MIXED SPECTRAL PROBLEMS FOR SCHRÖDINGER OPERATORS

A Thesis

by

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ABSTRACT

In this thesis, we discuss the mixed spectral problem for Schrödinger operators and study the invertibility properties of a certain Toeplitz operator. Further results of mixed spectral problems obtained by Borg, Hochstadt, Liberman and Horvath are mentioned.

In the first section, we summarize some basic facts of meromorphic inner functions, Herglotz functions, de Branges functions, and model spaces. We define the spectrum of a meromorphic inner function Θ by $\{\Theta = 1\}$. From the construction of a Weyl inner function of a Schrödinger operator, we show that the Dirichlet boundary condition can be replaced by any other boundary conditions with a trivial transform. Also, we discuss a chain structure of the de Branges spaces associated with de Branges functions E which are obtained from the solutions of a Schrödinger equation.

In the second section, we define the kernel of a Toeplitz operator and discuss the criterion of the triviality of the kernel. If we consider regular Schrödinger operators, after proving the ratio of the two meromorphic inner functions of different Schrödinger equations is a trivial factor, we can replace any non-trivial regular potentials by a trivial potential.

In the last section, we study the spectral problems for the Schrödinger operators. Firstly, we discuss the completeness problem of the model space in terms of the invertibility properties of a Toeplitz operator. Then we apply the Toeplitz kernels to the spectral problems. Especially, we characterize the mixed data spectral problem by a determination proposition of meromorphic inner functions $\Theta = \Phi\Psi$. Combined with our definition of the spectrum of a meromorphic inner function, we recover a meromorphic inner function Φ from the spectrum of Θ and the known factor Φ. Furthermore, we use this characterization to prove Hochstadt-Liberman's theorem and Horvath's theorem.

DEDICATION

To my mother, my father.

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1. BACKGROUND AND INTRODUCTION

In quantum mechanics, a Schrödinger equation describes a kind of motion with small distortion to the classical motion and the solutions of a Schrödinger equation determine a probability for finding a particle at some points. Solving the Schrödinger equations is a central topic in the study of quantum mechanics. The current methods are divided into two groups: matrix diagonalization and iteration numerical integration. In both case, the knowledge of the spectrum of Schrödinger operator enables the improvement of the characterization of a solution.

The spectral problems for differential operators attracted great deal of interest among mathematicians in the past. The classical spectral problems have two important branches: inverse and direct problems. The direct spectral problem is to find the spectra of the operator. In the case of Schrödinger operators, the problem is to determine the spectrum from the potential q. Also, the direct spectral problem asks for a spectrum measure of the operator. In inverse problem, we focus on the recovery of the operator from the information of the spectrum. In the case of Schrödinger operators, we try to recover the potential q from $\sigma(L)$, where $\sigma(L)$ denotes the spectrum of L. In this thesis, we mainly consider the inverse spectral problems for Schrödinger operators. Let us mention the important paper of Marchenko [1], where he explicitly states that knowing the information of the spectral measure can uniquely recover the potential q of a Schrödinger operator. Another important paper of Borg [2] states that given two spectra subject to different boundary conditions, the potential q can be uniquely recovered.

A very interesting group of inverse spectral problems is the mixed spectral problems. The mixed spectral problem asks for a description of the amount of information of partial spectrum and partial potential satisfying that the potential q can be uniquely recovered from the information. The amount of information is not quantifiable and the mixed information is case by case, from knowing a part of one spectrum to knowing parts of several spectra or from knowing the restriction of the potential to knowing additional smoothness of the potential. Our approach is in terms of the invertibility properties of a certain Toeplitz operator (whose symbol depends on the Weyl inner function of a Schrödinger operator). And the ideas of our approach enable a precise mathematical characterization of this problem. A special case of this problem was first explicitly solved by Hochstadt and Liberman in 1978 [3], where half of the potential plus one spectrum can recover the potential uniquely. Their statement is precise that missing a small amount of the information cannot recover the potential q. This result partially coincides Borg's theorem: the information of one spectrum gives half of the potential. For further deduction, the information of one-half of one spectrum gives one-fourth of the potential. In this thesis, we will study Hochstadt and Liberman's theorem and give a proof in terms of invertibility of certain Toeplitz operators. Let us emphasize a significant paper by Hruschev, Nikolskii, and Pavolv [4]. The idea to use Toeplitz operators to study of complex analysis was first mentioned.

Moreover, Gesztesy and Simon include direct information of the smoothness to recover the potential, e.g. [5, 6, 7, 8]. In [9], the mixed spectral problems were systematically treated by Horvath (see also [10]). His results generalize almost all former discussion of mixed spectral problem, where the idea to determine the condition of recovery in terms of closeness properties of known spectrum. In his discussion, the information of spectrum is selected from several spectra. In section 4.5, we give an equivalent statement of Horvath's theorem in terms of uniqueness sets in model space and sketch a proof for a special case. In paper [11], Makarov and Poltoratski obtain the most cutting-edge results which combine the uncertain principle with mixed spectral problem. Also, they discuss the three-interval case.

2. PRELIMINARIES

In this section, we discuss some harmonic functions on the upper half plane which will be used in our further discussions. Also, we discuss some spaces of harmonic functions: Hardy space $H^p(\mathbb{C}_+)$ and Smirnov class $N^+(\mathbb{C}_+)$; And we state the correspondence between Herglotz functions and meromorphic inner functions; In the end, we state the definition of de Branges functions and discuss an interesting chain structure of de Branges spaces. Some simple examples are included.

2.1 Harmonic functions on $\mathbb{C}$

Given a function $f(x) \in L^p(\mathbb{R}), 1 < p < \infty$, f can be instantly extended to a harmonic function on $\mathbb{C}_+$,

$$f(x+iy) := \frac{1}{\pi} \int \frac{yf(t)}{(x-t)^2 + y^2} dt.$$

This harmonic extension is called Poisson formula. Also, we have the following

$$\sup_{y>0} \int |f(x+iy)|^p dx < \infty. \tag{2.1}$$

The condition (2.1) characterizes the boundness of this extension. Conversely, if a harmonic function $f : \mathbb{C}_+ \to \mathbb{C}$ satisfies (2.1), then f can be obtained from a $L^p(\mathbb{R})$ integrable function by Poisson formula. All such functions form a normative space with the following norm

$$\|f(z)\|_p := (\lim_{y\to} \int |f(x+iy)|^p dx)^{\frac{1}{p}},$$

and denote as $h^p(\mathbb{C}_+)$. Given a function $f(z) \in h^p(\mathbb{C}_+)$, we can recover a $L^p(\mathbb{R})$ integrable function from $f(z)$ by taking vertical limits,

$$f(x) := \lim_{y \to 0} f(x + iy).$$

The limit is well-defined for almost every $x \in \mathbb{R}$. We usually call this limit as a boundary function.

Each function $f(z) \in h^p(\mathbb{C}_+)$ has a representation

$$f(x + iy) := ky + \frac{1}{\pi} + \pi \int \frac{y}{(x - t)^2 + y^2} \mu(dt),$$

where μ is some positive measure on the real line. The correspondence of function f and positive measure μ is one-to-one.

2.1.1 Hardy space H^p

If we take all the holomorphic function on the upper half plane satisfying condition (2.1), we get the Hardy space $H^p(\mathbb{C}_+)$, which is a linear subspace of $h^p(\mathbb{C}_+)$. The corresponding space of boundary functions is written as $H^p(\mathbb{R})$. Besides the Poisson formula, we can recover $f(z) \in H^p(\mathbb{C}_+)$ by the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int \frac{f(t)}{t - z} dt,$$

where $f(t)$ is L^p integrable on the real line. Moreover, for $1 < p < \infty$, the Cauchy operator $C : L^p(\mathbb{R}) \to H^p(\mathbb{R})$,

$$Cf(z) := \frac{1}{2\pi i} \int \frac{f(t)}{t - z} dt$$

is bounded and onto. When $p = 2$, the Cauchy operator defines an orthogonal projection.

2.1.2 Smirnov class N^+

The Smirnov class in the upper half plane is denoted by $N^+(\mathbb{C}_+)$. Functions in $N^+(\mathbb{C}_+)$ are ratios $\frac{H_1}{H_2}$, where $H_1, H_2 \in H^\infty(\mathbb{C}_+)$ and H_2 is outer. Furthermore, functions in the Smirnov class have positive real part on $\mathbb{C}_+$. Let Π denote the Poisson measure, i.e.

$$d\Pi(t) = \frac{dt}{1+t^2}.$$

The boundary functions $f(x)$ of $f(z) \in N^+(\mathbb{C}_+)$ is Poisson integrable,

$$f(t) \in L^1_\Pi = L^1(\frac{dt}{1+t^2}).$$

Conversely, if $f(z)$ has a Poisson integrable boundary function, then $f(z) \in N^+(\mathbb{C}_+)$.

Moreover, we have a complete description of the zeros sets of functions in Smirnov class. Suppose $\{z_k\}$ is the zeros set of $f(z) \in N^+(\mathbb{C}_+)$, then

$$\sum_k \frac{\Im(z_k)}{1+|z_k|^2} < \infty. \tag{2.2}$$

Given a point sets $\{z_k\}$ satisfying (2.2), there exists a function $b(z)$ in $N^+(\mathbb{C}_+)$ with zeros exactly in set $\{z_k\}$. We can construct the function as

$$b(z) := \prod \frac{1-\frac{z}{z_k}}{1-\frac{z}{\overline{z_k}}},$$

and $b(z)$ is known as the Blaschke product. It is easy to check that Blacschke product $|b(z)| \leq 1$ for $z \in \mathbb{C}_+$, and has vertical limit on $\mathbb{R}$. Besides having easy to describe zero sets, functions in Smirnov class have a factorization property. Let $f(z) \in N^+(\mathbb{C}_+)$, then

$$f(z) = Ce^{iaz}b(z)s_1(z)h(z), \tag{2.3}$$

where $|C| = 1$, $a \geq 0$, and

$$s_1(z) := exp(-\frac{1}{\pi}\int \frac{tz+1}{t-z}\frac{\mu_{1,s}(dt)}{1+t^2}),$$

and

$$h(z) := \exp((-\frac{1}{\pi}\int \frac{tz+1}{t-z}\frac{log|f(t)|}{1+t^2}).$$

This factorization is called inner-outer factorization, which is introduced by Burling.

Definition 2.1.1. *A holomorphic function f in the upper half plane is an inner function if $|f(z)| \leq 1$ for $z \in \mathbb{C}_+$ and $|f(z)| = 1$ for almost all $z \in \mathcal{R}$.*

Definition 2.1.2. *A function $H(z)$ in the upper half plane is an outer function if it takes the form*

$$H(z) = \exp(\mathcal{S}h), \quad h \in L^1_\Pi,$$

where $\mathcal{S}h(z) = \frac{1}{\pi i}\int \Big[\frac{1}{t-z} - \frac{t}{1+t^2}\Big]h(t)dt$ is the Schwarz integral of $h(z)$.

If $z \in \mathbb{R}$,

$$\mathcal{S}h(z) = \frac{1}{\pi i}\int \Big[\frac{1}{t-z} - \frac{t}{1+t^2}\Big]h(t)dt = h(z).$$

Hence, $H = e^h$ on $\mathbb{R}$. It is not hard to verify that H is in Simirnov class. In (2.3), $h(z)$ is outer, $s_1(z)$ and $b(z)$ in (2.3) are inner functions. To sum up, $f \in N^+(\mathbb{C}_+)$ can be factorized by: $f = IH$, where I is inner which is unimodular on the real line $\mathbb{R}$, and H is outer. Moreover, we have the inequality of $log|f|$:

$$\log|f(x+iy)| \leq \frac{1}{\pi}\int \frac{y\log|f(t)|}{(x-t)^2+y^2}dt, \quad y > 0. \tag{2.4}$$

Conversely, if a holomorphic function on $\mathcal{C}_+$ satisfies the inequality (2.4), then it must be in the Simirnov class. This characterization of Smirnov class yields that

$$H^p(\mathbb{R}) \subset N^+(\mathbb{C}_+), \ \ 1 \le p \le \infty.$$

Hence, every function in Simirnov class with L^p integrable boundary functions can recover a function in Hardy space H^p uniquely. In other word, we have

$$H^p = N^+(\mathbb{C}_+) \cap L^p(\mathbb{R}).$$

2.1.3 Herglotz functions

Definition 2.1.3. *A Herglotz function is a meromorphic function with a nonnegative imaginary part.*

In the Herglotz representation, a meromorphic Herglotz function can be represented by (b, a, μ),

$$m(z) = a + bz + i\frac{1}{\pi}\int \frac{tz+1}{t-z}\frac{\mu(dt)}{1+t^2}, \tag{2.5}$$

where a is a real number, b is non-negative, and μ is a positive Poisson integrable measure on the real line. The constant b is determined by the point mass of the measure at ∞:

$$b = \lim_{y\to\infty} \frac{\Im m(iy)}{y}.$$

Given a Herglotz function m, we can construct a corresponding meromorphic inner function Θ_m. The closed correspondence between these two functions is given by the Cayley transform

$$m(z) = i\frac{1+\Theta(z)}{1-\Theta(z)}.$$

Conversely, we have

$$\Theta(z) = \frac{m(z) - i}{m(z) + i}.$$

In this case, Θ is inner if and only if the measure μ in the representation of m is singular. From this construction, we can build a correspondence between inner functions and singular Poisson integrable measures with point mass at infinity. In other word, we have

$$\Re \frac{1 + \Theta(z)}{1 - \Theta(z)} = by + \frac{1}{\pi} \int \frac{y\mu(dt)}{(x-t)^2 + y^2}.$$

Also, given a meromorphic inner function Θ, Θ can be characterized by a pair (a, Λ),

$$\Theta = B_\Lambda e^{iaz},$$

where a is nonnegative and B_Λ is a Blaschke product with Λ satisfying the Blaschke condition:

$$\sum_\Lambda \frac{\Im(\lambda)}{1 + |\lambda|^2} < \infty.$$

Proposition 2.1.4. *If $\Theta(z)$ be a meromorphic inner function, then $\Theta(z)$ has an equivalent representation on $\mathbb{R}$,*

$$\Theta(z) = \exp(i\theta(z)),$$

where θ is a real increasing function.

We will see in section 2.2 that the meromorphic inner function Θ has a close relationship with mixed spectral problems. For now, we define the spectrum of an inner function and we will see in section 2.3 that the definition of spectrum reflects the *Dirichlet boundary condition* at one endpoint of the Schrödinger equation. For further reference, see [10].

Definition 2.1.5. *The spectral of an inner function J is*

$$\sigma(J) = \{J = 1\}.$$

If ∞ is in the spectrum, then $\sigma(\Theta) = \{\Theta = 1\} \cup \{\infty\}$.

We usually call the measure μ_Θ in the representation of m the *spectral measure* of Θ_m.

Example 1. *Let Θ_1 and Θ_2 be two inner functions, the corresponding spectral measures are μ_{Θ_1} and μ_{Θ_2}. If Θ is the inner function with spectral measure $\mu_\Theta = (\mu_{\Theta_1} + \mu_{\Theta_2})/2$, then Θ is given as*

$$\Theta = \frac{\Theta_1 + \Theta_2 - 2\Theta_1\Theta_2}{2 - \Theta_1 - \Theta_2}.$$

Definition 2.1.6. *The function ϕ is a factor of an inner function Θ if $\frac{\Theta}{\phi}$ is also an inner function, denoted by $\phi|\Theta$.*

We can see that if $\phi|\Theta_1$ and $\phi|\Theta_2$, then $\phi|\Theta$.

2.2 Schrödinger operators and meromorphic inner functions

2.2.1 Weyl inner functions

In the theory for spectral problems of second operator with compact resolvent, the meromorphic inner function is heavily used. See [12, 1] for the general reference of spectral theory. Here, we only consider Schrödinger operators.

Consider the Schrödinger equation

$$Lu(t) = -u''(t) + qu(t) = \lambda u(t), \quad t \in (a, b) \tag{2.6}$$

and suppose the potential q is L^1 integrable and a is finite. Let $u_\lambda(z)$ be a nontrivial solution of (2.6) with a fixed self-adjoint B.C. β at b, then the corresponding Weyl m-function is defined as

$$m^a_{b,\beta}(\lambda) = \frac{u'_\lambda(a)}{u_\lambda(a)}, \quad \lambda \notin \mathbb{R}.$$

We usually assume L has compact resolvent. Then the Weyl m-function m can be extended to a meromorphic Herglotz function. Thus, we have the corresponding meromorphic inner function

$$\Theta(z) = \frac{m(z) - i}{m(z) + i}.$$

In this case, Θ is also called the *Weyl* inner function. Since we construct the Weyl m-function m by fixing the boundary condition β at b, this function Θ is denoted as $\Theta^a_{b,\beta}$. Similarly, if we fixed the self-adjoint B.C. (α) at a and b is finite, we can define the Herglotz function m:

$$m^a_{a,\alpha}(\lambda) = -\frac{u'_\lambda(b)}{u_\lambda(b)}, \quad \lambda \notin \mathbb{R}.$$

Example 2. *Suppose the potential*

$$q_\nu(t) = \frac{\nu^2 - \frac{1}{4}}{t^2} \quad 0 < t < 1,$$

and $u_\lambda(t)$ be the solution subject to B.C. (α) at 0:

$$u_\lambda(t) = \sqrt{t} J_\nu(t\sqrt{\lambda})$$

of the Schrödinger equation (1.7). Obviously, when $\nu = -\frac{1}{2}$, α is Neumann condition; when $\nu = \frac{1}{2}$, α is Dirichlet condition. J_ν is the Bessel function of order ν. The corresponding Weyl m-*function is*

$$m_\nu(\lambda) = -\frac{\frac{1}{2} J_\nu(\sqrt{\lambda}) + \sqrt{\lambda} J'_\nu(\sqrt{\lambda})}{J_\nu(\sqrt{\lambda})},$$

and the Weyl inner function is

$$\Theta_\nu(\lambda) = \frac{(\frac{1}{2}+i)J_\nu(\sqrt{\lambda}) + \sqrt{\lambda}J'_\nu(\sqrt{\lambda})}{(\frac{1}{2}-i)J_\nu(\sqrt{\lambda}) + \sqrt{\lambda}J'_\nu(\sqrt{\lambda})}.$$

2.3 Model spaces and the modified Fourier transform

Each inner function $\Theta(z)$ defines a model space

$$K_\Theta := H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+) = \{F \in H^2 : \ \Theta\overline{F} \in H^2\}.$$

The H^2-model space K_Θ is a Banach space. Its reproducing kernel is

$$k_\lambda^\theta(z) = \frac{1}{2\pi i}\frac{1-\Theta(\lambda)\overline{\Theta(\lambda)}}{\overline{\lambda}-z}, \quad \lambda \in \mathbb{C}_+.$$

Moreover, if the inner function Θ is meromorphic, we can extend the definition of reproducing kernels to $\lambda \in \mathbb{R}$. Besides, if we consider the model spaces in $N^+(\mathbb{C}_+)$ and $H^p(\mathbb{C}_+)$, we have

$$K_\Theta^+ = N^+(\mathbb{C}_+) \ominus \Theta N^+(\mathbb{C}_+),$$

and

$$K_\Theta^p = H^p(\mathbb{C}_+) \ominus \Theta H^p(\mathbb{C}_+) = K_\Theta^+ \cup L^p(\mathbb{R})$$

Every function $f(z) \in K_\Theta$ can be extended analytically to the points where $\Theta(z)$ can be extended. An inner function $\Theta(z) = Ce^{iaz}b(z)s(z)$ can be extended to a meromorphic function on the complex plane if and only if $s(z)$ is unimodular on the real line and the zeros of $b(z)$ are not close to the real line. In this case, $f(z) \in K_\Theta$ can also be extended to a meromorphic function on the whole plane.

Another important theorem is given by Clark [13]. We briefly state the theorem without proof.

Theorem 2.3.1. *Let C_Θ be a restriction map from K_Θ to $L^2(\mu_\Theta)$,*

$$C_\Theta : \; f \to f|_{\sigma\Theta}.$$

Then, C_Θ is unitary.

Let S be the simplest inner function: $S(z) = e^{iz}$. Then the space $S^{-a}K[S^{2a}]$ is the space of Fourier transform of $L^2(-a,a)$, which is also called Paley-Wiener space PW_a. In other words, the Fourier transform identifies $L^2(-a,a)$ with $S^{-a}K[S^{2a}]$.

Intuitively, we try to construct a transform for the case of Schrödinger operator. In other word, the correspondence between $L^2(a,b)$ and K_Θ, and Θ is a Weyl inner function. For any $z \in \mathbb{C}$, there exists a solution $u_z(t)$ satisfying B.C. (β) at b. Thus, we can construct the transform $\mathcal{W}$ from L^2 to the model space K_Θ:

$$\mathcal{W} : \quad f(t) \to F(t) = \int_a^b f(t) \frac{u_z(t)}{u_z'(a) + iu_z(a)}.$$

This transform is also called Weyl-Titchmarch Fourier transform. It is not hard to show that $\mathcal{W}$ is indeed a unitary operator.

Corollary 2.3.2. *Let Θ be the meromorphic inner function of a Schrödinger operator. The composition of the modified Fourier transform and the Plancherel operator is a unitary operator:*

$$L^2(a,b) \xrightarrow{\mathcal{W}} K_\Theta \xrightarrow{C_\Theta} L^2_{\mu_\Theta}.$$

Remark 1. *This corollary is extremely important in our further discussion of the mixed spectrum problems for Schrödinger operators. We can view the invertible problem of a*

Weyl inner function as the completeness problem in space $L^2(a,b)$, which will be discussed in section 3.

We can see that the point mass of Weyl inner functions is finite. From our construction, we usually fix one boundary condition and construct the Weyl m-function m. Now let us explain the definition of the spectrum of an inner function.

Let $\Theta = \Theta^{a}_{b,\beta}$. If$\lambda \in \sigma(\Theta)$, then we have

$$\Theta(\lambda) = \frac{m-i}{m+i} = 1 \iff m_\Theta(\lambda) = \frac{u'_\lambda(a)}{u_\lambda(a)} = \infty \iff u_\lambda(a) = 0.$$

In other word, the spectrum of Θ is the spectrum of the Schrödinger operator with Dirichlet and Neumann boundary condition at two endpoints respectively. Similarly, we can see that $\sigma(-\Theta) = \sigma(q; N, \beta)$,

$$\Theta(\lambda) = \frac{m-i}{m+i} = -1 \iff m_\Theta(\lambda) = \frac{u'_\lambda(a)}{u_\lambda(a)} = 0 \iff u'_\lambda(a) = 0.$$

More generally, the boundary condition α at a is defined as

$$\cos\frac{\alpha}{2}u(a) + \sin\frac{\alpha}{2}u'(a) = 0.$$

Since

$$\Theta(\lambda) = \frac{m-i}{m+i} = e^{ia} \iff m_\Theta(\lambda) = \frac{u'_\lambda(a)}{u_\lambda(a)} = -\cot\frac{\alpha}{2} \iff \cos\frac{\alpha}{2}u(a)+\sin\frac{\alpha}{2}u'(a) = 0.$$

Hence, the spectrum of $e^{-ia}\Theta$ is the spectrum of the Schrödinger operator with α and β B.C. at two endpoints respectively

Remark 2. *From the above discussion, we can see that the definition of the spectral of a Weyl inner function is feasible. With a multiplication of constant of modular 1, any self-*

adjoint boundary conditions will be identified. Hence, we only consider the Dirichlet B.C. at endpoint a when fixing the B.C. (β) at endpoint b.

2.4 De Branges spaces

In this subsection, we discuss a class of entire functions with close relation with a Schrödinger operator.

Definition 2.4.1. *Cartwright class C_a is the space of entire functions $F(z)$ of exponential type $\leq a$ satisfying* $\log|F(t)| \in L^2_\Pi$.

An important theorem by Krein states a correspondence between the Simirnov class $N^+(\mathbb{C})$ and Cartwright class C_a.

Proposition 2.4.2. *An entire function $F(z) \in C_a$ if and only if*

$$\frac{F}{S^{-a}} \in N^+(\mathbb{C}), \quad and \quad \frac{F^\#}{S^{-a}} \in N^+(\mathbb{C}),$$

where $S(z) = e^{iz}$, and $F^\#(z) = \overline{F(\overline{z})}$.

If we consider entire functions with L^2 integrabal boundary functions, then we obtain a correspondence between the Hardy space $H^2(\mathbb{C}_+)$ and the Paley-Wiener space PW_a.

Proposition 2.4.3. *An entire function $F(z) \in PW_a$ if and only if*

$$\frac{F}{S^{-a}} \in H^2(\mathbb{C}), \quad and \frac{F^\#}{S^{-a}} \in H^2(\mathbb{C}).$$

The above statement of $f \in PW_a$ is a special case of the definition of Hermite-Biehler function.

Definition 2.4.4. *If an entire function E satisfies $E \neq 0$ on $\mathbb{R}$ and*

$$|E(z)| > |E(\overline{z})|, \quad z \in \mathbb{C}_+,$$

then such function E is a Hermite-Biehler function.

In [12], Louis de Branges introduced a correspondence of de Branges functions and meromorphic inner functions:

$$\Theta_E = \frac{E}{E}.$$

And we usually call de Branges functions with no real zeros Hermite-Biehler functions.

Usually we call an entire function real if it is real on $\mathbb{R}$. Any entire function F can be represented as $F = C + iD$ where C and D are real entire functions. It is well known that $E = C + iD$ is an Hermite-Biehler function if and only if the real function C and D have real alternating zeros. In the case $E = C + iD$ is a de Branges function, its real and imaginary parts can be viewed as analog of $\sin z$ and $\cos z$.

Example 3. *Let potential q be a summable function on interval (a, b). In this case, we assume a and b are finite. Given a self-adjoint boundary condition (α) at a, i.e.*

$$\cos\frac{\alpha}{2}u(a) + \sin\frac{\alpha}{2}u'(a) = 0.$$

Let u'_λ be a solution of a Schrödinger equation satisfying the following conditions

$$u_\lambda(a) = -\sin\frac{\alpha}{2}, \;\; u'_\lambda(a) = \cos\frac{\alpha}{2}.$$

And the Weyl inner function is

$$\Theta^b_{a,\alpha}(\lambda) = \frac{-u'_\lambda(b) - iu_\lambda(b)}{-u'_\lambda(b) + iu_\lambda(b)},$$

and the corresponding Hermite-Biehler function is

$$E(\lambda) = -u'_\lambda(b) + iu_\lambda(b).$$

Since we fix only one boundary condition, for $\lambda \in \mathbb{C}_+$, the function $u(\lambda)$ and $u'(\lambda)$ are holomorphic functions and $u_\lambda(b) + u'_\lambda(b) \neq 0$, $\lambda \in \mathbb{R}$.

Now, we can define the de Branges space.

Definition 2.4.5. *The de Branges space B_E associated with a Hermite-Biehler function $E(z)$ is the space of holomorphic functions $F(z)$ satisfying*

$$\frac{F}{E} \in H^2(\mathbb{C}), \ \ and \frac{F^\#}{E} \in H^2(\mathbb{C}).$$

Also, we can define a norm $\|F\|_E = \|F/E\|_2$, such that de Branges space is normative.

An important property of de Branges spaces is that we have an equivalent definition. Given a normative space H, then the following statements are equivalent:

1. $H = B_E$ for a de Branges function E;
2. If $F \in H$, $F(\lambda) = 0$, then $\frac{F(z)(z-\overline{\lambda})}{z-\lambda} \in H$ and it has the same norm with H;
3. For any λ with $\Im\lambda \neq 0$, point evaluation at λ is a bounded linear functional on H;
4. If $F \in H$, then $F^\#$ has the same norm.

Each de Branges space possesses a family of spectral measures ν, and the natural embedding $B(E) \to L^2(u)$ is a unitary operator. Now let us return to Schrödinger operators. The spectral measure of the Weyl inner function has closed relationship with the spectral measure of a de Branges space.

Suppose $u_\lambda(t)$ is a solution of Schrödinger equation with Neumann boundary conditions at a. Then the function $E_t(\lambda) = u_\lambda(t) + iu'_\lambda(t)$ is an Hermit-Biehler function. The spaces $B(E_t)$ form a chain, i.e., $B(E_t)$ is isometrically embedded into $B(E_s)$ for $t \leq s$. Let μ_- denote the spectral measure of $\Theta^a_{b,N}$. Then, μ_- is the spectral measure for de Branges space $B(E_b)$. For $t \in (a,b)$, the space $E(B_t)$ is isometrically embedded in $L^2(\mu_-)$.

Similarly, suppose $v_\lambda(t)$ is a solution of Schrödinger equation with the Neumann boundary conditions at b, $F_t(\lambda) = v_\lambda(t) - iv'_\lambda(t)$ is a Hermit-Biehler function. The spaces $B(E_t)$ form a chain, i.e., $B(F_t)$ is isometrically embedded into $B(E_s)$ for $t \geq s$. Let μ_+ denote the spectral measure of $\Theta^b_{a,N}$. Then, μ_+ is the spectral measure for de Branges space $B(E_b)$. For $t \in (a,b)$, the space $E(B_t)$ is isometrically embedded in $L^2(\mu_+)$. For further reference, see [11].

Indeed, the de Branges spaces $B(E_t)$ are equal to Paley-Wiener space $PW_{t/\pi}$ as sets. Similarly, $B(F_t) = PW_{(\pi-s)/\pi}$ as sets. For further reference of de Branges spaces, see [14].

3. TOEPLITZ KERNELS

In this section, we introduce the Toeplitz kernel, which is a main tool to study the mixed spectral problem in our discussion. We focus on the conditions when the Toeplitz kernel is trivial or non-trivial.

3.1 Introduction

Definition 3.1.1. *A Toeplitz operator T_U associated with a function $U \in L^\infty(\mathbb{R})$ is the map $T_U : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+)$ defined by*

$$T_U F := P_+(UF),$$

where P_+ is the orthogonal projection in $L^2(R)$ onto the Hardy space $H^2(\mathbb{C}_+)$.

In this section, we always assume U is unimodular, i.e.,

$$U = e^{i\gamma}, \quad \gamma : \mathbb{R} \to \mathbb{R}.$$

Let $N[U]$ denote the kernel of the Toeplitz kernel. The Toeplitz kernels in $N^+(\mathbb{C}_+)$ is

$$N^+[U] := kerT_U = \{f(z) \in N^+(\mathbb{R}_+) \cap L^1_{loc}(\mathbb{R}) \ : \ \bar{U}(t)\bar{f}(t) \in N^+(\mathbb{C}_+)\}.$$

Hence, the kernel in H^p, $0 < p \leq \infty$ is

$$N^p[U] = \{f(z) \in H^P(\mathbb{R}_+) \cap L^1_{loc}(\mathbb{R}) \ : \ \bar{U}(t)\bar{f}(t) \in H^P(\mathbb{C}_+)\}.$$

Especially, if U is a meromorphic inner function, we have a correspondence between the Toeplitz kernel of T_U and the model space $K_{\bar{U}}$,

$$N^+[\bar{\Theta}] = K_\Theta^+, \quad N^p[\bar{\Theta}] = K_\Theta^p.$$

3.2 Characterization of Toeplitz kernels

We usually multiply U by integer powers of Blaschke factor to characterize its kernel. Let b denote the Blaschke product

$$b(z) = \frac{i - z}{i + z}.$$

We can see that $b(z)$ is an inner function on the upper half plane, and the argument $2\arctan(b(t))$ increase from $-\pi$ at $-\infty$ to $+\pi$ at $+\infty$.

Lemma 3.2.1. $\dim N^p[U] = n + 1 \iff \dim N^p[b^n U] = 1$*, where* $n \in \mathbb{N}$.

Proof. Suppose the dimension of the kernel $N^p[U]$ is greater than 2, then there exists $H, G \in N^p[U]$, and G, F are linear independent. Then, the function

$$F = H(z)G(i) - H(i)G(z) \in N^p[U],$$

and we can see that i is a single zero of F. Hence,

$$\bar{b}F \in H^p, \quad (\bar{U}b)(\bar{b}F) = \bar{U}\bar{F} \in H^p,$$

where $\bar{U}\bar{F} \in H^p$ comes from $F \in N^p[U]$. From the definition of Toeplitz kernel, the function $\bar{b}F$ is in the kernel of Toeplitz operator T_{bU} and the dimension of the kernel is greater than 1.

On the other hand, given a non-trivial function F from the kernel $N^p[bU]$, then

$$F \in H^p, \ \ \bar{U}(\bar{b}\bar{F}) \in H^p.$$

Since $bF \in H^p$, we have $bF \in N^p$. Suppose $\bar{U}\bar{b}\bar{F} = G \in H^p$, then $\bar{U}\bar{F} = bG \in H^p$, which means that F is in the kernel $N^p[U]$. Since F and bF are linear independent, then $\dim N^p[U] \geq 2$. □

Let us consider the fractional power of the inner function b,

$$b^s = e^{2si \arctan x}, \ \ s \in \mathbb{R}.$$

The equality on $\mathbb{R}$

$$\bar{b}^s(1-b)^s = ((\bar{b}-1)^s - 1)^s$$

shows that $N^\infty[\bar{b}^s]$ is not trivial for non-negative s. Thus, for every unimodular function U, there exists $s_\star \in \mathbb{R}$ satisfying

$$N^p[\bar{b}^s U] \neq 0, \ \ \forall s > s_\star, \ \ N^p[\bar{b}^s U] = 0, \ \ \forall s < s_\star.$$

Now let us give a characterization of U in the case of non-trivial Toeplitz kernels. The following proposition is quite useful in our further discussion.

Proposition 3.2.2. $N[p] \neq 0$ *if and only if* U *equals to*

$$U = \bar{\Phi}\frac{\bar{H}}{H},$$

where Φ *is an inner function and* $H \in H^p(\mathbb{C}_+) \cap L^1(\mathbb{R})$ *is an outer function.*

Proof. If $F \in N^p[U]$, then $\bar{U}\bar{F} \in H^p$. Suppose $UF = \bar{G} \in \bar{H}^p$, then $|F| = |G|$ on the real line. Use the inner-outer factorization, we have

$$F = F_i F_e, \quad G = G_i G_e,$$

and $|F_e| = |G_e|$, which means $F_e = G_e$. Hence,

$$U = \frac{\bar{G}}{F} = (\bar{F}_i \bar{G}_i)\frac{\bar{F}_e}{F_e}.$$

□

Moreover, by Coburn's lemma, if $N[U]$ is trivial, then $N[\bar{U}]$ is not trivial; The opposite is the same.

3.3 Trivial factors

If we consider a Schrödinger operator on a fixed interval(a, b), its spectrum is subject to the B.C. at two endpoints and the potential q. From our approach, the problem can be simplified in the case $q \in L^1(\mathbb{R})$. In this subsection, we discuss the trivial factor of a Toeplitz kernel and prove the equivalence of trivial potentials and L^1 integrable potentials.

Definition 3.3.1. *A function V is called a trivial factor if*

$$N^p[UV] \neq 0 \iff N^p[U] \neq 0.$$

The trivial factors do not influence the invertible property of the Toeplitz operator. From the last section, we have the following proposition.

Proposition 3.3.2. *If $V = \frac{\bar{H}}{H}$ where $H^{\pm 1} \in H^\infty$, then V is a trivial factor.*

Let q be summable on (a, b) and let $(\alpha) \neq (D)$ be the boundary condition at $a = 0$. Then we construct $\Theta = \Theta^b_{a,\alpha}$. As mentioned before, we try to compare Θ with the Θ_N, where Θ_N is the corresponding Weyl inner function with trivial q and (N) B.C. at a.

Theorem 3.3.3. *The function $\frac{\Theta}{\Theta_N}$ is a trivial factor of the kernel of T_{Θ_N}.*

Proof. Recall that a Weyl inner function can be expressed by a de Branges function:

$$\Theta_E = \frac{E^\#}{E}.$$

The corresponding Hermite-Biehler function E_Θ is

$$E_N(\lambda) = \cos\sqrt{\lambda} - i\sqrt{\lambda}\sin\sqrt{\lambda}.$$

Suppose $u_\lambda(t)$ is a solution with $q \in L^1$, α boundary condition at $a = 0$, and initial value $u_\lambda(0) = 1$. Then the corresponding de Branges function is

$$_\Theta(\lambda) = -u_\lambda'(1) + iu_\lambda(1).$$

Hence, we have $\frac{\Theta}{\Theta_N} = \frac{\bar{H}}{H}$, where $H = E/E_N$. Both E and E_N are outer functions on the upper half plane. Hence, if we can show that $H \in H^\infty$, in other words,

$$|E| \asymp |E_N|, \quad z \in \mathbb{R},$$

then Θ/Θ_N is a trivial factor. From [10], we have the asymptotic formula for $u_\lambda(1)$ and $u_\lambda'(1)$:

$$|u_\lambda(1) - \cos\sqrt{\lambda}| = \mathcal{O}(\sqrt{\lambda}), \quad |u_\lambda'(1) + \sqrt{\lambda}\sin\sqrt{\lambda}| = \mathcal{O}(1),$$

when $|\lambda|$ goes to infinity.

In the case λ goes to positive infinity,

$$|E(\lambda)|^2, |E_N(\lambda)|^2 \asymp |\cos\sqrt{\lambda} + \mathcal{O}(\frac{1}{\sqrt{\lambda}})|^2 + \lambda|\sin\sqrt{\lambda} + \mathcal{O}(\frac{1}{\sqrt{\lambda}})|^2.$$

Thus, we can see that $\frac{E}{E_N}$ is bounded when $\lambda \to +\infty$.

If $\lambda \to -\infty$,

$$|\cos\sqrt{\lambda}| = |\frac{e^{i\sqrt{\lambda}} + e^{-i\sqrt{\lambda}}}{2}| \asymp \mathcal{O}(e^{\sqrt{-\lambda}}), \quad |\lambda\sin\sqrt{\lambda}| = |\lambda\frac{e^{i\sqrt{\lambda}} - e^{-i\sqrt{\lambda}}}{2i}| \asymp \mathcal{O}(\lambda e^{\sqrt{-\lambda}}).$$

Hence, $\frac{E}{E_N}$ is bounded when $\lambda \to -\infty$. □

3.4 Toeplitz kernels with real analytic symbols

In this subsection, we consider $U = e^{i\gamma}$ where γ is real analytic. In this case, functions in T_U are also real analytic.

Proposition 3.4.1. *If $\gamma \in C^\omega(\mathbb{R})$, then if $f(z) \in N^+[e^{i\gamma}]$, then $f(z)$ is real analytic.*

Proof. Take f from the Toeplitz kernel $N^+[U]$, then $UF = \bar{G}$ for some $G \in N^+$. We analytically extend the function UF to the lower half plane and denote as G_-. Since $U = e^{i\gamma} \neq 0$ on $\mathbb{R}$, then $F = U^{-1}G_-$ on $\mathbb{R}$, which means that F can be extended to real line $\mathbb{R}$. □

If $U = e^{i\gamma}$ for some real analytic γ, the Toeplitz kernel has a definite criterion of triviality.

Theorem 3.4.2. *Let γ be a real analytic function. Then $N^+[e^{i\gamma}] \neq 0$ if and only if γ has the following representation:*

$$\gamma = -f + \widetilde{h},$$

where f is an increasing real analytic function and $h \in L^1_\Pi$.

Proof. If $N^+[U] \neq 0$, from previous result, U has a representation:

$$U = \bar{I}\frac{\bar{F}}{F}, \quad z \in \mathbb{R},$$

for inner function I and real analytic function F. Also, F is outer. If F only has simple zeros on $\mathbb{R}$, then we choose J to be the Blaschke product with the zero set of $J-1$ exactly the zeros set of F. Hence, the outer function

$$H := \frac{F}{1-J}$$

has no zero on $\mathbb{R}$. Also, U can be represented by H:

$$U = \bar{I} \cdot \frac{\bar{H}}{H} \cdot \frac{1-\bar{J}}{1-J} = -\bar{I}\bar{J}\frac{\bar{H}}{H} := \bar{\Phi}\frac{\bar{H}}{H}.$$

If F has multiple zeros on $\mathbb{R}$, we do the same process and still get the same results. Hence, $N^+[U] \neq 0$ if and only if

$$U = \bar{\Phi}\frac{\bar{H}}{H},$$

and Φ is meromorphic inner and the outer function $H \in C^\omega(\mathbb{R}) \neq 0$ on the real line. $\square$

3.5 Twin inner functions

We discuss properties of twin inner functions. In the following sections, we apply the results of twin inner function to discuss spectral problems in the case of real spectrum.

Definition 3.5.1. *Let F and G be two meromorphic inner functions. If $\sigma(F) = \sigma(G)$, then we call F and G twin functions.*

Theorem 3.5.2. *If F and G are twin meromorphic inner functions, then $N[\bar{\Theta}J] = 0$.*

Proof. Suppose $N[\bar{\Theta}J] \neq 0$. From our previous results,

$$\bar{\Theta}J = \bar{\phi}\frac{\bar{H}}{H},$$

where ϕ is inner and $H \in H^2 \cap L^1_{loc}(\mathbb{R})$ is outer. Since $\sigma(\Theta) = \sigma(J) = \Lambda$, then $\phi(z) = 1$, $z \in \Lambda$. Hence, we have

$$\bar{\Theta}J = \frac{\bar{H}}{H},$$

and we can get

$$H = \frac{1-\Theta}{1-J}.$$

In this case, H is real analytic. If ∞ is not in the spectrum, then $H \notin L^2(\mathbb{R})$. If ∞ is in the spectrum, then

$$H(\infty) = \lim_{\infty} \frac{1-\Theta}{1-J} = \frac{\Theta'(\infty)}{J'(\infty)} \neq 0.$$

In this case, $H \notin L^2$. From our results, this contradicts to $H \in H^2 \cap L^1_{loc}(\mathbb{R})$. Hence, $N[\bar{\Theta}J] = 0$. $\square$

Remark 3.

- *From this theorem, we see that if* $\{\Theta = 1\} = \{J = 1\}$*, then* $N[\bar{\Theta}J] = 0$*. Also, if we require* $\infty \notin \sigma(\Theta)$*, then*

$$\sigma(\Theta) \subset \sigma(J) \quad \Rightarrow \quad N[\bar{\Theta}J] = 0.$$

- *If we consider function* $u = \bar{\Theta}J\bar{H}/H$ *for twin inner function* Θ *and* J *and outer function* F *with no zero on the real line, then if the dimension of* $N^p[\bar{H}/H]$ *is finite, then*

$$N^p[\bar{b}^s\bar{\Theta}J\bar{H}/H] \neq 0 \Longleftrightarrow (1-b)^s\frac{1-\Theta}{1-J}H \in H^p.$$

4. SUMMARY OF SPECTRAL PROBLEMS

In this section, we introduce some applications of Toeplitz kernels. Firstly, we discuss the completeness of a family of eigenfunctions of the Schrödinger operator in $L^2(a,b)$ by Toeplitz kernels; Then, we consider the spectral problems for Schrödinger operators and interpret the mixed spectral problems in terms of a determination statement. Some classical results are included, such as Borg's two-spectra theorem [2] and Hochstadt-Liberman's theorem [3]; In the end, we discuss Horvoth's results [9] and sketch a proof by our approach.

Let us mention the important paper of N. Makarov and A. Poltoratski [10], where the idea of using Toeplitz operators to explain Horvath's results was introduced. We will present these results in this section.

4.1 Uniqueness sets

Consider the Schrödinger equation

$$-u''(t) + q(t)u(t) = \lambda u(t), \ \ t \in (a,b),$$

where the potential $q \in L^1_{loc}(\mathbb{R})$ and a is finite. Fixing the B.C. (β) at b:

$$\cos\frac{\beta}{2}u(a) + \sin\frac{\beta}{2}u'(a) = 0.$$

For $\lambda \in \mathbb{C}$, we suppose u_λ is a solution satisfying the above boundary condition (uniquely up to a constant).

Definition 4.1.1. *For $\Lambda \subset \mathbb{C}$, the family of solutions $\{u_\lambda, \ \ \lambda \in \Lambda\}$ is complete in $L^2(a,b)$*

if

$$for\ f \in L^2(a,b),\ \ < f, u_\lambda >= 0,\ \ \lambda \in \Lambda \Rightarrow f \equiv 0.$$

For $\Lambda \subset \mathbb{C}$, let Λ_+ be the intersection of Λ and the upper half plane with boundary; Let Λ_- be the intersection of Λ and the lower half plane. From our previous construction, we denote $\Theta = \Theta^a_{b,\beta}$ as the Weyl inner function. For $\lambda \in \Lambda_+$, the reproducing kernel of K_Θ is

$$k^\Theta_\lambda(z) = \frac{1}{2\pi i}\frac{1 - \Theta(\bar{\lambda})\Theta(z)}{\bar{\lambda} - z},$$

and the dual reproducing kernel is

$$k^\star_\lambda(z) = \frac{1}{2\pi i}\frac{\Theta(\lambda) - \Theta(z)}{z - \lambda}.$$

And we have

$$\bar{\Theta} k^\Theta_\lambda = \bar{k^\star_\lambda}.$$

Also, let K^E_λ be the reproducing kernel of a de Branges space $B(E)$. From the discussion in [10], we have the following theorem.

Theorem 4.1.2. *If $\Lambda \in \mathbb{C}$ and $\{u_\lambda\}_\Lambda$ is the family of solutions of the Schrödinger equation, then the following completeness conditions are equivalent:*

- *The set of solutions $\{u_\lambda\}_\Lambda$ is complete in $L^2(a,b)$;*
- *The set of reproducing kernels$\{k^\star_\lambda\}_{\Lambda_+} \cup \{k_{\bar{\lambda}}\}_{\Lambda_-}$ is complete in K_Θ.*

Given $f \in K_\Theta$, if

$$f\ vanishes\ on\ \Lambda \iff f \equiv 0,$$

then the set Λ is unique in the model space K_Θ. If $\Lambda \subset \mathbb{C}_+ \cup \mathbb{R}$, we can extend the results to divisors.

Proposition 4.1.3. *Suppose $\Lambda = \Lambda_+$ and M is a point set in the upper half plane. The set of reproducing kernels $\{k^\star_\lambda\}_{\Lambda \in M} \cup \{k_\lambda\}_{\lambda \in \Lambda}$ is complete if and only if $\Lambda \cup M$ is a uniqueness divisor.*

Proof. If the set of reproducing kernels $\{k^\star_\lambda\}_{\Lambda \in M} \cup \{k_\lambda\}_{\lambda \in \Lambda}$ is not complete, then there exists F in the model space which is orthogonal to $\{k^\star_\lambda\}_{\Lambda \in M} \cup \{k_\lambda\}_{\lambda \in \Lambda}$. By the definition of model space, there exists $H \in K_\Theta$, such that

$$\bar{\Theta} F = \bar{H}, \quad z \in \mathbb{R}.$$

Hence, F vanishes on Λ. There exists a representation

$$H = B_M J,$$

where B_M is a Blaschke product of M and $J \in H^2$. Thus, we have

$$\bar{\Theta}(B_M F) = \bar{J}.$$

We see that the function $B_M F = 0$ on $\Lambda \cup M$ and $B_M F \in K_\Theta$. □

In the case Λ in the upper half plane, we have the following characterization of uniqueness sets of K_Θ.

Theorem 4.1.4. *If $\Lambda \in \mathbb{C}_+$, then Λ is a uniqueness set of model space with Θ if and only if the Toeplitz kernel of $T_{\bar{\Theta} B_\Lambda} = 0$.*

Proof. If $F \in K_\Theta$ vanishes on Λ, then we have

$$\Theta \bar{F} \in H^2, \quad \bar{B}_\Lambda F \in H^2.$$

Hence,

$$(\bar{\Theta}B_\Lambda)(\bar{B_\Lambda}F) = \bar{\Theta}F \in H^2,$$

which means that $B_\Lambda F \in N[\bar{\Theta}B_\Lambda]$. The opposite direction is the same. $\square$

If we consider the model space K^p_Θ, the result is the same. If Λ has no intersection with the lower half plane, then for $p > 0$, Λ is unique in the model space K^p_Θ if and only if $N^p[\bar{\Theta}B] = 0$.

4.2 Spectral problems

Consider a Schrödinger operator L on (a, b) with B.C. (α) at a and (β) at b and potential $q \in L^2_{loc}$. Pick c on the interval (a, b), $a < c < b$. We denote $q_- = q_{(a,c)}$ and $q_+ = q_{(c,b)}$. Let $\Theta_- = \Theta^c_{a,\alpha}$ and $\Theta_+ = \Theta^c_{b,\beta}$. Then, we have the following proposition about the spectrum of $\Theta_\pm$ and L.

Proposition 4.2.1. *The spectrum of L is the spectrum of* $(\Theta_-\Theta_+)$.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\Theta_-\Theta_+$, by definition, $\Theta_-(\lambda)\Theta_+(\lambda) = 1$. Then,

$$m_+(\lambda) + m_-(\lambda) = 0$$

or

$$m_-(\lambda)m_+(\lambda) = \infty.$$

In either case, we have the following equality

$$\frac{u'_{-,\lambda}(c)}{u_{-,\lambda}(c)} = \frac{u'_{+,\lambda}(c)}{u_{+,\lambda}(c)},$$

where $u_{-,\lambda}(z)$ and $u_{+,\lambda}(z)$ are non-trivial solutions. This means $\lambda \in \sigma(L)$. $\square$

As for the spectral problems with mixed data, we are given the information of potential

q on (a, c) and spectrum $\sigma(L)$. Can we uniquely recover the potential on (a, b)? From the above proposition, it is equivalent to recover Θ_+ from Θ_- and σ_L. According to the results of Borg and Marchenko [1]: the Herglotz function $m(z)$ determines the potential and the boundary condition. Hence, we can uniquely recover q from $\Theta_-\Theta_+$. So, we interpret our problem by the following statement.

Let $\Theta = \Psi\Phi$. In this case, we only discuss meromorphic inner functions. The information $[\Psi, \sigma(\Psi\Phi)]$ *determine* Φ if for any $\tilde{\Phi}$,

$$\sigma(\tilde{\Phi}\Psi) = \sigma(\Phi\Psi) \quad \Rightarrow \Phi = \tilde{\Phi},$$

where $\tilde{\Phi}$ and $\tilde{\Phi}\tilde{\Psi}$ are all meromorphic inner functions.

Then, we give a proposition in terms of the kernel of Toeplitz operators.

Theorem 4.2.2. *If* $N^\infty[\bar{\Phi}\Psi] \neq 0$*, then* $[\Psi, \sigma(\Phi\Psi)]$ *does not determine* Φ.

Proof. Take $a \in N[\bar{\Phi}\Psi]$ and $\|a\|_\infty < \frac{1}{2}$, then for some $b \in H^\infty$,

$$\bar{\Phi}\Psi a = \bar{\Theta}\Psi^2 a = \bar{b}.$$

Then, we have

$$\bar{\Theta}\Psi^2(a + b) = \bar{a} + \bar{b}.$$

Let $g = a + b$ and $f = \Psi g$, so

$$\bar{\Theta}\Psi f = \bar{\Psi}\bar{f},$$

i.e., $\bar{\Theta} f = \bar{f}$, $\Psi | f$ and $f \in H^\infty$. Then we can construct $\tilde{\Theta}$ as

$$\tilde{\Theta} = \frac{f + \Theta}{f + 1},$$

which Θ is an inner function:

$$|\bar{\Theta}|^2 = \frac{(f+\Theta)(\bar{f}+\bar{\Theta})}{(f+1)(\bar{f}+1} = 1.$$

Also, $\Psi|\Theta$. If λ is finite and $\lambda \in \sigma(\Theta)$, then $\tilde{\Theta}(\lambda) = 1$; If $\infty \in \sigma(\Theta)$, since

$$\tilde{\Theta} - 1 = \frac{\Theta - 1}{f+1} \in H^2,$$

then we have $\infty \in \sigma(\tilde{\Theta})$. $\square$

Proposition 4.2.3. *If $\exists p < 1$, such that $N^p_{\Pi}[\bar{\Phi}\Psi] = 0$, then $[\Psi, \sigma(\Psi\Phi)]$ determine Φ.*

Proof. Suppose there exists $\tilde{\Theta}$ such that $\sigma(\Theta) = \sigma(\tilde{\Theta})$ and $\Psi|\Theta$. Since $\Theta = \tilde{\Theta}$ on $\sigma(\Theta)$ and $(1-\tilde{\Theta}(z))^{-1} > 0$ for $z \in \mathbb{C}_+$, then we have

$$f = \frac{\Theta_1 - \Theta}{1-\Theta}$$

is in $H^p_{\Pi} \cap C^{\omega}(\mathbb{R})$ for all $p < 1$. Also, we have

$$\bar{\Theta}f = \bar{\Theta}\frac{\tilde{\Theta}-\Theta}{1-\tilde{\Theta}} = \frac{\tilde{\Theta}\bar{\Theta}-1}{1-\tilde{\Theta}} = \bar{f}.$$

Since $\Psi|\tilde{\Theta} - \Theta$, we can define $g = \bar{\Psi}f \in H^p_{\Pi} \cap C^{\omega}(\mathbb{R})$. Then, we have

$$\bar{\Phi}\Psi g = \Psi\bar{\Theta}f = \Psi\bar{f} = g.$$

Hence, $g \in N^p_{\Pi}[\bar{\Phi}\Psi] = 0$, which leads to $\Theta = \tilde{\Theta}$. $\square$

4.3 Defining sets

We introduce the *defining sets* of functions (especially of inner functions) and the relation to uniqueness sets in model space.

Definition 4.3.1. *Let* $\Phi = \exp(i\phi)$ *be an inner function and* Λ *be a subset of* $\mathbb{R}$. *We say* Λ *is defining for* Φ *if*

$$\tilde{\Phi} = \exp(i\tilde{\phi}), \quad \phi = \tilde{\phi} \text{ on } \Lambda \;\Rightarrow\; \Phi \equiv \tilde{\Phi}.$$

Now let us discuss two special cases.

(1) *Borg's two spectra case*

This is the case

$$\Lambda = \{\Theta = 1\} \cup \{\Theta = -1\}.$$

Fixing the boundary condition (β) at b, $\{\Theta = 1\}$ is the spectrum of (q, D, β) and $\{\Theta = -1\}$ is the spectrum of (q, N, β). This corresponds to the famous results by Borg, see [2].

Proposition 4.3.2. *Given a function* $\Theta = \Theta^a_{b,\beta}$. *An inner function* $\tilde{\Theta}$ *satisfies* $\{\Theta = 1\} = \{\tilde{\Theta} = 1\}$ *and* $\{\Theta = -1\} = \{\tilde{\Theta} = -1\}$ *if and only if for some* $c \in (-1.1)$,

$$\tilde{\Theta} = \frac{\Phi - c}{1 - c\Phi}.$$

(2) *General mixed data spectral problem*

Let $\Theta = \Psi\Phi$ and Λ be the spectrum of Θ. Instantly, we have

$(\Psi, \sigma(\Psi\Phi))$ determine $\Psi\Phi$ if and only if Λ is a defining set for function Φ.

Let $\{\lambda_n\}$ be the spectrum a Schrödinger operator (q, α, β) and $\lambda_n \leq \lambda_m$ for $n \leq m$. For $M \subset \mathbb{Z}$, suppose we are given partial spectrum $\{\lambda_n \; n \in M\}$. Then the mixed date spectral problem is whether the meromorphic inner function Ψ and $\{\lambda_n \; n \in M\}$ determine Θ. This is equivalent to discuss whether $\{\lambda_n \; n \in M\}$ is a defining set for Φ.

Furthermore, we can see that the condition is almost the condition of uniqueness sets. So, we have the following statements.

Proposition 4.3.3. *Let Λ be a point set and Φ be a meromorphic inner function. If there is a function $G \in K^{\infty}[\Phi]$ s.t.*

$$G = \bar{G} \ on \ \Lambda,$$

then Λ is not a defining set for Φ.

Also, we have the following observation: For $p > 1$,

$$\exists G \in K^p[\Phi], \ \ G \not\equiv const, \ \ G = \bar{G} \ on \ \Lambda,$$

if and only if

$$\exists F \in K^p[\Phi^2], \ \ F \not\equiv 0, \ \ F \ vanish \ on \ \Lambda.$$

For general p, the precise relation is an interesting question. Also, the condition is almost an if and only if condition. So, we can interpret the problems of defining sets of Φ by uniqueness sets in model space $K^{\infty}[\Phi^2]$.

4.4 Generalized Hochstadt-Liberman's theorem

In general, the potential q is uniquely determined by two spectra. By Hochstadt and Liberman [3], if given one half information of the potential q and a single spectrum, then the other half information of q is uniquely recovered. Also, we apply the Toeplitz kernels to extend the Hochstadt-Liberman's theorem. Let us state the original version first.

Theorem 4.4.1. *Consider a regular Schrödinger operator L with summable potential q. Let $\sigma(L)$ subject to boundary conditions α and β at two endpoints respectively; Also* *consider another operator $\tilde{L}$ with summable potential $\tilde{q}$ and*

$$q(x) = \tilde{q}(x),\ x \in (\frac{1}{2}, 1).$$

Suppose the spectrum of $\tilde{L}$ subject to the same boundary conditions is the same as $\sigma(L)$. Then,

$$q(x) = \tilde{q}(x)\ \ x \in (0, 1).$$

In this case, the potential $q \in L^1(0,1)$. From our construction of Θ_- and Θ_+, c is the mid-point of (a, b). Hence, we can interpret the Hochstadt-Liberman problems in terms of determination of meromorphic inner function.

Theorem 4.4.2. *Let L be a self-adjoint Schrödinger operator on (a, b) with B.C. $(\alpha \neq 0)$ at a and (β) at b. And the potential q is L^1 integrable. Let c be the midpoint of (a, b), then $(\Theta_-, \sigma(\Theta_-\Theta_+))$ determine Θ_+.*

Proof. Given the potential q on (a, c), we know half of the information of $\Theta_-\Theta_+$. Then this problem can be translated into the following lemma.

Lemma 4.4.3. *Suppose $\Theta = \Psi^2$, a function $\tilde{\Theta}$ satisfies $\Psi|\tilde{\Theta}$ and $\sigma(\tilde{\Theta}) = \sigma(\Theta)$ if and only $\exists r \in (-1, 1)$,*

$$\tilde{\Theta} = \Psi \frac{r + \Psi}{1 + r\Psi}.$$

Besides, $\dim(\{\tilde{\Theta}\}) = 1$.

We only prove the lemma in the case $\infty \notin \sigma(\Theta)$. Since $\Phi = \Psi$, so $N^\infty[\bar{\Phi}\Psi] \neq 0$. And by the second proposition in 3.2, the dimension of $\tilde{\Theta}$ is at least 1. If the $\dim(\{\tilde{\Theta}\}) \geq 2$, by the results of Toeplitz kernels, we have

$$N^\infty[b\bar{\Theta}J] \neq 0,$$

where Θ and J are inner functions satisfying $\{J = 1\} = \{\Theta = 1\}$. Therefore, we have

$$N[\bar{\Theta}J] \neq 0.$$

By assumption that $\infty \notin \sigma(\Theta)$, J and Θ are twin inner functions, which means

$$N[\bar{\Theta}J] = 0.$$

Hence, $\dim(\{\tilde{\Theta}\}) = 1$. By the construction of the second proposition in section 3.2,

$$\tilde{\Theta} = \Psi \frac{r + \Psi}{1 + r\Psi}, \quad r \in (-1, 1).$$

By Evritt's theorem [15], if m is a Weyl m-function of a Schrödinger operator, we have

$$m(z) = i\sqrt{z} + o(1), \quad z \to \infty.$$

Hence, if $\Theta = \Theta_m$, then

$$\Theta(z) = 1 - \frac{2}{\sqrt{z}} + \frac{2}{z} + o(z^{-1}), \quad z \to \infty.$$

All inner functions of the form $(r + \Psi)/(1 + r\Psi)$ with $r \neq 0$ do not satisfy the Evritt's asymptotic results. Hence, in this case $r = 0$, i.e. $\tilde{\Theta} = \Theta$. $\square$

Remark 4.

- *We can show the same statement without the assumption $\infty \notin \sigma(\Theta)$. Also, if we only require $\sigma(\tilde{\Theta}) \subset \sigma(\Theta)$, then the statement is also true.*
- *If we consider the Dirichlet boundary condition, the statement is also true, Also, if we consider non-regular $\tilde{L}$, we can still get the same statement.*

- *We can also apply the statement to the case of even potential on real line. In this case, we view the point* 0 *as the midpoint of the real line.*

4.5 Horvath's theorem

In a famous paper [9], Horvath gave precise conditions for a set of eigenvalues to determine the Schrödinger operator on a finite interval, which are closeness properties of the exponential system corresponding to the known eigenvalues. His results include nearly all former statement of inverse eigenvalue problem of the Schrödinger operator.

Let $\mathcal{FL}^r$ be the space of space of Fourier transform of L^r, we have

$$PW_2 = \mathcal{FL}^2 \subset \mathcal{FL}^1 \subset Cart_2 \cap L^\infty(\mathbb{R}).$$

We will introduce some selected Horvath' results, also see [10, 11]. Suppose $Schr(L^r, D) = \{L: \ \ L = (q, \alpha, \beta) \ \ q \in L^r(0, 1)\}$.

Theorem 4.5.1. *Let* $\sqrt{\Lambda}$ *be the square root of* Λ*, then*

- Λ *is defining in* $Schr(L^r, D)$ *if and only if* $\sqrt{\Lambda} \cup \{*, *\}$ *is a uniqueness set of* $\mathcal{FL}^r$*,* $\{*, *\}$ *is the set of any two points.*
- Λ *is defining in* $Schr(L^r, N)$ *if* $\sqrt{\Lambda}$ *is not a zero set of* $\mathcal{FL}^r$*.*

We will prove the following L^2 version of Horvath's theorem.

Proposition 4.5.2. Λ *is defining in* $Schr(L^2, D)$ *if and only if* $\sqrt{\Lambda}$ *plus any two points is a uniqueness set of* $\mathcal{FL}^2$*;*

Proof. Let q and $\tilde{q}$ be two potential from $L^2(\mathbb{R})$. We can always assume the Schrödinger operator $L = (q, D, N)$ and $\tilde{L} = (\tilde{L}, D, N)$ are positive on $(0, 1)$. Since by adding a positive constant to the potential, the property of uniqueness set is the same.

We denote θ and $\tilde{\theta}$ as the Weyl inner functions of L and $\tilde{L}$ after the square root transform. For $s > 0$, suppose $u_S(t)$ is the solution of the following equation

$$-u'' + qu = s^2 u, \quad u(0) = 0, \ u'(0) = 1.$$

Then u_s is the solution of the corresponding integral equation

$$u_s(x) = \sin sx + \frac{1}{s}\int_0^x \cos s(x-t)q(t)u_s(t)dt.$$

Take $x = 1$,

$$\begin{aligned}
u_s(1) &= \sin s + \frac{1}{s}\int_0^1 \cos s(1-x)q(x)u_s(t)dx \\
&= \sin s + \frac{1}{s}\int_0^1 \cos s(1-x)q(x)(\sin sx + \frac{1}{s}\int_0^x \cos s(x-t)q(t)u_s(t)dt)dx \\
&= \sin s + \frac{1}{s}\int_0^1 \cos s(1-t)q(t)\sin stdt \\
&+ \frac{1}{s^2}\int_0^1 \cos s(1-x)q(x)dx\int_0^x \cos s(x-t)q(t)u_s(t)dt
\end{aligned}$$

Let

$$F_1(s) = \int_0^1 \cos s(1-t)q(t)\sin stdt,$$

and

$$R_1(s) = \int_0^1 \cos s(1-x)q(x)dx\int_0^x \cos s(x-t)q(t)u_s(t)dt.$$

For $t \in (0,1)$,

$$|u_s(t)| \leq C_1.$$

Hence,

$$R_1(s) \leq const, \ \ \forall S.$$

Also, since $q(t) \in L^2(\mathbb{R})$, F_1 is the Fourier transform of a $L^2(\mathbb{R})$ integrable function. Hence,

$$F_1 \in L^2(\mathbb{R}).$$

From the formula of $u_s(x)$, we have

$$u_s'(x) = s\cos x + \frac{1}{s}q(x)u_s(x) - \int_0^x \sin s(x-t)q(t)u_s(t)dt.$$

Take $x = 1$,

$$\begin{aligned}
u_s'(1) &= s\cos s + \frac{1}{s}q(1)u_s(1) - \int_0^1 \sin s(1-t)q(t)u_s(t)dt \\
&= s\cos s + \frac{1}{s}q(1)u_s(1) - \int_0^1 \sin s(1-x)q(x)\sin sx dx \\
&+ \frac{1}{s}\int_0^1 \sin s(1-x)q(x)dx \int_0^x \cos s(x-t)q(t)u_s(t)dt \\
&= s\cos s + \frac{1}{s}q(1)u_s(1) - F_2(s) - \frac{1}{s}R_2(s)
\end{aligned}$$

Similarly, we have

$$R_2(s) \leq const, \quad \forall S,$$

and

$$F_2 \in L^2(\mathbb{R}).$$

Hence, the Weyl inner function is

$$\Theta(s^2) = \frac{-u_s'(1) - iu_s(1)}{-u_s'(1) + iu_s(1)},$$

and the square root transforms of Θ is

$$\theta = \frac{(s+1)\Theta(s^2) + (s-1)}{(s-1)\Theta(s^2) + (z+1)}.$$

After calculation, we have

$$\frac{\theta}{S^2} = \frac{\bar{H}}{H} \;\; on \;\; \mathbb{R}, \quad H^{\pm} \in H^{\infty},$$

and

$$x(\theta(x) - \tilde{\theta}(x)) \in L^2(\mathbb{R}).$$

Since we require the Neumann boundary condition at 1, we have

$$\tilde{\theta} = \theta \;\; on \;\; \{0\} \cup \sqrt{\Lambda}.$$

By the above asymptotic properties,

$$(z-1)(\theta - \tilde{\theta}) \in K[\theta\tilde{\theta}],$$

hence,

$$(z-1)(\theta(z) - \tilde{\theta}(z)) = 0 \;\; z \in \sqrt{\Lambda} \; or \; z = 0, 1.$$

Since

$$\frac{\theta}{S^2} = \frac{\bar{H}}{H} \;\; on \;\; \mathbb{R}, \quad H^{\pm} \in H^{\infty},$$

some function f from the model space of S^4 vanish on $\sqrt{\Lambda} \cup \{0, 1\}$. $\square$

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