A NON-LOCAL THIRD ORDER THEORY OF FUNCTIONALLY GRADED PLATES UNDER ELECTROMECHANICAL COUPLING EFFECT

A Dissertation

by

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ABSTRACT

In this research, a general nonlinear third-order plate theory has been developed using the principle of virtual displacements. The developed theory is based on geometric nonlinearity, size effects of structures in micro scale, functionally graded materials, and piezoelectric effects. The von Kármán nonlinear strains, i.e., small strain and moderate rotation, are considered to represent geometric nonlinearity. A modified couple stress theory is adopted to capture microstructure dependent size effect. A power law distribution is used to represent the variation of two material constituents through thickness. The developed plate theory is also specialized to classical, first order shear deformation, and Reddy third order plate theories.

Analytical solutions for the developed plate theory are presented using the Navier solution technique. All dependent variables are assumed to be forms of double trigonometric functions which satisfy the boundary conditions. The analytical solutions are limited to geometric linearity and simply supported plates. Examples of bending, buckling, and vibration problems are presented to show effects of the power-law distribution of two materials and the microstructure-dependent size parameter.

The nonlinear finite element model based on the developed plate theory is carried out to study of the static bending problems regarding the size effects of microstructure, geometric nonlinearity, and power-law variation of the material composition through the thickness. The principle of virtual work is utilized to develop a displacement based weak-form Galerkin finite element model which requires C^1 continuity of all dependent variables. A conforming element is implemented using Hermite type interpolation functions.

The piezoelectric effect is considered for functionally graded smart plates which have surface-mounted piezoelectric layers, and a functionally graded core layer. The formulation includes the coupling between mechanical deformations and the charge equations of electrostatics. In addition to the kinematic assumption of the developed plate theory, the potential function is assumed to be the combination of half cosine variation of electric potential and linear variation of applied voltage on outer surfaces. An analytical solution and a finite element model are obtained. A parametric study is presented to show effects of thickness ratio between core layer and piezoelectric layers in addition to material variation of core plate and micro structure size effects.

DEDICATION

To my beloved parents and brother

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1. INTRODUCTION

Plates and shells are three-dimensional (3D) bodies that have much smaller dimension in the out-of-plane direction than the in-plane directions. To study behavior of these structural elements, it is efficient to reduce 3D problems to 2D problems because of geometrical characteristics of plate- and shell- like bodies, the high computational costs and the complexity of the numerical analysis of 3D systems. The dimension can be reduced by separating volume integral into thickness (1D) and area integrals (2D), and performing thickness integral first. Overall properties of a plate are represented by equivalent single layer (ESL) properties. The plate and shell theories assume expansions of the field of dependent variables (stresses or displacements) through thickness directions based on hypotheses of each theories. The oldest and simplest plate theory is the classical plate theory (CPT, also known as Kirchhoff Love plate theory) based on the Kirchhoff hypothesis which assumes 1) the straight lines normal to the mid surface before deformation remain normal to the mid surface after deformation (normality), 2) straight lines before deformation remain straight after deformation (straight), and 3) the length of straight lines normal to mid surface remain the same length (inextensibility). As a result of using the Kirchhoff hypothesis, the CPT ignores transverse normal and shear strains. The simplest shear deformation plate theory is the first shear deformation theory (FSDT, also known as Mindlin-Reissner plate theory) which considers transverse shear strains and stresses by removing the normality condition from the Kirchhoff hypothesis. In the case of FSDT, however, the transverse shear strains and stresses are constants through thickness of plates, which does not agree with an exact solution (i.e. quadratic variation of transverse shear strain). To obtain an equivalent transverse shear forces to the exact solution, the shear stresses are modified using shear correction factors. To improve the transverse shear strains, higher order shear deformation theories (HSDTs) are introduced. In the case of HSDTs, in-plane displacement field is assumed to have more than quadratic variation; generally, a cubic variation is preferred through thickness direction. By using cubic or higher variations of in-plane displacement field, a parabolic variation of shear strain is promised. Because plates are fundamental and essential components in most structural systems, these theories have intensively and extensively grown with the introduction of advanced material systems, e.g., shape memory, functionally graded, piezoelectric, and bio materials. These advanced material systems are required for intelligence, functionality, and serviceability in the whole spectrum of magneto-, electro-, thermo-, and bio-mechanical couplings. Among various types of advanced materials, functionally graded materials (FGMs) have great advantages in applications of space and automobile structures, microelectromechanical systems (MEMS), and nanoelectromechanical systems (NEMS). In general, the volume fraction of two or more constituents in FGMs continuously varies in a body. A common FGM is made of two constituents to provide a certain functionality, e.g., thermal barriers. In the case of thermal barriers in space structures, the two constituents are ceramic and metal; the ceramic constituent with low thermal conductivity insulates structural systems against high temperatures, and the ductile metal constituent resists thermal fracture due to high temperature gradient. Because of the advantages of FGMs in the coupling of various spectrums, they are often used in micro- and nano-scale structural systems. When FGM plates are in those small scale systems, it is necessary to account for the microstructure size dependent effects, and piezoelectric materials are often used for actuating and sensing these plates. Because the plate and shell theories are based on classical elasticity, they are not able to capture the size dependent effects. These theories must be extended to account for size effects in addition to material gradation through thickness, piezoelectric effects, and geometric nonlinearity.

1.1 Motivation for proposed study

The numerous researchers have studied beam, plate, and shell theories based on assumptions of the displacement field, displacement and strain relations, and strain and stress relations. Those theories can be categorized into CPT, FSDT, and HSDT based on the assumptions of the displacement field, linear and nonlinear plate theories based on displacement and strain relations, and the consideration of materials based on strain and stress relations, e.g., linear and nonlinear elasticity; visco-elasticity; plasticity; and magneto-, electro-, thermo-, and bio-mechanical couplings. With developments of novel material systems such as functionally graded and piezoelectric materials, those theories should be extended to represent the new material systems. In the literature, functionally graded beam, plate and shell theories are based on CPT and FSDT due to their simplicity and acceptable accuracy in many applications, and only a few of them have been developed using HSDT.

FGMs are often used in micro- and nano-scale structural systems due to their advantages in the coupling problems, and piezoelectric materials are used for actuating or sensing the structural systems in the small scale. Many researchers have clearly shown that the size dependent effect should be included in the analysis of structures at micro- or nano-scale. However, a plate model does not exist that accounts for material variation in the thickness direction of plates, microstructure-dependent size effects, geometric nonlinearity, shear deformation without requiring shear correction factors, and the electromechanical coupling effect of piezoelectric materials. This very fact motivated the present study. The objective of this study is to develop a general third-order plate theory and to obtain its analytical solutions to bending, vibration, and buckling problems, and to develop a finite element model of the present plate theory that accounts for through-thickness power-law variation of a two material constituents, micro structure size effects using a modified couple stress theory, the bending-extensional coupling based on von Kármán nonlinearity, and electromechanical coupling effects.

1.2 Scope of the research

This research began at Texas A&M University in the Fall of 2011, and is mainly focused on developing a general third order plate theory that accounts for functionally graded and piezoelectric materials, geometrical nonlinearity, and micro structure size effects. The research includes analytical solutions using the Navier solution technique for simply supported square micro plates in bending, vibration, and buckling problems and a nonlinear finite element solution for various boundary conditions using a displacement based weak form Galerkin finite element model. In addition, a parametric study for the material variation through thickness, micro structure size effects, and piezoelectric effects is performed. The dissertation is organized as follows:

- In section 2, a general nonlinear third-order plate theory that accounts for geometric nonlinearity, microstructure-dependent size effects, and two-constituent material variation through the plate thickness is presented using the principle of virtual displacements. A detailed derivation of the equations of motion, using Hamilton's principle, is presented, and it is based on a modified couple stress theory, power-law variation of the material through the thickness, and the von Kármán nonlinear strains. The modified couple stress theory includes a material length scale parameter that can capture the size effect in a functionally graded material. The governing equations of motion derived herein for a general third-order theory with geometric nonlinearity, microstructure-dependent size effect, and material gradation through the thickness are specialized to classical and shear deformation plate theories available in the literature. The theory presented herein also can be used to develop finite element models and determine the effect of the geometric nonlinearity, microstructure-dependent size effects, and material grading through the thickness on bending and postbuckling response of elastic plates. The work presented in this section is from the published journal paper [1].

- In section 3, analytical solutions of the developed plate theory in section 2 are presented. The Navier solution technique is adopted to derive analytical solutions to simply supported rectangular plates and the analytical solution is limited to geometrically linear problems. The modulus of elasticity and the mass density are assumed to vary only through thickness of plate, and a single material length scale parameter of a modified couple stress theory captures the microstructure-dependent size effects. Examples of bending, buckling, and vibration problems are presented to show effects of the power-law distribution of two materials and the microstructure-dependent size parameter. The results presented in this section can be found in the published journal paper [2].

- In section 4, nonlinear finite element analyses of functionally graded plates based on the developed third order plate theory in section 2 is carried out to bring out the effects of couple stress, geometric nonlinearity, and power-law variation of the material composition through the thickness on static bending analyses of plates. The principle of virtual work is utilized to develop a displacement based weak-form Galerkin finite element model. The developed finite element model requires C^1 continuity of all dependent variables and no shear correction factors are needed. The micro-structural effects are captured using a length scale parameter via the modified couple stress theory. The variation of two-constituent material is assumed only through the thickness direction according to a power-law distribution. Numerical results are presented for static bending problems of rectangular plates with various boundary conditions to bring out the parametric effects of the power-law index and length scale parameter on the deflections. The results presented in this section can be found in the published journal paper [3].

- In section 5, the equation of motion for functionally graded plates with surface-mounted piezoelectric layers that accounts for the gradient elasticity through the modified couple stress model and linear piezoelectricity is developed using Hamilton's principle. The formulation includes the coupling between mechanical deformations and the charge equations of electrostatics. The mathematical model developed herein is an equivalent single-layer theory for mechanical displacement field and the potential functions. The displacement field of the general third order plate theory is used for kinematic assumption. The potential function is assumed as the combination of half cosine variation of electric potential and linear variation of applied voltage on outer surfaces. The approach described here is that standard plate models can be enhanced to include the coupling between the charge equations and the mechanical deformations as well as the size dependent effect of micro- and nano-scale structures. Analytical and finite element solutions of the developed plate model are presented. A parametric study is performed to study the effect of material variation through thickness of plates, length scale parameters to capture the size dependent effects, and the thickness ratio between piezoelectric layers and whole plate. The section introduces the work that can be found in the published journal paper [4].

- In section 6, concluding remarks of this dissertation and suggestions for future research directions are presented.

2. FORMULATION *

The next generation of material systems used in space and other structures as well as in MEMS and NEMS feature thermo-mechanical coupling, functionality, intelligence, and miniaturization. These systems may operate under varying conditions; they span the whole spectrum of magneto-electro-thermomechanical conditions. When functionally graded material systems are used in nano- and micro-devices, it is necessary to account for the microstructure-dependent size effect and the geometric nonlinearity. Since beam and plate structural elements are commonly used in these devices and structures, it is useful to develop refined theories of plates that account for size effects, material gradation through thickness, and geometric nonlinearity. The present study is focused on developing a general third-order theory with aforementioned effects. The following sections provide a background for the present study.

2.1 Higher-order plate theories

Plates are structural elements whose plane form dimensions are quite large compared to their thickness, supported at few points of the domain, and subjected to forces that make the structure stretch and bend under the action of external loads. Theories used to study the response of plates under external loads are obtained by reducing three-dimensional elasticity theory through a series of assumptions concerning the kinematics of deformation and constitutive behavior. The kinematic assumptions exploit the fact that such structures do not experience significant strains or stresses associated with the thickness direction. Thus, the solution of the three-dimensional elasticity problem associated with a plate is reformulated in terms of displacements or stresses whose form is presumed on the basis of

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an educated guess concerning the nature of the deformation.

Two major classes of two-dimensional plate theories can be found in the literature: one based on the assumed form of the displacement field and the other based on an assumed form of the stress field. In both cases, the fields are expanded in increasing powers of the thickness coordinate. Obviously, the higher-order terms would have diminishing returns compared to the lower-order terms due to the smallness of thickness compared to in-plane dimensions. Among the two classes, displacement-based theories have emerged as the preferred ones because one does not have to consider strain/stress compatibility conditions in addition to the kinematic and equilibrium conditions. It is determined that a third-order expansion of the displacement field is optimal because it gives quadratic variation of transverse strains and stresses, and require no shear correction factors compared to the first-order theory, where the transverse strains and stresses are constant through the plate thickness. To bring to light the the original contributions made in the development of plate theories over the years by others, a brief overview of research done in third-order plate theories is also included in here.

The simplest and oldest plate theory is the classical (Kirchhoff) plate theory (CPT) [5]. It is based on the kinematic assumptions that straight lines perpendicular to the plane of the undeformed plate remain straight and inextensible, and rotate such that they always remain perpendicular the midplane of the plate after deformation. These assumptions, known as the Kirchhoff hypothesis, amounts to neglecting both transverse shear and transverse normal strains [6, 7]. The assumed displacement field is of the form

$$u_1(x, y, z, t) = u(x, y, t) + z\theta_x(x, y, t) \tag{2.1}$$

$$u_2(x, y, z, t) = v(x, y, t) + z\theta_y(x, y, t) \tag{2.2}$$

$$u_3(x, y, z, t) = w(x, y, t) \tag{2.3}$$

where

$$\theta_x = -\frac{\partial w}{\partial x}, \quad \theta_y = -\frac{\partial w}{\partial y} \tag{2.4}$$

and z is the coordinate perpendicular to the undeformed midplane of the plate, and (x, y) coordinates lying in the plane. These assumptions simplify the three-dimensional problem significantly to a two-dimensional problem whose governing equations are expressed in terms of the three displacements (u, v, w) of a point on the midplane. The theory does not qualify to be called first-order because the first-order terms, θ_x and θ_y, are not independent of w. The theory is adequate in the vast majority of problems when thickness is very small (two orders of magnitude less than the smallest in-plane dimension) and transverse shear strains, γ_{xz} and γ_{yz}, are negligible. Finite element models of the CPT require C^1-continuity, i.e., continuity of the transverse displacement w as well as its derivatives - slopes θ_x and θ_y, and the development of CPT finite elements that satisfy all completeness and compatibility requirements is cumbersome [8, 9].

The simplest first-order shear deformation plate theory (FSDT), often referred to as the Mindlin plate theory [10], is based on the displacement expansion

$$u_1(x, y, z, t) = u(x, y, t) + z\theta_x(x, y, t) \tag{2.5}$$

$$u_2(x, y, z, t) = v(x, y, t) + z\theta_y(x, y, t) \tag{2.6}$$

$$u_3(x, y, z, t) = w(x, y, t) \tag{2.7}$$

where θ_x and θ_y are the rotations of a transverse normal line,

$$\theta_x = \frac{\partial u_1}{\partial z} \quad \text{and} \quad \theta_y = \frac{\partial u_2}{\partial z}. \tag{2.8}$$

The original idea of such expansion can be found in earlier works by Basset [11], Hencky [12] and Hildebrand, Reissner and Thomas [13].The first-order theory is based on the first two assumptions of the Kirchhoff hypothesis, and the normality of the assumption is not invoked, making the rotations θ_x and θ_y to be independent of (u, v, w). As a result the transverse shear strains γ_{xz} and γ_{yz} are nonzero but independent of z. This leads to the introduction of shear correction factors in the evaluation of the transverse shear forces. The finite element models of the theory require only C^0-continuity, i.e., the variables of the theory $(u, v, w, \theta_x, \theta_y)$ be continuous between elements; however, they can exhibit spurious transverse shear stiffness even in pure bending, known as the shear locking, as the plate becomes thin. The spurious transverse shear stiffness stems from an interpolation inconsistency that prevents the Kirchhoff conditions of Eq. (2.4) from being satisfied as the plate becomes thin. The shear locking phenomenon can be alleviated by using a reduced integration to evaluate transverse shear stiffness terms in the element stiffness matrix or by using higher-order approximations of the displacement field. Although the reduced integration solution is the most economical alternative, the process allows some elements to exhibit spurious displacement modes, i.e., deformation modes that result in zero strain at the Gaussian integration points.

Second-order and higher-order plate theories relax the Kirchhoff hypothesis further by allowing the straight lines normal to the midplane before deformation to become curves. However, most published theories still assume inextensibility of these lines. Second-order plate theories are not popular because of the fact that they too require shear correction factors and while not improving over FSDT. The third-order theories provide a slight increase in accuracy relative to the FSDT solution, at the expense of an increase in computational effort, and do not require shear correction factors. From the finite element model development standpoint, third-order plate theories requires C^1-continuity of the transverse displacement component and less sensitive to shear locking, depending on the specific third-order theory used.

Several third-order plate theories have been developed by different researchers [14–

22], and some of them are claimed to be new whereas they are not new, as pointed by Reddy [23], but only disguised in the form of the displacement expansions used. Various third-order plate theories developed over the years differ from each other based on the assumptions of their theories. The final equations developed depend on (1) the displacement field, (2) the strain-displacement relations (linear or nonlinear, if nonlinear, nature of the nonlinearity included - small strain but large displacements and rotations or moderate rotations, etc.), and (3) equilibrium (or equations of motion) adopted. Similar displacement fields were suggested by Schmidt [14], Murty [15], Lo, Christensen, and Wu [16, 17]. Since 1980 there appeared a number of papers which used third-order displacement fields of different types (see Levinson [18], Murthy [19], Kant [20], Reddy [21–24], Bhimaraddi and Stevens [25], Bose and Reddy [26, 27], among others). Several of these displacement fields look different while they are essentially the same, as pointed by Reddy [23]. The works of Schmidt [14], Murty [15], and Levinson [18] are restricted to isotropic plates. Schmidt [14] also accounted for the von Kármán non-linear strains which assumes small strains and moderate rotations.

Even when the displacement field used is the same, the equations of equilibrium used by various authors were different. Some used the equilibrium equations of the classical or first-order theory, because they are arrived using the vector approach, in which the equilibrium of a plate element is considered. Thus, these equations do not contain the effect of the higher-order terms in the form of higher-order stress resultants but they are included in the strains computed. These theories are not derivable from energy considerations and they result in unsymmetric stiffness coefficients even for a linear case (if the finite element method is used). The second approach is to use the principle of virtual displacements. The resulting set of equations for all theories higher than first-order are different from those arrived using the vector approach. The virtual work principle gives many more additional terms in the form of higher-order stress resultants. Reddy [21, 22] is the first one to develop the equilibrium equations of a third-order plate theory with vanishing tractions for laminated composite plates using the principle of virtual displacements. Reddy third-order plate theory [22] also accounts for the von Kármán strains.

A comment is in order on the development of finite element models of plates. When a continuum formulation is used, the finite element model includes the total nonlinearity, unless otherwise stated, and the kinematic assumptions are invoked at the time of approximating the displacement field (after the virtual work statement is derived). The continuum formulation inherently includes suitable measures of stress and strain (e.g., second Piola-Kirchhoff stress tensor and Green-Lagrange strain tensor) and geometric updates to determine the deformed configuration at any increment of the load. In contrast, finite element models developed using a given plate theory are based on the assumption that strains are small and rotations may be moderately large so that the geometry changes are neglected (i.e., there is no difference between the Cauchy stress and second Piola–Kirchhoff stress tensors). Thus, a plate theory with full nonlinearity should not be developed unless proper measures of stress and strain are used.

2.2 Modified couple stress theory

In recent years numerous papers that consider microstructure-dependent size effects in formulating the classical and first-order beam and plate theories have appeared (see Eringen [28], Anthoine [29], Yang, et al. [30], Park and Gao [31,32], Reddy [33–35], Lu et al. [36], Reddy and Pang [33], Ma, Gao, and Reddy [37–39], Aghababaei and Reddy [40], Tsiatas [41], Asghari, et al. [42, 43], Xia, Wang, and Yin [44], Ke and Wang [45], and Ke et al. [46]). As these studies have been motivated by the fact that beam-like and plate-like structural elements are commonly used in micro- and nano-scale devices and systems such as biosensors, atomic force microscopes, MEMS, and NEMS (see Li et al. [47], Lam et al. [48], Pei, Tian, and Thundat [49], and McFarland and Colton [50]), and their response may be influenced by certain microstructural parameters. The classical couple stress elasticity theory of Koiter [51] includes four material constants (two classical and two additional). The higher-order Bernoulli-Euler beam model developed by Papargyri-Beskou et al. [52] is based on the gradient elasticity theory with surface energy and it involves four elastic constants, two classical and two non-classical. In view of the difficulties in determining microstructure dependent length scale parameters (see Lam et al. [48]; Yang and Lakes [53]; Maranganti and Sharma [54]) and the approximate nature of beam and plate theories, refined beam and plate models that involve only one material length scale parameter are desirable. One such model has been developed for the Bernoulli-Euler beam by Park and Gao [31, 32] and for the Timoshenko and Reddy beam theories and Mindlin plates by Ma, Gao, and Reddy [37–39] and Reddy [35] using a modified couple stress theory proposed by Yang et al. [30], which contains only one material length scale parameter.

On the other hand, the nonlocal Bernoulli-Euler beam model by Peddieson, Buchanan, and McNitt [55], the nonlocal Timoshenko beam model by Wang et al. [56], the nonlocal Bernoulli-Euler, Timoshenko, Reddy, and Levinson beam models formulated by Reddy [34, 57] and Reddy and Pang [33] are developed using a constitutive equation suggested by Eringen [28]. The modified couple-stress models involve a length scale that is different from the nonlocal parameter used in the Eringen's model, and they have the opposite effect on the response. In addition, the nonlocal formulations of beams and plates based on Eringen's model yield governing equations that cannot be derived using the principle of virtual displacements. The present study is based on modified couple stress theory of Yang et al. [30].

2.3 Functionally graded materials

Structural designs are often based on maximum stress criteria. Elimination of stress concentrations in structural elements is mitigated by suitable design of materials. Functionally gradient materials (FGMs) are a class of materials that have a predetermined (often using an optimization procedure) variation of material properties from point surface to another (see Hasselman and Youngblood [58], and Koizumi [59]) These materials were proposed as thermal barrier materials for applications in space planes, space structures, nuclear reactors, turbine rotors, flywheels, and gears, to name only a few. These materials are often isotropic but nonhomogeneous. One reason for increased interest in FGMs is that it may be possible to create certain types of FGM structures capable of adapting to operating conditions.

A most common FGM is one in which two materials are mixed to achieve a composition that provides certain functionality. For example, for thermal-barrier structures, two-constituent FGMs are made of a mixture of ceramic and metals. The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture due to high temperature gradient in a very short period of time. The gradation in properties of the material reduces thermal stresses, residual stresses, and stress concentrations. The paper by Noda [60] provides an extensive review that covers a wide range of topics from thermo-elastic to thermo-inelastic problems. He also discussed the importance of temperature dependent properties on thermoelastics problems and presented analytical methods to handle transient heat conduction problems and indicates the necessity for the optimization of FGM properties. Tanigawa [61] compiled a comprehensive review on the thermoelastic analysis of FGMs.

A number of other investigations dealing with thermal stresses and deformations of beams, plates, and cylinders had been published in the literature (see, for example, Noda and Tsuji [62], Obata, Noda, and Tsuji [63], Reddy and Chin [64], Praveen and Reddy [65], Praveen, Chin, and Reddy [66], and Vel and Batra [67], among others). Among these studies that concern the thermo-elastic analysis of plates, beams or cylinders made of FGMs where the material properties have been considered temperature dependent are Noda and Tsuji [62], Praveen and Reddy [65], Praveen, Chin, and Reddy [66],Shen [68], Yang and Shen [69], and Kitipornchai, Yang, and Liew [70], among few others. The work of Praveen and Reddy [65], Reddy [71], and Aliaga and Reddy [72] also considered von Kármán nonlinearity (also see Reddy [6]) in functionally graded plates. The works of Aliaga and Reddy [72] also considered the third-order plate theory of Reddy [6, 21, 22]. Jin and Noda [73–75] suggested optimum material variation of metal-ceramic functionally gradient material by minimizing the thermal stress intensity factor and presented the steady state and the transient heat conduction problems in FGMs. Reddy and Berry [76] presented axisymmetric bending of functionally graded circular plates based on temperature dependent material properties and the von Kármán geometric nonlinearity. Saidi, Bodaghi, and Atashipour [77] presented Levy-type solution for bending-stretching analysis of thick functionally graded rectangular plates using Reddy third order plate theory [22].

2.4 Constitutive models

2.4.1 Material variation through the thickness

Consider a plate of total thickness, h. The x and y coordinates are taken in the mid-plane, denoted by Ω, and the z-axis is taken normal to the plate. We assume that the material of the plate is isotropic but varies from one kind of material on one side, $z = -h/2$, to another material on the other side, $z = h/2$. A typical material property of the FGM through the plate thickness is assumed to be represented by a power-law (see Praveen and Reddy [65] and Reddy [71])

$$P(z,T) = [P_c(T) - P_m(T)]\, f(z) + P_m(T), \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{2.9}$$

where P_c and P_m are the values of a typical material property, such as the modulus, density, and conductivity, of the ceramic material and metal, respectively; n denotes the volume fraction exponent, called power-law index. When $n = 0$, we obtain the single-material plate (with property P_c).

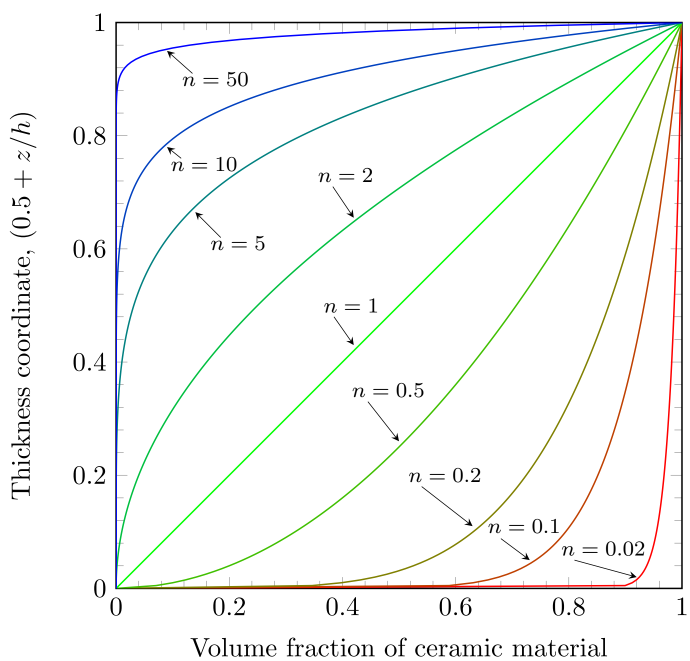

Figure 2.1: Volume fraction of P_c through the plate thickness for various values of power-law index [1]

Figure 2.1 shows the variation of the volume fraction of P_c through the plate thickness for various values of the power-law index n. Note that the volume fraction $f(z)$ decreases with increasing value of n. When FGMs are used in high-temperature environment, the material properties are temperature-dependent and they can be expressed as

$$P_\alpha(T) = c_0 \left(c_{-1}T^{-1} + 1 + c_1 T + c_2 T^2 + c_3 T^3\right), \quad \alpha = c \text{ or } m \tag{2.10}$$

where c_0 is a constant appearing in the cubic fit of the material property with temperature; and c_{-1}, c_1, c_2, and c_3 coefficients of T^{-1}, T, T^2, and T^3, obtained after factoring out c_0 from the cubic curve fit of the property. The modulus of elasticity, conductivity, and the coefficient of thermal expansion are considered to vary according to Eqs. (2.9) and (2.10).

2.4.2 Modified couple stress model

The couple stress theory proposed by Yang et al. [30] is a modification of the classical couple stress theory (see Koiter [51]). They established that the couple stress tensor is symmetric and the symmetric curvature tensor is the only proper conjugate strain measure to have a contribution to the total strain energy of the body. The two main advantages of the modified couple stress theory over the classical couple stress theory are the inclusion of a symmetric couple stress tensor and the involvement of only one length scale parameter, which is a direct consequence of the fact that the strain energy density function depends only on the strain and the symmetric part of the curvature tensor (see Ma, Gao, and Reddy [37]).

According to the modified couple stress theory, the virtual strain energy $\delta\mathcal{U}$ can be written as

$$\delta\mathcal{U} = \int_V (\delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma} + \delta\boldsymbol{\chi} : \mathbf{m})dv = \int_V (\sigma_{ij}\,\delta\varepsilon_{ij} + m_{ij}\,\delta\chi_{ij})\,dv \tag{2.11}$$

where summation on repeated indices is implied; here σ_{ij} denotes the cartesian components of (the symmetric part of) the stress tensor, ε_{ij} are the strain components, m_{ij} are the components of the deviatoric part of the symmetric couple stress tensor, and χ_{ij} are the components of the symmetric curvature tensor

$$\boldsymbol{\chi} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{\omega} + (\boldsymbol{\nabla}\boldsymbol{\omega})^{\mathrm{T}}\right], \quad \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{u} \tag{2.12}$$

or

$$\chi_{ij} = \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right), \quad i, j = 1, 2, 3 \tag{2.13}$$

and ω_i ($i = 1, 2, 3$) are the components of the rotation vector

$$\omega_x = \omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \tag{2.14}$$

$$\omega_y = \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \tag{2.15}$$

$$\omega_z = \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right). \tag{2.16}$$

Thus, we have

$$\chi_{xx} = \chi_{11} = \frac{1}{2}\left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_1 \partial x_3}\right) \tag{2.17}$$

$$\chi_{yy} = \chi_{22} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2 \partial x_3} - \frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right) \tag{2.18}$$

$$\chi_{zz} = \chi_{33} = \frac{1}{2}\left(\frac{\partial^2 u_2}{\partial x_1 \partial x_3} - \frac{\partial^2 u_1}{\partial x_2 \partial x_3}\right) \tag{2.19}$$

$$\chi_{xy} = 2\chi_{12} = \frac{1}{2}\left(\frac{\partial^2 u_3}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_2 \partial x_3} + \frac{\partial^2 u_1}{\partial x_1 \partial x_3} - \frac{\partial^2 u_3}{\partial x_1^2}\right) \tag{2.20}$$

$$\chi_{xz} = 2\chi_{13} = \frac{1}{2}\left(\frac{\partial^2 u_3}{\partial x_2 \partial x_3} - \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right) \tag{2.21}$$

$$\chi_{yz} = 2\chi_{23} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_3^2} - \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2} - \frac{\partial^2 u_1}{\partial x_2^2}\right). \tag{2.22}$$

2.4.3 Constitutive relations

For an isotropic, linear elastic material the 3-D stress-strain relations are

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\delta_{ij}\,\varepsilon_{kk} - \alpha\,(3\lambda + 2\mu)\,(T - T_0)\,\delta_{ij} \tag{2.23}$$

$$m_{ij} = 2\mu\,\ell^2\,\chi_{ij} \tag{2.24}$$

where μ and λ are the Lamé parameters (see Reddy [78])

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}\,, \quad 2\mu = \frac{E}{1+\nu}\,, \text{ and } \quad 3\lambda + 2\mu = \frac{E}{1-2\nu}$$

with E being Young's modulus and ν being Poisson's ratio, α is the coefficient of thermal expansion, and ΔT is the temperature increment from the room temperature, T_0, and ℓ is the material length scale parameter. The material length scale parameter is the square root of the ratio of the modulus of curvature to the modulus of shear, and it is a property measuring the effect of the couple stress. For a functionally graded material, μ and λ are functions of z and, possibly, T, as indicated in Eqs. (2.9) and (2.10).

2.5 General third-order plate theory

Consider a plate of total thickness h and composed of functionally graded material through the thickness. It is assumed that the material is isotropic, and the grading is assumed to be only through the thickness. The $xy-$plane is taken to be the undeformed midplane Ω of the plate with the z-axis positive upward from the midplane, as shown in Fig. 2.2. We denote the boundary of the midplane with Γ. The plate volume is denoted as $V = \Omega \times (-h/2, h/2)$. The plate is bounded by the top surface Ω^+, bottom surface Ω^-, and the lateral surface $S = \Gamma \times (-h/2, h/2)$.

Here develop a general third-order theory for the deformation of the plate first and then specialize to the well-known plate theories. We restrict the formulation to linear elastic material behavior, small strains, and moderate rotations and displacements, so that there is no geometric update of the domain, that is, the integrals posed on the deformed configuration are evaluated using the undeformed domain and there is no difference between the Cauchy stress tensor and the second Piola–Kirchhoff stress tensor.

The equations of motion are obtained using the principle of virtual displacements for the dynamic case (i.e., Hamilton's principle). The three-dimensional problem is reduced to two-dimensional one by assuming a displacement field that is explicit in the thickness coordinate. We make use of the fact that the volume integral of any sufficiently continuous function $f(x, y; z)$ that is explicit in z can be evaluated as

$$\int_V f(x, y; z)\, dV = \int_\Omega \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} f(x, y; z)\, dz \right) dxdy. \tag{2.25}$$

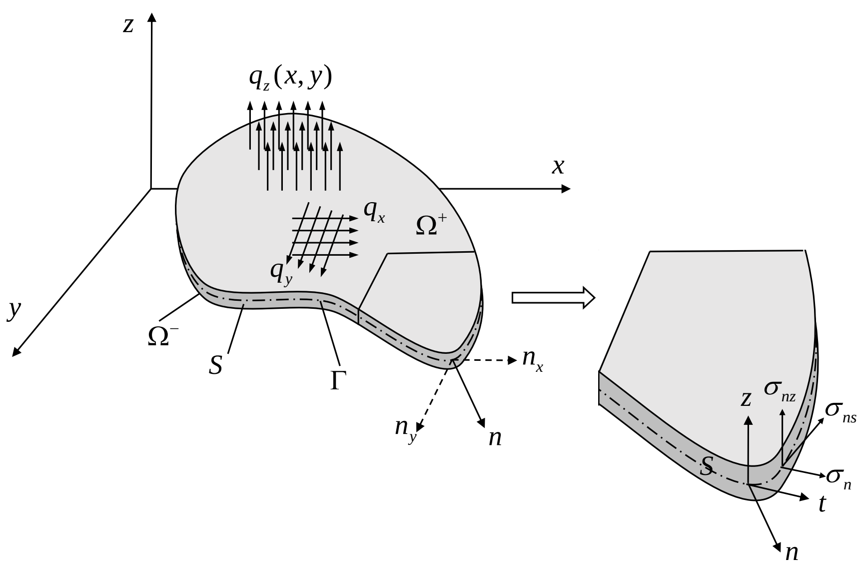

Figure 2.2: Domain and various boundary segments of the domain [1]

When thermal effects are considered, like in the case of thermomechanical loads, the temperature distribution, which is assumed to vary only in the thickness direction, i.e., $T = T(z)$, is determined by first solving a simple steady state heat transfer equation through the thickness of the plate, with specified temperature boundary conditions at the top and bottom of the plate. The energy equation for the temperature variation through the thickness is governed by

$$\rho c_v \frac{\partial T}{\partial t} - \frac{\partial}{\partial z}\left[k(z,T)\frac{\partial T}{\partial z}\right] = 0, \;\; -\frac{h}{2} \leq z \leq \frac{h}{2} \tag{2.26}$$

$$T(-h/2,t) = T_m(t), \;\; T(h/2,t) = T_c(t) \tag{2.27}$$

where $k(z,T)$ is assumed to vary according to Eqs.(2.9) and (2.10), while density ρ and specific heat c_v are assumed to be a constants.

2.5.1 Displacements and strains

We begin with the following displacement field:

$$u_1(x,y,z,t) = u(x,y,t) + z\theta_x + z^2\phi_x + z^3\psi_x \tag{2.28}$$

$$u_2(x,y,z,t) = v(x,y,t) + z\theta_y + z^2\phi_y + z^3\psi_y \tag{2.29}$$

$$u_3(x,y,z,t) = w(x,y,t) + z\theta_z + z^2\phi_z \tag{2.30}$$

where (u,v,w) are the displacements along the coordinate lines of a material point on the $xy-$plane, i.e., $u(x,y,t) = u_1(x,y,0,t)$, $v(x,y,t) = u_2(x,y,0,t)$, $w(x,y,t) = u_3(x,y,0,t)$, and

$$\theta_x = \left(\frac{\partial u_1}{\partial z}\right)_{z=0}, \quad \theta_y = \left(\frac{\partial u_2}{\partial z}\right)_{z=0}, \quad \theta_z = \left(\frac{\partial u_3}{\partial z}\right)_{z=0}$$
$$2\phi_x = \left(\frac{\partial^2 u_1}{\partial z^2}\right)_{z=0}, \quad 2\phi_y = \left(\frac{\partial^2 u_2}{\partial z^2}\right)_{z=0}, \quad 2\phi_z = \frac{\partial^2 u_3}{\partial z^2}$$
$$6\psi_x = \frac{\partial^3 u_1}{\partial z^3}, \quad 6\psi_y = \frac{\partial^3 u_2}{\partial z^3}.$$

The reason for expanding the in-plane displacements up to the cubic term and the transverse displacement up to the quadratic term in z is to obtain a quadratic variation of the transverse shear strains $\gamma_{xz} = 2\varepsilon_{xz}$ and $\gamma_{yz} = 2\varepsilon_{yz}$ through the plate thickness. Note that all three displacements contribute to the quadratic variation. In the most general case represented by the displacement field in Eqs. (2.28) to (2.30), there are 11 generalized displacements $(u, v, w, \theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z, \psi_x, \psi_y)$ and, therefore, 11 differential equations will be required to determine them.

If the transverse shear stresses, σ_{xz} and σ_{yz} are required to be zero on the top and bottom of the plate, i.e., $z = \pm h/2$, as in the Reddy third-order theory [6, 21, 78], it is necessary that γ_{xz} and γ_{yz} be zero at $z = \pm h/2$. This in turn yields

$$2\phi_x + \frac{\partial \theta_z}{\partial x} = 0 \tag{2.31}$$

$$2\phi_y + \frac{\partial \theta_z}{\partial y} = 0 \tag{2.32}$$

$$\theta_x + \frac{\partial w}{\partial x} + \frac{h^2}{4}\left(3\psi_x + \frac{\partial \phi_z}{\partial x}\right) = 0 \tag{2.33}$$

$$\theta_y + \frac{\partial w}{\partial y} + \frac{h^2}{4}\left(3\psi_y + \frac{\partial \phi_z}{\partial y}\right) = 0. \tag{2.34}$$

Thus, the variables $(\phi_x, \phi_y, \psi_x, \psi_y)$ can be expressed in terms of $(w, \theta_x, \theta_y, \theta_z, \phi_z)$, and thus reduce the number of generalized displacements from 11 to 7. In addition, if we set $\theta_z = \phi_z = 0$, we obtain the displacement field of the Reddy third-order theory, which has only 5 variables $(u, v, w, \theta_x, \theta_y)$.

The von Kármán nonlinear strain-displacement relations associated with the displacement field in Eqs. (2.28) to (2.30) can be obtained by assuming that the strains are small and rotations are moderately large; that is, we assume

$$\left(\frac{\partial u_\alpha}{\partial x}\right)^2 \approx 0\,, \quad \left(\frac{\partial u_\alpha}{\partial y}\right)^2 \approx 0,$$
$$\left(\frac{\partial u_3}{\partial x}\right)^2 \approx \left(\frac{\partial w}{\partial x}\right)^2, \quad \left(\frac{\partial u_3}{\partial y}\right)^2 \approx \left(\frac{\partial w}{\partial y}\right)^2, \quad \left(\frac{\partial u_3}{\partial x}\right)\left(\frac{\partial u_3}{\partial y}\right) \approx \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

for $\alpha = 1, 2$. Thus the nonzero strains of the general third-order theory with the von

Kármán nonlinearity are

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{zz}^{(0)} \\ 2\varepsilon_{xy}^{(0)} \\ 2\varepsilon_{xz}^{(0)} \\ 2\varepsilon_{yz}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ 2\varepsilon_{xy}^{(1)} \\ 2\varepsilon_{xz}^{(1)} \\ 2\varepsilon_{yz}^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \\ 2\varepsilon_{yz}^{(2)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ 0 \\ 2\varepsilon_{xy}^{(3)} \\ 0 \\ 0 \end{Bmatrix} \tag{2.35}$$

with

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \psi_x}{\partial x} \\ \frac{\partial \psi_y}{\partial y} \\ \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \theta_z \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix},$$

$$\begin{Bmatrix} \varepsilon_{zz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2\phi_z \\ 2\phi_x + \frac{\partial \theta_z}{\partial x} \\ 2\phi_y + \frac{\partial \theta_z}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3\psi_x + \frac{\partial \phi_z}{\partial x} \\ 3\psi_y + \frac{\partial \phi_z}{\partial y} \end{Bmatrix}$$

where $\left(\varepsilon_{xx}^{(0)}, \varepsilon_{yy}^{(0)}, \gamma_{xy}^{(0)}\right)$ are the membrane strains, $\left(\varepsilon_{xx}^{(1)}, \varepsilon_{yy}^{(1)}, \gamma_{xy}^{(1)}\right)$ are the flexural (bending) strains, $\left(\varepsilon_{xx}^{(2)}, \varepsilon_{yy}^{(2)}, \gamma_{xy}^{(2)}\right)$ and $\left(\varepsilon_{xx}^{(3)}, \varepsilon_{yy}^{(3)}, \gamma_{xy}^{(3)}\right)$ higher-order strains.

In view of the displacement field in Eqs. (2.28) to (2.30), the components of the rotation vector and curvature tensor take the form (with $\omega_1 = \omega_x$, $\omega_2 = \omega_y$, $\omega_3 = \omega_z$, $\chi_{11} = \chi_{xx}$, $\chi_{22} = \chi_{yy}$, and so on)

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{Bmatrix} \omega_x^{(0)} \\ \omega_y^{(0)} \\ \omega_z^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \omega_x^{(1)} \\ \omega_y^{(1)} \\ \omega_z^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \omega_x^{(2)} \\ \omega_y^{(2)} \\ \omega_z^{(2)} \end{Bmatrix} + z^3 \begin{Bmatrix} \omega_x^{(3)} \\ \omega_y^{(3)} \\ \omega_z^{(3)} \end{Bmatrix} \tag{2.36}$$

where

$$\begin{Bmatrix} \omega_x^{(0)} \\ \omega_y^{(0)} \\ \omega_z^{(0)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{\partial w}{\partial y} - \theta_y \\ \theta_x - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{Bmatrix} \tag{2.37}$$

$$\begin{Bmatrix} \omega_x^{(1)} \\ \omega_y^{(1)} \\ \omega_z^{(1)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{\partial \theta_z}{\partial y} - 2\phi_y \\ 2\phi_y - \frac{\partial \theta_z}{\partial x} \\ \frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y} \end{Bmatrix} \tag{2.38}$$

$$\begin{Bmatrix} \omega_x^{(2)} \\ \omega_y^{(2)} \\ \omega_z^{(2)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{\partial \phi_z}{\partial y} - 3\psi_y \\ 3\psi_x - \frac{\partial \phi_z}{\partial x} \\ \frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y} \end{Bmatrix} \tag{2.39}$$

$$\begin{Bmatrix} \omega_x^{(3)} \\ \omega_y^{(3)} \\ \omega_z^{(3)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial \psi_y}{\partial x} - \frac{\partial \theta_x}{\partial y} \end{Bmatrix}. \tag{2.40}$$

The curvature tensors (χ_{xx}, χ_{yy}, χ_{zz}, χ_{xy}, χ_{xz}, and χ_{yz}) are

$$\begin{Bmatrix} \chi_{xx} \\ \chi_{yy} \\ \chi_{zz} \\ \chi_{xy} \\ \chi_{xz} \\ \chi_{yz} \end{Bmatrix} = \begin{Bmatrix} \chi_{xx}^{(0)} \\ \chi_{yy}^{(0)} \\ \chi_{zz}^{(0)} \\ \chi_{xy}^{(0)} \\ \chi_{xz}^{(0)} \\ \chi_{yz}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \chi_{xx}^{(1)} \\ \chi_{yy}^{(1)} \\ \chi_{zz}^{(1)} \\ \chi_{xy}^{(1)} \\ \chi_{xz}^{(1)} \\ \chi_{yz}^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \chi_{xx}^{(2)} \\ \chi_{yy}^{(2)} \\ \chi_{zz}^{(2)} \\ \chi_{xy}^{(2)} \\ \chi_{xz}^{(2)} \\ \chi_{yz}^{(2)} \end{Bmatrix} + z^3 \begin{Bmatrix} \chi_{xx}^{(3)} \\ \chi_{yy}^{(3)} \\ \chi_{zz}^{(3)} \\ \chi_{xy}^{(3)} \\ \chi_{xz}^{(3)} \\ \chi_{yz}^{(3)} \end{Bmatrix} \tag{2.41}$$

where

$$\begin{Bmatrix} \chi_{xx}^{(0)} \\ \chi_{yy}^{(0)} \\ \chi_{zz}^{(0)} \\ \chi_{xy}^{(0)} \\ \chi_{xz}^{(0)} \\ \chi_{yz}^{(0)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y} - \theta_y\right) \\ \frac{\partial}{\partial y}\left(\theta_x - \frac{\partial w}{\partial x}\right) \\ \frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y} \\ \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} - \theta_y\right) + \frac{\partial}{\partial x}\left(\theta_x - \frac{\partial w}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial \theta_z}{\partial y} - 2\phi_y \\ \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + 2\phi_x - \frac{\partial \theta_z}{\partial x} \end{Bmatrix} \tag{2.42}$$

$$\begin{Bmatrix} \chi_{xx}^{(1)} \\ \chi_{yy}^{(1)} \\ \chi_{zz}^{(1)} \\ \chi_{xy}^{(1)} \\ \chi_{xz}^{(1)} \\ \chi_{yz}^{(1)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial \theta_z}{\partial y} - 2\phi_y\right) \\ \frac{\partial}{\partial y}\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right) \\ 2\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \theta_z}{\partial y} - 2\phi_y\right) + \frac{\partial}{\partial x}\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right) + 2\left(\frac{\partial \phi_z}{\partial_y} - 3\psi_y\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right) + 2\left(3\psi_x - \frac{\partial \phi_z}{\partial_x}\right) \end{Bmatrix} \tag{2.43}$$

$$\begin{Bmatrix} \chi_{xx}^{(2)} \\ \chi_{yy}^{(2)} \\ \chi_{zz}^{(2)} \\ \chi_{xy}^{(2)} \\ \chi_{xz}^{(2)} \\ \chi_{yz}^{(2)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial \phi_z}{\partial y} - 3\psi_y\right) \\ \frac{\partial}{\partial y}\left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right) \\ 3\left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \phi_z}{\partial y} - 3\psi_y\right) + \frac{\partial}{\partial x}\left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right) \end{Bmatrix} \tag{2.44}$$

$$\begin{Bmatrix} \chi_{xx}^{(3)} \\ \chi_{yy}^{(3)} \\ \chi_{zz}^{(3)} \\ \chi_{xy}^{(3)} \\ \chi_{xz}^{(3)} \\ \chi_{yz}^{(3)} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{\partial}{\partial x}\left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}\right) \end{Bmatrix}. \tag{2.45}$$

2.5.2 Equations of motion

The equations of motion can be derived using the principle of virtual displacements. In the derivation, we account for thermal effect with the understanding that the material properties are given functions of temperature, and that the temperature change ΔT is a known function of position from the solution of Eqs. (2.26) and (2.27). Thus, temperature field enters the formulation only through constitutive equations.

The principle of virtual displacements for the dynamic case requires that (see Reddy [79])

$$\int_0^T (\delta\mathcal{K} - \delta\mathcal{U} - \delta\mathcal{V})\, dt = 0 \tag{2.46}$$

where $\delta\mathcal{K}$ is the virtual kinetic energy, $\delta\mathcal{U}$ is the virtual strain energy, and $\delta\mathcal{V}$ is the virtual work done by external forces. Each of these quantities are derived next.

The virtual kinetic energy $\delta\mathcal{K}$ is

$$
\begin{aligned}
\delta\mathcal{K} &= \int_{\Omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\left(\frac{\partial u_1}{\partial t}\frac{\partial \delta u_1}{\partial t} + \frac{\partial u_2}{\partial t}\frac{\partial \delta u_2}{\partial t} + \frac{\partial u_3}{\partial t}\frac{\partial \delta u_3}{\partial t}\right) dz\, dxdy \\
&= \int_{\Omega}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\Big[\left(\dot{u} + z\dot{\theta}_x + z^2\dot{\phi}_x + z^3\dot{\psi}_x\right)\left(\delta\dot{u} + z\delta\dot{\theta}_x + z^2\delta\dot{\phi}_x + z^3\delta\dot{\psi}_x\right) \\
&\qquad + \left(\dot{v} + z\dot{\theta}_y + z^2\dot{\phi}_y + z^3\dot{\psi}_y\right)\left(\delta\dot{v} + z\delta\dot{\theta}_y + z^2\delta\dot{\phi}_y + z^3\delta\dot{\psi}_y\right) \\
&\qquad + \left(\dot{w} + z\dot{\theta}_z + z^2\dot{\phi}_z\right)\left(\delta\dot{w} + z\delta\dot{\theta}_z + z^2\delta\dot{\phi}_z\right)\Big] dz\, dxdy \\
&= \int_{\Omega}\Big[\left(m_0\dot{u} + m_1\dot{\theta}_x + m_2\dot{\phi}_x + m_3\dot{\psi}_x\right)\delta\dot{u} \\
&\qquad + \left(m_1\dot{u} + m_2\dot{\theta}_x + m_3\dot{\phi}_x + m_4\dot{\psi}_x\right)\delta\dot{\theta}_x \\
&\qquad + \left(m_2\dot{u} + m_3\dot{\theta}_x + m_4\dot{\phi}_x + m_5\dot{\psi}_x\right)\delta\dot{\phi}_x \\
&\qquad + \left(m_3\dot{u} + m_4\dot{\theta}_x + m_5\dot{\phi}_x + m_6\dot{\psi}_x\right)\delta\dot{\psi}_x \\
&\qquad + \left(m_0\dot{v} + m_1\dot{\theta}_y + m_2\dot{\phi}_y + m_3\dot{\psi}_y\right)\delta\dot{v} \\
&\qquad + \left(m_1\dot{v} + m_2\dot{\theta}_y + m_3\dot{\phi}_y + m_4\dot{\psi}_y\right)\delta\dot{\theta}_y \\
&\qquad + \left(m_2\dot{v} + m_3\dot{\theta}_y + m_4\dot{\phi}_y + m_5\dot{\psi}_y\right)\delta\dot{\phi}_y \\
&\qquad + \left(m_3\dot{v} + m_4\dot{\theta}_y + m_5\dot{\phi}_y + m_6\dot{\psi}_y\right)\delta\dot{\psi}_y \\
&\qquad + \left(m_0\dot{w} + m_1\dot{\theta}_z + m_2\dot{\phi}_z\right)\delta\dot{w} \\
&\qquad + \left(m_1\dot{w} + m_2\dot{\theta}_z + m_3\dot{\phi}_z\right)\delta\dot{\theta}_z \\
&\qquad + \left(m_2\dot{w} + m_3\dot{\theta}_z + m_4\dot{\phi}_z\right)\delta\dot{\phi}_z\Big] dxdy
\end{aligned} \tag{2.47}
$$

where the superposed dot on a variable indicates time derivative, e.g., $\dot{u} = \partial u/\partial t$, and m_i $(i = 0, 1, 2, \cdots, 6)$ are the mass moments of inertia

$$
m_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\,(z)^i\, dz. \tag{2.48}
$$

The virtual strain energy is given by [see Eq. (2.11)]

$$\begin{aligned}
\delta\mathcal{U} &= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} \\
&\qquad + \sigma_{xy}\delta\gamma_{xy} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{yz}\delta\gamma_{yz})\, dzdxdy \\
&+ \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} (m_{xx}\delta\chi_{xx} + m_{yy}\delta\chi_{yy} + m_{zz}\delta\chi_{zz} \\
&\qquad + m_{xy}\delta\chi_{xy} + m_{xz}\delta\chi_{xz} + m_{yz}\delta\chi_{yz})\, dzdxdy \\
&= \int_\Omega \bigg\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \bigg[\sigma_{xx}\Big(\sum_{i=0}^{3}(z)^i\delta\varepsilon_{xx}^{(i)}\Big) + \sigma_{yy}\Big(\sum_{i=0}^{3}(z)^i\delta\varepsilon_{yy}^{(i)}\Big) \\
&\qquad + \sigma_{xy}\Big(\sum_{i=0}^{3}(z)^i\delta\gamma_{xy}^{(i)}\Big) + \sigma_{zz}\Big(\sum_{i=0}^{2}(z)^i\delta\varepsilon_{zz}^{(i)}\Big) \\
&\qquad + \sigma_{xz}\Big(\sum_{i=0}^{2}(z)^i\delta\gamma_{xz}^{(i)}\Big) + \sigma_{yz}\Big(\sum_{i=0}^{2}(z)^i\delta\gamma_{yz}^{(i)}\Big)\bigg] dz \bigg\} dxdy \\
&+ \int_\Omega \bigg\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \bigg[m_{xx}\Big(\sum_{i=0}^{2}(z)^i\delta\chi_{xx}^{(i)}\Big) + m_{yy}\Big(\sum_{i=0}^{2}(z)^i\delta\chi_{yy}^{(i)}\Big) \\
&\qquad + m_{zz}\Big(\sum_{i=0}^{2}(z)^i\delta\chi_{zz}^{(i)}\Big) + m_{xy}\Big(\sum_{i=0}^{2}(z)^i\delta\chi_{xy}^{(i)}\Big) \\
&\qquad + m_{xz}\Big(\sum_{i=0}^{3}(z)^i\delta\chi_{xz}^{(i)}\Big) + m_{yz}\Big(\sum_{i=0}^{3}(z)^i\delta\chi_{yz}^{(i)}\Big)\bigg] dz \bigg\} dxdy. \qquad (2.49)
\end{aligned}$$

Next, we introduce thickness-integrated stress resultants as

$$M_{ij}^{(k)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k\sigma_{ij}\, dz \text{ and } \mathcal{M}_{ij}^{(k)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k m_{ij}\, dz, \quad (k = 0, 1, 2, 3).$$

Then the virtual strain energy can be expressed in terms of the stress resultants as

$$\delta\mathcal{U} = \int_\Omega \bigg[\sum_{i=0}^{3} \big(M_{xx}^{(i)}\delta\varepsilon_{xx}^{(i)} + M_{yy}^{(i)}\delta\varepsilon_{yy}^{(i)} + M_{xy}^{(i)}\delta\gamma_{xy}^{(i)}\big)$$

$$
\begin{aligned}
&+\sum_{i=0}^{2}\left(M_{zz}^{(i)}\delta\varepsilon_{zz}^{(i)}+M_{xz}^{(i)}\delta\gamma_{xz}^{(i)}+M_{yz}^{(i)}\delta\gamma_{yz}^{(i)}\right)\\
&+\sum_{i=0}^{2}\left(\mathcal{M}_{xx}^{(i)}\delta\chi_{xx}^{(i)}+\mathcal{M}_{yy}^{(i)}\delta\chi_{yy}^{(i)}+\mathcal{M}_{zz}^{(i)}\delta\chi_{zz}^{(i)}+\mathcal{M}_{xy}^{(i)}\delta\chi_{xy}^{(i)}\right)\\
&+\sum_{i=0}^{3}\left(\mathcal{M}_{xz}^{(i)}\delta\chi_{xz}^{(i)}+\mathcal{M}_{yz}^{(i)}\delta\chi_{yz}^{(i)}\right)\Bigg]dxdy. \qquad (2.50)
\end{aligned}
$$

Note that $M_{xx}^{(0)}$, $M_{yy}^{(0)}$, and $M_{xy}^{(0)}$ are the membrane forces (often denoted by N_{xx}, N_{yy}, and N_{xy}), $M_{xx}^{(1)}$, $M_{yy}^{(1)}$, and $M_{xy}^{(1)}$ are the bending moments (denoted by M_{xx}, M_{yy}, and M_{xy}), and $M_{xz}^{(0)}$ and $M_{yz}^{(0)}$ are the shear forces (denoted by Q_x and Q_y). The rest of the stress resultants are higher order generalized forces, which are often difficult to physically interpret. Because of their similarity to the generalized physical forces identified above, they are assumed to be zero when their counter parts (i.e., generalized displacements) are not specified.

The virtual work done by external forces consists of three parts: (1) virtual work done by the body forces and couples in $V = \Omega \times (-h/2, h/2)$, (2) virtual work done by surface tractions and couples acting on the top and bottom surfaces of the plate Ω^+ and Ω^-, and (3) virtual work done by the surface tractions and couples on the lateral surface $S = \Gamma \times (-h/2, h/2)$, where Ω^+ denotes the top surface of the plate, Ω the middle surface of the plate, Ω^- the bottom surface of the plate, and Γ is the boundary of the middle surface (see Fig. 2.2).

Let $\bar{f}_x$, $\bar{f}_y$, and $\bar{f}_z$ be the body forces (measured per unit volume), $\bar{t}_x$, $\bar{t}_y$, and $\bar{t}_z$ be the surface forces (measured per unit area) on S, and q_x^t, q_y^t, and q_z^t be the distributed forces (measured per unit area) on Ω^+, q_x^b, q_y^b, and q_z^b be the distributed forces (measured per unit area) on Ω^-, $\bar{c}_x$, $\bar{c}_y$, and $\bar{c}_z$ be the body couples (measured per unit volume), $\bar{s}_x$, $\bar{s}_y$, and $\bar{s}_z$ be the surface couples (measured per unit area) on S, and p_x^t, p_y^t, and p_z^t be the distributed couples (measured per unit area) on Ω^+, and p_x^b, p_y^b, and p_z^b be the distributed couples (measured per unit area) on Ω^- in the x, y, and z coordinate directions. Then the virtual work done by external forces is

$$
\begin{aligned}
\delta\mathcal{V} = -\Big[&\int_V \left(\bar{f}_x\delta u_1 + \bar{f}_y\delta u_2 + \bar{f}_z\delta u_3 + \bar{c}_x\delta\omega_x + \bar{c}_y\delta\omega_y + \bar{c}_z\delta\omega_z\right) dV \\
&+ \int_{\Omega^+} \Big(q_x^t\delta u_1 + q_y^t\delta u_2 + q_z^t\delta u_3 + p_x^t\delta\omega_x + p_y^t\delta\omega_y + p_z^t\delta\omega_z\Big) dxdy \\
&+ \int_{\Omega^-} \Big(q_x^b\delta u_1 + q_y^b\delta u_2 + q_z^b\delta u_3 + p_x^b\delta\omega_x + p_y^b\delta\omega_y + p_z^b\delta\omega_z\Big) dxdy \\
&+ \int_S (\bar{t}_x\delta u_1 + \bar{t}_y\delta u_2 + \bar{t}_z\delta u_3 + \bar{s}_x\delta\omega_x + \bar{s}_y\delta\omega_y + \bar{s}_z\delta\omega_z)\, dS\Big].
\end{aligned}
\tag{2.51}
$$

In view of the displacement field in Eqs. (2.28) to (2.30), $\delta\mathcal{V}$ can be expressed as

$$
\begin{aligned}
\delta\mathcal{V} = -\Big\{&\int_\Omega \Big(f_x^{(0)}\delta u + f_x^{(1)}\delta\theta_x + f_x^{(2)}\delta\phi_x + f_x^{(3)}\delta\psi_x + f_y^{(0)}\delta v + f_y^{(1)}\delta\theta_y \\
&+ f_y^{(2)}\delta\phi_y + f_y^{(3)}\delta\psi_y + f_z^{(0)}\delta w + f_z^{(1)}\delta\theta_z + f_z^{(2)}\delta\phi_z + c_x^{(0)}\delta\omega_x^{(0)} \\
&+ c_x^{(1)}\delta\omega_x^{(1)} + c_x^{(2)}\delta\omega_x^{(2)} + c_y^{(0)}\delta\omega_y^{(0)} + c_y^{(1)}\delta\omega_y^{(1)} + c_y^{(2)}\delta\omega_y^{(2)} \\
&+ c_z^{(0)}\delta\omega_z^{(0)} + c_z^{(1)}\delta\omega_z^{(1)} + c_z^{(2)}\delta\omega_z^{(2)} + c_z^{(3)}\delta\omega_z^{(3)}\Big) dxdy \\
&+ \int_\Omega \Big[\Big(q_x^t + q_x^b\Big)\delta u + \Big(q_y^t + q_y^b\Big)\delta v + \Big(q_z^t + q_z^b\Big)\delta w + \frac{h}{2}\Big(q_x^t - q_x^b\Big)\delta\theta_x \\
&+ \frac{h}{2}\Big(q_y^t - q_y^b\Big)\delta\theta_y + \frac{h}{2}\Big(q_z^t - q_z^b\Big)\delta\theta_z + \frac{h^2}{4}\Big(q_x^t + q_x^b\Big)\delta\phi_x \\
&+ \frac{h^2}{4}\Big(q_y^t + q_y^b\Big)\delta\phi_y + \frac{h^2}{4}\Big(q_z^t + q_z^b\Big)\delta\phi_z + \frac{h^3}{8}\Big(q_x^t - q_x^b\Big)\delta\psi_x \\
&+ \frac{h^3}{8}\Big(q_y^t - q_y^b\Big)\delta\psi_y + \Big(p_x^t + p_x^b\Big)\delta\omega_x^{(0)} + \Big(p_y^t + p_y^b\Big)\delta\omega_y^{(0)} \\
&+ \Big(p_z^t + p_z^b\Big)\delta\omega_z^{(0)} + \frac{h}{2}\Big(p_x^t - p_x^b\Big)\delta\omega_x^{(1)} + \frac{h}{2}\Big(p_y^t - p_y^b\Big)\delta\omega_y^{(1)} \\
&+ \frac{h}{2}\Big(p_z^t - p_z^b\Big)\delta\omega_z^{(1)} + \frac{h^2}{4}\Big(p_x^t + p_x^b\Big)\delta\omega_x^{(2)} + \frac{h^2}{4}\Big(p_y^t + p_y^b\Big)\delta\omega_y^{(2)} \\
&+ \frac{h^2}{4}\Big(p_z^t + p_z^b\Big)\delta\omega_z^{(2)} + \frac{h^3}{8}\Big(p_x^t - p_x^b\Big)\delta\omega_z^{(3)}\Big] dxdy \\
&+ \int_\Gamma \Big(t_x^{(0)}\delta u + t_x^{(1)}\delta\theta_x + t_x^{(2)}\delta\phi_x + t_x^{(3)}\delta\psi_x + t_y^{(0)}\delta v + t_y^{(1)}\delta\theta_y
\end{aligned}
$$

$$
\begin{aligned}
&+ t_y^{(2)}\delta\phi_y + t_y^{(3)}\delta\psi_y + t_z^{(0)}\delta w + t_z^{(1)}\delta\theta_z + t_z^{(2)}\delta\phi_z + s_x^{(0)}\delta\omega_x^{(0)} \\
&+ s_x^{(1)}\delta\omega_x^{(1)} + s_x^{(2)}\delta\omega_x^{(2)} + s_y^{(0)}\delta\omega_y^{(0)} + s_y^{(1)}\delta\omega_y^{(1)} + s_y^{(2)}\delta\omega_y^{(2)} \\
&+ s_z^{(0)}\delta\omega_z^{(0)} + s_z^{(1)}\delta\omega_z^{(1)} + s_z^{(2)}\delta\omega_z^{(2)} + s_z^{(3)}\delta\omega_z^{(3)}\Big)d\Gamma\Big\} \\
= -\Big\{\int_\Omega\Big[&F_x^{(0)}\delta u + F_y^{(0)}\delta v + F_z^{(0)}\delta w + F_x^{(1)}\delta\theta_x + F_y^{(1)}\delta\theta_y + F_x^{(2)}\delta\phi_x \\
&+ F_y^{(2)}\delta\phi_y + F_x^{(3)}\delta\psi_x + F_y^{(3)}\delta\psi_y + F_z^{(1)}\delta\theta_z + F_z^{(2)}\delta\phi_z \\
&+ \frac{1}{2}C_x^{(0)}\left(\frac{\partial\delta w}{\partial y} - \delta\theta_y\right) - \frac{1}{2}C_y^{(0)}\left(\frac{\partial\delta w}{\partial x} - \delta\theta_x\right) \\
&+ \frac{1}{2}C_z^{(0)}\left(\frac{\partial\delta v}{\partial x} - \frac{\partial\delta u}{\partial y}\right) + C_x^{(1)}\left(\frac{1}{2}\frac{\partial\delta\theta_z}{\partial y} - \delta\phi_y\right) \\
&- C_y^{(1)}\left(\frac{1}{2}\frac{\partial\delta\theta_z}{\partial x} - \delta\phi_x\right) + \frac{1}{2}C_z^{(1)}\left(\frac{\partial\delta\theta_y}{\partial x} - \frac{\partial\delta\theta_x}{\partial y}\right) \\
&+ \frac{1}{2}C_x^{(2)}\left(\frac{\partial\delta\phi_z}{\partial y} - 3\delta\psi_y\right) - \frac{1}{2}C_y^{(2)}\left(\frac{\partial\delta\phi_z}{\partial x} - 3\delta\psi_x\right) \\
&+ \frac{1}{2}C_z^{(2)}\left(\frac{\partial\delta\phi_y}{\partial x} - \frac{\partial\delta\phi_x}{\partial y}\right) + \frac{1}{2}C_z^{(3)}\left(\frac{\partial\delta\psi_y}{\partial x} - \frac{\partial\delta\psi_x}{\partial y}\right)\Big]dxdy \\
+ \int_\Gamma\Big[&t_x^{(0)}\delta u + t_y^{(0)}\delta v + t_z^{(0)}\delta w + t_x^{(1)}\delta\theta_x + t_y^{(1)}\delta\theta_y + t_z^{(1)}\delta\theta_z \\
&+ t_x^{(2)}\delta\phi_x + t_y^{(2)}\delta\phi_y + t_z^{(2)}\delta\phi_z + t_x^{(3)}\delta\psi_x + t_y^{(3)}\delta\psi_y \\
&+ \frac{1}{2}s_x^{(0)}\left(\frac{\partial\delta w}{\partial y} - \delta\theta_y\right) - \frac{1}{2}s_y^{(0)}\left(\frac{\partial\delta w}{\partial x} - \delta\theta_x\right) \\
&+ \frac{1}{2}s_z^{(0)}\left(\frac{\partial\delta v}{\partial x} - \frac{\partial\delta u}{\partial y}\right) + s_x^{(1)}\left(\frac{1}{2}\frac{\partial\delta\theta_z}{\partial y} - \delta\phi_y\right) \\
&- s_y^{(1)}\left(\frac{1}{2}\frac{\partial\delta\theta_z}{\partial x} - \delta\phi_x\right) + \frac{1}{2}s_z^{(1)}\left(\frac{\partial\delta\theta_y}{\partial x} - \frac{\partial\delta\theta_x}{\partial y}\right) \\
&+ \frac{1}{2}s_x^{(2)}\left(\frac{\partial\delta\phi_z}{\partial y} - 3\delta\psi_y\right) - \frac{1}{2}s_y^{(2)}\left(\frac{\partial\delta\phi_z}{\partial x} - 3\delta\psi_x\right) \\
&+ \frac{1}{2}s_z^{(2)}\left(\frac{\partial\delta\phi_y}{\partial x} - \frac{\partial\delta\phi_x}{\partial y}\right) + \frac{1}{2}s_z^{(3)}\left(\frac{\partial\delta\psi_y}{\partial x} - \frac{\partial\delta\psi_x}{\partial y}\right)\Big]d\Gamma\Big\} \qquad (2.52)
\end{aligned}
$$

where

$$f_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i \, \bar{f}_\xi \, dz, \;\; t_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i \, \bar{t}_\xi \, dz, \;\; c_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i \, \bar{c}_\xi \, dz,$$

$$s_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i \, \bar{s}_\xi \, dz, \;\; F_\xi^{(i)} = f_\xi^{(i)} + \left(\frac{h}{2}\right)^i \left[q_\xi^t + (-1)^i q_\xi^b\right],$$

$$\text{and } \; C_\xi^{(i)} = c_\xi^{(i)} + \left(\frac{h}{2}\right)^i \left[p_\xi^t + (-1)^i p_\xi^b\right]$$

for $i = 0, 1, 2, 3$ and $\xi = x, y, z$.

The equations of motion of the general third-order plate theory governing functionally graded plates accounting for modified couple stresses are obtained by substituting $\delta\mathcal{K}$, $\delta\mathcal{U}$, and $\delta\mathcal{V}$, from Eqs. (2.47), (2.50), and (2.52), respectively, into Eq. (2.46), applying the integration-by-parts to relieve all virtual generalized displacements of differentiations with respect to x, y and t, noting that all variations at the upper and lower time limits are zero, and invoking the fundamental lemma of the variational calculus (see Reddy [6,7,79]). We obtain (after a lengthy algebra and manipulations) the following equations:

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + F_x^{(0)} + \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial y}$$
$$= m_0\ddot{u} + m_1\ddot{\theta}_x + m_2\ddot{\phi}_x + m_3\ddot{\psi}_x \tag{2.53}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + F_y^{(0)} - \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial x}$$
$$= m_0\ddot{v} + m_1\ddot{\theta}_y + m_2\ddot{\phi}_y + m_3\ddot{\psi}_y) \tag{2.54}$$

$$\delta w: \quad \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right) + \frac{\partial M_{xz}^{(0)}}{\partial x}$$
$$+ \frac{\partial M_{yz}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}\right)$$

$$+ F_z^{(0)} + \frac{1}{2}\left(\frac{\partial C_y^{(0)}}{\partial x} - \frac{\partial C_x^{(0)}}{\partial y}\right) = m_0\ddot{w} + m_1\ddot{\theta}_z + m_2\ddot{\phi}_z \tag{2.55}$$

$$\delta\theta_x: \quad \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - M_{xz}^{(0)} + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(0)}}{\partial y}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) + F_x^{(1)} + \frac{1}{2}C_y^{(0)} + \frac{1}{2}\frac{\partial C_z^{(1)}}{\partial y}$$

$$= m_1\ddot{u} + m_2\ddot{\theta}_x + m_3\ddot{\phi}_x + m_4\ddot{\psi}_x \tag{2.56}$$

$$\delta\theta_y: \quad \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - M_{yz}^{(0)} - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(0)}}{\partial x}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) + F_y^{(1)} - \frac{1}{2}C_x^{(0)} - \frac{1}{2}\frac{\partial C_z^{(1)}}{\partial x}$$

$$= m_1\ddot{v} + m_2\ddot{\theta}_y + m_3\ddot{\phi}_y + m_4\ddot{\psi}_y \tag{2.57}$$

$$\delta\phi_x: \quad \frac{\partial M_{xx}^{(2)}}{\partial x} + \frac{\partial M_{xy}^{(2)}}{\partial y} - 2M_{xz}^{(1)} + \left(\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(1)}}{\partial y} - \mathcal{M}_{yz}^{(0)}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(2)}}{\partial y}\right) + F_x^{(2)} + C_y^{(1)} + \frac{1}{2}\frac{\partial C_z^{(2)}}{\partial y}$$

$$= m_2\ddot{u} + m_3\ddot{\theta}_x + m_4\ddot{\phi}_x + m_5\ddot{\psi}_x \tag{2.58}$$

$$\delta\phi_y: \quad \frac{\partial M_{xy}^{(2)}}{\partial x} + \frac{\partial M_{yy}^{(2)}}{\partial y} - 2M_{yz}^{(1)} - \left(\frac{\partial \mathcal{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(1)}}{\partial x} - \mathcal{M}_{xz}^{(0)}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(2)}}{\partial y}\right) + F_y^{(2)} - C_x^{(1)} - \frac{1}{2}\frac{\partial C_z^{(2)}}{\partial x}$$

$$= m_2\ddot{v} + m_3\ddot{\theta}_y + m_4\ddot{\phi}_y + m_5\ddot{\psi}_y \tag{2.59}$$

$$\delta\psi_x: \quad \frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y} + \frac{3}{2}\left(\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(2)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(2)}}{\partial y} - 2\mathcal{M}_{yz}^{(1)}\right)$$

$$- 3M_{xz}^{(2)} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(3)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(3)}}{\partial y}\right) + F_x^{(3)} + \frac{3}{2}C_y^{(2)} + \frac{1}{2}\frac{\partial C_z^{(3)}}{\partial y}$$

$$= m_3\ddot{u} + m_4\ddot{\theta}_x + m_5\ddot{\phi}_x + m_6\ddot{\psi}_x \tag{2.60}$$

$$\delta\psi_y: \quad \frac{\partial M_{xy}^{(3)}}{\partial x} + \frac{\partial M_{yy}^{(3)}}{\partial y} - \frac{3}{2}\left(\frac{\partial \mathcal{M}_{xx}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(2)}}{\partial x} - 2\mathcal{M}_{xz}^{(1)}\right)$$

$$- 3M_{yz}^{(2)} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(3)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(3)}}{\partial y}\right) + F_y^{(3)} - \frac{3}{2}C_x^{(2)} - \frac{1}{2}\frac{\partial C_z^{(3)}}{\partial x}$$

$$= m_3\ddot{v} + m_4\ddot{\theta}_y + m_5\ddot{\phi}_y + m_6\ddot{\psi}_y \tag{2.61}$$

$$\begin{aligned}\delta\theta_z: \quad & \frac{\partial M^{(1)}_{xz}}{\partial x} + \frac{\partial M^{(1)}_{yz}}{\partial y} - M^{(0)}_{zz} - \frac{1}{2}\frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{M}^{(1)}_{xx}}{\partial x} + \frac{\partial \mathcal{M}^{(1)}_{xy}}{\partial y}\Big) \\ & + \frac{1}{2}\frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{M}^{(1)}_{yy}}{\partial y} + \frac{\partial \mathcal{M}^{(1)}_{xy}}{\partial x}\Big) + \frac{1}{2}\Big(\frac{\partial \mathcal{M}^{(0)}_{xz}}{\partial y} - \frac{\partial \mathcal{M}^{(0)}_{yz}}{\partial x}\Big) \\ & + F^{(1)}_z + \frac{1}{2}\Big(\frac{\partial C^{(1)}_y}{\partial x} - \frac{\partial C^{(1)}_x}{\partial y}\Big) = m_1\ddot{w} + m_2\ddot{\theta}_z + m_3\ddot{\phi}_z \end{aligned} \tag{2.62}$$

$$\begin{aligned}\delta\phi_z: \quad & \frac{\partial M^{(2)}_{xz}}{\partial x} + \frac{\partial M^{(2)}_{yz}}{\partial y} - 2M^{(1)}_{zz} - \frac{1}{2}\frac{\partial}{\partial y}\Big(\frac{\partial \mathcal{M}^{(2)}_{xx}}{\partial x} + \frac{\partial \mathcal{M}^{(2)}_{xy}}{\partial y}\Big) \\ & + \frac{1}{2}\frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{M}^{(2)}_{yy}}{\partial y} + \frac{\partial \mathcal{M}^{(2)}_{xy}}{\partial x}\Big) + \frac{\partial \mathcal{M}^{(1)}_{xz}}{\partial y} - \frac{\partial \mathcal{M}^{(1)}_{yz}}{\partial x} \\ & + F^{(2)}_z + \frac{1}{2}\Big(\frac{\partial C^{(2)}_y}{\partial x} - \frac{\partial C^{(2)}_x}{\partial y}\Big) = m_2\ddot{w} + m_3\ddot{\theta}_z + m_4\ddot{\phi}_z. \end{aligned} \tag{2.63}$$

The boundary conditions involve specifying the following generalized forces that are dual to the generalized displacements $(u, v, w, \theta_x, \theta_y, \phi_x, \phi_y, \psi_x, \psi_y, \theta_z, \phi_z)$:

$$\delta u: \quad 0 = M^{(0)}_{xx}n_x + M^{(0)}_{xy}n_y + \frac{1}{2}\Big(\frac{\partial \mathcal{M}^{(0)}_{xz}}{\partial x} + \frac{\partial \mathcal{M}^{(0)}_{yz}}{\partial y}\Big)n_y + \frac{1}{2}C^{(0)}_z n_y + t^{(0)}_x \tag{2.64}$$

$$\delta v: \quad 0 = M^{(0)}_{xy}n_x + M^{(0)}_{yy}n_y - \frac{1}{2}\Big(\frac{\partial \mathcal{M}^{(0)}_{xz}}{\partial x} + \frac{\partial \mathcal{M}^{(0)}_{yz}}{\partial y}\Big)n_x - \frac{1}{2}C^{(0)}_z n_x + t^{(0)}_y \tag{2.65}$$

$$\begin{aligned}\delta w: \quad 0 = {} & \Big(\frac{\partial w}{\partial x}M^{(0)}_{xx} + \frac{\partial w}{\partial y}M^{(0)}_{xy}\Big)n_x + \Big(\frac{\partial w}{\partial x}M^{(0)}_{xy} + \frac{\partial w}{\partial y}M^{(0)}_{yy}\Big)n_y + M^{(0)}_{xz}n_x \\ & + M^{(0)}_{yz}n_y - \frac{1}{2}\Big(\frac{\partial \mathcal{M}^{(0)}_{xx}}{\partial x} + \frac{\partial \mathcal{M}^{(0)}_{xy}}{\partial y}\Big)n_y + \frac{1}{2}\Big(\frac{\partial \mathcal{M}^{(0)}_{yy}}{\partial y} + \frac{\partial \mathcal{M}^{(0)}_{xy}}{\partial x}\Big)n_x \\ & + \frac{1}{2}\left(C^{(0)}_y n_x - C^{(0)}_x n_y\right) + t^{(0)}_z \end{aligned} \tag{2.66}$$

$$\begin{aligned}\delta\theta_x: \quad 0 = {} & M^{(1)}_{xx}n_x + M^{(1)}_{xy}n_y + \frac{1}{2}C^{(1)}_z n_y + \frac{1}{2}\Big(\frac{\partial \mathcal{M}^{(1)}_{xz}}{\partial x} + \frac{\partial \mathcal{M}^{(1)}_{yz}}{\partial y}\Big)n_y \\ & + \frac{1}{2}\Big(\mathcal{M}^{(0)}_{xy}n_x + \mathcal{M}^{(0)}_{yy}n_y - \mathcal{M}^{(1)}_{zz}n_y\Big) + t^{(1)}_x \end{aligned} \tag{2.67}$$

$$\delta\theta_y: \quad 0 = M^{(1)}_{xy}n_x + M^{(1)}_{yy}n_y - \frac{1}{2}C^{(1)}_z n_x - \frac{1}{2}\Big(\frac{\partial \mathcal{M}^{(1)}_{xz}}{\partial x} + \frac{\partial \mathcal{M}^{(1)}_{yz}}{\partial y}\Big)n_x$$

$$
+\frac{1}{2}\left(\mathcal{M}^{(0)}_{xx}n_x+\mathcal{M}^{(0)}_{xy}n_y-\mathcal{M}^{(1)}_{zz}n_x\right)+t^{(1)}_y \tag{2.68}
$$

$$
\delta\phi_x:\quad 0=M^{(2)}_{xx}n_x+M^{(2)}_{xy}n_y+\frac{1}{2}C^{(2)}_z n_y+\left(\mathcal{M}^{(1)}_{xy}n_x+\mathcal{M}^{(1)}_{yy}n_y-\mathcal{M}^{(1)}_{zz}n_y\right)
$$

$$
+\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(2)}_{xz}}{\partial x}+\frac{\partial\mathcal{M}^{(2)}_{yz}}{\partial y}\right)n_y+t^{(2)}_x \tag{2.69}
$$

$$
\delta\phi_y:\quad 0=M^{(2)}_{xy}n_x+M^{(2)}_{yy}n_y-\frac{1}{2}C^{(2)}_z n_x-\left(\mathcal{M}^{(1)}_{xx}n_x+\mathcal{M}^{(1)}_{xy}n_y-\mathcal{M}^{(1)}_{zz}n_x\right)
$$

$$
-\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(2)}_{xz}}{\partial x}+\frac{\partial\mathcal{M}^{(2)}_{yz}}{\partial y}\right)n_x+t^{(2)}_y \tag{2.70}
$$

$$
\delta\psi_x:\quad 0=M^{(3)}_{xx}n_x+M^{(3)}_{xy}n_y+\frac{1}{2}C^{(3)}_z n_y+\frac{3}{2}\left(\mathcal{M}^{(2)}_{xy}n_x+\mathcal{M}^{(2)}_{yy}n_y-\mathcal{M}^{(2)}_{zz}n_y\right)
$$

$$
+\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(3)}_{xz}}{\partial x}+\frac{\partial\mathcal{M}^{(3)}_{yz}}{\partial y}\right)n_y+t^{(3)}_x \tag{2.71}
$$

$$
\delta\psi_y:\quad 0=M^{(3)}_{xy}n_x+M^{(3)}_{yy}n_y-\frac{1}{2}C^{(3)}_z n_x-\frac{3}{2}\left(\mathcal{M}^{(2)}_{xx}n_x+\mathcal{M}^{(2)}_{xy}n_y-\mathcal{M}^{(2)}_{zz}n_x\right)
$$

$$
-\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(3)}_{xz}}{\partial x}+\frac{\partial\mathcal{M}^{(3)}_{yz}}{\partial y}\right)n_x+t^{(3)}_y \tag{2.72}
$$

$$
\delta\theta_z:\quad 0=M^{(1)}_{xz}n_x+M^{(1)}_{yz}n_y+\frac{1}{2}\left(C^{(1)}_y n_x-C^{(1)}_x n_y\right)
$$

$$
+\frac{1}{2}\left(\mathcal{M}^{(0)}_{xz}n_y-\mathcal{M}^{(0)}_{yz}n_x\right)-\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(1)}_{xx}}{\partial x}+\frac{\partial\mathcal{M}^{(1)}_{xy}}{\partial y}\right)n_y
$$

$$
+\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(1)}_{xy}}{\partial x}+\frac{\partial\mathcal{M}^{(1)}_{yy}}{\partial y}\right)n_x+t^{(1)}_z \tag{2.73}
$$

$$
\delta\phi_z:\quad 0=M^{(2)}_{xz}n_x+M^{(2)}_{yz}n_y+\frac{1}{2}\left(C^{(2)}_y n_x-C^{(2)}_x n_y\right)+\mathcal{M}^{(1)}_{xz}n_y-\mathcal{M}^{(1)}_{yz}n_x
$$

$$
-\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(2)}_{xx}}{\partial x}+\frac{\partial\mathcal{M}^{(2)}_{xy}}{\partial x}\right)n_y+\frac{1}{2}\left(\frac{\partial\mathcal{M}^{(2)}_{xy}}{\partial x}+\frac{\partial\mathcal{M}^{(2)}_{yy}}{\partial y}\right)n_x+t^{(2)}_z. \tag{2.74}
$$

Here we assumed that

$$
\mathcal{M}^{(0)}_{xx}n_x+\mathcal{M}^{(0)}_{xy}n_y=0 \tag{2.75}
$$

$$
\mathcal{M}^{(0)}_{xy}n_x+\mathcal{M}^{(0)}_{yy}n_y=0 \tag{2.76}
$$

$$
\mathcal{M}^{(1)}_{xx}n_x+\mathcal{M}^{(1)}_{xy}n_y=0 \tag{2.77}
$$

$$\mathcal{M}_{xy}^{(1)} n_x + \mathcal{M}_{yy}^{(1)} n_y = 0 \tag{2.78}$$

$$\mathcal{M}_{xx}^{(2)} n_x + \mathcal{M}_{xy}^{(2)} n_y = 0 \tag{2.79}$$

$$\mathcal{M}_{xy}^{(2)} n_x + \mathcal{M}_{yy}^{(2)} n_y = 0 \tag{2.80}$$

$$\mathcal{M}_{xz}^{(0)} n_x + \mathcal{M}_{yz}^{(0)} n_y = 0 \tag{2.81}$$

$$\mathcal{M}_{xz}^{(1)} n_x + \mathcal{M}_{yz}^{(1)} n_y = 0 \tag{2.82}$$

$$\mathcal{M}_{xz}^{(2)} n_x + \mathcal{M}_{yz}^{(2)} n_y = 0 \tag{2.83}$$

$$\mathcal{M}_{xz}^{(3)} n_x + \mathcal{M}_{yz}^{(3)} n_y = 0. \tag{2.84}$$

2.5.3 Constitutive relations

Here we represent the profile for volume fraction variation by the expression in Eq. (2.9); we assume that moduli E and G, density ρ, density ρ, and thermal coefficient of expansion α vary according to Eq. (2.9), and ν is assumed to be a constant because the effect of the variation of Possion's ratio is negligible [80]. The linear constitutive relations are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \Lambda \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{Bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{Bmatrix}^{\mathrm{T}} \tag{2.85}$$

where $\Lambda = E/[(1+\nu)(1-2\nu)]$, α is the coefficients of thermal expansion, and ΔT is the temperature increment from a reference temperature T_0, $\Delta T = T - T_0$.

2.5.4 Plate constitutive equations

Here we relate the generalized forces ($M_{xx}^{(i)}$, $M_{yy}^{(i)}$, $M_{xy}^{(i)}$, $M_{xz}^{(i)}$, $M_{yz}^{(i)}$) and the generalized couples ($\mathcal{M}_{xx}^{(i)}$, $\mathcal{M}_{yy}^{(i)}$, $\mathcal{M}_{xy}^{(i)}$, $\mathcal{M}_{xz}^{(i)}$, $\mathcal{M}_{yz}^{(i)}$) to the generalized displacements (u, v, w, θ_x, θ_y, ϕ_x, ϕ_y, ψ_x, ψ_y, θ_z, ϕ_z). We have

$$\begin{Bmatrix} M_{xx}^{(i)} \\ M_{yy}^{(i)} \\ M_{zz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{11}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{12}^{(k)} & A_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(k-i)} \\ \varepsilon_{yy}^{(k-i)} \\ \varepsilon_{zz}^{(k-i)} \end{Bmatrix} - \begin{Bmatrix} X_T^{(i)} & Y_T^{(i)} & Z_T^{(i)} \end{Bmatrix}^{\mathrm{T}} \tag{2.86}$$

$$\begin{Bmatrix} M_{xy}^{(i)} \\ M_{xz}^{(i)} \\ M_{yz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} B_{11}^{(k)} & 0 & 0 \\ 0 & B_{11}^{(k)} & 0 \\ 0 & 0 & B_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^{(k-i)} \\ \gamma_{xz}^{(k-i)} \\ \gamma_{zz}^{(k-i)} \end{Bmatrix} \tag{2.87}$$

$$\begin{Bmatrix} \mathcal{M}_{xx}^{(i)} \\ \mathcal{M}_{yy}^{(i)} \\ \mathcal{M}_{zz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{zz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{2+i} \begin{bmatrix} \mathcal{B}_{11}^{(k)} & 0 & 0 \\ 0 & \mathcal{B}_{11}^{(k)} & 0 \\ 0 & 0 & \mathcal{B}_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \chi_{xx}^{(k-i)} \\ \chi_{yy}^{(k-i)} \\ \chi_{zz}^{(k-i)} \end{Bmatrix} \tag{2.88}$$

$$\begin{Bmatrix} \mathcal{M}_{xy}^{(i)} \\ \mathcal{M}_{xz}^{(i)} \\ \mathcal{M}_{yz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} m_{xy} \\ m_{xz} \\ m_{yz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} \mathcal{B}_{11}^{(k)} & 0 & 0 \\ 0 & \mathcal{B}_{11}^{(k)} & 0 \\ 0 & 0 & \mathcal{B}_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \chi_{xy}^{(k-i)} \\ \chi_{xz}^{(k-i)} \\ \chi_{yz}^{(k-i)} \end{Bmatrix} \tag{2.89}$$

where m_{ij} are couple stresses $m_{ij} = 2\mu l^2 \chi_{ij}$ and plate stiffness (A_{11},A_{12}, B_{11}, and $\mathcal{B}_{11}$)

are

$$A_{11}^{(k)} = \frac{1-\nu}{(1+\nu)(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz, \;\; A_{12}^{(k)} = \frac{\nu}{(1+\nu)(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz$$
$$B_{11}^{(k)} = \frac{1}{2(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz, \;\; \mathcal{B}_{11}^{(k)} = \frac{l^2}{(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz,$$

and the thermal generalized forces are $X_T^{(i)}$, $Y_T^{(i)}$, and $Z_T^{(i)}$ are defined by

$$X_T^{(i)} = Y_T^{(i)} = Z_T^{(i)} = \frac{1}{(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\alpha(z,T)\, \Delta T\, dz.$$

We note that $\varepsilon_{zz}^{(3)} = 0$, $\gamma_{xz}^{(3)} = 0$, and $\gamma_{xz}^{(3)} = 0$.

2.6 Special cases

The general third-order theory developed herein contains all of the existing plate theories but some of them have not been extended to contain the microstructure parameters and the von Kármán nonlinearity. They are summarized here.

2.6.1 A general third-order theory with tangential traction free on top and bottom surfaces

If the top and bottom surfaces of the plate are free of any tangential forces, i.e., $q_x^t = q_x^b = q_y^t = q_y^b = 0$, we can invoke the conditions in Eqs. (2.31) and (2.34) and eliminate ϕ_x, ϕ_y, ψ_x, and ψ_y:

$$\phi_x = -\frac{1}{2}\frac{\partial \theta_z}{\partial x}, \quad \psi_x = -\frac{1}{3}\frac{\partial \phi_z}{\partial x} - \frac{4}{3h^2}\left(\theta_x + \frac{\partial w}{\partial x}\right),$$
$$\phi_y = -\frac{1}{2}\frac{\partial \theta_z}{\partial y}, \quad \psi_y = -\frac{1}{3}\frac{\partial \phi_z}{\partial y} - \frac{4}{3h^2}\left(\theta_y + \frac{\partial w}{\partial y}\right).$$

Then the strains are given by Eq. (2.35) with

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix} \tag{2.90}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix} \tag{2.91}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \end{Bmatrix} = -\frac{1}{2}\begin{Bmatrix} \frac{\partial^2 \theta_z}{\partial x^2} \\ \frac{\partial^2 \theta_z}{\partial y^2} \\ 2\frac{\partial^2 \theta_z}{\partial x \partial y} \end{Bmatrix} \tag{2.92}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = -\frac{1}{3}\begin{Bmatrix} \frac{\partial^2 \phi_z}{\partial x^2} \\ \frac{\partial^2 \phi_z}{\partial y^2} \\ 2\frac{\partial^2 \phi_z}{\partial x \partial y} \end{Bmatrix} - c_1 \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial \theta_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{2.93}$$

$$\begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \theta_z \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix} \tag{2.94}$$

$$\begin{Bmatrix} \varepsilon_{zz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2\phi_z \\ 0 \\ 0 \end{Bmatrix} \tag{2.95}$$

$$\begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} = -c_2 \begin{Bmatrix} 0 \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix} \tag{2.96}$$

where $c_1 = 4/3h^2$ and $c_2 = 4/h^2$. The components of the rotation vector and curvature tensor are

$$\omega_x = \omega_x^{(0)} + z\omega_x^{(1)} + z^2\omega_x^{(2)} \tag{2.97}$$

$$\omega_y = \omega_y^{(0)} + z\omega_y^{(1)} + z^2\omega_y^{(2)} \tag{2.98}$$

$$\omega_z = \omega_z^{(0)} + z\omega_z^{(1)} + z^2\omega_z^{(2)} + z^3\omega_z^{(3)} \tag{2.99}$$

$$\chi_{xx} = \chi_{xx}^{(0)} + z\chi_{xx}^{(1)} + z^2\chi_{xx}^{(2)} \tag{2.100}$$

$$\chi_{yy} = \chi_{yy}^{(0)} + z\chi_{yy}^{(1)} + z^2\chi_{yy}^{(2)} \tag{2.101}$$

$$\chi_{zz} = \chi_{zz}^{(0)} + z\chi_{zz}^{(1)} + z^2\chi_{zz}^{(2)} \tag{2.102}$$

$$\chi_{xy} = \chi_{xy}^{(0)} + z\chi_{xy}^{(1)} + z^2\chi_{xy}^{(2)} \tag{2.103}$$

$$\chi_{xz} = \chi_{xz}^{(0)} + z\chi_{xz}^{(1)} + z^2\chi_{xz}^{(2)} + z^3\chi_{xz}^{(3)} \tag{2.104}$$

$$\chi_{yz} = \chi_{yz}^{(0)} + z\chi_{yz}^{(1)} + z^2\chi_{yz}^{(2)} + z^3\chi_{yz}^{(3)} \tag{2.105}$$

with

$$\begin{Bmatrix} \omega_x^{(0)} \\ \omega_y^{(0)} \\ \omega_z^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial y} - \theta_y\right) \\ \frac{1}{2}\left(\theta_x - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{Bmatrix} \tag{2.106}$$

$$\begin{Bmatrix} \omega_x^{(1)} \\ \omega_y^{(1)} \\ \omega_z^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_z}{\partial y} \\ -\frac{\partial \theta_z}{\partial x} \\ \frac{1}{2}\left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right) \end{Bmatrix} \tag{2.107}$$

$$\begin{Bmatrix} \omega_x^{(2)} \\ \omega_y^{(2)} \\ \omega_z^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_z}{\partial y} + \frac{c_2}{2}\left(\theta_y + \frac{\partial w}{\partial y}\right) \\ -\frac{\partial \phi_z}{\partial x} - \frac{c_2}{2}\left(\theta_x + \frac{\partial w}{\partial x}\right) \\ 0 \end{Bmatrix} \tag{2.108}$$

$$\left\{ \omega_z^{(3)} \right\} = \left\{ \frac{c_1}{2} \left(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x} \right) \right\} \tag{2.109}$$

$$\begin{Bmatrix} \chi_{xx}^{(0)} \\ \chi_{yy}^{(0)} \\ \chi_{zz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial y} - \theta_y \right) \\ \frac{1}{2} \frac{\partial}{\partial y} \left(\theta_x - \frac{\partial w}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y} \right) \end{Bmatrix} \tag{2.110}$$

$$\begin{Bmatrix} \chi_{xx}^{(1)} \\ \chi_{yy}^{(1)} \\ \chi_{zz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 \theta_z}{\partial x \partial y} \\ -\frac{\partial^2 \theta_z}{\partial x \partial y} \\ 0 \end{Bmatrix} \tag{2.111}$$

$$\begin{Bmatrix} \chi_{xx}^{(2)} \\ \chi_{yy}^{(2)} \\ \chi_{zz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x} \left[\frac{\partial \phi_z}{\partial y} + \frac{c_2}{2} \left(\theta_y + \frac{\partial w}{\partial y} \right) \right] \\ -\frac{\partial}{\partial y} \left[\frac{\partial \phi_z}{\partial x} + \frac{c_2}{2} \left(\theta_x + \frac{\partial w}{\partial x} \right) \right] \\ \frac{c_2}{2} \left(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x} \right) \end{Bmatrix} \tag{2.112}$$

$$\begin{Bmatrix} \chi_{xy}^{(0)} \\ \chi_{xz}^{(0)} \\ \chi_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \left[\frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} - \theta_y \right) + \frac{\partial}{\partial x} \left(\theta_x - \frac{\partial w}{\partial x} \right) \right] \\ \frac{\partial \theta_z}{\partial y} + \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ -\frac{\partial \theta_z}{\partial x} + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \end{Bmatrix} \tag{2.113}$$

$$\begin{Bmatrix} \chi_{xy}^{(1)} \\ \chi_{xz}^{(1)} \\ \chi_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial y} \left(\frac{\partial \theta_z}{\partial y} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \theta_z}{\partial x} \right) \\ 2 \frac{\partial \phi_z}{\partial y} + c_2 \left(\theta_y + \frac{\partial w}{\partial y} \right) + \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y} \right) \\ -2 \frac{\partial \phi_z}{\partial x} - c_2 \left(\theta_x + \frac{\partial w}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y} \right) \end{Bmatrix} \tag{2.114}$$

$$\begin{Bmatrix} \chi_{xy}^{(2)} \\ \chi_{xz}^{(2)} \\ \chi_{yz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial y} \left[\frac{\partial \phi_z}{\partial y} + \frac{c_2}{2} \left(\theta_y + \frac{\partial w}{\partial y} \right) \right] - \frac{\partial}{\partial x} \left[\frac{\partial \phi_z}{\partial x} + \frac{c_2}{2} \left(\theta_x + \frac{\partial w}{\partial x} \right) \right] \\ 0 \\ 0 \end{Bmatrix} \tag{2.115}$$

$$\begin{Bmatrix} \chi_{xz}^{(3)} \\ \chi_{yz}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \frac{c_1}{2} \frac{\partial}{\partial x} \left(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x} \right) \\ \frac{c_1}{2} \frac{\partial}{\partial y} \left(\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x} \right) \end{Bmatrix}. \tag{2.116}$$

Then the equations of motion become

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_x^{(0)} + \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial y}$$

$$= m_0\ddot{u} + m_1\ddot{\theta}_x - \frac{m_2}{2}\frac{\partial \ddot{\theta}_z}{\partial x} - m_3\left[c_1\left(\ddot{\theta}_x + \frac{\partial \ddot{w}}{\partial x}\right) + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial x}\right] \qquad (2.117)$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_y^{(0)} - \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial x}$$

$$= m_0\ddot{v} + m_1\ddot{\theta}_y - \frac{m_2}{2}\frac{\partial \ddot{\theta}_z}{\partial y} - m_3\left[c_1\left(\ddot{\theta}_y + \frac{\partial \ddot{w}}{\partial y}\right) + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial y}\right] \qquad (2.118)$$

$$\delta w: \quad \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right)$$

$$+ \frac{\partial \bar{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \bar{M}_{yz}^{(0)}}{\partial y} + c_1\left(\frac{\partial^2 M_{xx}^{(3)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(3)}}{\partial x \partial y} + \frac{\partial^2 M_{yy}^{(3)}}{\partial y^2}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \tilde{\mathcal{M}}_{xx}^{(0)}}{\partial x} + \frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \tilde{\mathcal{M}}_{yy}^{(0)}}{\partial y} + \frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial x}\right) + F_z^{(0)}$$

$$+ c_2\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial \tilde{C}_y^{(0)}}{\partial x} - \frac{\partial \tilde{C}_x^{(0)}}{\partial y}\right)$$

$$+ c_1\left(\frac{\partial f_x^{(3)}}{\partial x} + \frac{\partial f_y^{(3)}}{\partial y}\right) = c_1\Bigg[m_3\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right)$$

$$+ \hat{m}_4\left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y}\right) - c_1 m_6\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right)$$

$$- \frac{m_5}{2}\left(\frac{\partial^2 \ddot{\theta}_z}{\partial x^2} + \frac{\partial^2 \ddot{\theta}_z}{\partial y^2}\right) - \frac{m_6}{3}\left(\frac{\partial^2 \ddot{\phi}_z}{\partial x^2} + \frac{\partial^2 \ddot{\phi}_z}{\partial y^2}\right)\Bigg]$$

$$+ m_0\ddot{w} + m_1\ddot{\theta}_z + m_2\ddot{\phi}_z \qquad (2.119)$$

$$\delta\theta_x: \quad \frac{\partial \hat{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \hat{M}_{xy}^{(1)}}{\partial y} - \bar{M}_{xz}^{(0)} + \frac{1}{2}\left(\frac{\partial \bar{\mathcal{M}}_{xy}^{(0)}}{\partial x} + \frac{\partial \bar{\mathcal{M}}_{yy}^{(0)}}{\partial y} - \frac{\partial \bar{\mathcal{M}}_{zz}^{(0)}}{\partial y}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x} + \frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right) + c_2\mathcal{M}_{yz}^{(1)} + \hat{f}_x^{(1)} + \frac{1}{2}\bar{C}_y^{(0)} + \frac{1}{2}\frac{\partial \hat{C}_z^{(1)}}{\partial y}$$

$$= \hat{m}_1\ddot{u} + \hat{m}_2\ddot{\theta}_x - \frac{\hat{m}_3}{2}\frac{\partial \ddot{\theta}_z}{\partial x} - \hat{m}_4\left[c_1\left(\ddot{\theta}_x + \frac{\partial \ddot{w}_z}{\partial x}\right) + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial x}\right] \qquad (2.120)$$

$$\delta\theta_y : \quad \frac{\partial \hat{M}_{yy}^{(1)}}{\partial y} + \frac{\partial \hat{M}_{xy}^{(1)}}{\partial x} - \bar{M}_{yz}^{(0)} - \frac{1}{2}\left(\frac{\partial \bar{\mathcal{M}}_{xx}^{(0)}}{\partial x} + \frac{\partial \bar{\mathcal{M}}_{xy}^{(0)}}{\partial y} - \frac{\partial \bar{\mathcal{M}}_{zz}^{(0)}}{\partial x}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x} + \frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right) - c_2\mathcal{M}_{xz}^{(1)} + \hat{f}_y^{(1)} - \frac{1}{2}\bar{C}_x^{(0)} - \frac{1}{2}\frac{\partial \hat{C}_z^{(1)}}{\partial x}$$

$$= \hat{m}_1\ddot{v} + \hat{m}_2\ddot{\theta}_y - \frac{\hat{m}_3}{2}\frac{\partial \ddot{\theta}_z}{\partial y} - \hat{m}_4\left[c_1\left(\ddot{\theta}_y + \frac{\partial \ddot{w}_z}{\partial y}\right) + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial y}\right] \tag{2.121}$$

$$\delta\theta_z : \quad - M_{zz}^{(0)} + \frac{1}{2}\left(\frac{\partial^2 M_{xx}^{(2)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(2)}}{\partial x \partial y} + \frac{\partial M_{yy}^{(2)}}{\partial y^2}\right) + \frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial y}$$

$$- \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(1)}}{\partial y}\right)$$

$$+ \frac{1}{2}\left(\frac{\partial f_x^{(2)}}{\partial x} + \frac{\partial f_y^{(2)}}{\partial y}\right) - \frac{\partial C_x^{(1)}}{\partial y} + \frac{\partial C_y^{(1)}}{\partial x} + F_z^{(1)}$$

$$= \frac{1}{2}\left[m_2\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) + \hat{m}_3\left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y}\right)\right.$$

$$\left. - \frac{m_4}{2}\left(\frac{\partial^2 \ddot{\theta}_z}{\partial x^2} + \frac{\partial^2 \ddot{\theta}_z}{\partial y^2}\right)\right] - \frac{m_5}{2}\left[\frac{1}{3}\left(\frac{\partial^2 \ddot{\phi}_z}{\partial x^2} + \frac{\partial^2 \ddot{\phi}_z}{\partial y^2}\right)\right.$$

$$\left. + c_1\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right)\right] + m_1\ddot{w} + m_2\ddot{\theta}_z + m_3\ddot{\phi}_z \tag{2.122}$$

$$\delta\phi_z : \quad - 2M_{zz}^{(1)} + \frac{1}{3}\left(\frac{\partial^2 M_{xx}^{(3)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(3)}}{\partial x \partial y} + \frac{\partial M_{yy}^{(3)}}{\partial y^2}\right)$$

$$+ 2\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial y}\right)$$

$$+ \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(2)}}{\partial y}\right) + \frac{1}{3}\left(\frac{\partial f_x^{(3)}}{\partial x} + \frac{\partial f_y^{(3)}}{\partial y}\right) - \frac{\partial C_x^{(2)}}{\partial y}$$

$$+ \frac{\partial C_y^{(2)}}{\partial x} + F_z^{(2)} = \frac{1}{3}\left[m_3\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) + \hat{m}_4\left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y}\right)\right.$$

$$\left. - \frac{m_5}{2}\left(\frac{\partial^2 \ddot{\theta}_z}{\partial x^2} + \frac{\partial^2 \ddot{\theta}_z}{\partial y^2}\right)\right] - \frac{m_6}{3}\left[\frac{1}{3}\left(\frac{\partial^2 \ddot{\phi}_z}{\partial x^2} + \frac{\partial^2 \ddot{\phi}_z}{\partial y^2}\right)\right.$$

$$\left. + c_1\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right)\right] + m_2\ddot{w} + m_3\ddot{\theta}_z + m_4\ddot{\phi}_z \tag{2.123}$$

with

$$\hat{f}_\xi^{(i)} = f_\xi^{(i)} - c_1 f_\xi^{(i+2)},\ \ \hat{m}_i = m_i - c_1 m_{(i+2)},$$
$$\hat{C}_\xi = C_\xi^{(i)} - c_1 C_\xi^{(i+2)},\ \ \bar{C}_\xi = C_\xi^{(i)} - c_2 C_\xi^{(i+2)},\ \ \tilde{C}_\xi = C_\xi^{(i)} + c_2 C_\xi^{(i+2)},$$
$$\bar{M}_{\xi\eta}^{(i)} = M_{\xi\eta}^{(i)} - c_2 M_{\xi\eta}^{(i+2)},\ \ \hat{M}_{\xi\eta}^{(i)} = M_{\xi\eta}^{(i)} - c_1 M_{\xi\eta}^{(i+2)},$$
$$\hat{\mathcal{M}}_{\xi z}^{(i)} = \mathcal{M}_{\xi z}^{(i)} - c_1 \mathcal{M}_{\xi z}^{(i+2)},\ \ \bar{\mathcal{M}}_{\xi\eta}^{(i)} = \mathcal{M}_{\xi\eta}^{(i)} - c_2 \mathcal{M}_{\xi\eta}^{(i+2)},$$
$$\text{and}\ \ \tilde{\mathcal{M}}_{\xi\eta}^{(i)} = \mathcal{M}_{\xi\eta}^{(i)} + c_2 \mathcal{M}_{\xi\eta}^{(i+2)}$$

where ξ and η, both take on the symbols x, y, and z, and $i = 0, 1, \cdots$. The boundary conditions involve specifying the following generalized forces (including the Kirchhoff free-edge type boundary conditions; see Reddy [6, 7]):

$$\delta u: \quad 0 = M_{xx}^{(0)} n_x + M_{xy}^{(0)} n_y + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial y} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) n_y + \frac{1}{2} C_z^{(0)} n_y + t_x^{(0)} \tag{2.124}$$

$$\delta v: \quad ,0 = M_{xy}^{(0)} n_x + M_{yy}^{(0)} n_y - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) n_x - \frac{1}{2} C_z^{(0)} n_x + t_y^{(0)} \tag{2.125}$$

$$\begin{aligned}
\delta w: \quad 0 = & \left(\frac{\partial w}{\partial x} M_{xx}^{(0)} + \frac{\partial w}{\partial y} M_{xy}^{(0)}\right) n_x + \left(\frac{\partial w}{\partial y} M_{yy}^{(0)} + \frac{\partial w}{\partial x} M_{xy}^{(0)}\right) n_y \\
& + c_1 \left[\left(\frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y}\right) n_x + \left(\frac{\partial M_{xy}^{(3)}}{\partial x} + \frac{\partial M_{yy}^{(3)}}{\partial y}\right) n_y\right] \\
& - \frac{1}{2}\left(\frac{\partial \tilde{\mathcal{M}}_{xx}^{(0)}}{\partial x} + \frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial y}\right) n_y + \frac{1}{2}\left(\frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial x} + \frac{\partial \tilde{\mathcal{M}}_{yy}^{(0)}}{\partial y}\right) n_x \\
& + \left(\bar{M}_{xz}^{(0)} n_x + \bar{M}_{yz}^{(0)} n_y\right) + c_2 \left(\mathcal{M}_{xz}^{(1)} n_y - \mathcal{M}_{yz}^{(1)} n_x\right) \\
& + c_1 \left(f_x^{(3)} n_x + f_y^{(3)} n_y\right) + \frac{1}{2}\left(\tilde{C}_y^{(0)} n_x - \tilde{C}_x^{(0)} n_y\right) \\
& + c_1 \left[m_3 \ddot{u} + \hat{m}_4 \ddot{\theta}_x - \frac{m_5}{2}\frac{\partial \ddot{\theta}_z}{\partial x} - m_6 \left(c_1 \frac{\partial \ddot{w}}{\partial x} + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial x}\right)\right] n_x \\
& + c_1 \left[m_3 \ddot{v} + \hat{m}_4 \ddot{\theta}_y - \frac{m_5}{2}\frac{\partial \ddot{\theta}_z}{\partial y} - m_6 \left(c_1 \frac{\partial \ddot{w}}{\partial y} + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial y}\right)\right] n_y
\end{aligned}$$

$$+ c_1 \frac{\partial M_{ns}^{(3)}}{\partial s} + t_z^{(0)} \tag{2.126}$$

$$\frac{\partial \delta w}{\partial n}\colon \quad 0 = c_1 M_{nn}^{(3)} \tag{2.127}$$

$$\delta\theta_x\colon \quad 0 = \hat{M}_{xx}^{(1)} n_x + \hat{M}_{xy}^{(1)} n_y + \frac{1}{2}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x} + \frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right) n_y + \frac{1}{2}\hat{C}_z^{(1)} n_y + t_x^{(1)}$$

$$+ \frac{1}{2}\left(\bar{\mathcal{M}}_{xy}^{(0)} n_x + \bar{\mathcal{M}}_{yy}^{(0)} n_y - \bar{\mathcal{M}}_{zz}^{(0)} n_y\right) \tag{2.128}$$

$$\delta\theta_y\colon \quad 0 = \hat{M}_{yy}^{(1)} n_y + \hat{M}_{xy}^{(1)} n_x - \frac{1}{2}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x} + \frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right) n_x - \frac{1}{2}\hat{C}_z^{(1)} n_x + t_y^{(1)}$$

$$- \frac{1}{2}\left(\bar{\mathcal{M}}_{xx}^{(0)} n_x + \bar{\mathcal{M}}_{xy}^{(0)} n_y - \bar{\mathcal{M}}_{zz}^{(0)} n_x\right) \tag{2.129}$$

$$\delta\theta_z\colon \quad 0 = \frac{1}{2}\left[\left(\frac{\partial M_{xx}^{(2)}}{\partial x} n_x + \frac{\partial M_{xy}^{(2)}}{\partial y} n_y\right) + \left(\frac{\partial M_{xy}^{(2)}}{\partial x} n_x + \frac{\partial M_{yy}^{(2)}}{\partial y} n_y\right)\right]$$

$$- \mathcal{M}_{yz}^{(0)} n_x - \left(\frac{\partial \mathcal{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial y}\right) n_y + \left(\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(1)}}{\partial y}\right) n_x$$

$$+ \mathcal{M}_{xz}^{(0)} n_y + \frac{1}{2}\left(f_x^{(2)} n_x + f_y^{(2)} n_y\right) - C_x^{(1)} n_y + C_y^{(1)} n_x + t_z^{(1)}$$

$$+ \frac{1}{2}\left[m_2 \ddot{u} + \hat{m}_3 \ddot{\theta}_x - \frac{m_4}{2}\frac{\partial \ddot{\theta}_z}{\partial x} - m_5\left(c_1 \frac{\partial \ddot{w}}{\partial x} + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial x}\right)\right] n_x$$

$$+ \frac{1}{2}\left[m_2 \ddot{v} + \hat{m}_3 \ddot{\theta}_y - \frac{m_4}{2}\frac{\partial \ddot{\theta}_z}{\partial y} - m_5\left(c_1 \frac{\partial \ddot{w}}{\partial y} + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial y}\right)\right] n_y$$

$$+ \frac{1}{2}\frac{\partial M_{ns}^{(2)}}{\partial s} \tag{2.130}$$

$$\frac{\partial \delta\theta_z}{\partial n}\colon \quad 0 = \frac{1}{2} M_{nn}^{(2)} \tag{2.131}$$

$$\delta\phi_z\colon \quad 0 = \frac{1}{3}\left[\left(\frac{\partial M_{xx}^{(3)}}{\partial x} n_x + \frac{\partial M_{xy}^{(3)}}{\partial y} n_y\right) + \left(\frac{\partial M_{xy}^{(3)}}{\partial x} n_x + \frac{\partial M_{yy}^{(3)}}{\partial y} n_y\right)\right]$$

$$+ 2\left(\mathcal{M}_{xz}^{(1)} n_y - \mathcal{M}_{yz}^{(1)} n_x\right) - \left(\frac{\partial \mathcal{M}_{xx}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial y}\right) n_y$$

$$+ \left(\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(2)}}{\partial y}\right) n_x + \frac{1}{3}\frac{\partial M_{ns}^{(3)}}{\partial s}$$

$$+ \frac{1}{3}\left(f_x^{(3)} n_x + f_y^{(3)} n_y\right) - C_x^{(2)} n_y + C_y^{(2)} n_x + t_z^{(2)}$$

$$
\begin{aligned}
&+\frac{1}{3}\left[m_3\ddot{u}+\hat{m}_4\ddot{\theta}_x-\frac{m_5}{2}\frac{\partial\ddot{\theta}_z}{\partial x}-m_6\left(c_1\frac{\partial\ddot{w}}{\partial x}+\frac{1}{3}\frac{\partial\ddot{\phi}_z}{\partial x}\right)\right]n_x\\
&+\frac{1}{3}\left[m_3\ddot{v}+\hat{m}_4\ddot{\theta}_y-\frac{m_5}{2}\frac{\partial\ddot{\theta}_z}{\partial y}-m_6\left(c_1\frac{\partial\ddot{w}}{\partial y}+\frac{1}{3}\frac{\partial\ddot{\phi}_z}{\partial y}\right)\right]n_y
\end{aligned}
\tag{2.132}
$$

$$
\frac{\partial\delta\phi_z}{\partial n}:\quad 0=\frac{1}{3}M_{nn}^{(3)} \tag{2.133}
$$

where

$$
M_{ns}^{(i)}=\left(M_{yy}^{(i)}-M_{xx}^{(i)}\right)n_xn_y+M_{xy}^{(i)}\left(n_x^2-n_y^2\right),\quad M_{nn}^{(i)}=M_{xx}^{(i)}n_x^2+M_{yy}^{(i)}n_y^2+2M_{xy}^{(i)}n_xn_y
$$

for $i=1,2,3$. The conditions in Eqs. (2.75) to (2.84) are still assumed to hold. This third-order theory is an extension of the theory developed by Reddy [78] to modified couple stress theory.

2.6.2 The Reddy third-order theory

The Reddy third-order theory (Reddy [21, 22, 78]) is based on the displacement field in which $\theta_z=0$ and $\phi_z=0$; when the top and bottom surfaces of the plate are required to be free of any tangential forces, we obtain

$$
\phi_x=0 \tag{2.134}
$$

$$
\psi_x=-c_1\left(\theta_x+\frac{\partial w}{\partial x}\right) \tag{2.135}
$$

$$
\phi_y=0 \tag{2.136}
$$

$$
\psi_y=-c_1\left(\theta_y+\frac{\partial w}{\partial y}\right) \tag{2.137}
$$

Thus, the theory is a special case of the one derived in the previous section and it deduced by setting $\theta_z = 0$, $\phi_z = 0$, $\phi_x = 0$, and $\phi_y = 0$. Then the strains are

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{zz} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix}$$

where

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix},$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = -c_1 \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial \theta_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} = -c_2 \begin{Bmatrix} 0 \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix}.$$

It is clear that the transverse normal strain ε_{zz} is identically zero. Consequently, σ_{zz} does not enter the strain energy expression and we use the plane stress-reduced constitutive relations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \alpha \Delta T \\ \varepsilon_{yy} - \alpha \Delta T \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \quad (2.138)$$

and the stress resultants must be expressed in terms of the strains using the constitutive equations in Eq. (2.138).

The equations of motion and natural boundary conditions of the Reddy third-order theory are obtained from Eqs. (2.117)–(2.123). The equations of motion are

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_x^{(0)} + \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial y}$$
$$= m_0\ddot{u} + m_1\ddot{\theta}_x - m_3 c_1\left(\ddot{\theta}_x + \frac{\partial \ddot{w}}{\partial x}\right) \tag{2.139}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_y^{(0)} - \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial x}$$
$$= m_0\ddot{v} + m_1\ddot{\theta}_y - m_3 c_1\left(\ddot{\theta}_y + \frac{\partial \ddot{w}}{\partial y}\right) \tag{2.140}$$

$$\delta w: \quad \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right)$$
$$+ \frac{\partial \bar{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \bar{M}_{yz}^{(0)}}{\partial y} + c_1\left(\frac{\partial^2 M_{xx}^{(3)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(3)}}{\partial x \partial y} + \frac{\partial^2 M_{yy}^{(3)}}{\partial y^2}\right)$$
$$- \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \tilde{\mathcal{M}}_{xx}^{(0)}}{\partial x} + \frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \tilde{\mathcal{M}}_{yy}^{(0)}}{\partial y} + \frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial x}\right)$$
$$- \frac{1}{2}\left(\frac{\partial \tilde{C}_x^{(0)}}{\partial y} - \frac{\partial \tilde{C}_y^{(0)}}{\partial x}\right) + c_2\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial x}\right)$$
$$+ c_1\left(\frac{\partial f_x^{(3)}}{\partial x} + \frac{\partial f_y^{(3)}}{\partial y}\right) + F_z^{(0)} = m_0\ddot{w} + c_1\left[m_3\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right)\right.$$
$$\left. + \hat{m}_4\left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y}\right) - c_1 m_6\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right)\right]$$

$$\delta \theta_x: \quad \frac{\partial \hat{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \hat{M}_{xy}^{(1)}}{\partial y} - \bar{M}_{xz}^{(0)} + \frac{1}{2}\left(\frac{\partial \bar{\mathcal{M}}_{xy}^{(0)}}{\partial x} + \frac{\partial \bar{\mathcal{M}}_{yy}^{(0)}}{\partial y} - \frac{\partial \bar{\mathcal{M}}_{zz}^{(0)}}{\partial y}\right)$$
$$+ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x} + \frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right) + c_2\mathcal{M}_{yz}^{(1)} + \hat{f}_x^{(1)} + \frac{1}{2}\bar{C}_y^{(0)} + \frac{1}{2}\frac{\partial \hat{C}_z^{(1)}}{\partial y}$$
$$= \hat{m}_1\ddot{u} + \hat{m}_2\ddot{\theta}_x - \hat{m}_4 c_1\left(\ddot{\theta}_x + \frac{\partial \ddot{w}_z}{\partial x}\right) \tag{2.141}$$

$$\delta \theta_y: \quad \frac{\partial \hat{M}_{yy}^{(1)}}{\partial y} + \frac{\partial \hat{M}_{xy}^{(1)}}{\partial x} - \bar{M}_{yz}^{(0)} - \frac{1}{2}\left(\frac{\partial \bar{\mathcal{M}}_{xx}^{(0)}}{\partial x} + \frac{\partial \bar{\mathcal{M}}_{xy}^{(0)}}{\partial y} - \frac{\partial \bar{\mathcal{M}}_{zz}^{(0)}}{\partial x}\right)$$

$$
-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x}+\frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right)-c_2\mathcal{M}_{xz}^{(1)}+\hat{f}_y^{(1)}-\frac{1}{2}\bar{C}_x^{(0)}-\frac{1}{2}\frac{\partial \hat{C}_z^{(1)}}{\partial x}
$$

$$
=\hat{m}_1\ddot{v}+\hat{m}_2\ddot{\theta}_y-\hat{m}_4c_1\left(\ddot{\theta}_y+\frac{\partial \ddot{w}_z}{\partial y}\right). \tag{2.142}
$$

The boundary conditions are of the form

$$
\delta u: \quad 0=M_{xx}^{(0)}n_x+M_{xy}^{(0)}n_y+\frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial y}+\frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right)n_y+\frac{1}{2}C_z^{(0)}n_y+t_x^{(0)} \tag{2.143}
$$

$$
\delta v: \quad 0=M_{xy}^{(0)}n_x+M_{yy}^{(0)}n_y-\frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x}+\frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right)n_x-\frac{1}{2}C_z^{(0)}n_x+t_y^{(0)} \tag{2.144}
$$

$$
\begin{aligned}
\delta w: \quad 0=&\left(\frac{\partial w}{\partial x}M_{xx}^{(0)}+\frac{\partial w}{\partial y}M_{xy}^{(0)}\right)n_x+\left(\frac{\partial w}{\partial y}M_{yy}^{(0)}+\frac{\partial w}{\partial x}M_{xy}^{(0)}\right)n_y\\
&+\left(\bar{M}_{xz}^{(0)}n_x+\bar{M}_{yz}^{(0)}n_y\right)+c_1\frac{\partial M_{ns}^{(3)}}{\partial s}+t_z^{(0)}\\
&+c_1\left[\left(\frac{\partial M_{xx}^{(3)}}{\partial x}+\frac{\partial M_{xy}^{(3)}}{\partial y}\right)n_x+\left(\frac{\partial M_{xy}^{(3)}}{\partial x}+\frac{\partial M_{yy}^{(3)}}{\partial y}\right)n_y\right]\\
&-\frac{1}{2}\left(\frac{\partial \tilde{\mathcal{M}}_{xx}^{(0)}}{\partial x}+\frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial y}\right)n_y+\frac{1}{2}\left(\frac{\partial \tilde{\mathcal{M}}_{xy}^{(0)}}{\partial x}+\frac{\partial \tilde{\mathcal{M}}_{yy}^{(0)}}{\partial y}\right)n_x\\
&+c_2\left(\mathcal{M}_{xz}^{(1)}n_y-\mathcal{M}_{yz}^{(1)}n_x\right)+c_1\left(f_x^{(3)}n_x+f_y^{(3)}n_y\right)\\
&+\frac{1}{2}\left(\tilde{C}_y^{(0)}n_x-\tilde{C}_x^{(0)}n_y\right)+c_1\left[\left(m_3\ddot{u}+\hat{m}_4\ddot{\theta}_x-m_6c_1\frac{\partial \ddot{w}}{\partial x}\right)n_x\right.\\
&\left.+\left(m_3\ddot{v}+\hat{m}_4\ddot{\theta}_y-m_6c_1\frac{\partial \ddot{w}}{\partial y}\right)n_y\right]
\end{aligned} \tag{2.145}
$$

$$
\frac{\partial \delta w}{\partial n}: \quad 0=c_1M_{nn}^{(3)} \tag{2.146}
$$

$$
\begin{aligned}
\delta\theta_x: \quad 0=&\hat{M}_{xx}^{(1)}n_x+\hat{M}_{xy}^{(1)}n_y+\frac{1}{2}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x}+\frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right)n_y+\frac{1}{2}\hat{C}_z^{(1)}n_y+t_x^{(1)}\\
&+\frac{1}{2}\left(\bar{\mathcal{M}}_{xy}^{(0)}n_x+\bar{\mathcal{M}}_{yy}^{(0)}n_y-\bar{\mathcal{M}}_{zz}^{(0)}n_y\right)
\end{aligned} \tag{2.147}
$$

$$
\begin{aligned}
\delta\theta_y: \quad 0=&\hat{M}_{yy}^{(1)}n_y+\hat{M}_{xy}^{(1)}n_x-\frac{1}{2}\left(\frac{\partial \hat{\mathcal{M}}_{xz}^{(1)}}{\partial x}+\frac{\partial \hat{\mathcal{M}}_{yz}^{(1)}}{\partial y}\right)n_x-\frac{1}{2}\hat{C}_z^{(1)}n_x+t_y^{(1)}\\
&-\frac{1}{2}\left(\bar{\mathcal{M}}_{xx}^{(0)}n_x+\bar{\mathcal{M}}_{xy}^{(0)}n_y-\bar{\mathcal{M}}_{zz}^{(0)}n_x\right).
\end{aligned} \tag{2.148}
$$

The generalized forces ($M_{xx}^{(i)}$, $M_{yy}^{(i)}$, $M_{xy}^{(i)}$) and the generalized couples ($\mathcal{M}_{xx}^{(i)}$, $\mathcal{M}_{yy}^{(i)}$, $\mathcal{M}_{xy}^{(i)}$, $\mathcal{M}_{xz}^{(i)}$, $\mathcal{M}_{yz}^{(i)}$) are related to the generalized displacements $(u, v, w, \theta_x, \theta_y)$ by Eqs. (2.86) to (2.89) with

$$A_{11}^{(k)} = \frac{1}{(1-\nu^2)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz \tag{2.149}$$

$$A_{12}^{(k)} = \frac{\nu}{(1-\nu^2)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz \tag{2.150}$$

$$B_{11}^{(k)} = \frac{1}{2(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz. \tag{2.151}$$

2.6.3 The first-order plate theory

The well-known first-order plate theory is based on the displacement field in Eqs. (2.28) to (2.30) with $\phi_x = 0$, $\phi_y = 0$, $\psi_x = 0$, $\psi_y = 0$, $\theta_z = 0$, and $\phi_z = 0$. Then the strains in Eq. (2.35) simplify to

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{zz} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix} \tag{2.152}$$

with

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix}$$

The equations of motion and boundary conditions are obtained from Eqs. (2.53) – (2.63) by setting $\phi_x = 0$, $\phi_y = 0$, $\psi_x = 0$, $\psi_y = 0$, $\theta_z = 0$, and $\phi_z = 0$. The equations of motion are

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_x^{(0)} + \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial y}$$

$$= m_0\ddot{u} + m_1\ddot{\theta}_x \tag{2.153}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_y^{(0)} - \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial x}$$

$$= m_0\ddot{v} + m_1\ddot{\theta}_y \tag{2.154}$$

$$\delta w: \quad \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right) \tag{2.155}$$

$$+ \frac{\partial M_{xz}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}\right) + F_z^{(0)} + \frac{1}{2}\left(\frac{\partial C_y^{(0)}}{\partial x} - \frac{\partial C_x^{(0)}}{\partial y}\right) = m_0\ddot{w} \tag{2.156}$$

$$\delta\theta_x: \quad \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - M_{xz}^{(0)} + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(0)}}{\partial y}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) + f_x^{(1)} + \frac{1}{2}C_y^{(0)} + \frac{1}{2}\frac{\partial C_z^{(1)}}{\partial y} = m_1\ddot{u} + m_2\ddot{\theta}_x \tag{2.157}$$

$$\delta\theta_y: \quad \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - M_{yz}^{(0)} - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(0)}}{\partial x}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) + f_y^{(1)} - \frac{1}{2}C_x^{(0)} - \frac{1}{2}\frac{\partial C_z^{(1)}}{\partial x} = m_1\ddot{v} + m_2\ddot{\theta}_y. \tag{2.158}$$

The natural boundary conditions of the theory are

$$\delta u: \qquad 0 = M_{xx}^{(0)}n_x + M_{xy}^{(0)}n_y + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right)n_y + \frac{1}{2}C_z^{(0)}n_y + t_x^{(0)} \tag{2.159}$$

$$\delta v: \qquad 0 = M_{xy}^{(0)}n_x + M_{yy}^{(0)}n_y - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right)n_x - \frac{1}{2}C_z^{(0)}n_x + t_y^{(0)} \tag{2.160}$$

$$\delta w: \qquad 0 = \left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right)n_x + \left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right)n_y$$

$$+ M_{xz}^{(0)} n_x + M_{yz}^{(0)} n_y + t_z^{(0)} - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y} \right) n_y$$
$$+ \frac{1}{2}\left(\frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x} \right) n_x + \frac{1}{2}\left(C_y^{(0)} n_x - C_x^{(0)} n_y \right) \tag{2.161}$$

$$\delta\theta_x : \qquad 0 = M_{xx}^{(1)} n_x + M_{xy}^{(1)} n_y + \frac{1}{2} C_z^{(1)} n_y + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y} \right) n_y$$
$$+ \frac{1}{2}\left(\mathcal{M}_{xy}^{(0)} n_x + \mathcal{M}_{yy}^{(0)} n_y - \mathcal{M}_{zz}^{(1)} n_y \right) + t_x^{(1)} \tag{2.162}$$

$$\delta\theta_y : \qquad 0 = M_{xy}^{(1)} n_x + M_{yy}^{(1)} n_y - \frac{1}{2} C_z^{(1)} n_x - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y} \right) n_x$$
$$- \frac{1}{2}\left(\mathcal{M}_{xx}^{(0)} n_x + \mathcal{M}_{xy}^{(0)} n_y - \mathcal{M}_{zz}^{(1)} n_x \right) + t_y^{(1)} \tag{2.163}$$

The generalized forces and couples are related to the generalized displacements through Eqs. (2.86)–(2.89) with the coefficients $A_{11}^{(k)}$, $A_{12}^{(k)}$, and $B_{11}^{(k)}$ defined by Eqs. (2.149) to (2.151). Note that the shear correction factor must multiply the coefficient $B_{11}^{(k)}$ when computing the transverse shear forces.

2.6.4 The classical plate theory

The classical plate theory is obtained by setting

$$\theta_x = -\frac{\partial w}{\partial x}, \quad \theta_y = -\frac{\partial w}{\partial y} \tag{2.164}$$

and $\phi_x = \phi_y = \psi_x = \psi_y = \theta_z = \phi_z = 0$ in the displacement field of Eqs. (2.28) to (2.30), giving

$$u_1(x, y, z, t) = u(x, y, t) - z\frac{\partial w}{\partial x} \tag{2.165}$$
$$u_2(x, y, z, t) = v(x, y, t) - z\frac{\partial w}{\partial y} \tag{2.166}$$
$$u_3(x, y, z, t) = w(x, y, t). \tag{2.167}$$

The equations of motion are obtained from the Reddy third-order theory by invoking the conditions in Eqs. (2.134) to (2.137) and setting $c_1 = 1$, $c_2 = 0$, $\phi_x = 0$, and $\phi_y = 0$. The equations of motion are

$$\begin{aligned}
\delta u: \quad & \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_x^{(0)} + \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial y} \\
& = m_0\ddot{u} - m_1\frac{\partial \ddot{w}}{\partial x} \qquad (2.168) \\
\delta v: \quad & \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + f_y^{(0)} - \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial x} \\
& = m_0\ddot{v} - m_1\frac{\partial \ddot{w}}{\partial y} \qquad (2.169) \\
\delta w: \quad & \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right) \\
& + \frac{\partial^2 M_{xx}^{(1)}}{\partial x^2} + \frac{\partial^2 M_{yy}^{(1)}}{\partial y^2} + 2\frac{\partial^2 M_{xy}^{(1)}}{\partial x \partial y} - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y}\right) \\
& + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}\right) + \frac{\partial f_x^{(1)}}{\partial x} + \frac{\partial f_y^{(1)}}{\partial y} - \frac{\partial C_x^{(0)}}{\partial y} + \frac{\partial C_y^{(0)}}{\partial x} \\
& + F_z^{(0)} = m_0\ddot{w} + m_1\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) - m_2\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right) \qquad (2.170)
\end{aligned}$$

and the boundary conditions are

$$\delta u: \quad 0 = M_{xx}^{(0)}n_x + M_{xy}^{(0)}n_y + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right)n_y + \frac{1}{2}C_z^{(0)}n_y \qquad (2.171)$$

$$\delta v: \quad 0 = M_{xy}^{(0)}n_x + M_{yy}^{(0)}n_y - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right)n_x - \frac{1}{2}C_z^{(0)}n_x \qquad (2.172)$$

$$\begin{aligned}
\delta w: \quad 0 = & \left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right)n_x + \left(\frac{\partial w}{\partial y}M_{yy}^{(0)} + \frac{\partial w}{\partial x}M_{xy}^{(0)}\right)n_y \\
& + \left(\frac{\partial M_{xx}^{(1)}}{\partial x}n_x + \frac{\partial M_{xy}^{(1)}}{\partial y}n_x + \frac{\partial M_{xy}^{(1)}}{\partial x}n_x + \frac{\partial M_{yy}^{(1)}}{\partial y}n_y\right) + f_x^{(1)}n_x \\
& + f_y^{(1)}n_y - \left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y}\right)n_y + \left(\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y}\right)n_x
\end{aligned}$$

$$
\begin{aligned}
&+ C_y^{(0)} n_x - C_x^{(0)} n_y + m_1 \left(\ddot{u} n_x + \ddot{v} n_y \right) - m_2 \left(\frac{\partial \ddot{w}}{\partial x} n_x + \frac{\partial \ddot{w}}{\partial y} n_y \right) \\
&+ \frac{\partial M_{ns}^{(1)}}{\partial s} \qquad (2.173)
\end{aligned}
$$

$$
\frac{\partial \delta w}{\partial n}: \quad 0 = M_{nn}^{(1)}. \qquad (2.174)
$$

3. ANALYTICAL SOLUTION *

Analytical solutions of the general third order plate theory developed in the section 2 are presented using the Navier solution technique for bending, vibration, and buckling problems. The geometrical nonlinearity is not considered for the analytical solution since the Navier solution technique is limited to linear case and only for simply supported rectangular plates.

3.1 Linear strains

The linear strains are based on the assumption of small strain and small rotation. By this assumption, any higher order terms (i.e. higher than linear) are omitted. The linear strains of the general third order plate theory based on the displacement field of Eqs. (2.28) – (2.30) are

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{zz}^{(0)} \\ 2\varepsilon_{xy}^{(0)} \\ 2\varepsilon_{xz}^{(0)} \\ 2\varepsilon_{yz}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ 2\varepsilon_{xy}^{(1)} \\ 2\varepsilon_{xz}^{(1)} \\ 2\varepsilon_{yz}^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \\ 2\varepsilon_{yz}^{(2)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \varepsilon_{zz}^{(3)} \\ 2\varepsilon_{xy}^{(3)} \\ 2\varepsilon_{xz}^{(3)} \\ 2\varepsilon_{yz}^{(3)} \end{Bmatrix} \tag{3.1}$$

where

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$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{zz}^{(0)} \\ 2\varepsilon_{xy}^{(0)} \\ 2\varepsilon_{xz}^{(0)} \\ 2\varepsilon_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \theta_z \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix}, \qquad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ 2\varepsilon_{xy}^{(1)} \\ 2\varepsilon_{xz}^{(1)} \\ 2\varepsilon_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ 2\phi_z \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \\ 2\phi_x + \frac{\partial \theta_z}{\partial x} \\ 2\phi_y + \frac{\partial \theta_z}{\partial y} \end{Bmatrix} \tag{3.2}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \\ 2\varepsilon_{yz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ 0 \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \\ 3\psi_x + \frac{\partial \phi_z}{\partial x} \\ 3\psi_y + \frac{\partial \phi_z}{\partial y} \end{Bmatrix}, \qquad \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \varepsilon_{zz}^{(3)} \\ 2\varepsilon_{xy}^{(3)} \\ 2\varepsilon_{xz}^{(3)} \\ 2\varepsilon_{yz}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \psi_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ 0 \\ \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \\ 0 \\ 0 \end{Bmatrix}. \tag{3.3}$$

The rotation vectors and curvature tensors defined in Eqs. (2.36) – (2.45) do not changed for geometrically linear problems.

3.2 Isothermal constitutive equations of FGM plates

We assume isotropic plate with variation of two constituents through thickness. The constitutive relation in Eq. (2.85) is modified for the linear constitutive equations of the isothermal and isotropic plate:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \Lambda \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} \tag{3.4}$$

where $\Lambda = E/[(1+\nu)(1-2\nu)]$. In the equation (3.4), Young's modulus, E, varies through thickness direction according to the power law distribution in Eq. (2.9) and Possion's ratio, ν, is assumed as constant. The generalized forces and couples with the absence of thermal coupling can be expressed as

$$\begin{Bmatrix} M_{xx}^{(i)} \\ M_{yy}^{(i)} \\ M_{zz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{11}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{12}^{(k)} & A_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(k-i)} \\ \varepsilon_{yy}^{(k-i)} \\ \varepsilon_{zz}^{(k-i)} \end{Bmatrix} \tag{3.5}$$

$$\begin{Bmatrix} M_{xy}^{(i)} \\ M_{xz}^{(i)} \\ M_{yz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} B_{11}^{(k)} & 0 & 0 \\ 0 & B_{11}^{(k)} & 0 \\ 0 & 0 & B_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^{(k-i)} \\ \gamma_{xz}^{(k-i)} \\ \gamma_{zz}^{(k-i)} \end{Bmatrix} \tag{3.6}$$

$$\begin{Bmatrix} \mathcal{M}_{xx}^{(i)} \\ \mathcal{M}_{yy}^{(i)} \\ \mathcal{M}_{zz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{zz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{2+i} \begin{bmatrix} \mathcal{B}_{11}^{(k)} & 0 & 0 \\ 0 & \mathcal{B}_{11}^{(k)} & 0 \\ 0 & 0 & \mathcal{B}_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \chi_{xx}^{(k-i)} \\ \chi_{yy}^{(k-i)} \\ \chi_{zz}^{(k-i)} \end{Bmatrix} \tag{3.7}$$

$$\begin{Bmatrix} \mathcal{M}_{xy}^{(i)} \\ \mathcal{M}_{xz}^{(i)} \\ \mathcal{M}_{yz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} m_{xy} \\ m_{xz} \\ m_{yz} \end{Bmatrix} (z)^i \, dz = \sum_{k=i}^{3+i} \begin{bmatrix} \mathcal{B}_{11}^{(k)} & 0 & 0 \\ 0 & \mathcal{B}_{11}^{(k)} & 0 \\ 0 & 0 & \mathcal{B}_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \chi_{xy}^{(k-i)} \\ \chi_{xz}^{(k-i)} \\ \chi_{yz}^{(k-i)} \end{Bmatrix} \tag{3.8}$$

where m_{ij} are couple stresses $m_{ij} = 2\mu\ell^2\chi_{ij}$ and plate stiffness $(A_{11}$,A_{12}, B_{11}, and $\mathcal{B}_{11})$ are

$$A_{11}^{(k)} = \frac{1-\nu}{(1+\nu)(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z) \, dz \tag{3.9}$$

$$A_{12}^{(k)} = \frac{\nu}{(1+\nu)(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z) \, dz \tag{3.10}$$

$$B_{11}^{(k)} = \frac{1}{2(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z) \, dz \tag{3.11}$$

$$\mathcal{B}_{11}^{(k)} = \frac{\ell^2}{(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z) \, dz. \tag{3.12}$$

The resultants of moduli are following

$$E_k = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^{(k)} E(z) \, dz \tag{3.13}$$

$$E_0 = E_b h \frac{Mr+n}{n+1} \tag{3.14}$$

$$E_1 = \frac{E_b h^2 (Mr-1) n}{2(n+2)(n+1)} \tag{3.15}$$

$$E_2 = \frac{E_b h^3 (n^3 + 3Mrn^2 + 3n^2 + 3Mrn + 8n + 6Mr)}{12(3+n)(n+2)(n+1)} \tag{3.16}$$

$$E_3 = \frac{(n^2+3n+8)(Mr-1) h^4 n E_b}{8(n+4)(3+n)(n+2)(n+1)} \tag{3.17}$$

$$E_4 = \frac{E_b h^5 (5Mr\Delta_1 + n\Delta_2)}{80(5+n)(n+4)(3+n)(n+2)(n+1)} \tag{3.18}$$

$$E_5 = \frac{E_b n h^6 (n^4 + 10n^3 + 55n^2 + 110n + 184)(Mr-1)}{32(6+n)(5+n)(n+4)(3+n)(n+2)(n+1)} \tag{3.19}$$

$$E_6 = \frac{E_b h^7 \left(7Mr\Delta_3 + n\Delta_4\right)}{448\left(7+n\right)\left(6+n\right)\left(5+n\right)\left(n+2\right)\left(n+4\right)\left(3+n\right)\left(n+1\right)} \tag{3.20}$$

where

$$E\left(z\right) = \left(E_t - E_b\right) f\left(z\right) + E_b, f\left(z\right) = \left(\frac{1}{2} + \frac{z}{h}\right)^n$$

$$\Delta_1 = n^4 + 6n^3 + 23n^2 + 18n + 24$$

$$\Delta_2 = n^4 + 10n^3 + 55n^2 + 110n + 184$$

$$\Delta_3 = n^6 + 15n^5 + 115n^4 + 405n^3 + 964n^2 + 660n + 720$$

$$\Delta_4 = n^6 + 21n^5 + 217n^4 + 1155n^3 + 3934n^2 + 6384n + 8448$$

$$Mr = \frac{E_t}{E_b}$$

and the subscripts t and b indicate top$\left(z = \frac{h}{2}\right)$ and bottom$\left(z = -\frac{h}{2}\right)$ surfaces. The figure 3.1 shows the variation of the non-dimensional resultant of Young's modulus in terms of power law index n. It is clear that the maximum value occurs at $n = 0$ and the resultant of Young's modulus decreases with larger value of n since we assume that material on top surface is stiffer than one on bottom surface.

The equations of motion of the general third order plate theory has been derived in the section 2 using the principle of virtual displacements (2.46). The derivation is based on the nonlinear kinematic relation, a modified couple stress theory, and the variation of two materials through thickness. The nonlinear kinematic relation is simplified to linear strains in Eqs (3.1)–(3.2). Substituting the kinematic relation (3.1) and the constitutive relation (3.4) into the generalized forces (3.5) to (3.8), we obtain the equation of motion of the general third order plate theory in terms of the generalized displacements. The derived

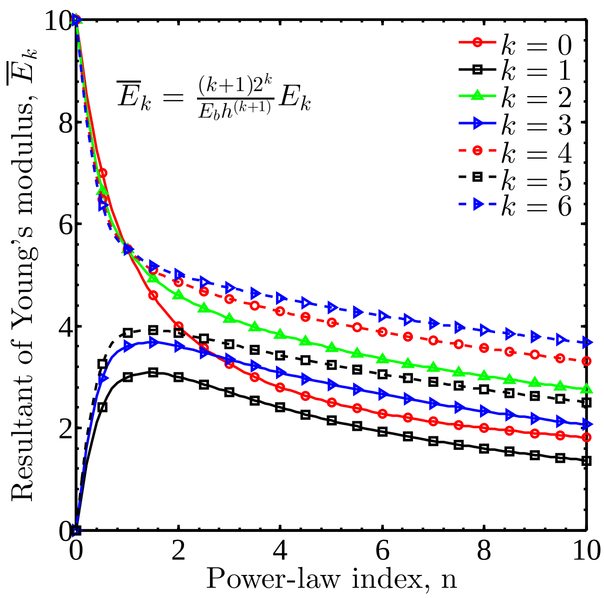

Figure 3.1: The non-dimensional resultants of modulus [2]

linear equation of motion can be expressed as following

$$
\begin{aligned}
\delta u: \quad & A_{11}^{(0)}\frac{\partial^2 u}{\partial x^2} + A_{12}^{(0)}\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial \theta_z}{\partial x}\right) + A_{11}^{(1)}\frac{\partial^2 \theta_x}{\partial x^2} + A_{12}^{(1)}\left(\frac{\partial^2 \theta_y}{\partial x \partial y} + 2\frac{\partial \phi_z}{\partial x}\right) \\
& + A_{11}^{(2)}\frac{\partial^2 \phi_x}{\partial x^2} + A_{12}^{(2)}\frac{\partial^2 \phi_y}{\partial x \partial y} + A_{11}^{(3)}\frac{\partial^2 \psi_x}{\partial x^2} + A_{12}^{(3)}\frac{\partial^2 \psi_y}{\partial x \partial y} \\
& + B_{11}^{(0)}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + B_{11}^{(1)}\left(\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial x \partial y}\right) \\
& + B_{11}^{(2)}\left(\frac{\partial^2 \phi_x}{\partial y^2} + \frac{\partial^2 \phi_y}{\partial x \partial y}\right) + B_{11}^{(3)}\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right) \\
& + \frac{1}{4}\left\{\mathcal{B}_{11}^{(0)}\left[\frac{\partial^4 v}{\partial x^3 \partial y} - \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial x \partial y^3} - \frac{\partial^4 u}{\partial y^4} - 2\left(\frac{\partial^2 \phi_y}{\partial x \partial y} - \frac{\partial \phi_x}{\partial y^2}\right)\right]\right. \\
& + \mathcal{B}_{11}^{(1)}\left[\frac{\partial^4 \theta_y}{\partial x^3 \partial y} - \frac{\partial^4 \theta_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \theta_y}{\partial x \partial y^3} - \frac{\partial^4 \theta_x}{\partial y^4} - 6\left(\frac{\partial^2 \psi_y}{\partial x \partial y} - \frac{\partial^2 \psi_x}{\partial y^2}\right)\right] \\
& + \mathcal{B}_{11}^{(2)}\left(\frac{\partial^4 \phi_y}{\partial x^3 \partial y} - \frac{\partial^4 \phi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi_y}{\partial x \partial y^3} - \frac{\partial^4 \phi_x}{\partial y^4}\right) \\
& \left. + \mathcal{B}_{11}^{(3)}\left(\frac{\partial^4 \psi_y}{\partial x^3 \partial y} - \frac{\partial^4 \psi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_y}{\partial x \partial y^3} - \frac{\partial^4 \psi_x}{\partial y^4}\right)\right\} + F_x^{(0)} + \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial y} \\
& = m_0 \ddot{u} + m_1 \ddot{\theta}_x + m_2 \ddot{\phi}_x + m_3 \ddot{\psi}_x
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
\delta v: \quad & A_{11}^{(0)}\frac{\partial^2 v}{\partial y^2} + A_{12}^{(0)}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial \theta_z}{\partial y}\right) + A_{11}^{(1)}\frac{\partial^2 \theta_y}{\partial y^2} + A_{12}^{(1)}\left(\frac{\partial^2 \theta_x}{\partial x \partial y} + 2\frac{\partial \phi_z}{\partial y}\right) \\
& + A_{11}^{(2)}\frac{\partial^2 \phi_y}{\partial y^2} + A_{12}^{(2)}\frac{\partial^2 \phi_x}{\partial x \partial y} + A_{11}^{(3)}\frac{\partial^2 \psi_y}{\partial y^2} + A_{12}^{(3)}\frac{\partial^2 \psi_x}{\partial x \partial y} \\
& + B_{11}^{(0)}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^v}{\partial x^2}\right) + B_{11}^{(1)}\left(\frac{\partial^2 \theta_x}{\partial x \partial y} + \frac{\partial^2 \theta_y}{\partial x^2}\right) \\
& + B_{11}^{(2)}\left(\frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial x^2}\right) + B_{11}^{(3)}\left(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right) \\
& - \frac{1}{4}\left\{\mathcal{B}_{11}^{(0)}\left[\frac{\partial^4 v}{\partial x^4} - \frac{\partial^4 u}{\partial x^2 \partial y} + \frac{\partial^4 v}{\partial x^2 \partial y^2} - \frac{\partial^4 u}{\partial x \partial y^3} - 2\left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y}\right)\right]\right. \\
& + \mathcal{B}_{11}^{(1)}\left[\frac{\partial^4 \theta_y}{\partial x^4} - \frac{\partial^4 \theta_x}{\partial x^2 \partial y} + \frac{\partial^4 \theta_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \theta_x}{\partial x \partial y^3} - 6\left(\frac{\partial^2 \psi_y}{\partial x^2} - \frac{\partial^2 \psi_x}{\partial x \partial y}\right)\right] \\
& + \mathcal{B}_{11}^{(2)}\left(\frac{\partial^4 \phi_y}{\partial x^4} - \frac{\partial^4 \phi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \phi_x}{\partial x \partial y^3}\right) \\
& \left. + \mathcal{B}_{11}^{(3)}\left(\frac{\partial^4 \psi_y}{\partial x^4} - \frac{\partial^4 \psi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \psi_x}{\partial x \partial y^3}\right)\right\} + F_y^{(0)} - \frac{1}{2}\frac{\partial C_z^{(0)}}{\partial x} \\
& = m_0 \ddot{v} + m_1 \ddot{\theta}_y + m_2 \ddot{\phi}_y + m_3 \ddot{\psi}_y
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
\delta w: \quad & B_{11}^{(0)}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y}\right) + 2B_{11}^{(1)}\left(\frac{\partial^2 \theta_z}{\partial x^2} + \frac{\partial^2 \theta_z}{\partial y^2}\right. \\
& \left. + \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) + 3B_{11}^{(2)}\left(\frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) \\
& - \frac{1}{4}\left\{\mathcal{B}_{11}^{(0)}\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} - \frac{\partial^3 \theta_x}{\partial x^3} - \frac{\partial^3 \theta_y}{\partial y^3}\right)\right. \\
& + \mathcal{B}_{11}^{(1)}\left[\frac{\partial^4 \theta_z}{\partial x^4} + \frac{\partial^4 \theta_z}{\partial y^4} - 2\left(\frac{\partial^3 \phi_x}{\partial x^3} + \frac{\partial^3 \phi_y}{\partial y^3}\right)\right] \\
& \left. + \mathcal{B}_{11}^{(2)}\left[\frac{\partial^4 \phi_z}{\partial x^4} + \frac{\partial^4 \phi_z}{\partial y^4} - 3\left(\frac{\partial^3 \psi_x}{\partial x^3} + \frac{\partial^3 \psi_y}{\partial y^3}\right)\right]\right\} \\
& + F_z^{(0)} + \frac{1}{2}\left(\frac{\partial C_y^{(0)}}{\partial x} + \frac{\partial C_x^{(0)}}{\partial y}\right) = m_0 \ddot{w} + m_1 \ddot{\theta}_z + m_2 \ddot{\phi}_z
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
\delta \theta_x: \quad & A_{11}^{(1)}\frac{\partial^2 u}{\partial x^2} + A_{12}^{(1)}\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial \theta_z}{\partial x}\right) + A_{11}^{(2)}\frac{\partial^2 \theta_x}{\partial x^2} + A_{12}^{(2)}\left(\frac{\partial^2 \theta_y}{\partial x \partial y} + 2\frac{\partial \phi_z}{\partial x}\right) \\
& + A_{11}^{(3)}\frac{\partial^2 \phi_x}{\partial x^2} + A_{12}^{(3)}\frac{\partial^2 \phi_y}{\partial x \partial y} + A_{11}^{(4)}\frac{\partial^2 \psi_x}{\partial x^2} + A_{12}^{(4)}\frac{\partial^2 \psi_y}{\partial x \partial y} \\
& + B_{11}^{(1)}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + B_{11}^{(2)}\left(\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial x \partial y}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ B_{11}^{(3)}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)+B_{11}^{(4)}\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}\right)\\
&- B_{11}^{(0)}\left(\theta_x+\frac{\partial w}{\partial x}\right)-B_{11}^{(1)}\left(2\phi_x+\frac{\partial\theta_z}{\partial x}\right)-B_{11}^{(2)}\left(3\psi_x+\frac{\partial\phi_z}{\partial x}\right)\\
&+\frac{1}{4}\Bigg\{\mathcal{B}_{11}^{(1)}\left[\frac{\partial^4 v}{\partial x^3\partial y}-\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 v}{\partial x\partial y^3}-\frac{\partial^4 u}{\partial y^4}-2\left(\frac{\partial^2\phi_y}{\partial x\partial y}-\frac{\partial\phi_x}{\partial y^2}\right)\right]\\
&+\mathcal{B}_{11}^{(2)}\left[\frac{\partial^4\theta_y}{\partial x^3\partial y}-\frac{\partial^4\theta_x}{\partial x^2\partial y^2}+\frac{\partial^4\theta_y}{\partial x\partial y^3}-\frac{\partial^4\theta_x}{\partial y^4}-6\left(\frac{\partial^2\psi_y}{\partial x\partial y}-\frac{\partial^2\psi_x}{\partial y^2}\right)\right]\\
&+\mathcal{B}_{11}^{(3)}\left(\frac{\partial^4\phi_y}{\partial x^3\partial y}-\frac{\partial^4\phi_x}{\partial x^2\partial y^2}+\frac{\partial^4\phi_y}{\partial x\partial y^3}-\frac{\partial^4\phi_x}{\partial y^4}\right)\\
&+\mathcal{B}_{11}^{(4)}\left(\frac{\partial^4\psi_y}{\partial x^3\partial y}-\frac{\partial^4\psi_x}{\partial x^2\partial y^2}+\frac{\partial^4\psi_y}{\partial x\partial y^3}-\frac{\partial^4\psi_x}{\partial y^4}\right)\Bigg\}\\
&-\frac{1}{4}\Bigg\{\mathcal{B}_{11}^{(0)}\left[\frac{\partial^3 w}{\partial x^3}-\frac{\partial^2\theta_x}{\partial x^2}-2\left(\frac{\partial^2\theta_x}{\partial y^2}-\frac{\partial^2\theta_y}{\partial x\partial y}\right)\right]\\
&+\mathcal{B}_{11}^{(1)}\left[\frac{\partial^3\theta_z}{\partial x^3}-2\frac{\partial^2\phi_x}{\partial x^2}-4\left(\frac{\partial^2\phi_x}{\partial y^2}-\frac{\partial^2\phi_y}{\partial x\partial y}\right)\right]\\
&+\mathcal{B}_{11}^{(2)}\left[\frac{\partial^3\phi_z}{\partial x^3}-3\frac{\partial^2\psi_x}{\partial x^2}-6\left(\frac{\partial^2\psi_y}{\partial x\partial y}-\frac{\partial^2\psi_x}{\partial y^2}\right)\right]\Bigg\}\\
&+F_x^{(1)}+\frac{1}{2}C_y^{(0)}+\frac{1}{2}\frac{\partial C_z^{(1)}}{\partial y}=m_1\ddot{u}+m_2\ddot{\theta}_x+m_3\ddot{\phi}_x+m_4\ddot{\psi}_x \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
\delta\theta_y:\quad & A_{11}^{(1)}\frac{\partial^2 v}{\partial y^2}+A_{12}^{(1)}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial\theta_z}{\partial y}\right)+A_{11}^{(2)}\frac{\partial^2\theta_y}{\partial y^2}+A_{12}^{(2)}\left(\frac{\partial^2\theta_x}{\partial x\partial y}+2\frac{\partial\phi_z}{\partial y}\right)\\
&+A_{11}^{(3)}\frac{\partial^2\phi_y}{\partial y^2}+A_{12}^{(3)}\frac{\partial^2\phi_x}{\partial x\partial y}+A_{11}^{(4)}\frac{\partial^2\psi_y}{\partial y^2}+A_{12}^{(4)}\frac{\partial^2\psi_x}{\partial x\partial y}\\
&+B_{11}^{(1)}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^v}{\partial x^2}\right)+B_{11}^{(2)}\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)\\
&+B_{11}^{(3)}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)+B_{11}^{(4)}\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^2\psi_y}{\partial x^2}\right)\\
&-B_{11}^{(0)}\left(\theta_y+\frac{\partial w}{\partial y}\right)-B_{11}^{(1)}\left(2\phi_y+\frac{\partial\theta_z}{\partial y}\right)-B_{11}^{(2)}\left(3\psi_y+\frac{\partial\phi_z}{\partial y}\right)\\
&-\frac{1}{4}\Bigg\{\mathcal{B}_{11}^{(1)}\left[\frac{\partial^4 v}{\partial x^4}-\frac{\partial^4 u}{\partial x^2\partial y}+\frac{\partial^4 v}{\partial x^2\partial y^2}-\frac{\partial^4 u}{\partial x\partial y^3}-2\left(\frac{\partial^2\phi_y}{\partial x^2}-\frac{\partial^2\phi_x}{\partial x\partial y}\right)\right]\\
&+\mathcal{B}_{11}^{(2)}\left[\frac{\partial^4\theta_y}{\partial x^4}-\frac{\partial^4\theta_x}{\partial x^2\partial y}+\frac{\partial^4\theta_y}{\partial x^2\partial y^2}-\frac{\partial^4\theta_x}{\partial x\partial y^3}-6\left(\frac{\partial^2\psi_y}{\partial x^2}-\frac{\partial^2\psi_x}{\partial x\partial y}\right)\right]\\
&+\mathcal{B}_{11}^{(3)}\left(\frac{\partial^4\phi_y}{\partial x^4}-\frac{\partial^4\phi_x}{\partial x^2\partial y^2}+\frac{\partial^4\phi_y}{\partial x^2\partial y^2}-\frac{\partial^4\phi_x}{\partial x\partial y^3}\right)
\end{aligned}
$$

$$+ \mathcal{B}_{11}^{(4)} \left(\frac{\partial^4 \psi_y}{\partial x^4} - \frac{\partial^4 \psi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \psi_x}{\partial x \partial y^3} \right) \Bigg\}$$

$$+ \frac{1}{4} \Bigg\{ \mathcal{B}_{11}^{(0)} \left[\frac{\partial^3 w}{\partial y^3} - \frac{\partial^2 \theta_y}{\partial y^2} - 2 \left(\frac{\partial^2 \theta_y}{\partial x^2} - \frac{\partial^2 \theta_x}{\partial x \partial y} \right) \right]$$

$$+ \mathcal{B}_{11}^{(1)} \left[\frac{\partial^3 \theta_z}{\partial y^3} - 2 \frac{\partial^2 \phi_y}{\partial y^2} - 4 \left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y} \right) \right]$$

$$+ \mathcal{B}_{11}^{(2)} \left[\frac{\partial^3 \phi_z}{\partial y^3} - 3 \frac{\partial^2 \psi_y}{\partial x^2} - 6 \left(\frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial^2 \psi_x}{\partial x \partial y} \right) \right] \Bigg\}$$

$$+ F_y^{(1)} - \frac{1}{2} C_x(0) - \frac{1}{2} \frac{\partial C_z^{(1)}}{\partial x} = m_1 \ddot{v} + m_2 \ddot{\theta}_y + m_3 \ddot{\phi}_y + m_4 \ddot{\psi}_y \tag{3.25}$$

$$\delta\theta_z : \quad B_{11}^{(1)} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y} \right) - A_{11}^{(0)} \theta_z$$

$$+ 2B_{11}^{(2)} \left(\frac{\partial^2 \theta_z}{\partial x^2} + \frac{\partial^2 \theta_z}{\partial y^2} + \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) - A_{12}^{(0)} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$+ 3B_{11}^{(3)} \left(\frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) - 2A_{11}^{(1)} \phi_z$$

$$- A_{12}^{(1)} \left(\frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y} \right) - A_{12}^{(2)} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) - A_{12}^{(3)} \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right)$$

$$- \frac{1}{4} \Bigg\{ \mathcal{B}_{11}^{(1)} \left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} - \frac{\partial^3 \theta_x}{\partial x^3} - \frac{\partial^3 \theta_y}{\partial y^3} \right)$$

$$+ \mathcal{B}_{11}^{(2)} \left[\frac{\partial^4 \theta_z}{\partial x^4} + \frac{\partial^4 \theta_z}{\partial y^4} - 2 \left(\frac{\partial^3 \phi_x}{\partial x^3} + \frac{\partial^3 \phi_y}{\partial y^3} \right) \right]$$

$$+ \mathcal{B}_{11}^{(3)} \left[\frac{\partial^4 \phi_z}{\partial x^4} + \frac{\partial^4 \phi_z}{\partial y^4} - 3 \left(\frac{\partial^3 \psi_x}{\partial x^3} + \frac{\partial^3 \psi_y}{\partial y^3} \right) \right] \Bigg\}$$

$$+ \frac{1}{4} \Bigg\{ \mathcal{B}_{11}^{(0)} \left[\frac{\partial^2 \theta_z}{\partial y^2} + \frac{\partial^2 \theta_z}{\partial x^2} - 2 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) \right]$$

$$+ \mathcal{B}_{11}^{(1)} \left[2 \left(\frac{\partial^2 \phi_z}{\partial_x^2} + \frac{\partial^2 \phi_z}{\partial_y^2} \right) - 6 \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) \right] \Bigg\}$$

$$+ F_z^{(1)} + \frac{1}{2} \left(\frac{\partial C_y^{(1)}}{\partial x} + \frac{\partial C_x^{(1)}}{\partial y} \right) = m_1 \ddot{w} + m_2 \ddot{\theta}_z + m_3 \ddot{\phi}_z \tag{3.26}$$

$$\delta\phi_x : \quad A_{11}^{(2)} \frac{\partial^2 u}{\partial x^2} + A_{12}^{(2)} \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial \theta_z}{\partial x} \right) + A_{11}^{(3)} \frac{\partial^2 \theta_x}{\partial x^2} + A_{12}^{(3)} \left(\frac{\partial^2 \theta_y}{\partial x \partial y} + 2 \frac{\partial \phi_z}{\partial x} \right)$$

$$+ A_{11}^{(4)} \frac{\partial^2 \phi_x}{\partial x^2} + A_{12}^{(4)} \frac{\partial^2 \phi_y}{\partial x \partial y} + A_{11}^{(5)} \frac{\partial^2 \psi_x}{\partial x^2} + A_{12}^{(5)} \frac{\partial^2 \psi_y}{\partial x \partial y}$$

$$+ B_{11}^{(2)} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + B_{11}^{(3)} \left(\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial x \partial y} \right)$$

$$
\begin{aligned}
&+ B_{11}^{(4)}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)+B_{11}^{(5)}\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}\right)\\
&-2\left[B_{11}^{(1)}\left(\theta_x+\frac{\partial w}{\partial x}\right)+B_{11}^{(2)}\left(2\phi_x+\frac{\partial\theta_z}{\partial x}\right)+B_{11}^{(3)}\left(3\psi_x+\frac{\partial\phi_z}{\partial x}\right)\right]\\
&+\frac{1}{4}\left\{\mathcal{B}_{11}^{(2)}\left[\frac{\partial^4 v}{\partial x^3\partial y}-\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 v}{\partial x\partial y^3}-\frac{\partial^4 u}{\partial y^4}-2\left(\frac{\partial^2\phi_y}{\partial x\partial y}-\frac{\partial\phi_x}{\partial y^2}\right)\right]\right.\\
&+\mathcal{B}_{11}^{(3)}\left[\frac{\partial^4\theta_y}{\partial x^3\partial y}-\frac{\partial^4\theta_x}{\partial x^2\partial y^2}+\frac{\partial^4\theta_y}{\partial x\partial y^3}-\frac{\partial^4\theta_x}{\partial y^4}-6\left(\frac{\partial^2\psi_y}{\partial x\partial y}-\frac{\partial^2\psi_x}{\partial y^2}\right)\right]\\
&+\mathcal{B}_{11}^{(4)}\left(\frac{\partial^4\phi_y}{\partial x^3\partial y}-\frac{\partial^4\phi_x}{\partial x^2\partial y^2}+\frac{\partial^4\phi_y}{\partial x\partial y^3}-\frac{\partial^4\phi_x}{\partial y^4}\right)\\
&\left.+\mathcal{B}_{11}^{(5)}\left(\frac{\partial^4\psi_y}{\partial x^3\partial y}-\frac{\partial^4\psi_x}{\partial x^2\partial y^2}+\frac{\partial^4\psi_y}{\partial x\partial y^3}-\frac{\partial^4\psi_x}{\partial y^4}\right)\right\}\\
&-\frac{1}{2}\left\{\mathcal{B}_{11}^{(1)}\left[\frac{\partial^3 w}{\partial x^3}-\frac{\partial^2\theta_x}{\partial x^2}-2\left(\frac{\partial^2\theta_x}{\partial y^2}-\frac{\partial^2\theta_y}{\partial x\partial y}\right)\right]\right.\\
&+\mathcal{B}_{11}^{(2)}\left[\frac{\partial^3\theta_z}{\partial x^3}-2\frac{\partial^2\phi_x}{\partial x^2}-4\left(\frac{\partial^2\phi_x}{\partial y^2}-\frac{\partial^2\phi_y}{\partial x\partial y}\right)\right]\\
&\left.+\mathcal{B}_{11}^{(3)}\left[\frac{\partial^3\phi_z}{\partial x^3}-3\frac{\partial^2\psi_x}{\partial x^2}-6\left(\frac{\partial^2\psi_y}{\partial x\partial y}-\frac{\partial^2\psi_x}{\partial y^2}\right)\right]\right\}\\
&-\frac{1}{2}\left[\mathcal{B}_{11}^{(0)}\left(\frac{\partial^2 v}{\partial y\partial x}-\frac{\partial^2 u}{\partial y^2}+2\phi_x-\frac{\partial\theta_z}{\partial x}\right)\right.\\
&+\mathcal{B}_{11}^{(1)}\left(\frac{\partial^2\theta_y}{\partial x\partial y}-\frac{\partial^2\theta_x}{\partial y^2}+6\psi_x-2\frac{\partial\phi_z}{\partial_x}\right)\\
&\left.+\mathcal{B}_{11}^{(2)}\left(\frac{\partial^2\phi_y}{\partial x\partial y}-\frac{\partial^2\phi_x}{\partial y^2}\right)+\mathcal{B}_{11}^{(3)}\left(\frac{\partial^2\psi_y}{\partial x\partial y}-\frac{\partial^2\psi_x}{\partial y^2}\right)\right]\\
&+F_x^{(2)}+\frac{1}{2}C_y^{(1)}+\frac{1}{2}\frac{\partial C_z^{(2)}}{\partial y}=m_2\ddot{u}+m_3\ddot{\theta}_x+m_4\ddot{\phi}_x+m_5\ddot{\psi}_x \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
\delta\phi_y:\quad &A_{11}^{(2)}\frac{\partial^2 v}{\partial y^2}+A_{12}^{(2)}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial\theta_z}{\partial y}\right)+A_{11}^{(3)}\frac{\partial^2\theta_y}{\partial y^2}+A_{12}^{(3)}\left(\frac{\partial^2\theta_x}{\partial x\partial y}+2\frac{\partial\phi_z}{\partial y}\right)\\
&+A_{11}^{(4)}\frac{\partial^2\phi_y}{\partial y^2}+A_{12}^{(4)}\frac{\partial^2\phi_x}{\partial x\partial y}+A_{11}^{(5)}\frac{\partial^2\psi_y}{\partial y^2}+A_{12}^{(5)}\frac{\partial^2\psi_x}{\partial x\partial y}\\
&+B_{11}^{(2)}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^v}{\partial x^2}\right)+B_{11}^{(3)}\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)\\
&+B_{11}^{(4)}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)+B_{11}^{(5)}\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^2\psi_y}{\partial x^2}\right)\\
&-2\left[B_{11}^{(1)}\left(\theta_y+\frac{\partial w}{\partial y}\right)+B_{11}^{(2)}\left(2\phi_y+\frac{\partial\theta_z}{\partial y}\right)+B_{11}^{(3)}\left(3\psi_y+\frac{\partial\phi_z}{\partial y}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{4}\Bigg\{\mathcal{B}_{11}^{(2)}\left[\frac{\partial^4 v}{\partial x^4} - \frac{\partial^4 u}{\partial x^2 \partial y} + \frac{\partial^4 v}{\partial x^2 \partial y^2} - \frac{\partial^4 u}{\partial x \partial y^3} - 2\left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y}\right)\right] \\
&+ \mathcal{B}_{11}^{(3)}\left[\frac{\partial^4 \theta_y}{\partial x^4} - \frac{\partial^4 \theta_x}{\partial x^2 \partial y} + \frac{\partial^4 \theta_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \theta_x}{\partial x \partial y^3} - 6\left(\frac{\partial^2 \psi_y}{\partial x^2} - \frac{\partial^2 \psi_x}{\partial x \partial y}\right)\right] \\
&+ \mathcal{B}_{11}^{(4)}\left(\frac{\partial^4 \phi_y}{\partial x^4} - \frac{\partial^4 \phi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \phi_x}{\partial x \partial y^3}\right) \\
&+ \mathcal{B}_{11}^{(5)}\left(\frac{\partial^4 \psi_y}{\partial x^4} - \frac{\partial^4 \psi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \psi_x}{\partial x \partial y^3}\right)\Bigg\} \\
&+ \frac{1}{2}\Bigg\{\mathcal{B}_{11}^{(1)}\left[\frac{\partial^3 w}{\partial y^3} - \frac{\partial^2 \theta_y}{\partial y^2} - 2\left(\frac{\partial^2 \theta_y}{\partial x^2} - \frac{\partial^2 \theta_x}{\partial x \partial y}\right)\right] \\
&+ \mathcal{B}_{11}^{(2)}\left[\frac{\partial^3 \theta_z}{\partial y^3} - 2\frac{\partial^2 \phi_y}{\partial y^2} - 4\left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y}\right)\right] \\
&+ \mathcal{B}_{11}^{(3)}\left[\frac{\partial^3 \phi_z}{\partial y^3} - 3\frac{\partial^2 \psi_y}{\partial x^2} - 6\left(\frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial^2 \psi_x}{\partial x \partial y}\right)\right]\Bigg\} \\
&+ \frac{1}{2}\Bigg[\mathcal{B}_{11}^{(0)}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial \theta_z}{\partial y} - 2\phi_y\right) \\
&+ \mathcal{B}_{11}^{(1)}\left(\frac{\partial^2 \theta_y}{\partial x^2} - \frac{\partial^2 \theta_x}{\partial x \partial y} + 2\frac{\partial \phi_z}{\partial_y} - 6\psi_y\right) \\
&+ \mathcal{B}_{11}^{(2)}\left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y}\right) + \mathcal{B}_{11}^{(3)}\left(\frac{\partial^2 \psi_y}{\partial x^2} - \frac{\partial^2 \psi_x}{\partial x \partial y}\right)\Bigg] \\
&+ F_y^{(2)} - \frac{1}{2}c_x(1) - \frac{1}{2}\frac{\partial c_z^{(2)}}{\partial x} = m_2\ddot{v} + m_3\ddot{\theta}_y + m_4\ddot{\phi}_y + m_5\ddot{\psi}_y \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
\delta\phi_z: \quad & B_{11}^{(2)}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y}\right) - 2\left[A_{11}^{(1)}\theta_z + A_{12}^{(1)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \\
&+ 2B_{11}^{(3)}\left(\frac{\partial^2 \theta_z}{\partial x^2} + \frac{\partial^2 \theta_z}{\partial y^2} + \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) + 4A_{11}^{(2)}\phi_z \\
&+ 3B_{11}^{(4)}\left(\frac{\partial^2 \phi_z}{\partial x^2} + \frac{\partial^2 \phi_z}{\partial y^2} + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) + 2A_{12}^{(2)}\left(\frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y}\right) \\
&+ 2A_{12}^{(3)}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) + A_{12}^{(4)}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) \\
&- \frac{1}{4}\Bigg\{\mathcal{B}_{11}^{(2)}\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} - \frac{\partial^3 \theta_x}{\partial x^3} - \frac{\partial^3 \theta_y}{\partial y^3}\right) \\
&+ \mathcal{B}_{11}^{(3)}\left[\frac{\partial^4 \theta_z}{\partial x^4} + \frac{\partial^4 \theta_z}{\partial y^4} - 2\left(\frac{\partial^3 \phi_x}{\partial x^3} + \frac{\partial^3 \phi_y}{\partial y^3}\right)\right] \\
&+ \mathcal{B}_{11}^{(4)}\left[\frac{\partial^4 \phi_z}{\partial x^4} + \frac{\partial^4 \phi_z}{\partial y^4} - 3\left(\frac{\partial^3 \psi_x}{\partial x^3} + \frac{\partial^3 \psi_y}{\partial y^3}\right)\right]\Bigg\}
\end{aligned}
$$

$$+ \frac{1}{2}\left\{\mathcal{B}_{11}^{(1)}\left[\frac{\partial^2\theta_z}{\partial y^2} + \frac{\partial^2\theta_z}{\partial x^2} - 2\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right)\right]\right.$$

$$\left. + \mathcal{B}_{11}^{(2)}\left[2\left(\frac{\partial^2\phi_z}{\partial_x^2} + \frac{\partial^2\phi_z}{\partial_y^2}\right) - 6\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\right)\right]\right\}$$

$$+ F_z^{(2)} + \frac{1}{2}\left(\frac{\partial C_y^{(2)}}{\partial x} + \frac{\partial C_x^{(2)}}{\partial y}\right) = m_2\ddot{w} + m_3\ddot{\theta}_z + m_4\ddot{\phi}_z \tag{3.29}$$

$$\delta\psi_x: \quad A_{11}^{(3)}\frac{\partial^2 u}{\partial x^2} + A_{12}^{(3)}\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial\theta_z}{\partial x}\right) + A_{11}^{(4)}\frac{\partial^2\theta_x}{\partial x^2} + A_{12}^{(4)}\left(\frac{\partial^2\theta_y}{\partial x\partial y} + 2\frac{\partial\phi_z}{\partial x}\right)$$

$$+ A_{11}^{(5)}\frac{\partial^2\phi_x}{\partial x^2} + A_{12}^{(5)}\frac{\partial^2\phi_y}{\partial x\partial y} + A_{11}^{(6)}\frac{\partial^2\psi_x}{\partial x^2} + A_{12}^{(6)}\frac{\partial^2\psi_y}{\partial x\partial y}$$

$$+ B_{11}^{(3)}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}\right) + B_{11}^{(4)}\left(\frac{\partial^2\theta_x}{\partial y^2} + \frac{\partial^2\theta_y}{\partial x\partial y}\right)$$

$$+ B_{11}^{(5)}\left(\frac{\partial^2\phi_x}{\partial y^2} + \frac{\partial^2\phi_y}{\partial x\partial y}\right) + B_{11}^{(6)}\left(\frac{\partial^2\psi_x}{\partial y^2} + \frac{\partial^2\psi_y}{\partial x\partial y}\right)$$

$$- 3\left[B_{11}^{(2)}\left(\theta_x + \frac{\partial w}{\partial x}\right) + B_{11}^{(3)}\left(2\phi_x + \frac{\partial\theta_z}{\partial x}\right) + B_{11}^{(4)}\left(3\psi_x + \frac{\partial\phi_z}{\partial x}\right)\right]$$

$$+ \frac{1}{4}\left\{\mathcal{B}_{11}^{(3)}\left[\frac{\partial^4 v}{\partial x^3\partial y} - \frac{\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 v}{\partial x\partial y^3} - \frac{\partial^4 u}{\partial y^4} - 2\left(\frac{\partial^2\phi_y}{\partial x\partial y} - \frac{\partial\phi_x}{\partial y^2}\right)\right]\right.$$

$$+ \mathcal{B}_{11}^{(4)}\left[\frac{\partial^4\theta_y}{\partial x^3\partial y} - \frac{\partial^4\theta_x}{\partial x^2\partial y^2} + \frac{\partial^4\theta_y}{\partial x\partial y^3} - \frac{\partial^4\theta_x}{\partial y^4} - 6\left(\frac{\partial^2\psi_y}{\partial x\partial y} - \frac{\partial^2\psi_x}{\partial y^2}\right)\right]$$

$$+ \mathcal{B}_{11}^{(5)}\left(\frac{\partial^4\phi_y}{\partial x^3\partial y} - \frac{\partial^4\phi_x}{\partial x^2\partial y^2} + \frac{\partial^4\phi_y}{\partial x\partial y^3} - \frac{\partial^4\phi_x}{\partial y^4}\right)$$

$$\left. + \mathcal{B}_{11}^{(6)}\left(\frac{\partial^4\psi_y}{\partial x^3\partial y} - \frac{\partial^4\psi_x}{\partial x^2\partial y^2} + \frac{\partial^4\psi_y}{\partial x\partial y^3} - \frac{\partial^4\psi_x}{\partial y^4}\right)\right\}$$

$$- \frac{3}{4}\left\{\mathcal{B}_{11}^{(2)}\left[\frac{\partial^3 w}{\partial x^3} - \frac{\partial^2\theta_x}{\partial x^2} - 2\left(\frac{\partial^2\theta_x}{\partial y^2} - \frac{\partial^2\theta_y}{\partial x\partial y}\right)\right]\right.$$

$$+ \mathcal{B}_{11}^{(3)}\left[\frac{\partial^3\theta_z}{\partial x^3} - 2\frac{\partial^2\phi_x}{\partial x^2} - 4\left(\frac{\partial^2\phi_x}{\partial y^2} - \frac{\partial^2\phi_y}{\partial x\partial y}\right)\right]$$

$$\left. + \mathcal{B}_{11}^{(4)}\left[\frac{\partial^3\phi_z}{\partial x^3} - 3\frac{\partial^2\psi_x}{\partial x^2} - 6\left(\frac{\partial^2\psi_y}{\partial x\partial y} - \frac{\partial^2\psi_x}{\partial y^2}\right)\right]\right\}$$

$$- \frac{3}{2}\left[\mathcal{B}_{11}^{(1)}\left(\frac{\partial^2 v}{\partial y\partial x} - \frac{\partial^2 u}{\partial y^2} + 2\phi_x - \frac{\partial\theta_z}{\partial x}\right)\right.$$

$$+ \mathcal{B}_{11}^{(2)}\left(\frac{\partial^2\theta_y}{\partial x\partial y} - \frac{\partial^2\theta_x}{\partial y^2} + 6\psi_x - 2\frac{\partial\phi_z}{\partial_x}\right)$$

$$\left. + \mathcal{B}_{11}^{(3)}\left(\frac{\partial^2\phi_y}{\partial x\partial y} - \frac{\partial^2\phi_x}{\partial y^2}\right) + \mathcal{B}_{11}^{(4)}\left(\frac{\partial^2\psi_y}{\partial x\partial y} - \frac{\partial^2\psi_x}{\partial y^2}\right)\right]$$

$$+ F_x^{(3)} + \frac{3}{2} C_y^{(2)} + \frac{1}{2} \frac{\partial C_z^{(3))}}{\partial y} = m_3 \ddot{u} + m_4 \ddot{\theta}_x + m_5 \ddot{\phi}_x + m_6 \ddot{\psi}_x \tag{3.30}$$

$$\begin{aligned}
\delta\psi_y : \quad & A_{11}^{(3)} \frac{\partial^2 v}{\partial y^2} + A_{12}^{(3)} \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial \theta_z}{\partial y} \right) + A_{11}^{(4)} \frac{\partial^2 \theta_y}{\partial y^2} + A_{12}^{(4)} \left(\frac{\partial^2 \theta_x}{\partial x \partial y} + 2 \frac{\partial \phi_z}{\partial y} \right) \\
& + A_{11}^{(5)} \frac{\partial^2 \phi_y}{\partial y^2} + A_{12}^{(5)} \frac{\partial^2 \phi_x}{\partial x \partial y} + A_{11}^{(6)} \frac{\partial^2 \psi_y}{\partial y^2} + A_{12}^{(6)} \frac{\partial^2 \psi_x}{\partial x \partial y} \\
& + B_{11}^{(3)} \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^v}{\partial x^2} \right) + B_{11}^{(4)} \left(\frac{\partial^2 \theta_x}{\partial x \partial y} + \frac{\partial^2 \theta_y}{\partial x^2} \right) \\
& + B_{11}^{(5)} \left(\frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial x^2} \right) + B_{11}^{(6)} \left(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2} \right) \\
& - 3 \left[B_{11}^{(2)} \left(\theta_y + \frac{\partial w}{\partial y} \right) + B_{11}^{(3)} \left(2\phi_y + \frac{\partial \theta_z}{\partial y} \right) + B_{11}^{(4)} \left(3\psi_y + \frac{\partial \phi_z}{\partial y} \right) \right] \\
& - \frac{1}{4} \left\{ \mathcal{B}_{11}^{(3)} \left[\frac{\partial^4 v}{\partial x^4} - \frac{\partial^4 u}{\partial x^2 \partial y} + \frac{\partial^4 v}{\partial x^2 \partial y^2} - \frac{\partial^4 u}{\partial x \partial y^3} - 2 \left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y} \right) \right] \right. \\
& + \mathcal{B}_{11}^{(4)} \left[\frac{\partial^4 \theta_y}{\partial x^4} - \frac{\partial^4 \theta_x}{\partial x^2 \partial y} + \frac{\partial^4 \theta_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \theta_x}{\partial x \partial y^3} - 6 \left(\frac{\partial^2 \psi_y}{\partial x^2} - \frac{\partial^2 \psi_x}{\partial x \partial y} \right) \right] \\
& + \mathcal{B}_{11}^{(5)} \left(\frac{\partial^4 \phi_y}{\partial x^4} - \frac{\partial^4 \phi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \phi_x}{\partial x \partial y^3} \right) \\
& \left. + \mathcal{B}_{11}^{(6)} \left(\frac{\partial^4 \psi_y}{\partial x^4} - \frac{\partial^4 \psi_x}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_y}{\partial x^2 \partial y^2} - \frac{\partial^4 \psi_x}{\partial x \partial y^3} \right) \right\} \\
& + \frac{3}{4} \left\{ \mathcal{B}_{11}^{(2)} \left[\frac{\partial^3 w}{\partial y^3} - \frac{\partial^2 \theta_y}{\partial y^2} - 2 \left(\frac{\partial^2 \theta_y}{\partial x^2} - \frac{\partial^2 \theta_x}{\partial x \partial y} \right) \right] \right. \\
& + \mathcal{B}_{11}^{(3)} \left[\frac{\partial^3 \theta_z}{\partial y^3} - 2 \frac{\partial^2 \phi_y}{\partial y^2} - 4 \left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y} \right) \right] \\
& \left. + \mathcal{B}_{11}^{(4)} \left[\frac{\partial^3 \phi_z}{\partial y^3} - 3 \frac{\partial^2 \psi_y}{\partial x^2} - 6 \left(\frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial^2 \psi_x}{\partial x \partial y} \right) \right] \right\} \\
& + \frac{3}{2} \left[\mathcal{B}_{11}^{(1)} \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial \theta_z}{\partial y} - 2\phi_y \right) \right. \\
& + \mathcal{B}_{11}^{(2)} \left(\frac{\partial^2 \theta_y}{\partial x^2} - \frac{\partial^2 \theta_x}{\partial x \partial y} + 2 \frac{\partial \phi_z}{\partial_y} - 6\psi_y \right) \\
& \left. + \mathcal{B}_{11}^{(3)} \left(\frac{\partial^2 \phi_y}{\partial x^2} - \frac{\partial^2 \phi_x}{\partial x \partial y} \right) + \mathcal{B}_{11}^{(4)} \left(\frac{\partial^2 \psi_y}{\partial x^2} - \frac{\partial^2 \psi_x}{\partial x \partial y} \right) \right] \\
& + F_y^{(3)} - \frac{1}{2} C_x(2) - \frac{1}{2} \frac{\partial C_z^{(3)}}{\partial x} = m_3 \ddot{v} + m_4 \ddot{\theta}_y + m_5 \ddot{\phi}_y + m_6 \ddot{\psi}_y.
\end{aligned} \tag{3.31}$$

3.3 The Navier solution

Analytical solutions for a simply supported rectangular FGM plate is obtained using Navier solution technique. The dependent unknowns(generalized displacements): u, v, w, θ_x, θ_y, θ_z, ϕ_x, ϕ_y, ϕ_z, ψ_x, ψ_y and external load q_z are expanded in double trigonometric series. The trigonometric functions are selected according to the boundary conditions of problems. To derive algebraic relation in terms of generalized displacements and known coefficients, the expansions of dependent unknowns and the external load are substituted into equation of motions (3.21) to (3.31). The external load q_z is only defined for the bending analysis and other external forces and body forces are omitted. For eigenvalue problems, free vibration and buckling problems, q_z is also omitted from the equation of motion.

3.3.1 Boundary conditions

Figure 3.2 shows the geometry a rectangular plate. a and b denote the in-plane dimensions along x and y coordinate and h denotes the height of the plate.

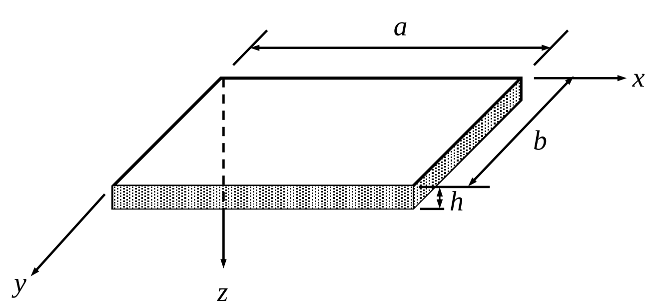

Figure 3.2: The geometry and coordinate system for a rectangular plate [2]

The boundary conditions of a simply supported rectangular plate can be expressed as

$$u(x,0) = u(x,b) = 0, \qquad \theta_x(x,0) = \theta_x(x,b) = 0,$$

$$\phi_x(x,0) = \phi_x(x,b) = 0, \qquad \psi_x(x,0) = \psi_x(x,b) = 0,$$

$$v(0,y) = v(a,y) = 0, \qquad \theta_y(0,y) = \theta_y(a,y) = 0,$$

$$\phi_y(0,y) = \phi_y(a,y) = 0, \qquad \psi_y(0,y) = \psi_y(a,y) = 0,$$

$$w(x,0) = w(x,b) = 0 \qquad w(0,y) = w(a,y) = 0,$$

$$\theta_z(x,0) = \theta_z(x,b) = 0, \qquad \theta_z(0,y) = \theta_z(a,y) = 0,$$

$$\phi_z(x,0) = \phi_z(x,b) = 0, \qquad \phi_z(0,y) = \phi_z(x,b) = 0,$$

$$M_{xx}^{(i)}(0,y) = M_{xx}^{(i)}(a,y) = 0 \qquad M_{yy}^{(i)}(x,0) = M_{yy}^{(i)}(x,b) = 0,$$

$$\mathcal{M}_{xy}^{(j)}(0,y) = \mathcal{M}_{xy}^{(j)}(a,y) = 0, \qquad \mathcal{M}_{xy}^{(j)}(x,0) = \mathcal{M}_{xy}^{(j)}(x,b) = 0$$

where i =0, 1, 2, 3 and j =0, 1, 2.

3.3.2 Expansions of displacements

The displacements are assumed as the series of double trigonometric functions that satisfy boundary conditions in section 3.3.1.

$$u(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Umn(t)\, cos(\alpha x)\, sin(\beta y) \tag{3.32}$$

$$v(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Vmn(t)\, sin(\alpha x)\, cos(\beta y) \tag{3.33}$$

$$w(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Wmn(t)\, sin(\alpha x)\, sin(\beta y) \tag{3.34}$$

$$\theta_x(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Theta_x mn(t)\, cos(\alpha x)\, sin(\beta y) \tag{3.35}$$

$$\theta_y(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Theta_y mn(t)\, sin(\alpha x)\, cos(\beta y) \tag{3.36}$$

$$\theta_z(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Theta_z mn(t)\,sin(\alpha x)\,sin(\beta y) \tag{3.37}$$

$$\phi_x(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_x mn(t)\,cos(\alpha x)\,sin(\beta y) \tag{3.38}$$

$$\phi_y(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_y mn(t)\,sin(\alpha x)\,cos(\beta y) \tag{3.39}$$

$$\phi_z(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_z mn(t)\,sin(\alpha x)\,sin(\beta y) \tag{3.40}$$

$$\psi_x(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_x mn(t)\,cos(\alpha x)\,sin(\beta y) \tag{3.41}$$

$$\psi_y(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_y mn(t)\,sin(\alpha x)\,cos(\beta y) \tag{3.42}$$

where $\alpha=\frac{m\pi}{a}$, $\beta=\frac{n\pi}{b}$. The generalized displacement coefficients $(Umn$, Vmn, Wmn, $\Theta_x mn$, $\Theta_y mn$, $\Theta_z mn$, $\Phi_x mn$, $\Phi_y mn$, $\Phi_z mn$, $\Psi_x mn$, and $\Psi_y)$ are treated as time independent variables for static bending and buckling problems.

3.3.3 Bending analysis

By substituting Eqs. (3.32) to (3.42) into Eqs. (3.21)-(3.31) and omitting the inertia terms, we have algebraic relations for bending problems in terms of the known coefficient matrix and load vector and the unknown generalized displacement vector. The distributed transverse load is also expanded in double trigonometric series in Eq. (3.43).

$$q_z=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Qmn(t)\,sin(\alpha x)\,sin(\beta y) \tag{3.43}$$

where

$$Qmn=\frac{4}{ab}\int_0^a\int_0^b q_z sin(\alpha x)\,sin(\beta y)\,dxdy. \tag{3.44}$$

The algebraic equations are

$$\begin{bmatrix} C_{0101} & C_{0102} & C_{0103} & \cdots & C_{0110} & C_{0111} \\ C_{0201} & C_{0202} & C_{0203} & \cdots & C_{0210} & C_{0211} \\ C_{0301} & C_{0302} & C_{0303} & \cdots & C_{0310} & C_{0311} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1001} & C_{1002} & C_{1003} & \cdots & C_{1010} & C_{1011} \\ C_{1101} & C_{1102} & C_{1103} & \cdots & C_{1110} & C_{1111} \end{bmatrix} \begin{Bmatrix} Umn \\ Vmn \\ Wmn \\ \Theta_x mn \\ \Theta_y mn \\ \Theta_z mn \\ \Phi_x mn \\ \Phi_y mn \\ \Phi_z mn \\ \Psi_x mn \\ \Psi_y mn \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Qmn \\ 0 \\ 0 \\ Qmn\frac{h}{2} \\ 0 \\ 0 \\ Qmn\frac{h^2}{4} \\ 0 \\ 0 \end{Bmatrix} \tag{3.45}$$

and to distinguish i=1 and j=11 from i=11 and j=1, the subscript i and j vary 01 to 11. The known coefficients C_{ij} in Eq. (3.45) are following

$$C_{0101} = A_{11}^{(0)}\alpha^2 + B_{11}^{(0)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(0)}\left(\beta^4 + \alpha^2\beta^2\right)$$

$$C_{0102} = A_{12}^{(0)}\alpha\beta + B_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(0)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0103} = 0$$

$$C_{0104} = A_{11}^{(1)}\alpha^2 + B_{11}^{(1)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0105} = A_{12}^{(1)}\alpha\beta + B_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0106} = -A_{12}^{(0)}\alpha$$

$$C_{0107} = A_{11}^{(2)}\alpha^2 + B_{11}^{(2)}\beta^2 + \frac{1}{2}\mathcal{B}_{11}^{(0)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0108} = A_{12}^{(2)}\alpha\beta + B_{11}^{(2)}\alpha\beta - \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0109} = -2A_{12}^{(1)}\alpha$$

$$C_{0110} = A_{11}^{(3)}\alpha^2 + B_{11}^{(3)}\beta^2 + \frac{3}{2}\mathcal{B}_{11}^{(1)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0111} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0201} = A_{12}^{(0)}\alpha\beta + B_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(0)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0202} = A_{11}^{(0)}\beta^2 + B_{11}^{(0)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(0)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0203} = 0$$

$$C_{0204} = A_{12}^{(1)}\alpha\beta + B_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0205} = A_{11}^{(1)}\beta^2 + B_{11}^{(1)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0206} = -A_{12}^{(0)}\beta$$

$$C_{0207} = A_{12}^{(2)}\alpha\beta + B_{11}^{(2)}\alpha\beta - \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0208} = A_{11}^{(2)}\beta^2 + B_{11}^{(2)}\alpha^2 + \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0209} = -2A_{12}^{(1)}\beta$$

$$C_{0210} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0211} = A_{11}^{(3)}\beta^2 + B_{11}^{(3)}\alpha^2 + \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0301} = 0$$

$$C_{0302} = 0$$

$$C_{0303} = B_{11}^{(0)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(0)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0304} = B_{11}^{(0)}\alpha - \frac{1}{4}\mathcal{B}_{11}^{(0)}\alpha^3$$

$$C_{0305} = B_{11}^{(0)}\beta - \frac{1}{4}\mathcal{B}_{11}^{(0)}\beta^3$$

$$C_{0306} = B_{11}^{(1)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0307} = 2B_{11}^{(1)}\alpha - \frac{1}{2}\mathcal{B}_{11}^{(1)}\alpha^3$$

$$C_{0308} = 2B_{11}^{(1)}\beta - \frac{1}{2}\mathcal{B}_{11}^{(1)}\beta^3$$

$$C_{0309} = B_{11}^{(2)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0310} = 3B_{11}^{(2)}\alpha - \frac{3}{4}\mathcal{B}_{11}^{(2)}\alpha^3$$

$$C_{0311} = 3B_{11}^{(2)}\beta - \frac{3}{4}\mathcal{B}_{11}^{(2)}\beta^3$$

$$C_{0401} = A_{11}^{(1)}\alpha^2 + B_{11}^{(1)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0402} = A_{12}^{(1)}\alpha\beta + B_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0403} = B_{11}^{(0)}\alpha - \frac{1}{4}\mathcal{B}_{11}^{(0)}\alpha^3$$

$$C_{0404} = A_{11}^{(2)}\alpha^2 + B_{11}^{(0)} + B_{11}^{(2)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(0)}\left(\alpha^2 + 2\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\beta^4 + \alpha^2\beta^2\right)$$

$$C_{0405} = A_{12}^{(2)}\alpha\beta + B_{11}^{(2)}\alpha\beta - \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0406} = -A_{12}^{(1)}\alpha + B_{11}^{(1)}\alpha - \frac{1}{4}\mathcal{B}_{11}^{(1)}\alpha^3$$

$$C_{0407} = A_{11}^{(3)}\alpha^2 + 2B_{11}^{(1)} + B_{11}^{(3)}\beta^2 + \frac{1}{2}\mathcal{B}_{11}^{(1)}\left(\alpha^2 + 3\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0408} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0409} = -2A_{12}^{(2)}\alpha + B_{11}^{(2)}\alpha - \frac{1}{4}\mathcal{B}_{11}^{(2)}\alpha^3$$

$$C_{0410} = A_{11}^{(4)}\alpha^2 + 3B_{11}^{(2)} + B_{11}^{(4)}\beta^2 + \frac{3}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^2 + 4\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0411} = A_{12}^{(4)}\alpha\beta + B_{11}^{(4)}\alpha\beta - 3\mathcal{B}_{11}^{(2)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0501} = A_{12}^{(1)}\alpha\beta + B_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0502} = A_{11}^{(1)}\beta^2 + B_{11}^{(1)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0503} = B_{11}^{(0)}\beta - \frac{1}{4}\mathcal{B}_{11}^{(0)}\beta^3$$

$$C_{0504} = A_{12}^{(2)}\alpha\beta + B_{11}^{(2)}\alpha\beta - \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0505} = A_{11}^{(2)}\beta^2 + B_{11}^{(0)} + B_{11}^{(2)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(0)}\left(2\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0506} = -A_{12}^{(1)}\beta + B_{11}^{(1)}\beta - \frac{1}{4}\mathcal{B}_{11}^{(1)}\beta^3$$

$$C_{0507} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0508} = A_{11}^{(3)}\beta^2 + 2B_{11}^{(1)} + B_{11}^{(3)}\alpha^2 + \frac{1}{2}\mathcal{B}_{11}^{(1)}\left(3\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0509} = -2A_{12}^{(2)}\beta + B_{11}^{(2)}\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\beta^3$$

$$C_{0510} = A_{12}^{(4)}\alpha\beta + B_{11}^{(4)}\alpha\beta - 3\mathcal{B}_{11}^{(2)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0511} = A_{11}^{(4)}\beta^2 + 3B_{11}^{(2)} + B_{11}^{(4)}\alpha^2 + \frac{3}{4}\mathcal{B}_{11}^{(2)}\left(4\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0601} = -A_{12}^{(0)}\alpha$$

$$C_{0602} = -A_{12}^{(0)}\beta$$

$$C_{0603} = B_{11}^{(1)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(1)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0604} = -A_{12}^{(1)}\alpha + B_{11}^{(1)}\alpha - \frac{1}{4}\mathcal{B}_{11}^{(1)}\alpha^3$$

$$C_{0605} = -A_{12}^{(1)}\beta + B_{11}^{(1)}\beta - \frac{1}{4}\mathcal{B}_{11}^{(1)}\beta^3$$

$$C_{0606} = A_{11}^{(0)} + B_{11}^{(2)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0607} = -A_{12}^{(2)}\alpha + 2B_{11}^{(2)}\alpha - \frac{1}{2}\mathcal{B}_{11}^{(2)}\alpha^3$$

$$C_{0608} = -A_{12}^{(2)}\beta + 2B_{11}^{(2)}\beta - \frac{1}{2}\mathcal{B}_{11}^{(2)}\beta^3$$

$$C_{0609} = 2A_{11}^{(1)} + B_{11}^{(3)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0610} = -A_{12}^{(3)}\alpha + 3B_{11}^{(3)}\alpha - \frac{3}{4}\mathcal{B}_{11}^{(3)}\alpha^3$$

$$C_{0611} = -A_{12}^{(3)}\beta + 3B_{11}^{(3)}\beta - \frac{3}{4}\mathcal{B}_{11}^{(3)}\beta^3$$

$$C_{0701} = A_{11}^{(2)}\alpha^2 + B_{11}^{(2)}\beta^2 + \frac{1}{2}\mathcal{B}_{11}^{(0)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0702} = A_{12}^{(2)}\alpha\beta + B_{11}^{(2)}\alpha\beta - \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0703} = 2B_{11}^{(1)}\alpha - \frac{1}{2}\mathcal{B}_{11}^{(1)}\alpha^3$$

$$C_{0704} = A_{11}^{(3)}\alpha^2 + 2B_{11}^{(1)} + B_{11}^{(3)}\beta^2 + \frac{1}{2}\mathcal{B}_{11}^{(1)}\left(\alpha^2 + 3\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0705} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0706} = -A_{12}^{(2)}\alpha + 2B_{11}^{(2)}\alpha - \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha - \frac{1}{2}\mathcal{B}_{11}^{(2)}\alpha^3$$

$$C_{0707} = A_{11}^{(4)}\alpha^2 + 4B_{11}^{(2)} + B_{11}^{(4)}\beta^2 + \mathcal{B}_{11}^{(0)} + \mathcal{B}_{11}^{(2)}\left(\alpha^2 + 3\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0708} = A_{12}^{(4)}\alpha\beta + B_{11}^{(4)}\alpha\beta - 3\mathcal{B}_{11}^{(2)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0709} = -2A_{12}^{(3)}\alpha + 2B_{11}^{(3)}\alpha - \mathcal{B}_{11}^{(1)}\alpha - \frac{1}{2}\mathcal{B}_{11}^{(3)}\alpha^3$$

$$C_{0710} = A_{11}^{(5)}\alpha^2 + 6B_{11}^{(3)} + B_{11}^{(5)}\beta^2 + 3\mathcal{B}_{11}^{(1)} + \frac{1}{2}\mathcal{B}_{11}^{(3)}\left(3\alpha^2 + 10\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{0711} = A_{12}^{(5)}\alpha\beta + B_{11}^{(5)}\alpha\beta - 5\mathcal{B}_{11}^{(3)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0801} = A_{12}^{(2)}\alpha\beta + B_{11}^{(2)}\alpha\beta - \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0802} = A_{11}^{(2)}\beta^2 + B_{11}^{(2)}\alpha^2 + \frac{1}{2}\mathcal{B}_{11}^{(0)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0803} = 2B_{11}^{(1)}\beta - \frac{1}{2}\mathcal{B}_{11}^{(1)}\beta^3$$

$$C_{0804} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0805} = A_{11}^{(3)}\beta^2 + 2B_{11}^{(1)} + B_{11}^{(3)}\alpha^2 + \frac{1}{2}\mathcal{B}_{11}^{(1)}\left(3\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0806} = -A_{12}^{(2)}\beta + 2B_{11}^{(2)}\beta - \frac{1}{2}\mathcal{B}_{11}^{(0)}\beta - \frac{1}{2}\mathcal{B}_{11}^{(2)}\beta^3$$

$$C_{0807} = A_{12}^{(4)}\alpha\beta + B_{11}^{(4)}\alpha\beta - 3\mathcal{B}_{11}^{(2)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0808} = A_{11}^{(4)}\beta^2 + 4B_{11}^{(2)} + B_{11}^{(4)}\alpha^2 + \mathcal{B}_{11}^{(0)} + \mathcal{B}_{11}^{(2)}\left(3\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0809} = -2A_{12}^{(3)}\beta + 2B_{11}^{(3)}\beta - \mathcal{B}_{11}^{(1)}\beta - \frac{1}{2}\mathcal{B}_{11}^{(3)}\beta^3$$

$$C_{0810} = A_{12}^{(5)}\alpha\beta + B_{11}^{(5)}\alpha\beta - 5\mathcal{B}_{11}^{(3)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{0811} = A_{11}^{(5)}\beta^2 + 6B_{11}^{(3)} + B_{11}^{(5)}\alpha^2 + 3\mathcal{B}_{11}^{(1)} + \frac{1}{2}\mathcal{B}_{11}^{(3)}\left(10\alpha^2 + 3\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{0901} = -2A_{12}^{(1)}\alpha$$

$$C_{0902} = -2A_{12}^{(1)}\beta$$

$$C_{0903} = B_{11}^{(2)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0904} = -2A_{12}^{(2)}\alpha + B_{11}^{(2)}\alpha - \frac{1}{4}\mathcal{B}_{11}^{(2)}\alpha^3$$

$$C_{0905} = -2A_{12}^{(2)}\beta + B_{11}^{(2)}\beta - \frac{1}{4}\mathcal{B}_{11}^{(2)}\beta^3$$

$$C_{0906} = 2A_{11}^{(1)} + B_{11}^{(3)}\left(\alpha^2 + \beta^2\right) + \frac{1}{2}\mathcal{B}_{11}^{(1)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0907} = -2A_{12}^{(3)}\alpha + 2B_{11}^{(3)}\alpha - \mathcal{B}_{11}^{(1)}\alpha - \frac{1}{2}\mathcal{B}_{11}^{(3)}\alpha^3$$

$$C_{0908} = -2A_{12}^{(3)}\beta + 2B_{11}^{(3)}\beta - \mathcal{B}_{11}^{(1)}\beta - \frac{1}{2}\mathcal{B}_{11}^{(3)}\beta^3$$

$$C_{0909} = 4A_{11}^{(2)} + B_{11}^{(4)}\left(\alpha^2 + \beta^2\right) + \mathcal{B}_{11}^{(2)}\left(\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^4 + \beta^4\right)$$

$$C_{0910} = -2A_{12}^{(4)}\alpha + 3B_{11}^{(4)}\alpha - 3\mathcal{B}_{11}^{(2)}\alpha - \frac{3}{4}\mathcal{B}_{11}^{(4)}\alpha^3$$

$$C_{0911} = -2A_{12}^{(4)}\beta + 3B_{11}^{(4)}\beta - 3\mathcal{B}_{11}^{(2)}\beta - \frac{3}{4}\mathcal{B}_{11}^{(4)}\beta^3$$

$$C_{1001} = A_{11}^{(3)}\alpha^2 + B_{11}^{(3)}\beta^2 + \frac{3}{2}\mathcal{B}_{11}^{(1)}\beta^2 + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{1002} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{1003} = 3B_{11}^{(2)}\alpha - \frac{3}{4}\mathcal{B}_{11}^{(2)}\alpha^3$$

$$C_{1004} = A_{11}^{(4)}\alpha^2 + 3B_{11}^{(2)} + B_{11}^{(4)}\beta^2 + \frac{3}{4}\mathcal{B}_{11}^{(2)}\left(\alpha^2 + 4\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{1005} = A_{12}^{(4)}\alpha\beta + B_{11}^{(4)}\alpha\beta - 3\mathcal{B}_{11}^{(2)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{1006} = -A_{12}^{(3)}\alpha + 3B_{11}^{(3)}\alpha - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha - \frac{3}{4}\mathcal{B}_{11}^{(3)}\alpha^3$$

$$C_{1007} = A_{11}^{(5)}\alpha^2 + 6B_{11}^{(3)} + 3\mathcal{B}_{11}^{(1)} + \frac{1}{2}\mathcal{B}_{11}^{(3)}\left(3\alpha^2 + 10\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{1008} = A_{12}^{(5)}\alpha\beta + B_{11}^{(5)}\alpha\beta - 5\mathcal{B}_{11}^{(3)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{1009} = -2A_{12}^{(4)}\alpha + 3B_{11}^{(4)}\alpha - 3\mathcal{B}_{11}^{(2)}\alpha - \frac{3}{4}\mathcal{B}_{11}^{(4)}\alpha^3$$

$$C_{1010} = A_{11}^{(6)}\alpha^2 + 9B_{11}^{(4)} + B_{11}^{(6)}\beta^2 + 9\mathcal{B}_{11}^{(2)} + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(9\alpha^2 + 30\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(6)}\left(\alpha^2\beta^2 + \beta^4\right)$$

$$C_{1011} = +A_{12}^{(6)}\alpha\beta + B_{11}^{(6)}\alpha\beta - \frac{15}{2}\mathcal{B}_{11}^{(4)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(6)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{1101} = A_{12}^{(3)}\alpha\beta + B_{11}^{(3)}\alpha\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^3\beta + \alpha\beta^3\right)$$

$$C_{1102} = A_{11}^{(3)}\beta^2 + B_{11}^{(3)}\alpha^2 + \frac{3}{2}\mathcal{B}_{11}^{(1)}\alpha^2 + \frac{1}{4}\mathcal{B}_{11}^{(3)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

$$C_{1103} = 3B_{11}^{(2)}\beta - \frac{3}{4}\mathcal{B}_{11}^{(2)}\beta^3$$

$$C_{1104} = A_{12}^{(4)}\alpha\beta + B_{11}^{(4)}\alpha\beta - 3\mathcal{B}_{11}^{(2)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^3\beta + \alpha\beta^3\right)$$
$$C_{1105} = A_{11}^{(4)}\beta^2 + 3B_{11}^{(2)} + B_{11}^{(4)}\alpha^2 + \frac{3}{4}\mathcal{B}_{11}^{(2)}\left(4\alpha^2 + \beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(\alpha^4 + \alpha^2\beta^2\right)$$
$$C_{1106} = -A_{12}^{(3)}\beta + 3B_{11}^{(3)}\beta - \frac{3}{2}\mathcal{B}_{11}^{(1)}\beta - \frac{3}{4}\mathcal{B}_{11}^{(3)}\beta^3$$
$$C_{1107} = A_{12}^{(5)}\alpha\beta + B_{11}^{(5)}\alpha\beta - 5\mathcal{B}_{11}^{(3)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^3\beta + \alpha\beta^3\right)$$
$$C_{1108} = A_{11}^{(5)}\beta^2 + 6B_{11}^{(3)} + B_{11}^{(5)}\alpha^2 + 3\mathcal{B}_{11}^{(1)} + \frac{1}{2}\mathcal{B}_{11}^{(3)}\left(10\alpha^2 + 3\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(5)}\left(\alpha^4 + \alpha^2\beta^2\right)$$
$$C_{1109} = -2A_{12}^{(4)}\beta + 3B_{11}^{(4)}\beta - 3\mathcal{B}_{11}^{(2)}\beta - \frac{3}{4}\mathcal{B}_{11}^{(4)}\beta^3$$
$$C_{1110} = A_{12}^{(6)}\alpha\beta + B_{11}^{(6)}\alpha\beta - \frac{15}{2}\mathcal{B}_{11}^{(4)}\alpha\beta - \frac{1}{4}\mathcal{B}_{11}^{(6)}\left(\alpha^3\beta + \alpha\beta^3\right)$$
$$C_{1111} = A_{11}^{(6)}\beta^2 + 9B_{11}^{(4)} + B_{11}^{(6)}\alpha^2 + 9\mathcal{B}_{11}^{(2)} + \frac{1}{4}\mathcal{B}_{11}^{(4)}\left(30\alpha^2 + 9\beta^2\right) + \frac{1}{4}\mathcal{B}_{11}^{(6)}\left(\alpha^4 + \alpha^2\beta^2\right)$$

and the coefficient for the transverse load, Qmn, is obtained evaluating Eq. (3.44) and examples are following

$$Qmn = \begin{cases} \dfrac{16q_o}{mn\pi^2} & \text{for uniformly distributed load} \\ \dfrac{4P}{ab} sin\left(\dfrac{m\pi}{2}\right) sin\left(\dfrac{n\pi}{2}\right) & \text{for point load at the center of plate} \end{cases} \tag{3.46}$$

where q_o and P are magnitude of the uniformly distributed load and the point load and $m = 1, 3, 5, \ldots$ and $n = 1, 3, 5, \ldots$. The unknown generalized displacement coefficients, Umn, Vmn, …, and Φmn, can be obtained from solving Eq. (3.45) for each m, and n. The displacements, $u, v, \ldots,$ and ψ_y, are evaluated by Eqs. (3.32)to (3.42).

3.3.4 Free vibration analysis

For the free vibration analysis, the time dependant variables in Eqs. (3.32)to (3.42) are assumed as

$$Umn\,(t) = Umn \times e^{-i\omega t} \tag{3.47}$$

$$\Theta_x mn\,(t) = \Theta_x mn \times e^{-i\omega t} \tag{3.48}$$

$$\Phi_x mn\,(t) = \Phi_x mn \times e^{-i\omega t} \tag{3.49}$$

$$V mn\,(t) = V mn \times e^{-i\omega t} \tag{3.50}$$

$$\Theta_y mn\,(t) = \Theta_y mn \times e^{-i\omega t} \tag{3.51}$$

$$\Phi_y mn\,(t) = \Phi_y mn \times e^{-i\omega t} \tag{3.52}$$

$$W mn\,(t) = W mn \times e^{-i\omega t} \tag{3.53}$$

$$\Theta_z mn\,(t) = \Theta_z mn \times e^{-i\omega t} \tag{3.54}$$

$$\Phi_z mn\,(t) = \Phi_z mn \times e^{-i\omega t} \tag{3.55}$$

$$\Psi_x mn\,(t) = \Psi_x mn \times e^{-i\omega t} \tag{3.56}$$

$$\Psi_y mn\,(t) = \Psi_y mn \times e^{-i\omega t} \tag{3.57}$$

where ω is the natural frequency. By substituting equations (3.47) to (3.57) and (3.32) to (3.42) into the equation of motion, we setup the eigenvalue problem to determine eigen-frequencies:

$$\left\{[C] - \omega_{mn}^2\,[M]\right\}\{U\} = \{0\}\,. \tag{3.58}$$

Here $[C]$ is the coefficient matrix, $[M]$ is the matrix of inertias, and U is the vector of generalized displacements. The matrix $[C]$ is the same as the one in bending analysis, and the coefficients of the matrix of inertias are

$$\begin{array}{llllll} M_{0101} = m_0 & M_{0102} = 0 & M_{0103} = 0 & M_{0104} = m_1 & M_{0105} = 0 & M_{0106} = 0 \\ M_{0107} = m_2 & M_{0108} = 0 & M_{0109} = 0 & M_{0110} = m_3 & M_{0111} = 0 & \\ M_{0201} = 0 & M_{0202} = m_0 & M_{0203} = 0 & M_{0204} = 0 & M_{0205} = m_1 & M_{0206} = 0 \\ M_{0207} = 0 & M_{0208} = m_2 & M_{0209} = 0 & M_{0210} = 0 & M_{0211} = m_3 & \end{array}$$

$$
\begin{array}{llllll}
M_{0301}=0 & M_{0302}=0 & M_{0303}=m_0 & M_{0304}=0 & M_{0305}=0 & M_{0306}=m_1 \\
M_{0307}=0 & M_{0308}=0 & M_{0309}=m_2 & M_{0310}=0 & M_{0311}=0 & \\
M_{0401}=m_1 & M_{0402}=0 & M_{0403}=0 & M_{0404}=m_2 & M_{0405}=0 & M_{0406}=0 \\
M_{0407}=m_3 & M_{0408}=0 & M_{0409}=0 & M_{0410}=m_4 & M_{0411}=0 & \\
M_{0501}=0 & M_{0502}=m_1 & M_{0503}=0 & M_{0504}=0 & M_{0505}=m_2 & M_{0506}=0 \\
M_{0507}=0 & M_{0508}=m_3 & M_{0509}=0 & M_{0510}=0 & M_{0511}=m_4 & \\
M_{0601}=0 & M_{0602}=0 & M_{0603}=m_1 & M_{0604}=0 & M_{0605}=0 & M_{0606}=m_2 \\
M_{0607}=0 & M_{0608}=0 & M_{0609}=m_3 & M_{0610}=0 & M_{0611}=0 & \\
M_{0701}=m_2 & M_{0702}=0 & M_{0703}=0 & M_{0704}=m_3 & M_{0705}=0 & M_{0706}=0 \\
M_{0707}=m_4 & M_{0708}=0 & M_{0709}=0 & M_{0710}=m_5 & M_{0711}=0 & \\
M_{0801}=0 & M_{0802}=m_2 & M_{0803}=0 & M_{0804}=0 & M_{0805}=m_3 & M_{0806}=0 \\
M_{0901}=0 & M_{0902}=0 & M_{0903}=m_2 & M_{0904}=0 & M_{0905}=0 & M_{0906}=m_3 \\
M_{0907}=0 & M_{0908}=0 & M_{0909}=m_4 & M_{0910}=0 & M_{0911}=0 & \\
M_{0807}=0 & M_{0808}=m_4 & M_{0809}=0 & M_{0810}=0 & M_{0811}=m_5 & \\
M_{1001}=m_2 & M_{1002}=0 & M_{1003}=0 & M_{1004}=m_3 & M_{1005}=0 & M_{1006}=0 \\
M_{1007}=m_4 & M_{1008}=0 & M_{1009}=0 & M_{1010}=m_5 & M_{1011}=0 & \\
M_{1101}=0 & M_{1102}=m_2 & M_{1103}=0 & M_{1104}=0 & M_{1105}=m_3 & M_{1106}=0 \\
M_{1107}=0 & M_{1108}=m_4 & M_{1109}=0 & M_{1110}=0 & M_{1111}=m_5 &
\end{array}
$$

where $m_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^i \rho\left(z\right) dz$. The rule of subscripts are same as known coefficients matrix, $[C]$. The scale of plate thickness may cause a badly scaled inertia matrix of GTPT since higher order of plate thickness terms in diagonal. In such cases, the governing equations (3.21) to (3.31) are needed to be dimensionless form.

3.3.5 Buckling analysis

We assume that only in-plane forces act on each sides of simply supported plate and all other mechanical and thermal loads are zero and inertial terms are omitted for the buckling analysis. For buckling analysis, the additional terms $\hat{N_{xx}}\frac{\partial^2 w}{\partial x^2} + \hat{N_{yy}}\frac{\partial^2 w}{\partial y^2}$ are added to the right side of the equation of motion (3.23) [6]. $\hat{N_{xx}}$ and $\hat{N_{yy}}$ are the compressive loads acting on edges of a simply supported square plate and the system equation of buckling analysis becomes

$$\begin{bmatrix} C_{0101} & C_{0102} & C_{0103} & \cdots & C_{0110} & C_{0111} \\ C_{0201} & C_{0202} & C_{0203} & \cdots & C_{0210} & C_{0211} \\ C_{0301} & C_{0302} & C_{0303} - N_0\left(\alpha^2 + k\beta^2\right) & \cdots & C_{0310} & C_{0311} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1001} & C_{1002} & C_{1003} & \cdots & C_{1010} & C_{1011} \\ C_{1101} & C_{1102} & C_{1103} & \cdots & C_{1110} & C_{1111} \end{bmatrix} \begin{Bmatrix} Umn \\ Vmn \\ Wmn \\ \Theta_x mn \\ \Theta_y mn \\ \Theta_z mn \\ \Phi_x mn \\ \Phi_y mn \\ \Phi_z mn \\ \Psi_x mn \\ \Psi_y mn \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.59}$$

where $N_0 = -\hat{N}_{xx}$, $k = \frac{\hat{N}_{yy}}{\hat{N}_{xx}}$. To determine N_0, the condensation of variables procedure is applied to Eq. (3.59), and all unknown coefficients except Wmn are eliminated. The equation (3.60) shows an expression of generalized displacements, (Umn, Vmn, $\Theta_x mn$, $\Theta_y mn$, $\Theta_z mn$, $\Phi_x mn$, $\Phi_y mn$, $\Phi_z mn$, $\Psi_x mn$, and Ψ_y) in terms of Wmn.

$$
\begin{Bmatrix} Umn \\ Vmn \\ \Theta_x mn \\ \Theta_y mn \\ \Theta_z mn \\ \Phi_x mn \\ \Phi_y mn \\ \Phi_z mn \\ \Psi_x mn \\ \Psi_y mn \end{Bmatrix} = [\bar{C}] \begin{Bmatrix} C_{0103} \\ C_{0203} \\ C_{0403} \\ C_{0503} \\ C_{0603} \\ C_{0703} \\ C_{0803} \\ C_{0903} \\ C_{1003} \\ C_{1103} \end{Bmatrix} \begin{Bmatrix} Wmn \end{Bmatrix} \tag{3.60}
$$

where

$$
[\bar{C}] = \begin{bmatrix} C_{0101} & C_{0102} & C_{0104} & C_{0105} & \cdots & C_{0110} & C_{0111} \\ C_{0201} & C_{0202} & C_{0204} & C_{0205} & \cdots & C_{0210} & C_{0211} \\ C_{0401} & C_{0402} & C_{0404} & C_{0405} & \cdots & C_{0410} & C_{0411} \\ C_{0501} & C_{0502} & C_{0504} & C_{0505} & \cdots & C_{0510} & C_{0511} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1001} & C_{1002} & C_{1004} & C_{1005} & \cdots & C_{1010} & C_{1011} \\ C_{1101} & C_{1102} & C_{1104} & C_{1105} & \cdots & C_{1110} & C_{1111} \end{bmatrix}^{-1} .
$$

After applying condensation of variables, we obtain the expression for the critical load N_0 as follows:

$$N_0 = \frac{1}{\alpha^2 + k\beta^2} \left(C_{0303} - \begin{Bmatrix} C_{0301} \\ C_{0302} \\ C_{0304} \\ C_{0305} \\ \vdots \\ C_{0310} \\ C_{0311} \end{Bmatrix}^T \left[\bar{C}\right] \begin{Bmatrix} C_{0103} \\ C_{0203} \\ C_{0403} \\ C_{0503} \\ \vdots \\ C_{1003} \\ C_{1103} \end{Bmatrix} \right). \tag{3.61}$$

3.4 Numerical results

Numerical examples of the analytical solution are obtained using the material properties and the dimension of the functionally graded plate from Reddy [35]. The dimensions of functionally graded plate in Figure 3.2 are $a = 20h$, $b = 20h$, and $h = 17.6 \times 10^{-6} m$. The moduli and mass densities of two constituents are $E_t = 14.4GPa$, $E_b = 1.44GPa$, $\rho_t = 12.2 \times 10^3 kg/m$, and $\rho_b = 1.22 \times 10^3 kg/m$.

3.4.1 Static bending

For bending example, The uniformly distributed load $q_0 = 1N/m^2$ is applied on top surface $z = \frac{h}{2}$. In this example, the number of summation in double trigonometric series in Eqs.(3.32) to (3.42) and (3.43) is taken up to 31 for bending problems. Figure 3.3 through 3.5 show non-dimensional deflections $\bar{w}\left(x, \frac{b}{2}, 0\right)$ with variation power-law index, n, in Eqs. (3.14) to (3.20) and the length scale parameter in Eq. (3.12). The microstructure effects that are included using length scale parameter, ℓ, makes the plate stiffer. It is clear that the functionally graded plate becomes stiffer along smaller power-law index, n since we assume that E_t is larger than E_b in this example.

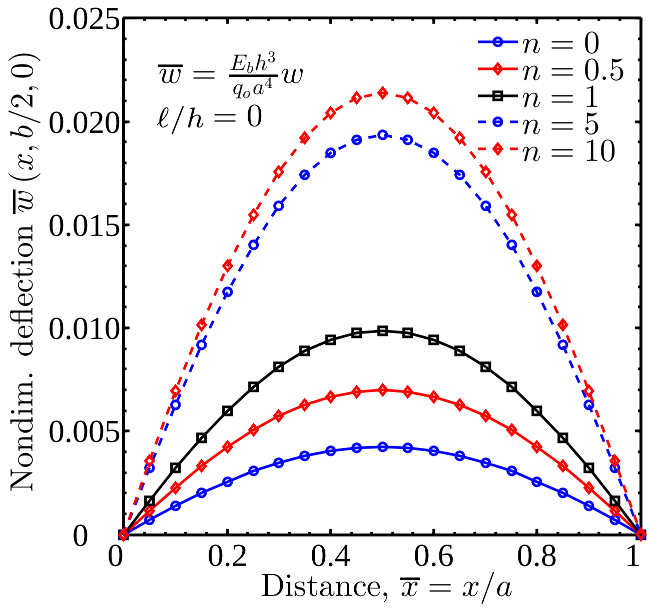

Figure 3.3: Non-dimensional deflection $\bar{w}\left(x, \frac{b}{2}, 0\right)$ versus distance along a FGM simply supported plate with various power-law index, n [2]

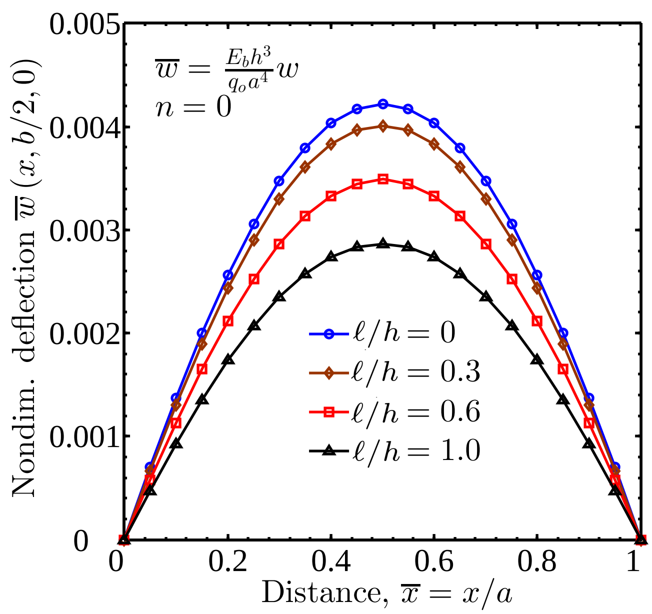

Figure 3.4: Non-dimensional deflection $\bar{w}\left(x, \frac{b}{2}, 0\right)$ versus distance along a homogeneous simply supported plate with various length scale parameter, ℓ [2]

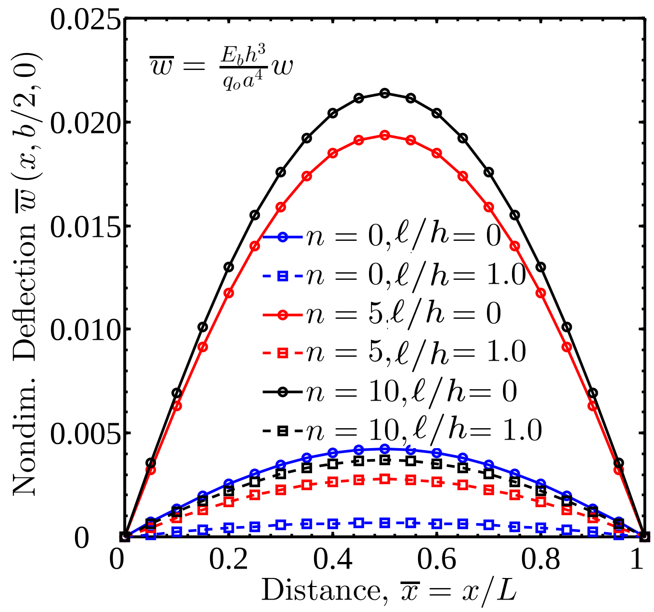

Figure 3.5: Non-dimensional deflection $\bar{w}\left(x, \frac{b}{2}, 0\right)$ versus distance along a FGM simply supported plate with various power-law index, n and length scale parameter, ℓ [2]

3.4.2 Natural vibration

The natural frequencies of the simply supported square FGM plates are obtained using Eq. (3.58). The fundamental frequencies are obtained when $m = 1$ and $n = 1$. Unlike homogeneous plate(n=0), the resultants of Young's modulus, E_1, E_3, and E_5, of FGM plates in Eqs.(3.15), (3.17), and (3.19) are nonzero and vary with power-law index, n. Figure 3.6 clearly shows the the natural frequency of FGM plates varies with power-law index,n in similar form with variation of those resultants of Young's modulus in the Fig. 3.1. Table 3.1 contains the non-dimensional natural frequencies for various length scale parameter, ℓ and power-law index, n. The consideration of microstructure effects increases the stiffness of the plate and the magnitude of the natural frequencies.

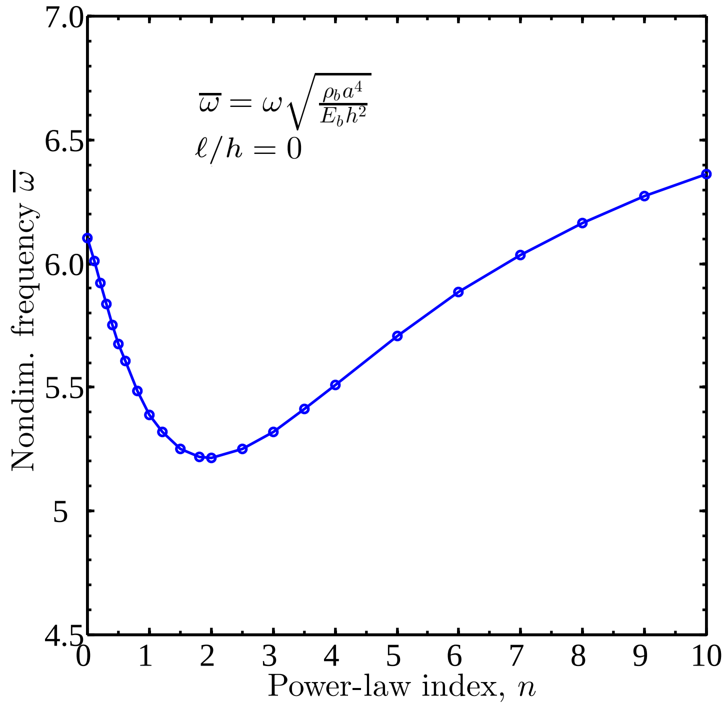

Figure 3.6: Non-dimensional fundamental frequency, $\bar{\omega}$ versus power-law index, n [2]

Table 3.1: Non-dimensional fundamental natural frequencies of simply supported plate $\left(\bar{\omega} = \omega\sqrt{\frac{\rho_b a^4}{E_b h^2}}\right)$ [2]

ℓ/h	Power-law index, n										
	0	1	2	3	4	5	6	7	8	9	10
0	6.10	5.39	5.22	5.32	5.51	5.71	5.88	6.04	6.17	6.27	6.36
0.5	9.31	8.87	8.77	8.83	8.95	9.07	9.18	9.28	9.36	9.43	9.49
1.0	15.25	15.00	14.95	14.98	15.05	15.12	15.18	15.24	15.28	15.32	15.36

3.4.3 Buckling

The critical loads for FGM simply supported square plates are obtained using Eq. (3.61). The same magnitude of compressive loads, $\hat{N}_{xx} = \hat{N}_{yy}$, are assumed for a square plate. The minimum critical load is obtained when $m = 1$ and $n = 1$ in double trigonometric function. The critical loads with various length scale parameters and power-law indices are shown in Figure 3.7 and Table 3.2. For larger length scale parameter, the FGM plates become stiffer and have larger critical load. It is shown that we can control the critical load for the FGM plates with predefined configuring material variation through thickness direction.

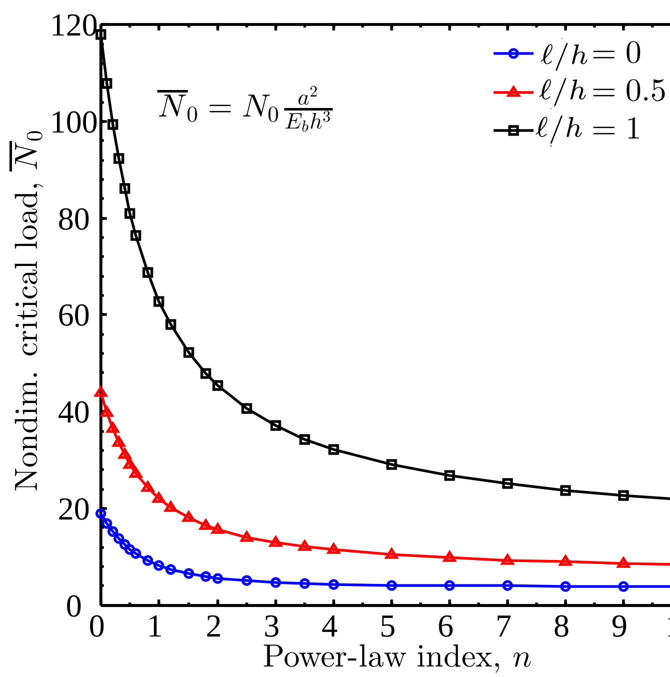

Figure 3.7: Non-dimensional critical load versus length scale parameter, ℓ along a FGM simply supported plate with various power-law index, n [2]

Table 3.2: The non-dimensional critical loads of simply supported plate $\left(\bar{N}_0 = \frac{N_0 a^2}{E_b h^3}\right)$ [2]

ℓ/h	Power-law index, n										
	0	1	2	3	4	5	6	7	8	9	10
0	18.90	8.11	5.53	4.67	4.32	4.13	4.02	3.93	3.86	3.80	3.74
0.5	43.96	21.95	15.61	12.87	11.38	10.44	9.79	9.30	8.91	8.59	8.32
1	118.0	62.82	45.37	37.05	32.20	29.01	26.75	25.05	23.72	22.66	21.78

4. FINITE ELEMENT MODEL *

4.1 Approximation of nodal variables

The displacement based weak form Galerkin finite element model for the general third order plate theory is developed using the principle of virtual displacements (2.46). The explicit forms of the virtual energies and the virtual work done by external forces are presented in the section 2. The principle of virtual displacement (2.46) contains the second derivative of dependent variables, $(u,\, v,\, w,\, \theta_x,\, \theta_y,\, \theta_z,\, \phi_x,\, \phi_y,\, \phi_z,\, \psi_x,\, \psi_y)$. Therefore, they can be approximated using the C^1 interpolation functions. The finite element approximation takes the form

$$U^{(I)}\left(x, y\right) = \sum_{j=1}^{4\times p} \bar{\Delta}_j^{(I)} \varphi_j^{(I)}\left(x, y\right) \tag{4.1}$$

where $U^{(I)}$ is the dependent variables, p is number of nodes in an element, the superscript I indicates I_{th} variable of a node, e.g. $U^{(1)} = u$, $U^{(2)} = v$, $U^{(3)} = w$, $U^{(4)} = \theta_x$, and so on. $\bar{\Delta}_j^{(I)}$ is the nodal variables and their derivatives of j^{th} node, and $\varphi_j^{(I)}\left(x, y\right)$ is C^1 Hermite type interpolation functions. In this study, we used a C^1 conforming element which has four degrees of freedom$\left(U^{(I)},\, \frac{\partial U^{(I)}}{\partial x},\, \frac{\partial U^{(I)}}{\partial y},\ \text{and}\ \frac{\partial^2 U^{(I)}}{\partial x \partial y}\right)$ per node [9]. Since the general third order plate theory has 11 dependent variables, the total degrees of freedom per node becomes 44. Figure 4.1 shows arrangement of typical nodal variable, $\bar{\Delta}_j^{(I)}$. The typical nodal variable is defined as

$$\bar{\Delta}_{k+1}^{(I)} = U_n^{(I)},\ \ \bar{\Delta}_{k+2}^{(I)} = \frac{\partial U_n^{(I)}}{\partial x},\ \ \bar{\Delta}_{k+3}^{(I)} = \frac{\partial U_n^{(I)}}{\partial y},\ \ \bar{\Delta}_{k+4}^{(I)} = \frac{\partial^2 U_n^{(I)}}{\partial x \partial y}$$

where $k = 4\left(n-1\right)$ and n is node number.

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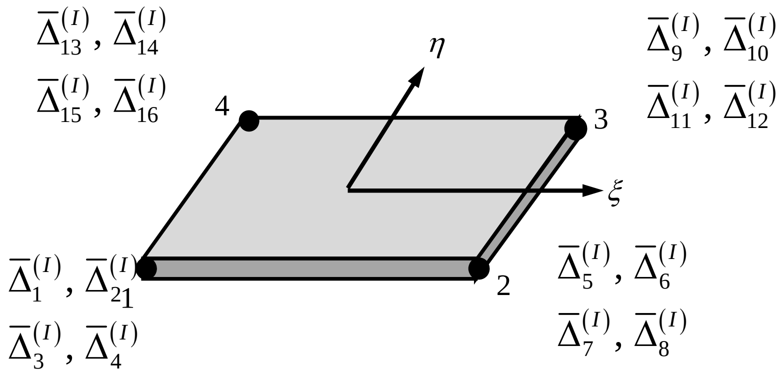

Figure 4.1: Arrangement of nodal variables for 4 node micro plate element

4.2 Displacement based weak-form Galerkin finite element model

By substituting the displacement field (2.28)–(2.30), the kinematic relation (2.35), (2.36), and (2.41), the constitutive relation (2.23) and (2.24), and the finite element approximation (4.1) into the principle of virtual displacement (2.46), we obtain the following the finite element model of the general third order plate theory in generic matrix form

$$[M]\{\ddot{U}\} + [K]\{U\} = \{F\} \tag{4.2}$$

where $[M]$ and $[K]$ are mass and stiffness matrices, and $\{\ddot{U}\}$ and $\{U\}$ are the acceleration and the displacement vectors respectively. $\{F\}$ is the force vector. The mass and stiffness matrices has 11 by 11 square submatrices, and the acceleration, the displacement, and the force vectors have 11 subvectors whose sizes are 4 times number of node in an element. The simplest rectangular element (4 node element) has 176 degrees of freedom per element. The explicit forms of nonzero mass are

$$M_{ij}^{0101} = \int_{\Omega} m^{(0)} \varphi_i^{(1)} \varphi_j^{(1)} dxdy \tag{4.3}$$

$$M_{ij}^{0104} = \int_{\Omega} m^{(1)} \varphi_i^{(1)} \varphi_j^{(4)} dxdy \tag{4.4}$$

$$M_{ij}^{0107} = \int_\Omega m^{(2)}\varphi_i^{(1)}\varphi_j^{(7)}dxdy \tag{4.5}$$

$$M_{ij}^{0110} = \int_\Omega m^{(3)}\varphi_i^{(1)}\varphi_j^{(10)}dxdy \tag{4.6}$$

$$M_{ij}^{0202} = \int_\Omega m^{(0)}\varphi_i^{(2)}\varphi_j^{(2)}dxdy \tag{4.7}$$

$$M_{ij}^{0205} = \int_\Omega m^{(1)}\varphi_i^{(2)}\varphi_j^{(5)}dxdy \tag{4.8}$$

$$M_{ij}^{0208} = \int_\Omega m^{(2)}\varphi_i^{(2)}\varphi_j^{(8)}dxdy \tag{4.9}$$

$$M_{ij}^{0211} = \int_\Omega m^{(3)}\varphi_i^{(2)}\varphi_j^{(11)}dxdy \tag{4.10}$$

$$M_{ij}^{0303} = \int_\Omega m^{(0)}\varphi_i^{(3)}\varphi_j^{(3)}dxdy \tag{4.11}$$

$$M_{ij}^{0306} = \int_\Omega m^{(1)}\varphi_i^{(3)}\varphi_j^{(6)}dxdy \tag{4.12}$$

$$M_{ij}^{0309} = \int_\Omega m^{(2)}\varphi_i^{(3)}\varphi_j^{(9)}dxdy \tag{4.13}$$

$$M_{ij}^{0401} = \int_\Omega m^{(1)}\varphi_i^{(4)}\varphi_j^{(1)}dxdy \tag{4.14}$$

$$M_{ij}^{0404} = \int_\Omega m^{(2)}\varphi_i^{(4)}\varphi_j^{(4)}dxdy \tag{4.15}$$

$$M_{ij}^{0407} = \int_\Omega m^{(3)}\varphi_i^{(4)}\varphi_j^{(7)}dxdy \tag{4.16}$$

$$M_{ij}^{0410} = \int_\Omega m^{(4)}\varphi_i^{(4)}\varphi_j^{(10)}dxdy \tag{4.17}$$

$$M_{ij}^{0502} = \int_\Omega m^{(1)}\varphi_i^{(5)}\varphi_j^{(2)}dxdy \tag{4.18}$$

$$M_{ij}^{0505} = \int_\Omega m^{(2)}\varphi_i^{(5)}\varphi_j^{(5)}dxdy \tag{4.19}$$

$$M_{ij}^{0508} = \int_\Omega m^{(3)}\varphi_i^{(5)}\varphi_j^{(8)}dxdy \tag{4.20}$$

$$M_{ij}^{0511} = \int_\Omega m^{(4)}\varphi_i^{(5)}\varphi_j^{(11)}dxdy \tag{4.21}$$

$$M_{ij}^{0603} = \int_\Omega m^{(1)}\varphi_i^{(6)}\varphi_j^{(3)}dxdy \tag{4.22}$$

$$M_{ij}^{0606} = \int_\Omega m^{(2)}\varphi_i^{(6)}\varphi_j^{(6)}dxdy \tag{4.23}$$

$$M_{ij}^{0609} = \int_\Omega m^{(3)}\varphi_i^{(6)}\varphi_j^{(9)}dxdy \tag{4.24}$$

$$M_{ij}^{0701} = \int_\Omega m^{(2)}\varphi_i^{(7)}\varphi_j^{(1)}dxdy \tag{4.25}$$

$$M_{ij}^{0704} = \int_\Omega m^{(3)}\varphi_i^{(7)}\varphi_j^{(4)}dxdy \tag{4.26}$$

$$M_{ij}^{0707} = \int_\Omega m^{(4)}\varphi_i^{(7)}\varphi_j^{(7)}dxdy \tag{4.27}$$

$$M_{ij}^{0710} = \int_\Omega m^{(5)}\varphi_i^{(7)}\varphi_j^{(10)}dxdy \tag{4.28}$$

$$M_{ij}^{0802} = \int_\Omega m^{(2)}\varphi_i^{(8)}\varphi_j^{(2)}dxdy \tag{4.29}$$

$$M_{ij}^{0805} = \int_\Omega m^{(3)}\varphi_i^{(8)}\varphi_j^{(5)}dxdy \tag{4.30}$$

$$M_{ij}^{0808} = \int_\Omega m^{(4)}\varphi_i^{(8)}\varphi_j^{(8)}dxdy \tag{4.31}$$

$$M_{ij}^{0811} = \int_\Omega m^{(5)}\varphi_i^{(8)}\varphi_j^{(11)}dxdy \tag{4.32}$$

$$M_{ij}^{0903} = \int_\Omega m^{(2)}\varphi_i^{(9)}\varphi_j^{(3)}dxdy \tag{4.33}$$

$$M_{ij}^{0906} = \int_\Omega m^{(3)}\varphi_i^{(9)}\varphi_j^{(6)}dxdy \tag{4.34}$$

$$M_{ij}^{0909} = \int_\Omega m^{(4)}\varphi_i^{(9)}\varphi_j^{(9)}dxdy \tag{4.35}$$

$$M_{ij}^{1001} = \int_\Omega m^{(3)}\varphi_i^{(10)}\varphi_j^{(1)}dxdy \tag{4.36}$$

$$M_{ij}^{1004} = \int_\Omega m^{(4)}\varphi_i^{(10)}\varphi_j^{(4)}dxdy \tag{4.37}$$

$$M_{ij}^{1007} = \int_\Omega m^{(5)}\varphi_i^{(10)}\varphi_j^{(7)}dxdy \tag{4.38}$$

$$M_{ij}^{1010} = \int_\Omega m^{(6)}\varphi_i^{(10)}\varphi_j^{(10)}dxdy \tag{4.39}$$

$$M_{ij}^{1102} = \int_\Omega m^{(3)}\varphi_i^{(11)}\varphi_j^{(2)}dxdy \tag{4.40}$$

$$M_{ij}^{1105} = \int_\Omega m^{(4)}\varphi_i^{(11)}\varphi_j^{(5)}dxdy \tag{4.41}$$

$$M_{ij}^{1108} = \int_\Omega m^{(5)}\varphi_i^{(11)}\varphi_j^{(8)}dxdy \tag{4.42}$$

$$M_{ij}^{1111} = \int_{\Omega} m^{(6)} \varphi_i^{(11)} \varphi_j^{(11)} dxdy \tag{4.43}$$

where the resultants of mass $\left(m^{(k)}\right)$ are

$$m^{(k)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k \rho(z)\, dz. \tag{4.44}$$

The stiffness matrices of the developed finite element model are

$$\begin{aligned} K_{ij}^{0101} = \int_{\Omega} & A_{11}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial y} \\ & + \frac{1}{4} \left(R_{44}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} + R_{55}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy \end{aligned} \tag{4.45}$$

$$\begin{aligned} K_{ij}^{0102} = \int_{\Omega} & A_{12}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial y} + B_{66}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial x} \\ & - \frac{1}{4} \left(R_{44}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} + R_{55}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \right) dxdy \end{aligned} \tag{4.46}$$

$$\begin{aligned} K_{ij}^{0103} = \frac{1}{2} \int_{\Omega} & \frac{\partial w}{\partial x} \left(A_{11}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} \right) \\ & + \frac{\partial w}{\partial y} \left(A_{12}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} \right) dxdy \end{aligned} \tag{4.47}$$

$$\begin{aligned} K_{ij}^{0104} = \int_{\Omega} & A_{11}^{(1)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(1)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial y} \\ & + \frac{1}{4} \left(R_{44}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(4)}}{\partial y^2} + R_{55}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y} \right) dxdy \end{aligned} \tag{4.48}$$

$$\begin{aligned} K_{ij}^{0105} = \int_{\Omega} & A_{12}^{(1)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial y} + B_{66}^{(1)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial x} \\ & - \frac{1}{4} \left(R_{44}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y} + R_{55}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(5)}}{\partial x^2} \right) dxdy \end{aligned} \tag{4.49}$$

$$K_{ij}^{0106} = \int_{\Omega} A_{13}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} \varphi_j^{(6)} + \frac{1}{4} \left(R_{44}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial \varphi_j^{(6)}}{\partial x} - R_{55}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial \varphi_j^{(6)}}{\partial y} \right) dxdy \tag{4.50}$$

$$
\begin{aligned}
K_{ij}^{0107} = \int_\Omega & A_{11}^{(2)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(7)}}{\partial x} + B_{66}^{(2)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(7)}}{\partial y} - \frac{1}{2} R_{44}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \varphi_j^{(7)} \\
& + \frac{1}{4} \left(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(7)}}{\partial y^2} + R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(7)}}{\partial x \partial y} \right) dxdy
\end{aligned} \tag{4.51}
$$

$$
\begin{aligned}
K_{ij}^{0108} = \int_\Omega & A_{12}^{(2)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(8)}}{\partial y} + B_{66}^{(2)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(8)}}{\partial x} + \frac{1}{2} R_{55}^{(0)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \varphi_j^{(8)} \\
& - \frac{1}{4} \left(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(8)}}{\partial x \partial y} + R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(8)}}{\partial x^2} \right) dxdy
\end{aligned} \tag{4.52}
$$

$$
K_{ij}^{0109} = \int_\Omega 2A_{13}^{(1)} \frac{\partial \varphi_i^{(1)}}{\partial x} \varphi_j^{(9)} + \frac{1}{2} \left(R_{44}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial \varphi_j^{(9)}}{\partial x} - R_{55}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial \varphi_j^{(9)}}{\partial y} \right) dxdy \tag{4.53}
$$

$$
\begin{aligned}
K_{ij}^{0110} = \int_\Omega & A_{11}^{(3)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial x} + B_{66}^{(3)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial y} - \frac{3}{2} R_{44}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \varphi_j^{(10)} \\
& + \frac{1}{4} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(10)}}{\partial y^2} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y} \right) dxdy
\end{aligned} \tag{4.54}
$$

$$
\begin{aligned}
K_{ij}^{0111} = \int_\Omega & A_{12}^{(3)} \frac{\partial \varphi_i^{(1)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial y} + B_{66}^{(3)} \frac{\partial \varphi_i^{(1)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial x} + \frac{3}{2} R_{55}^{(1)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \varphi_j^{(11)} \\
& - \frac{1}{4} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(1)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(1)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(11)}}{\partial x^2} \right) dxdy
\end{aligned} \tag{4.55}
$$

$$
\begin{aligned}
K_{ij}^{0201} = \int_\Omega & A_{21}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial y} \\
& - \frac{1}{4} \left(R_{44}^{(0)} \frac{\partial^2 \varphi_i^{(2)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} + R_{55}^{(0)} \frac{\partial^2 \varphi_i^{(2)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy
\end{aligned} \tag{4.56}
$$

$$
\begin{aligned}
K_{ij}^{0202} = \int_\Omega & A_{22}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial y} + B_{66}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial x} \\
& + \frac{1}{4} \left(R_{44}^{(0)} \frac{\partial^2 \varphi_i^{(2)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} + R_{55}^{(0)} \frac{\partial^2 \varphi_i^{(2)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \right) dxdy
\end{aligned} \tag{4.57}
$$

$$
K_{ij}^{0203} = \frac{1}{2} \int_\Omega \frac{\partial w}{\partial y} \left(A_{22}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} \right)
$$

$$+\frac{\partial w}{\partial x}\left(A_{21}^{(0)}\frac{\partial\varphi_i^{(2)}}{\partial y}\frac{\partial\varphi_j^{(3)}}{\partial x}+B_{66}^{(0)}\frac{\partial\varphi_i^{(2)}}{\partial x}\frac{\partial\varphi_j^{(3)}}{\partial y}\right)dxdy \tag{4.58}$$

$$K_{ij}^{0204}=\int_\Omega A_{21}^{(1)}\frac{\partial\varphi_i^{(2)}}{\partial y}\frac{\partial\varphi_j^{(4)}}{\partial x}+B_{66}^{(1)}\frac{\partial\varphi_i^{(2)}}{\partial x}\frac{\partial\varphi_j^{(4)}}{\partial y}$$
$$-\frac{1}{4}\left(R_{44}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(4)}}{\partial y^2}+R_{55}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial^2\varphi_j^{(4)}}{\partial x\partial y}\right)dxdy \tag{4.59}$$

$$K_{ij}^{0205}=\int_\Omega A_{22}^{(1)}\frac{\partial\varphi_i^{(2)}}{\partial y}\frac{\partial\varphi_j^{(5)}}{\partial y}+B_{66}^{(1)}\frac{\partial\varphi_i^{(2)}}{\partial x}\frac{\partial\varphi_j^{(5)}}{\partial x}$$
$$+\frac{1}{4}\left(R_{44}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(5)}}{\partial x\partial y}+R_{55}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial^2\varphi_j^{(5)}}{\partial x^2}\right)dxdy \tag{4.60}$$

$$K_{ij}^{0206}=\int_\Omega A_{23}^{(0)}\frac{\partial\varphi_i^{(2)}}{\partial y}\varphi_j^{(6)}-\frac{1}{4}\left(R_{44}^{(0)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial\varphi_j^{(6)}}{\partial x}-R_{55}^{(0)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial\varphi_j^{(6)}}{\partial y}\right)dxdy \tag{4.61}$$

$$K_{ij}^{0207}=\int_\Omega A_{21}^{(2)}\frac{\partial\varphi_i^{(2)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial x}+B_{66}^{(2)}\frac{\partial\varphi_i^{(2)}}{\partial x}\frac{\partial\varphi_j^{(7)}}{\partial y}+\frac{1}{2}R_{44}^{(0)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\varphi_j^{(7)}$$
$$-\frac{1}{4}\left(R_{44}^{(2)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(7)}}{\partial y^2}+R_{55}^{(2)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial^2\varphi_j^{(7)}}{\partial x\partial y}\right)dxdy \tag{4.62}$$

$$K_{ij}^{0208}=\int_\Omega A_{22}^{(2)}\frac{\partial\varphi_i^{(2)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial y}+B_{66}^{(2)}\frac{\partial\varphi_i^{(2)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial x}-\frac{1}{2}R_{55}^{(0)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\varphi_j^{(8)}$$
$$+\frac{1}{4}\left(R_{44}^{(2)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(8)}}{\partial x\partial y}+R_{55}^{(2)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial^2\varphi_j^{(8)}}{\partial x^2}\right)dxdy \tag{4.63}$$

$$K_{ij}^{0209}=\int_\Omega 2A_{23}^{(1)}\frac{\partial\varphi_i^{(2)}}{\partial y}\varphi_j^{(9)}-\frac{1}{2}\left(R_{44}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial\varphi_j^{(9)}}{\partial x}-R_{55}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial\varphi_j^{(9)}}{\partial y}\right)dxdy \tag{4.64}$$

$$K_{ij}^{0210}=\int_\Omega A_{21}^{(3)}\frac{\partial\varphi_i^{(2)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial x}+B_{66}^{(3)}\frac{\partial\varphi_i^{(2)}}{\partial x}\frac{\partial\varphi_j^{(10)}}{\partial y}+\frac{3}{2}R_{44}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\varphi_j^{(10)}$$
$$-\frac{1}{4}\left(R_{44}^{(3)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(10)}}{\partial y^2}+R_{55}^{(3)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial^2\varphi_j^{(10)}}{\partial x\partial y}\right)dxdy \tag{4.65}$$

$$K_{ij}^{0211}=\int_\Omega A_{22}^{(3)}\frac{\partial\varphi_i^{(2)}}{\partial y}\frac{\partial\varphi_j^{(11)}}{\partial y}+B_{66}^{(3)}\frac{\partial\varphi_i^{(2)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial x}-\frac{3}{2}R_{55}^{(1)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\varphi_j^{(11)}$$
$$+\frac{1}{4}\left(R_{44}^{(3)}\frac{\partial^2\varphi_i^{(2)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(11)}}{\partial x\partial y}+R_{55}^{(3)}\frac{\partial^2\varphi_i^{(2)}}{\partial x^2}\frac{\partial^2\varphi_j^{(11)}}{\partial x^2}\right)dxdy \tag{4.66}$$

$$K_{ij}^{0301} = \int_\Omega \frac{\partial w}{\partial x} \left(A_{11}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial y} \right)$$

$$+ \frac{\partial w}{\partial y} \left(A_{21}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial y} \right) dxdy \tag{4.67}$$

$$K_{ij}^{0302} = \int_\Omega \frac{\partial w}{\partial y} \left(A_{22}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial y} + B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial x} \right)$$

$$+ \frac{\partial w}{\partial x} \left(A_{12}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial y} + B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial x} \right) dxdy \tag{4.68}$$

$$K_{ij}^{0303} = \int_\Omega \frac{1}{2} \left[\left(\frac{\partial w}{\partial x} \right)^2 \left(A_{11}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} \right) \right.$$

$$+ 2B_{44}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} + 2B_{55}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x}$$

$$+ \left(\frac{\partial w}{\partial y} \right)^2 \left(A_{22}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} \right)$$

$$+ \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \left(A_{12}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} + A_{21}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} \right.$$

$$\left.\left. + \ B_{66}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} \right) \right] + \frac{1}{4} \left[R_{11}^{(0)} \frac{\partial^2 \varphi_i^{(3)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y} + R_{22}^{(0)} \frac{\partial^2 \varphi_i^{(3)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y} \right.$$

$$+ R_{66}^{(0)} \left(\frac{\partial^2 \varphi_i^{(3)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} + \frac{\partial^2 \varphi_i^{(3)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} - \frac{\partial^2 \varphi_i^{(3)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} \right.$$

$$\left.\left. - \frac{\partial^2 \varphi_i^{(3)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} \right) \right] dxdy \tag{4.69}$$

$$K_{ij}^{0304} = \int_\Omega \frac{\partial w}{\partial x} \left(A_{11}^{(1)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(1)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial y} \right)$$

$$+ \frac{\partial w}{\partial y} \left(A_{21}^{(1)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(1)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial y} \right) + B_{55}^{(0)} \frac{\partial \varphi_i^{(3)}}{\partial x} \varphi_j^{(4)}$$

$$+ \frac{1}{4} \left[R_{66}^{(0)} \left(\frac{\partial^2 \varphi_i^{(3)}}{\partial y^2} \frac{\partial \varphi_j^{(4)}}{\partial x} - \frac{\partial^2 \varphi_i^{(3)}}{\partial x^2} \frac{\partial \varphi_j^{(4)}}{\partial x} \right) - R_{22}^{(0)} \frac{\partial^2 \varphi_i^{(3)}}{\partial x \partial y} \frac{\partial \varphi_j^{(4)}}{\partial y} \right] dxdy \tag{4.70}$$

$$K_{ij}^{0305} = \int_\Omega \frac{\partial w}{\partial y} \left(A_{22}^{(1)} \frac{\partial \varphi_i^{(3)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial y} + B_{66}^{(1)} \frac{\partial \varphi_i^{(3)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial x} \right)$$

$$+\frac{\partial w}{\partial x}\left(A_{12}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(5)}}{\partial y}+B_{66}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(5)}}{\partial x}\right)+B_{44}^{(0)}\frac{\partial\varphi_i^{(3)}}{\partial y}\varphi_j^{(5)}$$

$$+\frac{1}{4}\left[R_{66}^{(0)}\left(\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial\varphi_j^{(5)}}{\partial y}-\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial\varphi_j^{(5)}}{\partial y}\right)-R_{11}^{(0)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial\varphi_j^{(5)}}{\partial x}\right]dxdy \quad (4.71)$$

$$K_{ij}^{0306}=\int_\Omega\frac{\partial w}{\partial x}A_{13}^{(0)}\frac{\partial\varphi_i^{(3)}}{\partial x}\varphi_j^{(6)}+\frac{\partial w}{\partial y}A_{23}^{(0)}\frac{\partial\varphi_i^{(3)}}{\partial y}\varphi_j^{(6)}+B_{44}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y}$$

$$+B_{55}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(6)}}{\partial x}+\frac{1}{4}\left[R_{11}^{(1)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(6)}}{\partial x\partial y}+R_{22}^{(1)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(6)}}{\partial x\partial y}\right.$$

$$+R_{66}^{(1)}\left(\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial^2\varphi_j^{(6)}}{\partial x^2}+\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial^2\varphi_j^{(6)}}{\partial y^2}-\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial^2\varphi_j^{(6)}}{\partial y^2}\right.$$

$$\left.\left.-\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial^2\varphi_j^{(6)}}{\partial x^2}\right)\right]dxdy \quad (4.72)$$

$$K_{ij}^{0307}=\int_\Omega\frac{\partial w}{\partial x}\left(A_{11}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(7)}}{\partial x}+B_{66}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y}\right)$$

$$+\frac{\partial w}{\partial y}\left(A_{21}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial x}+B_{66}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(7)}}{\partial y}\right)+2B_{55}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial x}\varphi_j^{(7)}$$

$$+\frac{1}{2}\left[R_{66}^{(1)}\left(\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial\varphi_j^{(7)}}{\partial x}-\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial\varphi_j^{(7)}}{\partial x}\right)-R_{22}^{(1)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y}\right]dxdy \quad (4.73)$$

$$K_{ij}^{0308}=\int_\Omega\frac{\partial w}{\partial y}\left(A_{22}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial y}+B_{66}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial x}\right)$$

$$+\frac{\partial w}{\partial x}\left(A_{12}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial y}+B_{66}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x}\right)+2B_{44}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial y}\varphi_j^{(8)}$$

$$+\frac{1}{2}\left[R_{66}^{(1)}\left(\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial\varphi_j^{(8)}}{\partial y}-\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial\varphi_j^{(8)}}{\partial y}\right)-R_{11}^{(1)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x}\right]dxdy \quad (4.74)$$

$$K_{ij}^{0309}=\int_\Omega 2\left(\frac{\partial w}{\partial x}A_{13}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial x}\varphi_j^{(9)}+\frac{\partial w}{\partial y}A_{23}^{(1)}\frac{\partial\varphi_i^{(3)}}{\partial y}\varphi_j^{(9)}\right)+B_{44}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(9)}}{\partial y}$$

$$+B_{55}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(9)}}{\partial x}+\frac{1}{4}\left[R_{66}^{(2)}\left(\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial^2\varphi_j^{(9)}}{\partial x^2}+\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial^2\varphi_j^{(9)}}{\partial y^2}\right.\right.$$

$$\left.-\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial^2\varphi_j^{(9)}}{\partial y^2}-\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial^2\varphi_j^{(9)}}{\partial x^2}\right)+R_{11}^{(2)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(9)}}{\partial x\partial y}$$

$$+\ R_{22}^{(2)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(9)}}{\partial x\partial y}+\Bigg]\,dxdy \tag{4.75}$$

$$\begin{aligned}
K_{ij}^{0310} = \int_\Omega &\frac{\partial w}{\partial x}\left(A_{11}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(10)}}{\partial x}+B_{66}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y}\right)\\
&+\frac{\partial w}{\partial y}\left(A_{21}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial x}+B_{66}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(10)}}{\partial y}\right)\\
&+3B_{55}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial x}\varphi_j^{(10)}+\frac{3}{4}\Bigg[R_{66}^{(2)}\left(\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial\varphi_j^{(10)}}{\partial x}-\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial\varphi_j^{(10)}}{\partial x}\right)\\
&-R_{22}^{(2)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y}\Bigg]\,dxdy
\end{aligned} \tag{4.76}$$

$$\begin{aligned}
K_{ij}^{0311} = \int_\Omega &\frac{\partial w}{\partial y}\left(A_{22}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(11)}}{\partial y}+B_{66}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial x}\right)\\
&+\frac{\partial w}{\partial x}\left(A_{12}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial y}+B_{66}^{(3)}\frac{\partial\varphi_i^{(3)}}{\partial y}\frac{\partial\varphi_j^{(11)}}{\partial x}\right)\\
&+3B_{44}^{(2)}\frac{\partial\varphi_i^{(3)}}{\partial y}\varphi_j^{(11)}+\frac{3}{4}\Bigg[R_{66}^{(2)}\left(\frac{\partial^2\varphi_i^{(3)}}{\partial x^2}\frac{\partial\varphi_j^{(11)}}{\partial y}-\frac{\partial^2\varphi_i^{(3)}}{\partial y^2}\frac{\partial\varphi_j^{(11)}}{\partial y}\right)\\
&-R_{11}^{(2)}\frac{\partial^2\varphi_i^{(3)}}{\partial x\partial y}\frac{\partial\varphi_j^{(11)}}{\partial x}\Bigg]\,dxdy
\end{aligned} \tag{4.77}$$

$$\begin{aligned}
K_{ij}^{0401} = \int_\Omega &A_{11}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(1)}}{\partial x}+B_{66}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(1)}}{\partial y}\\
&+\frac{1}{4}\left(R_{44}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\frac{\partial^2\varphi_j^{(1)}}{\partial y^2}+R_{55}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(1)}}{\partial x\partial y}\right)dxdy
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
K_{ij}^{0402} = \int_\Omega &A_{12}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(2)}}{\partial y}+B_{66}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(2)}}{\partial x}\\
&-\frac{1}{4}\left(R_{44}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\frac{\partial^2\varphi_j^{(2)}}{\partial x\partial y}+R_{55}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(2)}}{\partial x^2}\right)dxdy
\end{aligned} \tag{4.79}$$

$$\begin{aligned}
K_{ij}^{0403} = \frac{1}{2}\int_\Omega &\frac{\partial w}{\partial x}\left(A_{11}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(3)}}{\partial x}+B_{66}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(3)}}{\partial y}\right)\\
&+\frac{\partial w}{\partial y}\left(A_{12}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(3)}}{\partial y}+B_{66}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(3)}}{\partial x}\right)+2B_{55}^{(0)}\varphi_i^{(4)}\frac{\partial\varphi_j^{(3)}}{\partial x}
\end{aligned}$$

$$+ \frac{1}{2}\left[R_{66}^{(0)}\left(\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial^2\varphi_j^{(3)}}{\partial y^2} - \frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial^2\varphi_j^{(3)}}{\partial x^2}\right) - R_{22}^{(0)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial^2\varphi_j^{(3)}}{\partial x\partial y}\right] dxdy \quad (4.80)$$

$$K_{ij}^{0404} = \int_\Omega A_{11}^{(2)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(4)}}{\partial x} + B_{66}^{(2)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(4)}}{\partial y} + B_{55}^{(0)}\varphi_i^{(4)}\varphi_j^{(4)}$$
$$+ \frac{1}{4}\left(R_{22}^{(0)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(4)}}{\partial y} + R_{33}^{(0)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(4)}}{\partial y} + R_{66}^{(0)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(4)}}{\partial x}\right.$$
$$\left. + R_{44}^{(2)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\frac{\partial^2\varphi_j^{(4)}}{\partial y^2} + R_{55}^{(2)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(4)}}{\partial x\partial y}\right) dxdy \quad (4.81)$$

$$K_{ij}^{0405} = \int_\Omega A_{12}^{(2)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(5)}}{\partial y} + B_{66}^{(2)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(5)}}{\partial x} - \frac{1}{4}\left(R_{33}^{(0)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(5)}}{\partial x}\right.$$
$$\left. + R_{66}^{(0)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(5)}}{\partial y} + R_{44}^{(2)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\frac{\partial^2\varphi_j^{(5)}}{\partial x\partial y} + R_{55}^{(2)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(5)}}{\partial x^2}\right) dxdy \quad (4.82)$$

$$K_{ij}^{0406} = \int_\Omega A_{13}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial x}\varphi_j^{(6)} + B_{55}^{(1)}\varphi_i^{(4)}\frac{\partial\varphi_j^{(6)}}{\partial x}$$
$$+ \frac{1}{4}\left[R_{66}^{(1)}\left(\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial^2\varphi_j^{(6)}}{\partial y^2} - \frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial^2\varphi_j^{(6)}}{\partial x^2}\right)\right.$$
$$\left. - R_{22}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial^2\varphi_j^{(6)}}{\partial x\partial y} + R_{44}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\frac{\partial\varphi_j^{(6)}}{\partial x} - R_{55}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y}\right] dxdy \quad (4.83)$$

$$K_{ij}^{0407} = \int_\Omega A_{11}^{(3)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(7)}}{\partial x} + B_{66}^{(3)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y} + 2B_{55}^{(1)}\varphi_i^{(4)}\varphi_j^{(7)}$$
$$+ \frac{1}{2}\left[R_{22}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y} + R_{33}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y} - R_{44}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\varphi_j^{(7)}\right.$$
$$\left. + R_{66}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(7)}}{\partial x} + \frac{1}{2}\left(R_{44}^{(3)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\frac{\partial^2\varphi_j^{(7)}}{\partial y^2}R_{55}^{(3)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(7)}}{\partial x\partial y}\right)\right] dxdy \quad (4.84)$$

$$K_{ij}^{0408} = \int_\Omega A_{12}^{(3)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial y} + B_{66}^{(3)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x}$$
$$- \frac{1}{2}\left[R_{33}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x} - R_{55}^{(1)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\varphi_j^{(8)} + R_{66}^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial y}\right.$$
$$\left. + \frac{1}{2}\left(R_{44}^{(3)}\frac{\partial^2\varphi_i^{(4)}}{\partial y^2}\frac{\partial^2\varphi_j^{(8)}}{\partial x\partial y} + R_{55}^{(3)}\frac{\partial^2\varphi_i^{(4)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(8)}}{\partial x^2}\right)\right] dxdy \quad (4.85)$$

$$
\begin{aligned}
K_{ij}^{0409} = \int_\Omega & 2A_{13}^{(2)} \frac{\partial \varphi_i^{(4)}}{\partial x} \varphi_j^{(9)} + B_{55}^{(2)} \varphi_i^{(4)} \frac{\partial \varphi_j^{(9)}}{\partial x} \\
& + \frac{1}{4} \left[R_{66}^{(2)} \left(\frac{\partial \varphi_i^{(4)}}{\partial x} \frac{\partial^2 \varphi_j^{(9)}}{\partial y^2} - \frac{\partial \varphi_i^{(4)}}{\partial x} \frac{\partial^2 \varphi_j^{(9)}}{\partial x^2} \right) - R_{22}^{(2)} \frac{\partial \varphi_i^{(4)}}{\partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} \right. \\
& \left. + 2 \left(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(4)}}{\partial y^2} \frac{\partial \varphi_j^{(9)}}{\partial x} - R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(4)}}{\partial x \partial y} \frac{\partial \varphi_j^{(9)}}{\partial y} \right) \right] dxdy
\end{aligned} \tag{4.86}
$$

$$
\begin{aligned}
K_{ij}^{0410} = \int_\Omega & A_{11}^{(4)} \frac{\partial \varphi_i^{(4)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial x} + B_{66}^{(4)} \frac{\partial \varphi_i^{(4)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial y} + 3B_{55}^{(2)} \varphi_i^{(4)} \varphi_j^{(10)} \\
& + \frac{3}{4} \left[R_{22}^{(2)} \frac{\partial \varphi_i^{(4)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial y} + R_{33}^{(2)} \frac{\partial \varphi_i^{(4)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial y} \right. \\
& - 2R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(4)}}{\partial y^2} \varphi_j^{(10)} + R_{66}^{(2)} \frac{\partial \varphi_i^{(4)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial x} \\
& \left. + \frac{1}{3} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(4)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(10)}}{\partial y^2} + R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(4)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y} \right) \right] dxdy
\end{aligned} \tag{4.87}
$$

$$
\begin{aligned}
K_{ij}^{0411} = \int_\Omega & A_{12}^{(4)} \frac{\partial \varphi_i^{(4)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial y} + B_{66}^{(4)} \frac{\partial \varphi_i^{(4)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial x} \\
& - \frac{3}{4} \left[R_{33}^{(2)} \frac{\partial \varphi_i^{(4)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial x} - 2R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(4)}}{\partial x \partial y} \varphi_j^{(11)} + R_{66}^{(2)} \frac{\partial \varphi_i^{(4)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial y} \right. \\
& \left. + \frac{1}{3} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(4)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y} + R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(4)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(11)}}{\partial x^2} \right) \right] dxdy
\end{aligned} \tag{4.88}
$$

$$
\begin{aligned}
K_{ij}^{0501} = \int_\Omega & A_{21}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial y} \\
& - \frac{1}{4} \left(R_{44}^{(1)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} + R_{55}^{(1)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy
\end{aligned} \tag{4.89}
$$

$$
\begin{aligned}
K_{ij}^{0502} = \int_\Omega & A_{22}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial y} + B_{66}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial x} \\
& + \frac{1}{4} \left(R_{44}^{(1)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} + R_{55}^{(1)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \right) dxdy
\end{aligned} \tag{4.90}
$$

$$
K_{ij}^{0503} = \frac{1}{2} \int_\Omega \frac{\partial w}{\partial x} \left(A_{21}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} \right)
$$

$$+ \frac{\partial w}{\partial y}\left(A_{22}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(3)}}{\partial x}\right) + 2B_{44}^{(0)}\varphi_i^{(5)}\frac{\partial \varphi_j^{(3)}}{\partial y}$$

$$+ \frac{1}{2}\left[R_{66}^{(0)}\left(\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} - \frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial^2 \varphi_j^{(3)}}{\partial y^2}\right) - R_{11}^{(0)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y}\right] dxdy \quad (4.91)$$

$$K_{ij}^{0504} = \int_\Omega A_{21}^{(2)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(2)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(4)}}{\partial y} - \frac{1}{4}\left(R_{33}^{(0)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(4)}}{\partial y}\right.$$

$$\left. + R_{66}^{(0)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(4)}}{\partial x} + R_{44}^{(2)}\frac{\partial \varphi_i^{(5)}}{\partial x \partial y}\frac{\partial \varphi_j^{(4)}}{\partial y^2} + R_{55}^{(2)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x^2}\frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y}\right) dxdy \quad (4.92)$$

$$K_{ij}^{0505} = \int_\Omega A_{22}^{(2)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(5)}}{\partial y} + B_{66}^{(2)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(5)}}{\partial x} + B_{44}^{(0)}\varphi_i^{(5)}\varphi_j^{(5)}$$

$$+ \frac{1}{4}\left(R_{11}^{(0)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(5)}}{\partial x} + R_{33}^{(0)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(5)}}{\partial x} + R_{66}^{(0)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(5)}}{\partial y}\right.$$

$$\left. + R_{44}^{(2)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y}\frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y} + R_{55}^{(2)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x^2}\frac{\partial^2 \varphi_j^{(5)}}{\partial x^2}\right) dxdy \quad (4.93)$$

$$K_{ij}^{0506} = \int_\Omega A_{23}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial y}\varphi_j^{(6)} + B_{44}^{(1)}\varphi_i^{(5)}\frac{\partial \varphi_j^{(6)}}{\partial y}$$

$$+ \frac{1}{4}\left[R_{66}^{(1)}\left(\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial^2 \varphi_j^{(6)}}{\partial x^2} - \frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial^2 \varphi_j^{(6)}}{\partial y^2}\right) - R_{11}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial^2 \varphi_j^{(6)}}{\partial x \partial y}\right.$$

$$\left. - R_{44}^{(1)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y}\frac{\partial \varphi_j^{(6)}}{\partial x} + R_{55}^{(1)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x^2}\frac{\partial \varphi_j^{(6)}}{\partial y}\right] dxdy \quad (4.94)$$

$$K_{ij}^{0507} = \int_\Omega A_{21}^{(3)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(7)}}{\partial x} + B_{66}^{(3)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(7)}}{\partial y}$$

$$- \frac{1}{2}\left[R_{33}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(7)}}{\partial y} - R_{44}^{(1)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y}\varphi_j^{(7)} + R_{66}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(7)}}{\partial x}\right.$$

$$\left. + \frac{1}{2}\left(R_{44}^{(3)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y}\frac{\partial^2 \varphi_j^{(7)}}{\partial y^2} + R_{55}^{(3)}\frac{\partial^2 \varphi_i^{(5)}}{\partial x^2}\frac{\partial^2 \varphi_j^{(7)}}{\partial x \partial y}\right)\right] dxdy \quad (4.95)$$

$$K_{ij}^{0508} = \int_\Omega A_{22}^{(3)}\frac{\partial \varphi_i^{(5)}}{\partial y}\frac{\partial \varphi_j^{(8)}}{\partial y} + B_{66}^{(3)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(8)}}{\partial x} + 2B_{44}^{(1)}\varphi_i^{(5)}\varphi_j^{(8)}$$

$$+ \frac{1}{2}\left[R_{11}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(8)}}{\partial x} + R_{33}^{(1)}\frac{\partial \varphi_i^{(5)}}{\partial x}\frac{\partial \varphi_j^{(8)}}{\partial x}\right.$$

$$- R_{55}^{(1)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \varphi_j^{(8)} + R_{66}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(8)}}{\partial y}$$

$$+ \frac{1}{2} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(8)}}{\partial x \partial y} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(8)}}{\partial x^2} \right) \Bigg] dxdy \tag{4.96}$$

$$K_{ij}^{0509} = \int_\Omega 2A_{23}^{(2)} \frac{\partial \varphi_i^{(5)}}{\partial y} \varphi_j^{(9)} + B_{44}^{(2)} \varphi_i^{(5)} \frac{\partial \varphi_j^{(9)}}{\partial y}$$

$$+ \frac{1}{4} \left[R_{66}^{(2)} \left(\frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial x^2} - \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial y^2} \right) - R_{11}^{(2)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} \right.$$

$$- 2 \left(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y} \frac{\partial \varphi_j^{(9)}}{\partial x} - R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \frac{\partial \varphi_j^{(9)}}{\partial y} \right) \Bigg] dxdy \tag{4.97}$$

$$K_{ij}^{0510} = \int_\Omega A_{21}^{(4)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial x} + B_{66}^{(4)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial y}$$

$$- \frac{3}{4} \left[R_{33}^{(2)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial y} + R_{66}^{(2)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial x} - 2R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y} \varphi_j^{(10)} \right.$$

$$+ \frac{1}{3} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(10)}}{\partial y^2} + R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y} \right) \Bigg] dxdy \tag{4.98}$$

$$K_{ij}^{0511} = \int_\Omega A_{22}^{(4)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial y} + B_{66}^{(4)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial x} + 3B_{44}^{(2)} \varphi_i^{(5)} \varphi_j^{(11)}$$

$$+ \frac{3}{4} \left[R_{11}^{(2)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial x} + R_{33}^{(2)} \frac{\partial \varphi_i^{(5)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial x} \right.$$

$$- 2R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \varphi_j^{(11)} + R_{66}^{(2)} \frac{\partial \varphi_i^{(5)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial y}$$

$$+ \frac{1}{3} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y} + R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(5)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(11)}}{\partial x^2} \right) \Bigg] dxdy \tag{4.99}$$

$$K_{ij}^{0601} = \int_\Omega A_{31}^{(0)} \varphi_i^{(6)} \frac{\partial \varphi_j^{(1)}}{\partial x} + \frac{1}{4} \left(R_{44}^{(0)} \frac{\partial \varphi_i^{(6)}}{\partial x} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} - R_{55}^{(0)} \frac{\partial \varphi_i^{(6)}}{\partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy \tag{4.100}$$

$$K_{ij}^{0602} = \int_\Omega A_{32}^{(0)} \varphi_i^{(6)} \frac{\partial \varphi_j^{(2)}}{\partial y} - \frac{1}{4} \left(R_{44}^{(0)} \frac{\partial \varphi_i^{(6)}}{\partial x} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} - R_{55}^{(0)} \frac{\partial \varphi_i^{(6)}}{\partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \right) dxdy \tag{4.101}$$

$$K_{ij}^{0603} = \frac{1}{2} \int_\Omega \frac{\partial w}{\partial x} A_{31}^{(0)} \varphi_i^{(6)} \frac{\partial \varphi_j^{(3)}}{\partial x} + \frac{\partial w}{\partial y} A_{32}^{(0)} \varphi_i^{(6)} \frac{\partial \varphi_j^{(3)}}{\partial y}$$

$$
+ 2\left(B_{44}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial y}\frac{\partial\varphi_j^{(3)}}{\partial y} + B_{55}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial x}\frac{\partial\varphi_j^{(3)}}{\partial x}\right)
$$

$$
+ \frac{1}{2}\left[R_{11}^{(1)}\frac{\partial^2\varphi_i^{(6)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(3)}}{\partial x\partial y} + R_{22}^{(1)}\frac{\partial^2\varphi_i^{(6)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(3)}}{\partial x\partial y} + R_{66}^{(1)}\left(\frac{\partial^2\varphi_i^{(6)}}{\partial x^2}\frac{\partial^2\varphi_j^{(3)}}{\partial x^2}\right.\right.
$$

$$
\left.\left. + \frac{\partial^2\varphi_i^{(6)}}{\partial y^2}\frac{\partial^2\varphi_j^{(3)}}{\partial y^2} - \frac{\partial^2\varphi_i^{(6)}}{\partial x^2}\frac{\partial^2\varphi_j^{(3)}}{\partial y^2} - \frac{\partial^2\varphi_i^{(6)}}{\partial y^2}\frac{\partial^2\varphi_j^{(3)}}{\partial x^2}\right)\right] dxdy \tag{4.102}
$$

$$
K_{ij}^{0604} = \int_\Omega A_{31}^{(1)}\varphi_i^{(6)}\frac{\partial\varphi_j^{(4)}}{\partial x} + B_{55}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial x}\varphi_j^{(4)}
$$

$$
+ \frac{1}{4}\left[R_{66}^{(1)}\left(\frac{\partial^2\varphi_i^{(6)}}{\partial y^2}\frac{\partial\varphi_j^{(4)}}{\partial x} - \frac{\partial^2\varphi_i^{(6)}}{\partial x^2}\frac{\partial\varphi_j^{(4)}}{\partial x}\right)\right.
$$

$$
\left. - R_{22}^{(1)}\frac{\partial^2\varphi_i^{(6)}}{\partial x\partial y}\frac{\partial\varphi_j^{(4)}}{\partial y} + R_{44}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial x}\frac{\partial^2\varphi_j^{(4)}}{\partial y^2} - R_{55}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial y}\frac{\partial^2\varphi_j^{(4)}}{\partial x\partial y}\right] dxdy \tag{4.103}
$$

$$
K_{ij}^{0605} = \int_\Omega A_{32}^{(1)}\varphi_i^{(6)}\frac{\partial\varphi_j^{(5)}}{\partial y} + B_{44}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial y}\varphi_j^{(5)}
$$

$$
+ \frac{1}{4}\left[R_{66}^{(1)}\left(\frac{\partial^2\varphi_i^{(6)}}{\partial x^2}\frac{\partial\varphi_j^{(5)}}{\partial y} - \frac{\partial^2\varphi_i^{(6)}}{\partial y^2}\frac{\partial\varphi_j^{(5)}}{\partial y}\right)\right.
$$

$$
\left. - R_{11}^{(1)}\frac{\partial^2\varphi_i^{(6)}}{\partial x\partial y}\frac{\partial\varphi_j^{(5)}}{\partial x} - R_{44}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial x}\frac{\partial^2\varphi_j^{(5)}}{\partial x\partial y} + R_{55}^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial y}\frac{\partial^2\varphi_j^{(5)}}{\partial x^2}\right] dxdy \tag{4.104}
$$

$$
K_{ij}^{0606} = \int_\Omega A_{33}^{(0)}\varphi_i^{(6)}\varphi_j^{(6)} + B_{44}^{(2)}\frac{\partial\varphi_i^{(6)}}{\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y} + B_{55}^{(2)}\frac{\partial\varphi_i^{(6)}}{\partial x}\frac{\partial\varphi_j^{(6)}}{\partial x}
$$

$$
+ \frac{1}{4}\left[R_{11}^{(2)}\frac{\partial^2\varphi_i^{(6)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(6)}}{\partial x\partial y} + R_{22}^{(2)}\frac{\partial^2\varphi_i^{(6)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(6)}}{\partial x\partial y} + R_{44}^{(0)}\frac{\partial\varphi_i^{(6)}}{\partial x}\frac{\partial\varphi_j^{(6)}}{\partial x}\right.
$$

$$
+ R_{55}^{(0)}\frac{\partial\varphi_i^{(6)}}{\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y} + R_{66}^{(2)}\left(\frac{\partial^2\varphi_i^{(6)}}{\partial x^2}\frac{\partial^2\varphi_j^{(6)}}{\partial x^2} + \frac{\partial^2\varphi_i^{(6)}}{\partial y^2}\frac{\partial^2\varphi_j^{(6)}}{\partial y^2}\right.
$$

$$
\left.\left. - \frac{\partial^2\varphi_i^{(6)}}{\partial x^2}\frac{\partial^2\varphi_j^{(6)}}{\partial y^2} - \frac{\partial^2\varphi_i^{(6)}}{\partial y^2}\frac{\partial^2\varphi_j^{(6)}}{\partial x^2}\right)\right] dxdy \tag{4.105}
$$

$$
K_{ij}^{0607} = \int_\Omega A_{31}^{(2)}\varphi_i^{(6)}\frac{\partial\varphi_j^{(7)}}{\partial x} + 2B_{55}^{(2)}\frac{\partial\varphi_i^{(6)}}{\partial x}\varphi_j^{(7)} + \frac{1}{2}\left[R_{66}^{(2)}\left(\frac{\partial^2\varphi_i^{(6)}}{\partial y^2}\frac{\partial\varphi_j^{(7)}}{\partial x}\right.\right.
$$

$$
\left. - \frac{\partial^2\varphi_i^{(6)}}{\partial x^2}\frac{\partial\varphi_j^{(7)}}{\partial x}\right) - R_{22}^{(2)}\frac{\partial^2\varphi_i^{(6)}}{\partial x\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y} - R_{44}^{(0)}\frac{\partial\varphi_i^{(6)}}{\partial x}\varphi_j^{(7)}
$$

$$+ \frac{1}{2}\left(R_{44}^{(2)}\frac{\partial \varphi_i^{(6)}}{\partial x}\frac{\partial^2 \varphi_j^{(7)}}{\partial y^2} - R_{55}^{(2)}\frac{\partial \varphi_i^{(6)}}{\partial y}\frac{\partial^2 \varphi_j^{(7)}}{\partial x \partial y}\right)\Bigg] dxdy \tag{4.106}$$

$$K_{ij}^{0608} = \int_\Omega A_{32}^{(2)}\varphi_i^{(6)}\frac{\partial \varphi_j^{(8)}}{\partial y} + 2B_{44}^{(2)}\frac{\partial \varphi_i^{(6)}}{\partial y}\varphi_j^{(8)} + \frac{1}{2}\Bigg[R_{66}^{(2)}\left(\frac{\partial^2 \varphi_i^{(6)}}{\partial x^2}\frac{\partial \varphi_j^{(8)}}{\partial y}\right.$$

$$\left. - \frac{\partial^2 \varphi_i^{(6)}}{\partial y^2}\frac{\partial \varphi_j^{(8)}}{\partial y}\right) - R_{11}^{(2)}\frac{\partial^2 \varphi_i^{(6)}}{\partial x \partial y}\frac{\partial \varphi_j^{(8)}}{\partial x} - R_{55}^{(0)}\frac{\partial \varphi_i^{(6)}}{\partial y}\varphi_j^{(8)}$$

$$- \frac{1}{2}\left(R_{44}^{(2)}\frac{\partial \varphi_i^{(6)}}{\partial x}\frac{\partial^2 \varphi_j^{(8)}}{\partial x \partial y} - R_{55}^{(2)}\frac{\partial \varphi_i^{(6)}}{\partial y}\frac{\partial^2 \varphi_j^{(8)}}{\partial x^2}\right)\Bigg] dxdy \tag{4.107}$$

$$K_{ij}^{0609} = \int_\Omega 2A_{33}^{(1)}\varphi_i^{(6)}\varphi_j^{(9)} + B_{44}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial y}\frac{\partial \varphi_j^{(9)}}{\partial y} + B_{55}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial x}\frac{\partial \varphi_j^{(9)}}{\partial x}$$

$$+ \frac{1}{4}\Bigg[R_{11}^{(3)}\frac{\partial^2 \varphi_i^{(6)}}{\partial x \partial y}\frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} + R_{22}^{(3)}\frac{\partial^2 \varphi_i^{(6)}}{\partial x \partial y}\frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} + 2\left(R_{44}^{(1)}\frac{\partial \varphi_i^{(6)}}{\partial x}\frac{\partial \varphi_j^{(9)}}{\partial x}\right.$$

$$\left. + R_{55}^{(1)}\frac{\partial \varphi_i^{(6)}}{\partial y}\frac{\partial \varphi_j^{(9)}}{\partial y}\right) + R_{66}^{(3)}\left(\frac{\partial^2 \varphi_i^{(6)}}{\partial x^2}\frac{\partial^2 \varphi_j^{(9)}}{\partial x^2} + \frac{\partial^2 \varphi_i^{(6)}}{\partial y^2}\frac{\partial^2 \varphi_j^{(9)}}{\partial y^2}\right.$$

$$\left. - \frac{\partial^2 \varphi_i^{(6)}}{\partial x^2}\frac{\partial^2 \varphi_j^{(9)}}{\partial y^2} - \frac{\partial^2 \varphi_i^{(6)}}{\partial y^2}\frac{\partial^2 \varphi_j^{(9)}}{\partial x^2}\right)\Bigg] dxdy \tag{4.108}$$

$$K_{ij}^{0610} = \int_\Omega A_{31}^{(3)}\varphi_i^{(6)}\frac{\partial \varphi_j^{(10)}}{\partial x} + 3B_{55}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial x}\varphi_j^{(10)} + \frac{3}{4}\Bigg[R_{66}^{(3)}\left(\frac{\partial^2 \varphi_i^{(6)}}{\partial y^2}\frac{\partial \varphi_j^{(10)}}{\partial x}\right.$$

$$\left. - \frac{\partial^2 \varphi_i^{(6)}}{\partial x^2}\frac{\partial \varphi_j^{(10)}}{\partial x}\right) - R_{22}^{(3)}\frac{\partial^2 \varphi_i^{(6)}}{\partial x \partial y}\frac{\partial \varphi_j^{(10)}}{\partial y} + \frac{1}{3}\left(R_{44}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial x}\frac{\partial^2 \varphi_j^{(10)}}{\partial y^2}\right.$$

$$\left. - R_{55}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial y}\frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y}\right) - 2R_{44}^{(1)}\frac{\partial \varphi_i^{(6)}}{\partial x}\varphi_j^{(10)}\Bigg] dxdy \tag{4.109}$$

$$K_{ij}^{0611} = \int_\Omega A_{32}^{(3)}\varphi_i^{(6)}\frac{\partial \varphi_j^{(11)}}{\partial y} + 3B_{44}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial y}\varphi_j^{(11)} + \frac{3}{4}\Bigg[R_{66}^{(3)}\left(\frac{\partial^2 \varphi_i^{(6)}}{\partial x^2}\frac{\partial \varphi_j^{(11)}}{\partial y}\right.$$

$$\left. - \frac{\partial^2 \varphi_i^{(6)}}{\partial y^2}\frac{\partial \varphi_j^{(11)}}{\partial y}\right) - R_{11}^{(3)}\frac{\partial^2 \varphi_i^{(6)}}{\partial x \partial y}\frac{\partial \varphi_j^{(11)}}{\partial x} - \frac{1}{3}\left(R_{44}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial x}\frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y}\right.$$

$$\left. - R_{55}^{(3)}\frac{\partial \varphi_i^{(6)}}{\partial y}\frac{\partial^2 \varphi_j^{(11)}}{\partial x^2}\right) - 2R_{55}^{(1)}\frac{\partial \varphi_i^{(6)}}{\partial y}\varphi_j^{(11)}\Bigg] dxdy \tag{4.110}$$

$$K_{ij}^{0701} = \int_\Omega A_{11}^{(2)}\frac{\partial \varphi_i^{(7)}}{\partial x}\frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(2)}\frac{\partial \varphi_i^{(7)}}{\partial y}\frac{\partial \varphi_j^{(1)}}{\partial y} + \frac{1}{4}\left(R_{44}^{(2)}\frac{\partial^2 \varphi_i^{(7)}}{\partial y^2}\frac{\partial^2 \varphi_j^{(1)}}{\partial y^2}\right.$$

$$+ R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(7)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \Bigg) - \frac{1}{2} R_{44}^{(0)} \varphi_i^{(7)} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} dxdy \tag{4.111}$$

$$K_{ij}^{0702} = \int_\Omega A_{12}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial y} + B_{66}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial x} - \frac{1}{4} \Bigg(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(7)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y}$$

$$+ R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(7)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \Bigg) + \frac{1}{2} R_{44}^{(0)} \varphi_i^{(7)} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} dxdy \tag{4.112}$$

$$K_{ij}^{0703} = \frac{1}{2} \int_\Omega \frac{\partial w}{\partial x} \left(A_{11}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} \right)$$

$$+ \frac{\partial w}{\partial y} \left(A_{12}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} \right) + 4 B_{55}^{(1)} \varphi_i^{(7)} \frac{\partial \varphi_j^{(3)}}{\partial x}$$

$$+ R_{66}^{(1)} \left(\frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} - \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} \right) - R_{22}^{(1)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y} dxdy \tag{4.113}$$

$$K_{ij}^{0704} = \int_\Omega A_{11}^{(3)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(3)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial y} + 2 B_{55}^{(1)} \varphi_i^{(7)} \varphi_j^{(4)}$$

$$+ \frac{1}{2} \Bigg[R_{22}^{(1)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial y} + R_{33}^{(1)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial y}$$

$$- R_{44}^{(1)} \varphi_i^{(7)} \frac{\partial^2 \varphi_j^{(4)}}{\partial y^2} + R_{66}^{(1)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial x}$$

$$+ \frac{1}{2} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(7)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(4)}}{\partial y^2} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(7)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y} \right) \Bigg] dxdy \tag{4.114}$$

$$K_{ij}^{0705} = \int_\Omega A_{12}^{(3)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial y} + B_{66}^{(3)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial x}$$

$$- \frac{1}{2} \Bigg[R_{33}^{(1)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial x} - R_{44}^{(1)} \varphi_i^{(7)} \frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y} + R_{66}^{(1)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial y}$$

$$+ \frac{1}{2} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(7)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(7)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(5)}}{\partial x^2} \right) \Bigg] dxdy \tag{4.115}$$

$$K_{ij}^{0706} = \int_\Omega A_{13}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} \varphi_j^{(6)} + 2 B_{55}^{(2)} \varphi_i^{(7)} \frac{\partial \varphi_j^{(6)}}{\partial x} + \frac{1}{2} \Bigg[R_{66}^{(2)} \Bigg(\frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial^2 \varphi_j^{(6)}}{\partial y^2}$$

$$- \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial^2 \varphi_j^{(6)}}{\partial x^2} \Bigg) - R_{22}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial^2 \varphi_j^{(6)}}{\partial x \partial y} - R_{44}^{(0)} \varphi_i^{(7)} \frac{\partial \varphi_j^{(6)}}{\partial x}$$

$$+ \frac{1}{2}\left(R_{44}^{(2)}\frac{\partial^2\varphi_i^{(7)}}{\partial y^2}\frac{\partial\varphi_j^{(6)}}{\partial x} - R_{55}^{(2)}\frac{\partial^2\varphi_i^{(7)}}{\partial x\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y}\right)\Bigg] dxdy \quad (4.116)$$

$$K_{ij}^{0707} = \int_\Omega A_{11}^{(4)}\frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial\varphi_j^{(7)}}{\partial x} + B_{66}^{(4)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y} + 4B_{55}^{(2)}\varphi_i^{(7)}\varphi_j^{(7)}$$
$$+ R_{22}^{(2)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y} + R_{33}^{(2)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y} + R_{66}^{(2)}\frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial\varphi_j^{(7)}}{\partial x}$$
$$+ R_{44}^{(0)}\varphi_i^{(7)}\varphi_j^{(7)} - \frac{1}{2}R_{44}^{(2)}\left(\frac{\partial^2\varphi_i^{(7)}}{\partial y^2}\varphi_j^{(7)} + \varphi_i^{(7)}\frac{\partial^2\varphi_j^{(7)}}{\partial y^2}\right)$$
$$+ \frac{1}{4}\left(R_{44}^{(4)}\frac{\partial^2\varphi_i^{(7)}}{\partial y^2}\frac{\partial^2\varphi_j^{(7)}}{\partial y^2} + R_{55}^{(4)}\frac{\partial^2\varphi_i^{(7)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(7)}}{\partial x\partial y}\right) dxdy \quad (4.117)$$

$$K_{ij}^{0708} = \int_\Omega A_{12}^{(4)}\frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial y} + B_{66}^{(4)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x} - R_{33}^{(2)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x}$$
$$- R_{66}^{(2)}\frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial y} + \frac{1}{2}\Bigg[R_{44}^{(2)}\varphi_i^{(7)}\frac{\partial^2\varphi_j^{(8)}}{\partial x\partial y} + R_{55}^{(2)}\frac{\partial^2\varphi_i^{(7)}}{\partial x\partial y}\varphi_j^{(8)}$$
$$- \frac{1}{2}\left(R_{44}^{(4)}\frac{\partial^2\varphi_i^{(7)}}{\partial y^2}\frac{\partial^2\varphi_j^{(8)}}{\partial x\partial y} + R_{55}^{(4)}\frac{\partial^2\varphi_i^{(7)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(8)}}{\partial x^2}\right)\Bigg] dxdy \quad (4.118)$$

$$K_{ij}^{0709} = \int_\Omega 2A_{13}^{(3)}\frac{\partial\varphi_i^{(7)}}{\partial x}\varphi_j^{(9)} + 2B_{55}^{(3)}\varphi_i^{(7)}\frac{\partial\varphi_j^{(9)}}{\partial x}$$
$$+ \frac{1}{2}\Bigg[R_{66}^{(3)}\left(\frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial^2\varphi_j^{(9)}}{\partial y^2} - \frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial^2\varphi_j^{(9)}}{\partial x^2}\right) - R_{22}^{(3)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial^2\varphi_j^{(9)}}{\partial x\partial y}$$
$$+ R_{44}^{(3)}\frac{\partial^2\varphi_i^{(7)}}{\partial y^2}\frac{\partial\varphi_j^{(9)}}{\partial x} - R_{55}^{(3)}\frac{\partial^2\varphi_i^{(7)}}{\partial x\partial y}\frac{\partial\varphi_j^{(9)}}{\partial y} - 2R_{44}^{(1)}\varphi_i^{(7)}\frac{\partial\varphi_j^{(9)}}{\partial x}\Bigg] dxdy \quad (4.119)$$

$$K_{ij}^{0710} = \int_\Omega A_{11}^{(5)}\frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial\varphi_j^{(10)}}{\partial x} + B_{66}^{(5)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y} + 6B_{55}^{(3)}\varphi_i^{(7)}\varphi_j^{(10)}$$
$$+ 3R_{44}^{(1)}\varphi_i^{(7)}\varphi_j^{(10)} + \frac{3}{2}\Bigg[R_{22}^{(3)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y} + R_{33}^{(3)}\frac{\partial\varphi_i^{(7)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y}$$
$$- R_{44}^{(3)}\frac{\partial^2\varphi_i^{(7)}}{\partial y^2}\varphi_j^{(10)} + R_{66}^{(3)}\frac{\partial\varphi_i^{(7)}}{\partial x}\frac{\partial\varphi_j^{(10)}}{\partial x} + \frac{1}{6}\Bigg(R_{44}^{(5)}\frac{\partial^2\varphi_i^{(7)}}{\partial y^2}\frac{\partial^2\varphi_j^{(10)}}{\partial y^2}$$
$$+ R_{55}^{(5)}\frac{\partial^2\varphi_i^{(7)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(10)}}{\partial x\partial y}\Bigg) - \frac{1}{3}R_{44}^{(3)}\varphi_i^{(7)}\frac{\partial^2\varphi_j^{(10)}}{\partial y^2}\Bigg] dxdy \quad (4.120)$$

$$K_{ij}^{0711} = \int_\Omega A_{12}^{(5)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial y} + B_{66}^{(5)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial x} - \frac{3}{2} \left[R_{33}^{(3)} \frac{\partial \varphi_i^{(7)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial x} \right.$$

$$- R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(7)}}{\partial x \partial y} \varphi_j^{(11)} + R_{66}^{(3)} \frac{\partial \varphi_i^{(7)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial y} - \frac{1}{3} R_{44}^{(3)} \varphi_i^{(7)} \frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y}$$

$$\left. + \frac{1}{6} \left(R_{44}^{(5)} \frac{\partial^2 \varphi_i^{(7)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y} + R_{55}^{(5)} \frac{\partial^2 \varphi_i^{(7)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(11)}}{\partial x^2} \right) \right] dxdy \tag{4.121}$$

$$K_{ij}^{0801} = \int_\Omega A_{21}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial y} + \frac{1}{2} R_{55}^{(0)} \varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y}$$

$$- \frac{1}{4} \left(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} + R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy \tag{4.122}$$

$$K_{ij}^{0802} = \int_\Omega A_{22}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial y} + B_{66}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial x} - \frac{1}{2} R_{55}^{(0)} \varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2}$$

$$+ \frac{1}{4} \left(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} + R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \right) dxdy \tag{4.123}$$

$$K_{ij}^{0803} = \frac{1}{2} \int_\Omega \frac{\partial w}{\partial x} \left(A_{21}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} \right)$$

$$+ \frac{\partial w}{\partial y} \left(A_{22}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} \right) + 4 B_{44}^{(1)} \varphi_i^{(8)} \frac{\partial \varphi_j^{(3)}}{\partial y}$$

$$+ R_{66}^{(1)} \left(\frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} - \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} \right) - R_{11}^{(1)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y} dxdy \tag{4.124}$$

$$K_{ij}^{0804} = \int_\Omega A_{21}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial y}$$

$$- \frac{1}{2} \left[R_{33}^{(1)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial y} - R_{55}^{(1)} \varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y} + R_{66}^{(1)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial x} \right.$$

$$\left. + \frac{1}{2} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(4)}}{\partial y^2} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y} \right) \right] dxdy \tag{4.125}$$

$$K_{ij}^{0805} = \int_\Omega A_{22}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial y} + B_{66}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial x} + 2 B_{44}^{(1)} \varphi_i^{(8)} \varphi_j^{(5)}$$

$$+ \frac{1}{2} \left[R_{11}^{(1)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial x} + R_{33}^{(1)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial x} \right.$$

$$
\begin{aligned}
& - R_{55}^{(1)} \varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(5)}}{\partial x^2} + R_{66}^{(1)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial y} \\
& + \frac{1}{2} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(5)}}{\partial x^2} \right) \Bigg] \, dxdy
\end{aligned} \tag{4.126}
$$

$$
\begin{aligned}
K_{ij}^{0806} = & \int_\Omega A_{23}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \varphi_j^{(6)} + 2B_{44}^{(2)} \varphi_i^{(8)} \frac{\partial \varphi_j^{(6)}}{\partial y} + \frac{1}{2} \Bigg[R_{66}^{(2)} \Bigg(\frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial^2 \varphi_j^{(6)}}{\partial x^2} \\
& - \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial^2 \varphi_j^{(6)}}{\partial y^2} \Bigg) - R_{11}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial^2 \varphi_j^{(6)}}{\partial x \partial y} - R_{55}^{(0)} \varphi_i^{(8)} \frac{\partial \varphi_j^{(6)}}{\partial y} \\
& - \frac{1}{2} \left(R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial \varphi_j^{(6)}}{\partial x} - R_{55}^{(2)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial \varphi_j^{(6)}}{\partial y} \right) \Bigg] \, dxdy
\end{aligned} \tag{4.127}
$$

$$
\begin{aligned}
K_{ij}^{0807} = & \int_\Omega A_{21}^{(4)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(7)}}{\partial x} + B_{66}^{(4)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(7)}}{\partial y} - R_{33}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(7)}}{\partial y} \\
& - R_{66}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(7)}}{\partial x} + \frac{1}{2} \Bigg[R_{44}^{(2)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \varphi_j^{(7)} + R_{55}^{(2)} \varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(7)}}{\partial x \partial y} \\
& - \frac{1}{2} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(7)}}{\partial y^2} + R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(7)}}{\partial x \partial y} \right) \Bigg] \, dxdy
\end{aligned} \tag{4.128}
$$

$$
\begin{aligned}
K_{ij}^{0808} = & \int_\Omega A_{22}^{(4)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(8)}}{\partial y} + B_{66}^{(4)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(8)}}{\partial x} + 4B_{44}^{(2)} \varphi_i^{(8)} \varphi_j^{(8)} \\
& + R_{55}^{(0)} \varphi_i^{(8)} \varphi_j^{(8)} + R_{11}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(8)}}{\partial x} + R_{33}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(8)}}{\partial x} \\
& + R_{66}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(8)}}{\partial y} - \frac{1}{2} R_{55}^{(2)} \left(\varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(8)}}{\partial x^2} + \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \varphi_j^{(8)} \right) \\
& + \frac{1}{4} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(8)}}{\partial x \partial y} + R_{55}^{(4)} \frac{\partial^2 \varphi_j^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \right) dxdy
\end{aligned} \tag{4.129}
$$

$$
\begin{aligned}
K_{ij}^{0809} = & \int_\Omega 2A_{23}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial y} \varphi_j^{(9)} + 2B_{44}^{(3)} \varphi_i^{(8)} \frac{\partial \varphi_j^{(9)}}{\partial y} - R_{55}^{(1)} \varphi_i^{(8)} \frac{\partial \varphi_j^{(9)}}{\partial y} \\
& - \frac{1}{2} \Bigg[R_{11}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} + R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial \varphi_j^{(9)}}{\partial x} - R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial \varphi_j^{(9)}}{\partial y} \\
& + R_{66}^{(3)} \left(\frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial y^2} - \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial x^2} \right) \Bigg] \, dxdy
\end{aligned} \tag{4.130}
$$

$$K_{ij}^{0810} = \int_\Omega A_{21}^{(5)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial x} + B_{66}^{(5)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial y} - \frac{3}{2} \Bigg[R_{33}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial y}$$
$$- R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \varphi_j^{(10)} + R_{66}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial x} - \frac{1}{3} R_{55}^{(3)} \varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y}$$
$$+ \frac{1}{6} \left(R_{44}^{(5)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(10)}}{\partial y^2} + R_{55}^{(5)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y} \right) \Bigg] dxdy \tag{4.131}$$

$$K_{ij}^{0811} = \int_\Omega A_{22}^{(5)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial y} + B_{66}^{(5)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial x} + 6 B_{44}^{(3)} \varphi_i^{(8)} \varphi_j^{(11)}$$
$$+ 3 R_{55}^{(1)} \varphi_i^{(8)} \varphi_j^{(11)} + \frac{3}{2} \Bigg[R_{11}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial x} + R_{33}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial x} \frac{\partial \varphi_j^{(11)}}{\partial x}$$
$$- R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \varphi_j^{(11)} + R_{66}^{(3)} \frac{\partial \varphi_i^{(8)}}{\partial y} \frac{\partial \varphi_j^{(11)}}{\partial y} - \frac{1}{3} R_{55}^{(3)} \varphi_i^{(8)} \frac{\partial^2 \varphi_j^{(11)}}{\partial x^2}$$
$$+ \frac{1}{6} \left(R_{44}^{(5)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y} + R_{55}^{(5)} \frac{\partial^2 \varphi_i^{(8)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(11)}}{\partial x^2} \right) \Bigg] dxdy \tag{4.132}$$

$$K_{ij}^{0901} = \int_\Omega 2 A_{31}^{(1)} \varphi_i^{(9)} \frac{\partial \varphi_j^{(1)}}{\partial x} + \frac{1}{2} \left(R_{44}^{(1)} \frac{\partial \varphi_i^{(9)}}{\partial x} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} - R_{55}^{(1)} \frac{\partial \varphi_i^{(9)}}{\partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy \tag{4.133}$$

$$K_{ij}^{0902} = \int_\Omega 2 A_{32}^{(1)} \varphi_i^{(9)} \frac{\partial \varphi_j^{(2)}}{\partial y} - \frac{1}{2} \left(R_{44}^{(1)} \frac{\partial \varphi_i^{(9)}}{\partial x} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} - R_{55}^{(1)} \frac{\partial \varphi_i^{(9)}}{\partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \right) dxdy \tag{4.134}$$

$$K_{ij}^{0903} = \int_\Omega \frac{\partial w}{\partial x} A_{31}^{(1)} \varphi_i^{(9)} \frac{\partial \varphi_j^{(3)}}{\partial x} + \frac{\partial w}{\partial y} A_{32}^{(1)} \varphi_i^{(9)} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{44}^{(2)} \frac{\partial \varphi_i^{(9)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y}$$
$$+ B_{55}^{(2)} \frac{\partial \varphi_i^{(9)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} + \frac{1}{4} \Bigg[R_{66}^{(2)} \Bigg(\frac{\partial^2 \varphi_i^{(9)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} + \frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} \frac{\partial^2 \varphi_i^{(9)}}{\partial y^2}$$
$$- \frac{\partial^2 \varphi_i^{(9)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} - \frac{\partial^2 \varphi_i^{(9)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} \Bigg) + R_{11}^{(2)} \frac{\partial^2 \varphi_i^{(9)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y}$$
$$+ R_{22}^{(2)} \frac{\partial^2 \varphi_i^{(9)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y} \Bigg] dxdy \tag{4.135}$$

$$K_{ij}^{0904} = \int_\Omega 2 A_{31}^{(2)} \varphi_i^{(9)} \frac{\partial \varphi_j^{(4)}}{\partial x} + B_{55}^{(2)} \frac{\partial \varphi_i^{(9)}}{\partial x} \varphi_j^{(4)}$$
$$- R_{22}^{(2)} \frac{\partial^2 \varphi_i^{(9)}}{\partial x \partial y} \frac{\partial \varphi_j^{(4)}}{\partial y} + \frac{1}{4} \Bigg[R_{66}^{(2)} \left(\frac{\partial^2 \varphi_i^{(9)}}{\partial y^2} \frac{\partial \varphi_j^{(4)}}{\partial x} - \frac{\partial^2 \varphi_i^{(9)}}{\partial x^2} \frac{\partial \varphi_j^{(4)}}{\partial x} \right)$$

$$+ 2\left(R_{44}^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial x}\frac{\partial^2\varphi_j^{(4)}}{\partial y^2} - R_{55}^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial y}\frac{\partial^2\varphi_j^{(4)}}{\partial x\partial y}\right)\Bigg] dxdy \tag{4.136}$$

$$K_{ij}^{0905} = \int_\Omega 2A_{32}^{(2)}\varphi_i^{(9)}\frac{\partial\varphi_j^{(5)}}{\partial y} + B_{44}^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial y}\varphi_j^{(5)}$$

$$+ \frac{1}{4}\Bigg[R_{66}^{(2)}\left(\frac{\partial^2\varphi_i^{(9)}}{\partial x^2}\frac{\partial\varphi_j^{(5)}}{\partial y} - \frac{\partial^2\varphi_i^{(9)}}{\partial y^2}\frac{\partial\varphi_j^{(5)}}{\partial y}\right) - R_{11}^{(2)}\frac{\partial^2\varphi_i^{(9)}}{\partial x\partial y}\frac{\partial\varphi_j^{(5)}}{\partial x}$$

$$- 2\left(R_{44}^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial x}\frac{\partial^2\varphi_j^{(5)}}{\partial x\partial y} - R_{55}^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial y}\frac{\partial^2\varphi_j^{(5)}}{\partial x^2}\right)\Bigg] dxdy \tag{4.137}$$

$$K_{ij}^{0906} = \int_\Omega 2A_{33}^{(1)}\varphi_i^{(9)}\varphi_j^{(6)} + B_{44}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y} + B_{55}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial x}\frac{\partial\varphi_j^{(6)}}{\partial x}$$

$$+ \frac{1}{4}\Bigg[R_{11}^{(3)}\frac{\partial^2\varphi_i^{(9)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(6)}}{\partial x\partial y} + R_{22}^{(3)}\frac{\partial^2\varphi_i^{(9)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(6)}}{\partial x\partial y} + 2\Bigg(R_{44}^{(1)}\frac{\partial\varphi_i^{(9)}}{\partial x}\frac{\partial\varphi_j^{(6)}}{\partial x}$$

$$+ R_{55}^{(1)}\frac{\partial\varphi_i^{(9)}}{\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y}\Bigg) + R_{66}^{(3)}\Bigg(\frac{\partial^2\varphi_i^{(9)}}{\partial x^2}\frac{\partial^2\varphi_j^{(6)}}{\partial x^2} + \frac{\partial^2\varphi_i^{(9)}}{\partial y^2}\frac{\partial^2\varphi_j^{(6)}}{\partial y^2}$$

$$- \frac{\partial^2\varphi_i^{(9)}}{\partial x^2}\frac{\partial^2\varphi_j^{(6)}}{\partial y^2} - \frac{\partial^2\varphi_i^{(9)}}{\partial y^2}\frac{\partial^2\varphi_j^{(6)}}{\partial x^2}\Bigg)\Bigg] dxdy \tag{4.138}$$

$$K_{ij}^{0907} = \int_\Omega 2A_{31}^{(3)}\varphi_i^{(9)}\frac{\partial\varphi_j^{(7)}}{\partial x} + 2B_{55}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial x}\varphi_j^{(7)}$$

$$+ \frac{1}{2}\Bigg[R_{66}^{(3)}\left(\frac{\partial^2\varphi_i^{(9)}}{\partial y^2}\frac{\partial\varphi_j^{(7)}}{\partial x} - \frac{\partial^2\varphi_i^{(9)}}{\partial x^2}\frac{\partial\varphi_j^{(7)}}{\partial x}\right) - R_{22}^{(3)}\frac{\partial^2\varphi_i^{(9)}}{\partial x\partial y}\frac{\partial\varphi_j^{(7)}}{\partial y}$$

$$+ R_{44}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial x}\frac{\partial^2\varphi_j^{(7)}}{\partial y^2} - R_{55}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial y}\frac{\partial^2\varphi_j^{(7)}}{\partial x\partial y} - 2R_{44}^{(1)}\frac{\partial\varphi_i^{(9)}}{\partial x}\varphi_j^{(7)}\Bigg] dxdy \tag{4.139}$$

$$K_{ij}^{0908} = \int_\Omega 2A_{32}^{(3)}\varphi_i^{(9)}\frac{\partial\varphi_j^{(8)}}{\partial y} + 2B_{44}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial y}\varphi_j^{(8)}$$

$$+ \frac{1}{2}\Bigg[R_{66}^{(3)}\left(\frac{\partial^2\varphi_i^{(9)}}{\partial x^2}\frac{\partial\varphi_j^{(8)}}{\partial y} - \frac{\partial^2\varphi_i^{(9)}}{\partial y^2}\frac{\partial\varphi_j^{(8)}}{\partial y}\right) - R_{11}^{(3)}\frac{\partial^2\varphi_i^{(9)}}{\partial x\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x}$$

$$- R_{44}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial x}\frac{\partial^2\varphi_j^{(8)}}{\partial x\partial y} + R_{55}^{(3)}\frac{\partial\varphi_i^{(9)}}{\partial y}\frac{\partial^2\varphi_j^{(8)}}{\partial x^2} - 2R_{55}^{(1)}\frac{\partial\varphi_i^{(9)}}{\partial y}\varphi_j^{(8)}\Bigg] dxdy \tag{4.140}$$

$$K_{ij}^{0909} = \int_\Omega 4A_{33}^{(2)}\varphi_i^{(9)}\varphi_j^{(9)} + B_{44}^{(4)}\frac{\partial\varphi_i^{(9)}}{\partial y}\frac{\partial\varphi_j^{(9)}}{\partial y} + B_{55}^{(4)}\frac{\partial\varphi_i^{(9)}}{\partial x}\frac{\partial\varphi_j^{(9)}}{\partial x}$$

$$+ R_{44}^{(2)} \frac{\partial \varphi_i^{(9)}}{\partial x} \frac{\partial \varphi_j^{(9)}}{\partial x} + R_{55}^{(2)} \frac{\partial \varphi_i^{(9)}}{\partial y} \frac{\partial \varphi_j^{(9)}}{\partial y} + \frac{1}{4} \left[R_{11}^{(4)} \frac{\partial^2 \varphi_i^{(9)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} \right.$$

$$+ R_{22}^{(4)} \frac{\partial^2 \varphi_i^{(9)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} + R_{66}^{(4)} \left(\frac{\partial^2 \varphi_i^{(9)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(9)}}{\partial x^2} + \frac{\partial^2 \varphi_i^{(9)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(9)}}{\partial y^2} \right.$$

$$\left. \left. - \frac{\partial^2 \varphi_i^{(9)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(9)}}{\partial y^2} - \frac{\partial^2 \varphi_i^{(9)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(9)}}{\partial x^2} \right) \right] dxdy \tag{4.141}$$

$$K_{ij}^{0910} = \int_\Omega 2A_{31}^{(4)} \varphi_i^{(9)} \frac{\partial \varphi_j^{(10)}}{\partial x} - 3 \left(R_{44}^{(2)} \frac{\partial \varphi_i^{(9)}}{\partial x} \varphi_j^{(10)} - B_{55}^{(4)} \frac{\partial \varphi_i^{(9)}}{\partial x} \varphi_j^{(10)} \right)$$

$$- \frac{3}{4} \left[R_{22}^{(4)} \frac{\partial^2 \varphi_i^{(9)}}{\partial x \partial y} \frac{\partial \varphi_j^{(10)}}{\partial y} + R_{66}^{(4)} \left(\frac{\partial^2 \varphi_i^{(9)}}{\partial x^2} \frac{\partial \varphi_j^{(10)}}{\partial x} - \frac{\partial^2 \varphi_i^{(9)}}{\partial y^2} \frac{\partial \varphi_j^{(10)}}{\partial x} \right) \right.$$

$$\left. - \frac{2}{3} \left(R_{44}^{(4)} \frac{\partial \varphi_i^{(9)}}{\partial x} \frac{\partial^2 \varphi_j^{(10)}}{\partial y^2} - R_{55}^{(4)} \frac{\partial \varphi_i^{(9)}}{\partial y} \frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y} \right) \right] dxdy \tag{4.142}$$

$$K_{ij}^{0911} = \int_\Omega 2A_{32}^{(4)} \varphi_i^{(9)} \frac{\partial \varphi_j^{(11)}}{\partial y} + 3 \left(B_{44}^{(4)} \frac{\partial \varphi_i^{(9)}}{\partial y} \varphi_j^{(11)} - R_{55}^{(2)} \frac{\partial \varphi_i^{(9)}}{\partial y} \varphi_j^{(11)} \right)$$

$$- \frac{3}{4} \left[R_{11}^{(4)} \frac{\partial^2 \varphi_i^{(9)}}{\partial x \partial y} \frac{\partial \varphi_j^{(11)}}{\partial x} + R_{66}^{(4)} \left(\frac{\partial^2 \varphi_i^{(9)}}{\partial y^2} \frac{\partial \varphi_j^{(11)}}{\partial y} - \frac{\partial^2 \varphi_i^{(9)}}{\partial x^2} \frac{\partial \varphi_j^{(11)}}{\partial y} \right) \right.$$

$$\left. + \frac{2}{3} \left(R_{44}^{(4)} \frac{\partial \varphi_i^{(9)}}{\partial x} \frac{\partial^2 \varphi_j^{(11)}}{\partial x \partial y} - R_{55}^{(4)} \frac{\partial \varphi_i^{(9)}}{\partial y} \frac{\partial^2 \varphi_j^{(11)}}{\partial x^2} \right) \right] dxdy \tag{4.143}$$

$$K_{ij}^{1001} = \int_\Omega A_{11}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial y} - \frac{3}{2} R_{44}^{(1)} \varphi_i^{(10)} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2}$$

$$+ \frac{1}{4} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(10)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(10)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy \tag{4.144}$$

$$K_{ij}^{1002} = \int_\Omega A_{12}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial x} \frac{\partial \varphi_j^{(2)}}{\partial y} + B_{66}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial x} + \frac{3}{2} R_{44}^{(1)} \varphi_i^{(10)} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y}$$

$$- \frac{1}{4} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(10)}}{\partial y^2} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(10)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \right) dxdy \tag{4.145}$$

$$K_{ij}^{1003} = \frac{1}{2} \int_\Omega \frac{\partial w}{\partial x} \left(A_{11}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} \right)$$

$$+ \frac{\partial w}{\partial y} \left(A_{12}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} \right)$$

$$+ 6B_{55}^{(2)}\varphi_i^{(10)}\frac{\partial \varphi_j^{(3)}}{\partial x} + \frac{3}{2}\left[R_{66}^{(2)}\left(\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} - \frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial^2 \varphi_j^{(3)}}{\partial x^2}\right)\right.$$

$$\left. - R_{22}^{(2)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y}\right] dxdy \tag{4.146}$$

$$K_{ij}^{1004} = \int_\Omega A_{11}^{(4)}\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(4)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(4)}}{\partial y} + 3B_{55}^{(2)}\varphi_i^{(10)}\varphi_j^{(4)}$$

$$+ \frac{3}{4}\left[R_{22}^{(2)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(4)}}{\partial y} + R_{33}^{(2)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(4)}}{\partial y}\right.$$

$$- 2R_{44}^{(2)}\varphi_i^{(10)}\frac{\partial^2 \varphi_j^{(4)}}{\partial y^2} + R_{66}^{(2)}\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial \varphi_j^{(4)}}{\partial x}$$

$$\left. + \frac{1}{3}\left(R_{44}^{(4)}\frac{\partial^2 \varphi_i^{(10)}}{\partial y^2}\frac{\partial^2 \varphi_j^{(4)}}{\partial y^2} + R_{55}^{(4)}\frac{\partial^2 \varphi_i^{(10)}}{\partial x \partial y}\frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y}\right)\right] dxdy \tag{4.147}$$

$$K_{ij}^{1005} = \int_\Omega A_{12}^{(4)}\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial \varphi_j^{(5)}}{\partial y} + B_{66}^{(4)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(5)}}{\partial x}$$

$$- \frac{3}{4}\left[R_{33}^{(2)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(5)}}{\partial x} + R_{66}^{(2)}\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial \varphi_j^{(5)}}{\partial y} - 2R_{44}^{(2)}\varphi_i^{(10)}\frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y}\right.$$

$$\left. + \frac{1}{3}\left(R_{44}^{(4)}\frac{\partial^2 \varphi_i^{(10)}}{\partial y^2}\frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y} + R_{55}^{(4)}\frac{\partial^2 \varphi_i^{(10)}}{\partial x \partial y}\frac{\partial^2 \varphi_j^{(5)}}{\partial x^2}\right)\right] dxdy \tag{4.148}$$

$$K_{ij}^{1006} = \int_\Omega A_{13}^{(3)}\frac{\partial \varphi_i^{(10)}}{\partial x}\varphi_j^{(6)} + 3B_{55}^{(3)}\varphi_i^{(10)}\frac{\partial \varphi_j^{(6)}}{\partial x} + \frac{3}{4}\left[R_{66}^{(3)}\left(\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial^2 \varphi_j^{(6)}}{\partial y^2}\right.\right.$$

$$\left. - \frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial^2 \varphi_j^{(6)}}{\partial x^2}\right) - R_{22}^{(3)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial^2 \varphi_j^{(6)}}{\partial x \partial y} + \frac{1}{3}\left(R_{44}^{(3)}\frac{\partial^2 \varphi_i^{(10)}}{\partial y^2}\frac{\partial \varphi_j^{(6)}}{\partial x}\right.$$

$$\left.\left. - R_{55}^{(3)}\frac{\partial^2 \varphi_i^{(10)}}{\partial x \partial y}\frac{\partial \varphi_j^{(6)}}{\partial y}\right) - 2R_{44}^{(1)}\varphi_i^{(10)}\frac{\partial \varphi_j^{(6)}}{\partial x}\right] dxdy \tag{4.149}$$

$$K_{ij}^{1007} = \int_\Omega A_{11}^{(5)}\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial \varphi_j^{(7)}}{\partial x} + B_{66}^{(5)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(7)}}{\partial y} + 6B_{55}^{(3)}\varphi_i^{(10)}\varphi_j^{(7)}$$

$$+ 3R_{44}^{(1)}\varphi_i^{(10)}\varphi_j^{(7)} + \frac{3}{2}\left[R_{22}^{(3)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(7)}}{\partial y} + R_{33}^{(3)}\frac{\partial \varphi_i^{(10)}}{\partial y}\frac{\partial \varphi_j^{(7)}}{\partial y}\right.$$

$$- R_{44}^{(3)}\varphi_i^{(10)}\frac{\partial^2 \varphi_j^{(7)}}{\partial y^2} + R_{66}^{(3)}\frac{\partial \varphi_i^{(10)}}{\partial x}\frac{\partial \varphi_j^{(7)}}{\partial x} - \frac{1}{3}R_{44}^{(3)}\frac{\partial^2 \varphi_i^{(10)}}{\partial y^2}\varphi_j^{(7)}$$

$$+\frac{1}{6}\left(R_{44}^{(5)}\frac{\partial^2\varphi_i^{(10)}}{\partial y^2}\frac{\partial^2\varphi_j^{(7)}}{\partial y^2}+R_{55}^{(5)}\frac{\partial^2\varphi_i^{(10)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(7)}}{\partial x\partial y}\right)\Bigg]\,dxdy \tag{4.150}$$

$$\begin{aligned}K_{ij}^{1008}=\int_\Omega & A_{12}^{(5)}\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial y}+B_{66}^{(5)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x}-\frac{3}{2}\Bigg[R_{33}^{(3)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial\varphi_j^{(8)}}{\partial x}\\ &-R_{44}^{(3)}\varphi_i^{(10)}\frac{\partial^2\varphi_j^{(8)}}{\partial x\partial y}+R_{66}^{(3)}\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial\varphi_j^{(8)}}{\partial y}-\frac{1}{3}R_{55}^{(3)}\frac{\partial^2\varphi_i^{(10)}}{\partial x\partial y}\varphi_j^{(8)}\\ &+\frac{1}{6}\left(R_{44}^{(5)}\frac{\partial^2\varphi_i^{(10)}}{\partial y^2}\frac{\partial^2\varphi_j^{(8)}}{\partial x\partial y}+R_{55}^{(5)}\frac{\partial^2\varphi_i^{(10)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(8)}}{\partial x^2}\right)\Bigg]\,dxdy\end{aligned} \tag{4.151}$$

$$\begin{aligned}K_{ij}^{1009}=\int_\Omega & 2A_{13}^{(4)}\frac{\partial\varphi_i^{(10)}}{\partial x}\varphi_j^{(9)}-3\left(R_{44}^{(2)}\varphi_i^{(10)}\frac{\partial\varphi_j^{(9)}}{\partial x}-B_{55}^{(4)}\varphi_i^{(10)}\frac{\partial\varphi_j^{(9)}}{\partial x}\right)\\ &-\frac{3}{4}\Bigg[R_{22}^{(4)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial^2\varphi_j^{(9)}}{\partial x\partial y}+R_{66}^{(4)}\left(\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial^2\varphi_j^{(9)}}{\partial x^2}-\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial^2\varphi_j^{(9)}}{\partial y^2}\right)\\ &-\frac{2}{3}\left(R_{44}^{(4)}\frac{\partial^2\varphi_i^{(10)}}{\partial y^2}\frac{\partial\varphi_j^{(9)}}{\partial x}-R_{55}^{(4)}\frac{\partial^2\varphi_i^{(10)}}{\partial x\partial y}\frac{\partial\varphi_j^{(9)}}{\partial y}\right)\Bigg]\,dxdy\end{aligned} \tag{4.152}$$

$$\begin{aligned}K_{ij}^{1010}=\int_\Omega & A_{11}^{(6)}\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial\varphi_j^{(10)}}{\partial x}+B_{66}^{(6)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y}+9\Big(B_{55}^{(4)}\varphi_i^{(10)}\varphi_j^{(10)}\\ &+R_{44}^{(2)}\varphi_i^{(10)}\varphi_j^{(10)}\Big)-\frac{3}{2}\Bigg[R_{44}^{(4)}\left(\varphi_i^{(10)}\frac{\partial^2\varphi_j^{(10)}}{\partial y^2}+\frac{\partial^2\varphi_i^{(10)}}{\partial y^2}\varphi_j^{(10)}\right)\\ &-\frac{3}{2}\left(R_{22}^{(4)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y}+R_{33}^{(4)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial\varphi_j^{(10)}}{\partial y}+R_{66}^{(4)}\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial\varphi_j^{(10)}}{\partial x}\right)\\ &-\frac{1}{6}\left(R_{44}^{(6)}\frac{\partial^2\varphi_i^{(10)}}{\partial y^2}\frac{\partial^2\varphi_j^{(10)}}{\partial y^2}+R_{55}^{(6)}\frac{\partial^2\varphi_i^{(10)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(10)}}{\partial x\partial y}\right)\Bigg]\,dxdy\end{aligned} \tag{4.153}$$

$$\begin{aligned}K_{ij}^{1011}=\int_\Omega & A_{12}^{(6)}\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial y}+B_{66}^{(6)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial\varphi_j^{(11)}}{\partial x}\\ &+\frac{3}{2}\Bigg[\left(R_{44}^{(4)}\varphi_i^{(10)}\frac{\partial^2\varphi_j^{(11)}}{\partial x\partial y}+R_{55}^{(4)}\frac{\partial^2\varphi_i^{(10)}}{\partial x\partial y}\varphi_j^{(11)}\right)\\ &-\frac{3}{2}\left(R_{33}^{(4)}\frac{\partial\varphi_i^{(10)}}{\partial y}\frac{\partial\varphi_j^{(11)}}{\partial x}+R_{66}^{(4)}\frac{\partial\varphi_i^{(10)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial y}\right)\\ &-\frac{1}{6}\left(R_{44}^{(6)}\frac{\partial^2\varphi_i^{(10)}}{\partial y^2}\frac{\partial^2\varphi_j^{(11)}}{\partial x\partial y}+R_{55}^{(6)}\frac{\partial^2\varphi_i^{(10)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(11)}}{\partial x^2}\right)\Bigg]\,dxdy\end{aligned} \tag{4.154}$$

$$K_{ij}^{1101} = \int_\Omega A_{21}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(1)}}{\partial x} + B_{66}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(1)}}{\partial y} + \frac{3}{2} R_{55}^{(1)} \varphi_i^{(11)} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y}$$
$$- \frac{1}{4} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} + R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(1)}}{\partial x \partial y} \right) dxdy \tag{4.155}$$

$$K_{ij}^{1102} = \int_\Omega A_{22}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(2)}}{\partial y} + B_{66}^{(3)} \frac{\partial \varphi_j^{(2)}}{\partial x} \frac{\partial \varphi_i^{(11)}}{\partial x} - \frac{3}{2} R_{55}^{(1)} \varphi_i^{(11)} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2}$$
$$+ \frac{1}{4} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_j^{(2)}}{\partial x \partial y} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} + R_{55}^{(3)} \frac{\partial^2 \varphi_j^{(2)}}{\partial x^2} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \right) dxdy \tag{4.156}$$

$$K_{ij}^{1103} = \frac{1}{2} \int_\Omega \frac{\partial w}{\partial x} \left(A_{21}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial x} + B_{66}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial y} \right)$$
$$+ \frac{\partial w}{\partial y} \left(A_{22}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} + B_{66}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} \right)$$
$$+ 6 B_{44}^{(2)} \varphi_i^{(11)} \frac{\partial \varphi_j^{(3)}}{\partial y} + \frac{3}{2} \left[R_{66}^{(2)} \left(\frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial x^2} - \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial^2 \varphi_j^{(3)}}{\partial y^2} \right) \right.$$
$$\left. - R_{11}^{(2)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial^2 \varphi_j^{(3)}}{\partial x \partial y} \right] dxdy \tag{4.157}$$

$$K_{ij}^{1104} = \int_\Omega A_{21}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial x} + B_{66}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial y}$$
$$- \frac{3}{4} \left[R_{33}^{(2)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(4)}}{\partial y} - 2 R_{55}^{(2)} \varphi_i^{(11)} \frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y} + R_{66}^{(2)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(4)}}{\partial x} \right.$$
$$\left. + \frac{1}{3} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(4)}}{\partial y^2} + R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(4)}}{\partial x \partial y} \right) \right] dxdy \tag{4.158}$$

$$K_{ij}^{1105} = \int_\Omega A_{22}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial y} + B_{66}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial x} + 3 B_{44}^{(2)} \varphi_i^{(11)} \varphi_j^{(5)}$$
$$+ \frac{3}{4} \left[R_{11}^{(2)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial x} + R_{33}^{(2)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(5)}}{\partial x} \right.$$
$$- 2 R_{55}^{(2)} \varphi_i^{(11)} \frac{\partial^2 \varphi_j^{(5)}}{\partial x^2} + R_{66}^{(2)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(5)}}{\partial y}$$
$$\left. + \frac{1}{3} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(5)}}{\partial x \partial y} + R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(5)}}{\partial x^2} \right) \right] dxdy \tag{4.159}$$

$$K_{ij}^{1106} = \int_{\Omega} A_{23}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} \varphi_j^{(6)} + 3B_{44}^{(3)} \varphi_i^{(11)} \frac{\partial \varphi_j^{(6)}}{\partial y} + \frac{3}{4} \left[R_{66}^{(3)} \left(\frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial^2 \varphi_j^{(6)}}{\partial x^2} \right.\right.$$

$$\left. - \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial^2 \varphi_j^{(6)}}{\partial y^2} \right) - R_{11}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial^2 \varphi_j^{(6)}}{\partial x \partial y} - \frac{1}{3} \left(R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \frac{\partial \varphi_j^{(6)}}{\partial x} \right.$$

$$\left.\left. - R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \frac{\partial \varphi_j^{(6)}}{\partial y} \right) - 2R_{55}^{(1)} \varphi_i^{(11)} \frac{\partial \varphi_j^{(6)}}{\partial y} \right] dxdy \tag{4.160}$$

$$K_{ij}^{1107} = \int_{\Omega} A_{21}^{(5)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(7)}}{\partial x} + B_{66}^{(5)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(7)}}{\partial y} - \frac{3}{2} \left[R_{33}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(7)}}{\partial y} \right.$$

$$- R_{55}^{(3)} \varphi_i^{(11)} \frac{\partial^2 \varphi_j^{(7)}}{\partial x \partial y} + R_{66}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(7)}}{\partial x} - \frac{1}{3} R_{44}^{(3)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \varphi_j^{(7)}$$

$$\left. + \frac{1}{6} \left(R_{44}^{(5)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(7)}}{\partial y^2} + R_{55}^{(5)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(7)}}{\partial x \partial y} \right) \right] dxdy \tag{4.161}$$

$$K_{ij}^{1108} = \int_{\Omega} A_{22}^{(5)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(8)}}{\partial y} + B_{66}^{(5)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(8)}}{\partial x} + 6B_{44}^{(3)} \varphi_i^{(11)} \varphi_j^{(8)}$$

$$+ 3R_{55}^{(1)} \varphi_i^{(11)} \varphi_j^{(8)} + \frac{3}{2} \left[R_{11}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(8)}}{\partial x} + R_{33}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(8)}}{\partial x} \right.$$

$$- R_{55}^{(3)} \varphi_i^{(11)} \frac{\partial^2 \varphi_j^{(8)}}{\partial x^2} + R_{66}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(8)}}{\partial y} - \frac{1}{3} R_{55}^{(3)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \varphi_j^{(8)}$$

$$\left. + \frac{1}{6} \left(R_{44}^{(5)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \frac{\partial^2 \varphi_j^{(8)}}{\partial x \partial y} + R_{55}^{(5)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \frac{\partial^2 \varphi_j^{(8)}}{\partial x^2} \right) \right] dxdy \tag{4.162}$$

$$K_{ij}^{1109} = \int_{\Omega} 2A_{23}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial y} \varphi_j^{(9)} + 3 \left(B_{44}^{(4)} \varphi_i^{(11)} \frac{\partial \varphi_j^{(9)}}{\partial y} - R_{55}^{(2)} \varphi_i^{(11)} \frac{\partial \varphi_j^{(9)}}{\partial y} \right)$$

$$- \frac{3}{4} \left[R_{11}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial^2 \varphi_j^{(9)}}{\partial x \partial y} + R_{66}^{(4)} \left(\frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial y^2} - \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial^2 \varphi_j^{(9)}}{\partial x^2} \right) \right.$$

$$\left. + \frac{2}{3} \left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \frac{\partial \varphi_j^{(9)}}{\partial x} - R_{55}^{(4)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x^2} \frac{\partial \varphi_j^{(9)}}{\partial y} \right) \right] dxdy \tag{4.163}$$

$$K_{ij}^{1110} = \int_{\Omega} A_{21}^{(6)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial x} + B_{66}^{(6)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial y} + \frac{3}{2} \left[\left(R_{44}^{(4)} \frac{\partial^2 \varphi_i^{(11)}}{\partial x \partial y} \varphi_j^{(10)} \right.\right.$$

$$\left. + R_{55}^{(4)} \varphi_i^{(11)} \frac{\partial^2 \varphi_j^{(10)}}{\partial x \partial y} \right) - \frac{3}{2} \left(R_{33}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial x} \frac{\partial \varphi_j^{(10)}}{\partial y} + R_{66}^{(4)} \frac{\partial \varphi_i^{(11)}}{\partial y} \frac{\partial \varphi_j^{(10)}}{\partial x} \right)$$

$$- \frac{1}{6}\left(R_{44}^{(6)}\frac{\partial^2\varphi_i^{(11)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(10)}}{\partial y^2} + R_{55}^{(6)}\frac{\partial^2\varphi_i^{(11)}}{\partial x^2}\frac{\partial^2\varphi_j^{(10)}}{\partial x\partial y}\right)\Bigg] dxdy \tag{4.164}$$

$$\begin{aligned} K_{ij}^{1111} = &\int_\Omega A_{22}^{(6)}\frac{\partial\varphi_i^{(11)}}{\partial y}\frac{\partial\varphi_j^{(11)}}{\partial y} + B_{66}^{(6)}\frac{\partial\varphi_i^{(11)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial x} + 9\left(B_{44}^{(4)}\varphi_i^{(11)}\varphi_j^{(11)}\right. \\ &\left. + R_{55}^{(2)}\varphi_i^{(11)}\varphi_j^{(11)}\right) - \frac{3}{2}\left[R_{55}^{(4)}\left(\varphi_i^{(11)}\frac{\partial^2\varphi_j^{(11)}}{\partial x^2} + \frac{\partial^2\varphi_i^{(11)}}{\partial x^2}\varphi_j^{(11)}\right)\right. \\ &- \frac{3}{2}\left(R_{11}^{(4)}\frac{\partial\varphi_i^{(11)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial x} + R_{33}^{(4)}\frac{\partial\varphi_i^{(11)}}{\partial x}\frac{\partial\varphi_j^{(11)}}{\partial x} + R_{66}^{(4)}\frac{\partial\varphi_i^{(11)}}{\partial y}\frac{\partial\varphi_j^{(11)}}{\partial y}\right) \\ &\left. - \frac{1}{6}\left(R_{44}^{(6)}\frac{\partial^2\varphi_i^{(11)}}{\partial x\partial y}\frac{\partial^2\varphi_j^{(11)}}{\partial x\partial y} + R_{55}^{(6)}\frac{\partial^2\varphi_i^{(11)}}{\partial x^2}\frac{\partial^2\varphi_j^{(11)}}{\partial x^2}\right)\right] dxdy \end{aligned} \tag{4.165}$$

where the plate stiffness are $\left(A_{ij}^{(k)}, B_{mm}^{(k)}, \text{and } R_{nn}^{(k)}\right)$, are

$$A_{ij}^{(k)} = \frac{1-\nu}{(1+\nu)(1-2\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz, \quad i = j \tag{4.166}$$

$$A_{ij}^{(k)} = \frac{\nu}{(1+\nu)(1-2\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz, \quad i \neq j \tag{4.167}$$

$$B_{mm}^{(k)} = \frac{1}{2(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz \tag{4.168}$$

$$R_{ii}^{(k)} = \frac{l^2}{(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz \tag{4.169}$$

$$R_{mm}^{(k)} = \frac{l^2}{2(1+\nu)}\int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z)\, dz \tag{4.170}$$

for $i = 1, 2, 3$, $j = 1, 2, 3$, $m = 4, 5, 6$, and $k = 1, 2, ..., 6$. The superscripts of mass and stiffness matrices vary 01 to 11 to distinguish I=1 and J=11 from I=11 and J=1. The explicit form of force vector, $\{F\}$, is

$$F_i^1 = \int_\Omega f_x^{(0)}\varphi_i^{(1)} - \frac{1}{2}c_z^{(0)}\frac{\partial\varphi_i^{(1)}}{\partial y} dxdy + \int_\Gamma t_x^{(0)}\varphi_i^{(1)} - \frac{1}{2}s_z^{(0)}\frac{\partial\varphi_i^{(1)}}{\partial y} ds \tag{4.171}$$

$$F_i^2 = \int_\Omega f_y^{(0)}\varphi_i^{(2)} + \frac{1}{2}c_z^{(0)}\frac{\partial\varphi_i^{(2)}}{\partial x}dxdy + \int_\Gamma t_y^{(0)}\varphi_i^{(2)} + \frac{1}{2}s_z^{(0)}\frac{\partial\varphi_i^{(2)}}{\partial x}ds \tag{4.172}$$

$$\begin{aligned} F_i^3 = &\int_\Omega f_z^{(0)}\varphi_i^{(3)} + \frac{1}{2}\left(c_x^{(0)}\frac{\partial\varphi_i^{(3)}}{\partial y} - c_y^{(0)}\frac{\partial\varphi_i^{(3)}}{\partial x}\right)dxdy \\ &+ \int_\Gamma t_z^{(0)}\varphi_i^{(3)} + \frac{1}{2}\left(s_x^{(0)}\frac{\partial\varphi_i^{(3)}}{\partial y} - s_y^{(0)}\frac{\partial\varphi_i^{(3)}}{\partial x}\right)ds \end{aligned} \tag{4.173}$$

$$\begin{aligned} F_i^4 = &\int_\Omega f_x^{(1)}\varphi_i^{(4)} + \frac{1}{2}\left(c_y^{(0)}\varphi_i^{(4)} - c_z^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\right)dxdy \\ &+ \int_\Gamma t_x^{(1)}\varphi_i^{(4)} + \frac{1}{2}\left(s_y^{(0)}\varphi_i^{(4)} - s_z^{(1)}\frac{\partial\varphi_i^{(4)}}{\partial y}\right)ds \end{aligned} \tag{4.174}$$

$$\begin{aligned} F_i^5 = &\int_\Omega f_y^{(1)}\varphi_i^{(5)} - \frac{1}{2}\left(c_x^{(0)}\varphi_i^{(5)} - c_z^{(1)}\frac{\partial\varphi_i^{(5)}}{\partial x}\right)dxdy \\ &+ \int_\Gamma t_y^{(1)}\varphi_i^{(5)} - \frac{1}{2}\left(s_x^{(0)}\varphi_i^{(5)} - s_z^{(1)}\frac{\partial\varphi_i^{(5)}}{\partial x}\right)ds \end{aligned} \tag{4.175}$$

$$\begin{aligned} F_i^6 = &\int_\Omega f_z^{(1)}\varphi_i^{(6)} + \frac{1}{2}\left(c_x^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial y} - c_y^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial x}\right)dxdy \\ &+ \int_\Gamma t_z^{(1)}\varphi_i^{(6)} + \frac{1}{2}\left(s_x^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial y} - s_y^{(1)}\frac{\partial\varphi_i^{(6)}}{\partial x}\right)ds \end{aligned} \tag{4.176}$$

$$\begin{aligned} F_i^7 = &\int_\Omega f_x^{(2)}\varphi_i^{(7)} + \frac{1}{2}\left(2c_y^{(1)}\varphi_i^{(7)} - c_z^{(2)}\frac{\partial\varphi_i^{(7)}}{\partial y}\right)dxdy \\ &+ \int_\Gamma t_x^{(2)}\varphi_i^{(7)} + \frac{1}{2}\left(2s_y^{(1)}\varphi_i^{(7)} - s_z^{(2)}\frac{\partial\varphi_i^{(7)}}{\partial y}\right)ds \end{aligned} \tag{4.177}$$

$$\begin{aligned} F_i^8 = &\int_\Omega f_y^{(2)}\varphi_i^{(8)} - \frac{1}{2}\left(2c_x^{(1)}\varphi_i^{(8)} - c_z^{(2)}\frac{\partial\varphi_i^{(8)}}{\partial x}\right)dxdy \\ &+ \int_\Gamma t_y^{(2)}\varphi_i^{(8)} - \frac{1}{2}\left(2s_x^{(1)}\varphi_i^{(8)} - s_z^{(2)}\frac{\partial\varphi_i^{(8)}}{\partial x}\right)ds \end{aligned} \tag{4.178}$$

$$\begin{aligned} F_i^9 = &\int_\Omega f_z^{(2)}\varphi_i^{(9)} + \frac{1}{2}\left(c_x^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial y} - c_y^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial x}\right)dxdy \\ &+ \int_\Gamma t_z^{(2)}\varphi_i^{(9)} + \frac{1}{2}\left(s_x^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial y} - s_y^{(2)}\frac{\partial\varphi_i^{(9)}}{\partial x}\right)ds \end{aligned} \tag{4.179}$$

$$F_i^{10} = \int_\Omega f_x^{(3)}\varphi_i^{(10)} + \frac{1}{2}\left(3c_y^{(2)}\varphi_i^{(10)} - c_z^{(3)}\frac{\partial\varphi_i^{(10)}}{\partial y}\right)dxdy$$

$$+\int_{\Gamma} t_x^{(3)}\varphi_i^{(10)} + \frac{1}{2}\left(3s_y^{(2)}\varphi_i^{(10)} - s_z^{(3)}\frac{\partial\varphi_i^{(10)}}{\partial y}\right) ds \tag{4.180}$$

$$F_i^{11} = \int_{\Omega} f_y^{(3)}\varphi_i^{(11)} - \frac{1}{2}\left(3c_x^{(2)}\varphi_i^{(11)} - c_z^{(3)}\frac{\partial\varphi_i^{(11)}}{\partial x}\right) dxdy$$

$$+\int_{\Gamma} t_y^{(3)}\varphi_i^{(11)} - \frac{1}{2}\left(3s_x^{(2)}\varphi_i^{(11)} - s_z^{(3)}\frac{\partial\varphi_i^{(11)}}{\partial x}\right) ds \tag{4.181}$$

where

$$f_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i\ \bar{f}_\xi\, dz + \left(\frac{h}{2}\right)^i \left[q_\xi^t + (-1)^i q_\xi^b\right], \qquad t_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i\ \bar{t}_\xi\, dz,$$

$$c_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i\ \bar{c}_\xi\, dz + \left(\frac{h}{2}\right)^i \left[p_\xi^t + (-1)^i p_\xi^b\right], \qquad s_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i\ \bar{s}_\xi\, dz$$

for $i = 0, 1, 2, 3$ and $\xi = x, y, z$.

4.3 Numerical examples

Here we present the results of a static bending analysis using the developed finite element model. For the purpose of illustration, we take following material properties. The Young's moduli of top and bottom surfaces are $E_t = 14.4 \times 10^9 N/m^2$ and $E_b = 1.44 \times 10^9 N/m^2$, respectively. The poisson's ratio, ν, is assumed as 0.38 for both materials. The plate thickness, h, is assumed to be $17.6 \times 10^{-6} m$, and the length of a square plate, a, is assumed to be $20h$. The full plate is used as computational domain shown in Figure 4.2. The 16 by 16 mesh is used to analysis the micro plate. Simply supported and clamped boundary conditions are applied to $x = \pm\frac{a}{2}$ and $y = \pm\frac{b}{2}$. In the case of the simply supported boundary conditions, SS1 and SS3 types [9] are considered. SS1 type boundary condition is that the in-plane displacement (u_1 or u_2) whose direction is parallel to the normal direction on a side of plates is free to move. SS3 type boundary condition is that all bending displacements with respect to in-plane coordinates are free to move on all sides of plates. The clamped boundary condition is that all in-plane displacements are

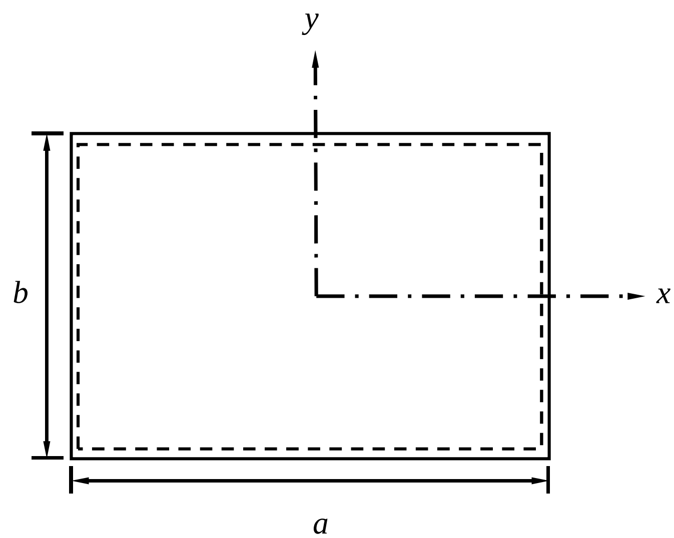

Figure 4.2: Computational domain of a full plate [3]

constrained on all sides of plates. For all types of boundary conditions, we constrain the transverse defection on middle surface, w, on all sides of plates. Since the developed finite element model is based on a modified couple stress theory, we need to define additional higher order boundary conditions (derivatives of displacements) in addition to boundary conditions of classical plate models. We keep the meaning of each boundary conditions, and define additions boundary conditions with respect to the rotation vector (2.36). For SS1 type, the rigid body rotation, ω_i, whose direction is parallel to the tangential direction of the side of plates is free to move, and other rotations are constrained. For example, only ω_y is not constrained at $x = \pm\frac{a}{2}$. Note that the rigid body rotation, ω_y and the bending rotations (θ_x, ϕ_x, and ψ_x) of u_1 are in the same direction. For SS3 type, the rigid body rotations with respect to in-plane coordinates are free to move, and all rigid body rotation are constrained for the clamped boundary condition. We assume that the rigid body rotation with respect to transverse direction, z, is constrained for all boundary cases. To compare with a classical model (the first order shear deformable plate model), SS1-0 type boundary condition which does not include any higher order terms is considered. The constrained degree of freedoms are shown in Table 4.1. Note that u_i and ω_j are defined in Eqs. (2.28)–(2.30) and (2.37)–(2.40) in terms of generalized displacements and their derivatives, $i = 1, 2, 3$ and $j = x, y, z$.

Table 4.1: Constrained degree of freedoms of each boundary conditions [3]

SS1-0	$x = \pm a/2$	$u_2,$	w				
	$y = \pm b/2$	$u_1,$	w				
SS1	$x = \pm a/2$	$u_2,$	$w,$	$\omega_x,$	ω_z		
	$y = \pm b/2$	$u_1,$	$w,$	$\omega_y,$	ω_z		
SS3	$x = \pm a/2$	$u,$	$v,$	$w,$	ω_z		
	$y = \pm b/2$	$u,$	$v,$	$w,$	ω_z		
Clamped	$x = \pm a/2,$ $y = \pm b/2$	$u_1,$	$u_2,$	$w,$	$\omega_x,$	$\omega_y,$	ω_z

To clearly see nonlinear behavior of a micro plate, $q_z^t = 5.4 \times 10^6 N/m^2$ is incrementally applied through 20 load steps. The Newton iteration scheme is used to solve the nonlinear equations [9]. Figures 4.3–4.6 show comparisons of the first order shear deformation plate theory (FSDT) using 8 by 8 quadratic element and the general third order plate theory (GTPT) using 16 by 16 cubic element in the case of homogenous material $(n = 0)$. The displacement, $\bar{u}_3 = \frac{u_3}{h}$, through thickness direction at the center of the plate and the middle plane deflection, $\bar{w} = \frac{w}{h}$, versus the load parameter, $\bar{q} = \frac{q_z^t a^4}{E_b h^4}$, are shown in Figs. 4.3 and 4.4. Since the general third order plate theory considers an extensible plate thickness, the deflection through thickness shows a quadratic variation (shown in Figure 4.3). Since the SS1 boundary condition constrains more degree of freedoms (derivatives of dependent variables), the system becomes slightly stiffer than SS1-0 boundary condition. Figures 4.5 and 4.6 show the comparison of bending and shear stresses of FSDT and GTPT at load parameter $\bar{q} = 50$.

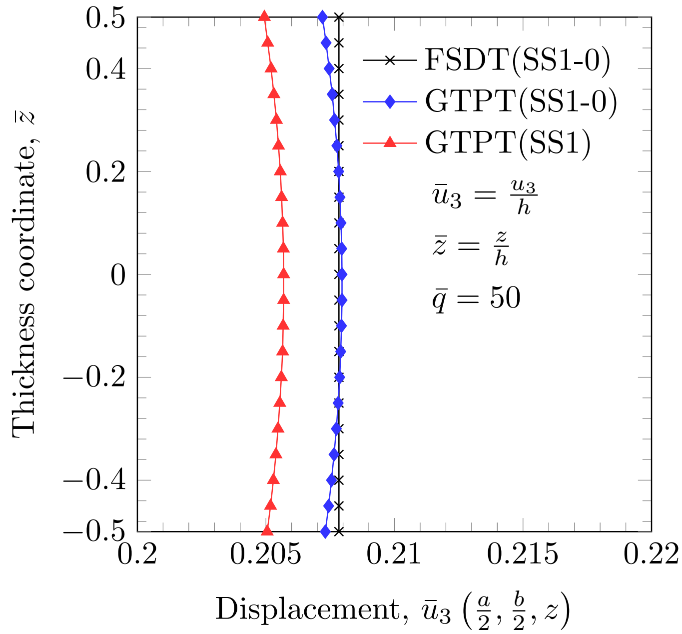

Figure 4.3: Comparison of center deflections of FSDT (8 by 8 quad.) and GTPT (16 by 16 cubic) through thickness [3]

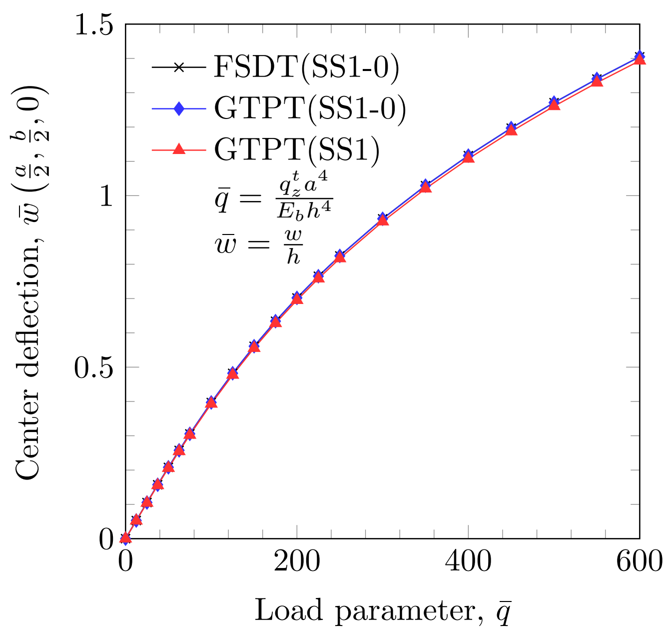

Figure 4.4: Comparison of middle plane deflections of FSDT (8 by 8 quad.) and GTPT (16 by 16 cubic) verse load parameter [3]

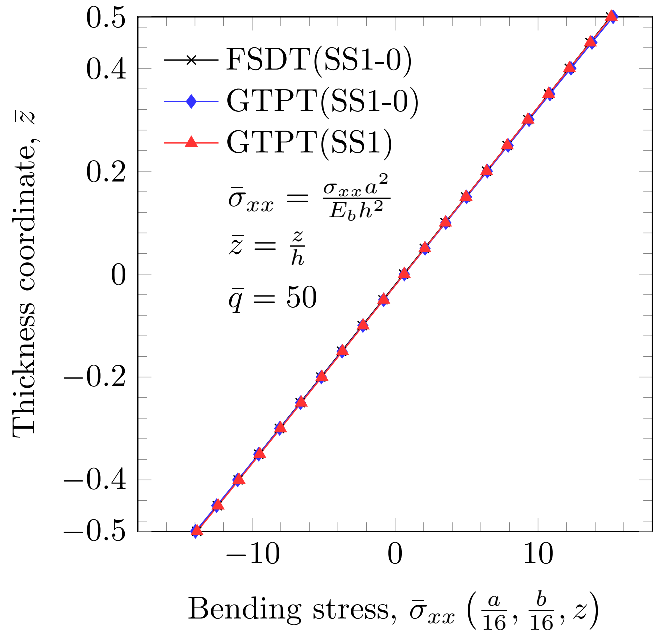

Figure 4.5: Comparison of bending stress of FSDT (8 by 8 quad.) and GTPT (16 by 16 cubic) through thickness [3]

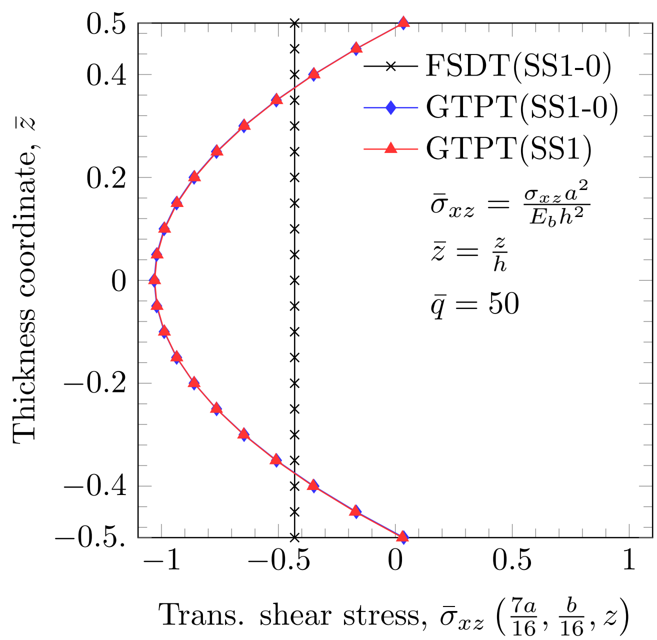

Figure 4.6: Comparison of shear stress of FSDT (8 by 8 quad.) and GTPT (16 by 16 cubic) through thickness [3]

The stresses of the GTPT are computed at nodal points instead of integration points since the displacement gradients are computed at nodes. To compare with the FSDT (8 by 8 quadratic element), the stresses of the FSDT are computed using one point Gaussian quadrature rule. The bending stress, σ_{xx}, and the transverse shear stress, σ_{xz}, are computed at $\left(\frac{a}{16}, \frac{b}{16}, z\right)$, and at $\left(\frac{7a}{16}, \frac{b}{16}, z\right)$ respectively. The bending stresses of the GTPT do not much differ from the bending stresses of the FSDT (shown in Fig. 4.5). Since the variation of transverse shear strains of the GTPT has a form of quadratic variation through thickness, the transverse shear stress in the case of homogenous material shows a quadratic variation (shown in Fig. 4.6). Note that we do not force the transverse shear strain to be zero on top and bottom surfaces and the transverse shear stresses on top and bottom surfaces are not exactly zero. Figures 4.7–4.9 show center deflections of middle plane, $w\left(\frac{a}{2}, b2, 0\right)$,

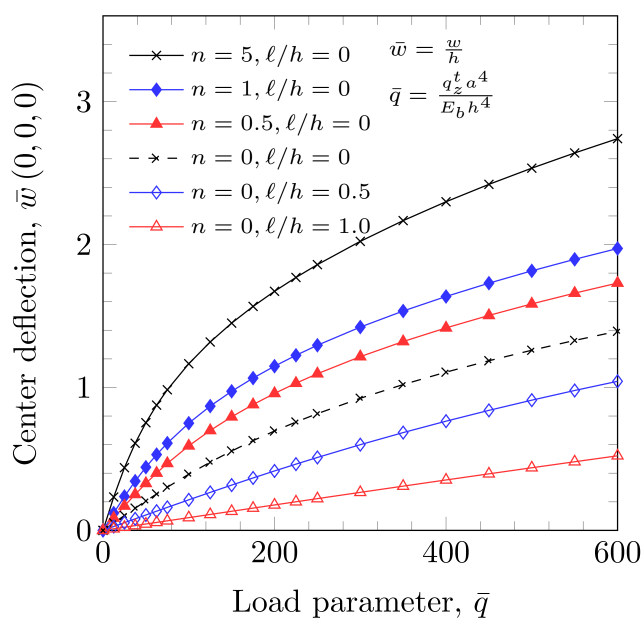

Figure 4.7: Middle plane deflection versus load parameter with SS1 boundary condition [3]

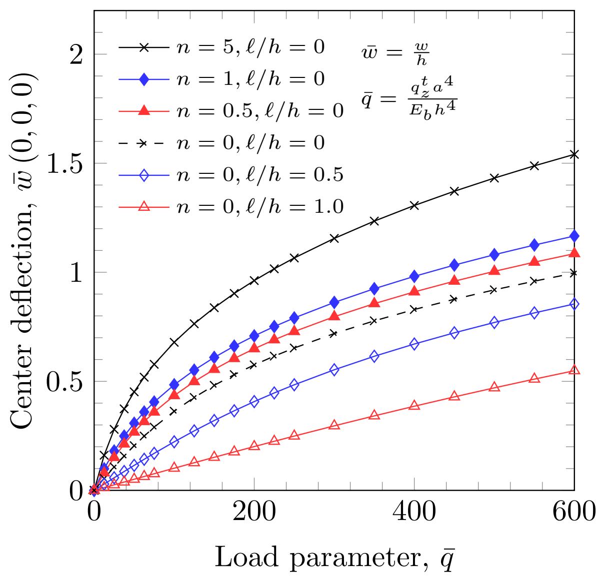

Figure 4.8: Middle plane deflection versus load parameter with SS3 boundary condition [3]

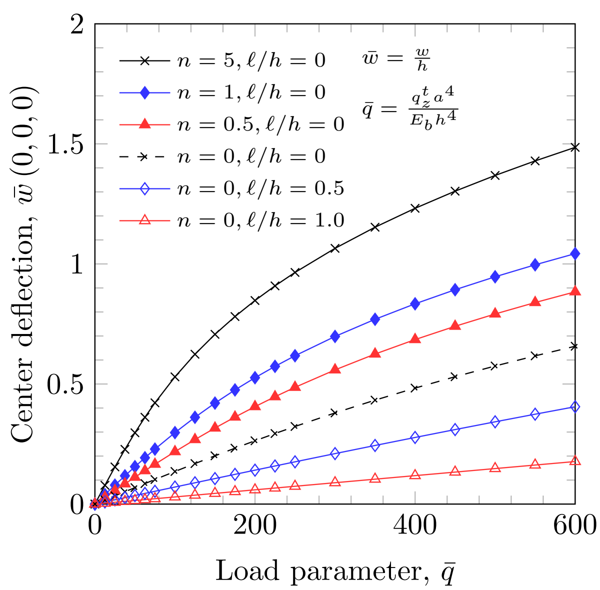

Figure 4.9: Middle plane deflection versus load parameter with clamped boundary condition [3]

versus load parameter, $\bar{q}$, with SS1, SS3, and clamped boundary conditions, respectively. When the power-law index is larger, the volume fraction of top surface material which is stiffer material decreases, and the plate stiffness in Eqs. (4.166)–(4.170) become softer, therefore the larger displacements are obtained for larger power-law index. In the case of considering the micro structure size effect, the stiffness matrices become stiffer due to the effect of the couple stress related terms, $R_{nn}^{(k)}, n = 1, 2, ..., 6$ in Eqs. (4.169) and (4.170), and smaller deflections are presented. Figures 4.10–4.15 show the bending and shear stresses through thickness direction with various boundary conditions at $\bar{q} = 50$.

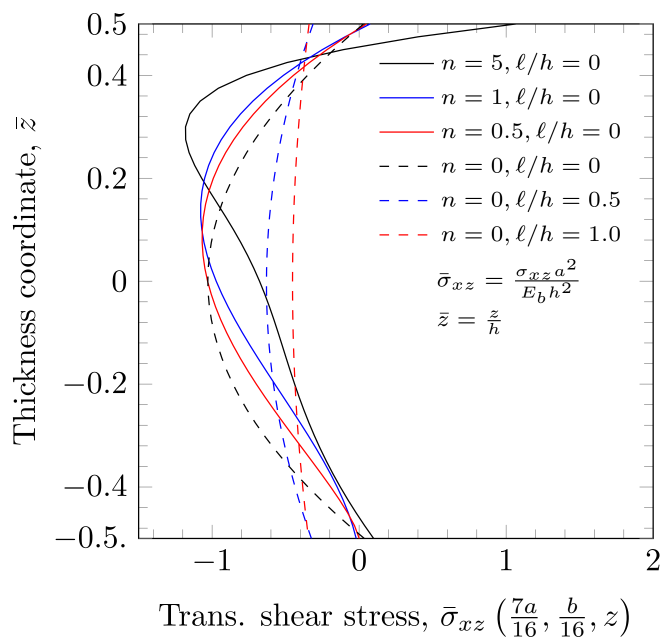

Figure 4.10: Bending stress through thickness with SS1 boundary condition [3]

The bending stresses and shear stresses are computed at $\left(\frac{a}{16}, \frac{b}{16}, z\right)$, and at $\left(\frac{7a}{16}, \frac{b}{16}, z\right)$ respectively. Unlike the homogenous plate, the variation of stresses of FGM plate not only depends on the variation of strains but also the variation of the material properties

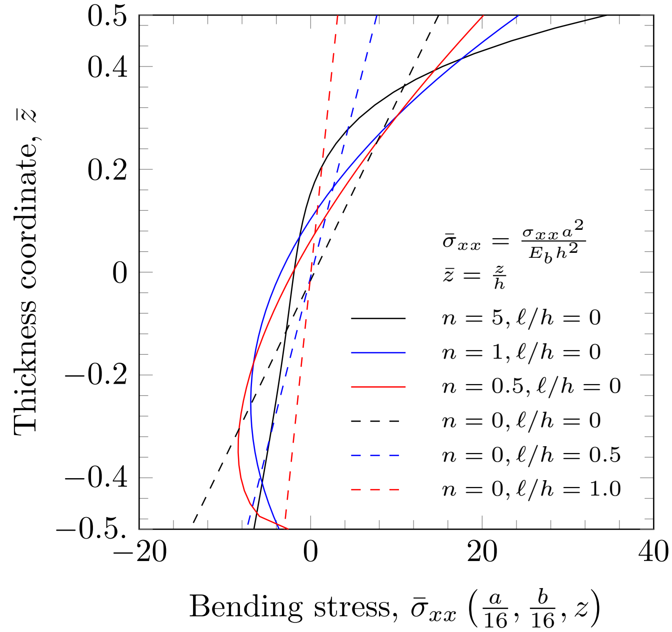

Figure 4.11: Transverse shear stress through thickness with SS1 boundary condition [3]

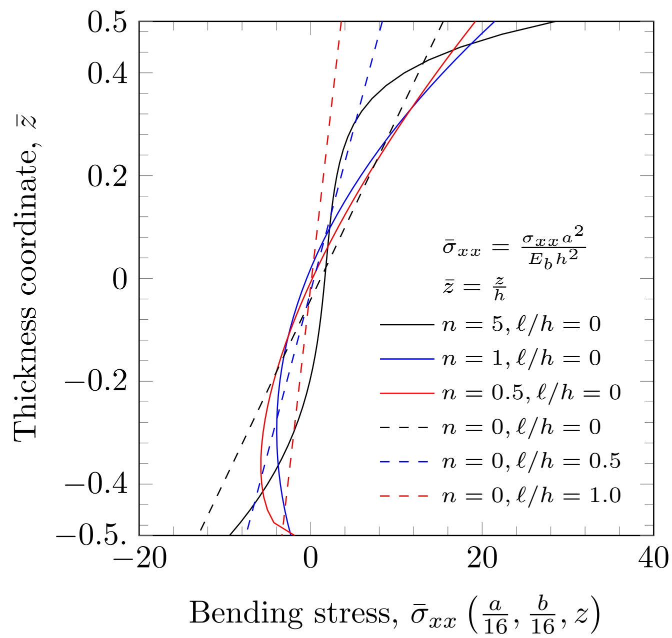

Figure 4.12: Bending stress through thickness with SS3 boundary condition [3]

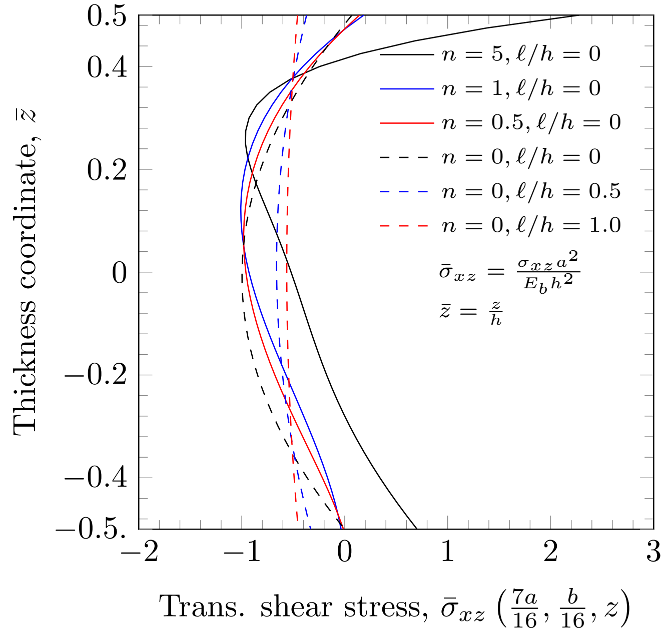

Figure 4.13: Transverse shear stress through thickness with SS3 boundary condition [3]

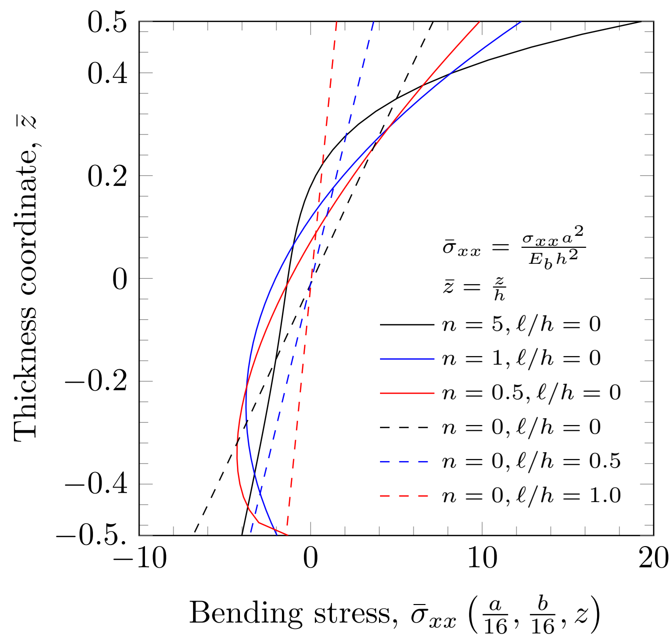

Figure 4.14: Bending stress through thickness with clamped boundary condition [3]

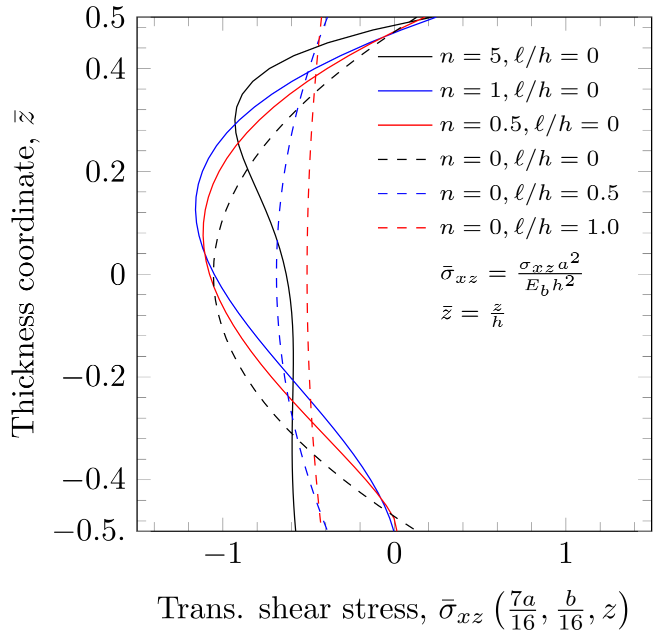

Figure 4.15: Transverse shear stress through thickness with clamped boundary condition [3]

in a body. For nonzero power-law index, the odd order of plate stiffnesses, $\left(A_{ij}^{(k)}, B_{mm}^{(k)}, \text{and } R_{nn}^{(k)}\right), k = 1, 3, 5$ are nonzero and contribute to system equation (4.2). When the odd order plate stiffnesses are nonzero, the contribution of the higher order generalized displacements to overall displacements increases. The effect of this phenomena makes larger nonzero transverse shear stresses on top and bottom surfaces.

5. PIEZOELECTRIC SMART PLATE *

Beams, plates, and shells are the most common structural elements used in a wide range of applications, including systems used for strain sensing and actuating. The response of such systems can be quite complex when there is magneto-electro-thermo-mechanical coupling. Through-thickness, functionally graded structures provide advantages over traditional laminated composite structures. Piezoelectric layers are surface mounted to these structural parts to monitor or control their functionality and structural integrity. Thus, it is useful to develop refined theories of plates that account for coupling of size effects, material gradation, and piezoelectric effects.

Many researchers have studied the behavior of structures under the effects of electric and/or magnetic fields in addition to thermomechanical loadings [81–83]. Crawley and de Luis [81] studied static and dynamic behavior of surface bonded and embedded piezoelectric actuator using Euler-Bernoulli beam model. Im and Atluri [82] presented a study of beam-column structure with two piezoelectric layers bonded on the upper and lower surfaces. Finite element models to study static and dynamic responses of beams with a bonded piezoelectric actuator is developed using four different displacement-based one-dimensional beam theories by Robinson and Reddy [83]. Tzou and Zhong [84] developed piezoelectric shell models using the Hamilton's principle. They presented a first order shear deformation shell model that accounts for electromechanical effects, and simplified the developed model to a classical shell model. Mitchell and Reddy [85] developed refined hybrid theory to study piezoelectric plates. They used an equivalent single-layer theory based on the displacement field of Reddy third order shear deformation theory [22] and a layerwise theory for the electric potential. Recently, Komijani, Reddy, and Eslami [86] investigated a nonlinear thermo-electro-mechanical response of functionally graded piezoelectric actuators. In their study, the Timoshenko beam model that accounts for the von Kármán nonlinearity, variation of two materials in transverse direction of the beam, and size effects of micro structure was presented. In this study, we develop a functionally graded smart plate model using a general third order plate theory that accounts for gradient elasticity effects, functionally graded materials and linear piezoelectricity. We assume functionally graded smart plates that consist of a functionally graded core layer and two piezoelectric layers bonded to top and bottom surfaces.

5.1 Formulation

In this study, Hamilton's principle is used to derive a set of governing equations of a plate with surface-mounted piezoelectric layers. The charge equations of electrostatics are coupled to the mechanical deformation by using a modified energy density function given by [85]

$$L = \int_v \left[\frac{1}{2} \rho \dot{u}_i \dot{u}_i - H\left(\varepsilon_{ij}, \chi_{ij}, E_i\right) \right] dxdydz \tag{5.1}$$

where $H\left(\varepsilon_{ij}, \chi_{ij}, E_i\right)$ is called the electric enthalpy density function having the strain tensor ε_{ij}, the symmetric part of curvature tensor χ_{ij}, and electric field E_i as arguments, ρ is the mass density, and $\dot{u}_i$ is the time derivative of the i^{th} displacement component. Numerous field theories can be derived based upon the particular selection of $H\left(\varepsilon_{ij}, \chi_{ij}, E_i\right)$. Of course, they must all follow according to the conservation laws of linear and angular momenta as well as energy. For this study, $H\left(\varepsilon_{ij}, \chi_{ij}, E_i\right)$ is taken as

$$H\left(\varepsilon_{ij}, \chi_{ij}, E_i\right) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \left(\varepsilon_{kl} - \delta_{ij} \alpha \Delta T\right) + \mu \ell^2 \chi_{ij} \chi_{ij} - e_{ijk} E_i \varepsilon_{jk} - \frac{1}{2} k_{ij} E_i E_j \tag{5.2}$$

where C_{ijkl}, μ, ℓ, e_{ijk}, and k_{ij} are called the elastic constant, the shear modulus, the material length scale parameter, piezoelectric, and dielectric permittivity constants, respectively [87]. The δ_{ij}, α, and ΔT, are the Kronecker delta, the thermal expansion coefficient, and temperature change. The electric field, E_i, is derivable from a scalar potential functions ϕ by

$$E_i = -\frac{\partial \phi}{\partial x_i}. \tag{5.3}$$

A complete theory of plates with surface mounted piezoelectric layers is developed with the help of Hamilton's principle

$$\begin{aligned} 0 = \int_0^T \Bigg\{ & \int_V \left[\rho \dot{u}_i \delta \dot{u}_i - \sigma_{ij} \delta \varepsilon_{ij} - m_{ij} \delta \chi_{ij} - D_i \left(\delta \phi \right)_{,i} \right] dV dt \\ & + \int_V \left(\bar{f}_i \delta u_i + \bar{c}_i \delta \omega_i + \bar{q} \delta \phi \right) dV + \int_S \left(\bar{t}_i \delta u_i + \bar{s}_i \delta \omega_i + \bar{d} \delta \phi \right) dS \\ & + \int_{\Omega^t} \left(q_i^t \delta u_i + p_i^t \delta \omega_i \right) d\Omega^t + \int_{\Omega^b} \left(q_i^b \delta u_i + p_i^b \delta \omega_i \right) d\Omega^b \Bigg\} dt \end{aligned} \tag{5.4}$$

where σ_{ij}, m_{ij}, and D_i are the symmetric parts of stress tensor and the deviatoric part of couple stress tensor, and the component of electric displacement vector which are derived from

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}} = C_{ijkl} \left(\varepsilon_{kl} - \delta_{ij} \alpha \Delta T \right) - e_{kij} E_k \tag{5.5}$$

$$m_{ij} = \frac{\partial H}{\partial \chi_{ij}} = 2\mu_{12} \ell^2 \chi_{ij} \tag{5.6}$$

$$D_i = -\frac{\partial H}{\partial E_i} = e_{ijk} \varepsilon_{jk} + k_{ij} E_j \tag{5.7}$$

and $\bar{f}_i$, $\bar{c}_i$, and $\bar{q}$ are the body forces and couples and the electric body charge,and $\bar{t}_i$, $\bar{s}_i$, and $\bar{d}$ are the surface forces, couples, and charges on the side surfaces, and q_i^α and p_i^α are the surface forces and couples on top $(\alpha = t)$ and bottom $(\alpha = b)$ surfaces.

5.2 Electric potential

The mostly used assumption of electric potential is an uniform distribution in the in-plane direction and a linear distribution in the transverse direction of plates, and this assumption violates Maxwell static electricity equation [88, 89]. Wang et al. [89] have been proposed a half-cosine electric potential distribution along transverse direction and numerically verified it using FEM. Wang [90] has been proposed a combination of a half-cosine and a linear variation of the electric potential. Based on his work, the assumed electric potential takes form of

$$\phi\left(x,y,z\right)=-\cos\left(\frac{\pi z_p}{h_p}\right)\bar{\phi}\left(x,y\right)+\frac{2z_pV_0}{h_p} \tag{5.8}$$

where z_p is measured from the geometrical center of the piezoelectric layer, h_p is the thickness of the piezoelectric layer, $\bar{\phi}\left(x,y\right)$ is a variation of electric potential in in-plane direction and it is measured at $z_p=0$, and V_0 is an externally applied electric voltage. The electric fields can be obtained by substituting Eq.(5.8) into Eq.(5.3)

$$\begin{aligned}
E_x &= \cos\left(\frac{\pi z_p}{h_p}\right)\frac{\partial\bar{\phi}\left(x,y\right)}{\partial x}\\
E_y &= \cos\left(\frac{\pi z_p}{h_p}\right)\frac{\partial\bar{\phi}\left(x,y\right)}{\partial y}\\
E_z &= -\frac{\pi}{h_p}\sin\left(\frac{\pi z_p}{h_p}\right)\bar{\phi}\left(x,y\right)-\frac{2}{h_p}V_0.
\end{aligned}$$

5.3 Constitutive relation

In this study, we consider a transversely isotropic materials including piezoelectric effect such as lead zirconate-titanate(PZT). The constitutive relation is given as following

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \varepsilon^T_{xx} \\ \varepsilon_{yy} - \varepsilon^T_{yy} \\ \varepsilon_{zz} - \varepsilon^T_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}
$$

$$
+ \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{Bmatrix} \tag{5.9}
$$

$$
\begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{zz} \\ m_{xz} \\ m_{yz} \\ m_{xy} \end{Bmatrix} = \begin{bmatrix} r_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & r_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{66} \end{bmatrix} \begin{Bmatrix} \chi_{xx} \\ \chi_{yy} \\ \chi_{zz} \\ \chi_{yz} \\ \chi_{xz} \\ \chi_{xy} \end{Bmatrix} \tag{5.10}
$$

$$
\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{13} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} - \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{11} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{Bmatrix} \tag{5.11}
$$

where $r_{11} = r_{22} = r_{33} = 2\mu_{12}\ell^2$ and $r_{44} = r_{55} = r_{66} = \mu_{12}\ell^2$, and the length scale parameter, ℓ, has meaning of the ratio of shear moduli. The non-piezoelectric laminae is assumed as a functionally graded material which of the material properties vary only through thickness direction. The variation of material properties are modeled using power-law distribution (2.9).

5.4 Equations of motion

The equations of motion is derived from Hamilton's principle (5.4). In the derivation, the temperature change ΔT is a known function of position. Thus, temperature field enters the formulation only through constitutive equations (5.9). Hamilton's principle (5.4) can be expressed as following (see [79])

$$\int_0^T (\delta\mathcal{K} - \delta\mathcal{U} - \delta\mathcal{E} - \delta\mathcal{V})\, dt = 0 \tag{5.12}$$

where $\delta\mathcal{K}$ is the virtual kinetic energy, $\delta\mathcal{U}$ is the virtual strain energy, $\delta\mathcal{E}$ is the contribution of the electric field, and $\delta\mathcal{V}$ is the virtual work done by external forces. The virtual kinetic energy, $\delta\mathcal{K}$ and the virtual strain energy, $\delta\mathcal{U}$ are given in Eqs. (2.47) and (2.49).

The contribution of the electric field, $\delta\mathcal{E}$, is given by

$$\begin{aligned}\delta\mathcal{E} &= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(D_1 \frac{\partial \delta\phi}{\partial x} + D_2 \frac{\partial \delta\phi}{\partial y} + D_3 \frac{\partial \delta\phi}{\partial z} \right) dzdxdy \\ &= \int_\Omega \left(P_1 \frac{\partial \delta\bar{\phi}\,(x,y)}{\partial x} + P_2 \frac{\partial \delta\bar{\phi}\,(x,y)}{\partial y} - P_3 \delta\bar{\phi}\,(x,y) \right) dxdy \end{aligned} \tag{5.13}$$

where

$$P_\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} D_\alpha \cos\left(\frac{\pi z_p}{h_p}\right) dz \;\text{ for }\; \alpha = 1,2, \;\text{ and }\; P_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} D_3 \frac{\pi}{h_p} \sin\left(\frac{\pi z_p}{h_p}\right) dz$$

where h_p is thickness of a piezoelectric layer. Note that z_p should be replaced with the global coordinate, z, and the range of the piezoelectric layer based on a specific problems.

The virtual work done by external forces and charge density is

$$
\begin{aligned}
\delta\mathcal{V} = &-\bigg[\int_V \left(\bar{f}_i\delta u_i + \bar{c}_i\delta\omega_i + \bar{q}\delta\phi\right) dV + \int_S \left(\bar{t}_i\delta u_i + \bar{s}_i\delta\omega_i + \bar{d}\delta\phi\right) dS \\
&+ \int_{\Omega^t} \left(q_i^t\delta u_i + p_i^t\delta\omega_i\right) d\Omega^t + \int_{\Omega^b} \left(q_i^b\delta u_i + p_i^b\delta\omega_i\right) d\Omega^b\bigg] \\
= &-\bigg\{\int_\Omega \bigg[F_x^{(0)}\delta u + F_y^{(0)}\delta v + F_z^{(0)}\delta w + F_x^{(1)}\delta\theta_x + F_y^{(1)}\delta\theta_y + F_x^{(2)}\delta\phi_x \\
&+ F_y^{(2)}\delta\phi_y + F_x^{(3)}\delta\psi_x + F_y^{(3)}\delta\psi_y + F_z^{(1)}\delta\theta_z + F_z^{(2)}\delta\phi_z \\
&+ \frac{1}{2}c_x^{(0)}\left(\frac{\partial\delta w}{\partial y} - \delta\theta_y\right) - \frac{1}{2}c_y^{(0)}\left(\frac{\partial\delta w}{\partial x} - \delta\theta_x\right) \\
&+ \frac{1}{2}c_z^{(0)}\left(\frac{\partial\delta v}{\partial x} - \frac{\partial\delta u}{\partial y}\right) + c_x^{(1)}\left(\frac{1}{2}\frac{\partial\delta\theta_z}{\partial y} - \delta\phi_y\right) \\
&- c_y^{(1)}\left(\frac{1}{2}\frac{\partial\delta\theta_z}{\partial x} - \delta\phi_x\right) + \frac{1}{2}c_z^{(1)}\left(\frac{\partial\delta\theta_y}{\partial x} - \frac{\partial\delta\theta_x}{\partial y}\right) \\
&+ \frac{1}{2}c_x^{(2)}\left(\frac{\partial\delta\phi_z}{\partial y} - 3\delta\psi_y\right) - \frac{1}{2}c_y^{(2)}\left(\frac{\partial\delta\phi_z}{\partial x} - 3\delta\psi_x\right) \\
&+ \frac{1}{2}c_z^{(2)}\left(\frac{\partial\delta\phi_y}{\partial x} - \frac{\partial\delta\phi_x}{\partial y}\right) + \frac{1}{2}c_z^{(3)}\left(\frac{\partial\delta\psi_y}{\partial x} - \frac{\partial\delta\psi_x}{\partial y}\right) + \hat{q}\bar{\phi}\bigg] dxdy \\
&+ \int_\Gamma \bigg[t_x^{(0)}\delta u + t_y^{(0)}\delta v + t_z^{(0)}\delta w + t_x^{(1)}\delta\theta_x + t_y^{(1)}\delta\theta_y + t_z^{(1)}\delta\theta_z \\
&+ t_x^{(2)}\delta\phi_x + t_y^{(2)}\delta\phi_y + t_z^{(2)}\delta\phi_z + t_x^{(3)}\delta\psi_x + t_y^{(3)}\delta\psi_y + \hat{d}\bar{\phi}\bigg] d\Gamma\bigg\}
\end{aligned}
\tag{5.14}
$$

where

$$
\begin{aligned}
f_\xi^{(i)} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i\, \bar{f}_\xi\, dz, \quad t_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i\, \bar{t}_\xi\, dz, \\
c_\xi^{(i)} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i\, \bar{c}_\xi\, dz + \left(\frac{h}{2}\right)^i \left[q_\xi^t + (-1)^i q_\xi^b\right], \\
F_\xi^{(i)} &= f_\xi^{(i)} + \left(\frac{h}{2}\right)^i \left[q_\xi^t + (-1)^i q_\xi^b\right],
\end{aligned}
$$

$$\hat{q} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{q} \cos\left(\frac{\pi z_p}{h_p}\right)\, dz, \;\; \hat{d} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{d} \cos\left(\frac{\pi z_p}{h_p}\right)\, dz$$

for $i = 0, 1, 2, 3$ and $\xi = x, y, z$.

The equations of motion are obtained by substituting $\delta\mathcal{K}$, $\delta\mathcal{U}$, $\delta\mathcal{E}$, and $\delta\mathcal{V}$, into Eq. (5.12) and applying the integration-by-parts to relieve all virtual generalized displacements and potential function of differentiations with respect to x, y and t.

$$\begin{aligned}
\delta u: \quad & \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + F_x^{(0)} + \frac{1}{2}\frac{\partial c_z^{(0)}}{\partial y} \\
& = m_0 \ddot{u} + m_1 \ddot{\theta}_x + m_2 \ddot{\phi}_x + m_3 \ddot{\psi}_x \qquad (5.15)
\end{aligned}$$

$$\begin{aligned}
\delta v: \quad & \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) + F_y^{(0)} - \frac{1}{2}\frac{\partial c_z^{(0)}}{\partial x} \\
& = m_0 \ddot{v} + m_1 \ddot{\theta}_y + m_2 \ddot{\phi}_y + m_3 \ddot{\psi}_y \qquad (5.16)
\end{aligned}$$

$$\begin{aligned}
\delta w: \quad & \frac{\partial M_{xz}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y}\right) \\
& + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}\right) + F_z^{(0)} + \frac{1}{2}\left(\frac{\partial c_y^{(0)}}{\partial x} - \frac{\partial c_x^{(0)}}{\partial y}\right) \\
& = m_0 \ddot{w} + m_1 \ddot{\theta}_z + m_2 \ddot{\phi}_z \qquad (5.17)
\end{aligned}$$

$$\begin{aligned}
\delta \theta_x: \quad & \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - M_{xz}^{(0)} + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(0)}}{\partial y}\right) \\
& + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) + F_x^{(1)} + \frac{1}{2}c_y^{(0)} + \frac{1}{2}\frac{\partial c_z^{(1)}}{\partial y} \\
& = m_1 \ddot{u} + m_2 \ddot{\theta}_x + m_3 \ddot{\phi}_x + m_4 \ddot{\psi}_x \qquad (5.18)
\end{aligned}$$

$$\begin{aligned}
\delta \theta_y: \quad & \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - M_{yz}^{(0)} - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(0)}}{\partial x}\right) \\
& - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) + F_y^{(1)} - \frac{1}{2}c_x^{(0)} - \frac{1}{2}\frac{\partial c_z^{(1)}}{\partial x} \\
& = m_1 \ddot{v} + m_2 \ddot{\theta}_y + m_3 \ddot{\phi}_y + m_4 \ddot{\psi}_y \qquad (5.19)
\end{aligned}$$

$$\delta\phi_x: \quad \frac{\partial M_{xx}^{(2)}}{\partial x} + \frac{\partial M_{xy}^{(2)}}{\partial y} - 2M_{xz}^{(1)} + \left(\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(1)}}{\partial y} - \mathcal{M}_{yz}^{(0)}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(2)}}{\partial y}\right) + F_x^{(2)} + c_y^{(1)} + \frac{1}{2}\frac{\partial c_z^{(2)}}{\partial y}$$

$$= m_2\ddot{u} + m_3\ddot{\theta}_x + m_4\ddot{\phi}_x + m_5\ddot{\psi}_x \tag{5.20}$$

$$\delta\phi_y: \quad \frac{\partial M_{xy}^{(2)}}{\partial x} + \frac{\partial M_{yy}^{(2)}}{\partial y} - 2M_{yz}^{(1)} - \left(\frac{\partial \mathcal{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(1)}}{\partial x} - \mathcal{M}_{xz}^{(0)}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(2)}}{\partial y}\right) + F_y^{(2)} - c_x^{(1)} - \frac{1}{2}\frac{\partial c_z^{(2)}}{\partial x}$$

$$= m_2\ddot{v} + m_3\ddot{\theta}_y + m_4\ddot{\phi}_y + m_5\ddot{\psi}_y \tag{5.21}$$

$$\delta\psi_x: \quad \frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y} - 3M_{xz}^{(2)} + \frac{3}{2}\left(\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(2)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(2)}}{\partial y} - 2\mathcal{M}_{yz}^{(1)}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xz}^{(3)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(3)}}{\partial y}\right) + F_x^{(3)} + \frac{3}{2}c_y^{(2)} + \frac{1}{2}\frac{\partial c_z^{(3)}}{\partial y}$$

$$= m_3\ddot{u} + m_4\ddot{\theta}_x + m_5\ddot{\phi}_x + m_6\ddot{\psi}_x \tag{5.22}$$

$$\delta\psi_y: \quad \frac{\partial M_{xy}^{(3)}}{\partial x} + \frac{\partial M_{yy}^{(3)}}{\partial y} - 3M_{yz}^{(2)} - \frac{3}{2}\left(\frac{\partial \mathcal{M}_{xx}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial y} - \frac{\partial \mathcal{M}_{zz}^{(2)}}{\partial x} - 2\mathcal{M}_{xz}^{(1)}\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{xz}^{(3)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(3)}}{\partial y}\right) + F_y^{(3)} - \frac{3}{2}c_x^{(2)} - \frac{1}{2}\frac{\partial c_z^{(3)}}{\partial x}$$

$$= m_3\ddot{v} + m_4\ddot{\theta}_y + m_5\ddot{\phi}_y + m_6\ddot{\psi}_y \tag{5.23}$$

$$\delta\theta_z: \quad \frac{\partial M_{xz}^{(1)}}{\partial x} + \frac{\partial M_{yz}^{(1)}}{\partial y} - M_{zz}^{(0)} - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial y}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{yy}^{(1)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial y} - \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial x}\right) + F_z^{(1)}$$

$$+ \frac{1}{2}\left(\frac{\partial c_y^{(1)}}{\partial x} - \frac{\partial c_x^{(1)}}{\partial y}\right) = m_1\ddot{w} + m_2\ddot{\theta}_z + m_3\ddot{\phi}_z \tag{5.24}$$

$$\delta\phi_z: \quad \frac{\partial M_{xz}^{(2)}}{\partial x} + \frac{\partial M_{yz}^{(2)}}{\partial y} - 2M_{zz}^{(1)} - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{M}_{xx}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial y}\right)$$

$$+ \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{M}_{yy}^{(2)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x}\right) + \frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial y} - \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial x} + F_z^{(2)}$$

$$+ \frac{1}{2}\left(\frac{\partial c_y^{(2)}}{\partial x} - \frac{\partial c_x^{(2)}}{\partial y}\right) = m_2\ddot{w} + m_3\ddot{\theta}_z + m_4\ddot{\phi}_z \tag{5.25}$$

$$\delta\bar{\phi}: \quad \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} - P_3 + \hat{q} = 0. \tag{5.26}$$

The boundary conditions involve specifying the following generalized forces that are dual to the generalized displacements and electric potentials (u, v, w, θ_x, θ_y, ϕ_x, ϕ_y, ψ_x, ψ_y, θ_z, ϕ_z, $\bar{\phi}$):

$$\delta u: \qquad 0 = M_{xx}^{(0)} n_x + M_{xy}^{(0)} n_y + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) n_y + \frac{1}{2} c_z^{(0)} n_y + t_x^{(0)} \tag{5.27}$$

$$\delta v: \qquad 0 = M_{xy}^{(0)} n_x + M_{yy}^{(0)} n_y - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial y}\right) n_x - \frac{1}{2} c_z^{(0)} n_x + t_y^{(0)} \tag{5.28}$$

$$\begin{aligned} \delta w: \qquad 0 = {} & M_{xz}^{(0)} n_x + M_{yz}^{(0)} n_y + t_z^{(0)} - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xx}^{(0)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial y}\right) n_y \\ & + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{yy}^{(0)}}{\partial y} + \frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}\right) n_x + \frac{1}{2}\left(c_y^{(0)} n_x - c_x^{(0)} n_y\right) \end{aligned} \tag{5.29}$$

$$\begin{aligned} \delta\theta_x: \qquad 0 = {} & M_{xx}^{(1)} n_x + M_{xy}^{(1)} n_y + \frac{1}{2} c_z^{(1)} n_y + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) n_y \\ & + \frac{1}{2}\left(\mathcal{M}_{xy}^{(0)} n_x + \mathcal{M}_{yy}^{(0)} n_y - \mathcal{M}_{zz}^{(1)} n_y\right) + t_x^{(1)} \end{aligned} \tag{5.30}$$

$$\begin{aligned} \delta\theta_y: \qquad 0 = {} & M_{xy}^{(1)} n_x + M_{yy}^{(1)} n_y - \frac{1}{2} c_z^{(1)} n_x - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial y}\right) n_x \\ & + \frac{1}{2}\left(\mathcal{M}_{xx}^{(0)} n_x + \mathcal{M}_{xy}^{(0)} n_y - \mathcal{M}_{zz}^{(1)} n_x\right) + t_y^{(1)} \end{aligned} \tag{5.31}$$

$$\begin{aligned} \delta\phi_x: \qquad 0 = {} & M_{xx}^{(2)} n_x + M_{xy}^{(2)} n_y + \frac{1}{2} c_z^{(2)} n_y + \left(\mathcal{M}_{xy}^{(1)} n_x + \mathcal{M}_{yy}^{(1)} n_y - \mathcal{M}_{zz}^{(1)} n_y\right) \\ & + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(2)}}{\partial y}\right) n_y + t_x^{(2)} \end{aligned} \tag{5.32}$$

$$\begin{aligned} \delta\phi_y: \qquad 0 = {} & M_{xy}^{(2)} n_x + M_{yy}^{(2)} n_y - \frac{1}{2} c_z^{(2)} n_x - \left(\mathcal{M}_{xx}^{(1)} n_x + \mathcal{M}_{xy}^{(1)} n_y - \mathcal{M}_{zz}^{(1)} n_x\right) \\ & - \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(2)}}{\partial y}\right) n_x + t_y^{(2)} \end{aligned} \tag{5.33}$$

$$\begin{aligned} \delta\psi_x: \qquad 0 = {} & M_{xx}^{(3)} n_x + M_{xy}^{(3)} n_y + \frac{1}{2} c_z^{(3)} n_y + \frac{3}{2}\left(\mathcal{M}_{xy}^{(2)} n_x + \mathcal{M}_{yy}^{(2)} n_y - \mathcal{M}_{zz}^{(2)} n_y\right) \\ & + \frac{1}{2}\left(\frac{\partial \mathcal{M}_{xz}^{(3)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(3)}}{\partial y}\right) n_y + t_x^{(3)} \end{aligned} \tag{5.34}$$

$$\delta\psi_y: \quad 0 = M_{xy}^{(3)} n_x + M_{yy}^{(3)} n_y - \frac{1}{2} c_z^{(3)} n_x - \frac{3}{2}\Big(\mathcal{M}_{xx}^{(2)} n_x + \mathcal{M}_{xy}^{(2)} n_y - \mathcal{M}_{zz}^{(2)} n_x\Big)$$
$$- \frac{1}{2}\Big(\frac{\partial \mathcal{M}_{xz}^{(3)}}{\partial x} + \frac{\partial \mathcal{M}_{yz}^{(3)}}{\partial y}\Big) n_x + t_y^{(3)} \tag{5.35}$$
$$\delta\theta_z: \quad 0 = M_{xz}^{(1)} n_x + M_{yz}^{(1)} n_y + \frac{1}{2}\Big(c_y^{(1)} n_x - c_x^{(1)} n_y\Big) + \frac{1}{2}\left(\mathcal{M}_{xz}^{(0)} n_y - \mathcal{M}_{yz}^{(0)} n_x\right)$$
$$- \frac{1}{2}\Big(\frac{\partial \mathcal{M}_{xx}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial y}\Big) n_y + \frac{1}{2}\Big(\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(1)}}{\partial y}\Big) n_x + t_z^{(1)} \tag{5.36}$$
$$\delta\phi_z: \quad 0 = M_{xz}^{(2)} n_x + M_{yz}^{(2)} n_y + \frac{1}{2}\Big(c_y^{(2)} n_x - c_x^{(2)} n_y\Big) + \mathcal{M}_{xz}^{(1)} n_y - \mathcal{M}_{yz}^{(1)} n_x$$
$$- \frac{1}{2}\Big(\frac{\partial \mathcal{M}_{xx}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x}\Big) n_y + \frac{1}{2}\Big(\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} + \frac{\partial \mathcal{M}_{yy}^{(2)}}{\partial y}\Big) n_x + t_z^{(2)} \tag{5.37}$$
$$\delta\bar{\phi}: \quad 0 = P_1 n_x + P_2 n_y + \hat{d}. \tag{5.38}$$

Note that the charge equation (5.26) and the corresponding boundary condition (5.38) are a general form in this study. When more than one piezoelectric layers are surface-mounted or embedded in the structural system, the charge equation should be written for each piezoelectric layers and the electric potential, $\bar{\phi}$, becomes $\bar{\phi}^k$ where k is k^{th} piezoelectric layer.

5.5 Analytical solution

A simply supported plate that consists of a functionally graded core layer and two piezoelectric layer on the top and bottom surfaces are analyzed. Figure 5.1 shows the geometry of the smart plate that is used for the numerical example. We assume that the poling direction is opposite to the positive transverse direction(z). a and b denote the in-plane dimensions along x and y coordinate and H and h denote the height of the total plate and the height of piezoelectric layers, respectively. An analytical solution is obtained using Navier solution technique.

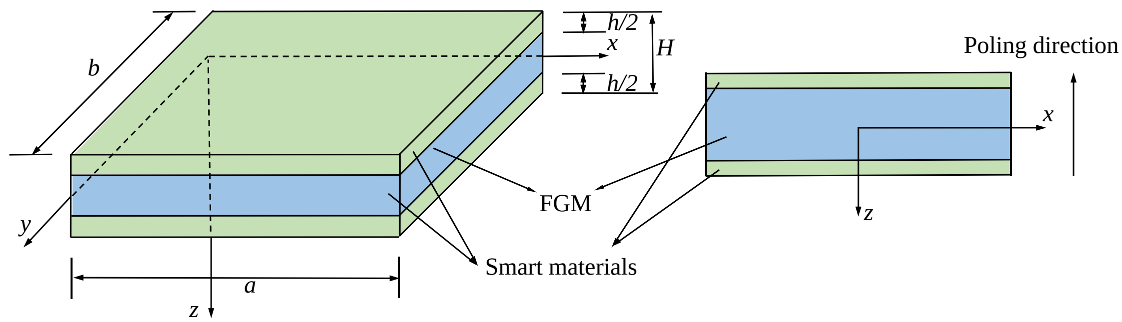

Figure 5.1: Smart plate geometry

The boundary conditions of a simply supported rectangular plate can be expressed as

$$
\begin{aligned}
&u\left(x,0\right)=u\left(x,b\right)=0, \quad \theta_x\left(x,0\right)=\theta_x\left(x,b\right)=0, \quad \phi_x\left(x,0\right)=\phi_x\left(x,b\right)=0,\\
&\psi_x\left(x,0\right)=\psi_x\left(x,b\right)=0, \quad v\left(0,y\right)=v\left(a,y\right)=0, \quad \theta_y\left(0,y\right)=\theta_y\left(a,y\right)=0,\\
&\phi_y\left(0,y\right)=\phi_y\left(a,y\right)=0, \quad \psi_y\left(0,y\right)=\psi_y\left(a,y\right)=0, \quad w\left(x,0\right)=w\left(x,b\right)=0,\\
&w\left(0,y\right)=w\left(a,y\right)=0, \quad \theta_z\left(x,0\right)=\theta_z\left(x,b\right)=0, \quad \theta_z\left(0,y\right)=\theta_z\left(a,y\right)=0,\\
&\phi_z\left(x,0\right)=\phi_z\left(x,b\right)=0, \quad \phi_z\left(0,y\right)=\phi_z\left(x,b\right)=0, \quad \bar{\phi}^t\left(x,0\right)=\bar{\phi}^t\left(x,b\right)=0,\\
&\bar{\phi}^t\left(0,y\right)=\bar{\phi}^t\left(a,y\right)=0, \quad \bar{\phi}^b\left(x,0\right)=\bar{\phi}^b\left(x,b\right)=0, \quad \bar{\phi}^b\left(0,y\right)=\bar{\phi}^b\left(a,y\right)=0,\\
&M_{xx}^{(i)}\left(0,y\right)=M_{xx}^{(i)}\left(a,y\right)=0, \quad M_{yy}^{(i)}\left(x,0\right)=M_{yy}^{(i)}\left(x,b\right)=0,\\
&\mathcal{M}_{xy}^{(j)}\left(0,y\right)=\mathcal{M}_{xy}^{(j)}\left(a,y\right)=0, \quad \mathcal{M}_{xy}^{(j)}\left(x,0\right)=\mathcal{M}_{xy}^{(j)}\left(x,b\right)=0
\end{aligned}
$$

where i =0, 1, 2, 3, j =0, 1, 2. The superscript t and b indicate top and bottom piezoelectric layers. In this example, two piezoelectric layers are bonded to top and bottom surface of core FGM plate.

The displacements and electric potentials are assumed as the series of double trigonometric functions that satisfy the simply supported boundary conditions.

$$u\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Umncos\left(\alpha x\right)sin\left(\beta y\right) \tag{5.39}$$

$$v\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Vmnsin\left(\alpha x\right)cos\left(\beta y\right) \tag{5.40}$$

$$w\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Wmnsin\left(\alpha x\right)sin\left(\beta y\right) \tag{5.41}$$

$$\theta_x\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Theta_x mncos\left(\alpha x\right)sin\left(\beta y\right) \tag{5.42}$$

$$\theta_y\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Theta_y mnsin\left(\alpha x\right)cos\left(\beta y\right) \tag{5.43}$$

$$\theta_z\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Theta_z mnsin\left(\alpha x\right)sin\left(\beta y\right) \tag{5.44}$$

$$\phi_x\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_x mncos\left(\alpha x\right)sin\left(\beta y\right) \tag{5.45}$$

$$\phi_y\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_y mnsin\left(\alpha x\right)cos\left(\beta y\right) \tag{5.46}$$

$$\phi_z\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_z mnsin\left(\alpha x\right)sin\left(\beta y\right) \tag{5.47}$$

$$\psi_x\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_x mncos\left(\alpha x\right)sin\left(\beta y\right) \tag{5.48}$$

$$\psi_y\left(x,y,z\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_y mnsin\left(\alpha x\right)cos\left(\beta y\right) \tag{5.49}$$

$$\bar{\phi}^t\left(x,y\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\phi}^t mnsin\left(\alpha x\right)sin\left(\beta y\right) \tag{5.50}$$

$$\bar{\phi}^b\left(x,y\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\phi}^b mnsin\left(\alpha x\right)sin\left(\beta y\right) \tag{5.51}$$

where $\alpha = \frac{m\pi}{a}$, $\beta = \frac{n\pi}{b}$. $\bar{\phi}^t$ and $\bar{\phi}^b$ are assumed electric potential for top and bottom piezoelectric layers and they are measured at center of each piezoelectric layers. By substitute Equations (5.39) to (5.39) in the governing equations (5.15) to (5.26), we have algebraic relations for bending problems. Numerical examples of the analytical solution are obtained using the dimensions of the functionally graded plate from Reddy [35], functionally graded material properties from Liew et al. [91] and piezoelectric properties from Mitchell and Reddy [85]. The dimensions of the plate in Fig. 5.1 are $H = 17.6 \times 10^{-6} m$, $a/H = b/H = 5, 10, 20$, and h varies to see the effect of thickness ratio between smart materials and FGMs. Table 5.1 shows the material properties for the functionally graded and piezoelectric materials.

Table 5.1: Material properties of FGM and piezoelectric layers

Properties	Zirconia [91]	Aluminium [91]	PZT [85]
Young's modulus(GPa)	151	70	-
Possion's ratio	0.3	0.3	-
Elastic coefficient C_{11} (GPa)	-	-	148
Elastic coefficient C_{33} (GPa)	-	-	131
Elastic coefficient C_{12} (GPa)	-	-	76.2
Elastic coefficient C_{13} (GPa)	-	-	74.2
Elastic coefficient C_{44} (GPa)	-	-	25.4
Elastic coefficient C_{66} (GPa)	-	-	35.9
Piezoelectric constant e_{31} ($\frac{c}{m^2}$)	-	-	-2.1
Piezoelectric constant e_{33} ($\frac{c}{m^2}$)	-	-	9.5
Piezoelectric constant e_{15} ($\frac{c}{m^2}$)	-	-	9.2
Dielectric coefficient k_{11} ($\frac{F}{m}$)	-	-	4.07 $\times 10^{-9}$
Dielectric coefficient k_{33} ($\frac{F}{m}$)	-	-	2.08 $\times 10^{-9}$

The assumed electric potential (5.8) is for a typical piezoelectric layer. Since we consider two piezoelectric layers, the electric potential should be written for each layer. To obtain the static bending responses, unit positive and negative voltages are applied to top (V_0^t) and bottom (V_0^b) surfaces of the plate, respectively.

Figures 5.2–5.4 show comparison between current study and the work of Mitchell and Reddy [85]. In their study, thickness change of the plate is not considered in the kinematic assumption and tangential tractions on top and bottom surfaces are assumed to be zero. The electric potential is modeled using a layerwise theory. When the ratio of plate side-to-thickness become smaller (i.e. plate becomes thicker), the difference between two study reduces.

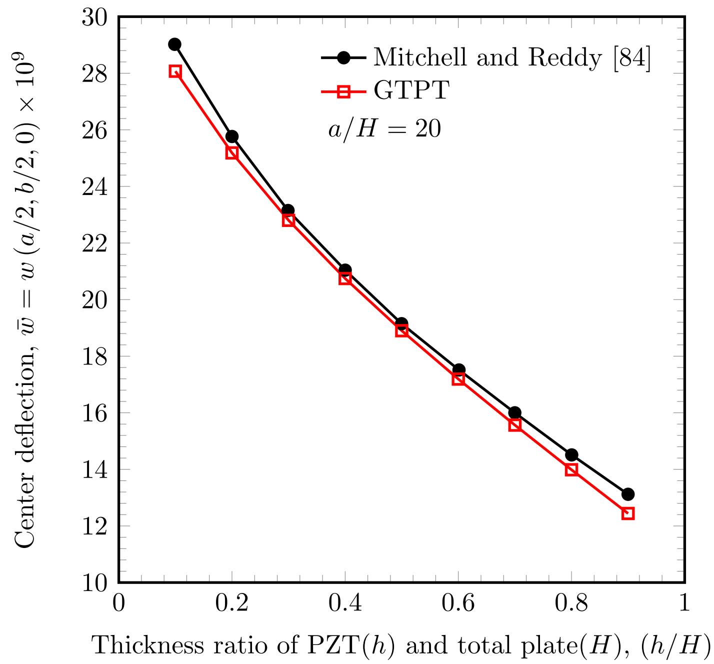

Figure 5.2: Center deflection of simply supported smart plate versus the thickness ratio of piezoelectric layer (Aluminium and PZT, $a/H = 20$)

The ratio of the side to thickness of the plate is taken as $a/H = 20$ to show effects of the power-law index and the length scale parameter. Figure 5.5 shows center transverse deflections of FGM plates for various values of the power-law index and the ratio between the piezoelectric layers and total plate thickness. As the plate thickness ratio (h/H) is

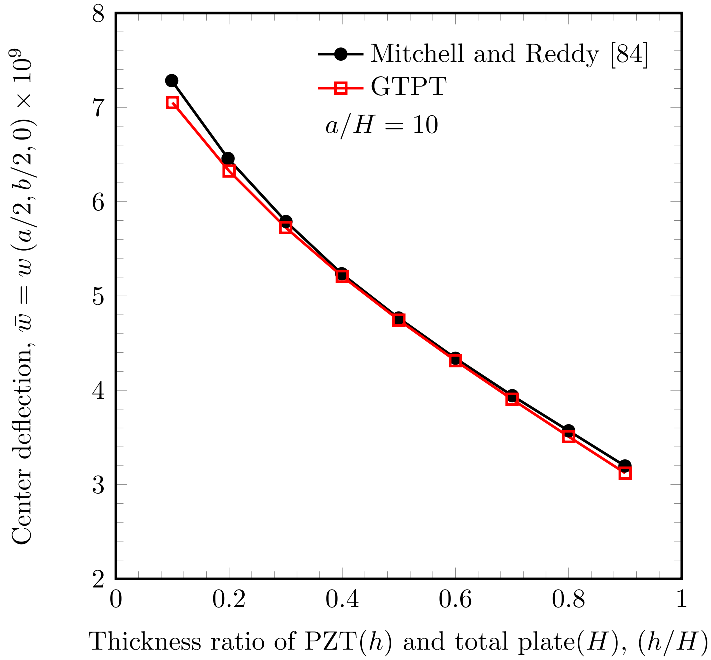

Figure 5.3: Center deflection of simply supported smart plate versus the thickness ratio of piezoelectric layer (Aluminium and PZT, $a/H = 10$)

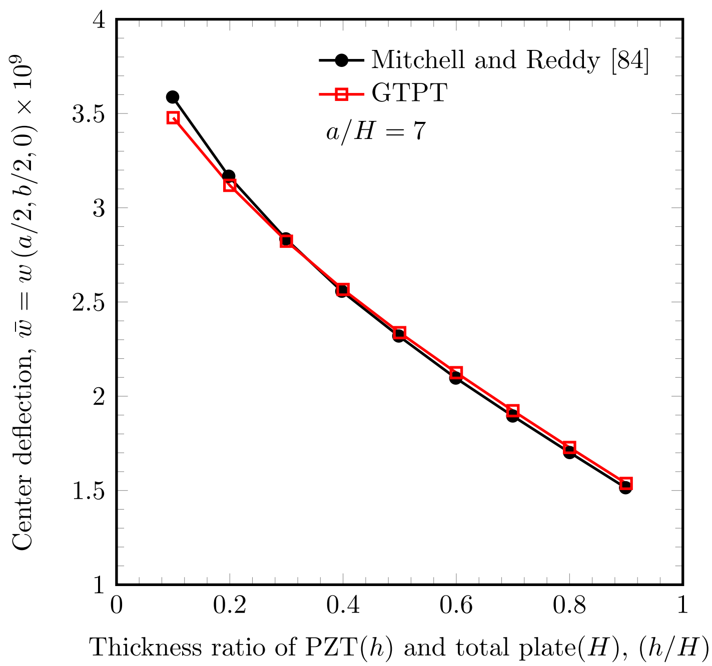

Figure 5.4: Center deflection of simply supported smart plate versus the thickness ratio of piezoelectric layer (Aluminium and PZT, $a/H = 7$)

increased, the plate behavior is close to that of a homogeneous plate, and the effect of power-law index vanishes. Due to the piezoelectric effect, in-plane stresses occur in top and bottom surfaces, and they create a bending moment which is a quadratic function of thickness of the piezoelectric layers. The elastic stiffness of Zirconia is slightly larger than PZT and, therefore, the plate becomes slightly softer when increasing the thickness of the piezoelectric layers. Since the change of plate stiffness is not significant, the variation of center deflection with change of thickness ratio is a parabolic variation, that is, it follows the variation of the bending moment. For the case of Aluminium, the plate becomes stiffer when increasing the thickness of the piezoelectric layers, and the deflection decreases. This can be clearly seen from Fig. 5.5. The size dependent effect is presented in Figs. 5.6 and 5.7. For larger values of the length scale parameter (ℓ), plates become stiffer and center deflection decreases. In addition to the size-dependent stiffening effect, the effect of the plate thickness ratio (h/H) diminishes with larger values of the length scale parameter. Note that we use the same values of length scale parameter for both the functionally graded and piezoelectric layers, but the effect of the length scale parameter may differ for each layer because the length scale parameter has the meaning of the shear moduli ratio at a material point. The functionally graded material is an isotropic material but the piezoelectric material is represented as a transversely isotropic material.

Figures 5.8, 5.9, and 5.10 show the center transverse deflection of the smart plates along the x-axis. When the power-law index become larger the functionally graded layer become softer (more Aluminium) and defection become larger. The size dependent stiffening effects are clearly shown in Figs. 5.9 and 5.10. To see the effects of the functionally graded material and the length scale parameter, the thickness ration, h/H is taken as 0.05.

The effect of the various values of the ratio of plate side-to-thickness of the smart plate is shown in Fig. 5.11. When the ratio becomes smaller, the plate become stiffer and the effect of thickness ratio decreases. For geometrically stiffer plate, that is small ratios, the

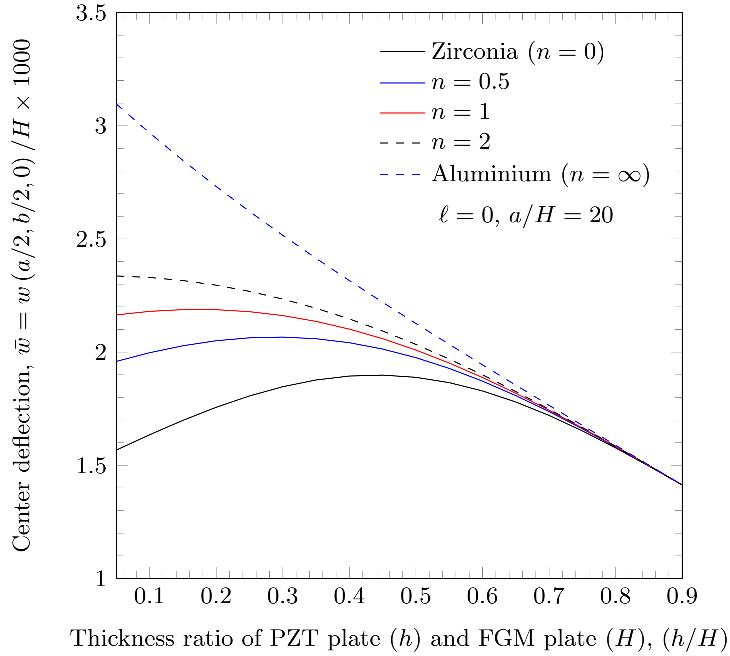

Figure 5.5: Center deflection versus piezoelectric thickness ratio for various power-law index ($\ell = 0$, $a/H = 20$) [4]

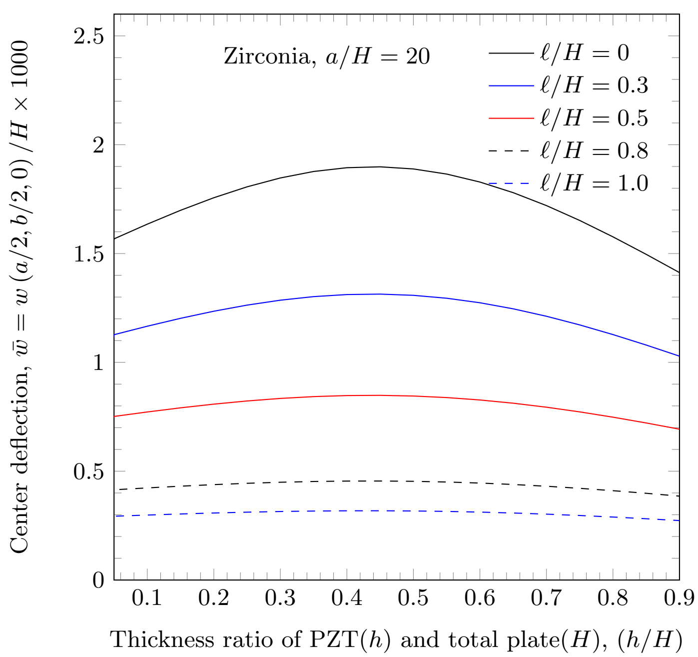

Figure 5.6: Center deflection versus piezoelectric thickness ratio for various length scale parameter (Zirconia, $a/H = 20$) [4]

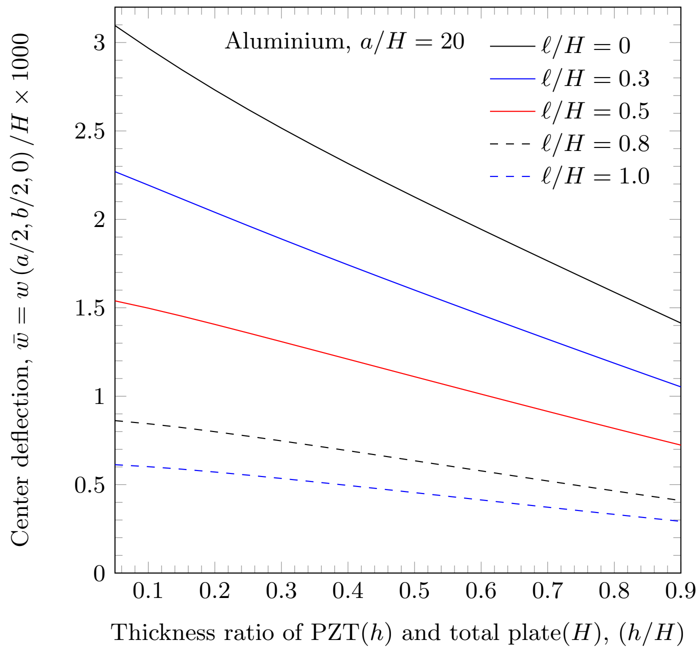

Figure 5.7: Center deflection versus piezoelectric thickness ratio for various length scale parameter (Aluminium, $a/H = 20$) [4]

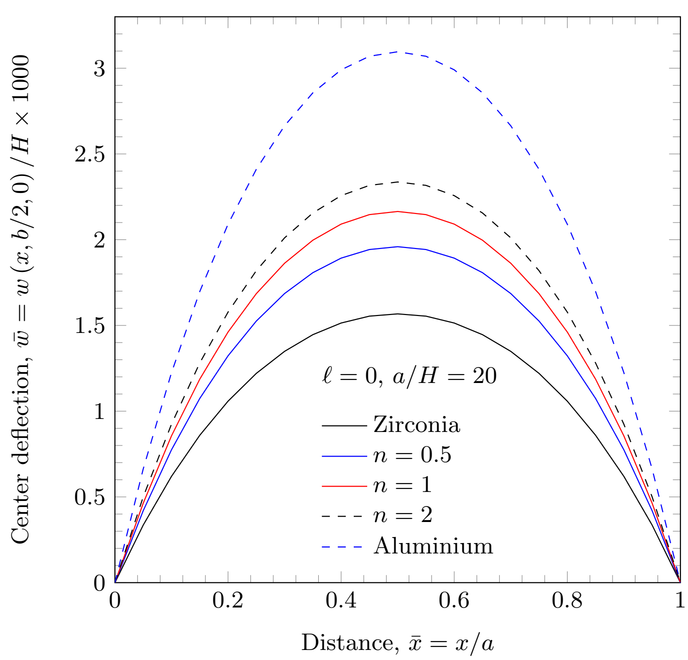

Figure 5.8: Transverse deflection along the x-axis for various values of the power-law index ($\ell = 0$, $a/H = 20$) [4]

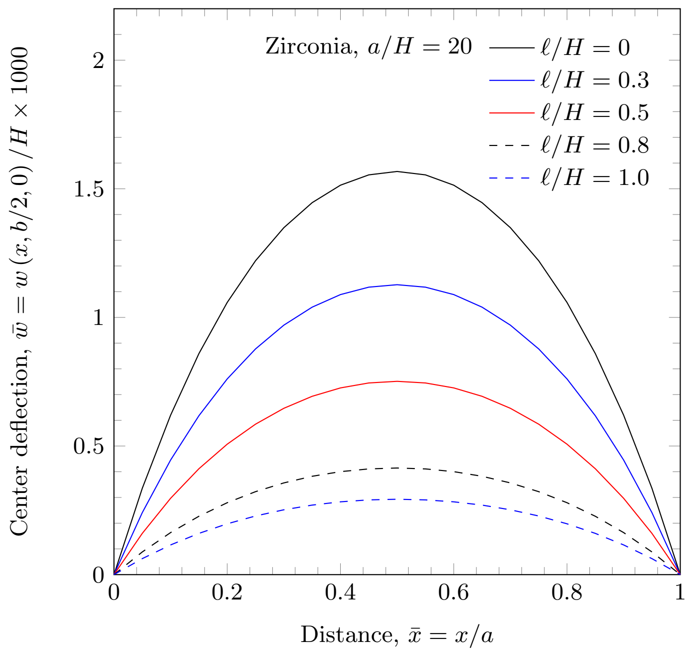

Figure 5.9: Transverse deflection along the x-axis for various values of the length scale parameter (Zirconia, $a/H = 20$) [4]

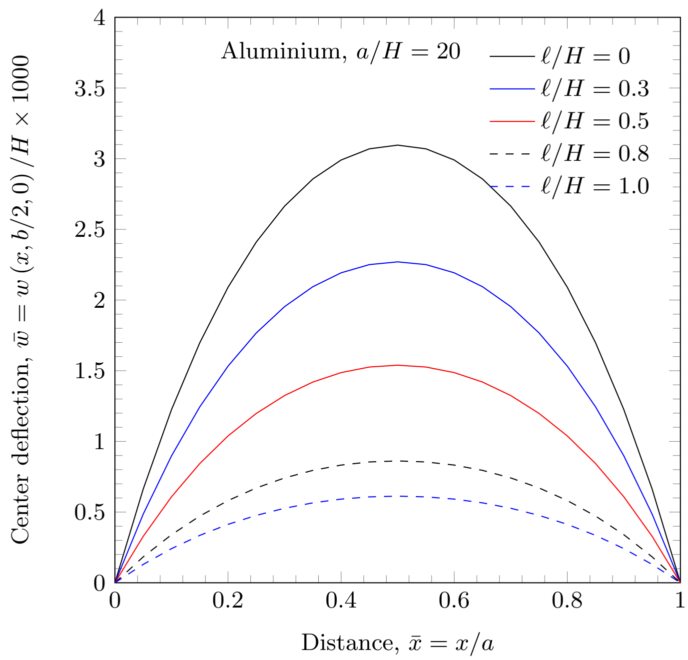

Figure 5.10: Transverse deflection along the x-axis for various values of the length scale parameter (Aluminium, $a/H = 20$) [4]

effects of material variation and thickness ratio decrease.

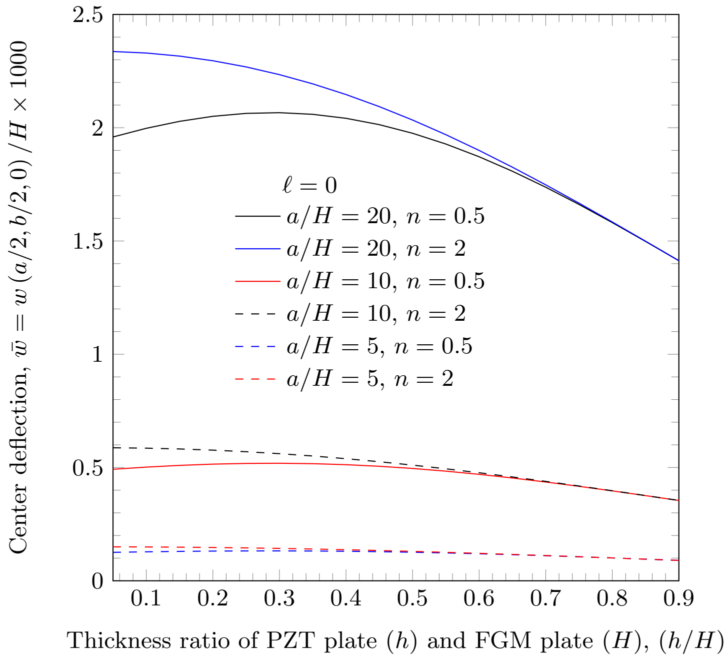

Figure 5.11: Center transverse deflection versus piezoelectric thickness ratio for various values of the ratio of plate side-to-thickness ratios of plates ($\ell = 0$) [4]

5.6 Finite element model

The finite element models for piezoelectric plates can be categorized into two groups. One does not consider electric degree of freedoms and the other one includes electric degree of freedoms in addition to kinematic nodal degree of freedoms. Without electric degree of freedoms, piezoelectric effects are considered using thermal analogy, i.e., with known temperature field, thermal effects are considered only as thermal forces. Among many researchers whose works do not consider electric degree of freedoms, Hwang and Park [92] and Shen and Sharpe Jr. [93] developed a finite element model for piezoelectric plate using classical plate theory, and Lam et al. [94] presented static bending and vibration of cantilevered plate using the classical plate theory. Chandrashekhara and his colleagues [95–97] presented finite element model for buckling and vibration problems using first shear deformation plate theory. Han and Lee [98] developed a finite element model to study actuation effectiveness of a cantilevered laminate composite plate using first order shear deformation theory. Hosseini-Hashemi, EsâĂŹhaghi, and Taher [99] used Reddy third order plate theory to study free vibration analysis of thick circular/annular plates. They used half sinusoidal distribution of the electric potential through thickness of the plate. A general third order plate theory with linear variation of electric potential was developed by Ray, Bhattacharyya, and Samanta [100] and they performed static bending analysis for laminate rectangular plate. Chen, Wang, and Liu [101] studied a vibration control and suppression of intelligent using classical plate theory with linear variation of electric potential. Carrera [102] presented a static analysis of multilayered plate using a first shear deformation plate theory with parabolic variation of electric potential.

In section 4, the displacement based weak form Galerkin finite element model for the general third order plate theory is developed using the principle of virtual displacements (2.46). For the smart functionally grade micro plate, the displacement field is assumed using C^1 Hermite interpolation functions, and the electric potential is assumed using C^0 Lagrange interpolation functions. Figure 5.12 shows nodal degree of freedoms in 4 node rectangular element. Equations (4.1) and (5.52) represent the assumed displacement and electric potential, respectively. The details of the assumed displacement are presented in the section 4.

$$\bar{\phi}^{(i)}\left(x,y\right)=\sum_{j=1}^{p}\phi_j^{(i)}\psi_j^{(i)}\left(x,y\right) \tag{5.52}$$

where $\bar{\phi}^{(i)}\left(x,y\right)$ is p is number of node in an element, the electric potential on i_{th} piezoelectric layer, $\phi_j^{(i)}$ is nodal value of the electric potential, and $\psi_j^{(i)}\left(x,y\right)$ is the Lagrange interpolation function. For the simplest rectangular element (4 node element), the finite

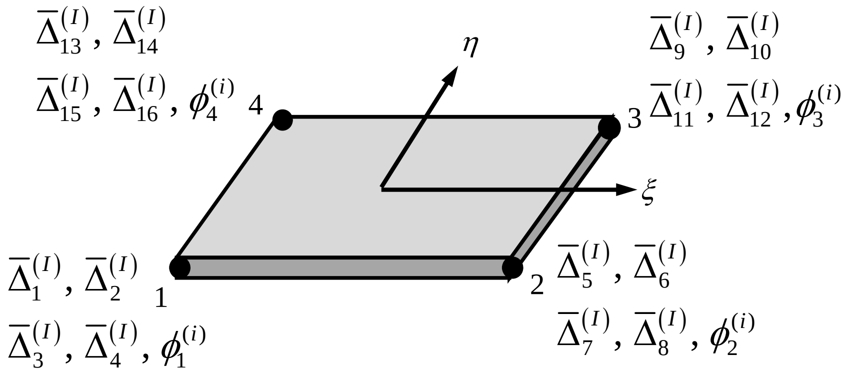

Figure 5.12: Arrangement of nodal variables for 4 node micro smart plate

element model has total 180 degree freedoms (176 degree of freedoms for displacements and their derivatives and 4 degree of freedoms for electric potential).

The finite element model for functionally graded micro plate is

$$[M]\{\ddot{U}\} + [K]\{U\} = \{F\} \tag{5.53}$$

where $[M]$ and $[K]$ are mass and stiffness matrix that has 13 by 13 sub-matrices when two piezoelectric layers are mounted. The number of sub-matrices depend on the number of the mounted piezoelectric layers because charge equation (5.26) must be written for each piezoelectric layers. The explicit form of nonzero mass matrices is derived in the section 4. The stiffness matrices derived in the section 2 still valid with the constitutive relation in Eqs. (5.5) to (5.5). In addition to the stiffness terms defined in the section 2, following stiffness terms are added to the system equation (5.53)

$$K_{ij}^{0112} = \int_{\Omega} -\hat{E}_{T31}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} \psi_j^{(12)} dxdy \tag{5.54}$$

$$K_{ij}^{0113} = \int_{\Omega} -\hat{E}_{B31}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} \psi_j^{(13)} dxdy \tag{5.55}$$

$$K_{ij}^{0212} = \int_{\Omega} -\hat{E}_{T32}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial y} \psi_j^{(12)} dxdy \tag{5.56}$$

$$K_{ij}^{0213} = \int_{\Omega} -\hat{E}_{B32}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial y} \psi_j^{(13)} dxdy \tag{5.57}$$

$$\begin{aligned} K_{ij}^{0312} = \int_{\Omega} & -\hat{E}_{T24}^{(0)} \frac{\varphi_i^{(3)}}{\partial y} \frac{\psi_j^{(12)}}{\partial y} - \hat{E}_{T15}^{(0)} \frac{\varphi_i^{(3)}}{\partial x} \frac{\psi_j^{(12)}}{\partial x} - \hat{E}_{T32}^{(0)} \frac{\partial w}{\partial y} \frac{\varphi_i^{(3)}}{\partial y} \psi_j^{(12)} \\ & - \hat{E}_{T31}^{(0)} \frac{\partial w}{\partial x} \frac{\varphi_i^{(3)}}{\partial x} \psi_j^{(12)} dxdy \end{aligned} \tag{5.58}$$

$$\begin{aligned} K_{ij}^{0313} = \int_{\Omega} & -\hat{E}_{B15}^{(0)} \frac{\varphi_i^{(3)}}{\partial x} \frac{\psi_j^{(13)}}{\partial x} - \hat{E}_{B24}^{(0)} \frac{\varphi_i^{(3)}}{\partial y} \frac{\psi_j^{(13)}}{\partial y} - \hat{E}_{B32}^{(0)} \frac{\partial w}{\partial y} \frac{\varphi_i^{(3)}}{\partial y} \psi_j^{(13)} \\ & - \hat{E}_{B31}^{(0)} \frac{\partial w}{\partial x} \frac{\varphi_i^{(3)}}{\partial x} \psi_j^{(13)} dxdy \end{aligned} \tag{5.59}$$

$$K_{ij}^{0412} = \int_{\Omega} -\hat{E}_{T15}^{(0)} \varphi_i^{(4)} \frac{\psi_j^{(12)}}{\partial x} - \hat{E}_{T31}^{(1)} \frac{\partial \varphi_i^{(4)}}{\partial x} \psi_j^{(12)} dxdy \tag{5.60}$$

$$K_{ij}^{0413} = \int_{\Omega} -\hat{E}_{B15}^{(0)} \varphi_i^{(4)} \frac{\psi_j^{(13)}}{\partial x} - \hat{E}_{B31}^{(1)} \frac{\partial \varphi_i^{(4)}}{\partial x} \psi_j^{(13)} dxdy \tag{5.61}$$

$$K_{ij}^{0512} = \int_{\Omega} -\hat{E}_{T24}^{(0)} \varphi_i^{(5)} \frac{\psi_j^{(12)}}{\partial y} - \hat{E}_{T32}^{(1)} \frac{\partial \varphi_i^{(4)}}{\partial y} \psi_j^{(12)} dxdy \tag{5.62}$$

$$K_{ij}^{0513} = \int_{\Omega} -\hat{E}_{B24}^{(0)} \varphi_i^{(5)} \frac{\psi_j^{(13)}}{\partial y} - \hat{E}_{B32}^{(1)} \frac{\partial \varphi_i^{(4)}}{\partial y} \psi_j^{(13)} dxdy \tag{5.63}$$

$$K_{ij}^{0612} = \int_{\Omega} -\hat{E}_{T33}^{(0)} \varphi_i^{(6)} \psi_j^{(12)} - \hat{E}_{T24}^{(1)} \frac{\partial \varphi_i^{(6)}}{\partial y} \frac{\psi_j^{(12)}}{\partial y} - \hat{E}_{T15}^{(1)} \frac{\partial \varphi_i^{(6)}}{\partial x} \frac{\psi_j^{(12)}}{\partial x} dxdy \tag{5.64}$$

$$K_{ij}^{0613} = \int_{\Omega} -\hat{E}_{B33}^{(0)} \varphi_i^{(6)} \psi_j^{(13)} - \hat{E}_{B24}^{(1)} \frac{\partial \varphi_i^{(6)}}{\partial y} \frac{\psi_j^{(13)}}{\partial y} - \hat{E}_{B15}^{(1)} \frac{\partial \varphi_i^{(6)}}{\partial x} \frac{\psi_j^{(13)}}{\partial x} dxdy \tag{5.65}$$

$$K_{ij}^{0712} = \int_{\Omega} -2\hat{E}_{T15}^{(1)} \varphi_i^{(7)} \frac{\psi_j^{(12)}}{\partial x} - \hat{E}_{T31}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} \psi_j^{(12)} dxdy \tag{5.66}$$

$$K_{ij}^{0713} = \int_{\Omega} -2\hat{E}_{B15}^{(1)} \varphi_i^{(7)} \frac{\psi_j^{(13)}}{\partial x} - \hat{E}_{B31}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} \psi_j^{(13)} dxdy \tag{5.67}$$

$$K_{ij}^{0812} = \int_{\Omega} -2\hat{E}_{T24}^{(1)} \varphi_i^{(8)} \frac{\psi_j^{(12)}}{\partial y} - \hat{E}_{T32}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \psi_j^{(12)} dxdy \tag{5.68}$$

$$K_{ij}^{0813} = \int_{\Omega} -2\hat{E}_{B24}^{(1)} \varphi_i^{(8)} \frac{\psi_j^{(13)}}{\partial y} - \hat{E}_{B32}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} \psi_j^{(13)} dxdy \tag{5.69}$$

$$K_{ij}^{0912} = \int_{\Omega} -2\hat{E}_{T33}^{(1)} \varphi_i^{(9)} \psi_j^{(12)} - \hat{E}_{T24}^{(2)} \frac{\varphi_i^{(9)}}{\partial y} \frac{\psi_j^{(12)}}{\partial y} - \hat{E}_{T15}^{(2)} \frac{\varphi_i^{(9)}}{\partial x} \frac{\psi_j^{(12)}}{\partial x} dxdy \tag{5.70}$$

$$K_{ij}^{0913} = \int_{\Omega} -2\hat{E}_{B33}^{(1)}\varphi_i^{(9)}\psi_j^{(13)} - \hat{E}_{B24}^{(2)}\frac{\varphi_i^{(9)}}{\partial y}\frac{\psi_j^{(13)}}{\partial y} - \hat{E}_{B15}^{(2)}\frac{\varphi_i^{(9)}}{\partial x}\frac{\psi_j^{(13)}}{\partial x}dxdy \tag{5.71}$$

$$K_{ij}^{1012} = \int_{\Omega} -3\hat{E}_{T15}^{(2)}\varphi_i^{(10)}\frac{\psi_j^{(12)}}{\partial x} - \hat{E}_{T31}^{(3)}\frac{\partial\varphi_i^{(10)}}{\partial x}\psi_j^{(12)}dxdy \tag{5.72}$$

$$K_{ij}^{1013} = \int_{\Omega} -3\hat{E}_{B15}^{(2)}\varphi_i^{(10)}\frac{\psi_j^{(13)}}{\partial x} - \hat{E}_{B31}^{(3)}\frac{\partial\varphi_i^{(10)}}{\partial x}\psi_j^{(13)}dxdy \tag{5.73}$$

$$K_{ij}^{1112} = \int_{\Omega} -3\hat{E}_{T24}^{(2)}\varphi_i^{(11)}\frac{\psi_j^{(12)}}{\partial y} - \hat{E}_{T32}^{(3)}\frac{\partial\varphi_i^{(11)}}{\partial y}\psi_j^{(12)}dxdy \tag{5.74}$$

$$K_{ij}^{1113} = \int_{\Omega} -3\hat{E}_{B24}^{(2)}\varphi_i^{(11)}\frac{\psi_j^{(13)}}{\partial y} - \hat{E}_{B32}^{(3)}\frac{\partial\varphi_i^{(11)}}{\partial y}\psi_j^{(13)}dxdy \tag{5.75}$$

$$K_{ij}^{1201} = \int_{\Omega} \hat{E}_{T31}^{(0)}\psi_i^{(12)}\frac{\partial\varphi_j^{(1)}}{\partial x}dxdy \tag{5.76}$$

$$K_{ij}^{1202} = \int_{\Omega} \hat{E}_{T32}^{(0)}\psi_i^{(12)}\frac{\partial\varphi_j^{(2)}}{\partial y}dxdy \tag{5.77}$$

$$\begin{aligned} K_{ij}^{1203} = \int_{\Omega} &\frac{1}{2}\left(\hat{E}_{T31}^{(0)}\psi_i^{(12)}\frac{\partial w}{\partial x}\frac{\partial\varphi_j^{(3)}}{\partial x} + \hat{E}_{T32}^{(0)}\frac{\partial w}{\partial y}\psi_i^{(12)}\frac{\partial\varphi_j^{(3)}}{\partial y}\right) \\ &+ \hat{E}_{T24}^{(0)}\frac{\partial\psi_i^{(12)}}{\partial y}\frac{\partial\varphi_j^{(3)}}{\partial y} + \hat{E}_{T15}^{(0)}\frac{\partial\psi_i^{(12)}}{\partial x}\frac{\partial\varphi_j^{(3)}}{\partial x}dxdy \end{aligned} \tag{5.78}$$

$$K_{ij}^{1204} = \int_{\Omega} \hat{E}_{T31}^{(1)}\psi_i^{(12)}\frac{\partial\varphi_j^{(4)}}{\partial x} + \hat{E}_{T15}^{(0)}\frac{\partial\psi_i^{(12)}}{\partial x}\varphi_j^{(4)}dxdy \tag{5.79}$$

$$K_{ij}^{1205} = \int_{\Omega} \hat{E}_{T24}^{(0)}\frac{\partial\psi_i^{(12)}}{\partial y}\varphi_j^{(5)} + \hat{E}_{T32}^{(1)}\psi_i^{(12)}\frac{\partial\varphi_j^{(5)}}{\partial y}dxdy \tag{5.80}$$

$$K_{ij}^{1206} = \int_{\Omega} \hat{E}_{T24}^{(1)}\frac{\partial\psi_i^{(12)}}{\partial y}\frac{\partial\varphi_j^{(6)}}{\partial y} + \hat{E}_{T33}^{(0)}\psi_i^{(12)}\varphi_j^{(6)} + \hat{E}_{T15}^{(1)}\frac{\partial\psi_i^{(12)}}{\partial x}\frac{\partial\varphi_j^{(6)}}{\partial x}dxdy \tag{5.81}$$

$$K_{ij}^{1207} = \int_{\Omega} 2\hat{E}_{T15}^{(1)}\frac{\partial\psi_i^{(12)}}{\partial x}\varphi_j^{(7)} + \hat{E}_{T31}^{(2)}\psi_i^{(12)}\frac{\partial\varphi_j^{(7)}}{\partial x}dxdy \tag{5.82}$$

$$K_{ij}^{1208} = \int_{\Omega} 2\hat{E}_{T24}^{(1)}\frac{\partial\psi_i^{(12)}}{\partial y}\varphi_j^{(8)} + \hat{E}_{T32}^{(2)}\psi_i^{(12)}\frac{\partial\varphi_j^{(8)}}{\partial y}dxdy \tag{5.83}$$

$$K_{ij}^{1209} = \int_{\Omega} 2\hat{E}_{T33}^{(1)}\psi_i^{(12)}\varphi_j^{(9)} + \hat{E}_{T24}^{(2)}\frac{\partial\psi_i^{(12)}}{\partial y}\frac{\partial\varphi_j^{(9)}}{\partial y} + \hat{E}_{T15}^{(2)}\frac{\partial\psi_i^{(12)}}{\partial x}\frac{\partial\varphi_j^{(9)}}{\partial x}dxdy \tag{5.84}$$

$$K_{ij}^{1210} = \int_{\Omega} 3\hat{E}_{T15}^{(2)}\frac{\partial\psi_i^{(12)}}{\partial x}\varphi_j^{(10)} + \hat{E}_{T31}^{(3)}\psi_i^{(12)}\frac{\partial\varphi_j^{(10)}}{\partial x}dxdy \tag{5.85}$$

$$K_{ij}^{1211} = \int_{\Omega} 3\hat{E}_{T24}^{(2)}\frac{\partial\psi_i^{(12)}}{\partial y}\varphi_j^{(11)} + \hat{E}_{T32}^{(3)}\psi_i^{(12)}\frac{\partial\varphi_j^{(11)}}{\partial y}dxdy \tag{5.86}$$

$$K_{ij}^{1212} = \int_{\Omega} \hat{D}_{T11} \frac{\partial \psi_i^{(12)}}{\partial x} \frac{\psi_j^{(12)}}{\partial x} + \hat{D}_{T22} \frac{\partial \psi_i^{(12)}}{\partial y} \frac{\psi_j^{(12)}}{\partial y} + \hat{D}_{T33} \psi_i^{(12)} \psi_j^{(12)} dxdy \tag{5.87}$$

$$K_{ij}^{1213} = 0 \tag{5.88}$$

$$K_{ij}^{1301} = \int_{\Omega} \hat{E}_{B31}^{(0)} \psi_i^{(13)} \frac{\partial \varphi_j^{(1)}}{\partial x} dxdy \tag{5.89}$$

$$K_{ij}^{1302} = \int_{\Omega} \hat{E}_{B32}^{(0)} \psi_i^{(13)} \frac{\partial \varphi_j^{(2)}}{\partial y} dxdy \tag{5.90}$$

$$\begin{aligned} K_{ij}^{1303} = \int_{\Omega} &\frac{1}{2} \left(\hat{E}_{B31}^{(0)} \frac{\partial w}{\partial x} \psi_i^{(13)} \frac{\partial \varphi_j^{(3)}}{\partial x} + \hat{E}_{B32}^{(0)} \frac{\partial w}{\partial y} \psi_i^{(13)} \frac{\partial \varphi_j^{(3)}}{\partial y} \right) \\ &+ \hat{E}_{B24}^{(0)} \frac{\partial \psi_i^{(13)}}{\partial y} \frac{\partial \varphi_j^{(3)}}{\partial y} + \hat{E}_{B15}^{(0)} \frac{\partial \psi_i^{(13)}}{\partial x} \frac{\partial \varphi_j^{(3)}}{\partial x} dxdy \end{aligned} \tag{5.91}$$

$$K_{ij}^{1304} = \int_{\Omega} \hat{E}_{B31}^{(1)} \psi_i^{(13)} \frac{\partial \varphi_j^{(4)}}{\partial x} + \hat{E}_{B15}^{(0)} \frac{\partial \psi_i^{(13)}}{\partial x} \varphi_j^{(4)} dxdy \tag{5.92}$$

$$K_{ij}^{1305} = \int_{\Omega} \hat{E}_{B24}^{(0)} \frac{\partial \psi_i^{(13)}}{\partial y} \varphi_j^{(5)} + \hat{E}_{B32}^{(1)} \psi_i^{(13)} \frac{\partial \varphi_j^{(5)}}{\partial y} dxdy \tag{5.93}$$

$$K_{ij}^{1306} = \int_{\Omega} \hat{E}_{B24}^{(1)} \frac{\partial \psi_i^{(13)}}{\partial y} \frac{\partial \varphi_j^{(6)}}{\partial y} + \hat{E}_{B33}^{(0)} \psi_i^{(13)} \varphi_j^{(6)} + \hat{E}_{B15}^{(1)} \frac{\partial \psi_i^{(13)}}{\partial x} \frac{\partial \varphi_j^{(6)}}{\partial x} dxdy \tag{5.94}$$

$$K_{ij}^{1307} = \int_{\Omega} 2\hat{E}_{B15}^{(1)} \frac{\partial \psi_i^{(13)}}{\partial x} \varphi_j^{(7)} + \hat{E}_{B31}^{(2)} \psi_i^{(13)} \frac{\partial \varphi_j^{(7)}}{\partial x} dxdy \tag{5.95}$$

$$K_{ij}^{1308} = \int_{\Omega} 2\hat{E}_{B24}^{(1)} \frac{\partial \psi_i^{(13)}}{\partial y} \varphi_j^{(8)} + \hat{E}_{B32}^{(2)} \psi_i^{(13)} \frac{\partial \varphi_j^{(8)}}{\partial y} dxdy \tag{5.96}$$

$$K_{ij}^{1309} = \int_{\Omega} 2\hat{E}_{B33}^{(1)} \psi_i^{(13)} \varphi_j^{(9)} + \hat{E}_{B24}^{(2)} \frac{\partial \psi_i^{(13)}}{\partial y} \frac{\partial \varphi_j^{(9)}}{\partial y} + \frac{\partial \psi_i^{(13)}}{\partial x} \hat{E}_{B15}^{(2)} \frac{\partial \varphi_j^{(9)}}{\partial x} dxdy \tag{5.97}$$

$$K_{ij}^{1310} = \int_{\Omega} 3\hat{E}_{B15}^{(2)} \frac{\partial \psi_i^{(13)}}{\partial x} \varphi_j^{(10)} + \hat{E}_{B31}^{(3)} \psi_i^{(13)} \frac{\partial \varphi_j^{(10)}}{\partial x} dxdy \tag{5.98}$$

$$K_{ij}^{1311} = \int_{\Omega} 3\hat{E}_{B24}^{(2)} \frac{\partial \psi_i^{(13)}}{\partial y} \varphi_j^{(11)} + \hat{E}_{B32}^{(3)} \psi_i^{(13)} \frac{\partial \varphi_j^{(11)}}{\partial y} dxdy \tag{5.99}$$

$$K_{ij}^{1312} = 0 \tag{5.100}$$

$$K_{ij}^{1313} = \int_{\Omega} \hat{D}_{B11} \frac{\partial \psi_i^{(13)}}{\partial x} \frac{\psi_j^{(13)}}{\partial x} + \hat{D}_{B22} \frac{\partial \psi_i^{(13)}}{\partial y} \frac{\psi_j^{(13)}}{\partial y} + \hat{D}_{B33} \psi_i^{(13)} \psi_j^{(13)} dxdy \tag{5.101}$$

where subscripts T and B indicate top and bottom piezoelectric layers. The coefficients

$\hat{E}_{Tij}^{(k)}$, $\hat{E}_{Bij}^{(k)}$, $\hat{D}_{Tii}$, and $\hat{D}_{Bii}$ are defined as

$$\hat{E}_{T3j}^{(n)} = \int_{\frac{h}{2}}^{\frac{h}{2}+h_t} -z^n e_{3j} \frac{\pi}{h_t} \sin\left(\frac{\pi\left(z - h/2 - h_t/2\right)}{h_t}\right) dz, \qquad j = 1,2,3 \quad (5.102)$$

$$\hat{E}_{B3j}^{(n)} = \int_{\frac{-h}{2}-h_b}^{\frac{-h}{2}} -z^n e_{3j} \frac{\pi}{h_b} \sin\left(\frac{\pi\left(z + h/2 + h_b/2\right)}{h_b}\right) dz, \qquad j = 1,2,3 \quad (5.103)$$

$$\hat{E}_{T\alpha\beta}^{(n)} = \int_{\frac{h}{2}}^{\frac{h}{2}+h_t} z^n e_{\alpha\beta} \cos\left(\frac{\pi\left(z - h/2 - h_t/2\right)}{h_t}\right) dz, \qquad \alpha\beta = 15 \text{ or } 24 \quad (5.104)$$

$$\hat{E}_{B\alpha\beta}^{(n)} = \int_{\frac{-h}{2}-h_b}^{\frac{-h}{2}} -z^n e_{\alpha\beta} \cos\left(\frac{\pi\left(z + h/2 + h_b/2\right)}{h_b}\right) dz, \qquad \alpha\beta = 15 \text{ or } 24 \quad (5.105)$$

$$\hat{D}_{Tii} = \int_{\frac{h}{2}}^{\frac{h}{2}+h_t} k_{ii} \cos^2\left(\frac{\pi\left(z - h/2 - h_t/2\right)}{h_t}\right) dz, \qquad i = 1,2,3 \quad (5.106)$$

$$\hat{D}_{Bii} = \int_{\frac{-h}{2}-h_b}^{\frac{-h}{2}} k_{ii} \cos^2\left(\frac{\pi\left(z + h/2 + h_t/2\right)}{h_b}\right) dz \qquad i = 1,2,3 \quad (5.107)$$

where h_t and h_b are thickness of top and bottom piezoelectric layers, and h is thickness of core plate, i.e., thickness of functionally graded plate. The forces due to piezoelectric effect are needed to add to the force vectors (4.171) – (4.181) in the section 2. The additional force vectors are

$$F_i^1 = \int_\Omega \tilde{E}_{T31}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} + \tilde{E}_{B31}^{(0)} \frac{\partial \varphi_i^{(1)}}{\partial x} dxdy \quad (5.108)$$

$$F_i^2 = \int_\Omega \tilde{E}_{T32}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial y} + \tilde{E}_{B32}^{(0)} \frac{\partial \varphi_i^{(2)}}{\partial y} dxdy \quad (5.109)$$

$$F_i^3 = 0 \quad (5.110)$$

$$F_i^4 = \int_\Omega \tilde{E}_{T31}^{(1)} \frac{\partial \varphi_i^{(4)}}{\partial x} + \tilde{E}_{B31}^{(1)} \frac{\partial \varphi_i^{(4)}}{\partial x} dxdy \quad (5.111)$$

$$F_i^5 = \int_\Omega \tilde{E}_{B32}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial y} + \tilde{E}_{T32}^{(1)} \frac{\partial \varphi_i^{(5)}}{\partial y} dxdy \quad (5.112)$$

$$F_i^6 = \int_\Omega \tilde{E}_{T33}^{(0)} \varphi_i^{(6)} + \tilde{E}_{B33}^{(0)} \varphi_i^{(6)} dxdy \quad (5.113)$$

$$F_i^7 = \int_\Omega \tilde{E}_{T31}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} + \tilde{E}_{B31}^{(2)} \frac{\partial \varphi_i^{(7)}}{\partial x} dxdy \tag{5.114}$$

$$F_i^8 = \int_\Omega \tilde{E}_{B32}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} + \tilde{E}_{T32}^{(2)} \frac{\partial \varphi_i^{(8)}}{\partial y} dxdy \tag{5.115}$$

$$F_i^9 = \int_\Omega 2\tilde{E}_{B33}^{(1)} \varphi_i^{(9)} + 2\tilde{E}_{T33}^{(1)} \varphi_i^{(9)} dxdy \tag{5.116}$$

$$F_i^{10} = \int_\Omega \tilde{E}_{B31}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial x} + \tilde{E}_{T31}^{(3)} \frac{\partial \varphi_i^{(10)}}{\partial x} dxdy \tag{5.117}$$

$$F_i^{11} = \int_\Omega \tilde{E}_{T32}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} + \tilde{E}_{B32}^{(3)} \frac{\partial \varphi_i^{(11)}}{\partial y} dxdy \tag{5.118}$$

$$F_i^{12} = 0 \tag{5.119}$$

$$F_i^{13} = 0 \tag{5.120}$$

where

$$\tilde{E}_{T3j}^{(n)} = \int_{\frac{h}{2}}^{\frac{h}{2}+h_t} z^n \frac{2}{h_t} V_0^t dz, \qquad j = 1, 2, 3 \tag{5.121}$$

$$\tilde{E}_{B3j}^{(n)} = \int_{\frac{-h}{2}-h_b}^{\frac{-h}{2}} z^n \frac{2}{h_b} V_0^b dz, \qquad j = 1, 2, 3. \tag{5.122}$$

To verify the finite element model, the solutions of finite element model are compared with the analytical solutions in Fig. 5.13. Both results are simply supported linear problem, since the Navier solution technique is only available for simply supported linear problems. The results of the finite element model show good agreement with the analytical solutions. Since the finite element model of the functionally graded smart plate accounts for geometrical nonlinearity, it is necessary to show nonlinear behavior of the finite element model. Figure 5.14 shows center defection of a simply supported smart plate versus applied external voltages. The voltages are applied to top and bottom surface in different directions.

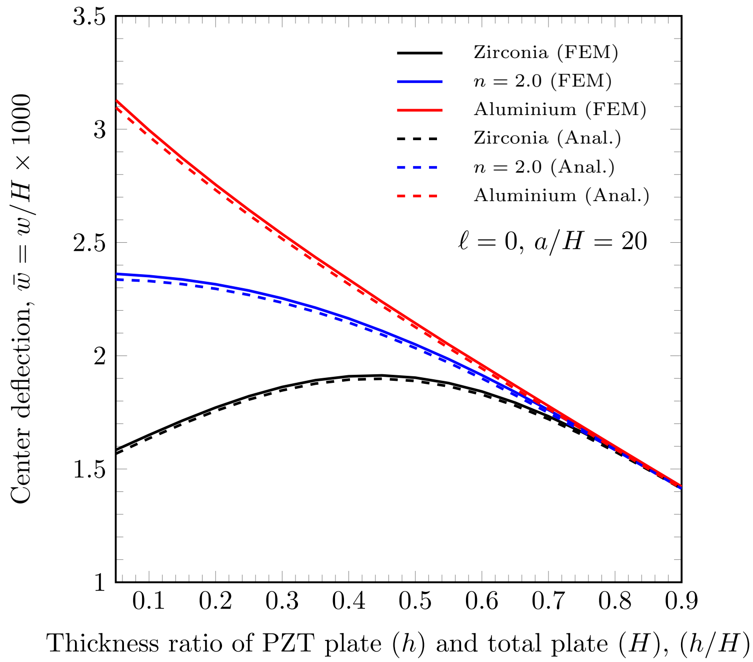

Figure 5.13: Comparison of analytical solution and finite element model solution [4]

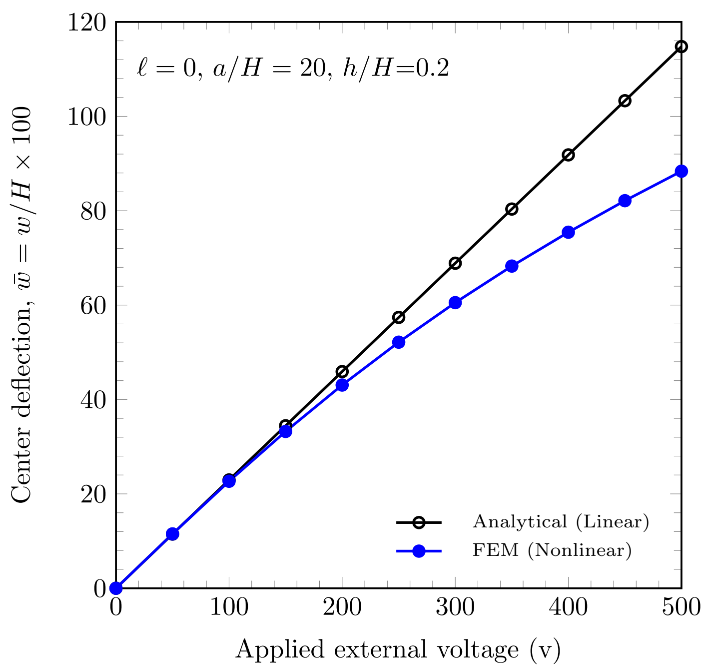

Figure 5.14: Center defection of a simply supported smart plate [4]

6. CONCLUSIONS

6.1 Concluding remarks

In this study, a general third-order plate theory that accounts for micro structure size effect, functionally graded and piezoelectric material behaviors, and geometrical nonlinearity is developed. Based on the proposed plate theory, analytical solutions and finite element solutions are presented. The parametric studies clearly show the effect of variation of materials through the thickness, length scale parameters, and electro-mechanical coupling effect.

In section 2, a general third-order theory of functionally graded plates with microstructure dependent length scale parameter and von Kármán nonlinearity is developed. The power-law distribution is used to model the functionally graded material, and modified couple stress theory is used to bring a microstructural length scale parameter. The equations of motions and associated force boundary conditions are derived using Hamilton's principle. The theory developed contains 11 generalized displacements. Three dimensional constitutive relations are used, consistent with the three dimensional strain field, to develop plate constitutive relations. Then the existing plate theories, namely, a third-order theory with vanishing surface tractions, the Reddy third-order plate theory, the first-order plate theory, and the classical plate theory are obtained as special cases of the developed general third-order plate theory.

In section 3, analytical solutions to a general third-order shear deformation plate theory that accounts for functionally graded material and modified couple stress theory are presented using the Navier solution technique. The variation of two constituents through the thickness of the FGM plate is considered using a power-law model, and the microstructure effects are considered using a length scale parameter. The solutions to static bending, free vibration, and buckling problems are presented to show the effect of FGMs and microstructure effects. It is clearly shown that smaller power-law index, n, makes plates stiffer based on the ratio of mechanical properties of the two constituents, and the length scale parameter, ℓ, has the ability to capture microstructure effects. The presented analytical solutions are limited to linear theory and for simply supported rectangular plates, but they are useful for the purpose of comparison with the numerical solutions (e.g., finite element models).

In section 4, a displacement based weak form Galerkin finite element model of a general third-order plate theory that accounts for the modified couple stress theory and the power-law variation of material through the thickness and von Kármán nonlinearity is developed. The micro structure size effect is captured by a length scale parameter through a modified couples stress theory. The finite element model requires C^1 continuity for all dependent variables. The 2D Hermite interpolation functions are used to represent the variables. The Newton iteration scheme is used to solve the resulting nonlinear finite element equations. Numerical results for rectangular plates with various boundary conditions are presented to study the effects of the power-law index and the length scale parameter in the static bending problems. The numerical results clearly show that the length scale parameter, ℓ, causes a stiffening effect in the plates. Since the plate theory does not explicitly imposes the vanishing of transverse shear stresses, they are not exactly zero; however, a quadratic variation of the transverse shear stresses is accounted for and no shear correction factors are used.

In section 5, a higher-order shear deformation plate model of functionally graded smart plates is developed using Hamilton's principle. The mechanical displacement field is assumed to be cubic variation for in-plane displacement and quadratic variations for transverse displacement. The electrical potential is assumed to be a combination of half cosine and linear variation of applied voltages. The presented model accounts for the power law variation of two materials through the thickness in core layer and piezoelectric effects on surface-mounted layers in addition to size dependent effects. The numerical examples for simply supported plates are presented using the Navier solution technique, and developed finite element model. The effects of the material variation through the thickness, the size dependent stiffening and the thickness ratio between piezoelectric layers and total plate are presented. The power-law distribution is used to model material variation through the transverse direction. The numerical examples presented herein clearly show that the effect of the piezoelectric layers does not only depend on its thickness but also upon the stiffness of the core. The size dependent stiffening effect is captured using a length scale parameter of the modified couple stress theory.

6.2 Recommendations

The encouraging results from the application of the general third order micro plate theory to functionally graded and piezoelectric materials open several interesting and challenging tasks to be carried out in future study.

For the development of plate theory, further studies can be carried out to satisfy traction boundary conditions, especially on top and bottom surfaces. In the literature, only Reddy third order theory [21] is developed to satisfy zero tangential tractions on top and bottom surfaces, and most displacement based plate theories do not consider the traction boundary conditions at the stage of development of displacement field. By neglecting traction conditions, most of plate theories result errors in the tractions on top and bottom surfaces of plates. Also, the proposed formulation can be extended to a higher order shell theory.

The present study accounts for the modified couple stress theory to capture the size effect of micro plates. The nonlocal theories requires additional material properties, e.g., length scale parameters, which are difficult to determine. Recently, Romanoff and Reddy [103] presented experimental validation of the modified couple stress Timoshenko beam theory for web-core sandwich panels. In their study, the length scale parameter was defined in terms of the dimensions of unit cell, and the analysis results showed good agreement with 3D finite element and experimental results. Explicit forms of those length scale parameters in nonlocal theories applied to specific problems should be investigated.

We present static bending, vibration, and buckling studies of functionally graded micro plates. It would be of great interest to perform transient analysis and post buckling analysis on functionally graded plates under thermomechanical coupling. In this study, we also presented static bending analysis of smart plates with surface mounted piezoelectric layers. The vibration and buckling studies using the developed finite element model of smart plates have yet to be carried out. Also, it would be of great interest to investigate distributed sensing performance of finite element models of piezoelectric plates.

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