A Kinematic Model for Surface Irrigation

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A kinematic wave model is developed to study surface irrigation. Depending on the variability of infiltration and the kinematic wave friction parameter, three cases are distinguished. Explicit analytical solutions are obtained for the case when infiltration is constant, and a possible approach is suggested for the case when it is not.

INTRODUCTION

Fluid flow in surface irrigation is an example of unsteady, nonuniform, gradually varied, free surface open channel flow over a porous bed with variable infiltration rate. Most models of surface irrigation can be grouped into three classes: (1) storage models, (2) kinematic models, and (3) dynamic models.

A storage model, often referred to as a volume balance model, is based on the equation of mass continuity and some assumptions. One of the earliest and most familiar storage models for the irrigation advance problem is perhaps the one developed by *Lewis and Milne* [1938]. The hypothesis underlying this model has since been incorporated in many investigations [*Hall,* 1956; *Davis,* 1961; *Philip and Farrell,* 1964; *Fok and Bishop,* 1965; *Wilke and Smerdon,* 1965; *Hart et al.*, 1968; *Anjaneyulu and Mishra,* 1972; *Lal and Pandya,* 1972; *Sastry and Agrawal,* 1972; *Collis-George,* 1974; *P. Singh and Chauhan,* 1972, 1975; *J. Singh,* 1975; *Cunge and Woolhiser,* 1975]. Traditionally, the solutions are obtained to relate the rate of advance in surface irrigation to variables of soil, crop, and topography through what are termed storage shape factors. There is disagreement regarding the functional form of these factors [*Chen,* 1970; *Fok and Bishop,* 1965; *Wilke and Smerdon,* 1965; *Hart et al.*, 1968; *P. Singh and Chauhan,* 1972]. Besides approximations in description of surface irrigation hydrodynamics, the major drawback of these investigations is their empirical nature. Although somewhat successfully applied to the design of irrigation systems, they contribute little to the understanding of irrigation phenomena.

Dynamic models use complete equations of motion (often referred to as de Saint Venant equations) to describe surface irrigation. Although they are the most accurate equations, they are also the most difficult to solve. The difficulty in their solution is compounded further by the fact that the velocity of advance and recession in surface irrigation is not known a priori, and so the boundaries of the flow region must be found as part of the solution. By making various assumptions regarding the boundaries of the flow region, investigators [*Kruger and Bassett,* 1965; *Tinney and Bassett,* 1961; *Chen and Hansen,* 1966; *Shreiber and Bassett,* 1967; *Kincaid et al.*, 1971; *Smith,* 1972; *Bassett and McCool,* 1973; *Sakkas and Strelkoff,* 1974] have solved these equations numerically. Some investigators [*Su,* 1961; *Olsen,* 1965; *Strelkoff,* 1972] have used simplified forms of these equations to reduce the complexity of numerical solutions.

Kinematic models utilize kinematic wave theory [*Lighthill and Whitham,* 1955] to describe surface irrigation [*Hart et al.*, 1968; *Chen,* 1970; *Smith,* 1972; *Cunge and Woolhiser,* 1975]. Analytical solutions have been obtained for constant infiltration and simple boundary conditions. The use of kinematic wave theory in the study of surface irrigation appears most promising: (1) the study of irrigation phenomena by kinematic wave theory is sufficiently rigorous and realistic and hence contributes to increased understanding of the phenomena, (2) the theory can be developed into an operational tool by which to design surface irrigation systems efficiently, (3) the theory can be utilized to interpret and evaluate constants of storage models physically, and (4) the theory can be used as a basis of comparison of storage models and dynamic models, thus evaluating the worth of complexity in the investigation of irrigation phenomena. For complex input and boundary conditions, analytical solutions may be unwieldy, but hybrid solutions can be easily developed, as is done in hydrologic studies [*V. P. Singh,* 1975*a*, *b*].

In the present investigation our objective is to study mathematically surface irrigation, utilizing kinematic wave theory under sufficiently general conditions. The conditions for which analytical solutions are tractable are specified.

IRRIGATION FLOW PROCESS

Surface irrigation involves movement of water as shallow flow over planes or in channels. When the inflow stream is introduced by the upstream end of the plane, water advances with a sharply defined wetting front down the slope toward the downstream end in what is referred to here as the advance phase of the irrigation flow process. This phase is characterized by downfield movement of the advancing water front and continues until the water reaches the lower end of the field. Assuming continued inflow after water has advanced to the downstream end, water will, if there is no downstream dam, flow out the end of the field and will continue to accumulate in the field in the storage phase. In this phase, water exists on the entire field, neither boundary moves, and inflow continues at the upper end of the field. The storage phase ends, and the depletion phase begins when the inflow ceases. The depletion phase continues until the depth of surface water at the upstream end is reduced to zero. This phase differs from the storage phase only in the absence of inflow into the field. The recession phase begins when the depth of surface water at the upstream decreases to zero. This marks the formation of the drying or recession front. The downfield movement of the drying front characterizes the recession phase of the flow. This phase continues until no water remains in the field and the irrigation is complete. It should be pointed out here that the above description of the irrigation cycle is an idealized one and is due to *Bassett and McCool* [1973].

The general surface irrigation process may thus be consid-

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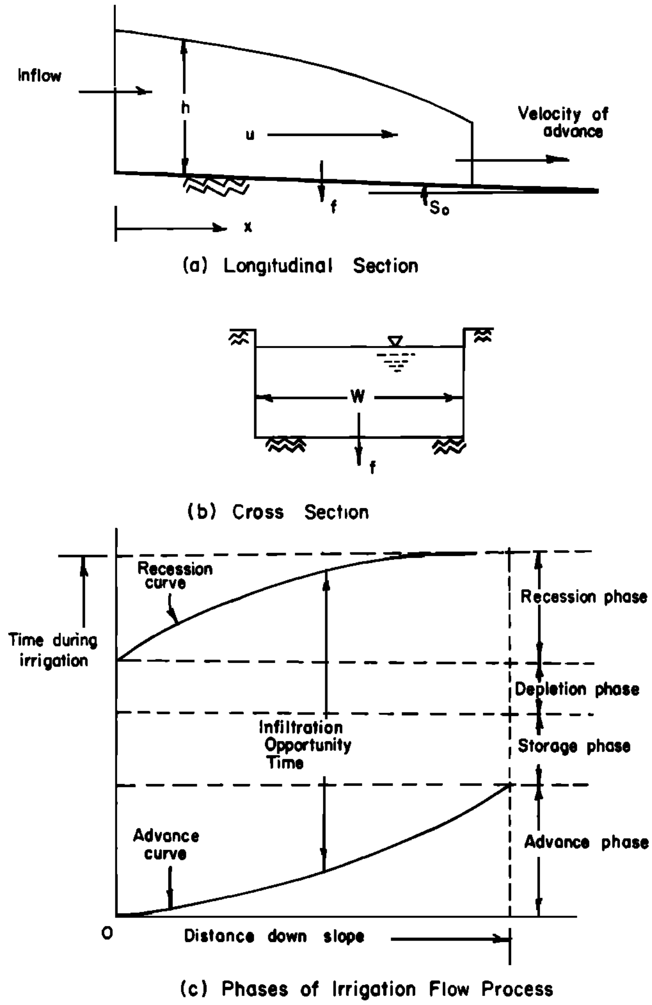

Fig. 1. Surface irrigation process.

ered to include four phases (advance, storage, depletion, and recession) as shown in Figure 1. These phases may occur sequentially, but if the inflow stream is stopped before advance is complete, the storage and even the depletion phase could be eliminated, and recession could occur concurrently with advance.

The advance curve (Figure 1) displays the time at which water arrives at various distances down the field during the advance phase. The recession curve displays the time at which water recedes from the field at various distances during the recession phase. The time interval during which infiltration of water into the soil can occur is bounded by the advance and recession functions and is often referred to as the infiltration opportunity time. Any influence designers have on the depth and uniformity of water application must be exerted through control of these two curves. The mathematical model developed here simulates and displays the advance and recession functions that would result under given irrigation conditions.

Before developing the model we make a few brief remarks on mathematical issues involved in this study.

COMMENTS ON MATHEMATICAL ASPECTS

We will be concerned with the following mathematical topics, which we wish to comment on briefly: quasi-linear first-order partial differential equations, free boundary problems, and differential-difference equations.

The quasi-linear first-order partial differential equation in two independent variables is

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{1}$$

Here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$. The Cauchy problem for (1) is the problem of solving (1) subject to specified values of $u(x, y)$ on a curve C in the (x, y) plane. More precisely, let C be given parametrically by $x = x(\tau)$, $y = y(\tau)$, and let $u = u(\tau)$ be the specified values of $u(x, y)$ on C. Thus we are to find the solution of (1) such that

$$u(\tau) = u(x(\tau), y(\tau)) \tag{2}$$

Interpreted geometrically, the solution $u = u(x, y)$ of (1) is a surface in (x, y, u) space; $x = x(\tau)$, $y = y(\tau)$, and $u = u(\tau)$ is the parameter representation of a space curve Γ in (x, y, u) space; and we wish to find the surface $u = u(x, y)$ which satisfies (1) and contains Γ, that is, satisfies (2). This Cauchy problem can be solved by the method of characteristics [*Courant and Hilbert*, 1962; *Garabedian*, 1964]. Consider the system of ordinary differential equations

$$\frac{dx}{dt} = a(x, y, u) \qquad \frac{dy}{dt} = b(x, y, u) \qquad \frac{du}{dt} = c(x, y, u) \tag{3}$$

These are the characteristic equations associated with (1). Corresponding to each point P in x, y, u space there is a unique solution $x(t, P)$, $y(t, P)$, $u(t, P)$ of (3) passing through P at $t = 0$, the characteristic curve through P. Let P be a point of Γ; since P is uniquely specified by the parameter τ, we may write the characteristic curve through P as

$$x = x(t, \tau) \qquad y = y(t, \tau) \qquad u = u(t, \tau) \tag{4}$$

Equation (4) is the collection of characteristic curves through Γ which taken together constitute a surface. Indeed, (4) is the parametric representation of this surface. It is easy to prove [*Courant and Hilbert*, 1962; *Garabedian*, 1964] that this surface solves the Cauchy problem (1) and (2). More precisely, if we solve the first two equations of (4) for t and τ in terms of x and y and insert in the third equation of (4), we get the solution $u = u(x, y)$ of (1) and (2).

Free boundary problems occur frequently throughout applied mathematics, e.g., in fluid mechanics [*Garabedian*, 1964], heat conduction with change of phase [*Carslaw and Jaeger*, 1959; *Friedman*, 1964], and probability and statistics [*Chernoff*, 1968]. In these problems, one or more partial differential equations are to be solved in some domain D subject to boundary conditions on the boundary of D. But D is not fully specified, and the determination of part or all of the boundary is part of the solution of the problem. An example, which is essentially the problem considered in this paper, is this: there is a channel of uniform cross section and uniform slope; x is the distance along the channel. At $x = 0$, appropriate boundary conditions, i.e., depth $h(0, t)$ or velocity $u(0, t)$ or both, are specified. Appropriate initial conditions are determined by the nature of the flow, subcritical or supercritical [*Stoker*, 1957, pp. 300–305], and also by the mathematical model. Since we use kinematic wave theory in the next section, u and h are functionally related, and therefore we need to specify only h. The channel is initially dry, $h(x, 0) = 0$, and when $h(x, t) > 0$, there is infiltration into the channel floor; the infiltration rate may be a function of x and t. There are equations of continuity and momentum for the depth $h(x, t)$ and velocity $u(x, t)$, but there is also the time history of the interface $s(t)$ between the covered ($x < s(t)$) and uncovered ($x > s(t)$) parts of the channel. The function $s(t)$ is not known and has to be determined along with u and h. In the (x, t) plane the domain D is

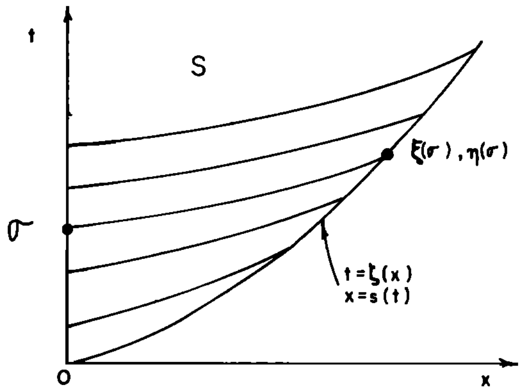

Fig. 2. Solution domain for (6) and (7).

bounded by the positive t axis and the curve $x = s(t)$; the equations of continuity and momentum are satisfied in D. There must be some further conditions specified in order that $s(t)$ be determined.

Differential-difference equations also occur frequently in applied mathematics [*Elsgolts*, 1966; *Hale*, 1971]. The essential difference between differential equations and differential-difference equations is this: in a differential equation the unknown function and its derivatives are evaluated at one value of the independent variable (or variables if the equation is partial), while in a differential-difference equation the unknown function and its derivatives are evaluated at more than one value of the independent variable (or variables). In the equation $y'(x) = ay(x)$, whose general solution is $y(x) = ce^{ax}$, both $y(x)$ and $y'(x)$ are evaluated at x. But the equation $y'(x) = ay(x - 1)$ is a differential-difference equation because the derivative y' is evaluated at x, while y is evaluated at $x - 1$. The following partial differential-difference equation occurs later in this paper:

$$\left[1 - n\left(\frac{h(x, \tau)}{h(x, 0)}\right)^{n-1}\right] h_\tau(x, \tau) + n\alpha h^{n-1}(x, \tau) h_x(x, \tau) = -f(\tau) \qquad (5)$$

In (5) the partial derivatives of h_τ and h_x are evaluated at (x, τ), but h, which appears in the coefficients, is evaluated both at (x, τ) and at $(x, 0)$.

THE KINEMATIC WAVE MODEL

We consider an irrigation channel of uniform cross section. Let x be the distance along the channel, which we assume extends indefinitely to the right of $x = 0$. At $x = 0$ we assume a time dependent depth of water, and we assume that initially there is no water in the channel. We assume that there is a front wall of water (this assumption is discussed in the last section of this paper) which advances to the right; let $x = s(t)$ or $t = \zeta(x)$ be the time history of that advancing front (Figure 2). Let $f(\tau)$ be the infiltration rate (volume per unit area) at time $\tau = t - \zeta(x)$; τ is the elasped time after the front wall of water covers x. We are assuming that the infiltration rate $f(\tau)$ is not x dependent. Then, if $h(x, t)$ is the depth of water and $Q(x, t)$ is the rate of flow (volume per unit length perpendicular to the direction of x), the equation of continuity is

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = -f[t - \zeta(x)] \qquad (6)$$

For the momentum equation we take the following simple equation [*Eagleson*, 1970, chapter 15]:

$$Q(x, t) = \alpha(x) h^n(x, t) \qquad (7)$$

Here $\alpha(x)$ is a friction coefficient, and $n > 1$. Let $u(x, t)$ be the velocity. Then, since $Q = uh$, we get

$$u(x, t) = \alpha(x) h^{n-1}(x, t) \qquad (8)$$

Thus the general momentum equation, which is a first-order partial differential equation, is replaced by the simple relation (8). Equations (6) and (7), taken together, constitute a kinematic wave model for the flow [*Lighthill and Whitham*, 1955]. Combining (6) and (7), we get

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[\alpha(x) h^n] = -f[t - \zeta(x)] \qquad (9)$$

The initial condition is

$$h(0, t) = g(t) \qquad t \geqq 0 \qquad (10)$$

We have yet to formulate an equation for the free boundary $x = s(t)$ or $t = \zeta(x)$; this is obtained from (8) by replacing x by $s(t)$ and observing that $u(s(t), t)$ is the velocity of the front wall of water. Thus

$$s'(t) = \alpha[s(t)] h^{n-1}(s(t), t) \qquad s(0) = 0 \qquad (11)$$

Since the inverse of $s(t)$ appears in (9), it is preferable to express the free boundary equation in terms of $\zeta(x)$ rather than $s(t)$. Referring to (8), we replace t by $\zeta(x)$ and use $\zeta'(x) = [s'(t)]^{-1}$ to obtain

$$\zeta'(x) = [\alpha(x) h^{n-1}(x, \zeta(x))]^{-1} \qquad \zeta(0) = 0 \qquad (12)$$

Equation (12) is valid when $h(x, t) > 0$. Thus the problem is formulated by (9), (10), and (12). We note that there are two unknown functions, $h(x, t)$ and $\zeta(x)$, which satisfy a partial differential equation (equation (9)) and an ordinary differential equation (equation (12)). Equation (9) is satisfied in the domain bounded by the positive t axis and the curve $t = \zeta(x)$.

We will consider the following cases: f and α constant, f constant and α not constant, and f not constant. With regard to $g(t)$ we first consider $g(t) > 0$, $t \geqq 0$, and then $g(t) > 0$, $0 \leqq t < T$, $g(t) = 0$, $t \geqq T$ for some $T > 0$.

CASE $f(t) = f = $ CONST AND $\alpha(x) = \alpha = $ CONST

In this case, (9) and (12) are uncoupled, and we can solve (9) and (10) for h. Considering first $g(t) > 0$ for all t, we apply the method of characteristics to (9) and (10). To this end, we take σ as the parameter on the t axis and t as the parameter along the characteristic curve. The characteristic curve $x(t, \sigma)$, $h(t, \sigma)$ passing through the points $(0, \sigma, g(\sigma))$ in (x, t, h) space satisfies

$$\frac{dx(t, \sigma)}{dt} = n\alpha h^{n-1}(t, \sigma) \qquad x(\sigma, \sigma) = 0$$
$$\frac{dh(t, \sigma)}{dt} = -f \qquad h(\sigma, \sigma) = g(\sigma) \qquad (13)$$

$x(t, \sigma)$ is the position at time t of the thin slab of water which is at $x = 0$ at time $t = \sigma$; $h(t, \sigma)$ is the height of that same slab at time t. The solution of (13) is, when $f > 0$,

$$h(t, \sigma) = g(\sigma) - f(t - \sigma)$$
$$x(t, \sigma) = \frac{\alpha}{f}\{g^n(\sigma) - [g(\sigma) - f(t - \sigma)]^n\} \qquad (14)$$

If $f = 0$, the solution of (13) is

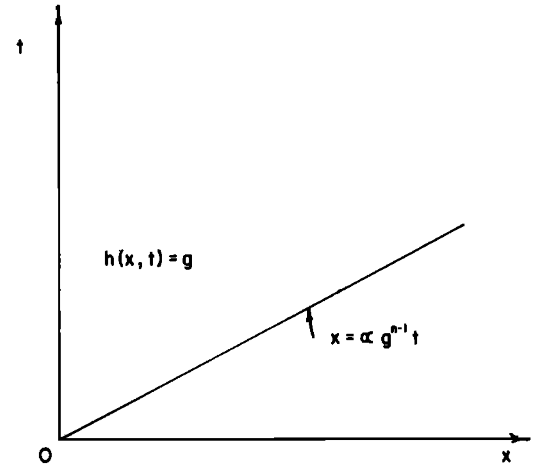

Fig. 3. Solution for the case when $f = 0$ and g and α are constant.

$$h(t, \sigma) = g(\sigma) \qquad x(t, \sigma) = n\alpha(t - \sigma)g^{n-1}(\sigma) \tag{15}$$

Since we are not concerned with negative values of x, $t \geq \sigma$ in (14) and (15). To find $h(x, t)$, we need to solve the equation for $x(t, \sigma)$ for σ, in terms of x and t, in (14) or (15) and substitute in the right side of the equation for $h(t, \sigma)$. In order to do this, $x(t, \sigma)$ must be, for fixed t, either an increasing function of σ or a decreasing function of σ. From

$$x_\sigma(t, \sigma) = \frac{\partial x(t, \sigma)}{\partial \sigma}$$

$$= \frac{\alpha n}{f} \left\{ g'(\sigma) \left[g^{n-1}(\sigma) - h^{n-1}(t, \sigma) \right] - f h^{n-1}(t, \sigma) \right\} \tag{16}$$

we see that $g'(\sigma) = dg/d\sigma \leq 0$ is a sufficient condition for $x_\sigma(t, \sigma) < 0$, that is, that $x(t, \sigma)$ is a decreasing function of σ for fixed t. The condition $g'(\sigma) \leq 0$ includes the case of principal interest to us, namely, $g(\sigma) =$ const. On the other hand, $x_\sigma(t, \sigma) > 0$ implies $g'(\sigma) > 0$; this case entails shock formation and is not discussed in this paper. Accordingly, we assume $g'(\sigma) \leq 0$. Under this condition, $x(t, \sigma)$ is a decreasing function of σ for fixed t in both (14) and (15), and therefore (14) defines $h(x, t)$ when $f > 0$, and (15) defines $h(x, t)$ when $f = 0$.

In order to find the free boundary $x = s(t)$ or $t = \zeta(x)$ it is convenient to express it in terms of the parameter σ introduced above. If we consider the curve $x(t, \sigma)$ in the (x, t) plane, it will intersect the free boundary at the point $(\xi(\sigma), \eta(\sigma))$ (Figure 2). Therefore $x = \xi(\sigma)$, $y = \eta(\sigma)$ is the parametric representation of the free boundary in terms of the parameter σ. If $f = 0$, we have

$$\begin{aligned} &x(\eta(\sigma), \sigma) = \xi(\sigma) = \eta\alpha g^{n-1}(\sigma) \left[\eta(\sigma) - \sigma \right] \\ &s'(t) = \frac{\xi'(\sigma)}{\eta'(\sigma)} = \alpha h^{n-1}(\eta(\sigma), \sigma) = \alpha g^{n-1}(\sigma) \qquad \eta(0) = 0 \end{aligned} \tag{17}$$

From (17) we get

$$\eta'(\sigma) + \frac{\eta g'(\sigma)}{g(\sigma)} \eta(\sigma) = \frac{n}{n-1} + \frac{n\sigma g'(\sigma)}{g(\sigma)} \qquad \eta(0) = 0 \tag{18}$$

The solution of (18) is

$$\eta(\sigma) = \sigma + \frac{1}{(n-1)g^n(\sigma)} \int_0^\sigma g^n(\gamma)\, d\gamma \tag{19}$$

and from the first equation of (17),

$$\xi(\sigma) = \frac{n\alpha}{(n-1)g(\sigma)} \int_0^\sigma g^n(\gamma)\, d\gamma \tag{20}$$

Equations (19) and (20) are the parametric representation of the free boundary in the case $f = 0$. If $g(\sigma) =$ const,

$$\xi(\sigma) = \frac{n\alpha\sigma g^{n-1}}{n-1} \qquad \eta(\sigma) = \frac{n\sigma}{n-1}$$

which gives $x = \alpha g^{n-1} t$; from (15) we get $h(x, t) = g$ (Figure 3).

To find the free boundary in the case $f > 0$, we use the same parametric representation as in the previous paragraph. Then, analogous to (17),

$$\begin{aligned} &\xi(\sigma) = \frac{\alpha}{f} \{ g^n(\sigma) - [g(\sigma) - f(\eta(\sigma) - \sigma)]^{n-1} \} \\ &\frac{\xi'(\sigma)}{\eta'(\sigma)} = \alpha \{ g(\sigma) - f[\eta(\sigma) - \sigma] \}^{n-1} \qquad \eta(0) = 0 \end{aligned} \tag{21}$$

From (21) we get

$$\eta'(\sigma) = \frac{n[g'(\sigma) + f]}{(n-1)f} - \frac{n g'(\sigma) g^{n-1}(\sigma)}{(n-1)f\{g(\sigma) - f[\eta(\sigma) - \sigma]\}^{n-1}}$$

$$\eta(0) = 0 \tag{22}$$

Equation (22) cannot, in general, be solved in terms of simple functions. If g is constant,

$$\xi(\sigma) = \frac{\alpha}{f} \left[g^n - \left(g - \frac{f\sigma}{n-1} \right)^n \right] \qquad \eta(\sigma) = \frac{n\sigma}{n-1}$$

which gives

$$x = \frac{\alpha}{f} \left[g^n - \left(g - \frac{ft}{n} \right)^n \right] \tag{23}$$

From (14) we get

$$h(x, t) = \left(g^n - \frac{fx}{\alpha} \right)^{1/n}$$

The solution in this case is shown in Figure 4.

We consider now $g(t) > 0$, $0 \leq t < T$, $g(t) = 0$, $t \geq T$. Let $g(t, \epsilon)$ be a continuous function of t coinciding with $g(t)$ on $0 \leq t < T$, $g(t, \epsilon) = 0$ on $t \geq T + \epsilon$ (Figure 5). If $f = 0$, we see from (15) that the lines $x = x(t, \sigma, \epsilon)$, $T \leq \sigma \leq T + \epsilon$, cover the region above $x = x(t, T)$ completely. From (15) we get

$$h(t, \sigma, \epsilon) = \left[\frac{x(t, \sigma, \epsilon)}{n\alpha(t - \sigma)} \right]^{1/(n-1)} \tag{24}$$

From (24) we derive, using $T \leq \sigma \leq T + \epsilon$,

$$\left[\frac{x}{n\alpha(t - T)} \right]^{1/(n-1)} \leq h(x, t, \epsilon) \leq \left[\frac{x}{n\alpha(t - T - \epsilon)} \right]^{1/(n-1)} \tag{25}$$

From (25) we get, letting $\epsilon \to 0$,

$$h(x, t) = \left[\frac{x}{n\alpha(t - T)} \right]^{1/(n-1)} \tag{26}$$

Equation (26) is valid in the domain D bounded by $x = x(t, T)$, $x = 0$, and the free boundary beyond the point P (Figure 6). To obtain the free boundary beyond the point P, we get from (19) and (20)

$$\left[\frac{(n-1)\xi(\sigma, \epsilon)}{n\alpha \int_0^\sigma g^n(\gamma, \epsilon)\, d\gamma} \right]^n = \left[\frac{(n-1)[\eta(\sigma, \epsilon) - \sigma]}{\int_0^\sigma g^n(\gamma, \epsilon)\, d\gamma} \right] \tag{27}$$

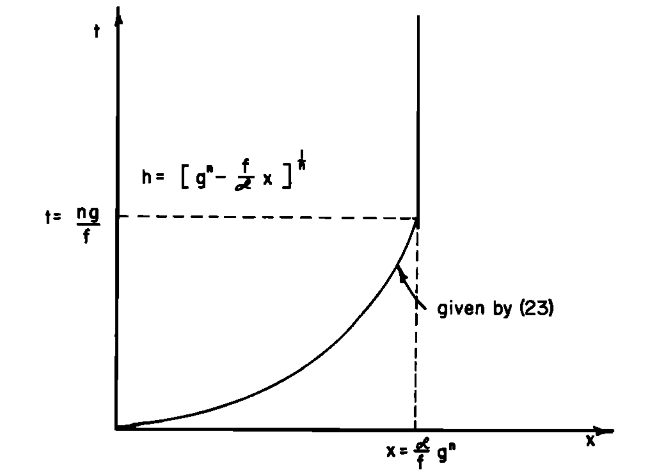

Fig. 4. Solution for the case when f, g, and α are constant, $f > 0$.

where $T \leq \sigma \leq T + \epsilon$. Letting $\epsilon \to 0$ in (19), we get

$$\left[\frac{(n-1)x}{n\alpha A}\right]^n = \frac{(n-1)(t-T)}{A} \qquad A = \int_0^T g^n(\gamma)\,d\gamma \tag{28}$$

Equation (28) is the free boundary beyond P. Equation (26) gives $h(x, t)$ in D. It is clear from (26) and (28) that the free boundary condition (12) is satisfied. It is clear from (26) that $h(x, t) > 0$, t fixed, from $x = 0$ to the free boundary (28). Along the free boundary (28), $h = \beta x^{-1}$, β being a constant.

If $f > 0$, it is plausible to expect another free boundary (FB3) to develop, starting at the point $t = T$, $x = 0$ (Figure 7). This will be the time history of the water edge as it recedes from $x = 0$. This free boundary is the locus $h(t, \sigma) = 0$. Using (14), we have

$$g(\sigma, \epsilon) - f(t - \sigma) = 0 \qquad T \leq \sigma \leq T + \epsilon$$

Thus

$$x(t, \sigma, \epsilon) = \alpha f^{n-1}(t - \sigma)^n \qquad T \leq \sigma \leq T + \epsilon$$

Letting $\epsilon \to 0$, we get the free boundary

$$x = \alpha f^{n-1}(t - T)^n \tag{29}$$

Between (29) and $x = 0$ we have $h = 0$. To obtain h to the right of (29), we have

$$x(t, \sigma, \epsilon) = \frac{\alpha}{f}\{[h(t, \sigma, \epsilon) + f(t - \sigma)]^n - h^n(t, \sigma, \epsilon)\} \qquad T \leq \sigma \leq T + \epsilon \tag{30}$$

Letting $\epsilon \to 0$ in (30), we get

$$x = \frac{\alpha}{f}\{[h + f(t - T)]^n - h^n\} \tag{31}$$

Referring to Figure 7, (31) describes $h(x, t)$ in D_2. In D_1, $h(x, t)$ is given by (14), FB3 is given by (29), and FB1 is given by the solution of (22) and the first equation of (21). The locus $h(x, t) = 0$ (FB2) can be obtained from (14); this is the time history of the front wall when the depth of that wall is 0. If $g(\sigma)$ is constant, we get, for FB2, $x = \alpha g^n/f$, and FB1 is given by (23) (Figure 8). In D_1,

$$h = \left[g^n - \frac{fx}{\alpha}\right]^{1/n}$$

If $g(\sigma)$ is not constant, we get the parametric representation of FB2:

$$x_{FB2} = \frac{\alpha}{f} g^n(\sigma) \qquad t_{FB2} = \frac{1}{f} g(\sigma) + \sigma$$

Thus

$$\frac{dx}{dt} = \frac{x'(\sigma)}{t'(\sigma)} = \frac{\alpha n g^{n-1}(\sigma) g'(\sigma)}{g'(\sigma) + f}$$

Since $g'(\sigma) \leq 0$, we have for FB2,

$$-f \leq g'(\sigma) \leq 0 \to dx/dt \leq 0$$

$$g'(\sigma) < -f \to dx/dt > 0$$

Figure 7 has been drawn in accordance with the condition $-f < g'(\sigma) < 0$, $0 < \sigma < T$; the front wall of water recedes ($dx/dt < 0$) when its depth is 0. If, in Figure 7, P and Q have x coordinates x_1 and x_2 and $x_1 < x < x_2$, then the front wall covers x as it advances with depth greater than 0 and uncovers it as it recedes with depth 0. For a given x the difference of the ordinates to FB3 and FB1 or FB2 and FB1 is the infiltration opportunity time. In Figure 8, $g'(\sigma) = 0$, $0 \leq \sigma \leq T$, and therefore the front wall of water is stationary ($dx/dt = 0$) when it has depth 0. In this case the infiltration opportunity time can be found explicitly from (23) and (29).

CASE $f(t) = f =$ CONST AND $\alpha(x)$ NOT CONSTANT

In this case, (9) becomes

$$h_t + n\alpha(x)h^{n-1}h_x = -f - \alpha'(x)h^n$$

and (13) becomes

$$\frac{dx(t, \sigma)}{dt} = n\alpha(x)h^{n-1}(t, \sigma) \qquad x(\sigma, \sigma) = 0$$
$$\frac{dh(t, \sigma)}{dt} = -f - \alpha'(x)h^n(t, \sigma) \qquad h(\sigma, \sigma) = g(\sigma) \tag{32}$$

Multiplying the first equation of (32) by dh/dt and the second by dx/dt and subtracting, we get

$$f\frac{dx}{dt} + \frac{d}{dt}\,\alpha(x)h^n = 0$$

or integrating from σ to t,

$$fx(t, \sigma) + \alpha(x)h^n(t, \sigma) = \alpha(0)g^n(\sigma)$$

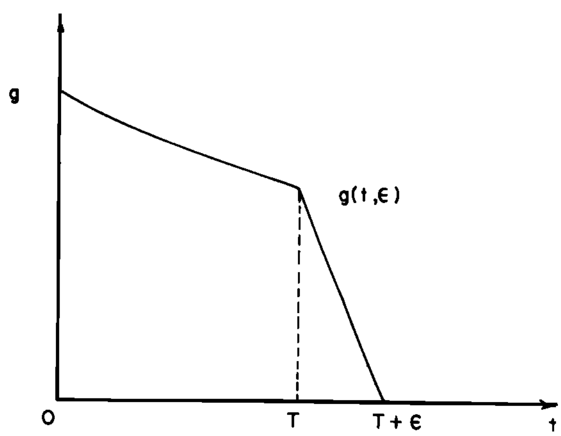

Fig. 5. Function $g(t, \epsilon)$.

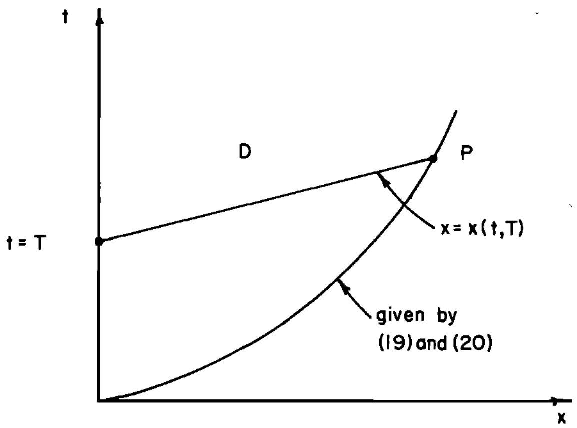

Fig. 6. Solution domain for the case when $f = 0$, $g(t) = 0$, $t \geq T$, and α is constant.

Thus

$$h(t, \sigma) = \left[\frac{\alpha(0)g^n(\sigma) - fx(t, \sigma)}{\alpha(x(t, \sigma))}\right]^{1/n} \tag{33}$$

Inserting (33) in the first equation of (32), we get

$$\frac{dx}{dt} = n\alpha(x)^{1/n}[\alpha(0)g^n(\sigma) - fx]^{(n-1)/n} \tag{34}$$

Integrating (34) gives

$$\int_0^x \frac{d\gamma}{n\alpha(\gamma)^{1/n}[\alpha(0)g^n(\sigma) - f\gamma]^{(n-1)/n}} = t - \sigma \tag{35}$$

Equation (35) implicity defines $x(t, \sigma)$. Thus (35) and (33) define $x(t, \sigma)$ and $h(t, \sigma)$.

We may pursue the matter further in the case $g(t) = \text{const}$. Then, from (33),

$$h(x, t) = \left[\frac{\alpha(0)g^n - fx}{\alpha(x)}\right]^{1/n} \tag{36}$$

Using (12) the free boundary $\zeta(x)$ is

$$\zeta(x) = \int_0^x \frac{d\gamma}{\alpha(\gamma)^{1/n}[\alpha(0)g^n - f\gamma]^{(n-1)/n}} \tag{37}$$

Thus (36) and (37) are the solution of (9), (10), and (12) in the case f constant, g constant, and $\alpha(x)$ a specified function of x. This solution contains, as a special case, the solution corresponding to $\alpha(x) = \alpha = \text{const}$ obtained in the previous section.

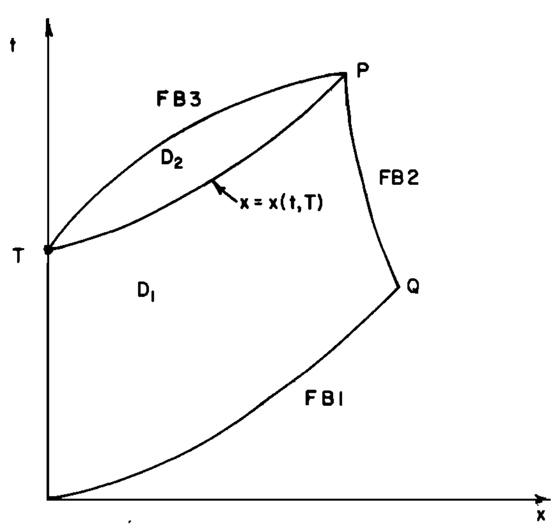

Fig. 7. Solution domain in the case when $f = \text{const} > 0$, $-f < g'(t) < 0$, $0 \leq t \leq T$, and α is constant.

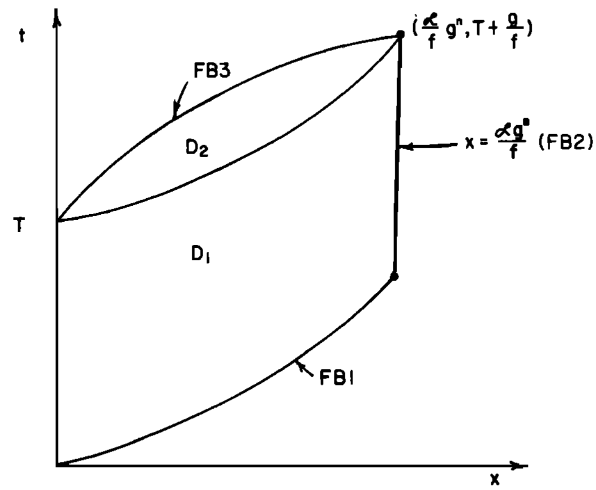

Fig. 8. Solution domain for the case when $f = \text{const} > 0$, $g(t) = \text{const}$, $t > 0$, $0 \leq t \leq T$, and α is constant.

CASE $f(t)$ NOT CONSTANT AND $\alpha(x) = \alpha = \text{CONST}$

In this case we introduce the new variable $\tau = t - \zeta(x)$ to replace t; x remains unchanged. The free boundary $t = \zeta(x)$ maps onto $\tau = 0$ (Figure 9). Then the problem takes the form ($h = h(x, \tau)$)

$$[1 - n\alpha h^{n-1}\zeta'(x)]h_\tau + n\alpha h^{n-1}h_x = -f(\tau) \qquad h(0, \tau) = g(\tau)$$

$$\zeta'(x) = [\alpha h^{n-1}(x, 0)]^{-1} \quad \zeta(0) = 0$$

This can be written as

$$\left[1 - n\left(\frac{h(x, \tau)}{h(x, 0)}\right)^{n-1}\right]h_\tau + n\alpha h^{n-1}(x, \tau)h_x$$

$$= -f(\tau) \qquad h(0, \tau) = g(\tau) \tag{38}$$

Equation (38) is a partial differential-difference equation. Such an equation will not, in general, be solvable explicitly. It is necessary to specify conditions under which (38) has a solution. A possible approach is as follows. We specify $h(x, 0) = \psi(x)$, $\psi(0) = g(0)$. Then we solve (38) with this specification of $h(x, \tau)$ on the x axis. Thus we write the characteristic equations:

$$\frac{d\tau(x, \xi)}{dx} = \frac{1 - n[h(x, \xi)/\psi(\xi)]^{n-1}}{n\alpha h^{n-1}(x, \xi)} \qquad \tau(\xi, \xi) = 0$$

$$\frac{dh(x, \xi)}{dx} = \frac{-f(\tau)}{n\alpha h^{n-1}(x, \xi)} \qquad h(\xi, \xi) = \psi(\xi) \tag{39}$$

The solution $\tau(x, \xi)$, $h(x, \xi)$ is the characteristic of (38) (with $h(x, 0) = \psi(x)$) passing through the point $(\xi, 0, \psi(\xi))$ of (x, τ, h) space. Writing the solution $h(x, \xi; \psi)$, $\tau(x, \xi; \psi)$ to emphasize the dependence on ψ, we have

$$h(0, \xi; \psi) = g(\tau(0, \xi; \psi)) \tag{40}$$

Finally, we need to prove that for each fixed ψ, (40) has a unique positive solution $\psi(\xi)$. We consider, as an example, the case f constant. Then, from (39),

$$\alpha[\psi^n(\xi) - h^n(x, \xi)] = -f(\xi - x)$$

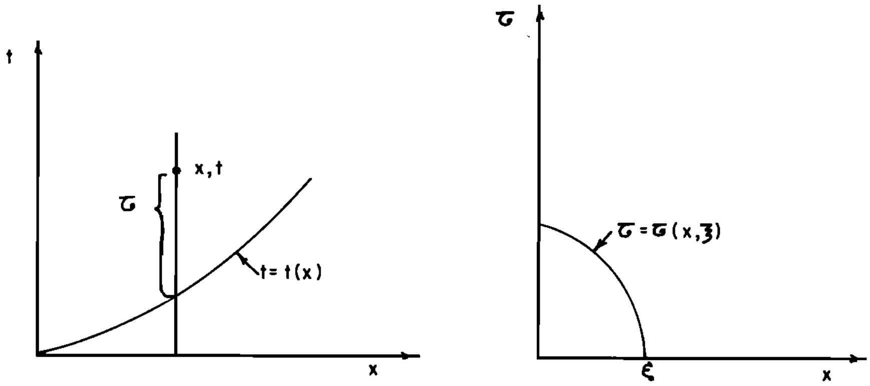

Fig. 9. Solution domain for the case when f is not constant.

and

$$\frac{d\tau(x,\xi)}{dx} = \frac{1}{n\alpha}\left[\psi^n(\xi) + \frac{f}{\alpha}(\xi - x)\right]^{-(n-1)/n} - \frac{1}{\alpha\psi^{n-1}(\xi)} \qquad \tau(\xi,\xi) = 0$$

Thus

$$\tau(x,\xi) = -\frac{1}{f}\left\{\left[\psi^n(\xi) + \frac{f}{\alpha}(\xi - x)\right]^{1/n} - \psi(\xi)\right\} + \frac{\xi - x}{\alpha\psi^{n-1}(\xi)}$$

$$h(x,\xi) = \left[\psi^n(\xi) + \frac{f}{\alpha}(\xi - x)\right]^{1/n} \tag{41}$$

From (41) we get

$$\tau(0,\xi) = -\frac{1}{f}\left[\left(\psi^n(\xi) + \frac{f\xi}{\alpha}\right)^{1/n} - \psi(\xi)\right] + \frac{\xi}{\alpha\psi^{n-1}(\xi)}$$

$$h(0,\xi) = \left[\psi^n(\xi) + \frac{f\xi}{\alpha}\right]^{1/n}$$

Thus, applying (40), we have to prove that

$$\left[\psi^n + \frac{f\xi}{\alpha}\right]^{1/n} = g\left\{-\frac{1}{f}\left[\left(\psi^n + \frac{f\xi}{\alpha}\right)^{1/n} - \psi\right] + \frac{\xi}{\alpha\psi^{n-1}}\right\} \tag{42}$$

has a unique solution ψ for fixed ξ. If g is constant, the solution is

$$\psi = \left[g^n - \frac{f\xi}{\alpha}\right]^{1/n} \qquad 0 \le \xi \le \frac{\alpha g^n}{f} \tag{43}$$

Equations (41) and (43) are the complete solution of the partial differential-difference equation (equation (38)) in the case when f, g, and α are all constant; since

$$h(x,\xi) = \left[g^n - \frac{f}{\alpha}x\right]^{1/n} \tag{44}$$

is independent of ξ, we have

$$h(x,\tau) = \left[g^n - \frac{f}{\alpha}x\right]^{1/n} \tag{45}$$

and (45) satisfies (38). If g is not constant, we proceed as follows. First, suppose $f = 0$. Then (42) becomes

$$\psi = g\left(\frac{(n-1)\xi}{\alpha n\psi^{n-1}}\right) \tag{46}$$

If we graph the left and right sides of (46) as functions of ψ, ξ fixed, we get Figure 10a. Since $g'(t) \le 0$, the right side of (46) is, for ξ fixed, a nondecreasing function of ψ tending to $g(0)$ as $\psi \to \infty$. Thus there is a unique intersection $\psi(\xi)$ of the graphs and therefore a unique solution of (46). If $f > 0$, the argument is similar; the left and right sides of (42) are graphed in Figure 10b as functions of ψ, ξ fixed. The graph of $g(F(\psi, \xi))$ is based on properties of $F(\psi, \xi)$, $\xi > 0$, defined by

$$F(\psi,\xi) = -\frac{1}{f}\left[\left(\psi^n + \frac{f\xi}{\alpha}\right)^{1/n} - \psi\right] + \frac{\xi}{n\psi^{n-1}}$$

namely, that $F(\psi, \xi)$ is a decreasing function of ψ for ξ fixed and that $F(0, \xi) = +\infty$, $F(\infty, \xi) = 0$. We omit the detailed proof of these properties (easily checked in the case $n = 2$). Again $g(F(\psi, \xi))$ is for fixed ξ an increasing function of ψ which has the value $g(\infty)$ when $\psi = 0$ and the value $g(0)$ when $\psi = +\infty$. There is therefore a unique intersection, as indicated in

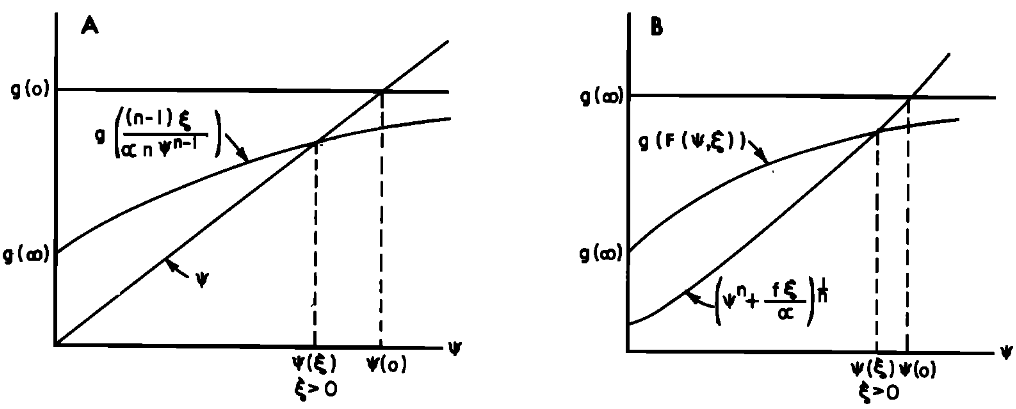

Fig. 10. (a) $f = 0$; (b) f = const > 0.

Figure 10*b*, and therefore a unique solution of (42).

In the general case when f is not a constant we have to conduct the argument without having the explicit solutions (41). Such an argument is probably difficult. On the other hand, the procedure suggested (that is, the specification of $h(x, 0) = \psi(x)$) is easily subject to numerical calculations. Of course, adjustments in $\psi(x)$ are necessary to get agreement with the prescribed $g(\tau)$, so an iterative scheme is necessary.

COMMENTS ON THE HYDRODYNAMIC MODEL

In the kinematic wave model used above, the advancing front wall of water is necessarily an advancing shock wave. But it is plausible to assume that the depth of water at the advancing front is 0. Since kinematic wave theory cannot accommodate itself to this assumption, the full momentum equation has to be used. The degree of accuracy of the kinematic wave solution can be assessed by comparing it with the solution based on the full momentum equation. It is not clear how this free boundary problem should be formulated when the full momentum equation is used. Aside from the condition $h(s(t), t) = 0$ on the free boundary it is not clear what further conditions are needed on the free boundary, if any, and what conditions are needed on $x = 0$. For further relevant remarks on these and other related matters we refer to *Stoker* [1957, pp. 305–342], *Woolhiser* [1970], *Cunge and Woolhiser* [1975], and *Henderson* [1963].

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