great deal of attention, especially in sciences. Although most of these applications have primarily been in physics, some of them relate to hydrological processes and will therefore have relevance in hydrological analysis and modeling. Koutsoyiannis (2005a, 2005b) was the first to employ the Tsallis entropy to characterize stochastic behavior of hydrological processes. Keylock (2005) introduced the Tsallis entropy and m-exponential distribution for deriving flood recurrence intervals. He reasoned that a distribution derived from power law considerations would be more appropriate than the power law itself. Hence, it can be argued that the Tsallis entropy has potential for much broader application in hydrology than is presented in this study.

Principle of Maximum Entropy

The principle of maximum entropy (POME) formulated by Jaynes (1957a, 1957b, 1958, 1982) says that the least biased probability distribution of I, $f(I)$, will be the one that will maximize $H(I)$ given by equation 1, subject to the given information on I expressed as constraints. In other words, if no information other than the given constraints is available, then the probability distribution should be selected such that it is least biased toward what is not known. Such a probability distribution is yielded by the maximization of the Tsallis entropy. Thus, one of the key points is to define constraints on I, for $f(I)$ depends on these constraints.

Constraints

Information on $I(t)$ can be obtained using the knowledge of soil physics and experimental observations. For a given soil, one frequently measures infiltration and then characterizes the soil infiltration and more particularly the time capacity rate of infiltration or infiltration curve for the soil under the condition that water supply is not a limiting factor. If infiltration capacity rate observations are available, then information on the infiltration capacity rate can be expressed in terms of constraints, C_r, $r = 0, 1, 2, ..., n$, as:

$$C_0 = \int_{I_L}^{I_U} f(I)dI = 1 \tag{2}$$

$$C_r = \int_{I_L}^{I_U} g_r(I)\, f(I)\, dI = \overline{g_r(I)},\ r = 1, 2, ..., n \tag{3}$$

where $g_r(I)$, $r = 1, 2, ..., n$, represent some functions of I, n denotes the number of constraints, and $\overline{g_r(I)}$ is the expectation of $g_r(I)$. The constraints are analogous to moments. For example, if $r = 1$ and $g_r(I) = I$, then equation 3 would correspond to the mean infiltration capacity rate; likewise, for $r = 2$ and $g_2(I) = (I - \bar{I})^2$, equation 3 would denote the variance of I. For most infiltration equations used in hydrology, more than two constraints are not needed. The role of constraints cannot be overemphasized. The type of probability distribution that one obtains by maximizing the entropy depends on the type of constraints that one defines. Thus, there is a one-to-one correspondence between the probability density function (PDF) and its constraints. Following the procedure discussed by Singh (1998), if a PDF is given, one can derive the corresponding constraints. Similarly, if constraints are specified, then they will lead to a unique PDF. In the case of deriving a specific infiltration equation, the problem becomes trickier, since its PDF is not known *a priori*. Hence, trial and error seems the only option in the beginning.

Maximization of Tsallis Entropy

In order to obtain the least biased $f(I)$, the entropy given by equation 1b is maximized, subject to equations 2 and 3, and one simple way to achieve maximization is the use of the method of Lagrange multipliers. To that end, the Lagrangean function L can be expressed as:

$$L = \frac{1}{m-1} \int_{I_L}^{I_U} f(I)\left\{1 - [f(I)]^{m-1}\right\} dI + \lambda_0 \left[\int_{I_L}^{I_U} f(I)dI - C_0\right] + \sum_{r=1}^{n} \lambda_r \left[\int_{I_L}^{I_U} f(I) g_r(I) dI - C_r\right] \tag{4}$$

where λ_r, $r = 0, 1, 2, ..., n$, are the Lagrange multipliers. Recalling the Euler-Lagrange equation of the calculus of variation, the least biased $f(I)$ is obtained by maximizing L, noting that f is variable and I is a parameter. Thus, differentiating equation 4 and equating the derivative to zero, one gets:

$$\frac{\partial L}{\partial f(I)} = 0 \Rightarrow \frac{1}{m-1}\left\{1 - [f(I)]^{m-1}\right\} - [f(I)]^{m-1} + \lambda_0 + \sum_{r=1}^{n} \lambda_r g_r(I) = 0 \tag{5}$$

Derivation of Probability Distribution and Maximum Entropy

Solution of equation 5 leads to the probability density function of I in terms of the given constraints:

$$f(I) = \left\{ \frac{1}{m} + \frac{(m-1)}{m} \left[\lambda_0 + \sum_{r=1}^{n} \lambda_r g_r(I)\right]\right\}^{1/(m-1)} \tag{6}$$

The Lagrange multipliers (λ_r values) can be determined with the use of equations 2 and 3. Equation 6 is the entropy-based probability density function of power type. The cumulative probability distribution function of I, $F(I)$, can be written as:

$$F(I) = \int_{I_L}^{I} \left\{ \frac{1}{m} + \frac{m-1}{m} \left[\lambda_0 + \sum_{r=1}^{m} \lambda_r g_r(I) \right] \right\}^{1/(m-1)} dI \quad (7)$$

Substituting equation 6 in equation 1, one obtains the maximum entropy of $f(I)$ or I:

$$H(I) = \frac{1}{m-1} \int_{I_L}^{I_U} \left\{ \frac{1}{m} + \frac{(m-1)}{m} \left[\lambda_0 + \sum_{r=1}^{n} \alpha_r g_r(I) \right] \right\}^{1/(m-1)} \cdot \left[1 - \left\{ \frac{1}{m} + \frac{(m-1)}{m} \cdot \left[\lambda_0 + \sum_{r=1}^{n} \lambda_r g_r(I) \right] \right\} \right] dI \quad (8)$$

Equations 1b, 2, 3, 6, and 8 constitute the building blocks of the entropy theory, which is now illustrated by deriving six popular infiltration equations as examples.

Derivation of General Infiltration Equation

Consider a dry soil element, as shown in figure 2, to which water is supplied without any limitation. The water infiltrates the soil element at a capacity rate of $I(t)$ and exits at a capacity rate of $I_c(t)$. The soil will have a maximum soil moisture retention capacity denoted S. For a dry soil, S will be equal to the soil porosity multiplied by the soil elemental volume minus the volume of pore spaces occupied by roots, earthworms, or other objects. The soil elemental volume is computed by choosing an appropriate length of the element, which depends on the soil type under consideration. In general, it is taken as the crop root zone depth, which may be about 100 cm or about three feet. In a dry soil with no macropores, the maximum amount of water retained will be the same as the cumulative infiltration J; that is, $0 \le J \le S$. If W is the amount of pore space available for infiltration of water at any time, then $W + J = S$.

The continuity equation for a soil element can be expressed as (Singh and Yu, 1990):

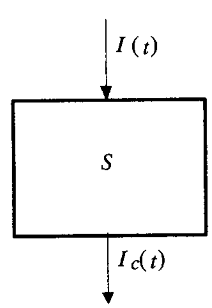

Figure 2. Soil element with infiltration: $I(t)$ = rate of infiltration, $I_c(t)$ = rate of infiltration exiting the element, and S = soil moisture retention capacity.

$$\frac{dJ}{dt} = I(t) - I_c(t) \quad (9)$$

where $J(t)$ defines the cumulative infiltration at time t. One can also express the continuity equation as:

$$J(t) = \int_0^t [I(t) - I_c(t)]dt \quad (10)$$

Strictly speaking, I_c varies in time; however, for the discussion in this article, it is assumed constant for two reasons, i.e., $I_c(t) = I_c$. First, the infiltration equations considered here assume a constant value of I_c. Second, measurements of I_c varying in time are usually not available.

It is hypothesized that the cumulative probability distribution of infiltration, $F(I)$, can be defined as the ratio of soil moisture potential (W) to the maximum soil moisture retention (S):

$$F(I) = \frac{W}{S} \quad (11)$$

$F(I)$ can also be defined as one minus the ratio of the cumulative infiltration to the maximum potential cumulative infiltration or maximum soil moisture retention, S:

$$F(I) = 1 - \frac{J}{S} \quad (12)$$

The hypothesis expressed by equation 12 needs to be validated using field data or experimental observations. Differentiation of equation 12 yields:

$$dF(I)dI = -\frac{dJ}{S};\ dF(I) = f(I) = -\frac{1}{S}\frac{dJ}{dI} \quad (13)$$

where $f(I)$ is the probability density function of $I(t)$, which is determined using the entropy theory.

Substitution of equation 7 in equation 13 and then integrating results in:

$$J = S \int_{I_U}^{I} \left\{ \frac{1}{m} + \frac{m-1}{m} \left[\lambda_0 + \sum_{r=1}^{n} \lambda_r g_r(I) \right] \right\}^{1/(m-1)} dI \quad (14)$$

Equation 14 expresses the relationship between cumulative infiltration and infiltration capacity rate and can be integrated. In a way, this equation describes what can be considered the infiltration rating curve. The explicit form of this relationship depends on the form of $gr(I)$, $r = 1, 2, .., n$. Then, noting equation 6 and expressing $I(t)$ as $dJ(t)/dt$, $J(t)$ can be determined. Thereafter, differentiation of $J(t)$ will lead to an expression for $I(t)$, which is what is desired. This suggests that the key to deriving an infiltration equation is to derive its associated probability density function, whose derivation depends on the constraints specific to that infiltration equation. Application of the theory is illustrated by deriving six popular infiltration equations, including the Horton, Kostiakov, Philip, Green-Ampt, Overton, and Holtan equations. The Horton equation is derived in what follows, and other infiltration equations are derived in Appendices A through E.

Horton Equation

Let the initial infiltration capacity rate be defined as I_0 and the steady or constant rate denoted as I_c. Thus, $I(t)$ will vary from I_c to I_0. The objective is to derive the rate of infiltration as a function of time.

Specification of constraints: The simplest constraint that $f(I)$ must satisfy is:

$$\int_{I_c}^{I_0} f(I)dI = 1 \tag{15}$$

Maximization of entropy: Applying POME and using the method of Lagrange multipliers (Singh, 1998), one obtains the Lagrangean function L as:

$$L = \frac{1}{m-1}\left\{\int_{I_c}^{I_0} f(I)\left(1-[f(I)]^{m-1}\right)\right\}dI + \lambda_0\left[\int_{I_c}^{I_0} f(I)dI - 1\right] \tag{16}$$

where λ_0 is the zeroth Lagrange multiplier. Recalling the Euler-Lagrange equation of calculus of variation and differentiating equation 16 with respect to f and keeping in mind that I is a parameter here, not a variable, and equating the derivative to zero, one gets:

$$\frac{\partial L}{\partial f} \Rightarrow 0 = \frac{1}{m-1} \cdot \left\{\int_{I_c}^{I_0}\left[1-[f(I)]^{m-1} - (m-1)[f(I)]^{m-1}\right]dI\right\} + \lambda_0\left[\int_{I_c}^{I_0} dI\right] \tag{17}$$

Equation 17 yields:

$$f(I) = \left\{\frac{m-1}{m}\left[\frac{1}{m-1} + \lambda_0\right]\right\}^{\frac{1}{m-1}} \tag{18}$$

Equation 18 is the Tsallis entropy-based probability density function and contains one unknown parameter: the zeroth Lagrange multiplier.

Determination of Lagrange multiplier: For simplicity, let:

$$\lambda_* = \lambda_0 + \frac{1}{m-1} \quad \text{and} \quad A = \left[\frac{m-1}{m}\lambda_*\right]^{\frac{1}{m-1}}$$

Equation 18 can be expressed as:

$$f(I) = \left\{\frac{m-1}{m}[\lambda_*]\right\}^{\frac{1}{m-1}} = A \tag{19}$$

Substituting equation 19 in equation 15, one obtains:

$$\int_{I_c}^{I_0} f(I)dI = 1 = \int_{I_c}^{I_0} dI = \frac{1}{A} \tag{20}$$

Equation 20 gives the Lagrange multiplier λ_0 as:

$$\lambda_0 = \frac{m}{m-1}\left(\frac{1}{I_0 - I_c}\right)^{m-1} - \frac{1}{m-1}$$

$$\lambda_* = \lambda_0 + \frac{1}{m-1} = \frac{m}{m-1}\left(\frac{1}{I_0 - I_c}\right)^{m-1} \tag{21}$$

Probability density function of infiltration: Substitution of equation 21 in equation 19 yields:

$$f(I) = \frac{1}{I_0 - I_c} \tag{22}$$

Equation 22 is the probability density function associated with the Horton equation, which is uniform and depends only on the initial and steady infiltration capacity rates. The cumulative distribution function of I would be linear, expressed as:

$$F(I) = \int_{I_c}^{I} f(I)dI = \int_{I_c}^{I} \frac{1}{I_0 - I_c}dI = \frac{I - I_c}{I_0 - I_c} \tag{23}$$

Infiltration equation: Combining equations 22 and 13, one obtains:

$$\frac{1}{I_0 - I_c}dI = -\frac{1}{S}dJ \tag{24}$$

Integrating equation 24, one obtains:

$$\frac{I - I_c}{I_0 - I_c} = 1 - \frac{J}{S} \tag{25}$$

Equation 25 can be recast as:

$$\frac{dJ}{dt} + \frac{J}{k} = I_0 - I_c, \quad k = \frac{I_0 - I_c}{S} \tag{26}$$

Solution of equation 26 yields the cumulative infiltration as:

$$J = (I_0 - I_c)k - (I_0 - I_c)\exp(-t/k) \tag{27}$$

Differentiating equation 27 with respect to t and recalling the continuity equation 9, one obtains the infiltration rate as:

$$I(t) = I_c + (I_0 - I_c)\exp(-t/k) \tag{28}$$

which is the Horton equation. Recall that:

$$k = \frac{S}{(I_0 - I_c)} \tag{29}$$

Equation 28 is the Horton equation derived using the entropy theory. Derivation of equation 28 shows that the Horton equation requires no constraint other than the total probability theorem, which is not a constraint in a true sense, for all probability distributions must satisfy it. Parameter k is expressed as the ratio of the maximum soil moisture retention and the initial infiltration capacity rate minus the steady-state infiltration rate. It has the dimension of time and indicates the time required for the infiltrated water to fill the maximum moisture retention space, if the capacity rate of infiltration were the initial infiltration rate (i.e., the maximum infiltration rate) minus the steady rate, or the initial excess infiltration capacity rate. Infiltration observations, under conditions of no limit on water supply, provide initial and steady infiltration capacity rates and for a given soil with knowledge of its porosity and its column height, the value of S (the maximum soil moisture retention) can be obtained. Thus, parameter k can be computed using equation 29 without any calibration. This also provides a physical interpretation of parameter k.

Entropy of Horton equation: The entropy of the probability distribution underlying the Horton equation or the infiltration rate can be expressed as:

$$H(I) = \frac{1}{m-1}\int_{I_c}^{I_0}\left\{1-[f(I)]^m\right\}dI$$

$$= \frac{1}{m-1}\left[(I_0 - I_c) - (I_0 - I_c)^{1-m}\right] \quad (30a)$$

For $m = 2$, equation 30a becomes:

$$H(I) = (I_0 - I_c) - \frac{1}{(I_0 - I_c)} \quad (30b)$$

Equation 30a states that the uncertainty of $f(I)$, or for that matter I, depends on the initial value of I, I_0, and steady rate I_c. This equation consists of two parts: $(I_0 - I_c)$ and $(I_0 - I_c)^{1-m}$. An important implication is that, for a given soil, the uncertainty of the Horton equation $m > 1$ is maximum when it is dry because that is when the initial infiltration will be maximum. As a result, the first part will be much greater than the second part, and hence the difference between these two parts will be greater, translating into greater entropy. This difference and hence entropy reduces as soil becomes wetter. This means that when sampling infiltration, greater care should be exercised in the beginning of infiltration and less toward the tail. This also means that infiltration observations should be more closely spaced temporally in the beginning, but the time interval between observations can be increased with the progress of infiltration.

Other Infiltration Equations

Kostiakov equation: The Kostiakov equation is derived in Appendix A and can be expressed as:

$$I(t) = 0.5at^{-0.5} \quad (31)$$

where a is parameter. From the entropy theory, one obtains $a = 2I_cS$, twice the product of steady infiltration rate (I_c) and maximum soil moisture retention (S), both of which can be determined for a given soil. This means that parameter a can be obtained from observations and does not need to be calibrated.

Philip two-term equation: The Philip two-term equation is derived in Appendix B and can be expressed as:

$$I(t) = a + 0.5(2aS)^{0.5}t^{-0.5} = a + bt^{-0.5}$$

$$b = 0.5(2aS)^{0.5} \quad (32)$$

where a and b are parameters. Parameter a is analogous to steady infiltration rate (or saturated hydraulic conductivity or a fraction thereof) and can be obtained without having any calibration. In general, a is between 0.5 to 0.7 of I_c. Parameter b can be expressed in terms of a and maximum soil moisture retention S, which also can be obtained from observations, as shown by equation 32. Thus, parameters a and b have physical meaning and need no calibration.

Green-Ampt (G-A) equation: The G-A equation is derived in Appendix C and can be expressed as:

$$t = \frac{1}{I_c}\left[J - \frac{a}{I_c}\log\left(1 + \frac{J}{a/I_c}\right)\right] \quad (33)$$

In equation 33, parameter I_c is the steady-state rate of infiltration and can be interpreted as almost equal to the saturated hydraulic conductivity. Parameter S is the maximum soil moisture retention, and $S = a/I_c$. Since I_c and S can be obtained from observations, $a = SI_c$ can also be obtained from observations. In the hydrologic literature, S is interpreted as equal to the product of the capillary suction at the wetting front and the initial moisture deficit (Singh, 1989). The entropy theory provides another interpretation of parameter S, and hence the G-A parameters can be estimated without calibration.

Overton equation: The Overton equation is derived in Appendix D and can be written as:

$$I(t) = I_c \sec^2\left[\sqrt{aI_c}\,(t_c - t)\right] \quad (34)$$

where t_c is the time to steady-state infiltration rate I_c; this time may be much smaller than the duration of the infiltration experiment or observations and can be obtained from observations. Parameter a is expressed as $(I_0 - I_c) = aS^2$, where I_0 is the initial infiltration capacity. Thus, parameters of the Overton equation can be obtained from observations, and calibration of these parameters may not be needed.

Holtan equation: The Holtan equation is derived in Appendix E and can be expressed as:

$$I = I_c + a\left[S^{1-n} - (1-n)at\right]^{\frac{n}{1-n}} \quad (35)$$

where a is a parameter expressed as:

$$a = \frac{(I_0 - I_c)}{S^n} \quad (36)$$

and

$$\overline{I - I_c} = B\frac{n}{1+n}(I_0 - I_c)^{\frac{1+n}{n}} \quad (37)$$

$$B = \frac{1}{n(I - I_c)^{1/n}} \quad (38)$$

Parameters a and n can be obtained from observations, as equations 36 and 37 show, and calibration may therefore not be needed.

Advantages of Entropy Theory

The derivations in the preceding section show that the entropy theory has several advantages: (1) The theory leads to a probabilistic characterization associated with each of the derived infiltration equations. It explicitly yields probability density functions and cumulative probability distributions. (2) It explicitly yields an expression of uncertainty associated with each probability density function and in turn each infiltration equation. (3) The uncertainty estimate can be gainfully employed for infiltration sampling. (4) The derivations of infiltration equations show that for the most part parameters of these equations can be expressed in terms of physically measurable quantities: initial infiltration, steady infiltration, and soil moisture retention capacity. Thus, this can be useful in hydrologic simulation and can make the simulation model more parsimonious. The fourth advantage is a remarkable finding.

Validation of Entropy Theory

Infiltration Data

Data on infiltration in field soils have been reported by Rawls et al. (1976) in a report published by the USDA Agriculture Research Service. Four data sets (labeled I, II, III, and IV) on infiltration in Robertsdale loamy sand, Stilson loamy sand, and Troupe sand in the Georgia Coastal Plain were obtained and used in this study. Characteristics of infiltration observations are given in table 1. In the table, D is the duration of the experiment; t_c is the time to the approximately constant rate of infiltration, which may be less than the duration of the experiment D; I_c is the constant (steady) rate of infiltration at the end of the infiltration experiment or the duration D applied to all the equations except for the Overton equation; I_c' is the constant rate of infiltration at time $t = t_c$ (which occurs before the end of the experiment) and is applied to the Overton equation; I_0 is the initial infiltration capacity rate given a few minutes later than the start of infiltration ($t = 0$) applied to all the six equations; $S1$ is the maximum soil moisture retention for the Green-Ampt equation and is determined by subtracting the initial soil moisture content from the final soil moisture content; $S2$ is the cumulative infiltration until time D applied to the Kostiakov and Philip equations; and $S3$ is the cumulative infiltration until time t_c applied to the Overton equation. S' is defined throughout the equation as $S' = S2 - I_c \times D$ and applied to the Horton and Holtan equations. It may be noted that the value of S differs from one infiltration equation to another because these equations are based on different assumptions and hypotheses. For illustrative purposes, data set IV for Troupe sand was selected.

For data set IV, the infiltration rate reached a lower value at $t = 110$ min and thereafter fluctuated round 4.37 cm h^{-1} (1.72 in. h^{-1}), corresponding to the cumulative infiltration of 11.21 cm (4.41 in.). Thus, in this case $t_c = 110$ min and $I_c' =$ 4.37 cm h^{-1} (1.72 in. h^{-1}). The initial infiltration capacity rate at $t = 4$ min was 11.60 cm h^{-1} (4.57 in. h^{-1}). The actual initial infiltration capacity rate (at $t = 0$) should be larger than 11.60 cm h^{-1}, which is the value at $t = 4$ min. For computation, the value of I_0 used was the value observed at $t = 4$ min. It is recognized that this is not the correct value, but no observations at $t = 0$ were available. It was assumed that the infiltration capacity rate at the end of the experiment reached the constant infiltration rate, and therefore the constant infiltration rate I_c was 4.37 cm h^{-1} (1.72 in. h^{-1}), which is the value of the infiltration rate at the end of the experiment. Since the connotation of parameter S may differ from one infiltration equation to another, it may have different values for different equations. Therefore, $S1$ was used to denote parameter S for the Green-Ampt equation as the maximum soil moisture retention determined by subtracting the intial soil moisture from the final soil moisture, while $S2$ was used to denote the cumulative infiltration until time D and applied to the Kostiakov and Philip equations. Likewise, $S3$ for the Overton equation was used to denote parameter S equal to the accumulated infiltration until t_c. S' was used to denote parameter S for the Horton equation, which was determined as $S' = S2 - I_c \times D$. These parameter values were obtained from observations and are given in table 1. In a similar manner, values of I_c, I'_c, I_0, I'_0, $S1$, $S2$, $S3$, and t_c were obtained for data sets I, II, and III and are shown in table 1. It is recognized that there is an element of subjectivity in the estimation of S for different equations, but experimental observations are a clear limitation in the reported data.

Validation of Infiltration Hypothesis

Equation 12 is a hypothesis fundamental to deriving the aforementioned infiltration equations and may be even for other equations. This hypothesis was tested for the above four data sets; it is shown in figure 3 for data set IV. The field data plotted approximately as a straight line, and it may be argued that the hypothesis is approximately valid but needs to be tested much more extensively. It may, however, be emphasized that the less than perfect validity of this hypothesis does not diminish the usefulness of the entropy theory.

Horton Equation

The Horton equation has three parameters, I_c, I_0, and k, as shown in equation 28. In the usual hydrologic practice, these parameters are obtained by calibration or fitting the Horton equation to infiltration observations. In the case of the entropy theory, parameters I_c and I_0 were obtained from observations. The value of S was also obtained from observations, where it was the difference between the maximum soil moisture and the initial soil moisture. Using

Table 1. Parameters from observations after Rawls (1976).

Soil Type	Code	ID	I_0 (cm h^{-1})	I_c (cm h^{-1})	I_c' (cm h^{-1})	$S1$ (cm)	$S2$ (cm)	$S3$ (cm)	S' (cm)	t_c (min)	Duration of Observations, D (min)
Robertsdale loamy sand	I	09091D	12.21	2.42	3.10	4.17	7.61	4.28	2.77	50	120
Robertsdale loamy sand	II	09091W	8.24	2.25	1.93	0.76	4.90	2.40	0.40	50	120
Stilson loamy sand	III	10101W	12.81	2.97	2.96	1.68	7.04	4.99	2.54	50	91
Troupe sand	IV	12112W	11.60	4.40	4.37	2.59	12.14	11.21	3.12	110	123

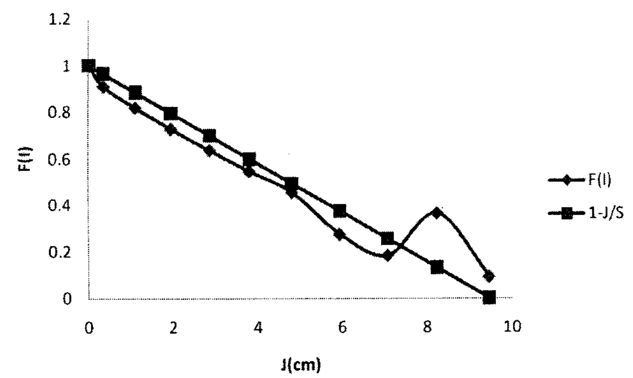

Figure 3. Relationship between cumulative probability distribution of infiltration rate and cumulative infiltration.

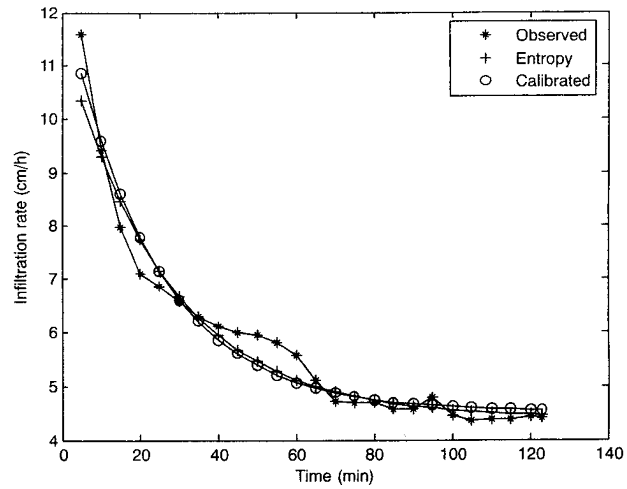

Figure 4. Comparison of infiltration rates computed using the Horton equation with parameters determined using entropy theory and by calibration with observed infiltration rates for date set IV.

these observed values of I_c, I_0, and S, parameter k was computed using equation 29, as shown in table 2. Thus, no calibration or fitting was done to obtain parameters I_c, I_0, and k. It may be noted that any error in data would directly translate into errors in the computed infiltration capacity rates. On the other hand, the three parameters were also obtained by calibration using the least square method in which the sum of squares of deviations between observed and computed infiltration rates was minimized. The Horton parameters obtained by calibration are shown in table 2. This was done for purposes of comparing the entropy theory-based infiltration capacity rates with the infiltration capacity rates obtained using calibrated parameter values.

With parameter values obtained from observations using the entropy theory and from calibration, the Horton equation was applied to all four data sets. For the sample data set (data set IV), the infiltration rates computed in the above two ways and observed capacity rates are shown in figure 4. The infiltration capacity rates computed using the entropy theory and calibration were in reasonable agreement with observed infiltration capacity rates. Clearly, the infiltration capacity rates obtained using the calibrated parameter values were in closer agreement with observed values. The average relative error (defined as the absolute difference between observed and computed capacity rates divided by the observed capacity rate) was under 13% for the entropy theory and under 8% for calibration. As expected, computed capacity rates improved as time progressed. For data sets I, II, and III, the absolute average relative error was, respectively, 12.6%, 10%, and 9.8% for the entropy theory and under 11.4%, 6.9%, and 2.6% for calibration. On average, the entropy theory performed remarkably well for all data sets, especially when there was no adjustment of parameters. It was observed that for data sets I and III, the maximum relative error (at a certain point in time) was significantly higher for the entropy theory than for calibration. However, two points need to be noted. First, for the most part, the relative error for the entropy theory was significantly lower, and thus the error was not as high as the maximum value of the error would lead one to infer. Second, a closer examination of data set I revealed that the infiltration rate started to fluctuate at t = 50 min all the way up to the end of the experiment, D = 120 min. This was also the case for data set III, where the infiltration rate started to fluctuate at t = 110 min. It was not clear what the reason for fluctuating infiltration rates was. It might have been small macropores or experimental errors.

Also computed was error equal to the square root of the mean of square of differences between computed and

Table 2. Equation parameters estimated by entropy theory and calibration.

	Horton Equation Parameters						Kostiakov Equation Parameters				Philip Two-Term Equation Parameters			
	Entropy			Calibration			Entropy		Calibration		Entropy		Calibration	
Data Set	k (h)	I_0 (cm h^{-1})	I_c (cm h^{-1})	k (h)	I_0 (cm h^{-1})	I_c (cm h^{-1})	a	b	a	b	a	b	a	b
ID = 09091D	0.28	12.21	2.42	0.13	16.21	3.02	6.07	-0.5	3.03	-0.46	1.21	2.14	0.53	2.55
ID = 09091W	0.07	8.24	2.25	0.03	52.90	2.19	4.70	-0.5	2.03	-0.42	1.13	1.66	0.71	1.41
ID = 10101W	0.26	12.81	2.97	0.13	30.82	3.06	6.46	-0.5	2.79	-0.74	1.48	2.29	-1.85	4.83
ID = 12112W	0.43	11.60	4.40	0.38	12.44	4.52	10.34	-0.5	5.31	-0.31	2.20	3.65	2.53	2.72

	Green-Ampt Equation Parameters				Overton Equation Parameters				Holtan Equation Parameters			
	Entropy		Calibration		Entropy		Calibration		Entropy (n = 1.5)		Calibration (n = 1.5)	
Data Set	I_c (cm h^{-1})	I_0×S (cm^2 h^{-1})	I_c (cm h^{-1})	I_0×S (cm^2 h^{-1})	I_c (cm h^{-1})	a	I_c (cm h^{-1})	a	I_c (cm h^{-1})	a	I_c (cm h^{-1})	a
ID = 09091D	2.42	10.08	1.49	7.72	3.10	0.50	2.83	0.63	2.42	2.13	2.93	6.36
ID = 09091W	2.25	1.71	1.07	2.90	1.93	1.10	1.33	1.56	2.25	23.57	2.16	50.79
ID = 10101W	2.97	4.97	-1.15	23.16	2.96	0.40	2.80	0.86	2.97	2.43	2.84	4.78
ID = 12112W	4.40	11.40	4.10	4.10	4.38	0.06	4.34	0.06	4.40	1.30	4.26	1.21

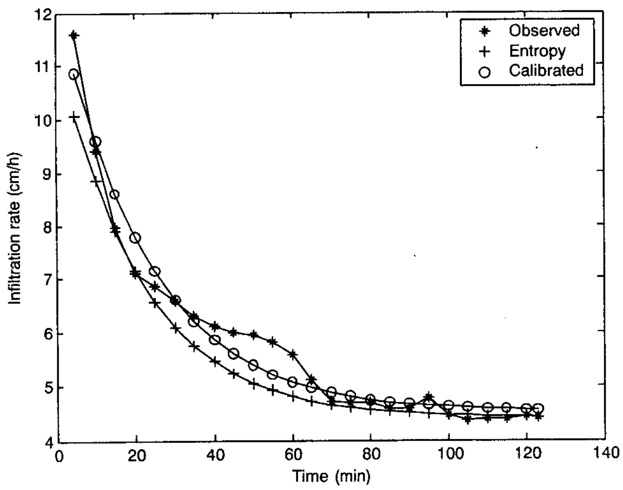

Figure 5. Sensitivity analysis of parameter for 0.8 *S* for data set IV.

observed infiltration rates. Using the entropy-based parameters, this error was 0.80, 0.83, 1.22, and 0.37 cm h^{-1}, respectively, for data sets I, II, III, and IV. For the calibrated parameters, the error was 0.57, 0.19, 0.16, and 0.35 cm h^{-1}. As expected, calibration produced infiltration rates closer to observed rates. Nevertheless, the entropy theory performed reasonably well. From now onwards, this error will be referred to as mean error.

Furthermore, the available value of parameter *S* may not be accurate, and hence a little bit of adjustment of the *S* value might lead to improved infiltration rates for the entropy theory. It was found that the value of *S* computed from the final and initial moisture content values did not match the accumulated infiltration. Hence, the *S* value used was not accurate. Parameter *S* was changed by plus or minus 10% to 40% with an increment of 10% in order to evaluate the sensitivity of infiltration rates to parameter *S*. Figure 5 shows infiltration rates for data set IV when *S* was reduced by 20% or $S = 0.8S0$ ($S0$ was the value from observations). The values computed by the entropy theory improved, indicating that more accurate observations would lead to improved infiltration rate estimates by the entropy theory.

Kostiakov Equation

This equation has only one parameter (*a*), which was obtained by calibration as well as directly from observations using equation A.13 due to the entropy theory, as shown in table 2. Figure 6 compares observed infiltration rates and the rates computed using the entropy theory and calibration for data set IV. The computed rates in both cases were higher than the observed rates for time equal to about 62 min. The absolute average relative error was, respectively, 12.23%, 35%, 14.4%, and 13% for data sets I, II, III, and IV for the entropy theory and 10%, 24.23%, 18.23%, and 3% for calibration. It may be noted that the value of parameter *a* as estimated for the entropy theory may be less than accurate, for the value of *S* as given in the data does not match the accumulated infiltration, i.e., the value of *S* is significantly less than the accumulated infiltration at the time when the rate of infiltration became almost constant. Perhaps this occurred either due to experimental errors or sudden appearance of macropores. A more accurate value of *S* would lead to a more accurate value of parameter *a* and hence to improved infiltration rates by the entropy theory.

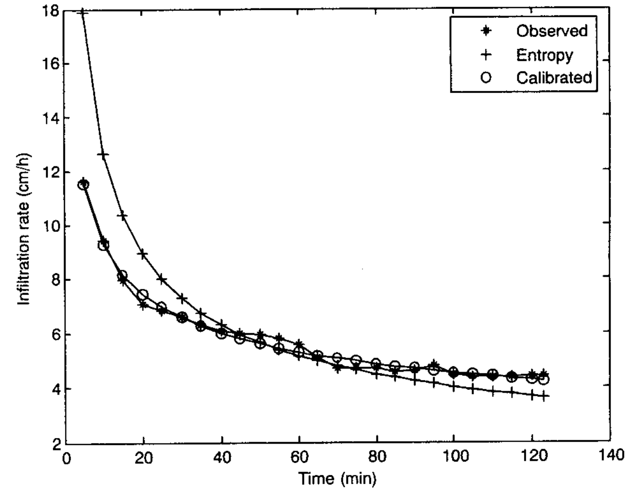

Figure 6. Comparison of infiltration rates computed using the Kostiakov with parameters determined using entropy theory and by calibration with observed infiltration rates for date set IV.

Philip Two-Term Equation

The Philip equation has two parameters (*a* and I_c), as shown in equation B.10. These parameters were estimated by calibration and from observations using equation B.10 for the entropy theory, as shown in table 2. Figure 7 compares observed infiltration capacity rates and the capacity rates computed using the entropy theory and calibration for data set IV. The figure shows that the entropy theory overestimated infiltration for the entire duration of the experiment, and the calibration method underestimated up to about 62 min and overestimated for the remainder of the experiment. The absolute average relative error was 9.6%, 35%, 4.7%, and 13%, respectively, for data sets I, II, III, and IV for the entropy theory and 8.5%, 21.33%, 20.6%, and 3.73% for calibration. Considering that there was no calibration for the entropy theory, it compared reasonably well with calibration. A more accurate value of *S* and/or *a* would lead to improved infiltration rate estimates.

Green-Ampt (G-A) Equation

The G-A equation has two parameters (*a* and *S*), as shown in equation C.12. These parameters were estimated by

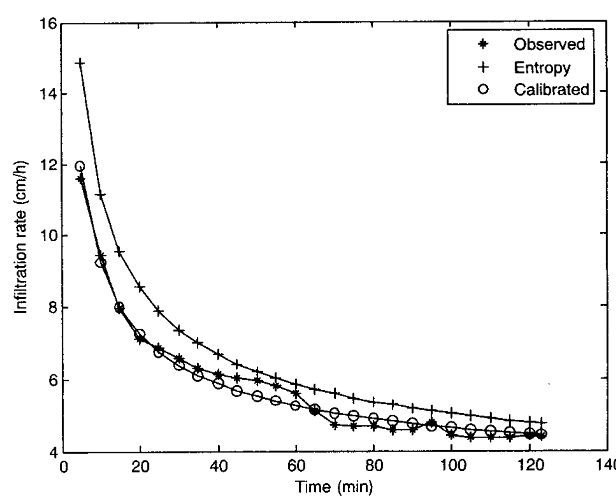

Figure 7. Comparison of infiltration rates computed using the Philip two-term equation with parameters determined using entropy theory and by calibration with observed infiltration rates for date set IV.

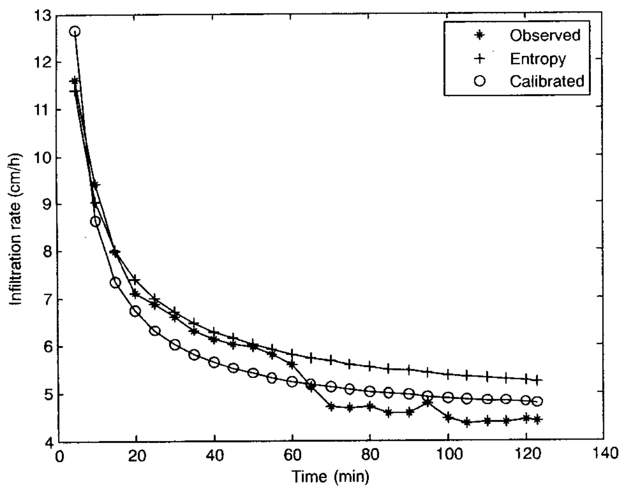

Figure 8. Comparison of infiltration rates computed using the Green-Ampt equation with parameters determined using entropy theory and by calibration with observed infiltration rates for date set IV.

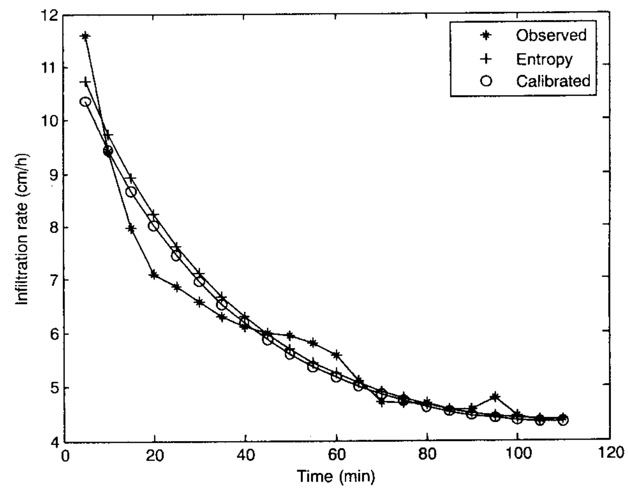

Figure 9. Comparison of infiltration rates computed using Overton equation with parameters determined using entropy theory and by calibration with observed infiltration rates for date set IV.

calibration and from observations using equation C.13 for the entropy theory, as shown in table 2. Figure 8 compares observed infiltration capacity rates and the capacity rates computed using the entropy theory and calibration for data set IV. The figure shows that the entropy theory consistently overestimated and the calibration method underestimated infiltration up to about 62 min, and then it overestimated. The absolute average relative error for data sets I, II, III, and IV was 28.6%, 29.1%, 25.4%, and 10.9%, respectively, for the entropy theory and 6.6%, 22.1%, 17.6%, and 8% for calibration. In this case, the entropy theory did not perform as well as it did for other equations. However, considering that there was no calibration of parameters, the performance was within error bounds that can be reduced. It was noticed that reducing the value of a through S and I_c would lead to improved infiltration estimates.

Overton Equation

The Overton equation has actually three parameters (a, I_c, and t_c), as shown in equation D.14. These parameters were estimated by calibration and from observations using equation D.13 for the entropy theory, as shown in table 2. Figure 9 compares observed infiltration rates and the rates computed using the entropy theory and calibration for data set IV. The figure shows that the entropy theory consistently overestimated infiltration capacity rate, and the calibration method underestimated between t = 20 min and t = 62 min and then overestimated. The absolute average error for data sets I, II, III, and IV was 6.5%, 36.2%, 20.5%, and 4.64%, respectively, for the entropy theory and below 11.2%, 21.8%, 5.2%, and 4.35% for calibration. Considering that there was no calibration for the entropy theory, it compared reasonably well with calibration. Reducing the value of a through S and I_c would lead to improved infiltration estimates.

Holtan Equation

The Holtan equation has three parameters (a, I_c, and n), as shown in equation E.12. These parameters were estimated by calibration and from observations using equation E.12 for the entropy theory, as shown in table 2. Their values were: I_c = 2.42 cm h^{-1}, a = 0.93, and n = 1.5, by entropy; and I_c = 2.82 cm h^{-1}, a = 3.14, and n = 1.5 by calibration, where n = 1.5 is fixed for both the methods. Figure 10 compares observed infiltration capacity rates and the capacity rates computed using the entropy theory and calibration for data set IV. The figure shows that both the entropy theory and the calibration method are comparable up to t = 62 min, first underestimating and then overestimating infiltration a little bit. The absolute relative error for data sets I, II, III, and IV was 8.3%, 10.9%, 13.2%, and 7.5%, respectively, for the entropy theory and 8.8%, 6.3%, 5.2%, and 3.9% for calibration. In this case, the entropy theory yielded not as good estimates as did calibration. However, considering that there was no calibration of parameters, the theory performed remarkably well. Reducing the value of a through S and I_c would lead to improved infiltration estimates.

Probability Distributions and Entropy of Infiltration Equations

For all six infiltration equations, CDFs and PDFs were determined both empirically and from the entropy theory (i.e., theoretically). The probability density function associated with the Horton equation is a uniform distribution

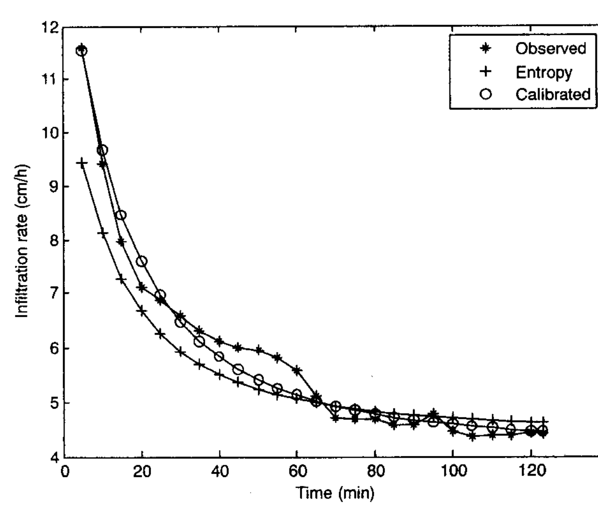

Figure 10. Comparison of infiltration rates computed using the Holtan equation with parameters determined using the entropy theory with n = 1.5 and by calibration with observed infiltration rates for date set IV.

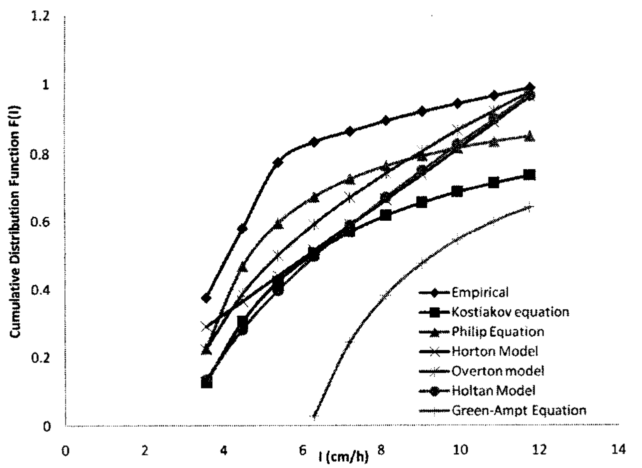

Figure 11. Cumulative probability distributions of infiltration equations.

over the length (I_0 - I_c), and hence the entropy of the Horton equation will be maximum over this length. This means that the larger the difference between initial infiltration capacity and the steady infiltration rate, the larger will be the entropy. In other words, there will be more uncertainty in the infiltration estimates. The implication is that more observations will be needed to better characterize infiltration. The probability density function of the Kostiakov equation is given by equation A.14, and its entropy is given by equation A.15. This shows that the uncertainty increases with increasing steady infiltration rate, and hence more observations will be needed to characterize infiltration. The probability density function of the Philip equation has the same shape as the Kostiakov equation, as seen in figure 11. The probability density function of the G-A equation is given by equation C.7, and its entropy is given by equation C.14. The probability density function of the Overton equation is given by equation D.8, and its entropy is given by equation D.15. The probability density function of the Holtan equation is given by equation E.4, and its entropy is given by equation E.14. Entropy values for the six infiltration equations were computed for all four data sets, as

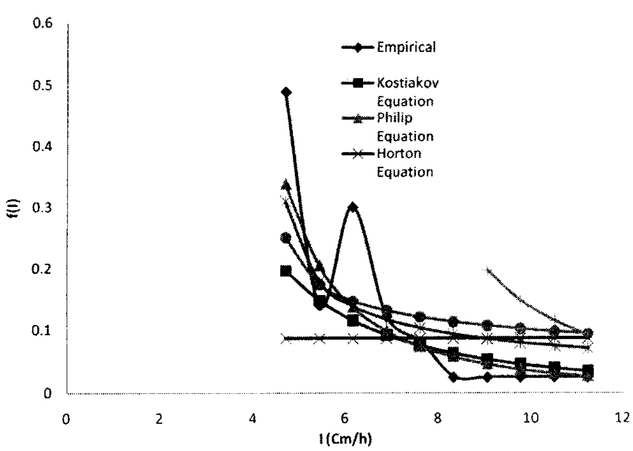

Figure 12. Probability density functions of infiltration equations.

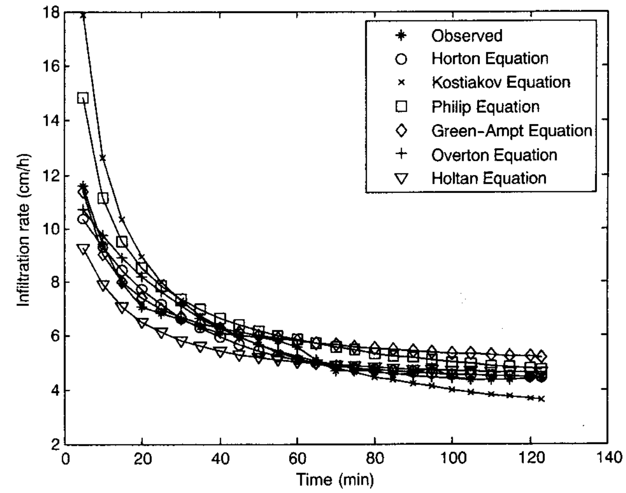

Figure 13. Comparison of different infiltration equations for data set IV.

show in table 3. It is seen that the Horton equation has the highest Tsallis entropy for $m = 2$ and hence more uncertainty. This may be explained by noting that the Horton equation has three parameters, and more observations will be needed to accurately determine these parameters. The Holtan equation has the second highest uncertainty, followed by the Overton equation. The remaining three equations (Kostiakov, Philip two-term, and Green-Ampt) have more or less the same entropy. It may also be noted that the entropy value changes from one data set to another, as it should.

For the sample data set (data set IV), these CDFs and PDFs are shown in figures 11 and 12. The theoretical CDFs and PDFs did not match the empirical CDFs and CDFs. All that the theory does is uncover the probability distributions underlying these equations. Another reason may be that infiltration observations are not entirely independently random. Furthermore, PDFs and CDFs of different equations are quite different from each other, reflecting the differences in the assumptions and hypotheses of these equations. The lack of agreement is not the weakness of the theory but is rather the weakness of the equations themselves. This can be explained as follows. The infiltration equations are only for flow in soil matrix and do not account for macropores or other structural features. In addition, the soil is uniform and homogeneous. Furthermore, it is implicit in these equations that the exit of air as the soil starts to become saturated does not exercise any influence on the rate of water entry. Thus, these equations only approximately yield infiltration capacity rates.

Comparison of Infiltration Equations

The infiltration equations with parameters estimated using the entropy theory were compared for all four data sets. For data set IV, a comparison of these equations along with observations is shown in figure 13. The Green-Ampt equation deviated more from observations than did the other equations. Because of the differences in the integration limits (or domains of solution), it is difficult to employ the entropy values for selecting the best equation for given sets of data.

Table 3. Entropy values associated with infiltration equations for different data sets.

Data Set	Entropy of Infiltration Equations					
	Horton	Kostiakov	Philip	Green-Ampt	Overton	Holtan
I	9.69	0.86	0.86	0.86	0.96	1.20
II	5.82	0.85	0.85	0.85	0.97	1.33
III	9.74	0.89	0.89	0.89	0.97	1.20
IV	7.06	0.92	0.92	0.92	0.88	1.28

CONCLUSIONS

The following conclusions are drawn from this study:

- Derivation of the infiltration equations of Horton, Kostiakov, Philip, Green and Ampt, Overton, and Holtan using the entropy theory leads to three fundamental parameters, including initial infiltration rate (I_0), steady-state or constant infiltration rate (I_c), and the maximum soil moisture retention (S). Parameters arising in these equations can be expressed in terms of these fundamental quantities, which all can be obtained from observations. In this manner, the entropy theory renders these infiltration equations non-parametric or parameter-free.
- In the case of the Overton equation, there is a time parameter that indicates the time at which the infiltration rate becomes constant. This parameter must be obtained from either observations or by calibration, and the Tsallis entropy theory provides no formulation for this time parameter.
- The infiltration rates computed by the six equations using the Tsallis entropy theory based parameters compare reasonably well with those computed using parameters obtained by calibration. In the case of the entropy theory, parameters are obtained from observations and no calibration is needed.
- The entropy theory provides a physical interpretation of infiltration equation parameters. Parameters of each infiltration equation are expressed in terms of physically measurable quantities, which have physical meaning. For example, the Horton parameter k can be interpreted as average travel time.

ACKNOWLEDGEMENTS

Mr. Zengchao Hao from the Department of Biological and Agricultural Engineering, Texas A&M University, helped with computations and graphs. His help is gratefully acknowledged.

REFERENCES

Awumah, K., I. Goulter, and S. K. Bhatt. 1991. Entropy-based redundancy measures in water distribution networks. *J. Hydraulic Eng. ASCE* 117(5): 595-614.

Barbé, D. E, J. F. Cruise, and V. P. Singh. 1991. Solution of three-constraint entropy-based velocity distribution. *J. Hydraulic Eng. ASCE* 117(10): 1389-1396.

Burgy, R. H., and J. N. Luthin. 1956. A test of the single and double ring types of infiltrometers. *Trans. American Geophysical Union* 27(2): 189-192.

Chiu, C. L. 1987. Entropy and probability concepts in hydraulics. *J. Hydraulic Eng. ASCE* 113(5): 583-600.

Chiu, C. L. 1988. Entropy and 2-D velocity distribution in open channels. *J. Hydraulic Eng. ASCE* 114(7): 738-756.

Chiu, C. L. 1989. Velocity distribution in open channel flow. *J. Hydraulic Eng. ASCE* 115(5): 576-594.

Chiu, C. L. 1991. Application of entropy concept in open channel flow study. *J. Hydraulic Eng. ASCE* 117(5): 615-628.

Crawford, N. H., and R. K. Linsley. 1966. Digital simulation in hydrology: Stanford watershed model IV. Technical Report No. 39. Palo Alto, Cal.: Stanford University.

Donigian, A. S., and J. Imhoff. 2006. Chapter 2: History and evolution of watershed modeling derived from the Stanford watershed model. In *Watershed Models*, 21-45. V. P. Singh and D. K. Frevert, eds. Boca Raton, Fla.: CRC Press.

Fiorentino, M., P. Claps, and V. P. Singh. 1993. An entropy-based morphological analysis of river basin networks. *Water Resources Res.* 29(4): 1215-1224.

Green, W. H., and C. A. Ampt. 1911. Studies on soil physics: I. Flow of air and water through soils. *J. Agric. Sci.* 4: 1-24.

Harmancioglu, N. B., O. Fistikoglu, S. D. Ozkul, V. P. Singh, and M. N. Alpaslan. 1999. *Water Quality Monitoring Network Design.* Boston, Mass.: Kluwer Academic.

Holtan, H. N. 1961. A concept of infiltration estimates in watershed engineering. ARS-41-51. Washington, D.C.: USDA Agricultural Research Service.

Horton, R. E. 1938. The interpretation and application of runoff plot experiments with reference to soil erosion problems. *SSSA Proc.* 3: 340-349.

Jaynes, E. T. 1957a. Information theory and statistical mechanics: I. *Physical Review* 106(4): 620-630.

Jaynes, E. T. 1957b. Information theory and statistical mechanics: II. *Physical Review* 108(2): 171-190.

Jaynes, E.T. 1958. Probability theory in science and engineering. Colloquium Lectures in Pure and Applied Science, No. 4. Dallas, Tex.: Socony Mobil Oil Company.

Jaynes, E. T. 1982. On the rationale of maximum-entropy methods. *Proc. IEEE* 70(9): 939-952.

Keylock, C. J. 2005. Describing the recurrence interval of extreme floods using nonextensive thermodynamics and Tsallis statistics. *Advances in Water Resources* 28(8): 773-778.

Klir, G. J. 2006. *Uncertainty and Information: Foundations of Generalized Information Theory.* New York, N.Y.: John Wiley and Sons.

Kostiakov, A. N. 1932. On the dynamics of the coefficient of water percolations in soils. Trans. 6th Congress Intl. Soil Sci. Soc., Part A: 15-21.

Koutsoyiannis, D. 2005a. Uncertainty, entropy, scaling, and hydrological stochastics: 1. Marginal distributional properties of hydrological processes and state scaling. *Hydrol. Sci. J.* 50(3): 381-404.

Koutsoyiannis, D. 2005b. Uncertainty, entropy, scaling, and hydrological stochastics: 2. Time dependence of hydrological processes and state scaling. *Hydrol. Sci. J.* 50(3): 405-426.

Maruyama, T., T. Kawachi, and V. P. Singh. 2005. Entropy-based assessment and clustering of potential water resources availability. *J. Hydrol.* 309: 104-113.

Overton, D. E. 1964. Mathematical refinement of an infiltration equation for watershed engineering. ARS-41-99. Washington, D.C.: USDA Agricultural Research Service.

Philip, J. R. 1957. Theory of infiltration, Parts 1 and 4. *Soil Sci.* 85(5): 345-357.

Rawls, W., P. Yates, and L. Asmussen. 1976. Calibration of selected infiltration equations for the Georgia plain. ARS-S-113. New Orleans, La.: USDA Agricultural Research Service.

Singh, V. P. 1989. *Hydrologic Systems: Vol. II. Watershed Modeling*. Englewood Cliffs, N.J.: Prentice Hall.
Singh, V. P., ed. 1995. *Computer Models of Watershed Hydrology.* Littleton, Colo.: Water Resources Publications.
Singh, V. P. 1997. The use of entropy in hydrology and water resources. *Hydrol. Proc.* 11(6): 587-626.
Singh, V. P. 1998. *Entropy-Based Parameter Estimation in Hydrology.* Dordrecht, The Netherlands: Kluwer Academic.
Singh, V. P., and D. K. Frevert, eds. 2002a. *Mathematical Models of Large Watershed Hydrology.* Highlands Ranch, Colo.: Water Resources Publications.
Singh, V. P., and D. K. Frevert, eds. 2002b. *Mathematical Models of Small Watershed Hydrology and Applications.* Highlands Ranch, Colo.: Water Resources Publications.
Singh, V. P., and D. K. Frevert, eds. 2006. *Watershed Models.* Boca Raton, Fla.: CRC Press.
Singh, V. P., and D. A. Woolhiser. 2002. Mathematical modeling of watershed hydrology. *J. Hydrol. Eng.* 7(4): 270-292.
Singh, V. P., and F. X. Yu. 1990. Derivation of infiltration equation using systems approach. *J. Irrig. and Drainage Eng.* 116(6): 837-858.
Tsallis, C. 1988. Possible generalization of Boltzmann-Gibbs statistics. *J. Statistical Physics* 52(1/2): 479-487.

APPENDICES

APPENDIX A: KOSTIAKOV EQUATION

Specification of constraints: Let the constraints be defined as equation 2 and:

$$\int_{I_c}^{\infty} I^{-2(m-1)} f(I)dI = E[I^{-2(m-1)}] = \overline{I^{-2(m-1)}} \tag{A.1}$$

where I_c is some small value equal to steady infiltration but tending to 0.

Maximization of entropy: Using POME and the method of Lagrange multipliers, the Lagrange function L becomes:

$$L = \frac{1}{m-1}\int_{I_c}^{\infty} f(I)\left\{1-[f(I)]^{m-1}\right\}dI + \lambda_0\left[\int_{I_c}^{\infty} f(I)dI - 1\right] + \lambda_1\left[\int_{I_c}^{\infty} I^{-2(m-1)} f(I)dI - \overline{I^{-2(m-1)}}\right] \tag{A.2}$$

Differentiating equation A.2 with respect to f and equating the derivative to 0, one obtains:

$$\frac{\partial L}{\partial f} \Rightarrow 0 = \frac{1}{m-1}\left\{\int_{I_c}^{\infty}\left[1-[f(I)]^{m-1}-[f(I)]^{m-1}\right]dI\right\} + \lambda_0\left[\int_{I_c}^{\infty} dI\right] + \lambda_1\left[\int_{I_c}^{\infty} I^{-2(m-1)}dI\right] \tag{A.3}$$

Probability density function: Solution of equation A.3 yields $f(I)$ as:

$$f(I) = \left[\frac{1}{m} + \frac{m-1}{m}(\lambda_0 + \lambda_1 I^{-2(m-1)})\right]^{\frac{1}{m-1}} \tag{A.4}$$

Let $\lambda_* = \lambda_0 + \frac{1}{m-1}$, $A = \frac{m-1}{m}\lambda_*$, and $B = \frac{m-1}{m}\lambda_1$.

Introducing these quantities in equation A.4, one obtains:

$$f(I) = \left[A + BI^{-2(m-1)}\right]^{\frac{1}{m-1}} \tag{A.5}$$

If it is assumed that $A = 0$ and $m = 2$, then:

$$f(I) = \frac{B}{I^2} \tag{A.6}$$

Equation A.4 will satisfy the total probability given by equation 2 if $B = I_c$. This means that $\lambda_1 = mI_c/(m - 1)$. If $m = 2$, then $\lambda_1 = 2I_c$.

Kostiakov infiltration equation: Combining equation A.6 with equation 13, the result with limits on I from I to ∞ and on J from J to 0 is:

$$\frac{I_c S}{I} = J \tag{A.7}$$

Recalling that $I = dJ/dt$, equation A.7 can be expressed as:

$$\frac{dJ}{dt} = \frac{I_c S}{J} \Rightarrow J = (2I_c S)^{0.5} t^{0.5} \tag{A.8}$$

Integration of equation A.6 yields:

$$J = (2I_c S)^{0.5} t^{0.5} \tag{A.9}$$

Differentiating equation A.9, one obtains the rate of infiltration:

$$I = \frac{1}{2}(2I_c S)^{0.5} t^{-0.5} \tag{A.10}$$

Equation A.9 can be recast as:

$$J = at^{0.5} \tag{A.11}$$

and equation A.12 as:

$$I(t) = 0.5at^{-0.5} \tag{A.12}$$

which is the Kostiakov equation with a as parameter expressed as:

$$a = 2I_c S \tag{A.13}$$

Thus, parameter a has physical meaning.

The probability density function of the Kostiakov equation can be expressed as:

$$f(I) = \frac{I_c}{I^2} \tag{A.14}$$

Entropy of Kostiakov equation: Substituting equation A.14 in equation 1b and with $m = 2$, the entropy of the Kostiakov equation can be written as:

$$H = 1 - \frac{1}{3I_c} \quad \text{(A.15)}$$

Appendix B: Philip Two-Term Equation

Specification of constraints: Let the infiltration rate be defined as $i = I - a$, where a is some constant value. Let the constraints be defined by equation 2 with limits as a to ∞, and:

$$\int_a^{\infty} i^{-2(m-1)} f(i) di = E\left[i^{-2(m-1)}\right] = \overline{i^{-2(m-1)}} \quad \text{(B.1)}$$

Maximization of entropy: Using POME and the method of Lagrange multipliers, the Lagrangean function L is:

$$L = \frac{1}{m-1} \cdot \int_a^{\infty} f(i)\left\{1 - [f(i)]^{m-1}\right\} di + \lambda_0 \left[\int_a^{\infty} f(i) di - 1\right] + \lambda_1 \left[\int_a^{\infty} i^{-2(m-1)} f(i) di - \overline{i^{-2(m-1)}}\right] \quad \text{(B.2)}$$

Differentiating equation B.2 with respect to f and equating the derivative to 0, one gets:

$$\frac{\partial L}{\partial f} \Rightarrow 0 = \frac{1}{m-1} \cdot \left\{\int_a^{\infty} \left[1 - [f(i)]^{m-1} - (m-1)[f(i)]^{m-1}\right]\right\} dI + \lambda_0 \left[\int_a^{\infty} di\right] + \lambda_1 \left[\int_a^{\infty} i^{-2(m-1)} di\right] \quad \text{(B.3)}$$

Probability density function: Solution of equation B.3 yields $f(i)$ as:

$$f(i) = \left[c + di^{\;2(m\;\;1)}\right]^{\frac{1}{m\;\;1}} \quad \text{(B.4)}$$

where $c = \frac{m-1}{m}\lambda_*$, $d = \frac{m-1}{m}\lambda_1$, and $\lambda_* = \frac{1}{m-1} + \lambda_0$.

If c is assumed zero, and $m = 2$, then:

$$f(i) = di^{-2} \quad \text{(B.5)}$$

Substituting equation B.3 in equation 2, one gets:

$$\int_a^{\infty} di^{-2} di = 1 \Rightarrow d = a;\ \lambda_1 = 2a \quad \text{(B.6)}$$

Equation B.4 is the probability density function of the Philip equation.

Philip infiltration equation: Combining equation B.5 with equation 13, and integrating with limits on i from i to ∞ and on J from J to 0, the result is:

$$\frac{aS}{i} = J \quad \text{(B.7)}$$

Recalling that $i = dJ/dt$, equation B.5 can be expressed as:

$$\frac{dJ}{dt} = \frac{aS}{J} \Rightarrow J = (2aS)^{0.5} t^{0.5} \quad \text{(B.8)}$$

Differentiating equation B.8, one obtains the rate of infiltration:

$$i = \frac{1}{2}(2aS)^{0.5} t^{-0.5} \quad \text{(B.9)}$$

Equation B.9 can be written in original terms as:

$$I(t) = a + 0.5(2aS)^{0.5} t^{-0.5} = a + bt^{-0.5}$$
$$b = 0.5(2aS)^{0.5} \quad \text{(B.10)}$$

which is the Philip two-term equation.

Entropy of Philip equation: Using equation B.6 in equation 1b one obtains the entropy of the Philip equation:

$$H = 1 - \frac{1}{3a} \quad \text{(B.11)}$$

Appendix C: Green-Ampt Equation

Specification of constraints: Let the constraints be defined by equation 2 with limits as b to c where b would tend to ∞, and c to I_c, and:

$$\int_c^{\infty} (I - I_c)^{-2(m-1)} f(I) dI = \overline{(I - I_c)^{-2(m-1)}} \quad \text{(C.1)}$$

Maximization of entropy: Using POME and the method of Lagrange multipliers, the Lagrangean function L is:

$$L = \frac{1}{m-1} \int_c^{\infty} f(I)\{1 - [f(I)]^{m-1}\} dI + \lambda_0 \left[\int_c^{\infty} f(I) dI - 1\right] + \lambda_1 \left[\int_c^{\infty} (I - I_c)^{-2(m-1)} f(I) dI - \overline{(I - I_c)^{-2(m-1)}}\right] \quad \text{(C.2)}$$

Differentiating equation C.2 with respect to f and equating the derivative to 0, one gets:

$$\frac{\partial L}{\partial f} \Rightarrow 0 = \frac{1}{m-1} \cdot \left\{\int_c^{\infty} \left[1 - [f(I)]^{m-1} - (m-1)[f(I)]^{m-1}\right]\right\} dI + \lambda_0 \left[\int_c^{\infty} dI\right] + \lambda_1 \left[\int_c^{\infty} (I - I_c)^{-2(m-1)} dI\right] \quad \text{(C.3)}$$

Probability density function: Equation C.3 yields $f(I)$ as:

$$f(I)=\left[\frac{m-1}{m}(\lambda_*+\lambda_1(I-I_c)^{-2(m-1)}\right]^{\frac{1}{m-1}} \quad (C.4)$$

Let $a=\frac{m-1}{m}\lambda_*,\ b=\frac{m-1}{m}\lambda_1,$ and $\lambda_*=\frac{1}{m-1}+\lambda_0.$

Equation C.4 becomes:

$$f(I)=\left[a+b(I-I_c)^{-2(m-1)}\right]^{\frac{1}{m-1}} \quad (C.5)$$

Taking $a = 0$ and $m = 2$, equation C.6 reduces to:

$$f(I)=\frac{b}{(I-I_c)^2} \quad (C.6)$$

In order for $f(I)$ to satisfy equation 2, $b = I_c$ or $\lambda_1 = mI/m - 1)$. If $m = 2$, then:

$$f(I)=\frac{I_c}{(I-I_c)^2} \quad (C.7)$$

Equation C.7 is the probability density function of the Green-Ampt equation. It should, however, be noted that this density function is valid only for $2I_c \le I < \infty$, not for the entire first quadrant.

G-A infiltration equation: Combining equation C.7 with equation 13, the result is:

$$\frac{I_c dI}{(I-I_c)^2}=-\frac{1}{S}\frac{dJ}{dI} \quad (C.8)$$

Integrating with limits for I from I to ∞ and for J from J to 0 yields:

$$\frac{I_c}{(I-I_c)}=-\frac{J}{S} \quad (C.9)$$

Recalling that $I = dJ/dt$, equation C.9 can be expressed as:

$$\frac{dJ}{dt}=\frac{SI_c}{J}+I_c \quad (C.10)$$

Solution of equation C.10, with the condition that $t = 0, J = 0$, is:

$$t=\frac{1}{I_c}\left[J-S\log\left(1+\frac{J}{S}\right)\right] \quad (C.11)$$

Equation C.11 can be expressed as:

$$t=\frac{1}{I_c}\left[J-\frac{a}{I_c}\log\left(1+\frac{J}{a/I_c}\right)\right] \quad (C.12)$$

where

$$a = SI_c \quad (C.13)$$

Equation C.12 is the Green-Ampt equation in which parameter I_c can be interpreted as equal to saturated hydraulic conductivity and parameter S equal to the product of the capillary suction at the wetting front and the initial moisture deficit.

Entropy of G-A equation: Entropy of the Green-Ampt equation can be written by substituting equation C.7 in equation 1b as:

$$H=1-\frac{1}{3I_c} \quad (C.14)$$

Appendix D: Overton Equation

Specification of constraints: Let the constraints be defined by equation 2 and:

$$\int_0^{i_0} i^{-0.5(m-1)} f(i)di=E\left[i^{-0.5(m-1)}\right]=\overline{i^{-0.5(m-1)}} \quad (D.1)$$

Maximization of entropy: Using POME and the method of Lagrange multipliers, the Lagrangean function L is:

$$L=\frac{1}{m-1}\int_0^{i_0} f(i)\left\{1-[f(i)]^{m-1}\right\}di + \lambda_0\left[\int_0^{i_0} f(i)di-1\right] + \lambda_1\left[\int_0^{i_0} i^{-0.5(m-1)} f(i)di-\overline{i^{-0.5(m-1)}}\right] \quad (D.2)$$

Differentiating equation D.2 with respect to f and equating the derivative to 0, one gets:

$$\frac{\partial L}{\partial f}\Rightarrow 0=\frac{1}{m-1}\left\{\int_0^{i_0}\left[1-[f(i)]^{m-1}-(m-1)[f(i)]^{m-1}\right]\right\}di + \lambda_0\left[\int_0^{i_0} di\right]+\lambda_1\left[\int_c^{\infty} i^{-0.5(m-1)}di\right] \quad (D.3)$$

Probability density function: Solution of equation D.3 yields $f(i)$ as:

$$f(I)=\left[\frac{m-1}{m}\left(\lambda_*+\lambda_1 i^{-0.5(m-1)}\right)\right]^{\frac{1}{m-1}} \quad (D.4)$$

Let $\lambda_*=\lambda_0+\frac{1}{m-1},\ A=\frac{m-1}{m}\lambda_*,$ and $B=\frac{m-1}{m}\lambda_1.$

Equation D.4 becomes:

$$f(I)=\left[A+Bi^{-0.5(m-1)}\right]^{\frac{1}{m-1}} \quad (D.5)$$

Assuming $A = 0$ and $m = 2$, equation D.5 becomes:

$$f(I)=Bi^{-0.5}$$

$$\int_{I_c}^{I_0} B(I - I_c)^{-0.5}\, dI = 1 \Rightarrow B = \frac{1}{2(I_0 - I_c)^{0.5}} \quad \text{(D.6)}$$

Inserting equation D.6 in equation D.4 yields:

$$f(I) = \frac{i^{-0.5}}{2i_0^{0.5}} \quad \text{(D.7)}$$

Equation D.7 can be cast as:

$$f(I) = \frac{(I - I_c)^{-0.5}}{2(I_0 - I_c)^{0.5}} \quad \text{(D.8)}$$

Equation D.8 is the probability density function of the Overton model.

Overton infiltration equation: Substituting equation D.8 in equation 12, one obtains:

$$-\frac{1}{S}\, dJ = \frac{0.5(I - I_c)^{-0.5}}{(I_0 - I_c)^{0.5}}\, dI \quad \text{(D.9)}$$

Integration of equation D.9 yields:

$$I = \frac{(I_0 - I_c)}{S^2} J^2 + I_c \quad \text{(D.10)}$$

Recalling equation 9, equation D.10 with limits on t from t to t_c and on J from J to J_c (constant) gives:

$$J = J_c - S\sqrt{\frac{I_c}{(I_0 - I_c)}} \tan\left[\frac{\sqrt{I_c(I_0 - I_c)}}{S}(t_c - t)\right] \quad \text{(D.11)}$$

Differentiating equation D.11 leads to:

$$I(t) = I_c \sec^2\left[\frac{\sqrt{I_c(I_0 - I_c)}}{S}(t_c - t)\right] \quad \text{(D.12)}$$

Let $\quad (I_0 - I_c) = aS^2 \quad$ (D.13)

Equation D.12 becomes:

$$I(t) = I_c \sec^2\left[\sqrt{aI_c}\,(t_c - t)\right] \quad \text{(D.14)}$$

Equation D.14 is the Overton model.

Entropy of Overton equation: Using equation D.8 in equation 1b, one obtains the entropy of the Overton equation:

$$H = 1 - \frac{1}{3}\frac{(I_c)^2}{(I_0 - I_c)^3} \quad \text{(D.15)}$$

Appendix E: Holtan Equation

Analogous to the Horton equation, let i define the excess infiltration rate ($I - I_c$) varying from 0 to i_0 where $i_0 = I_0 - I_c$.

Specification of constraints: Then, the constraints can be defined by equation 2 and equation D.1 (with proper infiltration rate in mind).

Probability density function: Using POME and the method of Lagrange multipliers, $f(i)$ is obtained as equation D.4 and eventually equation D.7:

$$f(I) = \left[A + Bi^{\left(\frac{1-n}{n}\right)(m-1)}\right]^{\frac{1}{m-1}} \quad \text{(E.1)}$$

where

$$A = \frac{m-1}{m}\lambda_*,\ B = \frac{m-1}{m}\lambda_1,\ \lambda_* = \frac{1}{m-1} + \lambda_0 \quad \text{(E.2)}$$

Let m = 2. Equation E.1 becomes:

$$f(I) = \left[A + Bi^{\left(\frac{1-n}{n}\right)}\right] \quad \text{(E.3)}$$

If A = 0, then equation E.3 can be recast as:

$$f(i) = Bi^{\frac{1-n}{n}} \quad \text{(E.4)}$$

Substituting equation E.4 in equation 2 yields:

$$B = \frac{1}{n(i_0)^{1/n}} \quad \text{(E.5)}$$

Equation E.4, in concert with equation E.5, is the probability density function of the Holtan equation.

Holtan infiltration equation: Substituting equation E.1 in equation 13, one obtains:

$$dJ = -Sbi^{\frac{1-n}{n}}\, di \quad \text{(E.6)}$$

Integration of equation E.6 yields:

$$S - J = Sbni^{\frac{1}{n}} \quad \text{(E.7)}$$

Equation E.7 can be expressed as:

$$(nbS)^n (S - J)^{-n}\, dJ = dt \quad \text{(E.8)}$$

Integrating equation E.8, one obtains:

$$J = S - \left[S^{1-n} - \frac{(1-n)}{(Sbn)^n} t\right]^{\frac{1}{1-n}} \quad \text{(E.9)}$$

Differentiation of equation E.9 with respect to t and simplification yield:

$$i = a\left[S^{1-n} - a(1-n)t\right]^{\frac{n}{1-n}} \quad \text{(E.10)}$$

where

$$a = \frac{i_0}{S^n} \quad \text{(E.11)}$$

Equation E.10 can be written in original terms as:

$$I = I_c + a\left[S^{1-n} - a(1-n)t\right]^{\frac{n}{1-n}}, \quad a = \frac{(I_0 - I_c)}{S^n} \qquad \text{(E.12)}$$

Equation E.12 is the Holtan equation with parameter a given by equation E.11.

Entropy of Holtan equation: Substituting equation E.4 in equation 1b yields the entropy of the Holtan equation:

$$H = 1 + \frac{1}{(2-n)(I_0 - I_c)} \qquad \text{(E.13)}$$

The Holtan equation has two parameters: a and n. Parameter a is expressed in terms of physically measurable quantities. Parameter m needs to be determined now, which can be done as follows.

Substitution of equation E.1 in equation 3 yields:

$$\overline{(I - I_c)} = B\frac{n}{1+n}(I_0 - I_c)^{\frac{1+n}{n}} \qquad \text{(E.14)}$$

Thus, n can also be expressed in terms of physically measurable quantities. In the simulation, parameter n was found to be 1.5.