THE POLYNOMIAL WARING PROBLEM AND THE DETERMINANT

A Dissertation

by

CAMERON LEE FARNSWORTH

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Chair of Committee,	J. M. Landsberg
Committee Members,	Frank Sottile
	Paulo Lima-Filho
	Jennifer Welch
Head of Department,	Emil Straube

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ABSTRACT

The symmetric rank of a polynomial P is the minimum number of d-th powers of linear forms necessary to sum to P. Questions pertaining to the rank and decomposition of symmetric forms or polynomials are of classic interest. Work on this topic dates back to the mid 1800's to J. J. Sylvester. Many questions have been resolved since Sylvester's work, yet many more questions have arisen. In recent years, certain polynomials including $\det_n$, the determinant of an $n \times n$ matrix of indeterminates, have become central in the study of rank problems. Symmetric border rank of a polynomial P is the minimum r such that P is in the Zariski closure of polynomials with symmetric rank r. It bounds and is closely related to rank. This dissertation demonstrates new lower bounds for the symmetric border rank of the polynomial $\det_n$. We prove this result using methods from algebraic geometry and representation theory. In addition to the lower bounds for symmetric border rank of $\det_n$, we present a lower bound for symmetric border rank of a related polynomial, perm_3. We conclude by giving future directions for continuing this project. The first direction is to use the methods from algebraic geometry and representation theory used in this dissertation to study perm_n. With the new lower bound on symmetric border rank of perm_3 we know that there are only 3 possible values for symmetric border rank of perm_3. One could ask which of the 3 possible values for symmetric border rank of perm_3 is the correct value.

DEDICATION

This dissertation is dedicated to my late grandparents, Grandpa Vernon and Grandma Jan.

ACKNOWLEDGEMENTS

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1. INTRODUCTION: THE WARING PROBLEM FOR THE DETERMINANT

The Waring problem emerged in the late 1700's, see [War82, p. 349]. The Waring problem is known to ask: if d is a natural number, is there a number $k(d)$ such that no more than $k(d)$ d-th powers of natural numbers are needed to sum to any $n \in \mathbb{N}$? This question remained open for over a century until David Hilbert answered the question in the affirmative in [Hil09]. For a survey on the Waring problem see [VW02].

Since the Waring problem was initially stated many analogues have arisen. In particular, it is known that any complex homogeneous degree d polynomial in n variables may be written as a sum of d-th powers of linear forms. This motivates the *polynomial Waring problem*:

Question 1.1 (polynomial Waring problem)**.** *Let P be a complex homogeneous degree d polynomial in n variables. What is the minimum r needed to write P as*

$$P = \sum_{i=1}^{r} (\alpha_{i_1} x_1 + \cdots + \alpha_{i_n} x_n)^d$$

where $\alpha_{i_j} \in \mathbb{C}$?

This question is still open for explicit polynomials. In the above question r is called the *symmetric rank* or *Waring rank* of P and is commonly called rank if it is clear by context.

Rank of polynomials is not semi-continuous and the Zariski closure of the set of homogeneous degree d polynomials in n variables of a fixed rank r can contain polynomials of rank greater than r. Symmetric border rank is a geometrically meaningful concept of rank addressing this phenomenon. See Definition 2.36 for a formal

definition of symmetric border rank. The set of homogeneous degree d polynomials in n variables with symmetric border rank at most r is an algebraic variety. This allows the powerful tools of algebraic geometry to be used to study border rank. Lower bounds for the symmetric border rank of P are obtained by evaluating defining equations of these varieties on P. Early examples of this method date back to Sylvester's work in the 1800's. Sylvester introduces *catalecticants* in [Syl51a] and observes a connection between their rank and the symmetric border rank of polynomials. This dissertation presents catalecticants as symmetric flattenings.

In [LO13], Landsberg and Ottaviani describe the method of Young Flattenings to obtain linear maps from a given polynomial P and which generalize catalecticants. Young flattenings provide equations on the variety of polynomials of border rank at most r and therefore find lower bounds on border rank of P.

This dissertation uses the method of Landsberg and Ottaviani to obtain lower bounds on the symmetric border rank of the determinant of an $n \times n$ matrix of indeterminates, $\det_n$, a homogeneous polynomial of degree n in n^2 variables. The determinant is a classic focus of research and many properties of the determinant are well understood, but the rank of $\det_n$ is unknown. Due to questions originating in complexity theory motivated by the work of Valiant in [Val79] and Mulmuley and Sohoni in [MS01] the determinant polynomial and a related polynomial, the permanent, have become increasingly important and popular foci of research.

Section two provides basic definitions and theorems needed to understand the Young flattenings of Landsberg and Ottaviani. This background section includes basics on tensor decomposition, and representation theory, and a brief discussion on secant varieties. Section three offers a brief summary of results and applications of tensor and polynomial rank. Section four covers lower bounds obtained via the use of Young flattenings. This section details a specific application of these flattenings

and describes how representation theory aided in calculating the rank of a linear map obtained as the Young flattening of $\det_n$. It also demonstrates how computer software calculated new lower bounds for small determinants and permanents. Section five summarizes this dissertation and outlines further research directions.

2. BACKGROUND

We begin with a discussion of tensors, including the concepts of rank and flattenings. Next, we introduce secant varieties and the border rank of a tensor. The section concludes with tools from representation theory crucial to understanding the proofs presented later. Much of the material in this section can be found in basic texts such as [Ful97, FH91, Ike13, Lan12]. Our base field is $\mathbb{C}$, the field of complex numbers.

Tensors play an important role in mathematics and its applications. The tensor is a universal object for multi-linear algebra. Our interest lies in tensors with symmetry, but the definitions and ideas in the general case for tensors can be easier to state and understand. The statements for the special case will follow and we may see the similarities.

Definition 2.1. A function $f\colon V_1^* \times \cdots \times V_k^* \to \mathbb{C}$ is k-*linear* if for every i

$$f(\ldots, \alpha_i + c\beta_i, \ldots) = f(\ldots, \alpha_i, \ldots) + cf(\ldots, \beta_i, \ldots).$$

Definition 2.2. Let $V_1, \ldots, V_k$ be vector spaces. The tensor product of the vector spaces $V_1, \ldots, V_k$ is the space of k-linear functions $f\colon V_1^* \times \cdots \times V_k^* \to \mathbb{C}$ and is denoted by $V_1 \otimes \cdots \otimes V_k$.

For $v_i \in V_i$ and $\alpha_i \in V_i^*$ where $i = 1, \ldots, k$, the map $v_1 \otimes \cdots \otimes v_k(\alpha_1, \ldots, \alpha_k) = \alpha_1(v_1) \cdots \alpha_k(v_k)$ defines the element $v_1 \otimes \cdots \otimes v_k \in V_1 \otimes \cdots \otimes V_k$. We call tensors $T \in V_1 \otimes \cdots \otimes V_k$ which may be written in this form *rank one*.

Definition 2.3. The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_k$ is the minimum r such that $T = \sum_{i=1}^r c_i \cdot v_{1,i} \otimes \cdots \otimes v_{k,i}$ where $c_i \in \mathbb{C}$ and $v_{j,i} \in V_j$.

That is, the rank of T is the minimal number of rank one tensors needed to sum to T.

Remark 2.4. For any choice of $i \in \{1, \ldots, k\}$, there exists an interpretation of $V_1 \otimes \cdots \otimes V_k$ as a $(k-1)$-linear map $T_{V_i} \colon V_1^* \times \cdots \times \widehat{V_i^*} \times \cdots \times V_k^* \to V_i$ where the $\widehat{V_i^*}$ should be interpreted as omitted.

Definition 2.5. Consider the tensor product $V_1 \otimes \cdots \otimes V_k$ and let $I \subseteq \{1, \ldots, k\}$ and let $J = \{1, \ldots, k\} \smallsetminus I$ and let $V_I^* = V_{i_1}^* \otimes \cdots \otimes V_{i_{|I|}}^*$ and $V_J = V_{j_1} \otimes \cdots \otimes V_{j_{|J|}}$ where $i_\ell \in I$ and $j_\ell \in J$ such that $i_m \neq i_n$ and $j_m \neq j_n$ when $m \neq n$. The linear map $T_{I,J} \colon V_I^* \to V_J$ is called a *flattening* of T.

Example 2.6. Let A, B, and C be vector spaces of dimension two and let a_i, b_i, and c_i be bases of vector spaces A, B, and C respectively and let α_i, β_i, and γ_i be bases of their dual spaces. The tensor $T = a_1 \otimes b_1 \otimes c_2 + 2a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2$ has the following flattenings:

$$
T_{\{2,3\},\{1\}} = \begin{array}{c} a_1 \\ a_2 \end{array} \begin{pmatrix} \overset{\beta_1 \otimes \gamma_1}{0} & \overset{\beta_1 \otimes \gamma_2}{1} & \overset{\beta_2 \otimes \gamma_1}{0} & \overset{\beta_2 \otimes \gamma_2}{0} \\ 2 & 0 & 0 & 1 \end{pmatrix},
$$

$$
T_{\{1,3\},\{2\}} = \begin{array}{c} b_1 \\ b_2 \end{array} \begin{pmatrix} \overset{\alpha_1 \otimes \gamma_1}{0} & \overset{\alpha_1 \otimes \gamma_2}{1} & \overset{\alpha_2 \otimes \gamma_1}{2} & \overset{\alpha_2 \otimes \gamma_2}{0} \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

and

$$T_{\{1,2\},\{3\}} = \begin{array}{c} \\ c_1 \\ c_2 \end{array} \begin{array}{cccc} \alpha_1 \otimes \beta_1 & \alpha_2 \otimes \beta_1 & \alpha_1 \otimes \beta_2 & \alpha_2 \otimes \beta_2 \\ \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

An application and generalization of tensor flattenings will be presented shortly.

Now we introduce secant varieties, important results, and definitions associated with such varieties. First, we introduce secant varieties in full generality, then move to a special case which provides an alternative to the tensor rank previously introduced.

Definition 2.7. Let $X \subset \mathbb{P}V$ be a variety. The *r-th secant variety* of X is

$$\sigma_r(X) = \overline{\bigcup_{x_1,\ldots,x_r \in X} \langle x_1, \ldots, x_r \rangle}.$$

Notice for every $x_1, \ldots, x_r$, $\langle x_1, \ldots, x_{r-1} \rangle \subseteq \langle x_1, \ldots, x_r \rangle$. This implies

$$\bigcup_{x_1,\ldots,x_{r-1} \in X} \langle x_1, \ldots, x_{r-1} \rangle \subseteq \bigcup_{x_1,\ldots,x_r \in X} \langle x_1, \ldots, x_r \rangle.$$

This implies $\sigma_{r-1}(X) \subseteq \sigma_r(X)$.

Definition 2.8. The *Segre variety* is the variety of rank one tensors. It is the image of the map

$$\mathrm{Seg} \colon \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k \to \mathbb{P}(V_1 \otimes \cdots \otimes V_k)$$

$$([v_1], \ldots, [v_k]) \mapsto [v_1 \otimes \cdots \otimes v_k].$$

In general, the set of rank r tensors is open and therefore does not define a projective variety. $\sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k))$ is a variety containing the tensors in

$V_1 \otimes \cdots \otimes V_k$ of rank r. Notice if $T \in \langle x_1, \ldots, x_r \rangle$ where $x_1, \ldots, x_r \in \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times$ $\mathbb{P}V_k)$, then T can be written as the sum of r rank one tensors; the r-th secant variety of the Segre variety is the Zariski closure of the set of such tensors. By taking the Zariski closure, more tensors than those of rank r may be in this secant variety. This motivates the following definition which provides a geometrically meaningful notion of rank.

Definition 2.9. For a tensor $T \in V_1 \otimes \cdots \otimes V_k$, the minimal r such that $T \in \sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k))$ is called the *border rank* of T and is denoted $\underline{R}(T)$.

The equations of the variety $\sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k))$ provide a method to test for the border rank of a tensor. If an equation vanishes on this variety but does not vanish on a tensor T, then T has border rank greater than r. Tensor flattenings provide some of the known equations for $\sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k))$. Given a tensor $T \in V_1 \otimes \cdots \otimes V_k$, if there exists a nonzero $(r+1) \times (r+1)$-minor of flattening $T_{I,J}$, then $\underline{R}(T) > r$ [Lan12, p. 76]. Unfortunately, in most cases the equations obtained from flattenings are incomplete and more equations are needed to understand these varieties.

Conditions on the rank of flattenings are useful for determining non-membership in the r-th secant variety of the Segre variety. A generalization of flattenings that implies lower bounds for the polynomial Waring problem, Question 1.1, is provided shortly.

We now discuss basics of representation theory. Tensors appear early on in the study of representations since there are natural actions of the general linear group and the symmetric group on tensors.

Definition 2.10. Let V be a finite dimensional vector space. $GL(V)$ denotes the group of all invertible linear transformations $f \colon V \to V$. This group is called the

general linear group of V. By fixing a basis of V, we identify $f \in GL(V)$ with an $n \times n$ matrix M_f with entries in $\mathbb{C}$. This identifies $GL(V)$ with the group of $n \times n$ invertible matrices with entries in $\mathbb{C}$ denoted GL_n. The *special linear group* of V is the subgroup $SL(V) \subset GL(V)$ of $f \in GL(V)$ such that $\det(M_f) = 1$.

Definition 2.11. Let V be finite dimensional vector space and G be a group. A *representation* of G is a group homomorphism

$$\rho \colon G \to GL(V).$$

We sometimes refer to V as a representation of G. In this case the reader should understand that there is a homomorphism $\rho \colon G \to GL(V)$ and the action of $g \in G$ on $v \in V$ is $g.v = \rho(g)(v)$.

Definition 2.12. Let V be a representation of the group G. A *subrepresentation* of V is a subspace $U \subseteq V$ such that $\forall u \in U$ and $\forall g \in G,\ g.u \in U$.

Definition 2.13. A group representation V is *irreducible* if its subrepresentations are $\{0\}$ or V.

Definition 2.14. Given a group G and representations V and W of G, a linear map $\varphi \colon V \to W$ is called a *G-module homomorphism* if $\varphi(g.v) = g.\varphi(v)$.

Understanding maps between two representations provides us with information about how the representations decompose. For instance, if we know that φ is a G-module homomorphism then we have the following lemmas about the maps kernel and image.

Lemma 2.15. *Let G be a group with representations V and W and let $\varphi \colon V \to W$ be a G-module homomorphism. Then $\ker(\varphi) \subseteq V$ is a G-module.*

Lemma 2.16. *Let G be a group with representations V and W and let $\varphi \colon V \to W$ be a G-module homomorphism. Then $\mathrm{Im}(\varphi) \subseteq W$ is a G-module.*

These two statements are essential for the proof of Schur's lemma, one of the most useful lemmas in representation theory.

Lemma 2.17 (Schur's lemma). *Let G be a group and let V and W be irreducible G-modules. Then $\varphi \colon V \to W$ is either identically zero or it is an isomorphism.*

Proof. In the statement of Schur's lemma V and W are irreducible G-modules. Since V is irreducible, $\ker(\varphi)$ is either $\{0\}$ and φ is an isomorphism or it is V and φ is identically zero. $\square$

Let V be a vector space of dimension n. Furthermore, let $V \cong V_i$ for every i and let $V^{\otimes k} = V_1 \otimes \cdots \otimes V_k$. The general linear group $GL(V)$ acts in the following way. Let $g \in GL(V)$, then $\forall w = v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$

$$g.w = g.v_1 \otimes \cdots \otimes g.v_k.$$

This extends to an action of $GL(V)$ on $V^{\otimes k}$ by the universal mapping property of tensor products.

Let $\mathfrak{S}_k$ denote the symmetric group on k letters. Let $\tau \in \mathfrak{S}_k$ and $w = v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$, then the action of $\mathfrak{S}_k$ on $V^{\otimes k}$ is defined as follows

$$\tau.w = v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(k)}$$

and extend linearly.

A tensor $T \in V^{\otimes k}$ is *symmetric* if $T(\alpha_1, \ldots, \alpha_k) = T(\alpha_{\tau(1)}, \ldots, \alpha_{\tau(k)})$ for every $\tau \in \mathfrak{S}_k$. $S^k V \subseteq V^{\otimes k}$ denotes the space of symmetric tensors or k-forms.

We identify $S^k V$ with the space of homogeneous degree k polynomials on V^* in the following way: Let $\overline{P} \in S^k V$ be a symmetric tensor. The homogeneous degree k polynomial $P \in S^k V$ associated to $\overline{P}$ is

$$P(\alpha) = \overline{P}(\alpha, \dots, \alpha).$$

Given a homogeneous degree k polynomial $P \in S^k V$, the symmetric tensor $\overline{P} \in S^k V$ associated with P is obtained via *polarization*, which is defined as follows:

$$\overline{P}(\alpha_1, \dots, \alpha_k) = \frac{1}{k!} \cdot \mathrm{coef}_{t_1 \cdots t_k}(P(t_1 \alpha_1 + \cdots + t_k \alpha_k)),$$

where $\mathrm{coef}_{t_1 \cdots t_k}(P)$ denotes the coefficient of $t_1 \cdots t_k$. The next example demonstrates both of these ideas.

Example 2.18. Let $P((\alpha, \beta)) = \alpha^2 \beta + 3\alpha\beta^2$, then

$$
\begin{aligned}
\overline{P}&((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)) = \\
=& \frac{1}{3!} \mathrm{coef}_{t_1 t_2 t_3}[P((t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3), (t_1\beta_1 + t_2\beta_2 + t_3\beta_3))] \\
=& \frac{1}{6} \mathrm{coef}_{t_1 t_2 t_3}[(2(\beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3) + 6(\beta_1\beta_2\alpha_3 + \beta_1\alpha_2\beta_3 + \alpha_1\beta_2\beta_3))t_1 t_2 t_3 \\
&+ (2\alpha_1\alpha_2\beta_1 + 3\alpha_2\beta_1^2 + \alpha_1^2\beta_2 + 6\alpha_1\beta_1\beta_2)t_1^2 t_2 + \cdots] \\
=& \frac{1}{6}[2(\beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3) + 6(\beta_1\beta_2\alpha_3 + \beta_1\alpha_2\beta_3 + \alpha_1\beta_2\beta_3)] \\
=& \frac{1}{3}(\beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3) + (\beta_1\beta_2\alpha_3 + \beta_1\alpha_2\beta_3 + \alpha_1\beta_2\beta_3).
\end{aligned}
$$

Reversing this process, we see

$$P((\alpha, \beta)) = \overline{P}((\alpha, \beta), (\alpha, \beta), (\alpha, \beta)) = \frac{1}{3}(3\alpha^2\beta) + 3\alpha\beta^2 = \alpha^2\beta + 3\alpha\beta^2.$$

Definition 2.19. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a weakly decreasing finite sequence of nonnegative numbers. The *length* of a partition equals the number of nonzero parts in the partition, denoted $\ell(\lambda)$, and $|\lambda| = \lambda_1 + \cdots + \lambda_d$. A *Young diagram* is a diagram of boxes left and top justified with λ_i boxes in the i-th row. If the boxes are filled with entries from $\{1, \ldots, n\}$, it is called a *Young tableau*. The *content* of a tableau T with fillings from $\{1, \ldots, n\}$ is an n-tuple $c(T) \in \mathbb{N}^n$, the i-th entry of $c(T)$ records the number of times the entry $i \in \{1, \ldots, n\}$ is an entry of T. The entries of a *standard* tableau strictly increase left to right across rows and down columns, while the entries of a *semistandard* tableau strictly increase down columns and weakly increase left to right across rows. See Figure 2.1 for examples.
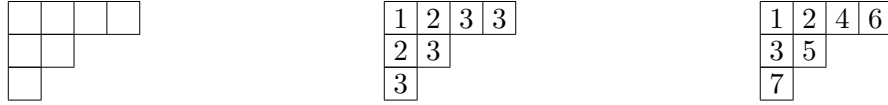
Figure 2.1: Examples of a Young diagram, a semistandard Young tableau, and a standard Young tableau corresponding to the partition $\lambda = (4, 2, 1)$.

If a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ is such that $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+k}$ where $1 \leq j+1 \leq j+k \leq d$ we may write $\lambda = (\lambda_1, \ldots, \lambda_j, \lambda_{j+1}^k, \lambda_{j+k+1}, \ldots, \lambda_d)$ to abbreviate notation. For example the partition $(4, 3, 2, 2, 2, 1)$ may be written $(4, 3, 2^3, 1)$.

Definition 2.20. Given a partition λ, its *conjugate partition* λ' is the partition whose i-th part is the length of the i-th column in the Young diagram of λ.

Example 2.21. Let $\lambda = (2, 1, 1)$ be a partition. The conjugate partition of λ is $\lambda' = (3, 1)$.

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Definition 2.22. Let λ be a partition and T be a standard tableau of shape λ with $c(T) = (1, \ldots, 1)$. The permutation $\tau \in \mathfrak{S}_{|\lambda|}$ acts on a standard tableau by permuting the entries of the tableau. Let $\mathfrak{S}_T(i) \subseteq \mathfrak{S}_{|\lambda|}$ consist of elements $\tau \in \mathfrak{S}_{|\lambda|}$ such that τ permutes only the entries of the i-th row of T and $\mathfrak{S}'_T(j) \subseteq \mathfrak{S}_{|\lambda|}$ consist of elements $\tau \in \mathfrak{S}_{|\lambda|}$ such that τ permutes only the entries of the j-th column of T. Let $\mathbb{C}[\mathfrak{S}_{|\lambda|}]$ denote the group algebra of $\mathfrak{S}_{|\lambda|}$ and for every $\tau \in \mathfrak{S}_{|\lambda|}$ let e_τ denote a basis element of $\mathbb{C}[\mathfrak{S}_{|\lambda|}]$. Let $\rho(T, i) = \sum_{\tau \in \mathfrak{S}_T(i)} e_\tau$ and define $\rho(T) = \rho(T, 1) \cdots \rho(T, \ell(\lambda))$. Similarly, define $\rho'(T, j) = \sum_{\tau \in \mathfrak{S}'_T(j)} \mathrm{sgn}(\tau) e_\tau$ and $\rho'(T) = \rho'(T, 1) \cdots \rho'(T, \ell(\lambda'))$. A *Young symmetrizer* corresponding to T is $Y(T) \in \mathbb{C}[\mathfrak{S}_{|\lambda|}]$ defined by $Y(T) = \rho(T)\rho'(T)$. For references on Young symmetrizers, see *e.g.*, [FH91, p. 46] and [Lan12, p. 142].

Definition 2.23. Let V be a vector space. Fix $H \subset GL(V)$ a maximal abelian diagonalizable subgroup. We usually work with a basis of V for which H is the subgroup of invertible diagonal matrices. Let W be a representation of $GL(V)$. For every group homomorphism $\lambda \colon H \to \mathbb{C}^*$, define $W_\lambda = \{w \in W \mid h.w = \lambda(h)w, \forall h \in H\}$. If $W_\lambda \neq 0$, we call it a *weight space* of W for H and λ is said to be its *weight*. Notice that the W_λ's are the (simultaneous) eigenspaces of H and λ's are their associated generalized eigenvalues.

Fix a basis where H is diagonal. In any irreducible $GL(V)$-representation W there is a unique line $\ell \subset W$ such $M.\ell = \ell$ for all upper triangular matrices M. This line ℓ is a weight space. We call the weight of ℓ a *highest weight* and ℓ a *highest weight line* see, *e.g.*, [Lan12, Ch. 6.8].

Definition 2.24. Let λ be partition. A *Schur module*, denoted $S_\lambda V$, is an irreducible $GL(V)$-module with highest weight λ and defined as the image of $V^{\otimes |\lambda|}$ under the projection determined by $Y(T)$ where T is a standard Young tableau with $c(T) = (1, \ldots, 1)$ and shape λ.

Let V be a vector space of dimension n. For a given Schur module $S_\lambda V$, we have the isomorphism $S_{\lambda^c} V^* \cong S_\lambda V$ as $SL(V)$-modules where λ^c is the complement of the Young diagram λ in the $n \times \lambda_1$ rectangle.

Example 2.25. Let V be a vector space of dimension three, then given $\lambda = (3, 1)$ then $\lambda^c = (3, 2)$.

The following is a discussion of the irreducible $GL(V)$-module $S_\lambda V$ as a quotient space of an abstract vector space of Young tableaux. This interpretation follows those found in [Ful97, p. 110] and [Ike13, p. 35f]. Let V be a vector space of dimension n and λ be a partition with $\ell(\lambda) \leq n$. Let $\mathbb{C}_\lambda(n)$ denote the vector space with the set of all Young tableaux of shape λ with fillings from $\{1, \ldots, n\}$ as its basis. Let $\mathbb{S}_1 \subseteq \mathbb{C}_\lambda(n)$ be the subspace generated by sums of tableaux $T + T'$ where T' is obtained from T by exchanging the entries of exactly two boxes of the same column. Sums of tableaux $\Gamma(T, i, j) := T - \sum_{T'} T'$ generate the subspace $\mathbb{S}_2 \subseteq \mathbb{C}_\lambda(n)$, where the sum $\sum_{T'} T'$ ranges over all T' that are obtained by exchanging the top j entries of the $(i + 1)$-st column of T with any choice of j entries in the i-th column of T while maintaining vertical order. Maintaining vertical order means if entries a and b are exchanged from column $i + 1$ to column i or column i to column $i + 1$ with entry a in a box above b, not necessarily immediately above, then a is in a box above b after the exchange. Let the subspace $\mathbb{S} \subseteq \mathbb{C}_\lambda(n)$ be defined $\mathbb{S} = \mathbb{S}_1 + \mathbb{S}_2$. The irreducible $GL(V)$-representation $S_\lambda V$ as a vector space can be defined as the quotient space of $\mathbb{C}_\lambda(n)/\mathbb{S}$ which has a basis of semistandard tableaux with fillings from $\{1, \ldots, n\}$. The *shuffling rules* Lemma 2.26 and Lemma 2.27 given below are relations among tableaux in the quotient space $\mathbb{C}_\lambda(n)/\mathbb{S}$. Through repeated applications of the shuffling rules a tableau $T \in \mathbb{C}_\lambda(n)/\mathbb{S}$ can be rewritten a sum of semistandard tableaux.

Lemma 2.26 (Shuffling rule 1)**.** *Let $T, T' \in \mathbb{C}_\lambda(n)$, with T' obtained by exchanging the entries of two boxes in the same column of T, then in the quotient space $\mathbb{C}_\lambda(n)/\mathbb{S}$ we have $[T'] = -[T]$, where $-T$ is an element of the vector space $\mathbb{C}_\lambda(n)$. See Figure 2.2 for an illustration.*

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\\hline 1 \\\cline{1-1}\end{array} = - \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\\hline 2 \\\cline{1-1}\end{array}$$

Figure 2.2: An example of the Lemma 2.26 with $\dim(V) \geq 3$.

Lemma 2.27 (Shuffling rule 2)**.** *Let $T \in \mathbb{C}_\lambda(n)$ and the tableaux $T' \in \mathbb{C}_\lambda(n)$ be tableaux obtained by exchanging the top j entries of the $(i+1)$-st column of T with any choice of j entries in the i-th column of T while maintaining vertical order. In the quotient space $\mathbb{C}_\lambda(n)/\mathbb{S}$, we have $[T] = \sum_{T'} [T']$. Figure 2.3 visually demonstrates this rule.*

$$\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\\hline 4 & 2 & 6 \\\hline 5 \\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\\hline 2 & 4 & 6 \\\hline 5 \\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\\hline 4 & 5 & 6 \\\hline 2 \\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 3 & 4 & 3 \\\hline 1 & 5 & 6 \\\hline 2 \\\cline{1-1}\end{array}$$

Figure 2.3: An example of the Lemma 2.27 with $\dim(V) \geq 6$.

Remark 2.28. As a result of Lemma 2.26, if tableau T has an entry repeated in the any given column then $[T] = [0] \in \mathbb{C}_\lambda(n)/\mathbb{S}$. Lemma 2.27 implies $[T] = [T'] \in \mathbb{C}_\lambda(n)/\mathbb{S}$ if T' is a tableau obtained from T by switching columns of the same size.

The tensor product $S_\lambda V \otimes S^k V$ decomposes as a $GL(V)$-module. This decomposition is governed by the Pieri rule.

Proposition 2.29 (Pieri rule). *Let V be a vector space of dimension n, then*

$$S_\lambda V \otimes S^k V = \bigoplus_\mu S_\mu V$$

where partitions μ are partitions with length at most n obtained from λ by adding k boxes to the diagram of λ with no two boxes added to the same column.

Let V be a vector space of dimension at least 4. An example of the Pieri rule, $S_{(2,1,1)}V \otimes S^3 V$, is demonstrated visually in Figure 2.4. We interpret Figure 2.4 as showing

$$S_{(2,1,1)}V \otimes S^3 V = S_{(5,1,1)}V \oplus S_{(4,2,1)}V \oplus S_{(4,1^3)}V \oplus S_{(3,2,1^2)}V.$$

On the left hand side we identify $S_{(2,1,1)}V$ with a Young diagram corresponding to the partition $\lambda = (2,1,1)$. We also identify $S^3 V$ and $S_{(3)}V$. Then $S_{(3)}V$ is identified with the Young diagram corresponding to the partition (3). The right hand side of the figure is a sum of Young diagrams which correspond to the partitions of each summand $S_\mu V$ occurring in the $GL(V)$-module decomposition of $S_{(2,1,1)}V \otimes S^3 V$.
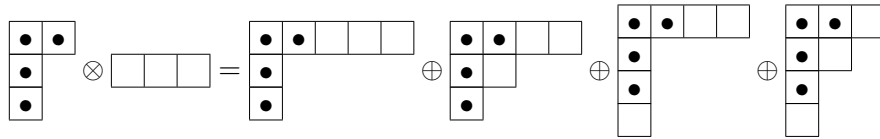
Figure 2.4: An example of the Pieri rule with $S_\lambda V$ where $\lambda = (2,1,1)$ and $S^3 V$.

The Pieri rule also governs the decomposition of $S_\lambda V \otimes \bigwedge^k V$ into irreducible $GL(V)$-modules. Given a vector space V of dimension n, this decomposition is as follows

$$S_\lambda V \otimes \bigwedge^k V = \bigoplus_\mu S_\mu V$$

where the partitions μ are partitions with length at most n obtained from λ by adding k boxes to the diagram of λ with no two boxes added to the same row. Figure 2.5 illustrates this version of the Pieri rule.
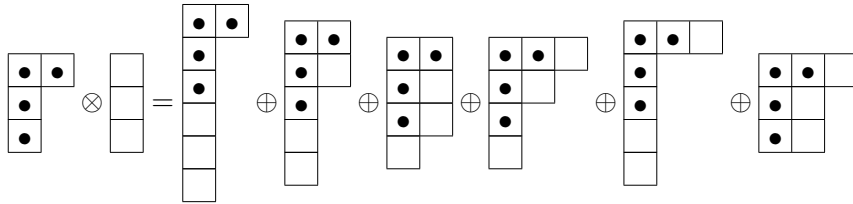
Figure 2.5: An example of the Pieri rule with $\lambda = (2, 1, 1)$ and the partition $(1, 1, 1)$.

On the left hand side of Figure 2.5, identifications similar to the identifications made in Figure 2.4 are made. The $GL(V)$-module $\bigwedge^3 V$ is identified with $S_{(1,1,1)}V$. The right hand side of the figure is a sum of Young diagrams which correspond to the partitions of each summand $S_\mu V$ occurring in the $GL(V)$-module decomposition of $S_{(2,1,1)}V \otimes \bigwedge^3 V$. Figure 2.5 is interpreted as showing

$$S_{(2,1,1)}V \otimes \bigwedge^3 V = S_{(2,1^5)}V \oplus S_{(2^2,1^3)}V \oplus S_{(2^3,1)}V$$

$$\oplus S_{(3,2,1^2)}V \oplus S_{(3,1^4)}V \oplus S_{(3,2^2)}V.$$

The Pieri rule is a special case of the Littlewood-Richardson rule. This gen-

eralization will not be explicitly stated as it is not used in this dissertation; however, the Littlewood-Richardson rule governs the decomposition of $S_\lambda V \otimes S_\mu V = \bigoplus_{|\nu|=|\lambda|+|\mu|} S_\nu V^{\oplus c_{\lambda\mu}^\nu}$ as $GL(V)$-representations [Lan12, p. 153].

In this dissertation we look at a vector space $V = A \otimes B$ where A and B are both of dimension n. In general, a $GL(V)$-representation for a vector space $V = A \otimes B$ decomposes as $GL(A) \times GL(B)$ irreducible modules as

$$S_\lambda(A \otimes B) = \bigoplus_{|\lambda|=|\mu|=|\nu|} (S_\mu A \otimes S_\nu B)^{\oplus g_{\lambda\mu\nu}}$$

where $g_{\lambda\mu\nu}$ is called a *Kronecker coefficient* [Lan12, p. 150].

Example 2.30. Let A and B be vector spaces of dimension n. The following examples, see *e.g.* [Lan12, p. 157], of Kronecker decompositions which will be useful later:

$$S^k(A \otimes B) = \bigoplus_{|\lambda|=k} S_\lambda A \otimes S_\lambda B$$

$$\wedge^k(A \otimes B) = \bigoplus_{|\lambda|=k} S_\lambda A \otimes S_{\lambda'} B.$$

Recall that given an order k tensor T and a subset I of k, a flattening of T was a linear map from V_I^* to V_J where $J = [k] \smallsetminus I$. The following defines an analogous map for symmetric tensors. The definition relies upon the observation that $S^k V \subset S^{k-d} V \otimes S^d V$ as a $GL(V)$-module for any $d < k$.

Remark 2.31. Let V be a vector space of dimension n. Let $\{x_1, \ldots, x_n\}$ be a basis of V and V^* be its dual space with dual basis $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$. Notice that $\partial_{x_i}(x_j) = \delta_j^i$. We extend this action of V^* on V to an action on $S^q V$ for any q by Leibniz's rule. We conclude that V^* can be identified with the space of linear differential operators

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on $\{x_1, \ldots, x_n\}$. Since $S^q V$ is a space of polynomials differentiation commutes and we conclude that $S^k V^*$ is the space of homogeneous order k differential operators with constant coefficients on $\{x_1, \ldots, x_n\}$.

Definition 2.32. Let $P \in S^k V$, let $P_{d,k-d} \in S^{k-d}V \otimes S^d V$ be the tensor P considered as an element of $S^{k-d}V \otimes S^d V$. The linear map $P_{d,k-d}$ is called a *(d-th) standard flattening* of P and is defined as follows

$$P_{d,k-d} \colon S^d V^* \to S^{k-d} V$$

$$\alpha \mapsto \alpha \lrcorner P$$

where $\alpha \lrcorner P$ is differentiation of P by α.

Remark 2.33. The catalecticants introduced by Sylvester in [Syl51a] are standard flattenings. There, Sylvester observes how the rank of standard flattenings has implications on Waring rank. This observation is explained at the end of this section.

Example 2.34. Let $V \cong \mathbb{C}^2$ and let $P = x^2 y + 3xy^2$. The matrix representing the linear map $P_{2,1} \colon S^2 V^* \otimes \to V$ is

$$P_{2,1} = \begin{matrix} & \begin{matrix} \partial_x^2 & \partial_x \partial_y & \partial_y^2 \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{pmatrix} 0 & 2 & 6 \\ 2 & 6 & 0 \end{pmatrix} \end{matrix}.$$

Earlier we defined the border rank for a tensor T via secant varieties. In particular, we were interested in secant varieties of the Segre variety. Attention is now turned towards an analogous definition for symmetric tensors $P \in S^k V$.

Definition 2.35. The Veronese variety is the image of the map

$$v_k \colon \mathbb{P}V \to \mathbb{P}S^k V$$

$$[x] \mapsto [x^k].$$

Definition 2.36. Let $P \in S^k V$. The *symmetric border rank* of P is the minimal r such that

$$P \in \sigma_r(v_k(\mathbb{P}V)).$$

Symmetric border rank of P is denoted $\underline{R}_s(P)$.

Remark 2.37. An element in $v_k(\mathbb{P}V)$ can be written in the form $[\ell^k]$ where $\ell \in V$. If P is a polynomial such that $P \in \bigcup_{x_1,\dots,x_r \in v_k(\mathbb{P}V)} \langle x_1, \dots, x_r \rangle$, then one may write

$$P = \sum_{i=1}^{r} \ell_i^k$$

where $\ell_i \in V$ is a linear form. If r is minimal for P then r is called its *symmetric rank* or *Waring rank* is denoted $R_s(P)$.

Since $\sigma_r(v_k(\mathbb{P}V)) = \overline{\bigcup_{x_1,\dots,x_r \in v_k(\mathbb{P}V)} \langle x_1, \dots, x_r \rangle}$, if $P \notin \sigma_r(v_k(\mathbb{P}V))$, then

$$P \neq \sum_{i=1}^{r} \ell_i^k$$

for any choice of linear forms $\ell_i \in V$. If we know a polynomial that vanishes on $\sigma_r(v_k(\mathbb{P}V))$ and that same polynomial does not vanish on P, then $P \notin \sigma_r(v_k(\mathbb{P}V))$. Therefore $\underline{R}_s(P) > r$ and furthermore $R_s(P) > r$.

It is not a coincidence that if $\underline{R}_s(P) > r$ then $R_s(P) > r$. In fact, it is known that for any polynomial $\underline{R}_s(P) \leq R_s(P)$.

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Remark 2.38. An alternate definition of $\underline{R}_s(P)$ is that it is the minimal r such that P can be written as a limit of polynomials P_t with $R_s(P_t) = r$. This is a consequence of the fact that the Euclidean closure of X_0, a Zariski open subset of a variety $X \subseteq \mathbb{P}V$, is X see, *e.g.*, [Mum95, p. 38].

Example 2.39. The simplest example of a polynomial P such that $\underline{R}_s(P) < R_s(P)$ is $P = x^2y$. We see that $P = \frac{1}{6}(x+y)^3 - \frac{1}{6}(x-y)^3 - \frac{1}{3}y^3$. It remains to be shown that $P \neq (ax+by)^3 + (cx+dy)^3$ for any $a, b, c, d \in \mathbb{C}$. Assuming there exists a choice of $a, b, c, d \in \mathbb{C}$ satisfying $P = (ax+by)^3 + (cx+dy)^3$, we get the following system of equations:

$$a^3 + c^3 = 0$$
$$a^2b + c^2d = 1$$
$$ab^2 + cd^2 = 0$$
$$b^3 + d^3 = 0$$

Let $I = (a^3 + c^3, a^2b + c^2d - 1, ab^2 + cd^2, b^3 + d^3) \subseteq \mathbb{C}[a, b, c, d]$ be an ideal. The variety in $\mathbb{C}[a, b, c, d]$ cut out by I is the set of solutions to this system. Define a row vector

$$M = \begin{pmatrix} a^3 + c^3 & a^2b + c^2d - 1 & ab^2 + cd^2 & b^3 + d^3 \end{pmatrix}.$$

Using Macaulay2 [GS14] we find a column vector

$$
N = \begin{pmatrix}
-a^2bc^3d^3 - ab^2c^2d - a^2bcd^2 \\
-a^2bc^4d^2 + a^3c^3d^3 + ab^2c^3 - 2a^2bc^2d + a^3cd^2 - a^2b - c^2d - 1 \\
2a^2bc^5d + a^3c^4d^2 + 2a^2bc^3 + 3a^3c^2d + a^3 + c^3 \\
-2a^3c^5d - 3a^3c^3
\end{pmatrix}
$$

such that the product $MN = (1)$. This implies $1 \in I$. Conclude that there does not exist a choice of a, b, c, d such that $P = (ax + by)^3 + (cx + dy)^3$. However,

$$
P = \lim_{t \to 0} \frac{1}{3t}[(x + ty)^3 - x^3] = \lim_{t \to 0} \frac{1}{3}y^3t^2 + xy^2t + x^2y = x^2y
$$

which shows that $P = x^2y$ is the limit of polynomials with symmetric rank 2 and demonstrates that P has symmetric border rank strictly less than its symmetric rank.

Rank conditions on flattenings of a tensor T provided information about border rank of a tensor T. There is an analogous idea for polynomials using standard flattenings. This idea appears as early as Sylvester's work in [Syl51a, Syl51b]. Let $P \in S^kV$, then $R_s(P) \geq \mathrm{rank}(P_{d,k-d})$. Assume

$$
P = \ell_1^k + \cdots + \ell_r^k.
$$

Flattening both the left hand side and right hand side we see

$$
P_{d,k-d} = [\ell_1^k]_{d,k-d} + \cdots + [\ell_r^k]_{d,k-d}.
$$

From basic linear algebra, if a linear map f may be written $f = g + h$ where g and

h are also linear maps, then $\operatorname{rank}(f) \leq \operatorname{rank}(g) + \operatorname{rank}(h)$. Applying this we get

$$\operatorname{rank}(P_{d,k-d}) \leq \operatorname{rank}([\ell_1^k]_{d,k-d}) + \cdots + \operatorname{rank}([\ell_r^k]_{d,k-d}).$$

We note that $\operatorname{rank}([\ell_i^k]_{d,k-d}) = \operatorname{rank}([x_1^k]_{d,k-d})$. The image of the flattening $[x_1^k]_{d,k-d}$ is in the span of x_1^{k-d} implying $\operatorname{rank}([x_1^k]_{d,k-d}) = 1$. This then shows that

$$\operatorname{rank}(P_{d,k-d}) \leq r.$$

Therefore $\underline{R}_s(P) \geq \operatorname{rank}(P_{d,k-d})$.

We now present a final generalization of flattenings offered by Landbserg and Ottaviani in [LO13]. Landsberg and Ottaviani demonstrated that these flattenings, like the previously defined flattenings, can be used to define equations for secant varieties of the Veronese variety and furthermore prove lower bounds on symmetric border rank of a polynomial.

Definition 2.40. Given a polynomial $P \in S^k V$ and a Schur module $S_\mu V \subset S_\lambda V \otimes S^k V$ a *Young flattening* is the linear map $\mathcal{F}_{\lambda,\mu}(P) \colon S_\lambda V \to S_\mu V$ obtained from the projection of the Pieri product $S_\lambda V \otimes P$ to $S_\mu V$.

Proposition 2.41. *[LO13, Prop. 4.1] Let $[x^k] \in v_k(\mathbb{P}V)$. Assume $\operatorname{rank}(\mathcal{F}_{\lambda,\mu}(x^k)) = t$. If $\underline{R}_s(P) \leq r$, then $\operatorname{rank}(\mathcal{F}_{\lambda,\mu}(P)) \leq rt$. In particular,*

$$\underline{R}_s(P) \geq \frac{\operatorname{rank}(\mathcal{F}_{\lambda,\mu}(P))}{\operatorname{rank}(\mathcal{F}_{\lambda,\mu}(x^k))}.$$

3. A BRIEF SURVEY OF RANK PROBLEMS

This section of the dissertation provides a summary of the progress made on the polynomial Waring problem and we describes the history of some related rank problems. The Waring problem was stated in the late 1700's by Edward Waring and remained open until David Hilbert answered the question in the affirmative in 1909. While existence had been answered, the particular number associated to any power d remained open.

The account presented here of the progress of the polynomial Waring problem and related rank problems shows that the variant of the Waring problem to polynomials follows similar developments. Even though a solution to the problem was provided for generic polynomials by Alexander and Hirschowitz in a series of papers concluding in 1995 with [AH95], many questions about Waring (symmetric) rank remain open. In particular, no efficient methods exist for identifying the rank of any particular polynomial P.

This section of the dissertation is broken up in the following way: Subsection 3.1 focuses on works leading up to the solution of the polynomial Waring problem for generic polynomials. That subsection also discusses articles which followed and simplified the original proofs. In subsection 3.2, applications of (symmetric) rank, (symmetric) border rank and tensor decomposition are discussed. Rank of a polynomial may be higher than the generic rank. Subsection 3.3 highlights questions of maximum rank. Subsection 3.4 covers known results on algorithms computing decompositions of symmetric tensors and uniqueness of such decompositions. When attention is restricted to polynomials with a low number of variables and low degree, much is known about questions of rank and decomposition. Additionally, restricting to particular

families of polynomials such as monomials or determinants is also an approach used to study rank and decomposition. Subsection 3.5 highlights what is known for rank problems for polynomials in low degree or with few variables, monomials, and the determinant.

3.1 The rank of generic polynomials

Among rank questions for polynomials, the case of the generic polynomial has a rich history. The full solution of this problem is given in a series of papers by authors Alexander and Hirschowitz [Ale88, AH92a, AH92b, AH95, Hir85]. The authors provide some simplifications to their earlier work in [AH97].

Remark 3.1. Let V be a vector space of dimension n, the affine dimension of $S^k V$ is $\binom{n+k-1}{k}$, as this is the number of monomials with degree k in n variables. The *expected dimension* of $\sigma_r(v_k(\mathbb{P}V))$ is $\min\{rn-1, \binom{n+k-1}{k}-1\}$ see, *e.g.*, [Lan12, p. 123]. This may be seen by counting the degrees of freedom involved in determining a point on a plane spanned by r points on $v_k(\mathbb{P}V)$. There are $\dim(v_k(\mathbb{P}V)) = n-1$ degrees of freedom in selecting each of the r points on $v_k(\mathbb{P}V)$ spanning the plane. The plane spanned by these points also contributes $r-1$ additional degrees of freedom. This totals to $r(n-1) + r - 1 = rn - 1$ degrees of freedom. The smallest r such that the expected dimension of $\sigma_r(v_k(\mathbb{P}V))$ is at least as large as $\binom{n+k-1}{k} - 1$ is called the *expected generic rank* of polynomials in $S^k V$. Therefore the expected generic rank of a polynomial in $S^k V$ is

$$\left\lceil \frac{\binom{n+k-1}{k}}{n} \right\rceil.$$

Theorem 3.2 (Alexander-Hirschowitz [AH95]). *Let V be a vector space of dimension n, the rank of a generic polynomial $P \in S^k V$ is the expected generic rank* $\left\lceil \frac{\binom{n+k-1}{k}}{n} \right\rceil$

except when $k = 2$ for any n, $k = 3$ with $n = 4$, and $k = 4$ with $n = 3, 4$, or 5.

Remark 3.3. In the exceptional cases where $k = 2$, the generic rank is n. In the remaining exceptional cases, the generic rank is $\left\lceil \frac{\binom{n+k-1}{k}}{n} \right\rceil + 1$.

Chandler streamlines the proof of the Alexander-Hirschowitz theorem in [Cha01, Cha02]. Postinghel provides an alternative proof in [Pos12]. For a more complete account of the history of the Alexander-Hirschowitz theorem and a more succinct proof of the case $k = 3$, one may look in [BO08].

3.2   Applications of (symmetric) tensor decomposition

Symmetric tensor decomposition, and tensor decomposition in general, is finding an increasingly large variety of applications outside of pure mathematics. These applications range from theoretical computer science to applications in statistics and engineering through signal processing. This section gives a brief summary of a few of these applications.

Tensors and the decompositions of tensors have found natural applications in statistics. In [DSS09] the authors discuss many of the ways these topics or closely related topics in algebraic geometry arise in algebraic statistics. One topic discussed in this dissertation was the Segre variety, viewed as the variety of rank one tensors. One idea in algebraic statistics is to look at tensors $P = \sum_{i_1, i_2} p_{i_1, i_2} = 1$ where $0 \leq p_{i_1, i_2}$. If X_1, X_2 are independent random variables, then their joint probabilities correspond to such P on the Segre variety [DSS09, p. 9]. Secant varieties also arise naturally in algebraic statistics as they relate to statistical models called *mixture models*. These mixture models are semi-algebraic sets contained inside the secant variety and are defined by taking convex combinations of points on a subset of a vector space as opposed to the entire span [DSS09, ch. 4]. In [GSS05], the authors study applications of algebraic geometry to Bayesian networks and discuss how secant

varieties of the Segre variety arise in this context.

The study of (symmetric) tensors is an area of increasing interest in computer science. Many applications towards complexity theory have been found involving tensors. For instance, matrix multiplication is a tensor, and the rank of this tensor corresponds to the number of multiplications needed to multiply a matrix. This application of tensors dates back to the work of Strassen [Str69] in 1969 with his proof that an algorithm for 2×2 matrix multiplication exists involving only 7 multiplications, as opposed to the naive algorithm which involved 8. This improvement to 7 multiplications leads to an algorithm for multiplying two $n \times n$ matrices with a complexity of $O(n^{2.8})$ and motivated numerous projects studying the matrix multiplication tensor and its rank. For further reference on applications of tensor decomposition towards matrix multiplication see [BCRL79, Lan06, LO15, LR15, Smi13, Smi15, Win71] and the references within.

It is also interesting to note the application of flattenings of (symmetric) tensors in complexity theory. For instance, [LO15] uses Young flattenings to prove a lower bound on the border rank of the matrix multiplication tensor. Flattenings can also be found in [NW97] where they are used to evaluate lower bounds for arithmetic circuit complexity. In [GKKS14, GKQ13, KS15] we see the application of Young flattenings to obtain various results on arithmetic circuits. In [GKKS14] the authors use shifted partials to gain understanding on the complexity of arithmetic circuits computing the permanent and determinant when the bottom fan-in is bounded, where the number of inputs of a gate is its fan-in. These shifted partials exhibited in [GKKS14] are examples of Young flattenings. Shifted partials are again used in the context of complexity theory in [ELSW15]. The authors of [ELSW15] prove that shifted partials alone cannot separate the padded $m \times m$ permanent in the GL_{n^2}-orbit closure of $\det_n$ for $m \leq 1.5n^2$. See Figure 3.1 for an example of the Young tableaux corre-

sponding to the shifted partial $P_{2,4[3]}$ where the notation $P_{2,4[3]}$ follows the notation presented in [ELSW15] and $P \in S^6V$ with $\dim(V) = 4$. This shifted partial maps
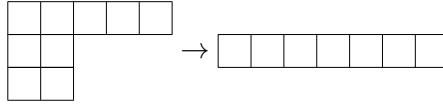
Figure 3.1: Example of the Young tableaux associated for the shifted partial map $P_{2,4[3]} \colon S^2V^* \otimes S^3V \to S^7V$, where $\dim(V) = 4$ and $P \in S^6V$.

$\alpha \otimes Q \mapsto (\alpha \lrcorner P)Q$ where $(\alpha \lrcorner P)Q \in S^7V$. Since $\dim(V) = 4$ the complement of the partition (2) is (2^3) and $S^2V^* \cong S_{(2^3)}V$ as $SL(V)$-modules. Observe that $S_{(5,2^2)}V$ is contained in $S_{(2^3)}V \otimes S^3V$. We note $S_{(9,2^3)}V \subseteq S_{(5,2^2)}V \otimes S^6V$ and as $SL(V)$-modules $S_{(9,2^3)}V \cong S^7V$. Thus we see that $S^7V \subseteq S_{(5,2^2)}V \otimes S^6V$ and therefore we see $P_{2,4[3]}$ is a Young flattening corresponding to the tableaux in Figure 3.1. It should be noted that there are two other Schur modules $S_{(4,2^2,1)}V$ and $S_{(3,2^3)}V$ which are both contained in $S_{(2^3)}V \otimes S^3V$. Furthermore, $S^7V \subseteq S_{(4,2^2,1)}V \otimes S^6V$ and $S^7V \subseteq S_{(3,2^3)}V \otimes S^6V$; however, the Young flattening from $S_{(5,2^2)}V \to S^7V$ corresponds to the shifted partial $P_{2,4[3]}$ as the completion of the first 2 columns of the partition $(5, 2^2)$ with two boxes from (6) corresponds to taking a second derivative of P. The flattenings $S_{(4,2^2,1)}V \to S^7V$ and $S_{(3,2^3)}V \to S^7V$ do not involve the differentiation $(\alpha \lrcorner P)$ where $\alpha \in S^2V^*$.

3.3 Maximum rank of forms

Polynomials may have rank higher than the generic rank. Studying how high this rank may be for a polynomial of degree k in n variables is interesting. Let V be a vector space of dimension n and let $r_{\text{gen}}(S^kV)$ be the rank of generic polynomials

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$P \in S^k V$ provided by the Alexander-Hirschowitz theorem. Let $\mathrm{r}_{\max}(S^k V)$ denote the maximum rank that could be achieved by a $P \in S^k V$. The problem of maximum rank has received recent attention.

Ballico and De Paris present an upper bound for maximum rank of

$$\mathrm{r}_{\max}(S^k V) \leq \binom{n+k-2}{k-1} - \binom{n+k-7}{k-4} - \binom{n+k-6}{k-3}$$

in [BD13]. This is shown by Ballico and De Paris using an induction similar to that presented in [BBS08, Jel14], which both present bounds for maximum rank.

In [BT15], Blekherman and Teitler provide a bound on maximum rank that asymptotically outperforms the previously mentioned bounds. This upper bound does not require the inductive framework developed and improved by Białynicka-Birula and Schinzel, Jelisiejew, and Ballico and De Paris to be proven and only requires the elementary observation by Blekherman and Teitler that any polynomial is the sum of two generic polynomials. This observation demonstrates

$$\mathrm{r}_{\max}(S^k V) \leq 2\,\mathrm{r}_{\mathrm{gen}}(S^k V).$$

3.4   Algorithms and uniqueness of decompositions

With the decomposition of (symmetric) tensors appearing in many applications, there is a growing interest in ways to algorithmically decompose these tensors. Brachat, Comon, Mourrain, and Tsigaridas present, in [BCMT10], an extension of Sylvester's classically known algorithm for decomposing binary forms to n-ary forms. However, in the paper's conclusion, the authors state the complexity of this algorithm has not been evaluated. Oeding and Ottaviani present another symmetric tensor decomposition algorithm in [OO13] which incorporates the use of algebraic

geometry and eigenvectors of tensors. Oeding and Ottaviani state that their algorithm only succeeds for forms of small enough rank; however, bounds on ranks where this algorithm is successful are provided by the authors.

A question that arises once a decomposition has been obtained is whether that decomposition is unique or whether other decompositions exist. In [RS00], Ranestad and Schreyer define a variety called the *variety of sums of powers* for a hypersurface in order to study this question. Later the variety of sums of powers is used by Buczyńska, Buczyński, and Teitler in [BBT13] to identify conditions determining the uniqueness of decompositions for monomials. Mella provides conditions on degree and number of variables for uniqueness of decompositions in [Mel09].

3.5   Ranks of specific forms

The question of determining the (symmetric) rank and (symmetric) border rank of any explicit tensor or polynomial is difficult. As a result, it is common to restrict attention to tensors or polynomials of a particular format. For instance, many researchers have restricted their attention to polynomials of a particular degree or those with a particular number of variables. Restricting attention to a particular family of polynomials such as monomials, determinants, or permanents is another method that has been used to produce results on questions of rank.

Many of the earliest results on decomposing symmetric tensors and determining the rank of symmetric tensors are attributed to the works of Sylvester. His results in [Syl51a, Syl51b, Syl86] which date back to the mid 1800's provided significant contributions to our understanding of polynomials in two variables.

Comas and Seiguer have a more modern approach to the study of symmetric rank and symmetric border rank of binary forms in [CS11]. This paper examines sets of forms with constant rank and shows that the set of degree k binary forms

with border rank r is the union of the set of degree k binary forms with symmetric rank r and the set of degree k binary forms with symmetric rank $k - r + 2$.

Remarkable progress has been made by restricting attention to the case of monomials. In [RS11], Ranestad and Schreyer prove that if $P = (x_1 \cdots x_n)^k$ then $R_s(P) = (k+1)^{n-1}$. Shortly following, a complete result for the symmetric rank of monomials was given by Carlini, Catalisano, and Geramita in [CCG12]. This later result showed

$$R_s(x_1^{k_1} \cdots x_n^{k_n}) = \frac{1}{k_1 + 1} \prod_{i=1}^{n} (k_i + 1)$$

where $k_1 \leq k_2 \leq \ldots \leq k_n$.

Results on symmetric rank of the determinant are of particular interest for this dissertation. Using catalecticants introduced by Sylvester in [Syl51a], classical lower bounds for the symmetric border rank of the $n \times n$ determinant polynomial were

$$\underline{R}_s(\det_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2.$$

There have been numerous improvements on lower bounds for symmetric rank and symmetric border rank of $\det_n$. In addition to other results, Landsberg and Teitler show in [LT10] that

$$R_s(\det_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2 + n^2 - (\lfloor n/2 \rfloor + 1)^2.$$

Further lower bounds on symmetric rank are provided by Shafiei in [Sha15] and by Derksen and Teitler in [DT15] which investigate the notion of *cactus rank*, a variant of rank defined via schemes denoted $\mathrm{krank}(P)$. These two papers prove lower bounds on $\mathrm{krank}(\det_n)$ and consequently provide lower bounds on the symmetric rank of the

determinant as it is known that

$$R_s(P) \geq \text{krank}(P).$$

This is similar to the inequality $R_s(P) \geq \underline{R}_s(P)$. While it is true that both symmetric border rank and cactus rank are, at most, as large as the symmetric rank, examples of polynomials exist having cactus rank larger than symmetric border rank, [BB15], but also examples of polynomials exist with symmetric border rank larger than cactus rank, [BR13]. In [Sha15], it is shown that

$$R_s(\text{det}_n) \geq \text{krank}(\text{det}_n) \geq \frac{1}{2}\binom{2n}{n}.$$

Derksen and Teitler show in [DT15] that

$$R_s(\text{det}_n) \geq \text{krank}(\text{det}_n) \geq \binom{2n}{n} - \binom{2n-2}{n-1}.$$

Derksen provides an upper bound on symmetric rank and therefore also an upper bound on symmetric border rank in [Der13]. In this paper he shows

$$R_s(\text{det}_n) \leq \left(\frac{5}{6}\right)^{\lfloor n/3 \rfloor} 2^{n-1} n!.$$

4. LOWER BOUNDS VIA KOSZUL-YOUNG FLATTENINGS[*]

This section focuses on results on the polynomial Waring problem of the determinant. Let $\det_n$ denote the polynomial obtained from taking the determinant of an $n \times n$ matrix of indeterminate forms. We write this polynomial

$$\det_n = \sum_{\tau \in \mathfrak{S}_n} \operatorname{sgn}(\tau) x^1_{\tau(1)} \cdots x^n_{\tau(n)}.$$

This section is adapted from a published work [Far16] by the author to serve as a section of this dissertation. The results presented here are lower bounds for $\underline{R}_s(\det_n)$ for all $n \geq 3$.

In subsection 4.1 we prove a lower bound for $\underline{R}_s(\det_4)$. The proof follows the same reasoning and techniques as for the case where $n \geq 5$. However, this case is smaller and easier to demonstrate, so it is presented before the larger case.

In subsection 4.2, we present lower bounds for the case when $n \geq 5$. This case is proven using similar techniques to those used in subsection 4.1; however, a few more details need to be addressed.

The case $n = 3$ is handled last as its proof technique differs from the case $n = 4$ and the case $n \geq 5$. This result using computer software provides a new lower bound for $\underline{R}_s(\det_3)$. This technique also calculates a new lower bound for $\underline{R}_s(\operatorname{perm}_3)$. The permanent is a polynomial defined similarly to the determinant making the

permanent interesting to study. This polynomial is defined as

$$\text{perm}_n = \sum_{\tau \in \mathfrak{S}_n} x^1_{\tau(1)} \cdots x^n_{\tau(n)}.$$

A corollary to the lower bound for $\underline{R}_s(\text{perm}_3)$ is $\underline{R}_s(\text{perm}_3)$ can be only one of three possible values.

4.1   The case $n = 4$

To make the method of proof clear, we present a preliminary result proving a lower bound for the case $n = 4$. By Proposition 2.41, to find a high lower bound for $\underline{R}_s(\det_n)$, we need to define a flattening such that $\text{rank}(\mathcal{F}(\det_n))$ is big and $\text{rank}(\mathcal{F}(x^n))$ is small. Given vector spaces A and B both of dimension n, and $\alpha \in S^d(A \otimes B)^*$, we will write $\alpha \lrcorner \det_n$ to denote differentiation of $\det_n$ by α.

Remark 4.1. If α is a minor of the determinant in the dual space $(A \otimes B)^*$, then $\alpha \lrcorner \det_n$ is a nonzero multiple of the minor on the complementary indices in the primal space.

For a tensor $\beta \in S^{n-d}(A \otimes B)$, let $\widehat{\beta} \in (A \otimes B) \otimes S^{n-d-1}(A \otimes B)$ be the image of β under partial polarization. Let $X^i_j = a_i \otimes b_j$ and for $I, J \subset [n]$ with $|I| = |J| = n - d$, let Δ^I_J denote the $(n-d) \times (n-d)$ minor on the indices in I and J.

Remark 4.2. $\widehat{\Delta}^I_J = \sum_{\substack{i \in I \\ j \in J}} (-1)^{i+j} X^i_j \otimes \Delta^{I \smallsetminus \{i\}}_{J \smallsetminus \{j\}}$

Remark 4.3. The 'standard' flattening of the determinant is $\det_{d,n-d} \colon S^d(A \otimes B)^* \longrightarrow S^{n-d}(A \otimes B)$ defined by $\alpha \mapsto \alpha \lrcorner \det_n$. Then $\text{Im}(\det_{d,n-d})$ is spanned by the $(n-d) \times (n-d)$ minors of the determinant.

Define the Young flattening

$$\det^{\wedge 1}_{d,n-d} \colon (A \otimes B) \otimes \textstyle\bigwedge^{n-d} A \otimes \bigwedge^{n-d} B \longrightarrow \bigwedge^2 (A \otimes B) \otimes \bigwedge^{n-d-1} A \otimes \bigwedge^{n-d-1} B$$

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on elements $v \otimes \Delta_J^I$ by

$$v \otimes \Delta_J^I \mapsto \sum_{\substack{i \in I \\ j \in J}} (-1)^{i+j} v \wedge X_j^i \otimes \Delta_{[n-d] \smallsetminus \{j\}}^{[n-d] \smallsetminus \{i\}}$$

and extend linearly.

Lemma 4.4. $\mathrm{Im}(\det_{d,n-d}^{\wedge 1})$ *is contained in*

$$S_{2,1^{n-d-1}}A \otimes S_{1^{n-d+1}}B \oplus S_{1^{n-d+1}}A \otimes S_{2,1^{n-d-1}}B$$

$$\oplus S_{2,1^{n-d-1}}A \otimes S_{2,1^{n-d-1}}B.$$

Proof. The decomposition of $\bigwedge^{n-d} A \otimes \bigwedge^{n-d} B \otimes A \otimes B$ as a $GL_n \times GL_n$-module is

$$S_{2,1^{n-d-1}}A \otimes S_{1^{n-d+1}}B \oplus S_{1^{n-d+1}}A \otimes S_{2,1^{n-d-1}}B$$

$$\oplus S_{2,1^{n-d-1}}A \otimes S_{2,1^{n-d-1}}B \oplus S_{1^{n-d+1}}A \otimes S_{1^{n-d+1}},$$

and $\bigwedge^{n-d-1} A \otimes \bigwedge^{n-d-1} B \otimes \bigwedge^2 (A \otimes B)$ as $GL_n \times GL_n$-module decomposes as

$$S_{1^{n-d+1}}A \otimes S_{3,1^{n-d-2}}B \oplus S_{2,1^{n-d-1}}A \otimes S_{3,1^{n-d-2}}B$$

$$\oplus S_{2,2,1^{n-d-3}}A \otimes S_{3,1^{n-d-2}}B \oplus S_{1^{n-d+1}}A \otimes S_{2,1^{n-d-1}}B$$

$$\oplus (S_{2,1^{n-d-1}}A \otimes S_{2,1^{n-d-1}}B)^{\oplus 2} \oplus S_{2,2,1^{n-d-3}}A \otimes S_{2,1^{n-d-1}}B$$

$$\oplus S_{3,1^{n-d-2}}A \otimes S_{1^{n-d+1}}B \oplus S_{3,1^{n-d-2}}A \otimes S_{2,1^{n-d-1}}B$$

$$\oplus S_{3,1^{n-d-2}}A \otimes S_{2,2,1^{n-d-3}}B \oplus S_{2,1^{n-d-1}}A \otimes S_{1^{n-d+1}}B$$

$$\oplus S_{2,1^{n-d-1}}A \otimes S_{2,2,1^{n-d-3}}B.$$

The irreducible modules in Lemma 4.4 are the only irreducible modules appearing in both decompositions. By Schur's lemma, we conclude that the module in

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Lemma 4.4 must contain $\mathrm{Im}(\det^{\wedge 1}_{d,n-d})$. $\qquad\qquad\square$

It must now be verified for each irreducible module in Lemma 4.4 that $\det^{\wedge 1}_{d,n-d}$ is not the zero map on the module. Since each irreducible module appears with multiplicity 1, for a given irreducible module with highest weight π, finding any highest weight vector $v \in (A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ of weight π such that $\det^{\wedge 1}_{d,n-d}(v) \neq 0$ proves $\det^{\wedge 1}_{d,n-d}$ is nonzero on the entire module.

Lemma 4.5. *The flattening $\det^{\wedge 1}_{d,n-d}$ is an isomorphism on the irreducible module $S_{2,1^{n-d-1}}A \otimes S_{2,1^{n-d-1}}B$.*

Proof. Let $a_1, \ldots, a_n$ be a basis of A and $b_1, \ldots, b_n$ be a basis of B. The irreducible module $S_{2,1^{n-d-1}}A \otimes S_{2,1^{n-d-1}}B$ has a highest weight vector $a_1 \wedge \cdots \wedge a_{n-d} \otimes a_1 \otimes b_1 \wedge \cdots \wedge b_{n-d} \otimes b_1$. The projection of this vector into $(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a nonzero multiple of

$$X_1^1 \otimes \Delta^{[n-d]}_{[n-d]}.$$

Then

$$
\begin{aligned}
&\det^{\wedge 1}_{d,n-d}(X_1^1 \otimes \Delta^{[n-d]}_{[n-d]}) \\
&= \sum_{\substack{i \in [n-d] \\ j \in [n-d]}} (-1)^{i+j} X_1^1 \wedge X_j^i \otimes \Delta^{[n-d] \smallsetminus \{i\}}_{[n-d] \smallsetminus \{j\}}.
\end{aligned}
$$

The term $X_1^1 \wedge X_2^1 \otimes \Delta^{[n-d] \smallsetminus \{1\}}_{[n-d] \smallsetminus \{2\}}$ will not cancel in the sum. $\qquad\square$

Lemma 4.6. *The flattening $\det^{\wedge 1}_{d,n-d}$ is an isomorphism on the irreducible module $S_{2,1^{n-d-1}}A \otimes S_{1^{n-d+1}}B$ and by symmetry on $S_{1^{n-d+1}}A \otimes S_{2,1^{n-d-1}}B$.*

Proof. Let $a_1, \ldots, a_n$ be a basis of A and $b_1, \ldots, b_n$ be a basis of B. The irreducible module $S_{2,1^{n-d-1}}A \otimes S_{1^{n-d+1}}B$ has a highest weight vector $a_1 \wedge \cdots \wedge a_{n-d} \otimes a_1 \otimes$

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$b_1 \wedge \cdots \wedge b_{n-d+1}$. The projection of this vector into $(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a nonzero multiple of

$$\sum_{j \in [n-d+1]} (-1)^j X_j^1 \otimes \Delta_{[n-d+1] \smallsetminus \{j\}}^{[n-d]}.$$

Then

$$\mathrm{det}_{d,n-d}^{\wedge 1}\left(\sum_{j \in [n-d+1]} (-1)^j X_j^1 \otimes \Delta_{[n-d+1] \smallsetminus \{j\}}^{[n-d]}\right)$$

$$= \sum_{\substack{j \in [n-d+1] \\ k \in [n-d+1] \smallsetminus \{j\}}} \sum_{i \in [n-d]} (-1)^j (-1)^{i+\tilde{k}} X_j^1 \wedge X_k^i \otimes \Delta_{[n-d+1] \smallsetminus \{j,k\}}^{[n-d] \smallsetminus \{i\}}$$

where

$$\tilde{k} = \begin{cases} k, & k < j \\ k-1, & j < k. \end{cases}$$

The term $X_1^1 \wedge X_2^1 \otimes \Delta_{[n-d+1] \smallsetminus \{1,2\}}^{[n-d] \smallsetminus \{1\}}$ does not cancel in the sum. □

Taking $d = \lfloor n/2 \rfloor$ gives:

Theorem 4.7. *For $n \geq 3$, the following are lower bounds on the symmetric border rank of the determinant, $\underline{R}_s(\det_n)$.*

For n even:

$$\underline{R}_s(\det_n) \geq \left(1 + \frac{4}{(-1+n)(2+n)^2}\right)\binom{n}{n/2}^2.$$

For n odd:

$$\underline{R}_s(\det_n) \geq \left(1 + \frac{8}{(-1+n)(3+n)^2}\right)\binom{n}{(n-1)/2}^2.$$

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Remark 4.8. Theorem 4.7 is only optimal for the Young flattenings we attempted when $n = 4$.

4.2   The case $n \geq 5$

We prove better lower bounds for $\underline{R}_s(\det_n)$ when $n \geq 5$ with a slightly different Koszul-Young flattening than that used for the proof of Theorem 4.7.

Theorem 4.9. *Let $n \geq 5$, then:*

For n even:

$$\underline{R}_s(\det_n) \geq \left(1 + \frac{8(-8 + 6n^2 + n^3)}{(-1 + n)(2 + n)(4 + n)^2(-2 + n^2)}\right)\binom{n}{n/2}^2.$$

For n odd:

$$\underline{R}_s(\det_n) \geq \left(1 + \frac{16(9 + 8n + n^2)}{(3 + n)(5 + n)^2(-2 + n^2)}\right)\binom{n}{(n-1)/2}^2.$$

Remark 4.10. Asymptotically, our bound is

$$\underline{R}_s(\det_n) \gtrsim \frac{2^{2n+1}}{\pi \cdot n} + \frac{2^{2n+1}}{\pi \cdot n^4}$$

whereas the previous lower bounds are approximately $\underline{R}_s(\det_n) \gtrsim \frac{2^{2n+1}}{\pi \cdot n}$.

To prove Theorem 4.9, we use the map

$$\det_{d,n-d}^{\wedge 2} \colon \wedge^2(A \otimes B) \otimes \wedge^{n-d} A \otimes \wedge^{n-d} B \longrightarrow \wedge^3(A \otimes B) \otimes \wedge^{n-d-1} A \otimes \wedge^{n-d-1} B$$

defined by

$$v \wedge w \otimes \Delta_J^I \mapsto \sum_{\substack{i \in I \\ j \in J}} (-1)^{i+j} v \wedge w \wedge X_j^i \otimes \Delta_{[n-d] \smallsetminus \{j\}}^{[n-d] \smallsetminus \{i\}}$$

and extended linearly. It remains to find the rank of $\det^{\wedge 2}_{d,n-d}$.

Lemma 4.11. $\mathrm{Im}(\det^{\wedge 2}_{d,n-d})$ *is contained in*

$$S_{3,1^{n-d-1}}A \otimes S_{1^{n-d+2}}B \oplus S_{1^{n-d+2}}A \otimes S_{3,1^{n-d-1}}B \oplus S_{3,1^{n-d-1}}A \otimes S_{2,1^{n-d}}B$$

$$\oplus S_{2,1^{n-d}}A \otimes S_{3,1^{n-d-1}}B \oplus S_{3,1^{n-d-1}}A \otimes S_{2,2,1^{n-d-2}}B$$

$$\oplus S_{2,2,1^{n-d-2}}A \otimes S_{3,1^{n-d-1}}B \oplus S_{2,1^{n-d+1}}A \otimes S_{2,1^{n-d+1}}B$$

$$\oplus S_{2,1^{n-d+1}}A \otimes S_{2,2,1^{n-d-1}}B \oplus S_{2,2,1^{n-d-1}}A \otimes S_{2,1^{n-d+1}}B.$$

Proof. Decomposing $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ and $\bigwedge^3(A \otimes B) \otimes \bigwedge^{n-d-1} A \otimes \bigwedge^{n-d-1} B$ as $GL_n \times GL_n$-modules, one sees that only the irreducibles listed in the lemma appear in both decompositions and that the minimum multiplicity each appears with is 1. By Schur's Lemma, no other irreducible may be in the image. $\square$

The above lemma gives an idea as to the largest lower bound that this particular flattening could achieve. To proceed, for each irreducible module in the lemma we find a highest weight vector and compute $\det^{\wedge 2}_{d,n-d}$ on this vector. Since each module appears with multiplicity 1, finding a single highest weight vector of the correct highest weight on which the flattening is nonzero is sufficient.

Lemma 4.12. *The flattening* $\det^{\wedge 2}_{d,n-d}$ *is an isomorphism on the irreducible module* $S_{3,1^{n-d-1}}A \otimes S_{1^{n-d+2}}B$ *and by symmetry on* $S_{1^{n-d+2}}A \otimes S_{3,1^{n-d-1}}B$.

Proof. Let $a_1, \ldots, a_n$ be a basis of A and $b_1, \ldots, b_n$ be a basis of B. The irreducible module $S_{3,1^{n-d-1}}A \otimes S_{1^{n-d+2}}B$ has a highest weight vector $a_1 \wedge \cdots \wedge a_{n-d} \otimes a_1 \otimes a_1 \otimes b_1 \wedge \cdots \wedge b_{n-d+2}$. The projection of this vector into $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a nonzero multiple of

$$\sum_{1 \leq i < j \leq n-d+2} (-1)^{i+j} X_i^1 \wedge X_j^1 \otimes \Delta^{[n-d]}_{[n-d+2] \smallsetminus \{i,j\}}.$$

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Then

$$\det_{d,n-d}^{\wedge 2}\left(\sum_{1\le i<j\le n-d+2}(-1)^{i+j}X_i^1\wedge X_j^1\otimes\Delta_{[n-d+2]\smallsetminus\{i,j\}}^{[n-d]}\right)$$

$$=\sum_{1\le i<j\le n-d+2}\left(\sum_{h=1}^{n-d}\sum_{k\in[n-d+2]\smallsetminus\{i,j\}}(-1)^{\tilde k+h}(-1)^{i+j}X_i^1\wedge X_j^1\wedge X_k^h\otimes\Delta_{[n-d+2]\smallsetminus\{i,j,k\}}^{[n-d]\smallsetminus\{h\}}\right)$$

where

$$\tilde k=\begin{cases} k, & k<i<j\\[4pt] k-1, & i<k<j\\[4pt] k-2, & i<j<k.\end{cases}$$

The term $X_1^1\wedge X_2^1\wedge X_3^1\otimes\Delta_{[n-d+2]\smallsetminus\{1,2,3\}}^{[n-d]\smallsetminus\{1\}}$ does not cancel in the sum. $\qquad\square$

Lemma 4.13. *The flattening* $\det_{d,n-d}^{\wedge 2}$ *is an isomorphism on the irreducible module* $S_{3,1^{n-d-1}}A\otimes S_{2,1^{n-d}}B$ *and by symmetry on* $S_{2,1^{n-d}}A\otimes S_{3,1^{n-d-1}}B$.

Proof. Let $a_1,\dots,a_n$ be a basis of A and $b_1,\dots,b_n$ be a basis of B. The irreducible module $S_{3,1^{n-d-1}}A\otimes S_{2,1^{n-d}}B$ has a highest weight vector $a_1\wedge\cdots\wedge a_{n-d}\otimes a_1\otimes a_1\otimes b_1\wedge\cdots\wedge b_{n-d+1}\otimes b_1$. The projection of this vector to $\bigwedge^2(A\otimes B)\otimes\bigwedge^{n-d}A\otimes\bigwedge^{n-d}B$ is a nonzero multiple of

$$\sum_{i=2}^{n-d+1}(-1)^iX_1^1\wedge X_i^1\otimes\Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]}.$$

Then

$$\det_{d,n-d}^{\wedge 2}\left(\sum_{i=2}^{n-d+1}(-1)^iX_1^1\wedge X_i^1\otimes\Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]}\right)$$

$$=\sum_{k=1}^{n-d}\sum_{i=2}^{n-d+1}\sum_{j\in[n-d+1]\smallsetminus\{i\}}(-1)^i(-1)^{\tilde j+k}X_1^1\wedge X_i^1\wedge X_j^k\otimes\Delta_{[n-d+1]\smallsetminus\{i,j\}}^{[n-d]\smallsetminus\{k\}}$$

where

$$\tilde{j} = \begin{cases} j, & j < i \\ j - 1, & i < j. \end{cases}$$

Now $X_1^1 \wedge X_3^1 \wedge X_2^1 \otimes \Delta_{[n-d+1]\smallsetminus\{2,3\}}^{[n-d]\smallsetminus\{1\}}$ does not cancel, proving the lemma. $\qquad\square$

Lemma 4.14. *The flattening $\det_{d,n-d}^{\wedge 2}$ is an isomorphism on the irreducible module $S_{3,1^{n-d-1}}A \otimes S_{2,2,1^{n-d-2}}B$ and by symmetry on $S_{2,2,1^{n-d-2}}A \otimes S_{3,1^{n-d-1}}B$.*

Proof. Let $a_1, \ldots, a_n$ be a basis of A and $b_1, \ldots, b_n$ be a basis of B. The irreducible module $S_{3,1^{n-d-1}}A \otimes S_{2,2,1^{n-d-2}}B$ has a highest weight vector $a_1 \wedge \cdots \wedge a_{n-d} \otimes a_1 \otimes a_1 \otimes b_1 \wedge \cdots \wedge b_{n-d} \otimes b_1 \wedge b_2$. The projection of this vector to $\wedge^2(A \otimes B) \otimes \wedge^{n-d} A \otimes \wedge^{n-d} B$ is a nonzero multiple of

$$X_1^1 \wedge X_2^1 \otimes \Delta_{[n-d]}^{[n-d]}.$$

Then

$$\det_{d,n-d}^{\wedge 2}\left(X_1^1 \wedge X_2^1 \otimes \Delta_{[n-d]}^{[n-d]}\right) = \sum_{i,j=1}^{n-d} (-1)^{j+i} X_1^1 \wedge X_2^1 \wedge X_j^i \otimes \Delta_{[n-d]\smallsetminus\{j\}}^{[n-d]\smallsetminus\{i\}}.$$

This is not zero since the term $X_1^1 \wedge X_2^1 \wedge X_3^1 \otimes \Delta_{[n-d]\smallsetminus\{3\}}^{[n-d]\smallsetminus\{1\}}$ appears in the sum only once. $\qquad\square$

Lemma 4.15. *The flattening $\det_{d,n-d}^{\wedge 2}$ is an isomorphism on the irreducible module $S_{2,1^{n-d}}A \otimes S_{2,1^{n-d}}B$.*

Proof. Let $a_1, \ldots, a_n$ be a basis of A and $b_1, \ldots, b_n$ be a basis of B. The irreducible module $S_{2,1^{n-d}}A \otimes S_{2,1^{n-d}}B$ has a highest weight vector $a_1 \wedge \cdots \wedge a_{n-d+1} \otimes a_1 \otimes b_1 \wedge$

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$\cdots \wedge b_{n-d+1} \otimes b_1$. The projection of this vector to $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a nonzero multiple of

$$\sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^1 \wedge X_j^i \otimes \Delta_{[n-d+1]\smallsetminus\{j\}}^{[n-d+1]\smallsetminus\{i\}} + \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^i \wedge X_j^1 \otimes \Delta_{[n-d+1]\smallsetminus\{j\}}^{[n-d+1]\smallsetminus\{i\}}.$$

Then

$$\det_{d,n-d}^{\wedge 2} \left(\sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^1 \wedge X_j^i \otimes \Delta_{[n-d+1]\smallsetminus\{j\}}^{[n-d+1]\smallsetminus\{i\}} \right.$$

$$\left. + \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} (-1)^{i+j} X_1^i \wedge X_j^1 \otimes \Delta_{[n-d+1]\smallsetminus\{j\}}^{[n-d+1]\smallsetminus\{i\}} \right)$$

$$= \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} \sum_{\substack{k \in [n-d+1]\smallsetminus\{i\} \\ l \in [n-d+1]\smallsetminus\{j\}}} (-1)^{i+j} (-1)^{\tilde{k}+\tilde{l}} X_1^1 \wedge X_j^i \wedge X_l^k \otimes \Delta_{[n-d+1]\smallsetminus\{j,l\}}^{[n-d+1]\smallsetminus\{i,k\}}$$

$$+ \sum_{i=1}^{n-d+1} \sum_{j=2}^{n-d+1} \sum_{\substack{k \in [n-d+1]\smallsetminus\{i\} \\ l \in [n-d+1]\smallsetminus\{j\}}} (-1)^{i+j} (-1)^{\tilde{k}+\tilde{l}} X_1^i \wedge X_j^1 \wedge X_l^k \otimes \Delta_{[n-d+1]\smallsetminus\{j,l\}}^{[n-d+1]\smallsetminus\{i,k\}}$$

where

$$\tilde{k} = \begin{cases} k, & k < i \\ k-1, & i < k \end{cases}$$

and

$$\tilde{l} = \begin{cases} l, & l < j \\ l-1, & j < l. \end{cases}$$

Since $X_1^1 \wedge X_2^1 \wedge X_1^2 \otimes \Delta_{[n-d+1]\smallsetminus\{2,1\}}^{[n-d+1]\smallsetminus\{1,2\}}$ does not cancel the lemma is proven. $\qquad\square$

Lemma 4.16. *The flattening* $\det_{d,n-d}^{\wedge 2}$ *is an isomorphism on the irreducible module* $S_{2,2,1^{n-d-2}} A \otimes S_{2,1^{n-d}} B$ *and by symmetry on* $S_{2,1^{n-d}} A \otimes S_{2,2,1^{n-d-2}} B$.

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Proof. Let $a_1, \ldots, a_n$ be a basis of A and $b_1, \ldots, b_n$ be a basis of B. The irreducible module $S_{2,2,1^{n-d-2}}A \otimes S_{2,1^{n-d}}B$ has a highest weight vector $a_1 \wedge \cdots \wedge a_{n-d} \otimes a_1 \wedge a_2 \otimes b_1 \wedge \cdots \wedge b_{n-d+1} \otimes b_1$ as a highest weight vector. The projection of this vector to $\bigwedge^2(A \otimes B) \otimes \bigwedge^{n-d} A \otimes \bigwedge^{n-d} B$ is a nonzero multiple of

$$\sum_{i=1}^{n-d+1}(-1)^i X_1^1 \wedge X_i^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]} + \sum_{i=1}^{n-d+1}(-1)^i X_i^1 \wedge X_1^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]}$$

Then

$$\mathrm{det}_{d,n-d}^{\wedge 2}\left(\sum_{i=1}^{n-d+1}(-1)^i X_1^1 \wedge X_i^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]} + \sum_{i=1}^{n-d+1}(-1)^i X_i^1 \wedge X_1^2 \otimes \Delta_{[n-d+1]\smallsetminus\{i\}}^{[n-d]}\right)$$

$$= \sum_{k=1}^{n-d}\sum_{i=1}^{n-d+1}\sum_{j\in[n-d+1]\smallsetminus\{i\}}(-1)^i(-1)^{\tilde{j}+k}X_1^1 \wedge X_i^2 \wedge X_j^k \otimes \Delta_{[n-d+1]\smallsetminus\{i,j\}}^{[n-d]\smallsetminus\{k\}}$$

$$+ \sum_{k=1}^{n-d}\sum_{i=1}^{n-d+1}\sum_{j\in[n-d+1]\smallsetminus\{i\}}(-1)^i(-1)^{\tilde{j}+k}X_i^1 \wedge X_1^2 \wedge X_j^k \otimes \Delta_{[n-d+1]\smallsetminus\{i,j\}}^{[n-d]\smallsetminus\{k\}}$$

where

$$\tilde{j} = \begin{cases} j, & j < i \\ j-1, & i < j. \end{cases}$$

Observing that $X_1^1 \wedge X_1^2 \wedge X_2^1 \otimes \Delta_{[n-d+1]\smallsetminus\{1,2\}}^{[n-d]\smallsetminus\{1\}}$ does not cancel proves the lemma. $\square$

In summary:

Lemma 4.17. *The image of $\mathrm{det}_{d,n-d}^{\wedge 2}$ consists of all of the irreducible modules in the decomposition in Lemma 4.11.*

Lemma 4.18. $\dim(\mathrm{Im}(\mathrm{det}_{d,n-d}^{\wedge 2}))$ *has a maximum at* $d = \lfloor \frac{n}{2} \rfloor$.

Proof. Factor $\dim(\mathrm{Im}(\mathrm{det}_{d,n-d}^{\wedge 2}))$ into the form $f(n,d)\binom{n}{d}^2$, where $f(n,d)$ is a rational function of n and d.

$$f(n,d) = \frac{(n+2)(n+1)(n-d)(d)(d-1)}{(n-d+2)^2(n-d+1)} + \frac{(n+2)(n+1)^2(n-d)(d)}{(n-d+2)^2}$$
$$+ \frac{(n+2)(n+1)^2(n-d)(n)(n-d-1)}{2(n-d+2)(n-d+1)} + \frac{(n+1)^2(n)(n-d-1)(d)}{(n-d+1)(n-d+2)}$$
$$+ \frac{(n+1)^2(d)^2}{(n-d+2)^2}.$$

Consider

$$f(n,d)\binom{n}{d}^2 - f(n,d+1)\binom{n}{d+1}^2$$

and rewrite it as

$$\left(f(n,d) - f(n,d+1)\frac{(n-d)^2}{(d+1)^2} \right)\binom{n}{d}^2.$$

Notice that $f(n,d) - f(n,d+1)\frac{(n-d)^2}{(d+1)^2} < 0$ for $d = \lfloor\frac{n}{2}\rfloor - 1$ and $f(n,d) - f(n,d+1)\frac{(n-d)^2}{(d+1)^2} > 0$ for $d = \lfloor\frac{n}{2}\rfloor$ and conclude. $\qquad\square$

Remark 4.19. The requirement for $n \geq 5$ in the main theorem, is so that the length of all partitions $S_{1^{n-d+2}}A$, $S_{2,1^{n-d}}A$, $S_{3,1^{n-d-1}}A$, and $S_{2,2,1^{n-d-2}}A$ (respectively B) do not exceed $\dim(A) = \dim(B) = n$. Hence, all of the irreducible modules in the decomposition in Lemma 4.11 occur when $d = \lfloor\frac{n}{2}\rfloor$.

Remark 4.20. $\mathrm{rank}([(X_j^i)^n]^{\wedge 2}_{d,n-d}) = \binom{n^2-1}{2}$: the of contraction $\alpha \in S^d(A \otimes B)*$ with $(X_j^i)^n$ is in the span of $(X_j^i)^{n-d}$ and $(\widehat{X_j^i})^{n-d}$ is in the span of $(X_j^i)^{n-d-1} \otimes X_j^i$. Hence $\mathrm{Im}([(X_j^i)^n]^{\wedge 2}_{d,n-d})) = \mathrm{Span}\{(X_j^i)^{n-d-1} \otimes X_j^i \wedge v \wedge w | v, w \notin \mathrm{Span}\{X_j^i\}\}$.

The main theorem follows by substituting $\lfloor\frac{n}{2}\rfloor$ into $f(n,d)$ from the proof of Lemma 4.18, dividing by $\binom{n^2-1}{2}$ which is the rank from Remark 4.20, and simplifying.

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4.3 The case $n = 3$ and a result on the permanent

Define the partitions $\pi_n = ((n-1)^{n+1}, (n-2)^{n+1}, \ldots, 1^{n+1})$ and $\tilde{\pi}_n = (n, \pi_n)$. For example, $\pi_3 = (2^4, 1^4)$ and $\tilde{\pi}_3 = (3, 2^4, 1^4)$. Note that $\dim(S_{\pi_3}\mathbb{C}^9) = \dim(S_{\tilde{\pi}_3}\mathbb{C}^9) = 1050$. For a polynomial $\phi \in S^3\mathbb{C}^9$, define the Young flattening

$$\mathcal{F}_{\pi_3, \tilde{\pi}_3}(\phi) \colon S_{\pi_3}\mathbb{C}^9 \to S_{\tilde{\pi}_3}\mathbb{C}^9$$

by the labeled Pieri product restricted to shape $\tilde{\pi}_3$

$$T_{\pi_3} \otimes \phi = \sum c_{T_{\pi_3}, \tilde{T}_{\tilde{\pi}_3}} \tilde{T}_{\tilde{\pi}_3}$$

where T_{π_3} and $\tilde{T}_{\tilde{\pi}_3}$ are semistandard fillings of tableaux of shape π_3 and $\tilde{\pi}_3$ respectively and where $c_{T_{\pi_3}, \tilde{T}_{\tilde{\pi}_3}}$ is obtained by adding boxes to π_3 to obtain a tableau of shape $\tilde{\pi}_3$ and for each monomial in ϕ, label the boxes with the variable names in all permutations and straighten. $c_{T_{\pi_3}, \tilde{T}_{\tilde{\pi}_3}}$ is the coefficient of $\tilde{T}_{\tilde{\pi}_3}$.

Consider $(x_{3,3})^3 \in S^3\mathbb{C}^9$. If T_{π_3} has any box labeled $x_{3,3}$, then T_{π_3} is in the kernel of $\mathcal{F}_{\pi_3, \tilde{\pi}_3}((x_{3,3})^3)$. Since this is the only restriction of tableaux,

$$\dim \operatorname{Im}(\mathcal{F}_{\pi_3, \tilde{\pi}_3}((x_{3,3})^3)) = \dim S_{\pi_3}\mathbb{C}^8 = 70.$$

By Proposition 2.41, if $[x^3] \in v_3(\mathbb{P}\mathbb{C}^9)$ has rank $\mathcal{F}_{\mu,\nu}(x^3) = p$, then for $[\phi] \in \mathbb{P}S^3\mathbb{C}^9$ with rank r, $\operatorname{rank}(\mathcal{F}_{\mu,\nu}(\phi)) \leq rp$. Thus the maximum lower bound on symmetric border rank on polynomial $\phi \in S^3\mathbb{C}^9$ this method may achieve is

$$\underline{R}_s(\phi) \geq 15$$

when $\dim \operatorname{Im}(\mathcal{F}_{\pi_3, \tilde{\pi}_3}(\phi)) = 1050$. Applying this flattening to $\det_3$ and perm_3 and

using the Macaulay2 [GS14] package **PieriMaps** developed by Steven Sam [Sam09]
we get

$$\dim \mathrm{Im}(\mathcal{F}_{\pi_3, \tilde{\pi}_3}(\det{}_3)) = 950$$

and

$$\dim \mathrm{Im}(\mathcal{F}_{\pi_3, \tilde{\pi}_3}(\mathrm{perm}_3)) = 934.$$

The ranks of these flattenings prove the following lower bounds.

Theorem 4.21. *The polynomials* $\det_3$ *and* perm_3 *have the following lower bounds on symmetric border rank*

$$\underline{R}_s(\det{}_3) \geq 14$$

and

$$\underline{R}_s(\mathrm{perm}_3) \geq 14.$$

These are improvements on the classical lower bounds for the determinant and the permanent of 9. In addition to the improvement to the lower bound on symmetric border rank of the determinant, we prove an interesting corollary.

Definition 4.22. Let $P \in S^d V$. We define the Chow rank of P, $\mathrm{rank}_{Chow}(P)$, as

$$\mathrm{rank}_{Chow}(P) = \min\{k : P = \sum_{i=1}^{k} \ell_{i1} \ldots \ell_{id} \mid \ell_{ij} \in V\}.$$

In [IT15] it is shown that $\mathrm{rank}_{Chow}(\mathrm{perm}_3) = 4$. Prior to this it was known

that $\text{rank}_{Chow}(\text{perm}_3) \leq 4$ [Gly10, Rys63]. Given $\text{rank}_{Chow}(\text{perm}_3) = 4$, results from [CCG12] and [BBT13] proving $\underline{R}_s(x_1 \cdots x_d) \leq 2^{d-1}$ show $\underline{R}_s(\text{perm}_3) \leq 16$. This observation is summarized by the following corollary.

Corollary 4.23. $14 \leq \underline{R}_s(\text{perm}_3) \leq 16$.

The Macaulay2 script used to compute the ranks of the flattenings for the $n = 3$ case of the determinant and the permanent is provided in Appendix A.

5. SUMMARY

The Waring problem is a classical problem motivated by questions of Edward Waring in the late 1700's [War82, p. 349]. Since the onset of the question it has been generalized in numerous ways.

Perhaps one of the most interesting generalizations to this centuries old question is to polynomials and their ranks. A significant case of the question was solved by Alexander-Hirschowitz in [AH95] yet the problem remains open for explicit polynomials. The determinant is an example of a polynomial for which this problem remains open. The work presented in this dissertation improves previously known lower bounds for symmetric border rank of this polynomial by using methods of [LO13]. The improvements demonstrated here for lower bounds on symmetric border rank of the determinant and those for the 3×3 permanent motivate further questions.

5.1 Further questions

Question 5.1. *What lower bounds may be proven for the symmetric border rank of the permanent by using the Young flattening method developed by Landsberg and Ottaviani in [LO13]?*

The Young flattening method was used in this dissertation to prove new lower bounds for the symmetric border rank of the $n \times n$ determinant polynomial for all values n. In addition to the results on the determinant, for the permanent we proved $\underline{R}_s(\text{perm}_3) \geq 14$ by computer calculations using Macaulay2 [GS14].

Young flattenings for the permanent differ from those of the determinant due to the flattenings of the permanent being a $\mathfrak{S}_n \times \mathfrak{S}_n$-module homomorphisms rather than $GL_n \times GL_n$-module homomorphisms. The irreducible modules for $\mathfrak{S}_n$ have much

smaller dimension than those of GL_n. As a consequence of the smaller dimensional irreducible modules, irreducible $\mathfrak{S}_n \times \mathfrak{S}_n$ modules appearing in the decomposition of a Young flattening of the permanent likely occur with high multiplicity. Understanding how to deal with these high multiplicities would be necessary to evaluate such Young flattenings.

Question 5.2. *We saw in Corollary 4.23 a result on the border rank of the 3×3 permanent stating:*

$$14 \leq \underline{R}_s(\mathrm{perm}_3) \leq 16.$$

What is the exact border rank of the 3×3 permanent?

This is the smallest nontrivial case of the permanent, yet surprisingly its symmetric border rank is still unknown. Previously, the list of possible border ranks was between 9 and 16, but now it is certain that the only possibilities are 14, 15, or 16. With such a small list of possibilities it is reasonable that one could identify the symmetric border rank of the 3×3 permanent.

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APPENDIX A

MACAULAY2 SCRIPT

This appendix provides the Macaulay2 script originally appearing in [Far16] to prove Theorem 4.21 of this dissertation. The script directly follows.

```
loadPackage"PieriMaps"
A=QQ[x_(0,0)..x_(2,2)]
time MX = pieri({3,2,2,2,2,1,1,1,1},{1,5,9},A);
rank diff(x_(0,0)^3,MX)
f = det genericMatrix(A,x_(0,0), 3,3)
rank diff(f,MX)
g = x_(0,2)*x_(1,1)*x_(2,0)+x_(0,1)*x_(1,2)*x_(2,0)+
    x_(0,2)*x_(1,0)*x_(2,1)+x_(0,0)*x_(1,2)*x_(2,1)+
    x_(0,1)*x_(1,0)*x_(2,2)+x_(0,0)*x_(1,1)*x_(2,2)
rank diff(g,MX)
```