NONLINEAR QUOTIENTS OF BANACH SPACES

A Dissertation

by

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ABSTRACT

We study quotients of Banach spaces in three nonlinear categories: Lipschitz, uniform and coarse. Following a brief review of what has been known for uniform and Lipschitz quotients of classical Banach spaces, we introduce the definition of coarse quotient and show that several results for uniform quotients also hold in the coarse setting. In particular, we prove that any Banach space that is a coarse quotient of $L_p \equiv L_p[0,1]$, $1 < p < \infty$, is isomorphic to a linear quotient of L_p. It is also proven, by applying a geometric notion of Rolewicz called property (β), that ℓ_q is not a coarse quotient of ℓ_p for $1 < p < q < \infty$, and c_0 is not a coarse quotient of any Banach space with property (β). On the other hand, we give a sharp distortion lower bound for embedding the countably branching tree into a Banach space with property (β). It is then shown how this work unifies and extends a series of results in the nonlinear quotient theory of Banach space.

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TABLE OF CONTENTS

1. INTRODUCTION: NONLINEAR GEOMETRY OF BANACH SPACES

In nonlinear geometric Banach space theory, Banach spaces are considered as metric spaces, while the morphisms between them are nonlinear maps rather than bounded linear operators. There are three different types of nonlinear maps that are widely studied, all of which can be defined using the modulus of continuity. Given a map $f : X \to Y$ between two metric spaces X and Y, the *modulus of continuity* of f is defined by

$$\omega_f(t) := \sup\{d_Y(f(x), f(y)) : d_X(x, y) \leq t\}.$$

We say that f is

(i) *Lipschitz* if $\omega_f(t) \leq Lt$ for some $L > 0$. The infimum of all such constants L is called the *Lipschitz constant* of f and denoted by $\mathrm{Lip}(f)$.

(ii) *uniformly continuous* if $\lim_{t\to 0} \omega_f(t) = 0$.

(iii) *coarsely continuous* if $\omega_f(t) < \infty$ for all $t > 0$.

Equivalence relations between metric spaces corresponding to these three notions of nonlinear maps can now be defined.

(i') X is *Lipschitz homeomorphic* to Y if there exists a one-to-one Lipschitz map from X onto Y whose inverse is also Lipschitz.

(ii') X is *uniformly homeomorphic* to Y if there exists a one-to-one uniformly continuous map from X onto Y whose inverse is also uniformly continuous.

(iii') X is *coarsely homeomorphic* to Y if there exist two coarsely continuous maps

$f : X \to Y$ and $g : Y \to X$ such that

$$\sup\{d_X(g \circ f(x), x) : x \in X\} < \infty,$$
$$\sup\{d_Y(f \circ g(y), y) : y \in Y\} < \infty.$$

Uniform homeomorphisms give information only about the small scale structure of metric spaces, while coarse homeomorphisms focus on the large-distance properties of metric spaces. A Lipschitz homeomorphism is the best equivalence between metric spaces because it preserves simultaneously the small and the large scale structure. As an example, let $|\cdot|$ be the usual Euclidean metric on $\mathbb{R}$. We define two different new metrics:

$$d(x, y) := \min\{|x - y|, 1\} \quad \text{and} \quad \rho(x, y) := |x - y| + 1, \quad x, y \in \mathbb{R}, x \neq y.$$

The metric d (resp. ρ) contains no information of distances larger (resp. less) than 1 with respect to the original metric $|\cdot|$ of $\mathbb{R}$. Indeed, $(\mathbb{R}, d)$ is uniformly homeomorphic but not coarsely homeomorphic to $(\mathbb{R}, |\cdot|)$, while $(\mathbb{R}, \rho)$ is coarsely homeomorphic but not uniformly homeomorphic to $(\mathbb{R}, |\cdot|)$.

The theme of nonlinear geometric Banach space theory is the study of nonlinear classification of Banach spaces. The main concern is that whether the linear structure of a Banach space can be determined by its nonlinear structure, namely, given a Banach space X, if Y is another Banach space that is Lipschitz (resp. uniformly, coarsely) homeomorphic to X, then when can we conclude that Y is isomorphic to X? In the Lipschitz and uniform categories, this problem is systematically addressed in the authoritative book of Benyamini and Lindenstrauss [8]. The coarse theory is analogous but not identical to the uniform theory; the only known example was given by Kalton [25], who showed that there are two coarsely homeomorphic Banach spaces that are not uniformly homeomorphic. Another major problem in this area is the stability of Banach space properties under nonlinear homeomorphisms, that is, if a Banach space X has a certain property (P) and X is Lipschitz (resp. uniformly, coarsely) homeomorphic to a Banach space Y, then does Y also possess property (P)? We refer to the survey papers [15, 24] for a summary of recent progress on these topics.

Nonlinear embedding is also a subject of importance. We say a metric space X *Lipschitz* (resp. *uniformly*, *coarsely*) *embeds* into a metric space Y provided X is Lipschitz (resp. uniformly, coarsely) homeomorphic to a subset of Y. Given two Banach spaces it is a natural question to ask whether it is possible to embed one into the other. For most classical Banach spaces the answer is known (see [15] for a collection of tables). On the other hand, in connection with geometric group theory and theoretical computer science, there is great interest in the problem of embedding discrete metric spaces, especially graphs, into Banach spaces. From the perspective of Banach space theory, one would like to characterize a particular Banach space property (P) in terms of non-embeddability of certain type of graphs. This problem, originating from the Ribe program [3, 38] that aims to find equivalent reformulations of Banach space concepts in the metric structure, has been studied for various local properties of Banach spaces. For example, superreflexivity can be characterized by the Lipschitz non-embeddability of binary trees [5, 9] or diamond graphs [23], and Banach space with nontrivial Rademacher type can be characterized by the Lipschitz non-embeddability of Hamming cubes [10], etc. The only known asymptotic property that has a metric characterization is Rolewicz's property (β) [39]. Under the assumption of reflexivity, the existence of an equivalent norm of property (β) can be characterized by the Lipschitz non-embeddability of countably branching trees [6, 13].

The notion dual to embedding is quotient, which is the heart of our work. In linear theory, A surjective bounded linear operator between Banach spaces is also called a linear quotient map, since by the Open Mapping Theorem such an operator must be open and hence a quotient map in the topological sense. The nonlinear analogue of linear quotient map was first studied by Bates, Johnson, Lindenstrauss, Preiss, and Schechtman [4]. In Section 2 we give a brief review of what has been known for uniform and Lipschitz quotients of classical Banach spaces such as L_p and ℓ_p. Rather than prove the theorems in detail, we explain the ideas of the proofs. Several important techniques, including linearization of Lipschitz maps as well as a delicate "fork argument" introduced by Lima and Randrianarivony [33], are discussed.

Section 3 is devoted to nonlinear quotients in the coarse category. We introduce the notion of coarse quotient, which is reasonable in the sense that the theory of coarse quotient is similar to that of uniform quotient. In particular, we apply a standard ultraproduct technique to give an isomorphic characterization of coarse quotients of L_p for $1 < p < \infty$. For ℓ_p spaces, we develop a coarse version of the fork argument to prove that ℓ_q is not a coarse quotient of ℓ_p for $1 < p < q < \infty$. This technique is also applied to show that c_0 is not a coarse quotient of any Banach space with property (β).

In the first part of the last section we digress from the nonlinear quotient theory by considering quantitative embeddings of countably branching trees into Banach spaces with property (β). It was shown implicitly in [6], as mentioned above, that countably branching trees do not Lipschitz embed into Banach spaces with property (β) with uniformly bounded distortion. We go further in this direction and obtain an explicit and optimal lower bound on the (β)-distortion of countably branching trees. More precisely, we show that it requires distortion at least of order $(\log h)^{1/p}$ for embedding the countably branching tree of height h into a Banach space admitting an equivalent norm satisfying property (β) with modulus of power type $p \in (1, \infty)$. Such a sharp estimate allows us to unify and extend a series of results about the stability under nonlinear quotients of the asymptotic structure of infinite-dimensional Banach spaces. This is presented in the latter part of the section.

Now we introduce the notations used throughout this dissertation. Given a metric space X, we denote by $B_X(x, r)$ the closed ball centered at $x \in X$ with radius $r > 0$. For a Banach space Y, B_Y and S_Y stand for the closed unit ball and unit sphere of Y, respectively. For a set S, the cardinality of S is denoted by $\mathrm{card}(S)$. The density character of a topological space T, denoted by $\mathrm{dens}(T)$, is the smallest cardinal $\mathfrak{m}$ such that T has a dense subset of cardinality $\mathfrak{m}$. Other omitted definitions and notational conventions from Banach space theory can be found in [19].

2. UNIFORM AND LIPSCHITZ QUOTIENTS OF BANACH SPACES: A REVISIT

Co-Lipschitz maps were considered first by Gromov [16] in the context of geometric group theory. Later, the terms "Lipschitz quotient" and "uniform quotient" were introduced and studied in the framework of Banach spaces by Bates, Johnson, Lindenstrauss, Preiss and Schechtman [4].

Definition 2.1 ([4])**.** A map $f : X \to Y$ between two metric spaces X and Y is called *co-uniformly continuous* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x \in X$,

$$f(B_X(x, \varepsilon)) \supseteq B_Y(f(x), \delta).$$

If δ can be chosen as ε/C for some constant $C > 0$ independent of ε, then f is said to be *co-Lipschitz*, and the infimum of all such constants C, denoted by $\mathrm{coLip}(f)$, is called the *co-Lipschitz constant* of f.

We say f is a *uniform* (resp. *Lipschitz*) *quotient map* if it is both uniformly continuous and co-uniformly continuous (resp. Lipschitz and co-Lipschitz). If in addition f is surjective, then Y is called a *uniform* (resp. *Lipschitz*) *quotient* of X.

Remark 2.2. Co-Lipschitz maps are automatically surjective. For a co-uniformly continuous map $f : X \to Y$, one can easily see that $f(X)$ is a clopen subset of Y. Consequently, f is surjective if the space Y is connected.

It is a well-known fact in linear theory that linear quotient maps are dual to isomorphic embeddings, namely, if T is an isomorphic embedding from a Banach space X into a Banach space Y, then $T^* : Y^* \to X^*$ is a linear quotient map; if $S : X \to Y$ is a linear quotient map, then S^* is an isomorphic embedding from

Y^* into X^*. However, in the nonlinear setting neither of the arguments holds. For instance, ℓ_1 can be Lipschitz embedded into c_0 (see [1]), but $\ell_\infty = \ell_1^*$ is not even a continuous image of $\ell_1 = c_0^*$. On the other hand, Johnson, Lindenstrauss, Preiss, and Schechtman [20] proved that if X is a separable Banach space that contains a subspace isomorphic to ℓ_1, then any separable Banach space is a Lipschitz quotient of X. In particular, ℓ_1 is a Lipschitz quotient of $C[0,1]$, but $\ell_\infty = \ell_1^*$ does not uniformly embed into the dual of $C[0,1]$. Nevertheless, it was shown in [4] that a local version of the dual argument holds under the assumption of superreflexivity.

Theorem 2.3 ([4]). *Let X and Y be two Banach spaces. Assume that X is superreflexive and Y is a Lipschitz quotient of X. Then Y^* is crudely finitely representable in X^*.*

Recall that a Banach space X is said to be crudely finitely representable in a Banach space Y if there exists $1 < \lambda < \infty$ so that for any finite-dimensional subspace $E \subseteq X$, there exists a finite-dimensional subspace $F \subseteq Y$ such that $d_{BM}(E, F) < \lambda$, where d_{BM} is the Banach-Mazur distance defined by

$$d_{BM}(E, F) := \inf\{\|T\|\|T^{-1}\| : T : E \to F \text{ is a surjective isomorphism}\}.$$

We say X is finitely representable in Y if X is crudely finitely representable in Y with constant λ for every $\lambda > 1$. A standard ultraproduct technique implies that Theorem 2.3 is also true if "Lipschitz quotient" is replaced by "uniform quotient".

One of the classical ways to linearize Lipschitz maps between Banach spaces is taking derivatives. Unfortunately the Gâteaux derivative, which is crucial in the study of Lipschitz embeddings, is insufficient for Lipschitz quotient maps. Indeed, for $1 \le p < \infty$, there exists a Lipschitz quotient map from ℓ_p onto itself whose Gâteaux derivative at zero is identically zero [4]. Thus rather than differentiation, the proof of Theorem 2.3 uses continuous affine functions to approximate Lipschitz maps. A technical notion called *uniform approximation by affine property* (UAAP) was introduced for this purpose.

Definition 2.4 ([4])**.** A pair of Banach spaces (X, Y) is said to have UAAP if for every $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ satisfying the following property: if B is a ball of radius $r > 0$ in X and $f : B \to Y$ is a Lipschitz map, then there exist a ball $B_1 \subseteq B$ of radius $r_1 > cr$ and a continuous affine function $g : X \to Y$ such that

$$\sup_{x \in B_1} \|f(x) - g(x)\| \leq \varepsilon r_1 \mathrm{Lip}(f).$$

Roughly speaking, UAAP guarantees that Lipschitz maps can be approximated locally by affine functions in a uniform way. It was then proven in the same paper that a pair of nonzero Banach spaces (X, Y) has UAAP if and only if one of them is superreflexive and the other is finite dimensional. This is the key ingredient in the proof of Theorem 2.3, and it also explains why the result is only relevant to finite-dimensional subspaces of the duals. Note that the assumption of superreflexivity is necessary since again $X = C[0, 1]$ and $Y = \ell_1$ gives a counterexample. Consequences of the theorem include an isomorphic characterization of uniform quotients of L_p $(1 < p < \infty)$ and Hilbert spaces: a uniform quotient of L_p is isomorphic to a linear quotient of L_p, and a uniform quotient of a Hilbert space is isomorphic to a Hilbert space.

Lipschitz quotients of ℓ_p spaces are more elusive than those of L_p. In [21], Johnson, Lindenstrauss, Preiss and Schechtman studied Lipschitz quotients of ℓ_p for $p > 2$; their approach is essentailly a differentiation argument. For Lipschitz quotient maps, although Gâteaux derivatives do not provide useful information, Fréchet derivatives work as well as affine functions. However, the existence of Fréchet derivatives is usually too much to ask for. It turns out that a weaker notion than Fréchet derivative, which is called ε-Fréchet derivative, is a suitable alternative in this context. Recall that a map f from a Banach space X to a Banach space Y is said to be *ε-Fréchet differentiable* at $x \in X$ for some $\varepsilon > 0$ if there exists a bounded linear operator $T : X \to Y$ and $\delta = \delta(\varepsilon) > 0$ such that for all $0 < \|u\| < \delta$,

$$\frac{\|f(x+u) - f(x) - T(u)\|}{\|u\|} < \varepsilon.$$

The operator T is called an ε-Fréchet derivative of f at x. Such an operator may not be unique, but it is not hard to check that if a Lipschitz quotient map has a point of ε-Fréchet differentiability for small enough ε, then any such ε-Fréchet derivative is a linear quotient map from X onto Y. Now the question is plain: when does a Lipschitz map $f : X \to Y$ have points of ε-Fréchet differentiability? It is known that there are points at which f is Gâteaux differentiable provided X is separable and Y has the Radon-Nikodým property (RNP) (see, e.g., Theorem 6.42 in [8]). Additional asymptotic structures are needed for the spaces to prove a similar existence theorem for ε-Fréchet derivatives.

Definition 2.5. Let $(X, \|\cdot\|)$ be a Banach space and $t > 0$. We define the *modulus of asymptotically uniform smoothness* of X by

$$\overline{\rho}_X(t) := \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| - 1$$

and the *modulus of asymptotically uniform convexity* of X by

$$\overline{\delta}_X(t) := \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \inf_{y \in S_Y} \|x + ty\| - 1.$$

The norm $\|\cdot\|$ is said to be *asymptotically uniformly smooth* (AUS) if $\overline{\rho}_X(t)/t \to 0$ as $t \to 0$. It is said to be *asymptotically uniformly convex* (AUC) if $\overline{\delta}_X(t) > 0$ for all $t > 0$.

Milman [37] first consider these moduli using different notations. Here we follow the notations in [21]. These notions are asymptotic analogues of uniform convexity (UC) and uniform smoothness (US), and it is immediate that UC implies AUC and US implies AUS. In particular, one can compute that for $1 \leq p < \infty$ and $0 < t \leq 1$,

$$\begin{aligned} \overline{\delta}_{\ell_p}(t) &= \overline{\rho}_{\ell_p}(t) = (1+t^p)^{\frac{1}{p}} - 1, \\ \overline{\delta}_{c_0}(t) &= \overline{\rho}_{c_0}(t) = 0. \end{aligned}$$

Hence ℓ_1 is AUC and c_0 is AUS. The following existence theorem for ε-Fréchet derivatives was proven in [21].

Theorem 2.6 ([21])**.** *Let $f : X \to Y$ be a Lipschitz map from a separable Banach space X to a Banach space Y with RNP. Suppose the norm of Y is AUC and for all $c > 0$,*

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{\overline{\delta}_Y(ct)} = 0.$$

Then for every $\varepsilon > 0$, there exists a point at which f is ε-Fréchet differentiable.

It follows from the theorem that if $1 \leq p < q < \infty$, then every Lipschit map from ℓ_q or c_0 to ℓ_p admits for every $\varepsilon > 0$ a point of ε-Fréchet differentiability. This together with the discussion above and the isomorphic structure of ℓ_p and c_0 show that for $1 \leq p < q < \infty$, ℓ_p is not a Lipschitz quotient of ℓ_q or c_0.

The classification of uniform quotients of ℓ_p is even more complicated since the classical differentiation technique does not work in the uniform setting. In [33], Lima and Randrianarivony had a breakthrough on this problem. They proved that for $1 < p < q < \infty$, ℓ_q is not a uniform quotient of ℓ_p. This result, in combination with the Johnson-Odell dichotomy theorem [22], imply that every uniform quotient of ℓ_p, $1 < p < 2$, is isomorphic to a linear quotient of ℓ_p. The following refinement of their result appears in [12].

Theorem 2.7 ([12]). *Let X be a linear quotient of a subspace of an ℓ_p-sum of finite-dimensional spaces, where $p \in (1, \infty)$. Assume that a Banach space Y is a uniform quotient of a subset of X, where the uniform quotient map is Lipschitz for large distances. Then Y does not contain a subspace isomorphic to ℓ_q for any $q > p$.*

The proof of the theorem relies on a geometric property introduced by Rolewicz [39] that is called property (β). We recall here the equivalent definition of property (β) given by Kutzarova [28].

Definition 2.8 ([28]). A Banach space X has *property (β)* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that for every elment $x \in B_X$ and every sequence $\{x_n\}_{n=1}^{\infty} \subseteq B_X$ with $\operatorname{sep}(\{x_n\}) := \inf\limits_{n \neq m} \|x_n - x_m\| \geq \varepsilon$, there exists an index i such that

$$\frac{\|x - x_i\|}{2} \leq 1 - \delta.$$

The (β)-modulus of X, denoted by $\bar{\beta}_X(\varepsilon)$, is the supremum of all $\delta > 0$ so that the above property is satisfied. $\bar{\beta}_X$ is said to have power type $p \in (1, \infty)$ if there exists $\gamma = \gamma(X) > 0$ such that $\bar{\beta}_X(\varepsilon) \geq \gamma \varepsilon^p$, and in this case we simply say that X has property (β_p).

Clearly property (β) is another asymptotic generalization of uniform convexity, which is weaker than but isomorphically different from UC. For example, the space $(\sum_{n=1}^{\infty} \ell_\infty^n)_{\ell_2}$ is not superreflexive but has property (β). Indeed, for $p \in (1, \infty)$, any

ℓ_p-sum of finite-dimensional Banach spaces, in particular ℓ_p, has property (β_p), and the (β)-modulus was computed explicitly in [2].

The main ingredient of the proof is a delicate and technical argument called the "fork argument". Note that the definition of property (β) is concerned with the geometry of the unit ball of a Banach space, in which the points are in a "fork" configuration: the line segment $[0, x]$ is the handle of the "fork" and the separating points x_n are the tips. The proof actually describes the behavior of a nonlinear lifting of points that are approximately in such a fork configuration. This behavior depends heavily on the asymptotic geometry of the spaces and can rule out the existence of uniform quotient maps. As explained in [33], the idea is to build in the target space a collection of points approximately in a fork configuration whose set of pre-images contains a fork of comparable size, and then use the quantification of property (β) to get a contradiction. This argument can also be applied to show that c_0 is not a uniform quotient of any Banach space with property (β).

The fork argument also plays an important role in the study of the stability of asymptotic structures under nonlinear quotient maps. In [14], Dilworth, Kutzarova and Randrianarivony proved that for separable Banach spaces, the (β)-renorming property is stable under uniform quotient maps. The proof requires, in addition to the fork argument, the construction of a variant of the Laakso graph (see, e.g. [29, 30]) that is called the parasol graph in [7]. Recently the same result was proven without the separability assumption in their joint paper with Lancien [13].

3. COARSE QUOTIENTS OF BANACH SPACES*

In this section we discuss nonlinear quotient maps in the coarse category following the presentation of [40].

3.1 Definition of coarse quotient

We start by giving the definition of coarse quotient map. For $K \geq 0$ and a subset A of a metric space X the notation A^K means the K-neighborhood of A, i.e.

$$A^K := \{x \in X : d_X(x, a) \leq K \text{ for some } a \in A\}.$$

Definition 3.1. Let $K \geq 0$ be a constant. A map $f : X \to Y$ between two metric spaces X and Y is called co-coarsely continuous with constant K if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that for every $x \in X$,

$$B_Y(f(x), \varepsilon) \subseteq f(B_X(x, \delta))^K. \tag{3.1}$$

f is said to be a coarse quotient map (with constant K) if f is both co-coarsely continuous (with constant K) and coarsely continuous; in this case we say Y is a coarse quotient of X.

The definition is justified by the following proposition.

Proposition 3.2. *If a map $f : X \to Y$ between two metric spaces X and Y is a coarse homeomorphism, then Y is a coarse quotient of X.*

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Proof. By the definition of coarse homeomorphism, there exist $M_1, M_2 \geq 0$ and a coarsely continuous map $g : Y \to X$ such that

$$\sup\{d_X(g \circ f(x), x) : x \in X\} \leq M_1,$$
$$\sup\{d_Y(f \circ g(y), y) : y \in Y\} \leq M_2.$$

To see that f is a coarse quotient map, we claim that in Definition 3.1 the constant K can be chosen as M_2, and for every $\varepsilon > 0$, $\delta = \delta(\varepsilon)$ can be chosen as $\omega_g(\varepsilon) + M_1$. Indeed, for every $x \in X$ and every $y \in B_Y(f(x), \varepsilon)$, one has

$$d_X(g \circ f(x), g(y)) \leq \omega_g(d_Y(f(x), y)) \leq \omega_g(\varepsilon).$$

Note that $d_X(g \circ f(x), x) \leq M_1$, so by the triangle inequality we get

$$d_X(g(y), x) \leq \omega_g(\varepsilon) + M_1 = \delta.$$

Hence the point $z := g(y)$ satisfies $z \in B_X(x, \delta)$ and

$$d_Y(y, f(z)) = d_Y(y, f \circ g(y)) \leq M_2 = K,$$

so the claim follows. □

A Lipschitz quotient map is also a coarse quotient map (with constant 0), but the converse is not true. For instance, the inclusion map $i : \mathbb{Z} \to \mathbb{R}$ is a coarse homeomorphism and hence a coarse quotient map, but it is not even a uniform quotient map since $i(B_{\mathbb{Z}}(0, \varepsilon)) = \{0\}$ for any $0 < \varepsilon < 1$.

Proposition 3.3. *Let X, Y and Z be metric spaces. If $f : X \to Y$ is a coarse* *quotient map with constant K_1 and $g : Y \to Z$ is a coarse quotient map with constant K_2, then $g \circ f : X \to Z$ is a coarse quotient map with constant $\omega_g(K_1) + K_2$.*

Proof. The coarse continuity of $g \circ f$ follows from the inequality that for all $t > 0$,

$$\omega_{g \circ f}(t) \leq \omega_g \circ \omega_f(t).$$

Also for $\varepsilon > 0$ and $x \in X$, since g is a co-coarsely continuous with constant K_2, there exists $\delta > 0$ depending on ε such that

$$B_Z(g \circ f(x), \varepsilon) \subseteq g(B_Y(f(x), \delta))^{K_2}.$$

Since f is co-coarsely continuous with constant K_1, there exists $\tilde{\delta} > 0$ depending on δ and hence on ε such that

$$B_Y(f(x), \delta) \subseteq f(B_X(x, \tilde{\delta}))^{K_1}.$$

Therefore,

$$B_Z(g \circ f(x), \varepsilon) \subseteq g\left(f(B_X(x, \tilde{\delta}))^{K_1}\right)^{K_2} \subseteq \left(g \circ f(B_X(x, \tilde{\delta}))\right)^{\omega_g(K_1)+K_2}.$$

This shows that $g \circ f$ is co-coarsely continuous with constant $\omega_g(K_1) + K_2$ and hence the proof is complete. $\square$

In general, coarse quotient maps are not necessarily surjective: (3.1) only implies that $f(X)$ is K-dense in Y, i.e. $Y = f(X)^K$. However in the Banach space setting, Johnson showed that one can always redefine a coarse quotient map to have constant $K = 0$. The proof of this surprising result relies on transfinite induction and the observation below.

Proposition 3.4. *If a Banach space Y is a coarse quotient of a Banach space X, then card$(X) \geq$ card(Y).*

Proof. In view of [18], it suffices to show that $\text{dens}(X) \geq \text{dens}(Y)$. Let $f : X \to Y$ be a coarse quotient map with constant K and S be a (infinite) dense subset of X. Then for $y \in Y$, since $f(X)$ is K-dense in Y, there exists $x \in X$ such that $\|y - f(x)\| \leq K$. The density of S in X implies that x is within distance 1 from some point $s \in S$, and hence $\|f(x) - f(s)\| \leq \omega_f(1)$. Thus by the triangle inequality we have

$$\|y - f(s)\| \leq \omega_f(1) + K := D,$$

which means the set $f(S)$ is D-dense in Y. Now by rescaling the set $\widetilde{S} := \bigcup_{n=1}^{\infty} \frac{f(S)}{n}$ is dense in Y, and the cardinality of $\widetilde{S}$ is at most the cardinality of S. The result then follows. $\square$

Let $\delta > 0$. We say that a subset N is a δ-net of a metric space X if it is δ-dense in X and δ-separated, i.e. $d(u, v) \geq \delta$ for all $u, v \in N$. To see that such a δ-net always exists, one can direct by inclusion all the δ-separated subsets of X and apply Zorn's lemma to find a maximal element, which is clearly a δ-net.

Lemma 3.5 (Johnson)**.** *Let X and Y be Banach spaces. Assume that Y is a coarse quotient of X. Then there exists a coarse quotient map with constant 0 from X onto Y.*

Proof. We use standard set theory language in this proof. Let $f : X \to Y$ be a coarse quotient map and N be a 1-net of X. Since N is coarsely equivalent to X, $f|_N : N \to Y$ is still a coarse quotient map, say, with constant $K \geq 0$. Thus for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that for every $x \in N$,

$$B_Y(f(x), \varepsilon) \subseteq f(B_X(x, \delta) \cap N)^K. \tag{3.2}$$

Consider the set

$$\Gamma = \{(x, \varepsilon, y) : x \in N, \varepsilon \in \mathbb{Q}, y \in B_Y(f(x), \varepsilon)\},$$

by Propsition 3.4 one has

$$\kappa := \mathrm{card}(\Gamma) \leq \max\{\mathrm{card}(N), \mathrm{card}(Y)\} \leq \mathrm{card}(X). \tag{3.3}$$

Fix a well-ordering $\preceq$ of Γ of order-type κ (i.e. each element has strictly fewer than κ predecessors); we will define by transfinite induction on $(x_\alpha, \varepsilon_\alpha, y_\alpha) := \alpha \in \Gamma$ new maps $g_\alpha \subseteq X \times Y$ such that $f|_N \subseteq g_\alpha \subseteq g_\beta$ for all $\alpha \preceq \beta$ in Γ. Then $g := \bigcup_{\alpha \in \Gamma} g_\alpha$ is the desired map, whose domain $\mathrm{dom} g := \widetilde{N}$ contains N as a subset.

Suppose g_β has been defined for all $\beta \prec \alpha$. By (3.2) there exist $\delta_\alpha = \delta(\varepsilon_\alpha) > 0$ and $u_\alpha \in B_X(x_\alpha, \delta_\alpha) \cap N$ so that $\|y_\alpha - f(u_\alpha)\| \leq K$. Note that $\delta_\alpha \geq \|x_\alpha - u_\alpha\| \geq 1$, so one has $B_X(u_\alpha, 1/2) \subseteq B_X(x_\alpha, 2\delta_\alpha)$. In view of (3.3), we can pick

$$v_\alpha \in B_X(u_\alpha, 1/2) \backslash \bigcup_{\beta \prec \alpha} \mathrm{dom} g_\beta$$

and define g_α by

$$g_\alpha = \bigcup_{\beta \prec \alpha} g_\beta \cup \{(v_\alpha, y_\alpha)\}.$$

Clearly g is surjective, and it follows from the choice of v_α and the triangle inequality that

$$\|f(v_\alpha) - g(v_\alpha)\| \leq \omega_f(1/2) + K,$$

so g is coarsely continuous. Moreover, g satisfies the local surjectivity condition at points of N, i.e. for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that for every $x \in N$,

$$B_Y(g(x), \varepsilon) \subseteq g(B_X(x, \delta) \cap \widetilde{N}).$$

Finally we need to extend g to all of X. Consider the selection map $p : X \to \widetilde{N}$ defined as $p(x) = x$ for $x \in \widetilde{N}$ and $p(x) = u_x$ for $x \in X \backslash \widetilde{N}$, where u_x is any point in N within distance 1 from x. Then one can easily check that the composition $g \circ p$ is a coarse quotient map with constant 0 from X onto Y. □

The next lemma is concerned with subsets of quotients and will be used repeatedly in the later sections. Roughly speaking, it says that a subset of a coarse quotient is actually a coarse quotient of a subset. Note that this argument is straightforward for uniform quotient, but in the coarse category it requires some effort.

Lemma 3.6. *Let X and Y be metric spaces and $f : X \to Y$ be a coarse quotient map with constant K. Then for any subset S of Y, there exist a subset Z of X and a coarse quotient map $g : Z \to S$ with constant $4K$.*

Proof. Consider the subset $Z := f^{-1}(S^K)$ of X. We first show that the restriction of f to Z, denoted by $\tilde{f}$, is a coarse quotient map with constant $2K$ from Z to S^K. Let $x \in Z$ and $\varepsilon > 0$. Since $f : X \to Y$ is co-coarsely continuous with constant K, there exists $\delta = \delta(\varepsilon) > 0$ so that

$$B_Y(f(x), \varepsilon) \subseteq f(B_X(x, \delta))^K.$$

If $K = 0$, then it follows that

$$B_S(f(x), \varepsilon) \subseteq f(B_Z(x, \delta))$$

and the proof is complete, so we may assume $K > 0$. For every $y \in B_{S^K}(f(x), \varepsilon)$, there exists $u \in B_X(x, \delta)$ such that $d_Y(y, f(u)) \leq K$. On the other hand, $y \in S^K$ implies that $d_Y(y, s) \leq K$ for some $s \in S$, so $s \in B_Y(f(u), 2K)$ by the triangle inequality. Now again apply the definition of co-coarse continuity of f to the point u, there exists a constant $\widetilde{K} > 0$ depending only on K such that

$$s \in B_Y(f(u), 2K) \subseteq f\left(B_X(u, \widetilde{K})\right)^K.$$

Thus there exists $v \in B_X(u, \widetilde{K})$ such that $d_Y(s, f(v)) \leq K$. It follows again by the triangle inequality that

$$v \in B_X(x, \widetilde{K} + \delta) \cap Z \quad \text{and} \quad d_Y(y, f(v)) \leq 2K.$$

Now we have shown for all $x \in Z$ and $\varepsilon > 0$ that

$$B_{S^K}(\tilde{f}(x), \varepsilon) \subseteq \tilde{f}\left(B_Z(x, \widetilde{K} + \delta)\right)^{2K}.$$

This implies that the map $\tilde{f} : Z \to S^K$ is co-coarsely continuous with constant $2K$. Moreover, as a restriction map it inherits the property of coarse continuity, so it is a coarse quotient map with constant $2K$.

Finally we define $p : S^K \to S$ by $p(a) = a$ if $a \in S$ and $p(a) = s_a$ otherwise, where s_a is any point in S within distance K from a. Then p is a coarse homeomorphism and hence a coarse quotient map (with constant 0). Therefore, by Proposition 3.3, the composition $g := p \circ \tilde{f} : Z \to S$ is a coarse quotient map with constant $4K$. $\square$

3.2 Coarse quotients of L_p

In this section we prove a coarse version of Theorem 2.3 and give an isomorphic characterization of coarse quotients of L_p for $1 < p < \infty$. Our approach is based on a standard ultraproduct technique. To this end, we need to figure out the large-distances behavior of coarse quotient maps. Given a map $f : X \to Y$ between two metric spaces X and Y, define for $t \geq 0$ the Lipschitz constant of f when distances are at least d by

$$\mathrm{Lip}_t(f) := \sup \left\{ \frac{d_Y(f(x), f(y))}{d_X(x, y)} : d_X(x, y) \geq t \right\}.$$

Then for any $t \geq 0$ and $s > 0$, one has

$$\omega_f(t) \leq \max\{\omega_f(s), \mathrm{Lip}_s(f) \cdot t\}.$$

Recall that a metric space X is called metrically convex if for every $x_0, x_1 \in X$ and $0 < \lambda < 1$, there is a point $x_\lambda \in X$ such that

$$d(x_0, x_\lambda) = \lambda d(x_0, x_1) \quad \text{and} \quad d(x_1, x_\lambda) = (1 - \lambda) d(x_0, x_1).$$

The following well-known lemma is due to Corson and Klee [11]. It says that a coarsely continuous map defined on a metrically convex space must be Lipschitz for large distances, i.e. $\mathrm{Lip}_t(f) < \infty$ for all $t > 0$. We present the proof here for later convenience.

Lemma 3.7 ([11])**.** *Let X and Y be two metric spaces and $f : X \to Y$ be a coarsely* *continuous map. Assume that X is metrically convex. Then for all $t > 0$,*

$$Lip_t(f) \leq \frac{2\omega_f(t)}{t}.$$

Consequently, for every $\varepsilon > 0$, there exists $L = L(\varepsilon) > 0$ such that for all $x \in X$ and $r \geq \varepsilon$,

$$f(B_X(x,r)) \subseteq B_Y(f(x), Lr).$$

Proof. For $x, y \in X$ with $d_X(x,y) \geq t$, let $n \in \mathbb{N}$ satisfy $(n-1)t \leq d_X(x,y) < nt$. Since X is metrically convex, there exist $\{u_i\}_{i=0}^n$ in X with $u_0 = x$ and $u_n = y$ such that $d_X(u_i, u_{i-1}) < t$ for all i. Thus $d_Y(f(u_i), f(u_{i-1})) \leq \omega_f(t)$ for all i. Note that $n \geq 2$, so by the triangle inequality we have

$$d_Y(f(x), f(y)) \leq \sum_{i=1}^{n} d_Y(f(u_i), f(u_{i-1})) < n \cdot \omega_f(t) \leq \frac{2\omega_f(t)}{t} \cdot d_X(x,y),$$

and hence the result follows.

To see that the inclusion holds, consider $r \geq \varepsilon$ and $y \in B_X(x,r)$. If $d_X(x,y) < \varepsilon$, then

$$d_Y(f(x), f(y)) \leq \omega_f(\varepsilon) \leq \frac{\omega_f(\varepsilon)}{\varepsilon} \cdot r.$$

If $\varepsilon \leq d_X(x,y) \leq r$, then

$$d_Y(f(x), f(y)) \leq \mathrm{Lip}_\varepsilon(f) \cdot d_X(x,y) \leq \frac{2\omega_f(\varepsilon)}{\varepsilon} \cdot r.$$

Therefore $L = L(\varepsilon)$ can be chosen as $2\omega_f(\varepsilon)/\varepsilon$. $\square$

Similarly, if the target space is metrically convex, then up to a constant, co-coarsely continuous maps behave like co-Lipschitz maps.

Lemma 3.8. *Let X and Y be two metric spaces and $f : X \to Y$ be a co-coarsely continuous map with constant K. Assume that Y is metrically convex. Then for every $\varepsilon > 2K$, there exists $C = C(\varepsilon) > 0$ such that for all $x \in X$ and $r \geq \varepsilon$,*

$$B_Y(f(x), r) \subseteq f\left(B_X\left(x, \frac{r}{C}\right)\right)^K.$$

Proof. For $\varepsilon > 2K$ and $r \geq \varepsilon$, let $n \in \mathbb{N}$ satisfy $(n-1)\varepsilon \leq r < n\varepsilon$ and assume that $y \in B_Y(f(x), r)$. Since Y is metrically convex, there exist $\{y_i\}_{i=0}^{2n}$ in Y with $y_0 = f(x)$ and $y_{2n} = y$ such that $d_Y(y_i, y_{i-1}) < \varepsilon/2$ for all i. Put $x_0 = x$. It follows from the co-coarse continuity of f that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$y_1 \in B_Y(f(x_0), \varepsilon) \subseteq f(B_X(x_0, \delta))^K,$$

so $d_Y(y_1, f(x_1)) \leq K$ for some $x_1 \in B_X(x_0, \delta)$, and hence it follows from the triangle inequality that $y_2 \in B_Y(f(x_1), \varepsilon)$. We proceed inductively to get $\{x_i\}_{i=1}^{2n}$ such that $d_X(x_i, x_{i-1}) \leq \delta$ and $d_Y(y_i, f(x_i)) \leq K$ for all i. This implies

$$y \in f(B_X(x, 2n\delta))^K \subset f\left(B_X\left(x, \frac{r}{C}\right)\right)^K,$$

where $C = C(\varepsilon) = \varepsilon/4\delta(\varepsilon)$. □

Remark 3.9. If $K > 0$, then Lemma 3.8 still holds for $\varepsilon = 2K$, i.e. there exists $\widetilde{C} = C(2K) > 0$ so that for all $x \in X$ and $r \geq 2K$,

$$B_Y(f(x), r) \subseteq f\left(B_X\left(x, \frac{r}{\widetilde{C}}\right)\right)^K.$$

Hence for all $x \in X$ and $r > 0$,

$$B_Y(f(x), r) \subseteq f\left(B_X\left(x, \frac{r}{\widetilde{C}}\right)\right)^{2K}.$$

Thus if the target space Y is metrically convex, then one can always assume, up to a larger constant $2K$, that the co-coarsely continuous map f is co-Lipschitz.

When dealing with Banach spaces, Lemma 3.7 and Lemma 3.8 allow us to pass from coarse quotients to Lipschitz quotients by taking ultrapowers. We refer to Appendix for definitions and results of ultraproduct of Banach spaces.

Proposition 3.10. *Let X and Y be Banach spaces and $\mathcal{U}$ be a free ultrafilter on the natural numbers $\mathbb{N}$. If Y is a coarse quotient of X, then $Y_{\mathcal{U}}$ is a Lipschitz quotient of $X_{\mathcal{U}}$.*

Proof. Let f be a coarse quotient map from X to Y with constant K. In view of Lemma 3.5 we may assume that $K = 0$. By Lemma 3.7 and Lemma 3.8, there are constants $L > 0$ and $C > 0$ such that for all $x \in X$ and $r \geq 1$,

$$\begin{aligned} f(B_X(x, r)) &\subseteq B_Y(f(x), Lr), \\ B_Y(f(x), r) &\subseteq f\left(B_X\left(x, \frac{r}{C}\right)\right). \end{aligned}$$

For each $n \in \mathbb{N}$, define $f_n : X \to Y$ by $f_n(x) = f(nx)/n$. Then for all $x \in X$ and $r \geq 1/n$,

$$\begin{aligned} f_n(B_X(x, r)) &\subseteq B_Y(f_n(x), Lr), \\ B_Y(f_n(x), r) &\subseteq f_n\left(B_X\left(x, \frac{r}{C}\right)\right). \end{aligned}$$

Define $T : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ by $T((x_n)_{\mathcal{U}}) = (f_n(x_n))_{\mathcal{U}}$ for $\tilde{x} = (x_n)_{\mathcal{U}} \in X_{\mathcal{U}}$. Then it follows easily that for each $\tilde{x} \in X_{\mathcal{U}}$ and $r > 0$,

$$T(B_{X_{\mathcal{U}}}(\tilde{x}, r)) \subseteq B_{Y_{\mathcal{U}}}(T\tilde{x}, Lr),$$
$$B_{Y_{\mathcal{U}}}(T\tilde{x}, r) \subseteq T\left(B_{X_{\mathcal{U}}}\left(\tilde{x}, \frac{r}{C}\right)\right).$$

Therefore T is a Lipschitz quotient map from $X_{\mathcal{U}}$ onto $Y_{\mathcal{U}}$. □

The next theorem is the coarse version of Theorem 2.3.

Theorem 3.11. *Let X and Y be two Banach spaces. Assume that X is superreflexive and Y is a coarse quotient of X. Then Y^* is crudely finitely representable in X^*.*

Proof. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. By Proposition 3.10, $Y_{\mathcal{U}}$ is a Lipschitz quotient of $X_{\mathcal{U}}$. Note that $X_{\mathcal{U}}$ is superreflexive, so it follows from Theorem 2.3 that $(Y^*)_{\mathcal{U}} = (Y_{\mathcal{U}})^*$ is crudely finitely representable in $(X_{\mathcal{U}})^* = (X^*)_{\mathcal{U}}$. Since Y^* can be viewed as a subspace of $(Y^*)_{\mathcal{U}}$ and $(X^*)_{\mathcal{U}}$ is finitely representable in X^*, the result follows. □

Corollary 3.12. *A Banach space that is a coarse quotient of a superreflexive Banach space is also superreflexive.*

Corollary 3.13. *A Banach space that is a coarse quotient of a Hilbert space is isomorphic to a Hilbert space.*

Proof. Let Y be a Banach space that is a coarse quotient of a Hilbert space. By Theorem 3.11, Y^* is crudely finitely representable in a Hilbert space and hence isomorphic to a Hilbert space. Thus Y is also isomorphic to a Hilbert space. □

Corollary 3.14. *If a Banach space Y is a coarse quotient of L_p, $1 < p < \infty$, then Y is isomorphic to a linear quotient of L_p.*

Proof. Note that Y is superreflexive and separable, so Y^* is separable. Also by Theorem 3.11, Y^* is crudely finitely representable in L_q, where q is the conjugate exponent of p, i.e. $1/p + 1/q = 1$. It follows that Y^* is isomorphic to a subspace of L_q (see [32]), i.e. Y is isomorphic to a linear quotient of L_p. $\square$

3.3 Coarse quotients of ℓ_p

The goal of this section is to give an isomorphic characterization of coarse quotients of ℓ_p for $1 < p < 2$. The proof is based on a coarse version of Lima and Randrianarivony's "fork argument" [33]. First we need a limiting argument. Let f be a coarse quotient map with constant K from a metric space X to a metric space Y. If Y is metrically convex, then by Lemma 3.8, for every $d > 2K$, there exists $C = C(d) > 0$ such that for all $x \in X$ and $r \geq d$,

$$B_Y(f(x), r) \subseteq f\left(B_X\left(x, \frac{r}{C}\right)\right)^K. \tag{3.4}$$

Denote by C_d the supremum of all C that satisfies (3.4). Clearly, C_d is nondecreasing with respect to d. Moreover, we have:

Lemma 3.15. *Let X and Y be two metric spaces and $f : X \to Y$ be a coarse quotient map with constant K. Assume that Y is metrically convex and f is Lipschitz for large distances. Then $\lim\limits_{d\to\infty} C_d < \infty$.*

Proof. It suffices to show that $\{C_d\}_{d>2K}$ is bounded. For $d > 2K$, let $C = C(d) > 0$ satisfy (3.4) for all $x \in X$ and $r \geq d$. Note that

$$f\left(B_X\left(x, \frac{r}{C}\right)\right)^K \subseteq B_Y\left(f(x), \omega_f\left(\frac{r}{C}\right)\right)^K = B_Y\left(f(x), \omega_f\left(\frac{r}{C}\right) + K\right),$$

one has

$$r \leq \omega_f\left(\frac{r}{C}\right) + K \leq \max\left\{\omega_f(1), \mathrm{Lip}_1(f) \cdot \frac{r}{C}\right\} + K.$$

Let $\eta := \max\{\omega_f(1), \mathrm{Lip}_1(f)\}$ and $r > C$. Then

$$r \leq \frac{\eta r}{C} + K,$$

It follows that

$$C \leq \frac{\eta r}{r-K} < 2\eta,$$

and hence $C_d \leq 2\eta$. □

Theorem 3.16. *Let $1 < p < q < \infty$. Assume that X is Banach space that has property (β_p). Then there is no coarse quotient map that is Lipschitz for large distances from any subset of X to ℓ_q. In particular, ℓ_q is not a coarse quotient of ℓ_p.*

Proof. Suppose that there exist a subset S of X and a coarse quotient map $f : S \to \ell_q$ with constant K so that f is Lipschitz for large distances. In view of Lemma 3.15, let C be the limit of C_d for the map f. Fix a small $0 < \varepsilon < 1$ and choose large d_0 so that

$$\frac{d_0}{3} > 2K \quad \text{ and } \quad C - \varepsilon < C_{d_0/3} \leq C_{d_0} \leq C < C + \varepsilon.$$

Since $C_{d_0} < C + \varepsilon$, by the definition of C_{d_0} as a supremum, there exist $z_\varepsilon \in S$ and $R \geq d_0$ such that

$$B_{\ell_q}(f(z_\varepsilon), R) \nsubseteq f\left(B_S\left(z_\varepsilon, \frac{R}{C+\varepsilon}\right)\right)^K,$$

so there exists $y_\varepsilon \in \ell_q$ satisfying $0 < \|y_\varepsilon - f(z_\varepsilon)\| := \gamma \leq R$ and

$$B_{\ell_q}(y_\varepsilon, K) \cap f\left(B_S\left(z_\varepsilon, \frac{R}{C+\varepsilon}\right)\right) = \emptyset. \tag{3.5}$$

Let m and M be two points on the line segment with endpoints y_ε and $f(z_\varepsilon)$ such that

$$\|y_\varepsilon - M\| = \|M - m\| = \|m - f(z_\varepsilon)\| = \frac{\gamma}{3}.$$

Since $C - \varepsilon < C_{d_0/3}$ and $R/3 \geq d_0/3$, by the definition of $C_{d_0/3}$ as a supremum we have

$$B_{\ell_q}\left(f(z_\varepsilon), \frac{R}{3}\right) \subseteq f\left(B_S\left(z_\varepsilon, \frac{R}{3(C-\varepsilon)}\right)\right)^K.$$

Note that $\|m - f(z_\varepsilon)\| = \gamma/3 \leq R/3$, so there exists $x \in S$ satisfying

$$\|x - z_\varepsilon\| \leq \frac{R}{3(C-\varepsilon)} \quad \text{and} \quad \|m - f(x)\| \leq K.$$

Let $(e_n)_{n=1}^\infty$ be the unit vector basis for ℓ_q, and denote by $(M-m)_N$ the truncation of $M - m$ to the first N coordinates, where $N \in \mathbb{N}$ is large enough so that

$$\|(M-m) - (M-m)_N\|_q < \frac{\varepsilon R}{3}.$$

For $n > N$, set

$$y_n := \frac{\varepsilon^{\frac{1}{q}} R}{3} e_n + (1-\varepsilon)^{\frac{1}{q}}(M-m)_N + m. \tag{3.6}$$

Then

$$\begin{aligned}
\|y_n - m\|^q &= \left\| \frac{\varepsilon^{\frac{1}{q}} R}{3} e_n + (1-\varepsilon)^{\frac{1}{q}}(M-m)_N \right\|^q \\
&= \varepsilon \left(\frac{R}{3}\right)^q + (1-\varepsilon)\|(M-m)_N\|^q \\
&\leq \varepsilon \left(\frac{R}{3}\right)^q + (1-\varepsilon)\left(\frac{\gamma}{3}\right)^q \leq \left(\frac{R}{3}\right)^q,
\end{aligned}$$

so $\|y_n - m\| \leq R/3$. Choose d_0 large enough so that $d_0\varepsilon \geq K$; we then have

$$\|y_n - f(x)\| \leq \|y_n - m\| + \|m - f(x)\| \leq \frac{R}{3} + K \leq \left(\frac{1}{3} + \varepsilon\right) R.$$

Since $\left(\frac{1}{3} + \varepsilon\right) R \geq d_0/3$, again by the definition of $C_{d_0/3}$ we have

$$B_{\ell_q}\left(f(x), \left(\frac{1}{3} + \varepsilon\right) R\right) \subseteq f\left(B_S\left(x, \frac{\left(\frac{1}{3} + \varepsilon\right) R}{C - \varepsilon}\right)\right)^K, \tag{3.7}$$

so there exists $z_n \in S$ satisfying

$$\|z_n - x\| \leq \frac{\left(\frac{1}{3} + \varepsilon\right) R}{C - \varepsilon} \quad \text{and} \quad \|y_n - f(z_n)\| \leq K.$$

Now we estimate $\|y_n - y_\varepsilon\|$:

$$\begin{aligned}
&\|y_n - y_\varepsilon\|^q \\
&= \left\| \frac{\varepsilon^{\frac{1}{q}} R}{3} e_n + (1-\varepsilon)^{\frac{1}{q}} (M-m)_N - (y_\varepsilon - m) \right\|^q \\
&= \left\| \frac{\varepsilon^{\frac{1}{q}} R}{3} e_n + (1-\varepsilon)^{\frac{1}{q}} (M-m)_N - 2(M-m) \right\|^q \\
&= \left\| \frac{\varepsilon^{\frac{1}{q}} R}{3} e_n + 2((M-m)_N - (M-m)) + (1-\varepsilon)^{\frac{1}{q}} (M-m)_N - 2(M-m)_N \right\|^q \\
&= \left\| \frac{\varepsilon^{\frac{1}{q}} R}{3} e_n + 2((M-m)_N - (M-m)) \right\|^q + \left(2 - (1-\varepsilon)^{\frac{1}{q}}\right)^q \|(M-m)_N\|^q \\
&\leq \left(\frac{\varepsilon^{\frac{1}{q}} R}{3} + \frac{2\varepsilon R}{3}\right)^q + (2 - (1-\varepsilon))^q \left(\frac{R}{3}\right)^q \\
&\leq 3^q \varepsilon \left(\frac{R}{3}\right)^q + (1+\varepsilon)^q \left(\frac{R}{3}\right)^q \\
&< (1 + 2 \cdot 3^q \varepsilon) \left(\frac{R}{3}\right)^q,
\end{aligned}$$

so

$$\|y_n - y_\varepsilon\| \le (1 + 2\cdot 3^q\varepsilon)^{\frac{1}{q}} \cdot \frac{R}{3} \le \left(1 + \frac{2\cdot 3^q\varepsilon}{q}\right)\frac{R}{3},$$

and hence

$$\begin{aligned}\|y_\varepsilon - f(z_n)\| &\le \|y_\varepsilon - y_n\| + \|y_n - f(z_n)\| \\ &\le \left(1 + \frac{2\cdot 3^q\varepsilon}{q}\right)\frac{R}{3} + K \\ &\le \left(1 + \frac{2\cdot 3^q\varepsilon}{q}\right)\frac{R}{3} + \varepsilon R = \left(\frac{1}{3} + \varepsilon + \frac{2\cdot 3^{q-1}\varepsilon}{q}\right)R := \rho_\varepsilon R.\end{aligned}$$

Since $\rho_\varepsilon R > R/3 \ge d_0/3$, again by the definition of $C_{d_0/3}$ we have

$$B_{\ell_q}(f(z_n), \rho_\varepsilon R) \subseteq f\left(B_S\left(z_n, \frac{\rho_\varepsilon R}{C-\varepsilon}\right)\right)^K, \tag{3.8}$$

so there exists $x_n \in S$ satisfying

$$\|x_n - z_n\| \le \frac{\rho_\varepsilon R}{C-\varepsilon} \quad \text{and} \quad \|y_\varepsilon - f(x_n)\| \le K.$$

In view of (3.5), we have

$$\|x_n - z_\varepsilon\| > \frac{R}{C+\varepsilon}.$$

Also note that $\rho_\varepsilon \downarrow \frac{1}{3}$ as $\varepsilon \downarrow 0$, so if ε is chosen small enough so that

$$\frac{1}{C+\varepsilon} - \frac{\rho_\varepsilon}{C-\varepsilon} > 0,$$

then by the triangle inequality one has

$$\|z_\varepsilon - z_n\| \ge \|z_\varepsilon - x_n\| - \|x_n - z_n\| \ge \frac{R}{C+\varepsilon} - \frac{\rho_\varepsilon R}{C-\varepsilon} > 0.$$

On the other hand, we could choose large d_0 so that

$$(2\varepsilon)^{\frac{1}{q}} \cdot \frac{d_0}{6} > \omega_f(1) + 2K.$$

Then for $k, n > N$ with $k \neq n$,

$$\begin{aligned}\omega_f(1) + 2K < (2\varepsilon)^{\frac{1}{q}} \cdot \frac{R}{3} &= \|y_n - y_k\| \\ &\leq \|y_n - f(z_n)\| + \|f(z_n) - f(z_k)\| + \|y_k - f(z_k)\| \\ &\leq 2K + \omega_f(\|z_n - z_k\|).\end{aligned}$$

Thus $\omega_f(\|z_n - z_k\|) > \omega_f(1)$ and it follows that $\|z_n - z_k\| > 1$ since $\omega_f(\cdot)$ is nondecreasing. Hence the Lipschitz for large distances property gives

$$(2\varepsilon)^{\frac{1}{q}} \cdot \frac{R}{3} = \|y_n - y_k\| \leq \mathrm{Lip}_1(f)\|z_n - z_k\| + 2K \leq \mathrm{Lip}_1(f)\|z_n - z_k\| + (2\varepsilon)^{\frac{1}{q}} \cdot \frac{R}{6},$$

which implies

$$\|z_n - z_k\| \geq (2\varepsilon)^{\frac{1}{q}} \cdot \frac{R}{6\mathrm{Lip}_1(f)}.$$

In summary, for all $n, k > N$ with $n \neq k$ we have

$$\|z_n - z_k\| \geq (2\varepsilon)^{\frac{1}{q}} \cdot \frac{R}{6\mathrm{Lip}_1(f)}, \qquad \|z_\varepsilon - z_n\| \geq \frac{R}{C+\varepsilon} - \frac{\rho_\varepsilon R}{C-\varepsilon},$$

$$\|z_\varepsilon - x\| \leq \frac{R}{3(C-\varepsilon)}, \qquad \|z_n - x\| \leq \frac{\left(\frac{1}{3}+\varepsilon\right)R}{C-\varepsilon}.$$

Assume that ε is small enough; by the definition of $\bar{\beta}_X$ we get

$$\bar{\beta}_X\left(\frac{(2\varepsilon)^{\frac{1}{q}}}{6\mathrm{Lip}_1(f)} \cdot \frac{C-\varepsilon}{\frac{1}{3}+\varepsilon}\right) \leq 1 - \frac{1}{2} \cdot \frac{C-\varepsilon}{\frac{1}{3}+\varepsilon}\left(\frac{1}{C+\varepsilon} - \frac{\rho_\varepsilon}{C-\varepsilon}\right). \tag{3.9}$$

Note that $\bar{\beta}_X(\cdot)$ is nondecreasing and has power type p, so if we started with small ε so that

$$\frac{C-\varepsilon}{1+3\varepsilon} > \frac{C}{2},$$

then

$$\text{left side of (3.9)} \geq \bar{\beta}_X\left(\frac{C}{2^{2-\frac{1}{q}}\mathrm{Lip}_1(f)} \cdot \varepsilon^{\frac{1}{q}}\right) \geq A\varepsilon^{\frac{p}{q}}$$

for some $A > 0$, whereas

$$\begin{aligned}
\text{right side of (3.9)} &\leq 1 - \left(\frac{1}{2} - \frac{\varepsilon}{C}\right) \cdot \frac{3}{1+3\varepsilon} + \frac{1}{2} + \frac{3^{q-1}\varepsilon}{\left(\frac{1}{3}+\varepsilon\right)q} \\
&\leq \frac{3}{2}\left(1 - \frac{1}{1+3\varepsilon}\right) + \frac{3\varepsilon}{C} + \frac{3^q\varepsilon}{q} \\
&\leq \left(\frac{9}{2} + \frac{3}{C} + \frac{3^q}{q}\right)\varepsilon,
\end{aligned}$$

so

$$\left(\frac{9}{2} + \frac{3}{C} + \frac{3^q}{q}\right)\varepsilon^{1-\frac{p}{q}} \geq A.$$

Since $1 < p < q < \infty$, we get a contradiction by letting $\varepsilon \to 0$. $\square$

A consequence of Theorem 3.16 is the following isomorphic characterization of coarse quotients of ℓ_p for $1 < p < 2$.

Corollary 3.17. *If a Banach space Y is a coarse quotient of ℓ_p, $1 < p < 2$, then Y is isomorphic to a linear quotient of ℓ_p.*

Proof. The proof of Corollary 3.14 shows that Y is isomorphic to a linear quotient of L_p. On the other hand, by Theorem 3.16, ℓ_2 is not isomorphic to a linear quotient of Y. Therefore the results follows from the Johnson-Odell dichotomy theorem [22]. $\square$

It follows from Corollary 3.12 that c_0 cannot be a coarse quotient of a superreflexive Banach space, but we know that there are Banach spaces with property (β) that are not superreflexive, e.g. $(\sum_{n=1}^{\infty} \ell_{\infty}^{n})_{\ell_2}$. Nevertheless, one can also prove using the same technique for Theorem 3.16 that c_0 cannot be a coarse quotient of a Banach space with property (β).

Theorem 3.18. *Let X be a Banach space with property (β). Then there is no coarse quotient map from any subset of X to c_0 that is Lipschitz for large distances. In particular, c_0 is not a coarse quotient of any Banach space with property (β).*

Proof. Suppose that there exist a subset S of X and a coarse quotient map $f : S \to c_0$ with constant K so that f is Lipschitz for large distances. Let C be the limit of C_d in Lemma 3.15 for the map f. Now we proceed the proof of Theorem 3.16 until (3.6), the choice of y_n. Instead, for $n > N$, set

$$y_n := \frac{R}{3} e_n + (M - m)_N + m.$$

Then

$$\|y_n - m\| = \left\| \frac{R}{3} e_n + (M - m)_N \right\| = \max\left\{ \frac{R}{3}, \|(M - m)_N\| \right\} = \frac{R}{3},$$

so as in Theorem 3.16 we can choose z_n by (3.7). Moreover,

$$\begin{aligned} \|y_n - y_\varepsilon\| &= \left\| \frac{R}{3} e_n + (M - m)_N - (y_\varepsilon - m) \right\| \\ &= \left\| \frac{R}{3} e_n + (M - m)_N - 2(M - m) \right\| \\ &= \left\| \frac{R}{3} e_n + 2((M - m)_N - (M - m)) - (M - m)_N \right\| \\ &= \max\left\{ \left\| \frac{R}{3} e_n + 2((M - m)_N - (M - m)) \right\|, \|(M - m)_N\| \right\} \\ &\le \max\left\{ \frac{R}{3} + \frac{2\varepsilon R}{3}, \frac{R}{3} \right\} = \frac{(1 + 2\varepsilon)R}{3}. \end{aligned}$$

Thus by the triangle inequality,

$$\|y_\varepsilon - f(z_n)\| \le \|y_\varepsilon - y_n\| + \|y_n - f(z_n)\| \le \frac{(1+2\varepsilon)R}{3} + K \le \frac{(1+5\varepsilon)R}{3} := \tilde{\rho}_\varepsilon R.$$

Note that $\tilde{\rho}_\varepsilon R > R/3 \ge d_0/3$, so similarly x_n can be chosen by (3.8). On the other hand, we could choose large d_0 so that $d_0/6 > \omega_f(1) + 2K$. Then for $k, n > N$ with $k \neq n$,

$$\omega_f(1) + 2K < \frac{R}{3} = \|y_n - y_k\| \le 2K + \omega_f(\|z_n - z_k\|),$$

which again implies that $\|z_n - z_k\| > 1$, and it follows from the Lipschitz for large distances property of f that

$$\frac{R}{3} = \|y_n - y_k\| \le \mathrm{Lip}_1(f)\|z_n - z_k\| + 2K \le \mathrm{Lip}_1(f)\|z_n - z_k\| + \frac{R}{6},$$

and hence

$$\|z_n - z_k\| \ge \frac{R}{6\mathrm{Lip}_1(f)}.$$

In summary, for all $n, k > N$ with $n \neq k$ we have

$$\|z_n - z_k\| \ge \frac{R}{6\mathrm{Lip}_1(f)}, \qquad \|z_\varepsilon - z_n\| \ge \frac{R}{C+\varepsilon} - \frac{\tilde{\rho}_\varepsilon R}{C-\varepsilon},$$

$$\|z_\varepsilon - x\| \le \frac{R}{3(C-\varepsilon)}, \qquad \|z_n - x\| \le \frac{\left(\frac{1}{3}+\varepsilon\right)R}{C-\varepsilon}.$$

If ε is chosen small enough so that $\varepsilon \le C/2$, then

$$\left\| \frac{(z_n - x) - (z_k - x)}{\frac{(\frac{1}{3}+\varepsilon)R}{C-\varepsilon}} \right\| \ge \frac{1}{6\mathrm{Lip}_1(f)} \cdot \frac{C-\varepsilon}{\frac{1}{3}+\varepsilon} \ge \frac{C}{16\mathrm{Lip}_1(f)} > 0.$$

Now by the definition of property (β), there exists $0 < \delta < 1$ independent of ε and an index $i > N$ such that

$$\|z_\varepsilon - z_i\| \leq \frac{(\frac{1}{3} + \varepsilon)R}{C - \varepsilon} \cdot (2 - 2\delta).$$

It follows that

$$\frac{R}{C + \varepsilon} - \frac{\tilde{\rho}_\varepsilon R}{C - \varepsilon} \leq \frac{\left(\frac{1}{3} + \varepsilon\right) R}{C - \varepsilon} \cdot (2 - 2\delta).$$

We get $2 \leq 2 - 2\delta$ by letting $\varepsilon \to 0$, a contradiction. $\square$

4. (β)-DISTORTION OF COUNTABLY BRANCHING TREES*

Let X and Y be two metric spaces. The *distortion* of a Lipschitz embedding $f : X \to Y$ is defined as

$$\mathrm{dist}(f) := \mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1}) = \sup_{x \neq y \in X} \frac{d_Y(f(x), f(y))}{d_X(x,y)} \cdot \sup_{x \neq y \in X} \frac{d_X(x,y)}{d_Y(f(x), f(y))},$$

and the Y-distortion of X is the smallest distortion needed to embed X into Y, i.e.

$$c_Y(X) := \inf\{\mathrm{dist}(f) : f : X \to Y \text{ is a Lipschitz embedding}\}.$$

If there is no Lipschitz embedding from X into Y then we set $c_Y(X) = \infty$.

The main concern in this section is the quantitative embedding theory of the countably branching trees into Banach spaces with property (β). More precisely, let $p \in (1, \infty)$, and define

$$\mathcal{C}_{(\beta_p)} := \{Y : Y \text{ has an equivalent norm with property } (\beta_p)\},$$

$$\mathcal{C}_{(\beta)} := \{Y : Y \text{ has an equivalent norm with property } (\beta)\}.$$

It was shown in [13] that X admits an equivalent norm with property (β) if and only if X admits an equivalent norm with property (β_p) for some $p \in (1, \infty)$. In other words, $\bigcup_{p \in (1,\infty)} \mathcal{C}_{(\beta_p)} = \mathcal{C}_{(\beta)}$. We denote the (β_p)-distortion of a metric space X by

$$c_{(\beta_p)}(X) := \inf\{c_Y(X) : Y \in \mathcal{C}_{(\beta_p)}\},$$

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which is a parameter that measures the best possible embedding of X into a Banach space admitting an equivalent norm with property (β_p). The case when $X = T_h^\omega$, the countably branching tree of height h, will be investigated in the first part of this section. It is proven that if $Y \in \mathcal{C}_{(\beta_p)}$, then $c_Y(T_h^\omega) = \Omega(\log(h)^{1/p})$, i.e. $c_Y(T_h^\omega) \gtrsim \log(h)^{1/p}$, where as usual the symbol $\gtrsim$ is meant to hide a constant independent of h. The proof combines an asymptotic version of the prong bending lemma from [27] (see also [34] for a similar argument) and a self-improvement argument of Johnson and Schechtman [23]. The optimality of the lower bound is also discussed. This quantitative approach provides a way to unify and extend a series of results in Section 3. We will see that in the latter part of this section.

4.1 A sharp distortion lower bound

Recall that a weighted connected simple graph is a connected graph $G = (V, E)$ with no multiple edges or self-loop, equipped with a positive weight function $w : E \to (0, \infty)$, where V and E are respectively the vertex set and edge set of the graph G. A graph is unweighted if every edge has unit weight. G will always be equipped with its canonical metric ρ_G on its set of vertices, where

$$\rho_G(x, y) := \inf \left\{ \sum_{e \in P} w(e) \colon P \text{ is a path connecting } x \text{ to } y \right\},$$

and the diameter of G is defined by

$$\operatorname{diam}(G) := \sup\{\rho_G(x, y) : x, y \in G\}.$$

A weighted tree is an acyclic weighted connected simple graph. In a tree two vertices are connected by a unique path and a leaf is a vertex of degree 1. For technical reasons we shall work with rooted trees. When we root a tree at an arbitrary vertex, the ancestor-descendant relationship between pairs of vertices is then well defined. The height of a vertex x of a rooted tree T, denoted by $h(x)$, is the number of edges separating x from the root. The height of a rooted tree T is then defined by $h(T) := \sup_{x\in T} h(x)$. The last common ancestor of two vertices x and y is denoted by $\mathrm{lca}(x, y)$. With this notation the canonical graph distance on an unweighted rooted tree is given explicitly by

$$\rho_T(x, y) = h(x) + h(y) - 2h(\mathrm{lca}(x, y)) = \rho_T(x, \mathrm{lca}(x, y)) + \rho_T(\mathrm{lca}(x, y), y).$$

For a positive integer h, T_h^ω denotes the (unweighted) countably branching tree of height h. The vertex set of T_h^ω can be identified with the elements in $\bigcup_{i=0}^{h} \mathbb{N}^h$, where by convention $\mathbb{N}^0 = \emptyset$ is the root. The notation $\bar{n} = (n_1, \ldots, n_r)$, for some $r \leq h$, designates a generic vertex of T_h^ω. Then the natural ordering on T_h^ω with respect to the ancestor-descendant relationship is defined by $\bar{n} \preceq \bar{m}$ (namely, $\bar{n}$ is an ancestor of $\bar{m}$) if $\bar{m} = (m_1, \ldots, m_k)$ extends $\bar{n} = (n_1, \ldots, n_r)$, i.e. $r \leq k$ and $n_i = m_i$ for all $1 \leq i \leq r$.

$K_{\omega,1}$ denotes the (unweighted) star graph with countably many branches, i.e. the bipartite graph that has a partition into exactly two classes, one consisting of a singleton called the center, the other consisting of countably many vertices called the leaves. In the sequel b will denote the center. An arbitrary leaf, denoted by r, is chosen, and a labeling $(t_i)_{i=1}^\infty$ of the (countably many) remaining leaves is fixed. With this labeling in mind $K_{\omega,1}$ can be seen as an "umbel" with countably many "pedicels", where r stands for root, b for the branching point on the "stem", and $(t_i)_{i=1}^\infty$ is a labeling of the tips of the "pedicels". As usual $K_{\omega,1}$ is equipped with the shortest path metric. The next lemma says that if the umbel is Lipschitz embedded into a Banach space with property (β), then at least one pedicel has to bend towards the root, and the distance from its tip to the root is shorter than expected. It can be seen as an asymptotic analogue of Lemma 2 in [27].

Lemma 4.1. *Let Y be a Banach space with property (β). Then for every Lipschitz embedding $f : K_{\omega,1} \to Y$ there exists $i_0 \in \mathbb{N}$ such that*

$$\|f(r) - f(t_{i_0})\| \leq 2Lip(f)\left(1 - \bar{\beta}_Y\left(\frac{2}{dist(f)}\right)\right). \tag{4.1}$$

Proof. One may assume after appropriate translation and rescaling that $f(b) = 0$ and $\mathrm{Lip}(f) = 1$. Let $x = f(r)$ and $y_i = f(t_i)$. Clearly $\|x\| \leq 1$, $\|y_i\| \leq 1$ for all $i \in \mathbb{N}$, and for $n \neq m$ we have

$$\|y_n - y_m\| \geq \frac{2}{\mathrm{dist}(f)} > 0.$$

Since the norm of Y satisfies property (β), there exists $i_0 \in \mathbb{N}$ such that

$$\left\|\frac{x - y_{i_0}}{2}\right\| \leq 1 - \bar{\beta}_Y\left(\frac{2}{\mathrm{dist}(f)}\right),$$

and hence the result follows. □

Remark 4.2. Note that the conclusion of Lemma 4.1 can be strengthened. Since (4.1) holds for all but finitely many indices i, there exists an infinite subset $\mathbb{M} \subseteq \mathbb{N}$ such that

$$\max\left\{\sup_{i\in\mathbb{M}}\|f(r) - f(t_i)\|, \sup_{i\neq j\in\mathbb{M}}\|f(t_i) - f(t_j)\|\right\} \leq 2\mathrm{Lip}(f)\left(1 - \bar{\beta}_Y\left(\frac{2}{\mathrm{dist}(f)}\right)\right).$$

The next proposition is a self-improvement argument of Johnson and Schechtman [23]. It shows that if the countably branching tree of a certain height embeds into a

Banach space with property (β), then the countably branching tree of roughly half the height embeds as well, but with a slightly better distortion.

Proposition 4.3. *Let Y be a Banach space with property (β). Let $k \in \mathbb{N}$, and assume that $T^{\omega}_{2^k}$ Lipschitz embeds into Y with distortion D. Then $T^{\omega}_{2^{k-1}}$ Lipschitz embeds into Y with distortion at most $D\left(1 - \bar{\beta}_Y\left(\frac{2}{D}\right)\right)$.*

Proof. Let $f : T^{\omega}_{2^k} \to Y$ be a Lipschitz embedding with distortion D. In order to define an embedding of $T^{\omega}_{2^{k-1}}$ into Y, one selects vertices located at even heights following a simple procedure. The set of all vertices of height at most 2 in the tree $T^{\omega}_{2^k}$ can be seen as being formed by countably many umbels. For every $n_1 \in \mathbb{N}$, consider the umbel whose root is the vertex $\emptyset$ and whose branching point is the vertex $(n_1) \in T^{\omega}_{2^k}$. By Lemma 4.1, there is a vertex located at level 2 which is "close" to the root of the umbel, i.e. there exists $t_{(n_1)} \in \mathbb{N}$ such that

$$\|f(\emptyset) - f((n_1, t_{(n_1)}))\| \leq 2\mathrm{Lip}(f)\left(1 - \bar{\beta}_Y\left(\frac{2}{D}\right)\right).$$

For every vertex $(n_1, t_{(n_1)})$ as above, and for every $n_2 \in \mathbb{N}$, consider the umbel whose root is the vertex $(n_1, t_{(n_1)})$, and whose branching point is the vertex $(n_1, t_{(n_1)}, n_2)$. Again select using Lemma 4.1, a level-4 vertex that is the tip of the bending pedicel, i.e. there exists $t_{(n_1,n_2)} \in \mathbb{N}$ such that

$$\|f((n_1, t_{(n_1)})) - f((n_1, t_{(n_1)}, n_2, t_{(n_1,n_2)}))\| \leq 2\mathrm{Lip}(f)\left(1 - \bar{\beta}_Y\left(\frac{2}{D}\right)\right).$$

Repeat this procedure until vertices located in the set of leaves of the tree are selected. To summarize we have chosen a collection of integers $(t_{\bar{n}})_{\bar{n} \in T^{\omega}_{2^{k-1}}}$ such that for every

$\bar{n} = (n_1, \ldots, n_r) \in T^{\omega}_{2^{k-1}}$ one has

$$\|f((n_1, t_{(n_1)}, \ldots, n_{r-1}, t_{(n_1,\ldots,n_{r-1})})) - f((n_1, t_{(n_1)}, \ldots, n_r, t_{(n_1,\ldots,n_r)}))\| \\ \leq 2\mathrm{Lip}(f)\left(1 - \bar{\beta}_Y\left(\frac{2}{D}\right)\right).$$

Finally define

$$g\colon T^{\omega}_{2^{k-1}} \to Y,\ \ \bar{n} = (n_1, \ldots, n_r) \mapsto \frac{f((n_1, t_{(n_1)}, \ldots, n_r, t_{(n_1,\ldots,n_r)}))}{2}$$

and $g(\emptyset) = \frac{1}{2}f(\emptyset)$. Since for a graph it is sufficient to consider adjacent vertices to estimate the Lipschitz constant, one can easily check that

$$\mathrm{dist}(g) \leq D\left(1 - \bar{\beta}_Y\left(\frac{2}{D}\right)\right).$$

$\square$

Theorem 4.4. *Let Y be a Banach space admitting an equivalent norm with property (β). Then*

$$\sup_{h \geq 1} c_Y(T^{\omega}_h) = \infty.$$

In particular, if Y is a Banach space with property (β_p) with $p \in (1, \infty)$ and constant $\gamma = \gamma(Y) > 0$, then

$$c_Y(T^{\omega}_h) \geq 2\gamma^{\frac{1}{p}} \log(\tfrac{h}{2})^{\frac{1}{p}}.$$

Proof. Let $D_h := c_Y(T^{\omega}_h)$ in the sequel, and assume that $\sup_{h \geq 1} D_h = D \in (0, \infty)$. According to Proposition 4.3, if Y has property (β), then for every $k \geq 1$,

$$D_{2^{k-1}} \leq D_{2^k}\left(1 - \bar{\beta}_Y\left(\frac{2}{D}\right)\right).$$

Taking the limit in k gives a contradiction.

Suppose that Y has (β_p) with $p \in (1, \infty)$ and constant $\gamma = \gamma(Y) > 0$. Let $k \in \mathbb{N}$ satisfy $2^k \leq h < 2^{k+1}$. Again it follows from Proposition 4.3 that for all $j \leq k$,

$$D_{2^{j-1}} \leq D_{2^j}\left(1 - \frac{2^p\gamma}{D_{2^j}^p}\right),$$

and hence

$$D_h \geq D_{2^k} \geq 2^p\gamma\sum_{j=1}^{k} D_{2^j}^{1-p} + D_1 \geq 2^p\gamma k D_h^{1-p}.$$

The conclusion follows easily. □

Bourgain [9] gave a Lipschitz embedding of the complete hyperbolic binary tree of height h into ℓ_2 with distortion $O(\sqrt{\log(h)})$. The next theorem is a slight modification of Bourgain's construction that gives a distortion upper bound of embedding any unweighted tree into ℓ_p space.

Theorem 4.5. *Let $p \in (1, \infty)$ and $T = (V, E)$ be an unweighted tree with $diam(T) > 1$. Then*

$$c_{\ell_p(E)}(T) = O\left(\log(diam(T))^{\frac{1}{p}}\right).$$

Proof. Denote by $\{u_e\}_{e \in E}$ the unit vector basis for $\ell_p(E)$. Let q be the conjugate exponent of p, i.e. $1/p + 1/q = 1$. Root the tree T at an arbitrary vertex r. For $x \in T$, let $P_x = \{e_1^x, e_2^x, ..., e_{h(x)}^x\} \subseteq E$ be the unique path connecting r and x. Now we show that the embedding of T into $\ell_p(E)$ is given by

$$f(x) = \sum_{k=1}^{h(x)}(1 + h(x) - k)^{\frac{1}{q}} u_{e_k^x}.$$

Let $x, y \in T$. Then

$$
\begin{aligned}
\|f(x)-f(y)\|^p &= \left\| \sum_{k=1}^{h(x)} (1+h(x)-k)^{\frac{1}{q}}\, u_{e_k^x} - \sum_{k=1}^{h(y)} (1+h(y)-k)^{\frac{1}{q}}\, u_{e_k^y} \right\|^p \\
&= \sum_{k=1}^{h(\mathrm{lca}(x,y))} \left|(1+h(x)-k)^{\frac{1}{q}} - (1+h(y)-k)^{\frac{1}{q}}\right|^p \qquad (C) \\
&+ \sum_{k=h(\mathrm{lca}(x,y))+1}^{h(x)} (1+h(x)-k)^{p-1} \qquad (A) \\
&+ \sum_{k=h(\mathrm{lca}(x,y))+1}^{h(y)} (1+h(y)-k)^{p-1} \qquad (B)
\end{aligned}
$$

The quantities A and B are easy to estimate and we record it in the following claim.

Claim 4.1. The following inequalities hold:

$$
\begin{aligned}
\left(\frac{h(x)-h(\mathrm{lca}(x,y))}{2}\right)^p &\le A \le (h(x)-h(\mathrm{lca}(x,y)))^p\,, \\
\left(\frac{h(y)-h(\mathrm{lca}(x,y))}{2}\right)^p &\le B \le (h(y)-h(\mathrm{lca}(x,y)))^p\,.
\end{aligned}
$$

Proof of Claim 4.1: First we note that

$$
A \le \sum_{k=h(\mathrm{lca}(x,y))+1}^{h(x)} (h(x)-h(\mathrm{lca}(x,y)))^{p-1} = (h(x)-h(\mathrm{lca}(x,y)))^p.
$$

To get the lower bound on A, let j be the largest integer in $[h(\mathrm{lca}(x,y))+1, h(x)]$ such that

$$
1+h(x)-j \ge \frac{1}{2}\left(h(x)-h(\mathrm{lca}(x,y))\right).
$$

Then one has

$$
\begin{aligned}
A &\geq \sum_{k=h(\mathrm{lca}(x,y))+1}^{j} (1+h(x)-k)^{p-1} \\
&\geq \sum_{k=h(\mathrm{lca}(x,y))+1}^{j} \left(\frac{h(x)-h(\mathrm{lca}(x,y))}{2}\right)^{p-1} \\
&= (j-h(\mathrm{lca}(x,y)))\left(\frac{h(x)-h(\mathrm{lca}(x,y))}{2}\right)^{p-1} \\
&\geq \left(\frac{h(x)-h(\mathrm{lca}(x,y))}{2}\right)^{p},
\end{aligned}
$$

where the last inequality follows from

$$
h(x)-j < \frac{1}{2}\left(h(x)-h(\mathrm{lca}(x,y))\right).
$$

By exchanging the role of x and y we get the same estimates for B.

Now we estimate $\|f(x)-f(y)\|$ from below:

$$
\begin{aligned}
\|f(x)-f(y)\| &\geq (A+B)^{\frac{1}{p}} \\
&\geq \left(\left(\frac{h(x)-h(\mathrm{lca}(x,y))}{2}\right)^{p}+\left(\frac{h(y)-h(\mathrm{lca}(x,y))}{2}\right)^{p}\right)^{\frac{1}{p}} \\
&\geq \frac{1}{2}\left(\frac{h(x)-h(\mathrm{lca}(x,y))}{2}+\frac{h(y)-h(\mathrm{lca}(x,y))}{2}\right) \\
&= \frac{1}{4}\rho_T(x,y).
\end{aligned}
$$

To estimate $\|f(x)-f(y)\|$ from above, we need an upper bound on C.

Claim 4.2. For all $x, y \in T$,

$$
C \leq 2^p \log(\mathrm{diam}(T))\rho_T(x,y)^p.
$$

Proof of Claim 4.2: First consider the case when $\mathrm{lca}(x,y)=x$ and $h(y)>h(x)$. It follows from the inequality $a^s-b^s\le a^{s-1}(a-b)$, which holds for every $s\in[0,1]$ and $a\ge b>0$, that

$$\begin{aligned}
C&=\sum_{k=1}^{h(x)}\left((1+h(y)-k)^{\frac{1}{q}}-(1+h(x)-k)^{\frac{1}{q}}\right)^p\\
&\le\sum_{k=1}^{h(x)}\left(\frac{h(y)-h(x)}{(1+h(y)-k)^{\frac{1}{p}}}\right)^p\\
&=\sum_{k=1}^{h(x)}\frac{\rho_T(x,y)^p}{1+h(y)-k}\\
&\le\rho_T(x,y)^p\sum_{k=2}^{h(y)}\frac{1}{k}\le\log(h(y))\rho_T(x,y)^p\le\log(\mathrm{diam}(T))\rho_T(x,y)^p.
\end{aligned}$$

Now for the general case one has

$$\begin{aligned}
C\le&\sum_{k=1}^{h(\mathrm{lca}(x,y))}2^p\left|(1+h(x)-k)^{\frac{1}{q}}-(1+h(\mathrm{lca}(x,y))-k)^{\frac{1}{q}}\right|^p\\
&+\sum_{k=1}^{h(\mathrm{lca}(x,y))}2^p\left|(1+h(y)-k)^{\frac{1}{q}}-(1+h(\mathrm{lca}(x,y))-k)^{\frac{1}{q}}\right|^p\\
&\le 2^p\log(\mathrm{diam}(T))(\rho_T(x,\mathrm{lca}(x,y))^p+\rho_T(y,\mathrm{lca}(x,y))^p)\\
&\le 2^p\log(\mathrm{diam}(T))\rho_T(x,y)^p.
\end{aligned}$$

This completes the proof of Claim 4.2.

Now combining the upper bound estimate of A,B and C we get

$$\begin{aligned}
\|f(x)-f(y)\|&=(A+B+C)^{\frac{1}{p}}\\
&\le(1+2^p\log(\mathrm{diam}(T)))^{\frac{1}{p}}\rho_T(x,y)\le 4\log(\mathrm{diam}(T))^{\frac{1}{p}}\rho_T(x,y).
\end{aligned}$$

Therefore, the distortion of f satisfies

$$\mathrm{dist}(f) \leq 16 \log(\mathrm{diam}(T))^{\frac{1}{p}},$$

and the proof is complete. □

Note that the upper bound in Theorem 4.5 is not optimal since it involves the diameter of the tree T. In the case when T is an infinite path, then it embeds isometrically into the real line $\mathbb{R}$, but $\mathrm{diam}(T) = \infty$ has no control on the distortion from above. Nevertheless, this upper bound is already sufficient to show that the lower bound obtained in Theorem 4.4 is sharp. In fact, since the edge set of T_h^ω is countable and the diameter of T_h^ω is $2h$, it follows from Theorem 4.5 that

$$c_{\ell_p}(T_h^\omega) = O\left(\log(h)^{\frac{1}{p}}\right).$$

In view of the fact that ℓ_p has property (β_p), the lower bound in Theorem 4.4 is indeed optimal up to constant factors.

Considering weighted trees is significantly more complicated (even for finite trees). Matoušek [34] introduced a combinatorial parameter called "caterpillar dimension" associated to a weighted tree, which is only related to the combinatorial structure of the tree but not the edge weights. He proved for finite weighted trees a similar upper bound as that in Theorem 4.5, with only the diameter of T replaced by the caterpillar dimension of T. It was also mentioned there that a similar result holds for infinite weighted trees. Later, Matoušek's upper bound was improved by Lee, Naor and Peres [31] using a coloring parameter that takes into account the edge weights. We also refer to [7] for a proof following the graph coloring approach in [31] for infinite weighted trees.

4.2 Applications in nonlinear quotient theory

We turn to applications of Theorem 4.4 and Theorem 4.5 in the nonlinear quotient theory of Banach spaces. In this part mainly nonlinear quotient maps defined on some subset of a metric space are considered. A metric space Y is said to be a *Lipschitz* (resp. *uniform*, *coarse*) *subquotient* of a metric space X if Y is a Lipschitz (resp. uniform, coarse) quotient of a subset of X. We will, in particular, emphasize a quantitative analysis of Lipschitz subquotients that is similar to the quantitative theory of Lipschitz embeddings.

Definition 4.6. Let X, Y be two metric spaces. Y is a said to be a Lipschitz subquotient of X with co-distortion $\alpha \in [1, \infty)$ (or simply Y is an α-Lipschitz subquotient of X) if there is a subset $Z \subseteq X$ and a Lipschitz quotient map $f : Z \to Y$ such that the co-distortion of f, defined as

$$\mathrm{codist}(f) := \mathrm{Lip}(f) \cdot \mathrm{coLip}(f),$$

satisfies $\mathrm{codist}(f) \leq \alpha$. We define the X-quotient co-distortion of Y as

$$qc_X(Y) := \inf\{\alpha : Y \text{ is an } \alpha\text{-Lipschitz subquotient of } X\}.$$

We set $qc_X(Y) = \infty$ if Y is not a Lipschitz quotient of any subset of X.

Remark 4.7. Lipschitz subquotients have already been implicitly touched upon (e.g. [14, 33, 36]). A "dual" notion was considered by Mendel and Naor [35], where given $\alpha \in [1, \infty)$ they say that X has an α-Lipschitz quotient in Y if there is a subset $S \subseteq Y$ and a Lipschitz quotient map $f : X \to S$ such that $\mathrm{codist}(f) \leq \alpha$.

Observe that if f is a Lipschitz embedding from X into Y, then f^{-1} is a Lipschitz quotient map from $f(X)$ onto X, with $\mathrm{codist}(f^{-1}) = \mathrm{dist}(f)$. Therefore we have $qc_Y(X) \le c_Y(X)$. A crucial observation for the ensuing discussion is that the inequality is actually an equality for trees.

Proposition 4.8. *Let Y be a metric space and T be a weighted tree. Then $qc_Y(T) = c_Y(T)$.*

Proof. Let Z be a subset of Y and $f : Z \to T$ be a Lipschitz quotient map. Equip T with its canonical graph metric ρ_T and root T at an arbitrary vertex r so that the height of the tree is well defined. By induction on the height of the tree it is possible to select a collection of points $(z_v)_{v\in T} \subseteq Z$ such that $f(z_v) = v$, and for every pair of adjacent vertices (v, w) one has

$$d_Y(z_v, z_w) \le \mathrm{coLip}(f) \cdot \rho_T(v, w).$$

Since for a weighted graph it is sufficient to consider pairs of adjacent vertices to estimate the Lipschitz constant of a map, the injective map $g : v \mapsto z_v$ is Lipschitz with $\mathrm{Lip}(g) \le \mathrm{coLip}(f)$. We conclude by simply observing that $\mathrm{Lip}(g^{-1}) \le \mathrm{Lip}(f)$, and hence $\mathrm{dist}(g) \le \mathrm{codist}(f)$. $\square$

The common feature of the proof of Theorem 2.7 and Theorem 3.16, as well as that of 3.18, is the implementation of the fork argument. We aim to circumvent this technical argument by the quantitative results obtained in the first part of this section and give a purely metric proof. It will be clear shortly that the alternative proof actually splits the proof mechanism of the fork argument into two distinct quantitative problems that can be treated by rather elementary techniques, and to a certain extent fits these theorems into the same framework. We start with uniform quotients. The following proposition serves as a "bridge", which guarantees the stability of quotient co-distortion of graphs under uniform quotient maps.

Proposition 4.9. *Let X and Y be Banach spaces such that Y is a uniform subquotient of X, where the uniform quotient map is Lipschitz for large distances. Let G be an unweighted connected simple graph. Then $qc_X(G) = O(qc_Y(G))$.*

Proof. Let Z be a subset of X and let $f : Z \to Y$ be a uniform quotient map that is Lipschitz for large distances. Assume that S is a subset of Y and $g : S \to G$ is a Lipschitz quotient map. By a scaling of the set S we may without loss of generality assume that $\mathrm{Lip}(g) = 1$. Let $\tilde{f}$ denote the restriction of f to $Z := f^{-1}(S)$. Then for every $t \in (0, \infty)$, $\mathrm{Lip}_t(\tilde{f}) \leq \mathrm{Lip}_t(f) < \infty$. We show next that $h := g \circ \tilde{f}$ is a Lipschitz quotient map from Z onto G.

Claim 4.3. There exists $\delta \in (0, \infty)$ such that $\mathrm{Lip}(h) \leq \mathrm{Lip}_\delta(f)$.

Proof of Claim 4.3. Since f is uniformly continuous, there exists $\delta \in (0, \infty)$ so that $\|f(x) - f(y)\| < 1$ whenever $\|x - y\| < \delta$. For every $x, y \in Z$ such that $\|x - y\| < \delta$, one has $h(x) = h(y)$ since

$$\rho_G(h(x), h(y)) \leq \|\tilde{f}(x) - \tilde{f}(y)\| < 1.$$

If $\|x - y\| \geq \delta$, then

$$\rho_G(h(x), h(y)) \leq \|\tilde{f}(x) - \tilde{f}(y)\| \leq \mathrm{Lip}_\delta(f)\|x - y\|.$$

This implies that $\mathrm{Lip}(h) \leq \mathrm{Lip}_\delta(f)$.

Claim 4.4. There exists $c \in (0, \infty)$ such that $\mathrm{coLip}(h) \leq (c + 1)\mathrm{coLip}(g)$.

Proof of Claim 4.4. Let $\mathrm{coLip}(g) := D \in [1, \infty)$. Since Y is a Banach space, it is metrically convex and hence f as well as its restriction to Z are co-Lipschitz for large distances. Thus there exists $c \in (0,\infty)$ such that for all $x \in Z$ and all $r \geq 1$,

$$B_S\left(\tilde{f}(x), \frac{r}{c}\right) \subseteq \tilde{f}(B_Z(x,r)).$$

For every $x \in Z$ one has

$$\begin{aligned} B_G(h(x),1) &\subseteq g(B_S(\tilde{f}(x), D)) \\ &\subseteq g\left(B_S\left(\tilde{f}(x), \frac{(c+1)D}{c}\right)\right) \subseteq h(B_Z(x,(c+1)D)). \end{aligned}$$

Since the graph G is connected, it follows that $\mathrm{coLip}(h) \leq (c+1)\mathrm{coLip}(g)$.

Therefore, h is a Lipschitz quotient map from Z onto G with co-distrotion

$$\mathrm{codist}(h) \leq (c+1)\mathrm{Lip}_\delta(f)\mathrm{codist}(g),$$

where the constant $(c+1)\mathrm{Lip}_\delta(f)$ depends only on f. $\square$

In view of Proposition 4.9 and Proposition 4.8, the alternative proof of (a stronger form) of Theorem 2.7 simply boils down to exhibiting a discrepancy between the Y-distortion and the X-distortion of the countably branching trees. This discrepancy is exhibited by comparing the lower bound from Theorem 4.4 with the upper bound from Theorem 4.5.

Theorem 4.10. *Let X be a Banach space admitting an equivalent norm with property (β_p) for some $p \in (1,\infty)$. Assume that a Banach space Y is a uniform subquotient of X, where the uniform quotient map is Lipschitz for large distances. Then ℓ_q is not a uniform subquotient of Y for any $q > p$ such that the uniform quotient map is Lipschitz for large distances.*

Proof. Suppose that ℓ_q is a uniform subquotient of Y for some $q > p$ such that the uniform quotient map is Lipschitz for large distances. Then it follows from Proposition 4.8 and Proposition 4.9 that for $h \in \mathbb{N}$,

$$c_X(T_h^\omega) = qc_X(T_h^\omega) \lesssim qc_Y(T_h^\omega) \lesssim qc_{\ell_q}(T_h^\omega) = c_{\ell_q}(T_h^\omega).$$

Since $X \in \mathcal{C}_{(\beta_p)}$, by Theorem 4.4 we have $c_X(T_h^\omega) \gtrsim \log(h)^{1/p}$, while Theorem 4.5 says that $c_{\ell_q}(T_h^\omega) \lesssim \log(h)^{1/q}$. This is a contradiction for h big enough. $\square$

Theorem 4.11. *c_0 is not a uniform subquotient of a Banach space admitting an equivalent norm with property (β) such that the uniform quotient map is Lipschitz for large distances.*

Proof. Assume that c_0 is a uniform subquotient of a Banach space X admitting an equivalent norm with property (β) such that the uniform quotient map is Lipschitz for large distances. Then it follows from Proposition 4.8 and Proposition 4.9 that for $h \in \mathbb{N}$,

$$c_X(T_h^\omega) = qc_X(T_h^\omega) \lesssim qc_{c_0}(T_h^\omega) = c_{c_0}(T_h^\omega).$$

It is easy to show using the summing basis of c_0 that $c_{c_0}(T_h^\omega) \leq 2$ (actually $c_{c_0}(T_h^\omega) = 1$ follows from Theorem 6.3 in [26]), but by Theorem 4.4, $c_X(T_h^\omega) \to \infty$ as $h \to \infty$. We again get a contradiction. $\square$

The case of coarse quotient is a bit more delicate. To prove a coarse analogue of Proposition 4.9, we need the following technical lemma, which is the large-distances version of Lemma 3.6.

Lemma 4.12. *Let X and Y be metric spaces and $f : X \to Y$ be a coarse quotient map with constant K. Assume that Y is metrically convex and S is a subset of Y. Then there exist a subset $Z \subseteq X$ and a map $g : Z \to S$ satisfying the following:*

(i) If $K = 0$, then for every $\varepsilon > 0$ there exists $c_1 = c_1(\varepsilon) > 0$ such that for all $x \in Z$ and $r \geq \varepsilon$,

$$B_S(g(x), r) \subseteq g(B_Z(x, c_1 r)). \tag{4.2}$$

(ii) If $K > 0$, then there exists $c_2 = c_2(K) > 0$ such that for all $x \in Z$ and $r > 0$,

$$B_S(g(x), r) \subseteq g(B_Z(x, c_2 r))^{4K}. \tag{4.3}$$

Proof. We follow the proof of Lemma 3.6. Since Y is metrically convex, by Lemma 3.8, for every $\varepsilon > 2K$, there exists $c = c(\varepsilon) > 0$ so that for all $x \in X$ and $r \geq \varepsilon$,

$$B_Y(f(x), r) \subseteq f(B_X(x, cr))^K.$$

Define $p : S^K \to S$ by $p(a) = a$ if $a \in S$ and $p(a) = s_a$ otherwise, where s_a is any point in S within distance K from a. We now show that in both cases (i) and (ii) one can take $Z := f^{-1}(S^K)$ and $g = p \circ \tilde{f}$, where $\tilde{f} : Z \to S^K$ is the restriction of f to Z.

Indeed, in case (i) when $K = 0$, the map p becomes the identity map on S and hence $g : Z \to S$ is the restriction of f to $Z = f^{-1}(S)$. Thus (4.2) follows with $c_1 = c_1(\varepsilon) = c$.

In case (ii) when $K > 0$, by Remark 3.9 there exists $\tilde{c} = c(2K) > 0$ so that for all $x \in X$ and $r \geq 2K$,

$$B_Y(f(x), r) \subseteq f(B_X(x, \tilde{c}r))^K.$$

Now for $x \in Z$ and $r \geq 2K$, suppose that $y \in B_{S^K}(\tilde{f}(x), r)$. Then there exists

$u \in B_X(x, \tilde{c}r)$ such that $d_Y(y, f(u)) \leq K$, and $y \in S^K$ implies that $d_Y(y, s) \leq K$ for some $s \in S$, so

$$s \in B_Y(f(u), 2K) \subseteq f(B_X(u, 2K\tilde{c}))^K.$$

Thus there exists $v \in B_X(u, 2K\tilde{c})$ such that $d_Y(s, f(v)) \leq K$. It follows by the triangle inequality that

$$v \in B_Z(x, 2K\tilde{c} + \tilde{c}r) \subseteq B_Z(x, 2\tilde{c}r) \quad \text{and} \quad d_Y(y, \tilde{f}(v)) \leq 2K.$$

Now we have shown that the map $\tilde{f} : Z \to S^K$ satisfies for all $x \in Z$ and $r \geq 2K$,

$$B_{S^K}(\tilde{f}(x), r) \subseteq \tilde{f}(B_Z(x, 2\tilde{c}r))^{2K}.$$

Therefore, for every $x \in Z$ and $r \geq 4K$,

$$\begin{aligned} B_S(g(x), r) \subseteq p(B_{S^K}(\tilde{f}(x), r + K)) &\subseteq p(B_{S^K}(\tilde{f}(x), 2r)) \\ &\subseteq p\left(\tilde{f}(B_Z(x, 4\tilde{c}r))^{2K}\right) \subseteq g(B_Z(x, 4\tilde{c}r))^{4K}. \end{aligned}$$

This implies that (4.3) holds for $c_2 = c_2(K) = 4\tilde{c}$. □

The next proposition is the analogue of Proposition 4.9 that is needed in the coarse case.

Proposition 4.13. *Let X and Y be Banach spaces such that Y is a coarse subquotient of X, where the coarse quotient map is Lipschitz for large distances. Then there exists $k \in \mathbb{N}$ (independent of n) so that $qc_X(T^{\omega}_{2^n}) = O(qc_Y(T^{\omega}_{2^{n+k}}))$ for all $n \in \mathbb{N}$.*

Proof. Let Z be a subset of X and let $f : Z \to Y$ be a coarse quotient map with constant K that is Lipschitz for large distances. We claim that k can be chosen as the smallest positive integer so that $2^k > \omega_f(1)+4K+1$. Assume that S_n is a subset of Y and $g_n : S_n \to T^{\omega}_{2^{n+k}}$ is a Lipschitz quotient map. By a scaling of the set S_n we may without loss of generality assume that $\mathrm{Lip}(g_n) = 1$. There exist a subset $T(n) \subseteq T^{\omega}_{2^{n+k}}$ in which the distance between points is at least 2^k, and a rescaled isometry $i_n : T(n) \to T^{\omega}_{2^n}$ so that for every $u, v \in T(n)$,

$$\rho_{T^{\omega}_{2^n}}(i_n(u), i_n(v)) = 2^{-k}\rho_{T^{\omega}_{2^{n+k}}}(u, v).$$

Let $\widetilde{S}_n := g_n^{-1}(T(n))$. By Lemma 4.12, there exists $c > 0$ depending only on K, so that for every $n \in \mathbb{N}$, there exist sets $Z_n \subseteq Z$ and coarse quotient maps $f_n : Z_n \to \widetilde{S}_n$ satisfying for all $x \in Z_n$ and $r \geq 4K + 1$,

$$B_{\widetilde{S}_n}(f_n(x), r) \subseteq f_n(B_{Z_n}(x, cr))^{4K}.$$

Consider the map $h_n := i_n \circ \tilde{g}_n \circ f_n : Z_n \to T^{\omega}_{2^n}$, where $\tilde{g}_n$ is the restriction of g_n to $\widetilde{S}_n$.

Claim 4.5. For every $n \in \mathbb{N}$, $\mathrm{Lip}(h_n) \leq 2^{-k}(2K + \mathrm{Lip}_1(f))$.

Proof of Claim 4.5. For every $x, y \in Z_n$ such that $\|x-y\| < 1$, one has $h_n(x) = h_n(y)$ since it follows from the proof of Lemma 4.12 that

$$\rho_{T^{\omega}_{2^n}}(h_n(x), h_n(y)) \leq 2^{-k}\|f_n(x) - f_n(y)\| \leq 2^{-k}(2K + \|f(x) - f(y)\|) < 1.$$

If $\|x - y\| \geq 1$, then

$$\rho_{T^{\omega}_{2^n}}(h_n(x), h_n(y)) \leq 2^{-k}(2K + \|f(x) - f(y)\|) \leq 2^{-k}(2K + \mathrm{Lip}_1(f))\|x - y\|.$$

Thus $\mathrm{Lip}(h_n) \leq 2^{-k}(2K + \mathrm{Lip}_1(f))$.

Claim 4.6. For every $n \in \mathbb{N}$, $\mathrm{coLip}(h_n) \leq 2^k c \cdot \mathrm{coLip}(g_n)$.

Proof of Claim 4.6. Let $\mathrm{coLip}(g_n) := D_n \in [1, \infty)$. For every $x \in Z_n$ one has

$$\begin{aligned} B_{T^{\omega}_{2^n}}(h_n(x), 1) &= i_n\left(B_{T(n)}(\tilde{g}_n \circ f_n(x), 2^k)\right) \subseteq i_n \circ \tilde{g}_n\left(B_{\widetilde{S}_n}(f_n(x), 2^k D_n)\right) \\ &\subseteq i_n \circ \tilde{g}_n\left(f_n\left(B_{Z_n}(x, 2^k D_n c)\right)^{4K}\right) \subseteq i_n\left(\left(\tilde{g}_n \circ f_n\left(B_{Z_n}(x, 2^k D_n c)\right)\right)^{4K}\right) \\ &= h_n(B_{Z_n}(x, 2^k D_n c)). \end{aligned}$$

This implies that $\mathrm{coLip}(h_n) \leq 2^k D_n c$.

Therefore, h_n is a Lipschitz quotient map from Z_n onto $T^{\omega}_{2^n}$ with co-distortion

$$\mathrm{codist}(h_n) \leq c(2K + \mathrm{Lip}_1(f))\mathrm{codist}(g_n),$$

where the constant $c(2K + \mathrm{Lip}_1(f))$ depends only on f and K. □

Now a combination of Proposition 4.13, Proposition 4.8, Theorem 4.5 and Theorem 4.4 implies the following (stronger form) of Theorem 3.16.

Theorem 4.14. *Let X be a Banach space admitting an equivalent norm with property (β_p) for some $p \in (1, \infty)$. Assume that a Banach space Y is a coarse subquotient of X, where the coarse quotient map is Lipschitz for large distances. Then ℓ_q is not a coarse subquotient of Y for any $q > p$ such that the coarse quotient map is Lipschitz for large distances.*

Proof. Suppose that ℓ_q is a coarse subquotient of Y for some $q > p$ such that the coarse quotient map is Lipschitz for large distances, then it follows from Proposition 4.8 and Proposition 4.13 that there exist $k_1, k_2 \in \mathbb{N}$ independent of $n \in \mathbb{N}$ such that

$$c_X(T^{\omega}_{2^n}) = q c_X(T^{\omega}_{2^n}) \lesssim q c_Y(T^{\omega}_{2^{n+k_1}}) \lesssim q c_{\ell_q}(T^{\omega}_{2^{n+k_1+k_2}}) = c_{\ell_q}(T^{\omega}_{2^{n+k_1+k_2}}).$$

Since $X \in \mathcal{C}_{(\beta_p)}$, by Theorem 4.4 we have $c_X(T^{\omega}_{2^n}) \gtrsim n^{1/p}$, while Theorem 4.5 implies that $c_{\ell_q}(T^{\omega}_{2^{n+k_1+k_2}}) \lesssim (n+k_1+k_2)^{1/q}$. This is a contradiction for n big enough. $\square$

Similarly, one can give an alternative proof for Theorem 3.18, which we omit here.

5. SUMMARY

This dissertation treats the nonlinear quotient theory of Banach spaces with an emphasis on the large scale geometry of the spaces. Definitions and results known for uniform and Lipschitz quotients of Banach spaces are developed in the coarse setting. We summarize our work as follows:

The notion of coarse quotient is introduced and coarse quotients of some classical Banach spaces are studied. More precisely, we give an isomorphic characterization of coarse quotients of the function spaces L_p for $1 < p < \infty$; namely, every coarse quotient of L_p is isomorphic to a linear quotient of L_p. It is also proven that every coarse quotient of the sequence space ℓ_p is isomorphic to a linear quotient of ℓ_p if $1 < p \leq 2$. Whether this is true when $p > 2$ is left open.

In connection with metric embedding, for $1 < p < q < \infty$ we show by comparing the ℓ_p-distortion and ℓ_q-distortion of the countably branching tree T_h^ω that there is no uniform or coarse quotient map from ℓ_p to ℓ_q. It is not known whether ℓ_p can be a uniform or coarse quotient of ℓ_q for the same range of p and q.

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APPENDIX A

ULTRAPRODUCT OF BANACH SPACES

This Appendix is a brief introduction to ultraproducts in Banach space theory. We will only list definitions and results. Detailed proof as well as thorough discussion of this topic can be found in [17].

Definition A.1. A filter $\mathcal{F}$ on an infinite set I is a subset of $\mathcal{P}(I)$ (the set of all subsets of I) satisfying the following conditions:

(1) $\emptyset \notin \mathcal{F}$;

(2) $\mathcal{F}$ is closed under finite intersections;

(3) If $A \in \mathcal{F}$, then $B \in \mathcal{F}$ for every $B \supseteq A$.

An ultrafilter $\mathcal{U}$ on I is a maximal filter with respect to inclusion. An ultrafilter is called free if the intersection of all the sets in it is empty.

Definition A.2. Let $\mathcal{U}$ be an ultrafilter on I and X be a topological space. We say that $(x_i)_{i\in I} \subseteq X$ converges to $x \in X$ through $\mathcal{U}$ and write $\lim_{\mathcal{U}} x_i = x$ if $\{i \in I : x_i \in U\} \in \mathcal{U}$ for every open neighborhood U of x.

Lemma A.3. *Let $\mathcal{U}$ be an ultrafilter on I and K be a compact topological space. Then any $(x_i)_{i\in I} \subseteq K$ converges to some $x \in K$ through $\mathcal{U}$. In particular, any bounded real-valued set $(x_i)_{i\in I}$ converges to some $x \in \mathbb{R}$ through $\mathcal{U}$.*

Let $(X_i)_{i\in I}$ be a family of Banach spaces and $\mathcal{U}$ be a free ultrafilter on I. Consider the ℓ_∞-sum of $(X_i)_{i\in I}$, i.e. the Banach space

$$\left(\sum_{i\in I} X_i\right)_{\ell_\infty} := \{(x_i)_{i\in I} : x_i \in X_i \text{ and } \sup_{i\in I} \|x_i\| < \infty\}$$

with the norm $\|(x_i)_{i\in I}\|_\infty = \sup_{i\in I}\|x_i\|$. In view of Lemma A.3, for each $(x_i)_{i\in I} \in (\sum_{i\in I} X_i)_{\ell_\infty}$, $\lim_{\mathcal{U}}\|x_i\|$ exists and defines a seminorm. It is not hard to see that the subspace of $(\sum_{i\in I} X_i)_{\ell_\infty}$ on which the seminorm is equal to 0, denoted by $N_{\mathcal{U}}$, is closed.

Definition A.4. The ultraproduct of $(X_i)_{i\in I}$ with respect to the free ultrafilter $\mathcal{U}$, denoted by $(\prod_{i\in I} X_i)_{\mathcal{U}}$, is the quotient space $(\sum_{i\in I} X_i)_{\ell_\infty}/N_{\mathcal{U}}$ with the norm $\|(x_i)_{\mathcal{U}}\| := \lim_{\mathcal{U}}\|x_i\|$, where $(x_i)_{\mathcal{U}}$ is the element in $(\prod_{i\in I} X_i)_{\mathcal{U}}$ corresponding to $(x_i)_{i\in I} \in (\sum_{i\in I} X_i)_{\ell_\infty}$. If all X_i's are the same Banach space X, then the ultraproduct is called an ultrapower of X and denoted by $X_{\mathcal{U}}$.

Proposition A.5. *Let X be a Banach space and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. If X is finite dimensional then $X_{\mathcal{U}}$ is isometrically isomorphic to X; if X is infinite dimensional then $X_{\mathcal{U}}$ is finitely representable in X.*

Proposition A.6. *Let X be a Banach space and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then $(X^*)_{\mathcal{U}} = (X_{\mathcal{U}})^*$ if and only if X is superreflexive.*