SPARSITY, RANDOMNESS AND CONVEXITY IN APPLIED ALGEBRAIC GEOMETRY

A Dissertation

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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August 2016

Major Subject: Mathematics

ABSTRACT

In this dissertation we study three problems in applied algebraic geometry. The first problem is to construct an algorithmically efficient approximation to the real part of the zero set of an exponential sum. We construct such a polyhedral approximation using techniques from tropical geometry. We prove precise distance bounds between our polyhedral approximation and the real part of the zero set. Our bounds depend on the number of terms of the exponential sum and the minimal distance between the exponents. Despite the computational hardness of the membership problem for the real part of the zero set, we prove that our polyhedral approximation can be computed by linear programing on the real BSS machine.

The second problem is to study the ratio of sums of squares polynomials inside the set of nonnegative polynomials. Our focus is on the effect of fixed monomial structure to the ratio of these two sets. We study this problem quantitatively by combining convex geometry and algebra. Some of our methods work for arbitrary Newton polytopes; however our main theorem is stated for multihomogenous polynomials. Our main theorem provides quantitative versions of some known algebraic facts, and also refines earlier quantitative results.

The third problem is to study the condition number of polynomial systems 'on average'. Condition number is a vital invariant of polynomial systems which controls their computational complexity. We analyze the condition number of random polynomial systems for a broad family of distributions. Our work shows that earlier results derived for the polynomial systems with real Gaussian independent random coefficients can be extended to the broader family of sub-Gaussian random variables allowing dependencies. Our results are near optimal for overdetermined systems but

there is room for improvement in the case of square systems of random polynomials.

The main idea binding our three problems is to observe structure and randomness phenomenon in the space of polynomials. We used combinatorial algebraic geometry to observe the 'structure' and convex geometric analysis to understand the 'randomness'. We believe results presented in this dissertation are just the first steps of the interaction between these two fields.

DEDICATION

To Betül, of course.

ACKNOWLEDGEMENTS

I would like present my sincere gratitude to my advisors Grigoris Paouris and J. Maurice Rojas. They are not only brilliant mathematicians but also exceedingly nice and thoughtful human beings. It is impossible to list all the things that I am thankful to Maurice and Grigoris for. I enjoyed the time we shared and hope they enjoyed it as well.

My wife Betül deserves the greatest of the thanks for always supporting me. And my little mischievous boys Bahadir and Hakan deserve a big thank you for making my life full of colors; joy, sadness, tiredness, sleeplessness and immense happiness. Without Bahadir and Hakan's help this dissertation would have been concluded with more results but less joy.

I thank NSF for funding my research. The NSF funding helped me to visit universities around the globe and to meet with excellent mathematicians. Among those, I thank Apostolos Giannopoulos, Peter Gritzmann, Greg Blekherman, Dan Bates and Lek-Heng Lim for their hospitality and Özgür Kisisel for being a good friend and mentor. Conversations with Özgür broadened my perspective about mathematics and life.

Last but not least, I thank my committee members David Kerr, Catherine Yan and Daniele Mortari for their help in preparation of this dissertation.

TABLE OF CONTENTS

Page

ABSTRACT . . . ii

DEDICATION . . . iv

ACKNOWLEDGEMENTS . . . v

TABLE OF CONTENTS . . . vi

LIST OF FIGURES . . . viii

1. INTRODUCTION . . . 1

1.1 Combinatorial Algebraic Geometry . . . 2
1.1.1 A-discriminants . . . 3
1.1.2 Amoebas of Hypersurfaces . . . 6
1.1.3 Tropical Geometry . . . 8
1.1.4 Real Part of the Roots of an Exponential Sum . . . 15
1.2 Modern Convex Geometry and Semidefinite Programing . . . 16
1.2.1 Santalo, Reverse Santalo and Uryshon Inequalities . . . 18
1.2.2 John's Theorem, Brascamp Lieb Inequality and Isotropic Measures . . . 20
1.2.3 Semidefinite Programing . . . 24
1.2.4 Quantitative Aspects of Hilbert's 17th Problem . . . 28
1.3 Efficient Polynomial System Solving . . . 29
1.3.1 Smale's 17th Problem . . . 30
1.3.2 Condition Number . . . 32
1.3.3 Small Ball Probabilities . . . 35
1.3.4 Condition Number of Random Polynomial Systems . . . 37

2. TROPICAL VARIETIES FOR EXPONENTIAL SUMS . . . 39

2.1 Introduction . . . 39
2.1.1 On the Computational Complexity of $\Re(Z(g))$ and $\mathrm{Trop}(g)$ . . . 42
2.2 Tropically Extending Classical Polynomials Root Bounds to Exponential Sums . . . 46
2.2.1 Basics on Roots of Univariate Exponential Sums . . . 46

2.2.2 Winding Numbers and Density of Roots in Rectangles and Vertical Strips 51
2.2.3 A Quick Distance Bound in Arbitrary Dimension 56
2.3 Small Ball Probability 58
2.4 Proof of Theorem 2.1.5 59
2.4.1 Proof of Assertion (0) 59
2.4.2 Proof of Assertion 1(a) 60
2.4.3 Proof of Assertion (2) 60
2.4.4 Proof of Assertion 1(b) 60
2.5 Proving Theorem 2.1.10 61

3. MULTIHOMOGENOUS NONNEGATIVE POLYNOMIALS AND SUMS OF SQUARES 63

3.1 Introduction 63
3.1.1 From All or Nothing to Something in Between 64
3.1.2 Our Results 65
3.2 Harmonic Polynomials and Euclidean Balls 68
3.3 The Cone of Nonnegative Polynomials 80
3.4 The Cone of Sums of Squares 86
3.5 The Cone of Powers of Linear Forms 90

4. CONDITION NUMBER OF RANDOM POLYNOMIAL SYSTEMS . . . 92

4.1 Introduction 92
4.2 Technical Background 99
4.3 Condition Number of Random Polynomial Systems 106
4.3.1 Introducing the Randomness 106
4.3.2 The Subgaussian Assumption and “Operator Norm" Bounds . 108
4.3.3 The Small Ball Assumption and Bounds for the “L" Quantity 111
4.3.4 The Condition Number Theorem and Consequences 117
4.4 Random Polynomial Systems With Given Support 132
4.4.1 Notation and the Basic Quantities 133
4.4.2 Bounding the Quantity $H(\mathbf{S})$ 136
4.4.3 The Main Result 139
4.4.4 Consequences of the Main Theorem 144

5. SUMMARY 150

REFERENCES 152

LIST OF FIGURES

FIGURE Page

1.1 Amoeba of $x_1 + x_2 = 1$ 6

1.2 Amoeba of $-1 + 5x_1 - 15x_2 + 10x_1x_2 + 3x_1^2 + 5x_2^2$ 7

1.3 Tropical curve of $x_1 + x_2 = 1$ 9

1.4 A quadratic tropical curve 10

1. INTRODUCTION

This introductory section will walk the reader through mathematical objects and problems that are present in this dissertation. Every subsection introduces tools from a branch of mathematics and concludes with a related research problem. Section 2 of the dissertation corresponds to the problem introduced at the end of the Combinatorial Algebraic Geometry subsection. Section 3 of the dissertation corresponds to the problem introduced at the end of Modern Convex Geometry and Semidefinite Programing subsection, finally the problem introduced at the end of Efficient Polynomial System Solving subsection corresponds to the Section 4.

A quick look at this section will reveal the diversity of the topics included in this dissertation. The common theme binding these topics is the desire to understand structure and randomness phenomenon in the space of multivariate polynomials.

Combinatorial algebraic geometry offers a beautiful frame to study the structure of sparse polynomials via combinatorial tools. Some of these tools will be introduced in the first subsection below. On the other hand, a generic real polynomial with n variables and degree d is by definition a high dimensional object, and one expects the high dimension to have regularity effects. Modern convex geometry offers strong tools to analyze these effects of regularity. Due to space limitations we will not be able introduce even some of the main ideas of modern convexity. However, we hope the second and the third subsections will provide a taste of the field.

The interaction between modern convex geometry and it's probabilistic tools with algebraic geometry is very important and fruitful. As first steps of this interaction, and to train our intuition, we studied problems arising from applied algebraic geometry. We hope the reader of this dissertation will find these problems interesting in their own right. We were happy that these problems were also challenging enough to force development of new theoretical tools.

1.1 Combinatorial Algebraic Geometry

We start with an example that we borrow from [100]. We consider the following system of polynomials

$$\begin{aligned} p_1(x,y,z) &= a_{11}x + a_{12}y^2 + a_{13}z^3 + a_{14}x^5y^6z^7 + a_{15}x^6y^7z^5 + a_{16}x^7y^5z^6 + a_{17}x^8y^9z^9 \\ &\quad + a_{18}x^{10}y^9z^9 + a_{19}x^9y^8z^9 + a_{110}x^9y^{10}z^9 + a_{111}x^9y^9z^{10} \\ p_2(x,y,z) &= a_{21}x + a_{22}y^2 + a_{23}z^3 + a_{24}x^5y^6z^7 + a_{25}x^6y^7z^5 + a_{26}x^7y^5z^6 \\ &\quad + a_{27}x^8y^9z^9 + a_{28}x^{10}y^9z^9 + a_{29}x^9y^8z^9 + a_{210}x^9y^{10}z^9 + a_{211}x^9y^9z^{10} \\ p_3(x,y,z) &= a_{31}x + a_{32}y^2 + a_{33}z^3 + a_{34}x^5y^6z^7 + a_{35}x^6y^7z^5 + a_{36}x^7y^5z^6 + a_{37}x^8y^9z^9 \\ &\quad + a_{38}x^{10}y^9z^9 + a_{39}x^9y^8z^9 + a_{310}x^9y^{10}z^9 + a_{311}x^9y^9z^{10} \end{aligned}$$

One might guess that for a generic set of coefficients a_{ij}, the polynomial system p_1, p_2, p_3 will have a finite number of common roots. The classical theorem of Bezout states that this finite number is bounded by multiplication of the degrees of polynomials i.e $28 \times 28 \times 28 = 21952$. As one can see from the statement of Bezout's theorem, this theorem does not take into account the structure of monomials; all the degree 28 polynomials are treated equivalently. This shortcoming is overcome by one of the fundamental objects of combinatorial algebraic geometry: toric varieties. The theory of toric varieties studies geometric objects encoded by combinatorial data. In our example 1.1 above, the correct combinatorial concept is the Newton polytope: for a polynomial $p = \sum_\alpha c_\alpha x^\alpha$ where $\alpha \in \mathbb{Z}^n$, the Newton polytope of p is defined as

$$Newt(p) = conv(\{\alpha : c_\alpha \neq 0\})$$

where $conv(*)$ stands for the convex hull of $*$. For the above example, we have

$$Newt(p_1) = conv(\{(1,0,0),(0,2,0),(0,0,3),(5,6,7),(6,7,5)$$
$$,(7,5,6),(8,9,9),(10,9,9),(9,8,9),(9,10,9),(9,9,10)\})$$

Using the compactification provided by theory of toric varieties, instead of classical projective space, helps to prove that the number of common roots are bounded by mixed volume of Newton polytopes [72]. In our example 1.1 above, this approach (Bernstein's theorem) gives the bound 321, and this number is exactly the number of common roots for a generic set of coefficients a_{ij}.

We will not work explicitly with toric varieties in this thesis but quite often it will be the underlying structure. For more information on this beautiful subject we refer the reader to [49] and [31].

1.1.1 A-discriminants

In this section we look a little more closely at the idea of generic set of coefficients. We start by considering the following example from high school algebra:

$$\mathcal{F} := \{q(x) = ax^2 + bx + c\}$$

Here $\mathcal{F}$ is the family of univariate quadratic equations. The reader surely knows that the discriminant corresponding to this family is $\Delta[2,1,0] := b^2 - 4ac$ and whenever $\Delta[2,10](a,b,c) = 0$ at a certain triple (a,b,c), the two roots of $q(x) = ax^2 + bx + c$ coincide. It is also known that whenever $\Delta[2,1,0](a,b,c) > 0$ there are two real roots of $ax^2 + bx + c$ and whenever $\Delta[2,10](a,b,c) < 0$ there are no real roots of $ax^2 + bx + c$.

Now we consider another family $\mathcal{H} := \{q(x) = ax^{29} + bx^{18} + cx^{7}\}$. We observe that the monomial set $\{29, 18, 7\}$ can be obtained by the set $\{2, 1, 0\}$ by scaling with 11 and shifting with 7. Thus, one expects the same algebraic rule Δ to govern this family of polynomials i.e $\Delta[29, 18, 7] = \Delta[2, 1, 0]$. We define the A-discriminants below, which is the expected, intuitive algebraic rule.

Definition 1.1.1. *Given $A := \{a_1, a_2, \ldots, a_m\} \subset \mathbb{Z}^n$ of cardinality m we define the family of polynomials $\mathcal{F}_A$ to be $\mathcal{F}_A := \{\sum_i c_i x^{a_i} : c_i \in \mathbb{C}\}$ then the A discriminant variety ∇_A is defined to be the following*

$$\nabla_A := \overline{\{[c_1 : c_2 : \ldots : c_m] \in \mathbb{P}(\mathbb{C})^{m-1} : p(x) = \sum_i c_i x^{a_i} \textit{ possesses a singularity in } (\mathbb{C}^*)^n\}}$$

where $\overline{V}$ is the Zariski closure of V. When ∇_A has codimension 1, Δ_A is the defining polynomial of ∇_A.

Within the perspective of the A-discriminant definition, a generic set of coefficients for the support A is a set of coefficients that does not lie in the A-discriminant variety.

The canonical reference for the theory of A-discriminants is [51]. The very first theorem (i.e Biduality Theorem, Thm 1.1) of [51] proves that A-discriminants are projective duals to the toric varieties parametrized by $\{[t^{a_1} : t^{a_2} : \ldots : t^{a_m}] : t \in (\mathbb{C}^*)^n\}$. We will not pursue this geometric direction further here; instead we continue with a concrete example that we borrow from [41]. Before going into the example, we note that via a standard method called the Cayley trick one can extend the definition of the A-discriminants to the polynomial systems [41]. Our example is an important family of polynomials called the Haas family [61].

Example 1.1.2. *We consider the following family of polynomials* $H_{a,b,d}$

$$f_1 = x^{2d} + ay^d - y$$
$$f_2 = y^{2d} + bx^d - x$$

then we have $A = \{(2d, 0), (0, d), (0, 1), (0, 2d), (d, 0), (1, 0)\}$ *and* $\Delta_A(1, a, -1, 1, b, -1)$ *happens to be a polynomial of degree 47 with 58 monomials and coefficients of the order* 10^{43}.

As one can see from the previous example, A-discriminants can easily become involved and computationally expensive. This clearly shows the need for more efficient ways of addressing the A-discriminant variety. Combinatorial algebraic geometry offers different tools for this issue. However in the most general form, finding an efficient (approximate) algorithm for A-discriminant variety membership can still be considered as an open problem. We close this section with a strong and simple theorem from [69].

Theorem 1.1.3. *(Horn-Kapranov Uniformization)* $A := \{a_1, a_2, \ldots, a_m\} \subset \mathbb{Z}^n$, *if* ∇_A *is of codimension then it is exactly the Zariski closure of*

$$\{[u_1 t^{a_1} : u_2 t^{a_2} : \ldots : u_m t^{a_m}] : u \in \mathbb{P}(\mathbb{C})^{m-1}, \mathcal{A}u = 0, t \in (\mathbb{C}^*)^n\}$$

where $\mathcal{A}$ *is the following matrix*

$$\mathcal{A} = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_m \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}$$

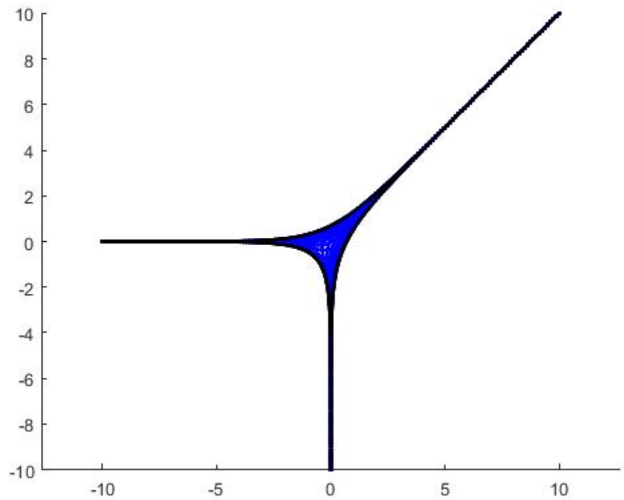

Figure 1.1: Amoeba of $x_1 + x_2 = 1$

1.1.2 Amoebas of Hypersurfaces

Assume we are interested in the magnitude of the points inside a variety X which lives in $(\mathbb{C}^*)^n$ and for this purpose we define the following map

$$Log : X \to \mathbb{R}^n$$

$$Log(z) = (\log(|z_1|), \log(|z_2|), \ldots, \log(|z_n|))$$

The image of X under this map is called amoeba of X. One may wonder the reason for this interesting name; we hope figures 1.1 and 1.2 provide a hint. Definition of the amoeba first appeared in [51]. Since then, it began to show up in different areas of pure and applied mathematics.

Amoebas carry a surprising amount of information from the original variety X. For instance, for a hypersurface X defined by $p = \sum_\alpha c_\alpha x^\alpha$ and it's amoeba $\mathcal{A}$, it was proven by Forsberg, Passare and Tsikh that there exists a one to one map from

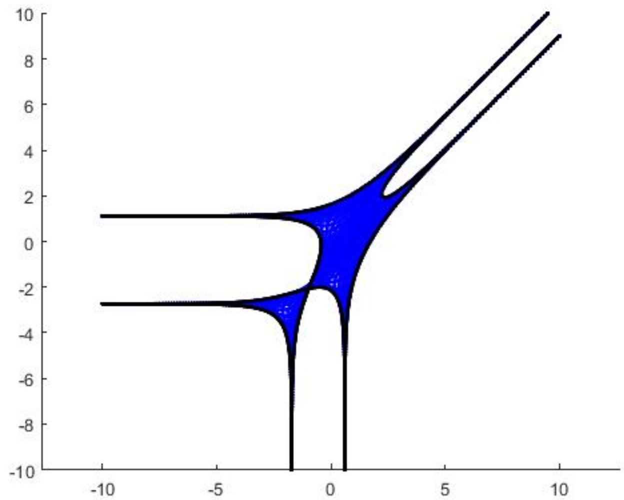

Figure 1.2: Amoeba of $-1 + 5x_1 - 15x_2 + 10x_1x_2 + 3x_1^2 + 5x_2^2$

connected components of the complement of $\mathcal{A}$ to the lattice points of $Newt(p)$ [91]. They proved in particular there are always connected components corresponding to vertices of $Newt(p)$, and other components of the complement correspond to inner lattice points.

Another surprising example is about the volume of two dimensional amoebae. Even though a two dimensional amoeba $\mathcal{A}$ of a hypersurface X is unbounded, the volume of $\mathcal{A}$ is bounded and the extremal examples are special plane curves. More precisely, when X is a hypersurface defined by $p(x, y) = \sum_\alpha c_\alpha x^\alpha$ where $\alpha \in \mathbb{Z}^2$, and $\mathcal{A}$ is the amoeba of X; we have the following inequality [92]:

$$Area(\mathcal{A}) \leq \pi^2 Area(Newt(p))$$

where $Area$ is the usual area on $\mathbb{R}^2$. Moreover, it was proven by Rullgard and Mikhalhin that the polynomials that achieve equality are precisely the extremal examples on Hilbert's 16th problem, namely the Harnack curves [84]. To the author's knowledge there is no multivariate analog of the area type result above.

After defining the amoeba and observing it's interesting properties, one natural question is to ask what happens if we modify the *Log* map by changing the basis from e to t? Especially, if we have a one parameter family of varieties V_t, what are the limit shapes of amoebae $Log_t(V_t)$ as $t \to \infty$? This question will be answered in the next section by tropical geometry.

1.1.3 Tropical Geometry

There are several ways to introduce tropical geometry. We prefer to start with introducing the tropical varieties on non-Archimedean fields. This is mainly due to the conceptual importance of Kapranov's Theorem. We begin by defining the valuation.

Definition 1.1.4. *(Valuation) Let $\mathbb{K}$ be a field and we denote the set of non-zero elements of the field by $\mathbb{K}^*$. A valuation on $\mathbb{K}$ is a function $val : K \longrightarrow \mathbb{R} \cup \infty$ satisfying the following three properties:*

1. $val(a) = 0$ *if and only if* $a = 0$
2. $val(ab) = val(a) + val(b)$
3. $val(a + b) \geq \min\{val(a), val(b)\}$ *for all* $a, b \in \mathbb{K}^*$

The following is a standard lemma.

Lemma 1.1.5. *If $val(a) \neq val(b)$ then $val(a + b) = \min\{val(a), val(b)\}$.*

Every valuation introduces a norm by $\|x\| = c^{-val(x)}$ for any $c > 0$. The extra property of this norm is that it satisfies the ultrametric inequality:

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

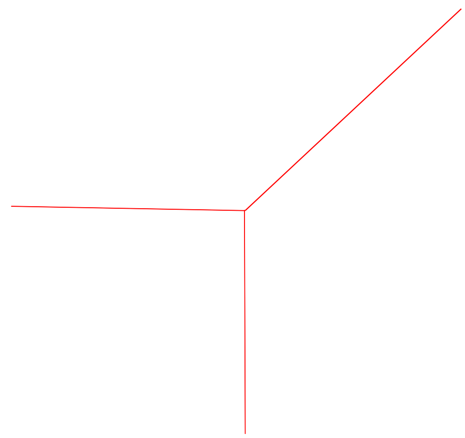

Figure 1.3: Tropical curve of $x_1 + x_2 = 1$

Normed fields that satisfy the ultra metric inequality are called non-Archimedean fields.

Definition 1.1.6. *If $v : K \longrightarrow \mathbb{R} \cup \infty$ is a valuation, then there are several objects that can be defined from it:*

1. *the valuation ring of v, denoted R_v is the set of elements $a \in K$ such that $v(a) \geq 0$, R_v is a ring in K.*

2. *the prime ideal of v (or the maximal ideal of v), denoted m_v is the set of elements $a \in K$ such that $v(a) > 0$, m_v is a maximal ideal of R_v.*

3. *the residue field of v, denoted k_v is R_v/m_v.*

For elements $a \in R_v$ the image of a under the canonical projection to the residue field k_v is denoted by $\bar{a}$.

Now let K be a non-Archimedean field, $f \in K[x_1, x_2, \ldots, x_n]$ and $V(f) := \{x \in K : f(x) = 0\}$. Then the analogue of amoeba in this non-Archimedean setup is the following:

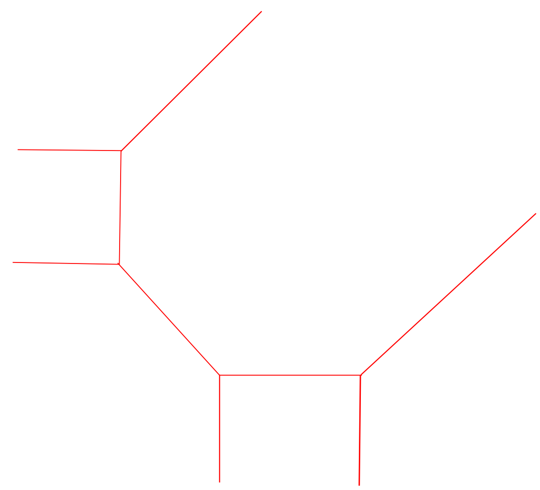

Figure 1.4: A quadratic tropical curve

$$\mathcal{A}_f = \{(val(z_1), val(z_2), \ldots, val(z_n)) : z \in V(f)\}$$

Let $f = \sum_i c_\alpha x^\alpha$, we set $trop(f)(w) = \min\{val(c_\alpha) + w.\alpha\}$ then we define the tropical variety as follows

$$Trop(f) = \{w \in \mathbb{R}^n : \text{the minimum in } trop(f) \text{ is attained at least twice }\}$$

Kapranov's Theorem will relate $\mathcal{A}_f$ and the $Trop(f)$. To state Kapranov's Theorem fully, we need to introduce one more object. To motivate the concept, we ask a basic question: how can we order monomials that appear in $K[x_1, x_2, \ldots, x_n]$? For the case $n = 1$, we all know that this can be done basically by the degree of the monomial i.e $x^i \prec x^j$ if $i \leq j$. Moreover, we also know that with this ordering $\prec$, $x^i \prec x^j$ implies $x^i | x^j$. This observation is the foundation of the euclidean division algorithm for univariate polynomials. Appropriately developing these ideas in multivariate setting leads to the theory of Gröbner basis. We don't cover the Gröbner basis theory here, but let us point out that there are infinitely many possible orders for the multivariate case. Let $w \in \mathbb{R}^n$ and define the following order; $x^\alpha \prec_w x^\beta$ if $\langle \alpha, w \rangle \leq \langle \beta, w \rangle$. Thus all $w \in \mathbb{R}^n$ provides an order for the monomials in $K[x_1, x_2, \ldots, x_n]$. Let $f \in K[x_1, x_2, \ldots, x_n]$, for a generic $w \in \mathbb{R}^n$ one hopes to have unique minimal monomial with respect to the order introduced by w. This intuition leads to the following definition.

Definition 1.1.7. *(Initial Term) For a fixed $w \in \mathbb{R}^n$ and $f = \sum_u c_u x^u$, let $W = trop(f)(w)$ then the initial term with respect w is*

$$in_w(f) = \sum_{u:val(c_u)+w.u=W} \overline{c_u} x^u$$

where $\bar{a}$ is as defined in 1.1.6.

We are now ready to state Kapranov's Theorem (see Thm 3.1.3, [79]).

Theorem 1.1.8. *(M. M. Kapranov) We use the notation developed in this section. Let K be a field with valuation and let $f \in K[x_1, x_2, \ldots, x_n]$, then the following three sets coincide*

1. *$Trop(f) \subset \mathbb{R}^n$*
2. *$w \in \mathbb{R}^n$ such that $in_w(f)$ is not a monomial*
3. *$\mathcal{A}_f \subset \mathbb{R}^n$*

There is a systematic algebraic way of defining the $Trop(f)$ from the polynomial f. This algebraic approach was developed by Brazilian computer scientist Imre Simon. Contemporary French mathematicians named the object Imre developed as tropical geometry, because everything produced in Brazil is exotic and tropical! Tropical Semiring is $(R \cup \infty, \oplus, \otimes)$ with the operations $\oplus, \odot$ defined as follows

$$a \oplus b := \min\{a, b\}$$

$$a \odot b := a + b$$

Then the tropicalization of a polynomial $f = \sum_\alpha c_\alpha x^\alpha$ is the following

$$trop(f)(w) = \oplus_\alpha val(c_\alpha) \odot x^\alpha$$

The tropical variety corresponding to f, can be also defined as follows

$$Trop(f) = \{w \in \mathbb{R}^n : trop(f) \text{ is not linear at } w\}$$

For the non-Archimedean fields, Kapranov's Theorem proves that the amoeba and the tropical variety coincide. Now, we would like turn our attention back to the Archimedean fields and the amoeba defined by the *Log* map. In the Archimedean fields we don't expect the coincidence of the two objects as in the case of Kapranov's Theorem. However, we expect the tropical variety to approximate the amoeba in some sense. First, we note that topologically tropical variety might be different than the amoeba; tropical variety is not always a deformation retract of the amoeba. Therefore one needs to search for a different notion of closeness. We ask a very basic question then; is the tropical variety always included inside the amoeba? Example 1.1.3 below will show that the answer to this basic question is negative.

Before continuing with the example, we would like to make several remarks. We note that, everything carried out with the min convention so far can be defined with the max convention as well. Same definitions with the max instead of the min, produce identical results. In the Archimedean setting, we would like to modify definitions of the objects as follows: max instead of min and *log* instead of *val*. That is, from this point on $trop(f)(w) = \max\{\log(|c_\alpha|) + w.\alpha\}$ and we define the tropical variety as follows

$$Trop(f) = \{w \in \mathbb{R}^n : \text{the max in } trop(f) \text{ is attained at least twice }\}$$

The example below shows that the Archimedean tropical variety is not always included inside the amoeba.

Example 1.1.9. *Define* $g(x) = 1 + x_1 + x_2 + \ldots + x_n$ *then we have* $Trop(g) \subsetneq \mathcal{A}_g$.

Proof. We first observe that $(\frac{-1}{n}, \frac{-1}{n}, \ldots, \frac{-1}{n})$ is a root of g. Hence $q = (\log(\left|\frac{1}{n}\right|), \ldots, \log(\left|\frac{1}{n}\right|))$ is a member of the amoeba. The second observation is that the Archimedean Newton polytope of g is

$$conv\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1, 0))\}$$

and the tropical variety is the set of outer normals of the Archimedean Newton polytope. Therefore, $Trop(g) \cap \mathbb{R}^{-n}$ is the boundary of the negative orthant. Hence the distance of q to the tropical variety is at least $\log(n)$. ■

Example was borrowed from [8]. The following theorem from [8], proves that the amoebae and the tropical varieties are metrically close.

Theorem 1.1.10. *(Avendaño, Kogan, Nisse, Rojas) Let* $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ *be a polynomial with* $t \geq n + 1$ *terms, and assume Newt(f) is of dimension* n*. Then, we have the following distance bounds*

1.

$$\sup_{r \in Amoeba(f)} \inf_{w \in Trop(f)} |r - w| \leq \log(t-1)$$

2.

$$\sup_{w \in Trop(f)} \inf_{r \in Amoeba(f)} |w - r| \leq (2t-3)\log(t-1)$$

A corollary of Theorem 1.1.10, will provide an answer to the limit shapes of the amoebae question which was posed in the previous section. In order to present the corollary, we need to define the Hausdorff distance.

Definition 1.1.11. *(Hausdorff Distance) For two subsets A and B of $\mathbb{R}^n$*

$$\Delta(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|\}$$

Corollary 1.1.12. *Let $f = \sum_u c_u x^u$ be a polynomial, and define $f_s = \sum_u c_u^{\log(s)} x^u$ be the family of polynomials polynomials parametrized by s. Then as $s \to \infty$ we have*

$$\Delta(Log_s(V(f_s)), Trop(f)) \to 0.$$

Proof. We first observe that $|c_u x^u| \geq |c_v x^v| \Leftrightarrow |c_u x^u|^{\log(s)} \geq |c_v x^v|^{\log(s)}$. Thus $Trop(f_s) = \log(s) Trop(f)$. Therefore by the theorem 1.1.10 we have

$$\Delta(\log(s) Log_s(V(f_s)), \log(s) Trop(f)) = \Delta(Log(V(f_s)), Trop(f_s)) \leq (2t-3)\log(t-1)$$

$$\Delta(Log_s(V(f_s)), Trop(f)) \leq \frac{(2t-3)\log(t-1)}{\log(s)}$$

■

Before passing to the next section, I would like to say a few words about the numerous applications of tropical geometry. Very first application of tropical geometry is to prove classical theorems of algebraic geometry via combinatorics. For instance, intersection theoretic properties of the initial variety carries through tropicalization very nicely , and this fact allows us to prove the theorems of Bezout and Bernstein just by sliding piecewise linear tropical curves [79] ! Approaching classical theorems of algebraic geometry with the tropical point of view has ben very fruitful; it expanded some classical theorems and it most cases provided new insights. This line of thought has far reaching conequences that we will not mention here. One other notorious application of the tropical geometry is due to Grigoriy Milhalkin. Milhalkin proved that the Gromov-Witten invariants of the plane can be studied using tropical geometry [83]. Finally, we would like mention that tropical geometry has interesting and deep links with the Berkovich Spaces [93]. This is by no means an exhaustive list, but we hope it gives a general idea about the applications of tropical geometry.

1.1.4 Real Part of the Roots of an Exponential Sum

In this section we would like to introduce our research problem on the metric relations between the amoebae and the tropical varieties for exponential sums. Exponential sums are a general family of functions that includes polynomials. We define an n-variate exponential sum, real part of it's zero set and the tropical variety as follows.

Definition 1.1.13. *We use the abbreviations* $[N]:=\{1,\ldots,N\}$*,* $w:=(w_1,\ldots,w_n)$*,* $z:=(z_1,\ldots,z_n)$*,* $w\cdot z:=w_1z_1+\cdots+w_nz_n$*, and* $\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$*. We also let* $\Re(z)$ *denote the vector whose* $i^{\underline{th}}$ *coordinate is the real part of* z_i*, and* $\Re(S):=\{\Re(z)\mid z\in S\}$ *for any subset* $S\subseteq\mathbb{C}^n$*. Henceforth, we let* $A:=\{a_1,\ldots,a_t\}\subset\mathbb{R}^n$ *have cardinality* $t\geq 2$*,* $b_j\in\mathbb{C}$ *for all* $j\in[t]$*, and set* $g(z):=\sum_{j=1}^t e^{a_j\cdot z+b_j}$*. We call* g *an* n-variate exponential t-sum *and call* A *the* spectrum *of* g. *We also call the* a_j *the* frequencies *of* g *and define their* minimal spacing *to be* $\delta(g) := \min_{p \neq q} |a_p - a_q|$ *where* $|\cdot|$ *denotes the standard* L^2*-norm on* $\mathbb{C}^n$. *Finally, let* $Z(g)$ *denote the zero set of* g *in* $\mathbb{C}^n$, *and define the* (Archimedean) tropical variety of g *to be*

$$\mathrm{Trop}(g) := \Re\left(\left\{z \in \mathbb{C}^n \ : \ \max_j \left|e^{a_j \cdot z + b_j}\right| \ \textit{is attained for at least two distinct } j\right\}\right). \quad \diamond$$

We hope at this point it is observable that the real part of the zero set for an exponential sum corresponds to the amoeba in the polynomial setting. This is mainly because the *Log* map is already 'built in' to the exponential sum and the real part corresponds to the magnitude.

We are interested in proving precise distance bounds between the real part of the zero set for an exponential sum and the tropical variety. The new subtlety is that the exponents in the spectrum can be arbitrarily close to each other. This brings new difficulties for approximating the real part of the zero set. For this reason, the $\delta(g)$ quantity is defined. Chapter 2 of this dissertation proves a theorem providing the desired distance bounds and also discuss it's algorithmic and theoretical applications.

1.2 Modern Convex Geometry and Semidefinite Programing

Optimization of multivariate polynomials is a fundamental problem that appears in numerous branches of basic science and engineering. Algorithmically, the optimization problem is equivalent to the certification of nonnegativity. Therefore finding algebraic certificates of nonnegativity is a central problem in both theory and applications.

One useful algebraic certificate of nonnegativity is the sums of squares representation. More precisely, if a polynomial $p(x) = \sum_i q_i(x)^2$ then p is called sums of squares of q_i. Clearly if a polynomial p is sums of squares of real polynomials then it is nonnegative on $\mathbb{R}^n$. The reverse of this assertion was studied by Hilbert [66].

Hilbert proved that every nonnegative polynomial with n variables and degree d is sums of squares of polynomials if and only if $n = 1$ or $d = 2$ or $n = 3, d = 4$.

An important aspect of the sums of squares representation is it's amenability to semidefinite programing. That is, whether a polynomial is sums of squares or not can be checked via semidefinite programing. We will introduce basics of semidefinite programing in the third subsection below and briefly discuss it's algorithmic efficiency.

Practical efficiency of the semidefinite programing motivates usage of the sums of squares representation as a relaxation to the optimization problem.

One clear question in this context is how much we loose by the sums of squares relaxation? Or what portion of non-negative polynomials are sums of squares? To make this question precise we need to introduce a bit of terminology.

Let $\mathbb{R}[\bar{x}] := \mathbb{R}[x_1, \ldots, x_n]$ denote the ring of real n-variate polynomials and let $P_{n,2k}$ denote the vector space of forms (i.e homogenous polynomials) of degree $2k$ in $\mathbb{R}[\bar{x}]$. A form $p \in P_{n,2k}$ is called *non-negative* if $p(\bar{x}) \geq 0$ for every $\bar{x} \in \mathbb{R}^n$. The set of *non-negative* forms in $P_{n,2k}$ is closed under nonnegative linear combinations and thus forms a cone. We denote the cone of nonnegative forms of degree $2k$ by $\mathrm{Pos}_{n,2k}$. Similarly, polynomials in $P_{n,2k}$ that can be represented as sums of squares of real polynomials form a cone that we denote by $\mathrm{Sq}_{n,2k}$.

Hilbert's Theorem proves that $\mathrm{Pos}_{n,2k} = \mathrm{Sq}_{n,2k}$ if and only if $n = 1$ or $d = 2$ or $n = 3, d = 4$. Hilbert's work in not constructive and for the cases of inequality it does not examine how close is $\mathrm{Sq}_{n,2k}$ to $\mathrm{Pos}_{n,2k}$. Gregoriy Blekherman developed a quantitative approach to the problem of comparing these two cones [17, 18]. In particular he proved the following theorem.

Theorem 1.2.1. *(G. Blekherman) Let* $C_{n,2k} := \{p \in P_{n,2k} \mid \int_S p \, d\sigma = 1\}$. *For any*

$X \subseteq P_{n,2k}$ we set $\mu(X) = \left(\frac{vol(X \cap C_{n,2k})}{vol(B)}\right)^{\frac{1}{D_{n,2k}}}$ where $D_{n,2k}$ is the dimension of $P_{n,2k}$ and B is the $D_{n,2k}$ dimensional Euclidean ball. Then the following estimates hold;

1.

$$\frac{1}{2\sqrt{4k+2}} n^{-\frac{1}{2}} \leq \mu(\mathrm{Pos}_{n,2k}) \leq 4\left(\frac{2k^2}{2k^2+n}\right)^{\frac{1}{2}}$$

2.

$$\frac{(k!)^2}{4^{2k}(2k)!\sqrt{24}} \frac{n^{\frac{k}{2}}}{(\frac{n}{2}+2k)^k} \leq \mu(\mathrm{Sq}_{n,2k}) \leq \frac{4^{2k}(2k)!\sqrt{24}}{k!} n^{-\frac{k}{2}}$$

In this dissertation we refine Blekherman's approach by considering the effect of monomial structure. Main novelty in our work is the modern convex geometry point of view. Following three subsections will introduce the necessary background from classical/modern convexity and semidefinite programing. The final subsection on quantitative aspects of Hilbert's 17th will introduce our research problem.

1.2.1 Santalo, Reverse Santalo and Uryshon Inequalities

Let K be a convex body (i.e convex and compact set with non-empty interior) in n dimensional real vector space V equipped with inner product $\langle \ , \ \rangle$. We define polar of a convex body K as follows.

$$K^\circ := \{x \in V : \langle x, y \rangle \leq 1 \text{ for every } y \in K\}$$

Let B_2^n be the Euclidean unit ball with respect to $\langle \ , \ \rangle$. Now we present a classical inequality of Santalo.

$$|K|\,|K^\circ| \leq |B_2^n|\,|B_2^n| \tag{1.1}$$

where $|.|$ denotes the n-dimensional volume.

It is also known that for any convex body, there exist a point z such that $|K|\,|(K-z)^{\circ}| \leq |B_2^n|\,|B_2^n|$. This point z is called the Santalo point. In the opposite direction, Mahler conjecture states that for every symmetric convex body the following statement holds.

$$|K|\,|K^{\circ}| \geq \frac{4^n}{n!} \tag{1.2}$$

Mahler's conjecture is verified in some special cases but in the general form it remains open. However, a remarkable inequality of Bourgain and Milman solves the conjecture asymptotically [24]. We present Bourgain and Milman's reverse Santalo inequality below.

Theorem 1.2.2. *There exists a constant $c>0$ such that for every convex body K in $\mathbb{R}^n$ that contains 0 in the interior the following holds*

$$(\frac{|K|}{|B_2^n|})^{\frac{1}{n}}(\frac{|K^{\circ}|}{|B_2^n|})^{\frac{1}{n}} \geq c \tag{1.3}$$

After the proof of Bourgain and Milman, several different proofs with improved universal constants are obtained. We refer the reader to the article [56] by Giannopoulos, Paouris and Vritsiou which nicely surveys related results and gives yet another proof of Reverse Santalo inequality. We would like to note that Santalo inequality also known as Blaschke-Santalo inequality is known since 1917 [16]. However the reverse form and it's numeruous proofs is discovered only after the development of modern convex geometry point of view.

Modern convex geometry or convex geometric analysis is an interplay of the ideas from the theory of Banach spaces and the ideas from classical convex geometry. The theory of Banach spaces is mainly concerned with the infinite dimensional normed spaces. Classical convex geometry is mainly concerned with convex objects in a finite dimensional Euclidean space. Around 60's, it was realized that there is a theory in 'between', which deals with finite but large dimensional Banach spaces and the effects of dimension as it grows. Since then, this approach uncovered deep relations between the analysis point of view of the Banach space theory and the geometric point of view of convex geometry. In this dissertation we'll be using different tools of convex geometric analysis without formally introducing basics of the field. For a nice survey by two masters of the subject, we refer the reader to [54].

Below, we present a classical inequality of Urysohn. As in the case of the Santalo inequality, reverse forms of Urysohn's inequality are studied by convex geometric analysis researchers [55].

Theorem 1.2.3. *(Urysohn Inequality) Let K be a convex body in $\mathbb{R}^n$ and let the support function of K be $h_K(u) = \max_{x \in K}\langle x, u\rangle$. Then we define width of K in the direction $u \in S^{n-1}$ as $w_K(u) = h_K(u) + h_K(-u)$. The mean width of K is defined as follows*

$$w(K) = \int_{S^{n-1}} w_K(u)\ \sigma(u) = 2 \int_{S^{n-1}} h_K(u)$$

.

Then we have the following inequality

$$(\frac{|K|}{|B_2^n|})^{\frac{1}{n}} \leq w(K)$$

1.2.2 John's Theorem, Brascamp Lieb Inequality and Isotropic Measures

Every convex body K includes an ellipsoid of maximal volume. This was proved by Fritz John with a complete characterization of the cases where this maximal ellipsoid is the Euclidean ball. For the proof of John's theorem and the basics of modern convex geometry, we refer the reader to a masterpiece of mathematical exposition by Keith Ball [10].

Theorem 1.2.4. *(John's Theorem) Every convex body K contains an ellipsoid of maximal volume. This ellipsoid is B_2^n if and only if the following conditions are satisfied: $B_2^n \subset K$, there are unit vectors $(u_i)_{i=1}^m$ on the boundary of K and positive real numbers c_i satisfying*

$$\sum_i c_i u_i = 0$$

and for all $x \in \mathbb{R}^n$

$$\sum_i c_i \langle x, u_i \rangle^2 = \|x\|_2^2$$

The first condition in John's theorem guarantees that the vectors u_i are not all in one side of the Euclidean ball. In other words, weighted average of the vectors is 0. As one can see from the lemma below, the second condition of John's Theorem can be viewed in different ways. The lemma below is standard and it's proof can be found in any textbook on frame theory.

Lemma 1.2.5. *We denote the map which sends x to $\langle x, u \rangle u$ by $u \otimes u$. Then the following are equivalent*

1. $$I = \sum_i c_i u_i \otimes u_i$$

2. *For every $x \in \mathbb{R}^n$*

$$x = \sum_i c_i \langle x, u_i \rangle u_i$$

3. For every $x \in \mathbb{R}^n$

$$\sum_i c_i \langle x, u_i \rangle^2 = \|x\|_2^2$$

Now we present an important functional inequality due to Brascamp and Lieb.

Theorem 1.2.6. *(Brascamp-Lieb Inequality) Let* $n, m \geq 1$ *and* $p_1, \ldots, p_m > 0$ *be such that* $\sum_i \frac{1}{p_i} = n$*. Let* $f_1, f_2, \ldots, f_m : \mathbb{R} \to \mathbb{R}^+$ *be integrable functions,* $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$ *then*

$$\int\limits_{\mathbb{R}^m} \prod_i f_i(\langle v_i, x \rangle)\, dx \leq D \prod_i \|f_i\|_{p_i}$$

where D *is universal and the equality is attained if* f_i *are all centered Gaussian functions.*

The theorem of Brascamp and Lieb is a very strong and important inequality. However, precise equality conditions of the inequality are quite complicated. Keith Ball had the observation that the conditions of John's theorem is amenable to the Brascamp-Lieb inequality. This observation led him to produce a geometric version of the Brascamp-Lieb inequality [11]. We state the theorem of Keith Ball below.

Theorem 1.2.7. *Let* $n, m \geq 1$ *and let* $u_1, u_2, \ldots, u_m \in S^{n-1}$*,* $c_1, c_2, \ldots, c_m > 0$ *be such that* $I = \sum_i c_i u_i \otimes u_i$*. Let* $f_i : \mathbb{R} \to \mathbb{R}^+$ *be integrable functions. Then the following holds true*

$$\int\limits_{\mathbb{R}^n} \prod_i^m f_i(\langle x, u_i \rangle)^{c_i}\, dx \leq \prod_i^m \left(\int\limits_{\mathbb{R}} f_i \right)^{c_i}$$

Remark 1.2.1. *In theorem 1.2.7 if we let* $f_i = e^{-\alpha t^2}$ *then by Lemma 1.2.5 we have that*

$$\prod_i f_i(\langle x, u_i\rangle)^{c_i} = \prod_i e^{-\alpha(c_i\langle x,u_i\rangle^2)} = e^{-\alpha\|x\|_2^2} = \prod_i^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}$$

Therefore for Theorem 1.2.7, $D = 1$ and the equality conditions are much simpler than the original Brascamp-Lieb inequality.

We feel compelled to mention that in [11] Ball also proves sharp reverse isoperimetric inequalities. His proof is based on 'isomorphic' point of view modern convexity and the theorem 1.2.7 proves very useful. For more details and precise statements, the reader is invited to read the very nice article of Ball.

Now, we would like to make a slight change in our point of view. We would like to see the vectors u_i and positive real numbers c_i in Lemma 1.2.5 as a discrete measure on the sphere. First we calculate the 'measure' of the sphere. The following observation is useful; if we take traces in both sides of the first item in Lemma 1.2.5 we have

$$n = Trace(I) = Trace(\sum_i c_i u_i \otimes u_i) = \sum_i c_i$$

Thus, we need to divide c_i by n to define a discrete probability measure on the sphere. The resulting measure will be supported only at points u_i and the measure of u_i will be $\frac{c_i}{n}$.

At this point, a clear question is what would be a continuous analog of the conditions of Lemma 1.2.5?

We believe that the following definition gives a satisfactory answer.

Definition 1.2.8. *(Isotropic Measure on the Sphere) A measure Z on S^{n-1} is* *isotropic if for every $x \in \mathbb{R}^n$ we have*

$$\|x\|_2^2 = \int_{S^{n-1}} \langle x, y\rangle^2 \, dZ(y)$$

Building on the ideas of Ball and Barthe, the three authors Lutwak, Yhang and Zhang proved the following theorem [78].

Theorem 1.2.9. *(Lutwak, Yhang, Zhang) If Z is an isotropic measure on S^{n-1} whose centroid is at the origin and $Z_\infty = \mathrm{Conv}(\mathrm{Supp}(Z))$, then we have*

$$\mathrm{Vol}(Z_\infty^\circ) \leq \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!}$$

Here the main intuition is that, with the correct inequalities (i.e continous versions of theorem 1.2.7) support of the isotropic measure behaves like the touching points of Jonh's ellipsoid. We are going to use Theorem 1.2.9 to prove volume bounds on the cone of nonnegative polynomials. Our main idea is to create an isotropic measure linked to the nonnegative polynomials.

1.2.3 Semidefinite Programing

In this section, we introduce basics of semidefinite programing and relate the technique to the polynomial optimization. As a refresher, let us recall the linear programing problem in standard form.

$$\begin{aligned} \text{minimize} \ \ & c^T x \\ \text{subject to} \ & Ax = b \\ & x \geq 0 \end{aligned}$$

In this formulation of linear programing, the feasible set of $x \in \mathbb{R}^n$ is the intersection of the positive orthant with the affine subspaces given by $Ax = b$. Thus, by definition the feasible set is a polyhedron. If the feasible set is bounded that means it is a polytope. Hence, linear programing problem is essentially optimization of a linear function over a polyhedron.

The idea of semidefinite programing is to change the variable $x \in \mathbb{R}^n$ to a matrix, and therefore optimize a linear function over a (preferably convex) set of matrices. To make this idea precise, we need to define linear inequalities on matrices and the corresponding polyhedra-like objects.

Definition 1.2.10. *(Linear Matrix Inequality) Let S_+^n be the set of $n \times n$ positive semidefinite matrices and $\succ$ be the order induced by S_+^n. That is $A \succ B$ if $A - B \in S_+^n$. Then a linear matrix inequality (LMI) has the form*

$$A_0 + \sum_i A_i x_i \succ 0$$

where $A_i \in S^n$ are $n \times n$ real symmetric matrices.

Definition 1.2.11. *(Spectrahedron) A set V is a spectrahedron if it has the following form*

$$V := \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : A_0 + \sum_{i=1}^{m} A_i x_i \succ 0\}$$

Spectrahedra will be the feasible set for semidefinite programs (SDP) as polyhedra is the feasible set to the linear programs (LP). Below we borrow an example from [89].

Example 1.2.12. *Consider the spectrahedron in $\mathbb{R}^2$ given by*

$$V := \{(x, y) : A(x, y) \succ 0\}$$

where $A(x,y)$ is given by

$$\begin{bmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{bmatrix}$$

Then calculating determinant of $A(x,y)$ shows that the boundary of V is given by elliptic curve

$$3 + x - x^3 - 3x^2 - 2y^2 = 0$$

and the complete description of the spectrahedron is given by the following inequlaties

1. $x + 5 \geq 0$

2. $-x^2 + 2x - y^2 + 7 \geq 0$

3. $3 + x - x^3 - 3x^2 - 2y^2 \geq 0$

■

A semidefinite program in it's standard form is the following.

$$\begin{aligned} &\text{minimize } \ \langle C, X \rangle \\ &\text{subject to } \langle A_i, X \rangle = b_i \text{ for } i = 1, 2, \ldots, m \\ &\qquad\qquad X \succ 0 \end{aligned}$$

Note that in the SDP formulation, we used inner product on matrices defined by $\langle A, B \rangle = Trace(A^T B)$.

The feasible set for a semidefinite program is intersection of the cone of positive semidefinite matrices with the hyperplanes defined by $\langle A_i, X \rangle = b_i$. Thus, by definition it is convex and actually it is defined by constraints of a spectrahedron. Hence, semidefinite programing is optimization of a linear functional over a spectrahedron. This brings a host of questions that we will not cover here; which sets can be represented as spectrahedrons? Are spectrahedrons closed under projection? If not, which sets are projected spectrahedra? Given a spectrahedra how do we efficiently compute the defining LMI's?

A new area of mathematics that strives to answer these questions is called convex algebraic geometry. Up to authors knowledge, there is only one book written in this area, and luckily it is a good book! We refer the reader to [19] for delving deep into convex algebraic geometry.

We would like to briefly mention algorithmic efficiency of semidefinite programing. On the theoretical side, ruling out doubly exponentially small solutions, ellipsoid method guarantees semidefinite programing to work in polynomial time. However, despite it's theoretical strength in practice ellipsoid method is often too slow. This fact led to development of SDP algorithms based on the interior point methods. Currently, there are several different SDP algorithms and implemented software packages. We refer the reader to [89] Section 2.3.1 for a survey of current software packages. In conclusion, theoretical complexity of semidefinite programing is not completely clear. However, in practice it is widely accepted as an efficient method.

Let p be a n-variate degree $2d$ polynomial such that $p = \sum_i q_i^2$ where q_i are real polynomials of degree d with n variables. We consider each q_i represented as a vector in the basis of n-variate monomials with degree at most d. Note that this basis has $\binom{n+d}{d}$ elements. We set the vector X_d to be vector of all these monomials. Then in this basis, q_i^2 is represented with a rank 1 matrix $A_i = q_i \otimes q_i$ with

$$q_i^2 = X_d^T A_i X_d$$

Therefore, in this basis p is represented with a positive semidefinite matrix $A = \sum_i A_i$ as $p = X_d^t A X_d$. We believe in the reader that she is able to observe this line of reasoning is reversible. Therefore the representation of a sums of squares polynomial with a positive semidefinite matrix is an if and only if criterion. Hence, certifying whether a polynomial is sums of squares can be done via semidefinite programing.

1.2.4 Quantitative Aspects of Hilbert's 17th Problem

We start with recalling a theorem of Rezncik [97, Thm. 1].

Theorem 1.2.13. *(Reznick) If* $p = \sum_{i=1}^{r} g_i^2$ *for some* $g_1, \ldots, g_r \in \mathbb{R}[\bar{x}]$ *then* $\mathrm{Newt}(g_i) \subseteq \frac{1}{2}\mathrm{Newt}(p)$ *for all* i*.*

Reznick's theorem clearly enables us to ask questions related to sums of squares decomposition in the more refined setting of Newton polytopes. Our aim is to compare the cone of nonnegative polynomials and the cone of sums of squares for polynomials with a given Newton polytope. As a first step, we considered the case of multihomogenous polynomials.

Definition 1.2.14. *Assume henceforth that* $n = n_1 + \cdots + n_m$ *and* $k = k_1 + \cdots + k_m$*, with* $k_i, n_i \in \mathbb{N}$ *for all* i*, and set* $N := (n_1, \ldots, n_m)$ *and* $K := (k_1, \ldots, k_m)$*. We will partition the vector* $\bar{x} = (x_1, \ldots, x_n)$ *into* m *sub-vectors* $\bar{x}_1, \ldots, \bar{x}_m$ *so that* $\bar{x}_i$ *consists of exactly* n_i *variables for all* i*, and say that* $p \in \mathbb{R}[\bar{x}]$ *is* homogenous of type (N, K) *if and only if* p *ishomogenous of degree* k_i *with respect to* $\bar{x}_i$ *for all* i*. Finally, let* $Q_{N,K} := Q_{n_1,k_1} \times \cdots \times Q_{n_m,k_m}$*.* $\diamond$

Example 1.2.15. $p(\bar{x}) := x_1^3 x_4^2 + x_1 x_2^2 x_5^2 + x_3^3 x_4 x_5$ *is homogenous of type* (N, K) *with* $N = (3, 2)$ *and* $K = (2, 3)$*. (So* $\bar{x}_1 = (x_1, x_2, x_3)$ *and* $\bar{x}_2 = (x_4, x_5)$*.) In particular,* $\mathrm{Newt}(p) \subseteq Q_{N,K} = Q_{3,2} \times Q_{2,3}$*.* $\diamond$

We aim to develop a quantitative comparison of the multihomogenous nonnegative polynomials and sums of squares. In particular, we would like to develop a localized version of the theorem 1.2.1. Third chapter of this dissertation presents our quantitative theorem on the multihomogenous nonnegative polynomials and sums of squares.

1.3 Efficient Polynomial System Solving

In theory, every linear algebra student knows how to solve the system of linear equations $Ax = b$. It is just the Gaussian elimination! In practice however, controlling the round of errors; exploiting structures such Toeplitz matrices; taking advantage of sparsity; and using the randomized methods; occupy a vast literature in numerical linear algebra. Now we consider the following example; which is a non-linear counterpart of the equation $Ax = b$.

Example 1.3.1.

$$f_1 = 10500t - t^2u^{492} - 3500u^{463}v^5w^5$$

$$f_2 = 10500t - t^2 - 3500u^{691}v^5w^5$$

$$f_3 = 14000t - 2t^2 + t^2u^{492} - 2500tv$$

$$f_4 = 14000t + 2t^2 - t^2u^{492} - 3500tw$$

In theory, students of undergraduate algebraic curves class know that the system of equations $f = (f_1, f_2, f_3, f_4)$ has at most $501 \times 701 \times 494 \times 494 = 85705687236$ common complex roots. Bernstein's Theorem mentioned in the first subsection of this section, improves this bound to 7663 on $(\mathbb{C}^*)^n$. If we ask a bound for the number of common roots in positive real orthant, Gale duality shows that it is only six [15]. In practice, efficiently finding these finitely many roots occupy a good portion of literature in computational algebraic geometry.

1.3.1 Smale's 17th Problem

Around the turn of millennium, Vladimir Arnold wrote to several mathematicians asking a list of problems for 21st century. Arnold's aim was to create a list for the 21st century mathematicians which can serve as Hilbert's list of problems for the mathematicians of 20th century. In his response to this call, Steve Smale included the following problem as the 17th [116].

Smale's 17th Problem. *Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?*

To make this problem precise, we need to define three words: approximate, average and uniform. Uniform is a technical term that is precisely defined in computational complexity literature. The reader can safely assume it means an algorithm that has precisely described steps which works for all polynomials with any number of variables n and any degree d. We will explain the 'average' word in the last subsection on random polynomial systems. We give Smale's definition of 'approximate root' below.

Definition 1.3.2. *(Approximate Root) Let $f = (f_1, f_2, \ldots, f_n)$ be a polynomial system, we consider the system as a map $f : \mathbb{C}^n \to \mathbb{C}^n$. Multivariate Newton iteration of f at x is defined as follows*

$$N(f, x) = x - Df(x)^{-1} f(x)$$

*where $Df(x)$ is Jacobian matrix of f at x. An **approximate zero** associated to ζ, $f(\zeta) = 0$ is a point x_0 such that*

1. *The sequence defined inductively by* $x_{i+1} = N(f, x_i)$ *is well defined*

2. $\|x_{i+1} - x_i\| \leq 2^{-2^i+1} \|x_1 - x_0\|$

3. $\lim_i x_i = \zeta$

In fact, Smale defined two types of approximate roots and this is the second one. We preferred to give the second definition because it is made efficient with consequent theoretical developments.

Shub and Smale developed tools to check if a given point z is an approximate root for a polynomial system f [109, 110, 111]. This tools are called Smale's α theory. Currently, there is a software package due to Sottile and Hauenstein which implements tools of α theory [65].

Steve Smale in collaboration with Mike Shub, wrote series of articles on the 17th problem [109, 110, 111]. These articles are titled Complexity of Bezout's Theorem I, II, III, IV and V. In these series of papers, several novel ideas are introduced including the mentioned development of α theory and a clarification of the metric structure of deformation paths between polynomial systems. We would like briefly explain the idea of deformation paths between polynomial systems.

Assume there is a polynomial system $g = (g_1, g_2, \ldots, g_n)$ for which it is easy to find common roots. For instance, g_i can be a polynomials with only two terms. Wlog say $deg(g_i) = d_i$. Now, suppose we want to solve a polynomial system $f = (f_1, f_2, \ldots, f_n)$ where $deg(f_i) = d_i$. We define $h_t = (1-t)g + tf$. At a given t, h_t is a polynomial system with $deg((h_t)_i) = d_i$, $h_0 = g$ and $h_1 = f$. Now let z_0 be a root of g. If z_0 does not possess multiplicity greater than 1, we know that $Dg(z_0)$ is full rank, and we have the implicit function theorem. Therefore, around z_0 there is a continuous function between coefficients of h_t and a root of h_t, say z_t. Iterating this idea, one can prove existence of a continuous path from (h_0, z_0) to (h_1, z_1). The only necessary condition is along the path, for every t, h_t should have all roots isolated (i.e multiplicity one). Using language of the discriminants, we can also say that the only condition along the path id h_t should not be a member of the discriminant variety.

For generic set of coefficients, by Bezout's theorem $g = h_0$ has $d_1 d_2 \ldots d_n$ many roots. Therefore, there will be $d_1 d_2 \ldots d_n$ many paths to follow between roots of h_0 to h_1. This is the reason for the title of the series of papers by Shub and Smale.

After this conceptual step, the clear question is how can we track all these root paths numerically? Shub and Smale proved that one can track these paths by the multivariate Newton iteration! This statement will be made precise in the next section. However, the following question should be clear; what is the complexity of the algorithm that tracks $d_1 d_2 \ldots d_n$ many paths using the multivariate Newton iteration? Answer to this question is given by a fundamental invariant of the polynomials on the path; the condition number of h_t. We invite the reader to the next section for an introduction to notion of conditioning in numerical analysis.

1.3.2 Condition Number

Computational complexity of a numerical problem depends on the sensitivity of the answer to the small changes in the input. This is formalized in the notion of condition number. Every numerical problem has a corresponding condition number, and this invariant is vital for any realistic complexity analysis of algorithms.

Consider the case of linear equation solving; $Ax = b$. For now, assume that we have a parametrized system with respect to t; $A(t)x(t) = b(t)$. Differentiating implicitly we have

$$\dot{A}x + A(\dot{x}) = \dot{b}$$

$$\dot{x} = A^{-1}\dot{b} - A^{-1}\dot{A}x$$

If we take norms of both sides and follow routine manipulations we have

$$\frac{\|\dot{x}\|}{\|x\|} \leq \left\|A^{-1}\right\|_2 \|A\|_F \left(\frac{\left\|\dot{A}\right\|_F}{\|A\|_F} + \frac{\left\|\dot{b}\right\|}{\|b\|} \right)$$

where we use $\|\|_2$ for the operator norm and $\|\|_F$ for the Frobenius norm. Therefore for the case of linear equation solving, the quantity $\kappa(A) = \|A^{-1}\|_2 \|A\|_F$ provides an upper bound to the relative change in the output in terms of the relative change in the input. Thus, condition number of a matrix A is defined to be $\kappa(A) = \|A^{-1}\|_2 \|A\|_F$.

For conceptual understanding, one would like to see the condition number quantity geometrically. Following theorem of Eckart and Young provides the desired geometric insight [44].

Theorem 1.3.3. *(Eckart-Young) Let A be an $n \times n$ complex matrix, then*

$$\kappa(A) = \frac{\|A\|_F}{d(A, \Sigma)}$$

where $\Sigma = \{M \in \mathbb{C}^{n^2} : det(M) = 0\}$ and d is the usual Euclidean distance.

Condition number of a problem being proportional to reciprocal of the distance to the set of ill posed problems is a general phenomenon in numerical analysis. For a nice exposition of this phenomenon, we refer the reader to [39]. In the case of linear equation solving the set of ill posed problems are the set of matrices that we can not take the inverse, which is precisely the set Σ.

For polynomial system solving, the set of ill posed problems are the polynomial systems that possesses a singularity i.e members of the discriminant variety. Therefore we expect a condition number notion which is proportional to reciprocal of distance to the discriminant variety. Shub and Smale defined such a condition number for polynomial system solving on the field of complex numbers [109]. However, in this thesis we are mainly interested in finding real roots of polynomial systems. Luckily a correct variant of condition number for real root finding is defined by Cucker, Krick, Malajovich and Wshebore in their series of papers [32, 33, 34].

Before closing this section we would like to present a theorem from Complexity of Bezout's Theorem VI by Mike Shub. Let H_D be the space of polynomial systems $h = (h_1, h_2, \ldots, h_n)$ where $deg(h_i) = d_i$. Let Σ be the discriminant variety in H_D and assume h_t is a path in $H_D \setminus \Sigma$. We perform Newton iteration on the path h_t as follows:

$$z_{a+t_0} = z_a - (Dh_{a+t_0}(z_a))^{-1} h_{a+t_0}(z_a)$$

We denote the condition number (in Shub-Smale sense) of system h_t at ζ_t as $\mu(h_t, \zeta_t)$. The following theorem of Shub proves that the number of Newton iterations to track a path from h_0 to h_1 is bounded by the 'condition number length' of the path.

Theorem 1.3.4. *(M. Shub) Let $h_t \subseteq H_D \setminus \Sigma$ where $t \in [a, b]$ be a C^1 path. If the steps $t_o, t_1, \ldots$ are correctly chosen, then approximate root of h_b is achieved at some point, namely there exists a k such that $\sum_{i=1}^{k} t_i = b - a$. Moreover, one can bound*

$$k \leq C d^{\frac{3}{2}} L_\kappa$$

where $d = \max_i d_i$, C is a universal constant, and

$$L_\kappa = \int_a^b \mu(h_t, z_t) \left\| (\dot{h_t}, \dot{z}_t) \right\| \, dt$$

Moreover the amount of arithmetic operations needed in each step is polynomial in terms of $\dim(H_D)$*, and hence the total complexity of following* h_t *is polynomial in* $\dim(H_D)$ *and linear in* L_κ*.*

1.3.3 Small Ball Probabilities

Our work on random polynomials and parts of our work in Chapter 2 relies on probabilistic estimates which are called the 'small ball probabilities'. In this section, we would like to briefly introduce the idea of small ball probabilities and present some basic small ball type estimates.

Perhaps the earliest appearance of small ball probabilities is in the work Littlewood and Offord, and later strengthened by Erdös [45]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent identically distributed random variables, $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, $\epsilon > 0$, the basic problem about small ball probabilities (Littlewood-Offord problem) is to estimate the following quantity:

$$p_\epsilon(a) = \sup_{u \in \mathbb{R}} \mathbb{P}\left\{ \left| \sum_i a_i \xi_i - u \right| < \epsilon \right\}$$

Erdös proved that if ξ_i are random signs which takes ∓ 1 with probability $\frac{1}{2}$ and $a \in \mathbb{R}^n$ is a determistic vector such that $|a_i| > 1$ then,

$$p_1(a) \sim n^{-\frac{1}{2}}.$$

It is known that for ξ_i centered Gaussian random variables with variance 1,

$$p_\epsilon(a) \sim \frac{\epsilon}{\|a\|_2}.$$

Work of Erdós was motivated by combinatorial problems and was not aimed to build up a probabilistic theory. Until recently, it was not so clear that small ball probabilities, i.e estimates on the function $p_\epsilon(a)$, is a very fundamental aspect of the random variables ξ_i. Small ball probabilities are now becoming common in different aspects of probability theory mainly with the contributions random matrix theory and asymptotic convex geometry researchers. I would like to list two different types of small ball probabilities as examples of the general framework. First example is due to Rudelson and Vershynin [101]. We need to introduce a piece of terminology.

Definition 1.3.5. *(Essential LCD) Let $\alpha \in (0,1)$ and $\kappa \geq 0$. The essential least common denominator $LCD_{\alpha,\kappa}(a)$ of a vector $a \in \mathbb{R}^n$ is defined as the infimum of $t > 0$ such that all except κ coordinates of the vector ta are of distance at most α from nonzero integers.*

Theorem 1.3.6. *(Rudelson-Vershynin) Let ξ_i be centered identically distributed random variables with variances at least one and third moment bounded by B. Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ such that there exists $K_1, K_2 > 0$ with*

$$K_1 < |a_i| < K_2$$

for all i. Let $\alpha \in (0,1)$ and $\kappa \in (0,n)$. Then for every $\epsilon \geq 0$ one has

$$p_\epsilon(a) \leq \frac{C}{\kappa}\left(\epsilon + \frac{1}{LCD_{\alpha,\kappa}(a)}\right) + Ce^{-c\alpha^2\kappa}$$

where $C, c > 0$ depend polynomially only on B, K_1, K_2.

Our second example concerns k dimensional subspaces of $\mathbb{R}^n$. Let $G_{n,k}$ be the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$, equipped with its unique rotation-invariant Haar probability measure $\mu_{n,k}$. Then the following 'small ball probability' type estimate holds.

Lemma 1.3.7. *[52, Lemma 3.2] Let* $1 \leq k \leq n-1$, $x \in \mathbb{R}^n$, *and* $\varepsilon \leq \frac{1}{\sqrt{e}}$. *Then*

$$\mu_{n,k}\left(\left\{F \in G_{n,k} \;\middle|\; |P_F(x)| \leq \varepsilon\sqrt{\frac{k}{n}}|x|\right\}\right) \leq \left(\sqrt{e}\varepsilon\right)^k,$$

where P_F *is the surjective orthogonal projection mapping* $\mathbb{R}^n$ *onto* F. ■

Even though we provided a reference for this lemma, it is actually folklore in convex geometric analysis literature and it is hard to find a reference with a complete proof.

1.3.4 Condition Number of Random Polynomial Systems

It has been known since the 80's that deciding whether a polynomial system has real root or not is NP-Hard (see, e.g., [27]). Therefore one does not expect to develop an efficient exact polynomial time algorithm for finding the real roots of a polynomial system. However, based on the numerical ideas introduced in Smale's 17th problem section it is possible to develop algorithms for real root finding. Note that this algorithms are always analyzed in terms of the corresponding condition number. Therefore, one expects the intrinsic complexity of the problem to be transferred into the corresponding condition number. The word 'average' in Smale's 17th problem gives a way out: one analyzes algorithms and thus the corresponding condition number 'on average'.

Explaining the word 'on average', Smale wrote the following [116]: " A probability measure must be put on the space of all such f, for each $d = (d_1, d_2, \ldots, d_n)$ and the time of the algorithm is averaged over the space of f ".

Smale did not specify the probability measure on the input space. In Smale 17th problem literature, the probability measure is always taken to be consisting of independent Gaussians coordinates with specially chosen variances so that the measure remains invariant under the action of orthogonal group. These strong assumptions makes it questionable that if the resulting complexity analysis is a property of the analyzed algorithm or more of a property of strongly symmetric probability measure.

In chapter 4 we provide estimates for (real) condition number of random polynomial systems for a broader family of distributions. We do not assume orthogonal invariance of the the distribution which allows refined analysis for sparse polynomials. Main novelty of our approach is usage of small ball probabilities instead of analytic formula for random fields.

2. TROPICAL VARIETIES FOR EXPONENTIAL SUMS

2.1 Introduction

Since the late 20th century (see, e.g., [123, 70, 72]) it has been known that many of the quantitative results relating algebraic sets and polyhedral geometry can be extended to more general analytic functions, including exponential sums. Here, we show that the recent estimates on the distance between amoebae and Archimedean tropical varieties from [8] admit such an extension. Metric estimates for amoebae of polynomials are useful for coarse approximation of solution sets of polynomial systems, as a step toward finer approximation via, say, homotopy methods (see, e.g., [5, 64]). Polynomial systems are ubiquitous in numerous applications, and via a logarithmic change of variables, are clearly equivalent to systems of exponential sums with integer frequencies. Exponential sums with real frequencies are important in Signal Processing, Model Theory, and 3-manifold invariants (see Remark 2.1.1 below).[1]

Definition 2.1.1. *We use the abbreviations* $[N]:=\{1,\ldots,N\}$, $w:=(w_1,\ldots,w_n)$, $z:=(z_1,\ldots,z_n)$, $w\cdot z:=w_1z_1+\cdots+w_nz_n$, *and* $\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$. *We also let* $\Re(z)$ *denote the vector whose* $i^{\underline{th}}$ *coordinate is the real part of* z_i, *and* $\Re(S):=\{\Re(z)\mid z\in S\}$ *for any subset* $S\subseteq\mathbb{C}^n$. *Henceforth, we let* $A:=\{a_1,\ldots,a_t\}\subset\mathbb{R}^n$ *have cardinality* $t\geq 2$, $b_j\in\mathbb{C}$ *for all* $j\in[t]$, *and set* $g(z):=\sum_{j=1}^{t}e^{a_j\cdot z+b_j}$. *We call* g *an* n-variate exponential t-sum *and call* A *the* spectrum *of* g. *We also call the* a_j *the* frequencies *of* g *and define their* minimal spacing *to be* $\delta(g):=\min_{p\neq q}|a_p-a_q|$ *where* $|\cdot|$ *denotes the standard* L^2*-norm on* $\mathbb{C}^n$. *Finally, let* $Z(g)$ *denote the zero set of* g *in* $\mathbb{C}^n$, *and*

[1] Lest there be any confusion, let us immediately clarify that we do *not* consider terms of the form $e^{p(x)}$ with p a polynomial of degree ≥ 2. The latter type of exponential sums are of great importance in analytic number theory and the study of zeta functions.

define the (Archimedean) tropical variety of g *to be*

$$\mathrm{Trop}(g) := \Re\left(\left\{z \in \mathbb{C}^n \ : \ \max_j \left|e^{a_j \cdot z + b_j}\right| \ \textit{is attained for at least two distinct } j\right\}\right). \quad \diamond$$

Note that while we restrict to real frequencies for our exponential sums, we allow complex coefficients. $\mathrm{Trop}(g)$ also admits an equivalent (and quite tractable) definition as the dual of a polyhedral subdivision of A depending on the real parts of the b_j (see Thm. 2.1.10 and Prop. 2.2.4 below).

Example 2.1.2. *When* $n=1$ *and* $g(z) = e^{\sqrt{2}z_1} + e^{\log(3)+\pi\sqrt{-1}}$, *we see that* $Z(g)$ *is a countable, discrete, and unbounded subset of the vertical line* $\left\{z_1 \in \mathbb{C} \mid \Re(z_1) = \frac{\log 3}{\sqrt{2}}\right\}$. *So* $\Re(Z(g)) = \left\{\frac{\log 3}{\sqrt{2}}\right\}$. $\diamond$

Example 2.1.3. *When* $g(z) := e^{a_1 z_1 + b_1} + e^{a_2 z_1 + b_2}$ *for some distinct* $a_1, a_2 \in \mathbb{R}$ *(and any* $b_1, b_2 \in \mathbb{C}$*) it is easily checked that* $\mathrm{Trop}(g) = \Re(Z(g)) = \left\{\frac{\Re(b_1 - b_2)}{a_2 - a_1}\right\}$. *More generally, for any* n*-variate exponential* 2*-sum* g, $\mathrm{Trop}(g)$ *and* $\Re(Z(g))$ *are the same affine hyperplane. However, the univariate exponential* 3*-sum* $g(z_1) := (e^{z_1} + 1)^2$ *gives us* $\mathrm{Trop}(g) = \{\pm \log 2\}$, *which is neither contained in, nor has the same number of points, as* $\Re(Z(g)) = \{0\}$. $\diamond$

When $A \subset \mathbb{Z}^n$, $\Re(Z(g))$ is the image of the complex zero set of the polynomial $\sum_{j=1}^t e^{b_j} x^{a_j}$ under the coordinate-wise log-absolute value map, i.e., an *amoeba* [51]. Piecewise linear approximations for amoebae date back to work of Viro [122] and, in the univariate case, Ostrowski [88]. More recently, Alessandrini has associated piecewise linear approximations to log-limit sets of semi-algebraic sets and definable sets in an o-minimal stucture [3]. However, other than Definition 2.1.1 here, we are unaware of any earlier formulation of such approximations for real parts of complex zero sets of n-variate exponential sums.

Our first main results are simple and explicit bounds for how well $\mathrm{Trop}(g)$ approximates $\Re(Z(g))$, in arbitrary dimension.

Definition 2.1.4. *Given any subsets* $R,S\subseteq\mathbb{R}^n$*, their* Hausdorff distance *is*

$$\Delta(R,S):=\max\left\{\sup_{r\in R}\inf_{s\in S}|r-s|,\sup_{s\in S}\inf_{r\in R}|r-s|\right\}. \qquad \diamond$$

Theorem 2.1.5. *For any* n*-variate exponential* t*-sum* $g(z):=\sum_{j=1}^{t}e^{a_j\cdot z+b_j}$ *with* $a_j\in\mathbb{R}^n$ *and* $b_j\in\mathbb{C}$ *for all* j*, let* d *be the dimension of the smallest affine subspace containing* $a_1,\ldots,a_t$*, and set* $\delta(g):=\min_{p\neq q}|a_p-a_q|$*. Then* $t\geq d+1$ *and*

(0) If $t=d+1$ *then* $\mathrm{Trop}(g)\subseteq\Re(Z(g))$ *(and thus* $\sup\limits_{w\,\in\,\mathrm{Trop}(g)}\inf\limits_{r\,\in\,\Re(Z(g))}|r-w|=0$*).*

(1) For $t\geq 2$ *we have:*

(a) $\sup\limits_{r\,\in\,\Re(Z(g))}\inf\limits_{w\,\in\,\mathrm{Trop}(g)}|r-w|\leq\log(t-1)/\delta(g)$

(b) $\Delta(\Re(Z(g)),\mathrm{Trop}(g))\leq\frac{\sqrt{ed}t^2(2t-3)\log 3}{\delta(g)}$.

(2) Defining the n*-variate exponential* t*-sum* $g_{t,n}(x):=(e^{\delta z_1}+1)^{t-n}+e^{\delta z_2}+\cdots+e^{\delta z_n}$*, we have*

$$\Delta(\Re(Z(g_{t,n})),\mathrm{Trop}(g_{t,n}))\geq\log(t-n)/\delta$$

for $t\geq n+1$ *and* $\delta>0$*.*

We prove Theorem 2.1.5 in Section 2.4. Fundamental results on the geometric and topological structure of $\Re(Z(g))$ have been derived in recent decades by Favorov and Silipo [46, 113]. However, we are unaware of any earlier explicit bounds for the distance between $\Re(Z(g))$ and $\mathrm{Trop}(g)$ when $A\not\subset\mathbb{Z}^n$.

The special case $A\subset\mathbb{Z}^n$ of Theorem 2.1.5 was known earlier, with a bound independent of n: Our $\mathrm{Trop}(g)$ agrees with the older definition of (Archimedean) tropical variety for the polynomial $f(x):=\sum_{j=1}^{t}e^{b_j}x^{a_j}$, and the simpler bound $\Delta(\mathrm{Amoeba}(f),\mathrm{Trop}(f))\leq(2t-3)\log(t-1)$ holds [8]. Earlier metric results for the special case $A\subset\mathbb{Z}$ date back to work of Ostrowski on Graeffe iteration [88]. Viro and Mikhalkin touched upon the special case $A\subset\mathbb{Z}^2$ in [122] and [83, Lemma 8.5, pg. 360].

We derive our distance bounds by using a projection trick arising from the study of random convex sets (see [52] and Section 2.3 below) to reduce to the $d=1$ case. The $d=1$ case then follows from specially tailored extensions of existing results for the polynomial case (see Section 2.2 below). This approach results in succinct proofs for our bounds. However, it is not yet clear if the dependence on d is actually necessary or just an artifact of our techniques.

A consequence of our approach is a refinement of an earlier estimate of Wilder (see [124], [123], and Section 2.2.2 below) on the number of roots of univariate exponential sums in infinite horizontal strips of $\mathbb{C}$: Theorem 2.2.10 (see Section 2.2.2) allows us to estimate the number of roots in certain axis-parallel rectangles in $\mathbb{C}$. A very special case of Theorem 2.2.10 is the fact that *all* the roots of g are confined to an explicit union of infinite *vertical* strips explicitly determined by $\mathrm{Trop}(g)$. In what follows, the *open ε-neighborhood* of a subset $X\subseteq\mathbb{R}$ is simply $\{x'\in\mathbb{R} \ : \ |x-x'|<\varepsilon \text{ for some } x\in X\}$.

Corollary 2.1.6. *Suppose g is any univariate t-sum with real spectrum and W is the open $\frac{\log 3}{\delta(g)}$-neighborhood of* $\mathrm{Trop}(g)$. *Then all the complex roots of g lie in $W\times\mathbb{R}$. In particular,* $\sup\limits_{r\,\in\,\Re(Z(g))}\ \inf\limits_{w\,\in\,\mathrm{Trop}(g)} |r-w| \leq \frac{\log 3}{\delta(g)}$ *in the univariate case.* ■

Unlike the distribution of roots of g in horizontal strips, where there is a kind of equidistribution (see, e.g., [123, 4] and Section 2.2 below), Corollary 2.1.6 tells us that the roots of g cluster only within certain deterministically predictable vertical strips.

Our next main results concern the complexity of deciding whether a given point lies in the real part of the complex zero set of a given exponential sum, and whether checking membership in a neighborhood of a tropical variety instead is more efficient.

2.1.1 On the Computational Complexity of $\Re(Z(g))$ *and* $\mathrm{Trop}(g)$

We have tried to balance generality and computational tractability in the family of functions at the heart of our paper. In particular, the use of arbitrary real inputs causes certain geometric and algorithmic subtleties. We will see below that these difficulties are ameliorated by replacing exact queries with approximate queries.

Remark 2.1.1. *"Polynomials" with real exponents — sometimes called* posinomials *— occur naturally in many applications. For example, the problem of finding the directions of a set of unknown signals, using a radar antenna built from a set of specially spaced sensors, can easily be converted to an instance of root-finding in the univariate case [47, 68]. Approximating roots in the higher-dimensional case is the fundamental computational problem of Geometric Programming [42, 29, 25]. Pathologies with the phases of complex roots can be avoided through a simple exponential change of variables, so this is one reason that exponential sums are more natural than posinomials. Among other applications, exponential sums occur in the calculation of* 3*-manifold invariants (see, e.g., [82, Appendix A] and [63]), and have been studied from the point of view of Model Theory and Diophantine Geometry (see, e.g., [125, 127, 128]).* $\diamond$

To precisely compare the computational complexity of $\Re(Z(g))$ and $\mathrm{Trop}(g)$ we will first need to fix a suitable model of computation: We will deal mainly with the *BSS model over* $\mathbb{R}$ [21]. This model naturally augments the classical *Turing machine* [90, 6, 114] by allowing field operations and comparisons over $\mathbb{R}$ in unit time. We are in fact forced to move beyond the Turing model since our exponential sums involve arbitrary real numbers, and the Turing model only allows finite bit strings as inputs. We refer the reader to [21] for further background.

We are also forced to move from *exact* equality and membership questions to questions allowing a margin of uncertainty. One reason is that exact arithmetic involving exponential sums still present difficulties, even for computational models allowing field operations and comparisons over $\mathbb{R}$.

Proposition 2.1.7. *The problem of determining, for an input* $(z_1, z_2) \in \mathbb{R}^2$*, whether* $z_1 = e^{z_2}$*, is undecidable*[2] *in the BSS model over* $\mathbb{R}$*, i.e., there is no algorithm terminating in finite time for all inputs.*

(We were unable to find a precise statement of Proposition 2.1.7 in the literature, so we provide a proof at the end of this section.) Note that when the input is restricted, deciding whether $z_1 = e^{z_2}$ can be tractable (and even trivially so). For instance, a famous result of Lindemann [76] tells us that e^{z_2} is transcendental if $z_2 \in \mathbb{C}$ is nonzero and algebraic.

Proposition 2.1.7 may be surprising in light of there being efficient iterations for approximating the exponential function [22, 2]. Determining which questions are tractable for expressions involving exponentials has in fact been an important impetus behind parts of Computational Algebra, Model Theory, and Diophantine Geometry in recent decades (see, e.g., [99, 125, 127, 62, 105]). As for the complexity of $\Re(Z(g))$, deciding membership turns out to be provably hard, already for the simplest bivariate exponential 3-sums.

Theorem 2.1.8. *Determining, for arbitrary input* $r_1, r_2 \in \mathbb{R}$ *whether* $(r_1, r_2) \in \Re(Z(1 - e^{z_1} - e^{z_2}))$ *is undecidable in the BSS model over* $\mathbb{R}$*.*

(We prove Theorem 2.1.8 at the end of this section.) The intractability asserted in Theorem 2.1.8 can be thought of as an amplification of the **NP**-hardness of deciding amoeba membership when $A \subset \mathbb{Z}$ [8, Thm. 1.9]. (See also [94] for an important precursor.) However, just as in Proposition 2.1.7, there are special cases of the membership problem from Theorem 2.1.8 that are perfectly tractable. For instance, when $e^{r_1}, e^{r_2} \in \mathbb{Q}$, deciding whether $(r_1, r_2) \in \Re(Z(1 - e^{z_1} - e^{z_2}))$ is in fact doable —

[2] [95] provides an excellent survey on undecidability, in the classical Turing model, geared toward non-experts in complexity theory.

even on a classical Turing machine — in polynomial-time (see, e.g., [117, 118] and [8, Thm. 1.9]).

More to the point, Theorem 2.1.8 above is yet another motivation for approximating $\Re(Z(g))$, and our final main result shows that membership queries (and even distance queries) involving $\mathrm{Trop}(g)$ are quite tractable in the BSS model over $\mathbb{R}$. We refer the reader to [60, 126, 37] further background on polyhedral geometry and subdivisions.

Definition 2.1.9. *For any n-variate exponential t-sum g, let $\Sigma(\mathrm{Trop}(g))$ denote the polyhedral complex whose cells are exactly the (possibly improper) faces of the closures of the connected components of $\mathbb{R}^n \setminus \mathrm{Trop}(g)$.* $\diamond$

Theorem 2.1.10. *Suppose n is fixed. Then there is a polynomial-time algorithm that, for any input $w \in \mathbb{R}^n$ and n-variate exponential t-sum g, outputs the closure — described as an explicit intersection of $O(t^2)$ half-spaces — of the unique cell σ_w of $\Sigma(\mathrm{Trop}(g))$ containing w.*

We prove Theorem 2.1.10 in Section 2.5. An analogue of Theorem 2.1.10, for the classical Turing model (assuming $A \subset \mathbb{Z}^n$ and $w \in \mathbb{Q}^n$) appears in [5, Thm. 1.5]. Extending to $A \subset \mathbb{R}^n$ and real coefficients, and using the BSS model over $\mathbb{R}$, in fact conceptually simplifies the underlying algorithm and helps us avoid certain Diophantine subtleties.

By applying the standard formula for point-hyperplane distance, and the well-known efficient algorithms for approximating square-roots (see, e.g., [22]), Theorem 2.1.10 implies that we can also efficiently check membership in any ε-neighborhood about $\mathrm{Trop}(g)$. This means, thanks to Theorem 2.1.5, that membership in a neighborhood of $\mathrm{Trop}(g)$ is a tractable and potentially useful relaxation of the problem of deciding membership in $\Re(Z(g))$.

For completeness, we now prove Proposition 2.1.7 and Theorem 2.1.8.

Proof of Proposition 2.1.7: The key is to consider the shape of the space of inputs $\mathcal{I}$ that lead to a "Yes" answer in a putative BSS machine deciding membership in the curve in $\mathbb{R}^2$ defined by $y=e^x$. In particular, [21, Thm. 1, Pg. 52] tells us that any set of inputs leading to a "Yes" answer in a BSS machine over $\mathbb{R}$ must be a countable union of semi-algebraic sets. So if $\mathcal{I}$ is indeed decidable relative to this model then $\mathcal{I}$ must contain a bounded connected neighborhood W of a real algebraic curve (since $\mathcal{I}$ has infinite length). Since $\mathcal{I}$ is the graph of e^x, W extends by analytic continuation to the graph of an entire algebraic function. But this impossible: One simple way to see this is that an entire algebraic function must have polynomial growth order. However, the function e^x clearly has non-polynomial growth order. ■

Proof of Theorem 2.1.8: Similar to our last argument, one can easily show that $\mathcal{I} := \Re(Z(1 - e^{z_1} - e^{z_2}))$ being decidable by a BSS machine over $\mathbb{R}$ implies that a neighborhood W of the boundary of I must be real algebraic. (We may in fact assume that W is the part of the boundary that lies in the curve defined by $y=\log(1-e^x)$.) So, via analytic continuation to $U := \mathbb{C} \setminus \{(2k+1)\sqrt{-1}\pi \mid k \in \mathbb{Z}\}$, it suffices to show that $\log(1-e^x)$ is not an algebraic function that is analytic on U. But this is easy since an algebraic function can only have finitely many branch points, whereas $\log(1-e^x)$ has infinitely many. (Moreover, each branch point of $\log(1-e^x)$ has infinite monodromy whereas algebraic functions can only have branch points with finite monodromy.) ■

2.2 Tropically Extending Classical Polynomials Root Bounds to Exponential Sums

2.2.1 Basics on Roots of Univariate Exponential Sums

Let $\#S$ denote the cardinality of a set S. It is worth noting that although $\#\mathrm{Trop}(g)$ and our bounds for $\Delta(\Re(Z(g)), \mathrm{Trop}(g))$ are independent of the maximal distance between frequencies $D := \max_{p,q} |a_p - a_q|$, the cardinality $\#\Re(Z(g))$ can certainly depend on D, and even be infinite for $n=1$.

Example 2.2.1. *For any integer $D \geq 2$, $g(z_1) := e^{Dz_1} + e^{z_1} + 1$ satisfies $\#\mathrm{Trop}(g) = 1$ but $\#\Re(Z(g)) = \lceil D/2 \rceil$. The latter cardinality is easily computed by observing that the non-real roots of the trinomial $f(x_1) := x_1^D + x_1 + 1$ occur in conjugate pairs, and at most 2 roots of f can have the same norm. (The latter fact is a very special case of [119, Prop. 4.3].)* $\diamond$

Example 2.2.2. *Considering the decimal expansion of $\sqrt{2}$, and the local continuity of the roots of $e^{Dz_1} + e^{z_1} + 1$ as a function of $D \in \mathbb{R}$, it is not hard to show that $X := \Re\left(Z\left(e^{\sqrt{2}z_1} + e^{z_1} + 1\right)\right)$ is in fact countably infinite, and Corollary 2.2.6 below tells us that X is also contained in the open interval $\left(-\frac{\log 2}{\sqrt{2}-1}, \frac{\log 2}{\sqrt{2}-1}\right)$.* $\diamond$

To derive our main results we will need the following variant of the Newton polytope, specially suited for studying real parts of roots of exponential sums.

Definition 2.2.3. *Let $\mathrm{Conv}(S)$ denote the convex hull of a subset $S \subseteq \mathbb{R}^n$, i.e., the smallest convex set containing S. Given any n-variate exponential t-sum $g(z) = \sum_{j=1}^{t} e^{a_j \cdot z + b_j}$ with real frequencies a_j, we then define its* Archimedean Newton polytope *to be $\mathrm{ArchNewt}(g) := \mathrm{Conv}\left(\{(a_j, -\Re(b_j))\}_{j \in [t]}\right)$. We also call any face of a polytope $P \subset \mathbb{R}^{n+1}$ having an outer-normal vector with* negative *last coordinate a* lower *face.* $\diamond$

Proposition 2.2.4. *For any n-variate exponential t-sum g with real spectrum we have*

$$\mathrm{Trop}(g) = \{w \mid (w, -1) \textit{ is an outer normal of a } \textit{positive-dimensional face of } \mathrm{ArchNewt}(g)\}$$

Furthermore, when $n=1$, $\mathrm{Trop}(g)$ is also the set of slopes of the lower edges of $\mathrm{ArchNewt}(g)$. ■

We refer the reader to [8] for further background on the polynomial cases of ArchNewt and Trop.

A key trick we will use is relating the points of $\mathrm{Trop}(g)$ to (vertical) half-planes of $\mathbb{C}$ where certain terms of the univariate exponential sum g dominate certain sub-summands of g.

Proposition 2.2.5. *Suppose $g(z_1) := \sum_{j=1}^{t} e^{a_j z_1 + b_j}$ satisfies $a_1 < \cdots < a_t$ and $b_j \in \mathbb{C}$ for all j. Suppose further that $w \in \mathrm{Trop}(g)$, ℓ is the unique index such that $(a_\ell, \Re(b_\ell))$ is the right-hand vertex of the lower edge of $\mathrm{ArchNewt}(g)$ of slope w, and let $\delta_\ell := \min_{p-q} |a_p - a_q|$.*
Then for any $N \in \mathbb{N}$ and $z_1 \in \left[w + \frac{\log(N+1)}{\delta_\ell}, \infty\right) \times \mathbb{R}$ we have $\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| < \frac{1}{N}\left|e^{a_\ell z_1 + b_\ell}\right|$.

Proof: First note that $2 \le \ell \le t$ by construction. Let $\beta_j := \Re(b_j)$, $r := \Re(z_1)$, and note that

$$\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| \le \sum_{j=1}^{\ell-1} \left|e^{a_j z_1 + b_j}\right| = \sum_{j=1}^{\ell-1} e^{a_j r + \beta_j} = \sum_{j=1}^{\ell-1} e^{a_j (r-w) + a_j w + \beta_j}$$

Now, since $a_{j+1} - a_j \ge \delta_\ell$ for all $j \in \{1, \ldots, \ell-1\}$, we obtain $a_j \le a_\ell - (\ell - j)\delta_\ell$. So for $r > w$ we have $\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| \le \sum_{j=1}^{\ell-1} e^{(a_\ell - (\ell-j)\delta_\ell)(r-w) + a_j w + \beta_j} \le \sum_{j=1}^{\ell-1} e^{(a_\ell - (\ell-j)\delta_\ell)(r-w) + a_\ell w + \beta_\ell}$, where the last inequality follows from Definition 2.1.1. So then

$$\begin{aligned}
\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| &\le e^{(a_\ell - (\ell-1)\delta_\ell)(r-w) + a_\ell w + \beta_\ell} \sum_{j=1}^{\ell-1} e^{(j-1)\delta_\ell (r-w)} \\
&= e^{(a_\ell - (\ell-1)\delta_\ell)(r-w) + a_\ell w + \beta_\ell} \left(\frac{e^{(\ell-1)\delta_\ell (r-w)} - 1}{e^{\delta_\ell (r-w)} - 1}\right) \\
&< e^{(a_\ell - (\ell-1)\delta_\ell)(r-w) + a_\ell w + \beta_\ell} \left(\frac{e^{(\ell-1)\delta_\ell (r-w)}}{e^{\delta_\ell (r-w)} - 1}\right) = \frac{e^{a_\ell r + \beta_\ell}}{e^{\delta_\ell (r-w)} - 1}
\end{aligned}$$

So to prove our desired inequality, it clearly suffices to enforce $e^{\delta_\ell(r-w)} - 1 \geq N$. The last inequality clearly holds for all $r \geq w + \frac{\log(N+1)}{\delta_\ell}$, so we are done. ■

It is then easy to prove that the largest (resp. smallest) point of $\Re(Z(g))$ can't be too much larger (resp. smaller) than the largest (resp. smallest) point of $\mathrm{Trop}(g)$. Put another way, we can give an explicit vertical strip containing all the complex roots of g.

Corollary 2.2.6. *Suppose g is a univariate exponential t-sum with real spectrum and minimal spacing $\delta(g)$, and $w_{\min}$ (resp. $w_{\max}$) is $\max \mathrm{Trop}(g)$ (resp. $\min \mathrm{Trop}(g)$). Then $\Re(Z(g))$ is contained in the open interval $\left(w_{\min} - \frac{\log 2}{\delta(g)}, w_{\max} + \frac{\log 2}{\delta(g)}\right)$.*

The $\log 2$ in Corollary 2.2.6 can not be replaced by any smaller constant: For $g(z_1) = e^{(t-1)z_1} - e^{(t-2)z_1} - \cdots - e^{z_1} - 1$ we have $\delta(g) = 1$, $\mathrm{Trop}(g) = \{0\}$, and it is easily checked that $\Re(Z(g))$ contains points approaching $\log 2$ as $t \longrightarrow \infty$. While the polynomial analogue of Corollary 2.2.6 goes back to work of Cauchy, Birkhoff, and Fujiwara predating 1916 (see [96, pp. 243–249, particularly bound 8.1.11 on pg. 247] and [48] for further background) we were unable to find an explicit bound for exponential sums like Corollary 2.2.6 in the literature. So we supply a proof below.

Proof of Corollary 2.2.6: Replacing z_1 by its negative, it clearly suffices to prove $\Re(Z(g)) \subset \left(-\infty, w_{\max} + \frac{\log 2}{\delta}\right)$. Writing $g(z_1) = \sum_{j=1}^{t} e^{a_j z_1 + b_j}$ with $a_1 < \cdots < a_t$, let ζ denote any root of g, $r := \Re(\zeta)$, and $\beta_j := \Re(b_j)$ for all j. Since we must have $\sum_{j=1}^{t-1} e^{a_j\zeta + b_j} = -e^{a_t\zeta + b_t}$, taking absolute values implies that $\left|\sum_{j=1}^{t-1} e^{a_j\zeta + b_j}\right| = \left|e^{a_t\zeta + b_t}\right|$. However, this equality is contradicted by Proposition 2.2.5 for $\Re(z_1) \geq w_{\max} + \frac{\log 2}{\delta}$. So we are done. ■

Another simple consequence of our term domination trick (Proposition 2.2.5 above) is that we can give explicit vertical strips in $\mathbb{C}$ free of roots of g.

Corollary 2.2.7. *Suppose* $g(z_1):=\sum_{j=1}^{t} e^{a_j z_1+b_j}$ *satisfies* $a_1<\cdots<a_t$, $b_j\in\mathbb{C}$ *for all* j, *and that* w_1 *and* w_2 *are* consecutive *points of* $\mathrm{Trop}(g)$ *satisfying* $w_2\geq w_1+\frac{2\log 3}{\delta(g)}$. *Let* ℓ *be the unique index such that* $(a_\ell,\Re(b_\ell))$ *is the vertex of* $\mathrm{ArchNewt}(g)$ *incident to lower edges of slopes* w_1 *and* w_2. *Then the vertical strip* $\left[w_1+\frac{\log 3}{\delta(g)}, w_2-\frac{\log 3}{\delta(g)}\right]\times\mathbb{R}$ *contains* no *roots of* g.

Proof: By Proposition 2.2.5, we have $\left|\sum_{j=1}^{\ell-1} e^{a_j z_1+b_j}\right|<\frac{1}{2}\left|e^{a_\ell z_1+b_\ell}\right|$ for all $z_1\in\left[w_1+\frac{\log 3}{\delta(g)},\infty\right)$ and (employing the change of variables $z_1\mapsto -z_1$) $\left|\sum_{j=\ell+1}^{t} e^{a_j z_1+b_j}\right|<\frac{1}{2}\left|e^{a_\ell z_1+b_\ell}\right|$ for all $z_1\in\left(-\infty,w_2-\frac{\log 3}{\delta(g)}\right]$. So we obtain $\left|\sum_{j\neq\ell} e^{a_j z_1+b_j}\right|<\left|e^{a_\ell z_1+b_\ell}\right|$ in the stated vertical strip, and this inequality clearly contradicts the existence of a root of g in $\left[w_1+\frac{\log 3}{\delta(g)}, w_2-\frac{\log 3}{\delta(g)}\right]\times\mathbb{R}$. ■

Remark 2.2.1. *Corollary 2.1.6 from the introduction follows immediately from Corollaries 2.2.6 and 2.2.7.* ◇

Let us now recall a result of Wilder [124] (later significantly refined by Voorhoeve [123]) that tightly estimates the number of roots of exponential sums in infinite horizontal strips of $\mathbb{C}$. Let $\Im(\alpha)$ denote the imaginary part of $\alpha\in\mathbb{C}$ and let $\langle x\rangle:=\min_{u\in\mathbb{Z}}|x-u|$ be the distance of x to the nearest integer.

Wilder-Voorhoeve Theorem. *[123, Thm. 5.3] For any univariate exponential* t*-sum* g *with real frequencies* $a_1<\cdots<a_t$ *and* $u\leq v$ *let* $H_{u,v}$ *denote the number of roots of* g, *counting multiplicity, in the infinite horizontal strip* $\{z_1\in\mathbb{C}\mid \Im(z_1)\in[u,v]\}$. *Then*

$$\left|H_{u,v}-\frac{v-u}{2\pi}(a_t-a_1)\right|\leq t-1-\sum_{j=2}^{t}\left\langle\frac{(v-u)(a_j-a_{j-1})}{2\pi}\right\rangle. \qquad ■$$

We will ultimately refine the Wilder-Voorhoeve Theorem into a *localized* deviation bound (Theorem 2.2.10 below) counting the roots of g in special axis parallel rectangles in $\mathbb{C}$. For this, we will need to look more closely at the variation of the argument of g on certain vertical and horizontal segments.

2.2.2 Winding Numbers and Density of Roots in Rectangles and Vertical Strips

To count roots of exponential sums in rectangles, it will be useful to observe a basic fact on winding numbers for *non*-closed curves.

Proposition 2.2.8. *Suppose $I \subset \mathbb{C}$ is any compact line segment and g and h are functions analytic on a neighborhood of I with $|h(z)| < |g(z)|$ for all $z \in I$. Then* $\left|\Im\left(\int_I \frac{g'+h'}{g+h}dz - \int_I \frac{g'}{g}dz\right)\right| < \pi$.

Proof: The quantity $V_1 := \Im\left(\int_I \frac{g'}{g}dz\right)$ (resp. $V_2 := \Im\left(\int_I \frac{g'+h'}{g+h}dz\right)$) is nothing more than the variation of the argument of g (resp. $g+h$) along the segment I. Since I is compact, $|g|$ and $|g+h|$ are bounded away from 0 on I by construction. So we can lift the paths $g(I)$ and $(g+h)(I)$ (in $\mathbb{C}^*$) to the universal covering space induced by the extended logarithm function. Clearly then, V_1 (resp. V_2) is simply a difference of values of $\Im(\mathrm{Log}(g))$ (resp. $\Im(\mathrm{Log}(g+h))$), evaluated at the endpoints I, where different branches of Log may be used at each endpoint. In particular, at any fixed endpoint z, our assumptions on $|g|$ and $|h|$ clearly imply that $g(z)+h(z)$ and $g(z)$ both lie in the open half-plane normal (as a vector in $\mathbb{R}^2$) to $g(z)$. So $|\Im(\mathrm{Log}(g(z)+h(z))) - \Im(\mathrm{Log}(g(z)))| < \frac{\pi}{2}$ at the two endpoints of I, and thus $|V_1 - V_2| < \frac{\pi}{2} + \frac{\pi}{2} = \pi$. ■

Re-examining Corollary 2.1.6 from the last section, one quickly sees that the vertical strips in $\mathbb{C}$ containing the roots of a univariate exponential sum g correspond exactly to clusters of "closely spaced" consecutive points of $\mathrm{Trop}(g)$. These clusters of points in $\mathrm{Trop}(g)$ in turn correspond to certain sub-summands of g. In particular, sets of consective "large" (resp. "small") points of $\mathrm{Trop}(g)$ correspond to sums of "high" (resp. "low") order terms of g. Our next step will then be to relate the roots of a high (or low) order summand of g to an *explicit portion* of the roots of g.

Lemma 2.2.9. *Let* $g(z_1) := \sum_{j=1}^{t} e^{a_j z_1 + b_j}$ *with* $a_1 < \cdots < a_t$ *and* $b_j \in \mathbb{C}$ *for all* j, $u \leq v$, *and let* $w_{\min}$ *(resp.* $w_{\max}$*) be* $\min \mathrm{Trop}(g)$ *(resp.* $\max \mathrm{Trop}(g)$*). Also let* w_1 *and* w_2 *be* consecutive *points of* $\mathrm{Trop}(g)$ *satisfying* $w_{\min} < w_1 < w_2 < w_{\max}$ *and let* ℓ *be the unique index such that* $(a_\ell, \Re(b_\ell))$ *is the vertex of* $\mathrm{ArchNewt}(g)$ *incident to lower edges of slopes* w_1 *and* w_2 *(so* $2 \leq \ell \leq t-1$*). Finally, assume* $w_2 - w_1 \geq \frac{2\log 3}{\delta(g)}$. *and let* $R^1_{u,v}$ *and* $R^2_{u,v}$ *respectively denote the number of roots of* g*, counting multiplicity, in the rectangles* $\left(w_{\min} - \frac{\log 2}{\delta(g)}, w_1 + \frac{\log 3}{\delta(g)}\right) \times [u,v]$ *and* $\left(w_2 - \frac{\log 3}{\delta(g)}, w_{\max} + \frac{\log 2}{\delta(g)}\right) \times [u,v]$. *Then*

$$\left|R^1_{u,v} - \frac{v-u}{2\pi}(a_\ell - a_1)\right| \leq \varepsilon_1 + 1 \quad \textit{and} \quad \left|R^2_{u,v} - \frac{v-u}{2\pi}(a_t - a_\ell)\right| \leq \varepsilon_2 + 1,$$

where $\varepsilon_1, \varepsilon_2 \geq 0$ *and* $\varepsilon_1 + \varepsilon_2 \leq t - 1 - \sum_{j=2}^{t} \left\langle \frac{(v-u)(a_j - a_{j-1})}{2\pi} \right\rangle$.

When $\mathrm{Trop}(g)$ has two adjacent points sufficiently far apart (as detailed above), Lemma 2.2.9 thus refines the Wilder-Voorhoeve Theorem. Lemma 2.2.9 also considerably generalizes an earlier root count for the polynomial case presented in [8, Lemma 2.8]: Rephrased in terms of the notation above, the older root count from [8, Lemma 2.8] becomes the equalities $R^1_{0,2\pi} = a_\ell - a_1$ and $R^2_{0,2\pi} = a_t - a_\ell$ for the special case $A \subset \mathbb{Z}$.

Proof of Lemma 2.2.9: By symmetry (with respect to replacing z_1 by $-z_1$) it clearly suffices to prove the estimate for $R^2_{u,v}$. Since g is analytic, the Argument Principle (see, e.g., [1]) tells us that

$$R^2_{u,v} = \frac{1}{2\pi\sqrt{-1}} \int_{I_- \cup I_+ \cup J_- \cup J_+} \frac{g'}{g} dz$$

where I_- (resp. I_+, J_-, J_+) is the oriented line segment from

$$\left(w_2 - \tfrac{\log 3}{\delta(g)}, v\right) \text{ (resp. } \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, u\right), \left(w_2 - \tfrac{\log 3}{\delta(g)}, u\right), \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, v\right))$$

to

$$\left(w_2 - \tfrac{\log 3}{\delta(g)}, u\right) \text{ (resp. } \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, v\right), \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, u\right), \left(w_2 - \tfrac{\log 3}{\delta(g)}, v\right)),$$

assuming no root of g lies on $I_- \cup I_+ \cup J_- \cup J_+$. By Corollaries 2.2.6 and 2.2.7, there can be no roots of g on $I_- \cup I_+$. So let assume temporarily that there are no roots of g on $J_- \cup J_+$.

Since $w_2 - \frac{\log 3}{\delta(g)} \geq w_1 + \frac{\log 3}{\delta(g)}$ by assumption, Proposition 2.2.5 tells us that

$$\frac{1}{2}\left|c_\ell e^{a_\ell\left(w_2-\frac{\log 3}{\delta(g)}+\sqrt{-1}v\right)+b_\ell}\right| > \left|\sum_{j=1}^{\ell-1} e^{a_j\left(w_2-\frac{\log 3}{\delta(g)}+\sqrt{-1}v\right)+b_j}\right|$$

and, by symmetry and another application of Proposition 2.2.5,

$$\frac{1}{2}\left|c_\ell e^{a_\ell\left(w_2-\frac{\log 3}{\delta(g)}+\sqrt{-1}v\right)+b_\ell}\right| > \left|\sum_{j=\ell+1}^{t} e^{a_j\left(w_2-\frac{\log 3}{\delta(g)}+\sqrt{-1}v\right)+b_j}\right|.$$

So we can apply Proposition 2.2.8 and deduce that $\left|\Im\left(\int_{I_-}\frac{g'}{g}dz - \int_{I_-}\frac{(e^{a_\ell z+b_\ell})'}{e^{a_\ell z+b_\ell}}dz\right)\right| < \pi$. So then, since the last integral has imaginary part easily evaluating to $a_\ell(u-v)\sqrt{-1}$, we clearly obtain $\left|\left(\frac{1}{2\pi\sqrt{-1}}\int_{I_-}\frac{g'}{g}dz\right) - a_\ell(u-v)\right| < \frac{1}{2}$. An almost identical argument (applying Propositions 2.2.5 and 2.2.8 again, but with the term $e^{a_t z+b_t}$ dominating instead) then also yields $\left|\left(\frac{1}{2\pi\sqrt{-1}}\int_{I_+}\frac{g'}{g}dz\right) - a_t(v-u)\right| < \frac{1}{2}$.

So now we need only prove sufficiently sharp estimates on $\frac{1}{2\pi\sqrt{-1}}\int_{J_\pm}\frac{g'}{g}dz$:

$$\begin{aligned}
\left|\int_{J_-\cup J_+}\Im\left(\frac{g'}{g}\right)dz\right| &= \left|\int_{w_2-\frac{\log 3}{\delta(g)}}^{w_{\max}+\frac{\log 2}{\delta(g)}}\Im\left(\frac{g'\left(z+u\sqrt{-1}\right)}{g\left(z+u\sqrt{-1}\right)} - \frac{g'\left(z+v\sqrt{-1}\right)}{g\left(z+v\sqrt{-1}\right)}\right)dz\right| \\
&\leq \int_{w_2-\frac{\log 3}{\delta(g)}}^{w_{\max}+\frac{\log 2}{\delta(g)}}\left|\Im\left(\frac{g'\left(z+u\sqrt{-1}\right)}{g\left(z+u\sqrt{-1}\right)} - \frac{g'\left(z+v\sqrt{-1}\right)}{g\left(z+v\sqrt{-1}\right)}\right)\right|dz \\
&=: K\left(w_2-\frac{\log 3}{\delta(g)}, w_{\max}+\frac{\log 2}{\delta(g)}; u, v; g\right).
\end{aligned}$$

A quantity closely related to $K(x_1, x_2; u, v; g)$ was, quite fortunately, already studied in Voorhoeve's 1977 Ph.D. thesis: In our notation, the proof of [123, Thm. 5.3] immediately yields $\lim_{x\to\infty} K(-x, x; u, v; g) = t-1-\sum_{j=2}^{t}\left\langle\frac{(v-u)(a_j-a_{j-1})}{2\pi}\right\rangle$. In particular, by the additivity of integration, the nonnegativity of the underlying integrands, and taking $\varepsilon_1 := K\left(w_{\min}-\frac{\log 2}{\delta(g)}, w_1+\frac{\log 3}{\delta(g)}; u, v; g\right)$ and $\varepsilon_2 := K\left(w_2-\frac{\log 3}{\delta(g)}, w_{\max}+\frac{\log 2}{\delta(g)}; u, v; g\right)$, we obtain $\left|\int_{J_-\cup J_+} \Im\left(\frac{g'}{g}\right) dz\right| \leq \varepsilon_2$, with $\varepsilon_1, \varepsilon_2 \geq 0$ and

$$\varepsilon_1+\varepsilon_2 \leq t-1-\sum_{j=2}^{t}\left\langle\frac{(v-u)(a_j-a_{j-1})}{2\pi}\right\rangle.$$

Adding terms and errors, we then clearly obtain $\left|R^2_{u,v}-\frac{v-u}{2\pi}(a_t-a_\ell)\right|<\varepsilon_2+1$, in the special case where no roots of g lie on $J_-\cup J_+$. To address the case where a root of g lies on $J_-\cup J_+$, note that the analyticity of g implies that the roots of g are a discrete subset of $\mathbb{C}$. So we can find arbitrarily small $\eta>0$ with the boundary of the slightly stretched rectangle $\left(w_2-\frac{\log 3}{\delta(g)}, w_{\max}+\frac{\log 2}{\delta(g)}\right)\times[u-\eta, v+\eta]$ not intersecting any roots of g. So then, by the special case of our lemma already proved, $\left|R^2_{u-\eta,v+\eta}-\frac{v-u+2\eta}{2\pi}(a_t-a_\ell)\right|<\varepsilon'_2+1$, with $\varepsilon'_1, \varepsilon'_2 \geq 0$ and $\varepsilon'_1+\varepsilon'_2 \leq t-1-\sum_{j=2}^{t}\left\langle\frac{(v-u+2\eta)(a_j-a_{j-1})}{2\pi}\right\rangle$. Clearly, $R^2_{u-\eta,v+\eta}=R^2_{u,v}$ for η sufficiently small, and the limit of the preceding estimate for $R^2_{u-\eta,v+\eta}$ tends to the estimate stated in our lemma. So we are done. ■

We at last arrive at our strongest refinement of the Wilder-Voorhoeve Theorem.

Theorem 2.2.10. *Suppose* $g(z_1)=\sum_{j=1}^{t} e^{a_j z_1+b_j}$, $a_1<\cdots<a_t$, *and* C *is any connected component of the open* $\frac{\log 3}{\delta(g)}$*-neighborhood of* $\mathrm{Trop}(g)$*. Also let* $w_{\min}(C)$ *(resp.* $w_{\max}(C)$*) be* $\min(\mathrm{Trop}(g)\cap C)$ *(resp.* $\max(\mathrm{Trop}(g)\cap C)$*) and let* i *(resp.* j*) be the unique index such that* $(a_i, \Re(b_i))$ *is the left-most (resp. right-most) vertex of the lower edge of* $\mathrm{ArchNewt}(g)$ *of slope* $w_{\min}(C)$ *(resp.* $w_{\max}(C)$*). Finally, let* $R_{C,u,v}$ *denote the number of roots of* g*, counting multiplicity, in the rectangle* $C\times[u,v]$*. Then*

$$\left|R_{C,u,v}-\frac{v-u}{2\pi}(a_j-a_i)\right| \leq \varepsilon_C+1,$$

where $\varepsilon_C \geq 0$ *and the sum of* ε_C *over all such connected components* C *is no greater than* $t-1-\sum_{j=2}^{t}\left\langle\frac{(v-u)(a_j-a_{j-1})}{2\pi}\right\rangle$.

Note that Lemma 2.2.9 is essentially the special case of Theorem 2.2.10 where C is the leftmost or rightmost connected component specified above. Note also that a special case of Theorem 2.2.10 implies that the fraction of roots of g lying in $C \times \mathbb{R}$ (i.e., the ratio $\lim\limits_{y\to\infty} \frac{R_{C,u,v}}{H_{u,v}}$, using the notation from our statement of the Wilder-Voorhoeve Theorem) is exactly $\frac{a_j - a_i}{a_t - a_1}$. This density of roots localized to a vertical strip can also be interpreted as the average value of the function 1, evaluated at all root of g in $C \times \mathbb{R}$. Soprunova has studied the average value of general analytic functions h, evaluated at the roots (in a sufficiently large vertical strip) of an exponential sum [115]. Theorem 2.2.10 thus refines the notion of the "average value of 1 over the roots of g in $\mathbb{C}$" in a different direction.

Proof of Theorem 2.2.10: The argument is almost identical to the proof of Lemma 2.2.9, save for the horizontal endpoints of the rectangle and the dominating terms in the application of Proposition 2.2.5 being slightly different. ■

A consequence of our development so far, particularly Corollary 2.1.6, is that every point of $\Re(Z(g))$ is close to some point of $\mathrm{Trop}(g)$. We now show that every point of $\mathrm{Trop}(g)$ is close to some point of $\Re(Z(g))$. The key trick is to break $\mathrm{Trop}(g)$ into clusters of closely spaced points, and use the fact that every connected component C (from Theorem 2.2.10) contains at least one real part of a complex root of g.

Theorem 2.2.11. *Suppose g is any univariate exponential t-sum with real spectrum and $t \geq 2$. Let s be the maximum cardinality of $\mathrm{Trop}(g) \cap C$ for any connected component C of the open $\frac{\log 3}{\delta(g)}$-neighborhood of $\mathrm{Trop}(g)$. (So $1 \leq s \leq t-1$ in particular.) Then for any $v \in \mathrm{Trop}(g)$ there is a root $z \in \mathbb{C}$ of g with $|\Re(z) - v| \leq \frac{(2s-1)\log 3}{\delta(g)}$.*

Proof: For convenience, for the next two paragraphs we will allow negative indices i for $\sigma_i \in \mathrm{Trop}(g)$ (but we will continue to assume σ_i is increasing in i).

Let us define R to be the largest j with $v, \sigma_1, \ldots, \sigma_j$ being consecutive points of $\mathrm{Trop}(g)$ in increasing order, $\sigma_1 - v \leq \frac{2\log 3}{\delta(g)}$, and $\sigma_{i+1} - \sigma_i \leq \frac{2\log 3}{\delta(g)}$ for all $i \in [j-1]$.

(We set $R=0$ should no such j exist.) Similarly, let us define L to be the largest j with $v, \sigma_{-1}, \ldots, \sigma_{-j} \in \mathrm{Trop}(g)$ being consecutive points of $\mathrm{Trop}(g)$ in decreasing order, $v - \sigma_{-1} \leq \frac{2\log 3}{\delta(g)}$, and $\sigma_{-i} - \sigma_{-i-1} \leq \frac{2\log 3}{\delta(g)}$ for all $i \in [j-1]$. (We set $L=0$ should no such j exist.) Note that $L+R+1 \leq s$.

By Theorem 2.2.10 there must then be at least one point of $\Re(Z(g))$ in the interval $\left[v-(2L+1)\frac{\log 3}{\delta(g)}, v+(2R+1)\frac{\log 3}{\delta(g)}\right]$. So there must be a point of $\Re(Z(g))$ within distance $(2\max\{L,R\}+1)\frac{\log 3}{\delta(g)}$ of v. Since $2L+2, 2R+2 \leq 2s$, we are done. ■

At this point, we are almost ready to prove our main theorems. The remaining fact we need is a generalization of Corollary 2.1.6 to arbitrary dimension.

2.2.3 A Quick Distance Bound in Arbitrary Dimension

Having proved an upper bound for the largest point of $\Re(Z(g))$, one may wonder if there is a *lower bound* for the largest point of $\Re(Z(g))$. Montel proved (in different notation) the univariate polynomial analogue of the assertion that the largest points of $\Re(Z(g))$ and $\mathrm{Trop}(g)$ differ by no more than $\log(t-1)$ [85]. One can in fact guarantee that *every* point of $\Re(Z(g))$ is close to some point of $\mathrm{Trop}(g)$, and in arbitrary dimension.

Lemma 2.2.12. *For any n-variate exponential t-sum g with real spectrum and $t \geq 2$ we have* $\sup\limits_{r \in \Re(Z(g))} \inf\limits_{w \in \mathrm{Trop}(g)} |r-w| \leq \log(t-1)/\delta(g)$.

Proof: Let $z \in Z(g)$ and assume without loss of generality that

$$\left|e^{a_1\cdot z+b_1}\right| \geq \left|e^{a_2\cdot z+b_2}\right| \geq \cdots \geq \left|e^{a_t\cdot z+b_t}\right|.$$

Since $g(z)=0$ implies that $\left|e^{a_1\cdot z+b_1}\right| = \left|e^{a_2\cdot z+b_2}+\cdots+e^{a_t\cdot z+b_t}\right|$, the Triangle Inequality immediately implies that $\left|e^{a_1\cdot z+b_1}\right| \leq (t-1)\left|e^{a_2\cdot z+b_2}\right|$. Taking logarithms, and letting

$\rho := \Re(z)$ and $\beta_i := \Re(b_i)$ for all i, we then obtain

$$a_1 \cdot \rho + \beta_1 \geq \cdots \geq a_t \cdot \rho + \beta_t \quad \text{and} \tag{2.1}$$

$$a_1 \cdot \rho + \beta_1 \leq \log(t-1) + a_2 \cdot \rho + \beta_2 \tag{2.2}$$

For each $i \in \{2, \ldots, t\}$ let us then define η_i to be the shortest vector such that

$$a_1 \cdot (\rho + \eta_i) + \beta_1 = a_i \cdot (\rho + \eta_i) + \beta_i.$$

Note that $\eta_i = \lambda_i(a_i - a_1)$ for some nonnegative λ_i since we are trying to affect the dot-product $\eta_i \cdot (a_1 - a_i)$. In particular, $\lambda_i = \frac{(a_1 - a_i) \cdot \rho + \beta_1 - \beta_i}{|a_1 - a_i|^2}$ so that $|\eta_i| = \frac{(a_1 - a_i) \cdot \rho + \beta_1 - \beta_i}{|a_1 - a_i|}$. (Indeed, Inequality (2.1) implies that $(a_1 - a_i) \cdot \rho + \beta_1 - \beta_i \geq 0$.)

Inequality (2.2) implies that $(a_1 - a_2) \cdot \rho + \beta_1 - \beta_2 \leq \log(t-1)$. We thus obtain $|\eta_2| \leq \frac{\log(t-1)}{|a_1 - a_2|} \leq \frac{\log(t-1)}{\delta(g)}$. So let $i_0 \in \{2, \ldots, t\}$ be any i minimizing $|\eta_i|$. We of course have $|\eta_{i_0}| \leq \log(t-1)/\delta(g)$, and by the definition of η_{i_0} we have

$$a_1 \cdot (\rho + \eta_{i_0}) + \beta_1 = a_{i_0} \cdot (\rho + \eta_{i_0}) + \beta_{i_0}.$$

Moreover, the fact that η_{i_0} is the shortest among the η_i implies that

$$a_1 \cdot (\rho + \eta_{i_0}) + \beta_1 \geq a_i \cdot (\rho + \eta_{i_0}) + \beta_i$$

for all i. Otherwise, we would have $a_1 \cdot (\rho + \eta_{i_0}) + \beta_1 < a_i \cdot (\rho + \eta_{i_0}) + \beta_i$ and $a_1 \cdot \rho + \beta_1 \geq a_i \cdot \rho + \beta_i$ (the latter following from Inequality (2.1)). Taking a convex linear combination of the last two inequalities, it is then clear that there must be a $\mu \in [0, 1)$ such that

$$a_1 \cdot (\rho + \mu\eta_{i_0}) + \beta_1 = a_i \cdot (\rho + \mu\eta_{i_0}) + \beta_i.$$

Thus, by the definition of η_i, we would obtain $|\eta_i| \leq \mu|\eta_{i_0}| < |\eta_{i_0}|$ — a contradiction.

We thus have the following: (i) $a_1 \cdot (\rho + \eta_{i_0}) - (-\beta_1) = a_{i_0} \cdot (\rho + \eta_{i_0}) - (-\beta_{i_0})$, (ii) $a_1 \cdot (\rho + \eta_{i_0}) - (-\beta_1) \geq a_i \cdot (\rho + \eta_{i_0}) - (-\beta_i)$ for all i, and (iii) $|\eta_{i_0}| \leq \log(t-1)/\delta(g)$.

Together, these inequalities imply that $\rho+\eta_{i_0}\in\mathrm{Trop}(g)$. In other words, we've found a point in $\mathrm{Trop}(g)$ sufficiently near ρ to prove our desired upper bound. ■

2.3 Small Ball Probability

Let $G_{n,k}$ be the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$, equipped with its unique rotation-invariant Haar probability measure $\mu_{n,k}$. The following "small ball probability" estimate holds.

Lemma 2.3.1. *[52, Lemma 3.2] Let $1\leq k\leq n-1$, $x\in\mathbb{R}^n$, and $\varepsilon\leq\frac{1}{\sqrt{e}}$. Then*

$$\mu_{n,k}\left(\left\{F\in G_{n,k}\;\middle|\;|P_F(x)|\leq\varepsilon\sqrt{\frac{k}{n}}|x|\right\}\right)\leq\left(\sqrt{e}\varepsilon\right)^k,$$

where P_F is the surjective orthogonal projection mapping $\mathbb{R}^n$ onto F. ■

An important precursor, in the context of bounding distortion under more general Euclidean embeddings, appears in [80].

A simple consequence of the preceding metric result is the following fact on the existence of projections mapping a high-dimensional point set onto a lower-dimensional subspace in a way that preserves the minimal spacing as much as possible.

Proposition 2.3.2. *Let $\gamma>0$ and $x_1,\ldots,x_N\in\mathbb{R}^n$ be such that $|x_i-x_j|\geq\gamma$ for all distinct i,j. Then, following the notation of Lemma 2.3.1, there exist $F\in G_{n,k}$ such that $|P_F(x_i)-P_F(x_j)|\geq\sqrt{\frac{k}{en}}\frac{\gamma}{N^{2/k}}$ for all distinct i,j.*

Proof: Let $z_{\{i,j\}}:=x_i-x_j$. Then our assumption becomes $z_{\{i,j\}}\geq\gamma$ for all distinct i,j and there are no more than $N(N-1)/2$ such pairs $\{i,j\}$. By Lemma 2.3.1 we have, for any fixed $\{i,j\}$, that $|P_F(z_{\{i,j\}})|\geq\varepsilon\sqrt{\frac{k}{n}}|z_{\{i,j\}}|$ with probability at least $1-(\sqrt{e}\varepsilon)^k$. So if $\varepsilon<\frac{1}{\sqrt{e}}\left(\frac{2}{N(N-1)}\right)^{1/k}$, the union bound for probabilities implies that, for all distinct i,j, we have $|P_F(z_{\{i,j\}})|\geq\varepsilon\sqrt{\frac{k}{n}}|z_{\{i,j\}}|\geq\varepsilon\gamma\sqrt{\frac{k}{n}}$ (and thus $|P_F(x_i)-P_F(x_j)|\geq\varepsilon\gamma\sqrt{\frac{k}{n}}$)

with probability at least $1-\frac{N(N-1)}{2}(\sqrt{e}\varepsilon)^k$. Since this lower bound is positive by construction, we can conclude by choosing $\varepsilon := \frac{1}{\sqrt{e}N^{2/k}}$. ■

2.4 Proof of Theorem 2.1.5

The assertion that $t \geq d+1$ is easy since any d-dimensional polytope always has at least $d+1$ vertices. So we now focus on the rest of the theorem. We prove Assertion 1(b) last.

In what follows, for any real $n \times n$ matrix M and $z \in \mathbb{R}^n$, we assume that z is a column vector when we write Mz. Also, for any subset $S \subseteq \mathbb{R}^n$, the notation $MS := \{Mz \mid z \in S\}$ is understood. The following simple functoriality properties of $\mathrm{Trop}(g)$ and $\Re(Z(g))$ will prove useful.

Proposition 2.4.1. *Suppose g_1 and g_2 are n-variate exponential t-sums, $\alpha \in \mathbb{C}^*$, $a \in \mathbb{R}^n$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$, and g_2 satisfies the identity $g_2(z) = \alpha e^{a\cdot z} g_1(z_1 + \beta_1, \ldots, z_n + \beta_n)$. Then $\Re(Z(g_2)) = \Re(Z(g_1)) - \Re(\beta)$ and $\mathrm{Trop}(g_2) = \mathrm{Trop}(g_1) - \Re(\beta)$. Also, if $M \in \mathbb{R}^{n\times n}$ and we instead have the identity $g_2(z) = g_1(Mz)$, then $M\Re(Z(g_2)) = \Re(Z(g_1))$ and $M\mathrm{Trop}(g_2) = \mathrm{Trop}(g_1)$.* ■

2.4.1 Proof of Assertion (0)

First note that, thanks to Proposition 2.4.1, an invertible linear change of variables allows us to reduce to the special case $A = \{\mathbf{O}, e_1, \ldots, e_n\}$, where $\mathbf{O}$ and $\{e_1, \ldots, e_n\}$ are respectively the origin and standard basis vectors of $\mathbb{R}^n$. But this special case is well known: One can either prove it directly, or avail to earlier work of Rullgård on the spines of amoebae (see, e.g., the remark following Theorem 8 on Page 33, and Theorem 12 on Page 36, of [104]). In fact, observing that our change of variables can in fact be turned into an isotopy (by the connectivity of $\mathbb{GL}_n^+(\mathbb{R})$), we can further assert that $\mathrm{Trop}(g)$ is a deformation retract of $\Re(Z(g))$ in this case.

■

2.4.2 Proof of Assertion 1(a)

This is simply Lemma 2.2.12, which was proved in Section 2.2. ■

2.4.3 Proof of Assertion (2)

The special case $\delta=1$ follows immediately from Assertion (2) of Theorem 1.5 of [8] (after setting $x_i = e^{z_i}$ in the notation there). Proposition 2.4.1 tells us that scaling the spectrum of g by a factor of δ scales $\Re(Z(g))$ and $\mathrm{Trop}(g)$ each by a factor of $1/\delta$. So we are done. ■

2.4.4 Proof of Assertion 1(b)

First note that the Hausdorff distance in question is invariant under rotation in $\mathbb{R}^n$. So we may in fact assume that g involves just the variables $z_1, \ldots, z_d$ and thus assume $d=n$.

By the $k=1$ case of Proposition 2.3.2 we deduce that there exists a unit vector $\theta \in \mathbb{R}^n$ such that

$$\min_{i \neq j} |a_i \cdot \theta - a_j \cdot \theta| \geq \frac{\delta(g)}{\sqrt{en}t^2} \tag{2.3}$$

Now let $v \in \mathrm{Trop}(g)$ and write $v = v_\theta \theta + v_\theta^\perp$ for some $v_\theta^\perp$ perpendicular to θ. Also let $u_\theta \in \mathbb{C}$ and $u \in \mathbb{C}^n$ satisfy $u = u_\theta \theta + v_\theta^\perp$. For $z_1 \in \mathbb{C}$ define the univariate exponential t-sum $\tilde{g}(z_1) = \sum_{j=1}^t e^{(a_j \cdot (z_1 \theta + v_\theta^\perp)) + b_i}$. By Inequality (2.3) we see that $\delta(\tilde{g}) \geq \frac{\delta(g)}{\sqrt{en}t^2}$. We also see that $\tilde{g}(u_\theta) = g(u)$ and $\tilde{g}(v_\theta) = g(v)$. By Theorem 2.2.11 there exists a value for u_θ such that $0 = \tilde{f}(u_\theta) = f(u)$ and $|\Re(u_\theta) - v_\theta| \leq \frac{(2t-3)\log 3}{\delta(\tilde{g})} \leq \frac{\sqrt{en}t^2(2t-3)\log 3}{\delta(g)}$. So $|\Re(u) - v| = |(\Re(u_\theta) - v_\theta)\theta| \leq \frac{\sqrt{en}t^2(2t-3)\log 3}{\delta(g)} = \frac{\sqrt{ed}t^2(2t-3)\log 3}{\delta(g)}$ since we've already reduced to the case $d=n$. ■

2.5 Proving Theorem 2.1.10

We will need some supporting results on linear programming before starting our proof.

Definition 2.5.1. *Given any matrix* $M \in \mathbb{R}^{N\times n}$ *with* $i^{\underline{th}}$ *row* m_i*, and* $c := (c_1, \ldots, c_N) \in \mathbb{R}^N$*, the notation* $Mx \leq c$ *means that* $m_1 \cdot x \leq c_1, \ldots, m_N \cdot x \leq c_N$ *all hold. These inequalities are called* constraints*, and the set of all* $x \in \mathbb{R}^N$ *satisfying* $Mx \leq c$ *is called the* feasible region *of* $Mx \leq c$*. We also call a constraint* active *if and only if it holds with equality. Finally, we call a constraint* redundant *if and only if the corresponding row of* M *and corresponding entry of* c *can be deleted without affecting the feasible region of* $Mx \leq c$*.* $\diamond$

Lemma 2.5.2. *Suppose* n *is fixed. Then, given any* $c \in \mathbb{R}^N$ *and* $M \in \mathbb{R}^{N\times n}$*, we can, in time polynomial in* N*, find a submatrix* M' *of* M*, and a subvector* c' *of* c*, such that the feasible regions of* $Mx \leq c$ *and* $M'x \leq c'$ *are equal, and* $M'x \leq c'$ *has no redundant constraints. Furthermore, in time polynomial in* N*, we can also enumerate all maximal sets of active constraints defining vertices of the feasible region of* $Mx \leq c$*.*

$\blacksquare$

Note that we are using the BSS model over $\mathbb{R}$ in the preceding lemma. In particular, we are only counting field operations and comparisons over $\mathbb{R}$ (and these are the only operations needed). We refer the reader to the excellent texts [107, 59, 58] for further background and a more leisurely exposition on linear programming.

Proof of Theorem 2.1.10: Let $w \in \mathbb{R}^n$ be our input query point. Using $O(t \log t)$ comparisons, we can isolate all indices such that $\max_j |e^{a_j \cdot z + b_j}|$ is attained, so let j_0 be any such index. Taking logarithms, we then obtain, say, J equations of the form $a_j \cdot w + \Re(b_j) = a_{j_0} \cdot w + \Re(b_{j_0})$ and K inequalities of the form $a_j \cdot w + \Re(b_j) > a_{j_0} \cdot w + \Re(b_{j_0})$ or $a_j \cdot w + \Re(b_j) < a_{j_0} \cdot w + \Re(b_{j_0})$.

Thanks to Lemma 2.5.2, we can determine the exact cell of $\mathrm{Trop}(f)$ containing w if $J \geq 2$. Otherwise, we obtain the unique cell of $\mathbb{R}^n \setminus \mathrm{Trop}(f)$ with relative interior containing w. Note also that an $(n-1)$-dimensional face of either kind of cell must be the dual of an edge of $\mathrm{ArchNewt}(g)$. Since every edge has exactly 2 vertices, there are at most $t(t-1)/2$ such $(n-1)$-dimensional faces, and thus σ_w is the intersection of at most $t(t-1)/2$ half-spaces. So we are done. ■

3. MULTIHOMOGENOUS NONNEGATIVE POLYNOMIALS AND SUMS OF SQUARES

3.1 Introduction

Let $\mathbb{R}[\bar{x}] := \mathbb{R}[x_1, \ldots, x_n]$ denote the ring of real n-variate polynomials and let $P_{n,2k}$ denote the vector space of forms (i.e homogenous polynomials) of degree $2k$ in $\mathbb{R}[\bar{x}]$. A form $p \in P_{n,2k}$ is called *non-negative* if $p(\bar{x}) \geq 0$ for every $\bar{x} \in \mathbb{R}^n$. The set of *non-negative* forms in $P_{n,2k}$ is closed under nonnegative linear combinations and thus forms a cone. We denote cone of nonnegative forms of degree $2k$ by $\mathrm{Pos}_{n,2k}$. A fundamental problem in polynomial optimization and real algebraic geometry is to efficiently certify non-negativity for real forms, i.e., membership in $\mathrm{Pos}_{n,2k}$.

If a real form can be written as a sum of squares of other real polynomials then it is evidently non-negative. Polynomials in $P_{n,2k}$ that can be represented as sums of squares of real polynomials form a cone that we denote by $\mathrm{Sq}_{n,2k}$. Clearly, $\mathrm{Sq}_{n,2k} \subseteq \mathrm{Pos}_{n,2k}$. We are then lead to the following natural question.

Question 3.1.1. *For which pairs of* (n, k) *do we have* $\mathrm{Sq}_{n,2k} = \mathrm{Pos}_{n,2k}$*?*

In his seminal 1888 paper [66] Hilbert showed that the answer to Question 3.1.1 is affirmative exactly for $(n, k) \in (\{2\} \times 2\mathbb{N}) \cup (\mathbb{N} \times \{2\}) \cup \{3, 4\}$. Hilbert's beatiful proof was not constructive: The first well known example of a non-negative form which is not sums of squares is due to Motzkin from around 1967: $x_3^6 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3x_3^2)$.

Hilbert stated a variation of Question 3.1.1 in his famous list of problems for 20^{th} century mathematicians:

Hilbert's 17th Problem. *Do we have, for every* n *and* k*, that every* $p \in \mathrm{Pos}_{n,2k}$ *is a sum of squares of* rational functions*?*

Artin and Schrier solved Hilbert's 17th Problem affirmatively around 1927 [7]. However there is no known efficient and general algorithm for finding the asserted collection of rational functions for a given input p. Despite the computational hardness of finding a representation as a sum of squares of rational functions, obtaining a representation as a sum of squares of *polynomials* (when possible) can be done efficiently via *semidefinite programming* (see, e.g., [74]). This connection to semidefinite programming (which has been used quite successfully in electrical engineering and optimization) strongly motivates a classification of which (n, k) have membership in $\mathrm{Sq}_{n,2k}$ occuring with high probability, relative to some natural probability measure μ on $\mathrm{Pos}_{n,2k}$.

3.1.1 From All or Nothing to Something in Between

Hilbert's 17th problem is essentially an algebraic problem. However methods from analysis have recently enabled some advances. The first example of this perspective is Gregoriy Blekherman's work: A consequence of his paper [18] is a probability measure μ on $\mathrm{Pos}_{n,2k}$ supported in an hyperplane, for which $\mu(\mathrm{Sq}_{n,2k}) \to 0$ as $n \to \infty$, for any fixed $k \geq 2$.

It is important to observe that for many problems of interest in algebraic geometry, forms with a special structure (e.g., sparse polynomials) behave differently from generic forms of degree $2k$. Precious little is known about Hilbert's 17^{th} Problem in the setting of sparse polynomials [40, 97, 20]. So let us first recall the notion of Newton polytope and then a theorem of Reznick: For any $p(\bar{x}) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\bar{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the *Newton polytope* of p is the convex hull $\mathrm{Newt}(p) := \mathrm{Conv}(\{\alpha \mid c_\alpha \neq 0\})$.

Theorem 3.1.1. *[97, Thm. 1] If* $p = \sum_{i=1}^r g_i^2$ *for some* $g_1, \ldots, g_r \in \mathbb{R}[\bar{x}]$ *then* $\mathrm{Newt}(g_i) \subseteq \frac{1}{2}\mathrm{Newt}(p)$ *for all* i.

This theorem enables us to refine the comparison of cones of *sums of squares* and *non-negative* polynomials to be more sensitive to monomial term structure.

Definition 3.1.2. *For any polytope* $Q \subset \mathbb{R}^n$ *with vertices in* $\mathbb{Z}^n$*, let* $N_Q := \#(Q \cap \mathbb{Z}^n)$*,* $c = (c_\alpha \mid \alpha \in Q \cap \mathbb{Z}^n)$*,* $p_c(\bar{x}) = \sum_{\alpha \in Q \cap \mathbb{Z}^n} c_\alpha \bar{x}^\alpha$ *and then define*

$$\mathrm{Pos}_Q := \{c \in \mathbb{R}^{N_Q} \mid p_c(\bar{x}) \geq 0 \textit{ for every } x \in \mathbb{R}^n\}$$

$$\mathrm{Sq}_Q := \{c \in \mathbb{R}^{N_Q} \mid p_c(\bar{x}) = \textstyle\sum_i q_i(\bar{x})^2 \textit{ where } \mathrm{Newt}(q_i) \subseteq \frac{1}{2}Q\} \qquad \diamond$$

In our notation here, Blekherman's paper [18] focused on volumetric estimates for the cones $\mathrm{Pos}_{Q_{n,2k}}$ and $\mathrm{Sq}_{Q_{n,2k}}$, where $Q_{n,2k}$ is the scaled $(n-1)$-simplex

$$\{\bar{x} \in \mathbb{R}^n \mid \textstyle\sum_{i=1}^n x_i = 2k\ ,\ x_1, \ldots, x_n \geq 0\}.$$

In this context the following problem arises naturally:

Weighted Polytopal SOS Problem. *Given a polytope* Q *and a probability measure* μ *on* Pos_Q*, estimate* $\mu\big(\mathrm{Sq}_Q\big)$*.* $\diamond$

Note that Hilbert's classic work [66] implies that, for any n and k and any continuous nonnegative probability measure on the cone $\mathrm{Pos}_{n,2k}$, we have $\mu\big(\mathrm{Sq}_{Q_{3,4}}\big) = \mu\big(\mathrm{Sq}_{Q_{2,2k}}\big) = \mu\big(\mathrm{Sq}_{Q_{n,2}}\big) = 1$. A related variant of the Weighted Polytopal SOS Problem was recently answered, as a consequence of the main theorem from [20]: There is now a complete combinatorial classification of those polytopes Q for which $\mathrm{Sq}_Q = \mathrm{Pos}_Q$.

3.1.2 Our Results

We focus on the *multihomogenous* case of the Weighted Polytopal SOS Problem. We have leaned toward general methods rather than ad hoc methods, in order to allow future study of arbitrary polytopes. We begin here with Cartesian products of scaled standard simplices.

Definition 3.1.3. *Assume henceforth that* $n = n_1 + \cdots + n_m$ *and* $k = k_1 + \cdots + k_m$*, with* $k_i, n_i \in \mathbb{N}$ *for all* i*, and set* $N := (n_1, \ldots, n_m)$ *and* $K := (k_1, \ldots, k_m)$*. We will partition the vector* $\bar{x} = (x_1, \ldots, x_n)$ *into* m *sub-vectors* $\bar{x}_1, \ldots, \bar{x}_m$ *so that* $\bar{x}_i$ *consists of exactly* n_i *variables for all* i*, and say that* $p \in \mathbb{R}[\bar{x}]$ *is* homogenous of type (N, K) *if and only if* p *ishomogenous of degree* k_i *with respect to* $\bar{x}_i$ *for all* i*. Finally, let* $Q_{N,K} := Q_{n_1,k_1} \times \cdots \times Q_{n_m,k_m}$*.* $\diamond$

Example 3.1.4. $p(\bar{x}) := x_1^3 x_4^2 + x_1 x_2^2 x_5^2 + x_3^3 x_4 x_5$ *is homogenous of type* (N, K) *with* $N = (3, 2)$ *and* $K = (2, 3)$*. (So* $\bar{x}_1 = (x_1, x_2, x_3)$ *and* $\bar{x}_2 = (x_4, x_5)$*.) In particular,* $\mathrm{Newt}(p) \subseteq Q_{N,K} = Q_{3,2} \times Q_{2,3}$*.* $\diamond$

Multihomogenous forms appeared before in the work of Choi, Lam and Reznick. In particular they proved the following theorem in [30]:

Theorem 3.1.5. *(Choi, Lam, Reznick) Let* $N = (n_1, n_2, \ldots, n_m)$ *and* $K = (2k_1, 2k_2, \ldots, 2k_m)$ *where* $n_i \geq 2$ *and* $k_i \geq 1$ *then* $\mathrm{Pos}_{Q_{N,K}} = \mathrm{Sq}_{Q_{N,K}}$ *if and only if* $m = 2$ *and* (N, K) *is either* $(2, n_2; 2k_1, 2)$ *or* $(n_1, 2; 2, 2k_2)$*.*

Our result can be viewed as a localized version of Blekherman's Theorem and also as a quantitative version of the theorem of Choi, Lam and Reznick [30]. In order to state our result we need to introduce the following function on subsets of $P_{N,K}$.

Definition 3.1.6. *Let* S^{n-1} *denote the standard unit* $(n-1)$*-sphere in* $\mathbb{R}^n$ *and define* $S := S^{n_1-1} \times \cdots \times S^{n_m-1}$*,* $P_{N,K} := \{p \in \mathbb{R}[\bar{x}]$ *homogeneous of type* $(N, K)\}$*, and* $C_{N,K} := \{p \in P_{N,K} \mid \int_S p \, d\sigma = 1\}$*. For any* $X \subseteq P_{N,K}$ *we set* $\mu(X) = \left(\frac{vol(X \cap C_{N,K})}{vol(B)}\right)^{\frac{1}{D_{N,K}}}$ *where* $D_{N,K}$ *is the dimension of* $P_{N,K}$ *and* B *is the* $D_{N,K}$ *dimensional ball.* $\diamond$

Our main theorem is the following.

Theorem 3.1.7. *Let $N = (n_1, n_2, \ldots, n_m)$ and $K = (2k_1, 2k_2, \ldots, 2k_m)$. We define $L_{Q_{N,K}} := \{p \in \mathrm{Pos}_{Q_{N,K}} : p = \sum_i l_{i1}^{2k_1} l_{i2}^{2k_2} \cdots l_{im}^{2k_m}$ where l_{ij} is a linear form in variables $\bar{x}_j\}$, then the following bounds hold*

$$\frac{1}{4^m\sqrt{\max_i n_i}} \prod_{i=1}^{m} (2k_i+1)^{-\frac{1}{2}} \leq \mu(\mathrm{Pos}_{Q_{N,K}}) \leq c_o$$

$$c_1 \prod_{i=1}^{m} (2k_i + \frac{n_i}{2})^{-\frac{k_i}{2}} \leq \mu(\mathrm{Sq}_{Q_{N,K}}) \leq c_2 \prod_{i=1}^{m} (\frac{ck_i}{n_i+k_i})^{\frac{k_i}{2}}$$

$$c_3 \prod_{i=1}^{m} (2k_i + \frac{n_i}{2})^{-\frac{k_i}{2}} \leq \mu(L_{Q_{N,K}}) \leq \sqrt{\max_i n_i} \prod_{i=1}^{m} 4(2k_i+1)^{\frac{1}{2}} \prod_{i=1}^{m} (\frac{n_i}{2k_i})^{-\frac{k_i}{2}}$$

where c_i are absolute constants with $c_o \leq 5$ and $c = 2^{10}e$.

To compare our bounds with Blekherman's bounds [18] let us consider the following special cases of Theorem 3.1.7:

Corollary 3.1.8. *1. For $N = (2, n-2)$ and $K = (2k-2, 2)$ we have the following bounds:*

$$c_1(2k-1)^{\frac{-k+1}{2}}(n+2)^{\frac{-1}{2}} \leq \frac{\mu\Big(\mathrm{Sq}_{Q_{N,K}}\Big)}{\mu\Big(\mathrm{Pos}_{Q_{N,K}}\Big)} \leq 1$$

2. Assume $n = k.n_1$, we partition into k groups by setting $N = (n_1, n_1, \ldots, n_1)$ and $K = (2, 2, \ldots, 2)$, then we have the following bounds:

$$c_1(2 + \frac{n}{2k})^{\frac{-k}{2}} \leq \frac{\mu\Big(\mathrm{Sq}_{Q_{N,K}}\Big)}{\mu\Big(\mathrm{Pos}_{Q_{N,K}}\Big)} \leq c_2(\frac{n}{ck})^{\frac{-k+1}{2}}$$

where c, c_1 and c_2 are absolute constants with $c = \frac{2^{10}e}{48}$.

Note that both cases considered above are contained in $Q_{n,2k}$. In particular, Blekherman's Theorems 4.1 and 6.1 from [18] give the following estimates:

$$\frac{n^{\frac{k+1}{2}}}{(\frac{n}{2}+2k)^k}\frac{c_1 k!(k-1)!}{4^{2k}(2k)!} \leq \frac{\mu\Big(\mathrm{Sq}_{Q_{n,2k}}\Big)}{\mu\Big(\mathrm{Pos}_{Q_{n,2k}}\Big)} \leq \frac{c_2 4^{2k}(2k)!\sqrt{k}}{k!} n^{\frac{-k+1}{2}}$$

where c_1 and c_2 are absolute constants.

The first case in Corollary 3.1.8, is a modest example to show the reflection of monomial structure in our bounds. Bounds in the second case is dependent on $\frac{n}{k}$ instead of n which shows the effect of underlying multihomogeneity. In particular in the cases that k and n are comparable bounds behave significantly different then the bounds of Blekherman.

In general, Theorem 3.1.7 proves that if we assume multihomogeneity on the set of variables $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$, bounds derived in Blekherman's work for the ratio of sums of squares to non-negative polynomials, holds locally for every set of variable $\bar{x}_j$.

The remainder of this paper is structured as follows: In Section 2 we define two different inner products, and investigate basic relations between geometries introduced by these two inner products. Section 2 also includes definiton of zonal harmonics and their basic properties. The hurried reader can see the definitions at the very beginning and then go to Lemmata 3.2.8, 3.2.11, 3.2.13 and 3.2.14. In Section 3 we prove volumetric bounds for $\mathrm{Pos}_{Q_{N,K}}$. A key step is discovering existence of an isotropic measure linked to the zonal harmonics. In Section 4 we give bounds for $\mathrm{Sq}_{Q_{N,2K}}$ via classical convex geometry. Section 5 is devoted to polynomials that are powers of linear forms. The bounds there are derived by a simple duality observation.

3.2 Harmonic Polynomials and Euclidean Balls

In this section we develop necessary background for the proof of Theorem 3.1.7. We are going to make use of two different inner products on $P_{N,K}$, mainly due to two useful notions of duality.

Definition 3.2.1. *For $f, g \in P_{N,K}$ the "usual" inner product is defined as* $\langle f, g\rangle := \int\limits_S fg\, d\sigma$. *For* $f(x) = \sum_\alpha c_\alpha x^\alpha \in P_{N,K}$ *with* $\alpha = (\alpha_1, \dots, \alpha_n)$ *we define the linear differential operator* $D[f] := \sum_\alpha c_\alpha \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\right)$, *and set* $\langle f, g\rangle_D := D[f](g)$. *This way of defining* $\langle f, g\rangle_D$, *introduces an inner product which we call the "differential" inner product.* $\diamond$

If $g(x) = \sum_\alpha b_\alpha x^\alpha \in P_{n,d}$ and $f(x) = \sum_\alpha c_\alpha x^\alpha \in P_{n,d}$ then it is easily checked that

$$\langle f, g\rangle_D = d! \sum_{\alpha_1+\cdots+\alpha_n=d} \frac{c_\alpha b_\alpha}{\binom{d}{\alpha_1,\dots,\alpha_n}}$$

where $\binom{d}{\alpha_1,\dots,\alpha_n}$ is the multinomial coefficient $\frac{d!}{\alpha_1!\cdots\alpha_n!}$.

Below we list some useful properties of the differential inner product. This inner product is actually the *Bombieri-Weyl inner product* (see, e.g., [112]) in our setting.

Lemma 3.2.2. *For any vectors $\bar{v}, \bar{x} \in S$ we define*

$$K_v(x) = (v_1x_1 + \cdots + v_{n_1}x_{n_1})^{2k_1}(v_{n_1+1}x_{n_1+1} + \cdots + v_n x_{n_1+n_2})^{2k_2} \\ \cdots (v_{n-n_m+1}x_{n-n_m+1} + \cdots + v_n x_n)^{2k_m}$$

Then

1. $\langle p, K_v(x)\rangle_D = p(v)$
2. $\langle pq, h\rangle_D = \frac{\deg(q)!}{\deg(h)!}\langle q, D[p](h)\rangle_D$
3. *Let* $s_i := \sum_{j=1}^{i} n_j$ *and* $s_0 := 0$ *let for any* i $\Delta_i = \frac{\partial^2}{\partial x^2_{s_{i-1}+1}} + \cdots + \frac{\partial^2}{\partial x^2_{s_i}}$ *and* $r_i = (x^2_{s_{i-1}+1} + \cdots + x^2_{s_i})^{\frac{1}{2}}$ *then* $(p+2)(p+1)\langle r_i^2 p, q\rangle_D = \langle p, \Delta_i(q)\rangle_D$

We would like to compare the geometry induced by the two different inner products. For instance how are the Euclidean balls with respect to two different products related to each other? To find out, we will need to introduce a generalization of harmonic polynomials that applies to our multihomogenous setting, and define corresponding linear operators acting on underlying vector spaces.

We call f Π-*harmonic* if $\Delta_1(f) = \Delta_2(f) = \cdots = \Delta_m(f) = 0$. One may suspect that Π-harmonicity is too strong a condition to be as natural as ordinary harmonicity. However, one observes that Π-harmonicity is a special case of Helgason's general theory of Harmonic polynomials (see Chapter 3, [67]) .

Let $H_{N,K}$ be the space of Π-harmonic polynomials in $P_{N,K}$. We observe that $H_{N,K}$ is a vector space by the linearity of the Δ_i operators. Also, for any vector $K = (k_1, \ldots, k_m) \in (\mathbb{N} \cup \{0\})^m$ we define

$$\mathcal{K}_K = \{Q \in (\mathbb{N} \cup \{0\})^m \mid Q = (q_1, \ldots, q_m),\ 2k_i \geq q_i \text{ and } q_i \text{ is even.}\}.$$

Lemma 3.2.3. *For $U, Q \in \mathcal{K}_K$ such that $Q \neq U$ we have that $H_{N,U}$ and $H_{N,Q}$ are orthogonal with respect to the usual inner product.*

Proof. Let $f \in H_{N,Q}$ and $g \in H_{N,U}$ with $Q \neq U$. Without loss of generality, assume $Q = (q_1, \ldots, q_m)$, $U = (u_1, \ldots, u_m)$ and $q_1 \neq u_1$. Then we have

$$(q_1 - u_1) \int\limits_{S^{n_1-1}} fg\, d\sigma_1 = \int\limits_{S^{n_1-1}} (q_1 fg - u_1 fg)\, d\sigma_1$$

$$\int\limits_{S^{n_1-1}} (q_1 fg - u_1 fg)\, d\sigma_1 = \int\limits_{S^{n_1-1}} (fD_n(g) - gD_n(f))\, d\sigma_1 = \int\limits_{B^{n_1-1}} (f\Delta_1 g - g\Delta_1 f) d\sigma_1 = 0.$$

So $\int\limits_{S^{n_1-1}} fg d\sigma_1 = 0$ and thus

$$\int\limits_S fg \, d\sigma = \int\limits_{S^{n_m-1}} \cdots \int\limits_{S^{n_1-1}} fg \, d\sigma_1 \cdots d\sigma_m = 0$$

■

To see the connection between our two inner products we use a variant of a map introduced by Reznick [98]: Let $T : P_{N,K} \to P_{N,K}$ be defined via

$$T(f)(x) = A^{-1} \int\limits_{S^{n_1-1}\times\ldots\times S^{n_m-1}} f(v)K(v,x) \, d\sigma(v)$$

where $A = \int\limits_{S^{n_1-1}\times S^{n_2-1}\ldots\times S^{n_m-1}} x_{n_1}^{2k_1} x_{n_1+n_2}^{2k_2} \cdots x_n^{2k_m}$ and $K(v,x)$ as defined in Lemma 3.2.2.

The following lemma shows that the operator T captures the relationship between our two inner products.

Lemma 3.2.4. *For all $f, g \in P_{N,K}$ we have $\langle T(f), g\rangle_D = A^{-1} 2k_1! \cdots 2k_m! \langle f, g\rangle$.*

Proof. The case of arbitrary m is only notationally more difficult than the $m=2$, thanks to induction. So we assume without loss of generality than $m=2$.

$$\langle T(f), g\rangle_D = \left\langle A^{-1} \int\limits_{S^{n_1-1}\times S^{n_2-1}} f(v)v^K \, d\sigma, g \right\rangle_D = A^{-1} \int\limits_{S^{n_1-1}\times S^{n_2-1}} \langle f(v)v^K, g\rangle_D \, d\sigma$$

Since $\langle v^K, g\rangle_D = 2k_1!2k_2!g(v)$ we have

$$A^{-1} \int\limits_{S^{n_1-1}\times S^{n_2-1}} \langle f(v)v^K, g\rangle_D \, d\sigma = A^{-1} \int\limits_{S^{n_1-1}\times S^{n_2-1}} 2k_1!2k_2! f(v)g(v) \, d\sigma$$

$$= \frac{2k_1!2k_2!}{A} \langle f, g\rangle$$

■

The lemma above immediately tells us T is one-to-one since for $f, h \in P_{N,K}$, assuming $T(f) = T(h)$ implies $\langle f, g\rangle = \langle h, g\rangle$ for all $g \in P_{N,K}$. To prove our next lemma we need to recall a theorem of Funk and Hecke.

Theorem 3.2.5. *[50] Given any measurable function K on $[-1, 1]$ such that the integral*

$$\int_{-1}^{1} |K(t)| \, (1-t^2)^{\frac{n-2}{2}} \, dt$$

is well-defined, for every function H that is harmonic on S^n we have

$$\int_{S^n} K(\sigma \cdot \zeta) H(\zeta) \, d\zeta = \left(\mathrm{Vol}\Big(S^{n-1}\Big) \int_{-1}^{1} K(t) P_{k,n}(t)(1-t^2)^{\frac{n-2}{2}} \right) H(\sigma)$$

where $P_{k,n}(t)$ is the classical Gegenbauer (ultraspherical) polynomial.

Gegenbauer polynomials are naturally introduced by zonal harmonics and they do exist more generally in spaces of sparse polynomials as well. Zonal harmonics and ultraspherical polynomials in our setting are introduced in Lemmata 3.2.12, 3.2.13, 3.2.14.

Lemma 3.2.6. *For $f \in H_{N,K}$ we have*

$$T(f)(x) = E_{N,K} f(x)$$

where $E_{N,K}$ depends only on N and K.

Proof. Defining $K_i(v, x) = (v_{s_{i-1}+1} x_{s_{i-1}+1} + \cdots + v_{s_{i-1}+n_i} x_{s_{i-1}+n_i})^{2k_i}$, we have

$$T(f)(x) = A^{-1} \int_{S^{n_m-1}} \cdots \int_{S^{n_1-1}} f(v) \prod_{i=1}^{m} K_i(v, x) \, d\sigma_i(v)$$

We observe that each K_i satisfies the assumptions of the Funk-Hecke formula. So we have

$$T(f)(x) = A^{-1} \int\limits_{S^{n_m-1}} \cdots \int\limits_{S^{n_2-1}} A_1 f_1(v,x) \prod_{i=2}^{m} K_i(v,x)\, d\sigma_i(v)$$

here $f_1(v,x)$ means first n_1 variables are fixed to $x_1, \ldots, x_{n_1}$ and the rest left as variables in the integral and $A_1 = \left(\frac{vol(S^{n_1-1})}{\dim(H_{n_1,2k_1})} \int\limits_{-1}^{1} t^{2k_1} P_{n_1,2k_1}(t)(1-t^2)^{\frac{n_1-2}{2}}\, dt \right)$ where $P_{n_1,2k_1}$ is the corresponding ultraspherical polynomial. Iterating the argument we have

$$\begin{aligned} T(f)(x) &= A^{-1} \prod_i A_i f(x) \\ &= A^{-1} \prod_i \left(\frac{vol(S^{n_i-1})}{\dim(H_{n_i,2k_i})} \int\limits_{-1}^{1} t^{2k_i} P_{n_i,2k_i}(t)(1-t^2)^{\frac{n_i-2}{2}}\, dt \right) f(x) \end{aligned}$$

■

Thanks to Lemma 3.2.6 we know that Π-harmonic polynomials are eigenvectors for T. We also know a relation between our two inner products thanks to Lemma 3.2.4, and the orthogonality of spaces of Π-harmonic polynomials with respect to usual inner product thanks to Lemma 3.2.3. We thus immediately obtain the following corollary.

Corollary 3.2.7. *For $U, Q \in \mathcal{K}_K$ such that $Q \neq U$ we have that $H_{N,U}$ and $H_{N,Q}$ are orthogonal with respect to the differential inner product.*

Remark 3.2.1. *For notational convenience we set $r = r_1 \cdots r_m$ where r_i is as defined in Lemma 3.2.2 and let $r^\alpha = r_1^{\alpha_1} \cdots r_m^{\alpha_m}$ for any $\alpha \in \mathbb{Z}^m$.*

For $g \in SO(n_1) \times \cdots \times SO(n_m)$ and $f \in P_{N,K}$ we define $g \circ f$ by setting $g \circ f(x) = f(g^{-1}(x))$. This gives a well defined group action on vector spaces of polynomials. We observe that the operator T commutes with $SO(n_1) \times \cdots \times SO(n_m)$ action.

This implies that $T(r^K) = ar^K$ for some constant a since the only polynomials that are fixed under the action are constant multiples of r^K. To compute a we check $1 = T(r^K)(e_{n_1} + \cdots + e_n) = ar^K(e_1 + \cdots + e_n) = a$. Hence r is fixed under T. Lemma 3.2.4 implies that the hyperplane orthogonal to r is fixed also. $\diamond$

Now we prove a decomposition lemma for $P_{N,K}$ in a Hilbert space sense.

Lemma 3.2.8. $P_{N,K} = \bigoplus_{\alpha \in \mathcal{K}_K} r^{K-\alpha} H_{N,\alpha}$

Proof. First observe that $r^{K-\alpha} H_{N,\alpha} \perp r^{K-\beta} H_{N,\beta}$ for any $\alpha \neq \beta$ by Lemma 3.2.3. Also via iterated usage of third property in Lemma 3.2.2 we observe $r^{K-\alpha} H_{N,\alpha} \perp_D r^{K-\beta} H_{N,\beta}$. So direct sum makes sense for both of the inner products. Let $E = \bigoplus_{\alpha \in \mathcal{K}_K} r^{K-\alpha} H_{N,\alpha}$ and assume $E \neq P_{N,K}$. This implies there exists $p \in P_{N,K}$ such that $p \perp_D E$. By assumption $p \notin H_{N,K}$ therefore there exist i such that $\Delta_i(p) \neq 0$. Wlog say $\Delta_1(p) = p_1 \neq 0$. Then if p_1 is Π-harmonic we have $\langle p, r_1^2 p_1 \rangle_D = \langle \Delta_1(p), p_1 \rangle_D = \langle p_1, p_1 \rangle_D \neq 0$. This yields a contradiction since $r_1^2 p_1 \in E$. If p_1 is not Π-harmonic there exist a j such that $\Delta_j(p_1) \neq 0$. Wlog say $\Delta_1(p_1) = p_2$. Repeating the same argument, we arrive to a contradiction surely since all polynomials of degree 0 are Π-harmonic! ■

Corollary 3.2.9. *$P_{N,K}(S) = \bigoplus_{\alpha \in K^m} H_{N,\alpha}(S)$ where $P_{N,K}(S)$ is restriction of $P_{N,K}$ to $S = S^{n_1-1} \times S^{n_2-1} \times S^{n_3-1} \times \ldots \times S^{n_m-1}$.*

From Lemma 3.2.8 we know how $P_{N,K}$ is decomposed into Π-harmonic polynomials. We also know $T(f) = E_{N,K} f$ for any $f \in H_{N,K}$. For $f \in r^{K-\alpha} H_{n,\alpha}$, since T is averaging over S where $r^{K-\alpha}$ is constant, repeating Funk-Hecke argument in Lemma 3.2.6 we observe $T(f) = C_{N,\alpha} f$ for some constant depending on N and α

only. We would like to compute these constants and write the operator T explicitly. Thankfully the integrals that gives the constants are well known and computed. (See for example Lemma 7.4 of [17]). We write the result without proof.

Lemma 3.2.10. *Let f_α be the projection of f onto the subspace $H_{N,\alpha}$ then we have*

$$T(f) = \sum_{\alpha \in \mathcal{K}_K} \prod_{i=1}^{m} \frac{k_i!\Gamma(\frac{n_i+2k_i}{2})}{(k_i - \frac{\alpha_i}{2})!\Gamma(\frac{n_i+2k_i+\alpha_i}{2})} r^{K-\alpha} f_\alpha.$$

Now let $f = \sum_{\alpha \in \mathcal{K}_K} r^{K-\alpha} f_\alpha$ that is $\|f\|^2 = \langle f, f\rangle = \sum_{\alpha \in \mathcal{K}_K} \|f_\alpha\|^2$. We set $a_\alpha = \prod_{i=1}^{m} \frac{k_i!\Gamma(\frac{n_i+2k_i}{2})}{(k_i - \frac{\alpha_i}{2})!\Gamma(\frac{n_i+2k_i+\alpha_i}{2})}$. Then define $A_{N,K} = \max_{\alpha \in \mathcal{K}_K} a_\alpha$, $B_{N,K} = \min_{\alpha \in \mathcal{K}_K} a_\alpha$ and $C_{N,K} = A^{-1} \prod_{i=1}^{m} 2k_i!$.

$$A_{N,K} C_{N,K} \|f\|^2 \geq \langle T(f), T(f)\rangle_D = C_{N,K}\langle f, T(f)\rangle \geq B_{N,K} C_{N,K} \|f\|^2$$

We denote the ball with respect to the usual inner product by B and the ball with respect to the differential inner product by B_D. The observation above implies

$$\frac{1}{\sqrt{A_{N,K}C_{N,K}}} T(B) \subseteq B_D \subseteq \frac{1}{\sqrt{B_{N,K}C_{N,K}}} T(B)$$

Lemma 3.2.11. *Let T be the operator on $P_{N,K}$ as defined before Lemma 3.2.4, let $A_{N,K}$, $B_{N,K}$, $C_{N,K}$ be defined as above , and let dim be the dimension of $P_{N,K}$, then we have the following*

$$A_{N,K} = 1$$

$$B_{N,K} = \left(\prod_{i=1}^{m} \binom{\frac{n_i}{2} + 2k_i}{k_i} \right)^{-1}$$

$$\prod_{i=1}^{m}\left(\frac{1}{2k_i+\frac{n_i}{2}}\right)^{\frac{k_i}{2}} \leq |\det(T)|^{\frac{1}{dim}} \leq \prod_{i=1}^{m}\left(\frac{1}{1+\frac{n_i}{2k_i}}\right)^{\frac{k_i}{2}}$$

$$\prod_{i=1}^{m}\left(\frac{1}{2k_i+\frac{n_i}{2}}\right)^{\frac{k_i}{2}} \leq \sqrt{C_{N,K}}(\frac{|B_D|}{|B|})^{\frac{1}{dim}} \leq e^{\frac{k}{2}}\prod_{i=1}^{m}\left(1+\frac{1}{\frac{n_i}{2k_i}+1}\right)^{\frac{k_i}{2}}$$

Proof.

$$a_\alpha = \prod_{i=1}^{m}\frac{k_i!(\frac{n_i}{2}+k_i)!}{(k_i-\frac{\alpha_i}{2})!(\frac{n_i}{2}+k_i+\frac{\alpha_i}{2})!}$$

It is quite clear that a_α is maximized for $\alpha = (0,0,\ldots,0)$ and minimized for $\alpha = K$. Thus $A_{N,K} = 1$ and $B_{N,K} = \prod_{i=1}^{m}(\binom{\frac{n_i}{2}+2k_i}{k_i})^{-1}$. Also T is a diagonal operator in the basis of harmonic polynomials and it's entries are a_α. Thus

$$\frac{|T(B)|}{|B|} = |\det(T)| = \left|\prod_{\alpha\in\mathcal{K}_K} {a_\alpha}^{dim(H_{n,\alpha})}\right|$$

We observe $|\mathcal{K}_K| = (k_1+1)(k_2+1)\cdots(k_m+1)$,

$$a_\alpha = \prod_{i=1}^{m}\frac{\binom{\frac{n_i}{2}+2k_i}{k_i-\frac{\alpha_i}{2}}}{\binom{\frac{n_i}{2}+2k_i}{k_i}}$$

this yields the formula

$$|\det(T)|^{\frac{1}{dim}} = B_{N,K}\prod_{\alpha\in\mathcal{K}_K}\left(\prod_{i=1}^{m}\binom{\frac{n_i}{2}+2k_i}{k_i-\frac{\alpha_i}{2}}\right)^{\frac{\dim(H_{N,\alpha})}{\dim(P_{N,K})}}$$

If we partition $\mathcal{K}_K$ into k_1+1 subsets by defining $\mathcal{K}_j := \{\alpha\in\mathcal{K}_K : \alpha_1 = 2k_1-2j\}$ then we have

$$|\det(T)|^{\frac{1}{dim}} = B_{N,K}\prod_{j=0}^{k_i}\prod_{\alpha\in\mathcal{K}_j}a_\alpha = B_{N,K}\prod_{j=0}^{k_i}\binom{\frac{n_i}{2}+2k_i}{j}^{\frac{1}{k_i+1}}A$$

For some A determined by n_i and k_i for $i\geq 2$. We repeat the same trick for A and do some housekeeping to arrive at the following formula

$$|\det(T)|^{\frac{1}{dim}} = B_{N,K}\prod_{i=1}^{m}\left(\prod_{j=0}^{k_i}\binom{\frac{n_i}{2}+2k_i}{j}\right)^{\frac{1}{k_i+1}} = \prod_{i=1}^{m}\left(\prod_{j=0}^{k_i}\frac{\binom{\frac{n_i}{2}+2k_i}{j}}{\binom{\frac{n_i}{2}+2k_i}{k_i}}\right)^{\frac{1}{k_i+1}}$$

$$|\det(T)|^{\frac{1}{dim}} = \prod_{i=1}^{m}\left(\prod_{j=0}^{k_i}\frac{k_1!(\frac{n_1}{2}+k_1)!}{j!(\frac{n_1}{2}+2k_1-j)!}\right)^{\frac{1}{k_i+1}} = \prod_{i=1}^{m}\prod_{j=1}^{k_i}\left(\frac{j}{\frac{n_1}{2}+2k_1-j+1}\right)^{\frac{j}{k_i+1}}$$

Applying the trivial bounds $(\frac{1}{\frac{n_i}{2}+2k_i})^j \le \frac{k_1!(\frac{n_1}{2}+k_1)!}{j!(\frac{n_1}{2}+2k_1-j)!}$ and $\frac{j}{\frac{n_1}{2}+2k_1-j+1} \le \frac{k_i}{\frac{n_i}{2}+k_i+1}$ we have

$$\prod_{i=1}^{m}\left(\frac{1}{\frac{n_i}{2}+2k_i}\right)^{\frac{k_i}{2}} \le |\det(T)|^{\frac{1}{dim}} \le \prod_{i=1}^{m}\left(\frac{k_i}{\frac{n_i}{2}+k_i+1}\right)^{\frac{k_i}{2}}$$

Let us note that

$$B_{N,K}^{-1/2} \le \prod_{i=1}^{m}\left(e.\frac{\frac{n_i}{2}+2k_i}{k_i}\right)^{\frac{k_i}{2}}$$

$$B_{N,K}^{-1/2}|\det(T)|^{\frac{1}{dim}} \le e^{\frac{k}{2}}\prod_{i=1}^{m}\left(1+\frac{1}{\frac{n_i}{2k_i}+1}\right)^{\frac{k_i}{2}}$$

■

We define a class of Π-harmonic polynomials which turns out to be very useful. Assume $y_1,\dots,y_M$ is an orthonormal basis of $H_{N,K}$. Then let $v \in S^{n_1-1}\times\cdots\times S^{n_m-1}$, $q_v = \sum_{i=1}^{M} y_i(v)y_i$. We observe $q_v \in H_{N,K}$ moreover $\langle f, q_v\rangle = f(v)$ for all $f \in H_{N,K}$. This special polynomial q_v is called the zonal harmonic corresponding to the vector v on $H_{N,K}$. The following lemma states basic properties of zonal harmonics:

Lemma 3.2.12.

1. *$q_v(w) = q_w(v)$ for all $v, w \in S$.*

2. *$q_w(v) = q_{T(w)}(T(v))$ for all $v, w \in S$ and $T \in O(n_1) \times O(n_2) \times O(n_3) \times \cdots \times O(n_m)$.*

3. *$q_v(v) = \|q_v\|^2 = \dim H_{N,K}$.*

4. *$|q_v(w)| \leq \dim H_{N,K}$.*

Proof. (1) Let $e_1, \ldots, e_l$ be an orthonormal basis for $H_{N,K}$. Then $q_v = \sum_{i=1}^{l} \langle e_i, q_v \rangle e_i = \sum_{i=1}^{m} e_i(v) e_i$. So $q_v(w) = \sum e_i(v) e_i(w) = q_w(v)$.
(2) For $p \in H_{N,K}$ we have

$$p(T(w)) = \langle p \circ T, q_w \rangle = \int_S p(T(x)) q_w(x)\, d\sigma(x) = \int_S p(x) q_w(T^{-1}(x))\, d\sigma(x)$$

Since the zonal harmonic is unique we deduce that $q_w \circ T^{-1} = q_{T(w)}$.
(3) By the notation of (1) and using (2) afterward

$$q_v(v) = \langle q_v, q_v \rangle_\Pi = \langle \sum e_i(v) e_i, \sum e_i(v) e_i \rangle_\Pi = \sum |e_i(v)|^2$$

$$q_v(v) = \int_S q_v(v)\, d\sigma(v) = \int_S \sum_{i=1}^{m} |e_i(v)|^2 d\sigma(v) = m = \dim H_{N,K}$$

(4) $|\langle q_v, q_w \rangle| = |q_v(w)| \leq \|q_v\| \|q_w\| = \dim H_{N,K}$. Thus $|q_v(w)| \leq \dim H_{N,K} = q_v(v)$.
■

Now we define, for any vector $v \in S$, the polynomial $p_v = \sum_{\alpha \in \mathcal{K}_K} r^{K-\alpha} q_{v,\alpha}$ where $q_{v,\alpha}$ is the zonal harmonic corresponding to v in $H_{N,\alpha}$. Let $\|f\|_\infty = \max_{v \in S} |f(v)|$.

We observe p_v inherits properties from zonal harmonics:

Lemma 3.2.13. *1. $f(v) = \langle f, p_v \rangle$ for every $f \in P_{N,K}$.*

2. $\|p_v\|^2 = \langle p_v, p_v \rangle = \sum \langle q_{v,\alpha}, q_{v,\alpha} \rangle = \sum \dim H_{N,\alpha} = \dim P_{N,K}$

3. $|f(v)| = |\langle f, p_v \rangle| \le \|f\| \|p_v\|$ and thus $\frac{\|f\|_\infty}{\|f\|} \le \|p_v\| = \sqrt{\dim P_{N,K}}$.

4. $|p_v(w)| \le p_v(v)$ for all $w \in S$ and $\frac{\|p_v\|_\infty}{\|p_v\|} = \sqrt{\dim P_{N,K}}$.

The third property turns out to be a characterization of the polynomials p_v:

Lemma 3.2.14. *Let $f \in P_{N,K}$ be such that $\frac{\|f\|_\infty}{\|f\|} \ge \frac{\|g\|_\infty}{\|g\|}$ for all $g \in P_{N,K}$. Then f is a constant multiple of p_v for some v.*

Proof. Assume $\|f\|_\infty = f(v) = C$ and let $T = \{g \in P_{N,K} : g(v) = C\}$. We observe for $g \in T$, $\|g\|_\infty \ge C$.

Using the assumption on f we deduce $\|g\| \ge \|f\|$ for all $g \in T$. Thus f is the shortest form on the hyperplane. We also observe $g(v) = \langle g, p_v \rangle$ from Lemma 3.2.13. This proves f to be a constant multiple of p_v. ■

Let us consider the hyperplane $L_C := \{q \in P_{N,K} : q(v) = C\} = \{q \in P_{N,K} : \langle q, p_v \rangle = C\}$. We define $SO(v) := \{g \in SO(n_1) \times \cdots \times SO(n_m) : g(v) = v\}$. Now observe that L_C is fixed under $SO(v)$ action. This implies p_v is fixed under $SO(v)$ action and thus, for every $c \in \mathbb{R}$ and $M_c =: \{x \in \mathbb{R}^n : \langle x, v \rangle = c\}$, p_v is constant on M_c. This implies $p_v(w) = q_{N,K}(\langle v, w \rangle)$ for some univariate polynomial $q_{N,K}$. This $q_{N,K}$ is the Gegenbauer or ultraspherical polynomial in our setting. Gegenbauer polynomial in our setting or the classical Gegenbauer polynomial will be both referred as ultraspherical polynomial throughout this paper.

3.3 The Cone of Nonnegative Polynomials

In this section we construct an isotropic measure introduced by the zonal harmonics, then use a theorem of Lutwak, Yhang and Zhang (Theorem 3.3.1 below) on the volume of convex hull of an isotropic measure supported on the sphere. Our upper bound for $\mathrm{Vol}\left(\overline{\mathrm{Pos}}_{N,K}\right)$ then follows from Theorem 3.3.1 via duality.

Let us start by defining isotropicity: A measure Z on S^{t-1} is isotropic if for every $x \in \mathbb{R}^t$ we have

$$\|x\|_2^2 = \int_{S^{t-1}} \langle x, y \rangle^2 \, dZ(y)$$

We need to introduce one more definition to state the theorem Lutwak, Yhang and Zhang. For a convex body $K \in \mathbb{R}^n$ the polar of K denoted by K° is defined as follows:

$$K^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$$

The main theorem of [78] is the following:

Theorem 3.3.1. *(Lutwak, Yhang, Zhang) [78] If Z is an isotropic measure on S^{t-1} whose centroid is at the origin and $Z_\infty = \mathrm{Conv}(\mathrm{Supp}(Z))$, then we have*

$$\mathrm{Vol}(Z_\infty^\circ) \leq \frac{t^{\frac{t}{2}}(t+1)^{\frac{t+1}{2}}}{t!}$$

The lemma below states our upper bound for $\mathrm{Vol}\left(\overline{\mathrm{Pos}}_{N,K}\right)$. As we derive our bounds we will find that $\overline{\mathrm{Pos}}_{N,K}$ is always in John's position in a sense we now describe.

Remark 3.3.1. *We will observe that $\overline{\mathrm{Pos}}_{N,K}$ is a dual body to the convex hull of an isotropic measure on the sphere. Condition of being an isotropic measure with* *centroid at the origin is actually a "continuous" version of the decomposition of identity in John's Theorem. This point of view is elaborated in [53], and essentially tells us that a section of the cone of nonnegative polynomials $\overline{\mathrm{Pos}}_{N,K}$ is in John's position. This fact will remain valid for cone of nonnegative polynomials supported with arbitrary Newton polytopes.*

Theorem 3.3.1 above uses a "continuous" version of John's theorem combined with the observation of Ball [11] that conditions of John's theorem are compatible with the Brascamp-Lieb inequality to derive their sharp estimates.

Barvinok and Blekherman [13] used the classical version of John's Theorem to approximate the volume of the convex hull of orbits of compact groups. The classical John's Theorem provides very good approximation for ellipsoid-like bodies but may not be sharp for convex bodies that do not resemble ellipsoids. For instance, as far as we are able to compute Barvinok and Blekherman's Theorem yields an upper bound of order $\sqrt{M}$ for the ratio $\left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Pos}}_{N,K}\right)}{\mathrm{Vol}(B)}\right)^{\frac{1}{M}}$. $\diamond$

Lemma 3.3.2.

$$\left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Pos}}_{N,K}\right)}{|B|}\right)^{\frac{1}{M}} \leq C$$

where M is the dimension of $\overline{\mathrm{Pos}}_{N,K}$, B is the M-dimensional ball with respect to usual inner product, and C is an absolute constant bounded from above by 5.

Proof. We identify (N,K)-homogenous polynomials with the corresponding vector space $P_{N,K}$ of dimension $\mathbb{R}^{\binom{n_1+2k_1-1}{2k_1}\binom{n_2+2k_2-1}{2k_2}}$ where $N=(n_1,n_2)$ and $K=(2k_1,2k_2)$.

We define a map $\Phi : S^{n_1-1} \times S^{n_2-1} \rightarrow P_{N,K}$ by

$$\Phi(v) = \frac{p_v - r}{\sqrt{\binom{n_1+2k_1-1}{2k_1}\binom{n_2+2k_2-1}{2k_2} - 1}}$$

where p_v is the polynomial corresponding to the vector v as in Lemma 3.2.13.

It is not hard to prove that Φ is Lipschitz and injective. Now let U be the subspace of $P_{N,K}$ defined by $U = \{p \in P_{N,K} : \langle p, r\rangle = 0\}$. We observe that for all $v \in S^{n_1-1} \times S^{n_2-1}$, $\Phi(v) \in U$ and $\|\Phi(v)\|_2 = 1$.
Now let $\sigma_1 \times \sigma_2$ be the product of uniform measures on S^{n_1-1} and S^{n_2-1}. We define the measure Z on the unit sphere of U, as the push-forward measure of $\sigma_1 \times \sigma_2$ under the map Φ. It follows directly that Z is well-defined, with $\mathrm{Supp}(Z) = \mathrm{Image}(\Phi)$, and satisfies the following property (see, e.g., [81] Theorem 1.19 Chapter 1)

$$\int g \, dZ = \int g(\Phi) \; \sigma_1 \times \sigma_2$$

Now for every $q \in U$ we have the following equality

$$\begin{aligned} \|q\|_2^2 = \int\limits_{S^{n_1-1}\times S^{n_2-1}} q(v)^2 \sigma_1 \times \sigma_2(v) &= \int\limits_{S^{n_1-1}\times S^{n_2-1}} M\langle q, \Phi(v)\rangle^2 \sigma_1 \times \sigma_2(v) \\ &= \int\limits_{S^{M-1}} M\langle q, v\rangle^2 dZ(v) \end{aligned}$$

where $M = \binom{n_1+2k_1-1}{2k_1}\binom{n_2+2k_2-1}{2k_2} - 1$. This simply implies MdZ is an isotropic measure on S^{M-1}!

To compute the centroid of Z let $q = \int\limits_{S^{n_1-1}\times S^{n_2-1}} p_v \; \sigma_1 \times \sigma_2(v)$. We observe q is invariant under the action of $SO(n_1) \times SO(n_2)$ as defined in Remark 3.2.1. This observation immediately yields $q = r$. Thus the centroid of Z is the origin. Now using Theorem 3.3.1 we deduce

$$\mathrm{Vol}(\mathrm{Conv}(Im(\Phi))^\circ) \leq \frac{M^{\frac{M}{2}}(M+1)^{\frac{M+1}{2}}}{M!}$$

We define $A = \text{Conv}(\{p_v - r : v \in S^{n_1-1} \times S^{n_2-1}\})$ where p_v is the polynomial corresponding to the vector v as defined in Lemma 3.2.13. We consider A in $\mathbb{R}^M$, and note that $A = \sqrt{M}\text{Conv}(\text{Image}(\Phi))$. Using the above estimate we have

$$|A^\circ| \leq \frac{M^{\frac{M}{2}}(M+1)^{\frac{M+1}{2}}}{M!M^{\frac{M}{2}}} \tag{3.1}$$

Now observe that for all $q \in P_{N,K}$ that satisfies $\int\limits_{S^{n_1} \times S^{n_2-1}} q = \langle q, r\rangle = 1$ we have

$$q(v) \geq 0 \text{ for all } v \in S^{n_1} \times S^{n_2-1} \Leftrightarrow (q-r)(v) \geq -1 \Leftrightarrow \langle r-q, p_v - r\rangle \leq 1$$

Thus $\overline{\text{Pos}}_{N,K} - r = A^\circ$. Hence by 3.1

$$\left(\frac{\left|\overline{\text{Pos}}_{N,K}\right|}{|B|}\right)^{\frac{1}{M}} \leq \left(\frac{M^{\frac{M}{2}}(M+1)^{\frac{M+1}{2}}}{M!M^{\frac{M}{2}}\,|B|}\right)^{\frac{1}{M}}$$

where B denotes the M dimensional ball.

$$\left(\frac{\left|\overline{\text{Pos}}_{N,K}\right|}{|B|}\right)^{\frac{1}{M}} \leq \frac{|B|^{\frac{-1}{M}}\,M^{\frac{1}{2}}}{\frac{M}{e}}$$

$$\left(\frac{\left|\overline{\text{Pos}}_{N,K}\right|}{|B|}\right)^{\frac{1}{M}} \leq \frac{e}{\sqrt{M}\,|B|^{\frac{1}{M}}} \leq 5$$

■

Remark 3.3.2. *Blekherman derived an upper bound for* $\left(\frac{\text{Vol}\left(\overline{\text{Pos}}_{N,K}\right)}{\text{Vol}(B)}\right)^{\frac{1}{M}}$ *in [18] for the usual homogenous polynomial setting with degree fixed. Blekherman's bounds seems sharper than ours for fixed degree homogenous polynomials, i.e., the special case where the underlying Newton polytope is a scaled standard simplex. However,* *Blekherman's methods do not apply to polynomials supported on more general Newton polytopes.* $\diamond$

The following lemma states our lower bounds for $\mathrm{Vol}\left(\overline{\mathrm{Pos}}_{N,K}\right)$. The construction carried out in the previous proof seems to indicate a lower bound via discretization and Vaaler's Inequality [120]. For now we give the following lower bound by using the Gauge function.

Lemma 3.3.3.

$$(\frac{\left|\overline{\mathrm{Pos}}_{N,K}\right|}{|B|})^{\frac{1}{M}} \geq \frac{1}{\sqrt{16\max\{n_1,n_2\}(2k_1+1)(2k_2+1)}}$$

Proof. To derive a lower bound for $\left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Pos}}_{N,K}\right)}{\mathrm{Vol}(B)}\right)^{\frac{1}{M}}$ we examine $\overline{\mathrm{Pos}}_{N,K} - r$. For any $q \in P_{N,K}$ such that $\langle q, r\rangle = 0$ we observe

$$q \in \overline{\mathrm{Pos}}_{N,K} - r \Leftrightarrow q(v) \geq -1 \text{ for all } v \in S^{n_1-1} \times S^{n_2-1}$$

That is for $f \in U$ and $G_{\overline{\mathrm{Pos}}_{N,K}-r}(f)$ the Gauge Function of $\overline{\mathrm{Pos}}_{N,K} - r$ we have

$$G_{\overline{\mathrm{Pos}}_{N,K}-r}(f) = |\min_{x\in S} f(x)|$$

We set $\|f\|_{\infty} = \max_{x\in S^{n_1-1}\times S^{n_2-1}} |f(x)|$ and let $S^{M-1} = \{f \in U : \|f\|_2 = 1\}$ then

$$\left(\frac{\left|\overline{\mathrm{Pos}}_{N,K}\right| - r}{|B|}\right)^{\frac{1}{M}} = \left(\int_{S^{M-1}} |\min_{x\in S^{n_1-1}\times S^{n_2-1}} f(x)|^{-M}\, df\right)^{1/M}$$

and

$$\left(\frac{\left|\overline{\mathrm{Pos}}_{N,K}\right| - r}{|B|}\right)^{\frac{1}{M}} \geq \left(\int_{S^{M-1}} \|f\|_{\infty}^{-d}\, df)^{1/d} \geq \int_{S^{M-1}} \|f\|_{\infty}^{-1}\, df \geq (\int_{S^{M-1}} \|f\|_{\infty}\, df\right)^{-1}$$

where the second line of inequalities is derived by consecutive applications of Jensen's inequality. Therefore to prove a lower bound for the volume of $\mathrm{Vol}\left(\overline{\mathrm{Pos}}_{N,K}\right)$, it suffices to prove an upper bound for $\int\limits_{S^{M-1}} \|f\|_\infty \, df$. To this end we invoke Theorem 3.1 from [13] for the compact group $G = SO(n_1) \times SO(n_2)$ and the vector space $V = (R^{n_1})^{\otimes 2k_1} \times (R^{n_2})^{\otimes 2k_2}$. Barvinok's theorem shows that for any $m > 0$ we have

$$\left(\int\limits_{S^{n_1-1}\times S^{n_2-1}} f(x)^{2m}\right)^{\frac{1}{2m}} \leq \|f\|_\infty \leq d_m^{\frac{1}{2m}} \left(\int\limits_{S^{n_1-1}\times S^{n_2-1}} f(x)^{2m}\right)^{\frac{1}{2m}}$$

where $d_m \leq \binom{n_1+2k_1m-1}{2k_1m}\binom{n_2+2k_2m-1}{2k_2m}$. This yields

$$\int\limits_{S^{M-1}} \|f\|_\infty \, df \leq d_m^{\frac{1}{2m}} \int\limits_{S^{M-1}} \left(\int\limits_{S^{n_1-1}\times S^{n_2-1}} f(v)^{2m} \, dv\right)^{\frac{1}{2m}} df$$

Using Hölder's inequality and Fubini's theorem we have

$$\int\limits_{S^{M-1}} \|f\|_\infty \, df \leq d_m^{\frac{1}{2m}} \left(\int\limits_{S^{n_1-1}\times S^{n_2-1}} \int\limits_{S^{M-1}} f(v)^{2m} \, df \, dv\right)^{\frac{1}{2m}}$$

The average inside the integral is independent of vector v, thus for a fixed v

$$\int\limits_{S^{M-1}} \|f\|_\infty \, df \leq d_m^{\frac{1}{2m}} \left(\int\limits_{S^{M-1}} \langle f, p_v\rangle^{2m}\right)^{\frac{1}{2m}}.$$

Note that we know $\|p_v\|_2 = \sqrt{M+1}$. So we obtain

$$\int\limits_{S^{M-1}} \|f\|_\infty \, df \leq {d_m}^{\frac{1}{2m}} \sqrt{M+1} \left(\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2}M)}{\sqrt{\pi}\Gamma(\frac{1}{2}M+m)}\right)^{\frac{1}{2m}}$$

$$\left(\frac{\Gamma(m+\frac{1}{2})}{\pi}\right)^{\frac{1}{2m}} \leq \sqrt{m} \text{ and } \left(\frac{\Gamma(\frac{1}{2}M)}{\Gamma(\frac{1}{2}M+m)}\right)^{\frac{1}{2m}} \leq \sqrt{\frac{2}{M}}$$

$$\int\limits_{S^{M-1}} \|f\|_\infty \, df \le \binom{n_1 + 2k_1 m - 1}{2k_1 m}^{\frac{1}{2m}} \binom{n_2 + 2k_2 m - 1}{2k_2 m}^{\frac{1}{2m}} \sqrt{M+1}\sqrt{m}\sqrt{\frac{2}{M}}$$

We set $h = \max\{n_1, n_2\}$, $m = h(2k_1+1)(2k_2+1)$, for the case $t = (2k_1+1)(2k_2+1) > h$ we have

$$\binom{n_1 + 2k_1 m - 1}{2k_1 m}^{\frac{1}{2m}} \binom{n_2 + 2k_2 m - 1}{2k_2 m}^{\frac{1}{2m}} \le (2k_1 m+1)^{\frac{n_1}{2m}}(2k_2 m+1)^{\frac{n_2}{2m}} \le t^{\frac{1}{t}}(th)^{\frac{2}{t}} \le 4$$

For the case $t = (2k_1+1)(2k_2+1) \le h$ we write $\binom{n_i+2k_i m-1}{2k_i m}^{\frac{1}{2m}} \le (n_i+1)^{\frac{2k_i}{2m}}$ then the rest of the proof follows similarly. Hence we have proved

$$\int\limits_{S^{M-1}} \|f\|_\infty \, df \le 4\sqrt{h(2k_1+1)(2k_2+1)}$$

■

Remark 3.3.3. *If we would like the bounds in Lemma 3.3.2 and Lemma 3.3.3 to be in terms of the body A that was introduced in the proof of Lemma 3.3.2 we have*

$$c_0 \le \left(\frac{|A|}{|B|}\right)^{\frac{1}{M}} \le 4\sqrt{\max\{n_1, n_2\}(2k_1+1)(2k_2+1)}$$

where c_0 is a constant. ◇

3.4 The Cone of Sums of Squares

In this section we prove our bounds for $\mathrm{Vol}\left(\overline{\mathrm{Sq}}_{N,K}\right)$. We start with the upper bound.

Lemma 3.4.1.

$$\left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Sq}}_{N,K}\right)}{\mathrm{Vol}(B)}\right)^{\frac{1}{M}} \leq \frac{9}{2}(2^{10}e)^{\frac{k_1+k_2}{2}}(\frac{k_1}{n_1+k_1})^{\frac{k_1}{2}}(\frac{k_2}{n_2+k_2})^{\frac{k_2}{2}}$$

where c is a constant with $c \leq 5$.

Proof. We define $C = \{p \in U_{N,K} : p + r \in \overline{\mathrm{Sq}}_{N,K}\}$. Let $h_C(f) = \max_{g \in C}\langle f, g\rangle$. We use Urysohn's Lemma [106] in order to bound volume of C. The mean width of C can be written as

$$W_C = 2\int_{S_{M-1}} h_C(f)\, d\sigma$$

where $S^{M-1} = \{f \in U_{N,K} : \|f\| = 1\}$. By Urysohn's Lemma we have

$$\left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Sq}}_{N,K}\right)}{\mathrm{Vol}(B)}\right)^{\frac{1}{M}} = \left(\frac{\mathrm{Vol}(C)}{\mathrm{Vol}(B)}\right)^{\frac{1}{M}} \leq \frac{W_C}{2}$$

Observe that the extreme points of C are of the form $g^2 - r$ where $g \in P_{N,K/2}$ and $\|g\| = 1$. Also observe for $f \in S^{M-1}$, we have $\langle f, r\rangle = \int_S f\, d\sigma = 0$ that is $\langle f, g^2 - r\rangle = \langle f, g^2\rangle$. Hence we could write $h_C(f) \leq \max_{g \in P_{N,K/2}, \|g\|=1}\langle f, g^2\rangle$.

$$\frac{W_C}{2} = \int_{S^{M-1}} h_C(f)\, d\sigma = \int_{S^{M-1}} \max_{g \in P_{N,K/2}, \|g\|=1}\langle f, g^2\rangle\, d\sigma \leq \int_{S^{M-1}} \max_{g \in P_{N,K/2}, \|g\|=1}|\langle f, g^2\rangle|\, d\sigma$$

For a fixed f, $\langle f, g^2\rangle$ is a quadratic form. So Theorem 3.1 of [13] or Barvinok's earlier inequality [14] for $q = \binom{n_1+k_1-1}{k_1}\binom{n_2+k_2-1}{k_2}$ yields

$$\frac{W_C}{2} \leq \int\limits_{S^{M-1}} (\int\limits_{S^{D-1}} \langle f, g^2\rangle^{2q})^{\frac{1}{2q}} d\sigma(g) d\sigma(f) \leq \binom{3q-1}{2q}^{\frac{1}{2q}} (\int\limits_{S^{D-1}} \int\limits_{S^{M-1}} \langle f, g^2\rangle^{2q} d\sigma(f) d\sigma(g))^{\frac{1}{2q}}$$

where $S^{D-1} = \{g \in P_{N,K/2} : \|g\| = 1\}$. Thanks to Reverse Hölder inequalities of J. Duoandikoetxea [43] we know $\|g^2\| \leq 2^{4(k_1+k_2)}$.

We follow the proof of Lemma 3.3.3 verbatim to arrive at the following estimate

$$\frac{W_C}{2} \leq \binom{3q-1}{2q}^{\frac{1}{2q}} 2^{4(k_1+k_2)} \sqrt{\frac{\binom{n_1+k_1-1}{k_1}\binom{n_2+k_2-1}{k_2}}{\binom{n_1+2k_1-1}{2k_1}\binom{n_2+2k_2-1}{2k_2}}}$$

After this point we apply classical bounds for binomial coefficients, hence

$$\left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Sq}}_{N,K}\right)}{\mathrm{Vol}(B)}\right)^{\frac{1}{M}} \leq \frac{9}{2}(2^8)^{\frac{k_1+k_2}{2}} (\frac{4ek_1}{n_1+k_1})^{\frac{k_1}{2}} (\frac{4ek_2}{n_2+k_2})^{\frac{k_2}{2}}$$

■

To prove our lower bound we need the following lemma which was essentially proved by Blekherman as Lemma 5.3 at [18]

Lemma 3.4.2. *(Blekherman)*

$$\mathrm{Sq}_{N,K}^{d*} \subseteq \mathrm{Sq}_{N,K}$$

where $\mathrm{Sq}_{N,K}^{d*}$ *is the dual cone with respect to the differential metric.*

Lemma 3.4.3.

$$\left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Sq}}_{N,K}\right)}{\mathrm{Vol}(B)}\right)^{\frac{1}{M}} \geq c \prod_{i=1}^{m} \left(\frac{1}{2k_i + \frac{n_i}{2}}\right)^{\frac{k_i}{2}}$$

Proof.

$$\langle T(r), r\rangle_D = \langle r, r\rangle_D = C_{N,K}\langle r, r\rangle = C_{N,K}$$

$$\overline{\mathrm{Sq}}_{N,K} = \{p \in \mathrm{Sq}_{N,K} : \langle p, r\rangle = 1\} := \{p \in \mathrm{Sq}_{N,K} : \langle p, r\rangle_D = C_{N,K}\}$$

$$A = \overline{\mathrm{Sq}}_{N,K} - r := \{p \in P_{N,K} : p + r \in \mathrm{Sq}_{N,K} \text{ and } \langle p, r\rangle_D = 0\}$$

$$A_d^{\circ} = \{q \in P_{N,K} : \langle q, r\rangle_D = 0, \langle q, p\rangle_D \leq 1 \; \forall p \in A\}$$

$$C_{N,K}^{-1} r - A_d^{\circ} = \{q \in P_{N,K} : \langle q, r\rangle_D = 1, \langle q, p\rangle_D \geq -1 \; \forall p \in A\}$$

$$C_{N,K}^{-1} r - A_d^{\circ} = \{q \in P_{N,K} : \langle q, r\rangle_D = 0, \langle q, p\rangle_D \leq 0 \; \forall p \in \overline{\mathrm{Sq}}_{N,K}\}$$

Observe that for any $f \in P_{N,K}$

$$\langle f, g\rangle_D \geq 0 \; \forall g \in \overline{\mathrm{Sq}}_{N,K} \Leftrightarrow \langle f, g\rangle_D \geq 0 \; \forall g \in \mathrm{Sq}_{N,K}$$

thus $C_{N,K}^{-1} r - A_d^{\circ} = B \subseteq \mathrm{Sq}_{N,K}^{d*} \subseteq \mathrm{Sq}_{N,K}$. Hence $B \subseteq \mathrm{Sq}_{N,K} = C_{N,K}^{-1}\overline{\mathrm{Sq}}_{N,K}$. By the Reverse Santalo inequality [24] we have

$$c \leq \left(\frac{\mathrm{Vol}(A)}{\mathrm{Vol}(B_D)}\right)^{\frac{1}{M}} \left(\frac{\mathrm{Vol}(A_d^{\circ})}{\mathrm{Vol}(B_D)}\right)^{\frac{1}{M}} \leq \left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Sq}}_{N,K}\right)}{\mathrm{Vol}(B_D)}\right)^{\frac{1}{M}} \left(\frac{\mathrm{Vol}\left(C_{N,K}^{-1}\overline{\mathrm{Sq}}_{N,K}\right)}{\mathrm{Vol}(B_D)}\right)^{\frac{1}{M}}$$

$$\sqrt{c \cdot C_{N,K}} \leq \left(\frac{\mathrm{Vol}\left(\overline{\mathrm{Sq}}_{N,K}\right)}{\mathrm{Vol}(B_D)} \right)^{\frac{1}{M}}$$

Using the bounds in Lemma 3.2.11 completes the proof.

■

3.5 The Cone of Powers of Linear Forms

This section develops quantitative bounds on the cone of even powers of linear forms. More precisely let $\overline{L}_{N,K} := \{p \in L_{Q_{N,K}} : \langle p, r \rangle = 1\}$, we prove upper and lower bounds on the volume of $\overline{L}_{N,K}$.

Now let us consider the image of p_v (as in Lemma 3.2.13) under the map T:

1. $\langle T(p_v), r \rangle_D = C_{N,K}$
2. For every $f \in P_{N,K}$ we have

$$\langle f, T(p_v) \rangle_D = C_{N,K} \langle f, p_v \rangle = C_{N,K} f(v)$$

Since for all $f \in P_{N,K}$, $\langle f, C_{N,K} K_v \rangle_D = C_{N,K} f(v)$ we have $T(p_v) = C_{N,K} K_v$. Now let $\acute{A} = \{p_v : v \in S\}$ and $A = \mathrm{Conv}(\acute{A})$. If we define the map Φ as in the non-negative polynomials section we observe $A = \sqrt{\dim(P_{N,K})}\,\mathrm{Conv}(\mathrm{Image}(\Phi))$. By Krein-Milman theorem and linearity of T we have

$$\overline{L}_{N,K} = \mathrm{Conv}(C_{N,K} K_v : v \in S) = \mathrm{Conv}(T(\acute{A})) = T(\mathrm{Conv}(\acute{A})) = T(A)$$

This implies that $\mathrm{Vol}\left(\overline{L}_{N,K}\right) = |\det(T)| \,\mathrm{Vol}(A)$. Therefore, from Remark 3.3.3 and the bounds derived in Lemma 3.2.11, we deduce the following estimate on the volume of $\overline{L}_{N,K}$.

Lemma 3.5.1.

$$c_o \prod_{i=1}^{m} \left(\frac{\frac{1}{k_i}}{2 + \frac{n_i}{2k_i}} \right)^{\frac{k_i}{2}} \leq \left(\frac{\left| \overline{L}_{N,K} \right|}{|B|} \right)^{\frac{1}{M}}$$
$$\leq 4\sqrt{\max\{n_1, n_2\}(2k_1+1)(2k_2+1)} \prod_{i=1}^{m} \left(\frac{1}{1 + \frac{n_i}{2k_i}} \right)^{\frac{k_i}{2}}$$

where c_0 *is a positive constant and* $c_0 \leq 5$.

4. CONDITION NUMBER OF RANDOM POLYNOMIAL SYSTEMS

4.1 Introduction

When designing algorithms for polynomial system solving, it quickly becomes clear that complexity is governed by more than simply the number of variables and degrees of the equations. Numerical solutions are meaningless without further information on the spacing of the roots, not to mention their sensitivity to perturbation. A mathematically elegant means of capturing this sensitivity is the notion of *condition number* (see, e.g., [21, 26], and the next section).

A subtlety behind complexity bounds incorporating the condition number is that it is probably as hard to compute the condition number — even if one allows a large multiplicative error — as it is to compute the numerical solution one seeks in the first place (see, e.g., [39] for a precise statement in the linear case). However, a growing body of results have shown that the condition number admits *probabilistic* bounds, thus enabling its use in average-case and/or high probability analysis of numerical algorithms. In fact, these probabilistic bounds have revealed that numerical solving can be done in polynomial-time on average, in spite of numerical solving having exponential *deterministic* complexity.

The numerical approximation of complex roots provides an instructive example of how one can profit from randomized input.

First, there are classical reductions showing that deciding the existence of complex roots for systems of polynomials in $\bigcup_{m,n\in\mathbb{N}}(\mathbb{Z}[x_1,\ldots,x_n])^m$ is **NP**-hard. However, classical algebraic geometry (Bertini's Theorem and Bézout's Theorem [108]) tells us that the number of complex roots of a *random* polynomial system $p:=(p_1,\ldots,p_m)\in\mathbb{C}[x_1,\ldots,x_n]$ (with each p_i having fixed positive degree d_i) is 0, $\prod_{i=1}^{n} d_i$, or infinite, according as $m > n$, $m = n$, or $m < n$. Any continuous positive probability measure on the coefficients will do in the preceding statement.

Secondly, examples like $p := (x_1 - x_2^2, x_2 - x_3^2, \ldots, x_{n-1} - x_n^2, (2x_n - 1)(3x_n - 1))$, which has affine roots $\left(2^{-2^{n-1}}, \ldots, 2^{-2^0}\right)$ and $\left(3^{-2^{n-1}}, \ldots, 3^{-2^0}\right)$, reveal that the number of digits necessary to distinguish the coordinates of roots of p may be exponential in n (among other parameters). However, it is now known (via earlier work on the condition number, or an application of discriminants [51] and Weyl's Tube Formula [57]) that *on average*, the number of digits needed to separate roots of p is polynomial in n. The key assumption on the underlying probability measure is that the probability of being near a certain discriminant variety be small. (See, e.g., [28] for more explicit results along these lines.) In particular, this discriminant distance criterion is equivalent to a suitable condition number being large. (We fully define condition numbers in the next section.)

In short, for the problem of numerically approximating a single complex root of a polynomial system, randomization enables polynomial-time average-case complexity when exponential deterministic complexity appears inevitable. The recent positive solution to Smale's 17th Problem provides a rigorous and explicit framework for this intuition.

It appears that similar speed-ups are possible for the harder problem of numerically approximating real roots for real polynomial systems. However, a new subtlety is that the number of real roots of n polynomials in n variables (of fixed degree) is no longer constant with probability 1, even if the probability measure for the coefficients is continuous and positive. In fact, small perturbations of the coefficients can sometimes change the number of real roots from positive to zero. Nevertheless, it is possible to define a condition number that enables useful average-case and high probability complexity estimates: This is accomplished in the seminal series of papers

[32, 33, 34], where the probability measure underlying the coefficients is a centered Gaussian distribution with specially chosen variances (and zero covariance).

Our main results (Theorems 4.3.9–4.4.14 below) show that useful condition number estimates can still be derived for a much broader class of probability measures, allowing dependence and non-Gaussian distributions. Unlike the existing literature, our methods do not use any imposed structure (by special variances) such as unitary invariance. This aspect allows localized analysis for sparse polynomials, where unitary action does not preserve sparsity.

We denote the condition number of a polynomial system $p = (p_1, p_2, \ldots, p_m)$ with $\kappa(p)$. Since the condition number notion is a bit technical, we defer the precise definition of $\kappa(p)$ until the next section. Our estimates cover overdetermined and square random polynomial systems for both generic and sparse cases. Up to our knowledge, our estimates are the first ones in literature for the overdetermined random systems. For square systems, the seminal work of Cucker, Malajovich, Krick and Wschebore [32, 33, 34] considers polynomial systems with Gaussian i.i.d random variables with specially chosen variances. We recall their result below.

Theorem 4.1.1. *Let $p = (p_1, \ldots, p_{n-1})$ be a system of homogenous polynomials where $p_i = \sum_{|\alpha|=d_i} \sqrt{\binom{d_i}{\alpha}} c_\alpha x^\alpha$ and c_α are i.i.d Gaussian random variables with mean zero and variance 1. Also let $D = \prod_i d_i$ where $d_i = \deg(p_i)$ and assume $d = \max_i d_i$ then the following inequalities hold:*

1. *Let $N = \sum_{i=1}^{n-1} \binom{n+d_i-1}{d_i}$, $K_n = 8d^2\sqrt{D}\sqrt{N}n^{\frac{5}{2}} + 1$ and $a > K_n$ then*
$$\mathbb{P}\{\kappa(p) \leq a\} \leq K_n \frac{(1+\log(a))^{\frac{1}{2}}}{a}.$$
2. $\mathbb{E}(\log \kappa(p)) \leq \log(K_n) + \log(K_n)^{\frac{1}{2}} + \log(K_n)^{-\frac{1}{2}}$.

Our results at the third section, provides estimates for a broader family of distributions allowing dependency among the coefficients. For the precise statement of our assumptions on the randomness, we refer the reader to the beginning of the section 3. Let us mention that known examples of distributions satisfying our assumptions include the uniform distributions on the Euclidean ball, the Euclidean sphere, and the B_p balls for $p > 2$. Random vectors which has independent subgaussian coordinates with bounded densities are also among the examples that satisfy our assumptions. To illustrate our results for generic polynomials systems, we recite a corollary of Theorem 4.3.9 for the square systems. In what follows, C is a universal constant and c_0, K are parameters of our assumptions on the randomness.

Theorem 4.1.2. *Let* $\mathbf{p}$ *be a random polynomial system as in Theorem 4.3.9 with* $m = n - 1$. *Then we have the following:*

1. *If* $d = \deg(p_i)$, $1 \leq i \leq m$, *set*

$$M := M(n, d, c_0, K) := C\sqrt{c_0}\sqrt{N}K\left(Cc_0Kd\log(ed)\right)^{n-2}.$$

then

(a)

$$\mathbb{P}\{\kappa(\mathbf{p}) \geq tM\} \leq \begin{cases} \frac{3}{t} & \text{if } 1 \leq t \leq e^{(n-1)\log(ed)} \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{(n-1)\log(ed)} \leq t \leq e^N \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}}\left(\frac{\log t}{N}\right)^{\frac{1}{2}} & \text{if } e^N \leq t \end{cases}$$

(b) *For all* $q \leq 1 - \frac{1}{2\log(ed)}$

$$\left(\mathbb{E}(\kappa(\mathbf{p})^q)\right)^{\frac{1}{q}} \leq Me^{\frac{1}{q}}.$$

Moreover

$$\mathbb{E}\log\kappa(p) \leq \log M + 1$$

2. If $\max_i \deg(p_i) = d$ *set*

$$M := M(n,d,c_0,K) := C\sqrt{c_0}\sqrt{N}K\left(Cc_0Kd^2\log(ed)\right)^{n-2}.$$

then

(a)

$$\mathbb{P}\{\kappa(p) \geq tM\} \leq \begin{cases} \frac{3}{t} & \text{if } 1 \leq t \leq e^{(n-1)\log(ed)} \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{(n-1)\log(ed)} \leq t \leq e^{N} \\ \frac{3}{t}\left(\frac{\log t}{N}\right)^{\frac{1}{2}}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{N} \leq t \end{cases}$$

(b) Then for all $q \leq 1 - \frac{1}{2\log(ed)}$

$$(\mathbb{E}(\kappa(p)^q))^{\frac{1}{q}} \leq Me^{\frac{1}{q}}.$$

Moreover

$$\mathbb{E}\log\kappa(p) \leq \log M + 1$$

As once can see from the corollary above our results are quite similar to the results of Cucker, Malajovich, Krick and Wschebore for Gaussian i.i.d polynomial systems. The only important difference is a factor of 2 in the expectation. We discuss this further at remarks 4.3.2 and 4.3.3.

Up to our knowledge, the only non-Gaussian result for condition number of polynomial systems is due to Nguyen [87]. The quantity that is analyzed in Nguyen's work is not the condition number and the degree of polynomials are assumed to be bounded by a small of number of the variables. Nevertheless, we borrow some of the ideas present in Nguyen's work, as we will mention in section 3.

In section 4, we consider random polynomial systems with a given monomial structure. Let $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ be a system of polynomials with a fixed set of monomials S_i for all p_i. At the beginning of section 4, we provide examples of S_i such that for a polynomial system $\mathbf{p}$ supported with $S = (S_1, S_2, \ldots, S_m)$, $\kappa(\mathbf{p})$ is always infinite independent of the coefficients. To rule out such examples, and to control the effect of the monomial structure on the conditioning, we develop a quantity depending on the set S that we denote by $H(S)$. In particular, we prove that if $\kappa(\mathbf{p}) < \infty$ then $H(S) < \infty$. And also if $H(S) < \infty$ then $\kappa(\mathbf{p}) < \infty$ with probability 1. To illustrate our results at section 4, we recite a corollary of Theorem 4.4.9.

Theorem 4.1.3. *There exists $C, c, \tilde{c} > 0$ such that for every $n \geq 3, d \geq 2$ and $\mathbf{p} := (p_1, \cdots, p_{n-1})$ be a random polynomial system in n-variables with degrees d_j, which satisfies the randomness assumptions with constants K, c_0 respectively and has (proper eligible) support $\mathbf{S} := (S, \cdots, S)$, the following holds*
In the case $d_j = d, 1 \leq j \leq m$ we set

$$M := \sqrt{I} cKH(S)(nc_0)^{\frac{1}{2}} \left(cc_0K\sqrt{m}H(S)d\log(ed)\right)^{n-2}$$

In the case $\max_{1\leq j\leq m} d_j = d$ we set

$$M := \sqrt{I} cKH(S)(nc_0)^{\frac{1}{2}} \left(cc_0K\sqrt{m}H(S)d^2\log(ed)\right)^{n-2}$$

We consider two cases:

1. In the case $I \geq (n-1)\log(ed)$, we have that

$$\mathbb{P}\{\kappa(p) \geq tM\} \leq \begin{cases} \frac{3}{t} & \text{if } 1 \leq t \leq e^{(n-1)\log(ed)} \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{(n-1)\log(ed)} \leq t \leq e^{I} \\ \frac{3}{t}\left(\frac{\log t}{I}\right)^{\frac{1}{2}}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{I} \leq t. \end{cases}$$

2. In the case $I \leq (n-1)\log(ed)$, we have that

$$\mathbb{P}\{\kappa(p) \geq tM\} \leq \begin{cases} \frac{3}{t} & \text{if } 1 \leq t \leq e^{I} \\ \frac{3}{t}\left(\frac{\log t}{I}\right)^{\frac{(n-1)}{2}} & \text{if } e^{I} \leq t. \end{cases}$$

Moreover,

$$\mathbb{E}\log(\kappa(\mathbf{p})) \leq \log M + 1.$$

We observe that estimates for the sparse polynomial systems are better in terms of I being possibly much less than N. However, $H(S)$ can get quite big for certain sparse systems. Currently, trade off between the loss from the $H(S)$ and the gain from I is not completely clear. However, we are still able to prove some optimistic results as in the proposition 4.4.15. For more details on this trade-off, please see the remark 4.4.1.

As one can easily observe, our estimates for the same degree case and the mixed degree case are always different. This is mainly due to one of the tools that we developed, which might be of independent interest: Theorem 4.2.3. Theorem 4.2.3 is an extension of Kellog's classical theorem on polynomials, to the systems of polynomials. This theorem bounds the Lipschitz constant of a homogenous polynomial system, in terms of the supremum of the system on the sphere. For the precise statement and the proof of Theorem 4.2.3, we invite the reader to the next section.

4.2 Technical Background

We start by defining an inner product structure on spaces of polynomial systems. For n-variate degree d homogenous polynomials f and g, say $f = \sum_{|\alpha|=d} b_\alpha x^\alpha$ and $g = \sum_{|\alpha|=d} c_\alpha x^\alpha$ the Weyl-Bombieri inner product is defined as follows

$$\langle f, g \rangle_W = \sum_{|\alpha|=d} \frac{b_\alpha c_\alpha}{\binom{d}{\alpha}}$$

It is known that for $U \in O(n)$

$$\langle f \circ U, g \circ U \rangle_W = \langle f, g \rangle_W$$

Now let $D = (d_1, \ldots, d_m)$ and set H_D to be the space of homogenous n-variate polynomial systems with degrees $d_1, \ldots, d_m$ respectively. Then for $f = (f_1, \ldots, f_m) \in H_D$ and $g = (g_1, \ldots, g_m) \in H_D$ the Weyl-Bombieri inner product is defined as follows:

$$\langle f, g \rangle_W = \sum_{i=1}^{m} \langle f_i, g_i \rangle_W$$

Systems of n-variate homogeneous polynomials can be divided into three categories: overdetermined ($m > n-1$), underdetermined ($m < n-1$), and square ($m = n-1$). We provide estimates for the condition number of overdetermined and square systems.

How to define the condition number for a given numerical problem is already a non-trivial question. In their seminal work [109, 110, 111, 112] Shub and Smale defined the condition number of the complex root finding problem for square systems. We are interested in real root finding, and this problem requires a different notion of conditioning. A correct variant of the condition number for real root finding was defined in [32] as follows:

Let $p = (p_1, \ldots, p_m)$ be a system of n-variate polynomials with degrees $d_1, \ldots, d_m$ respectively, let $\Delta(d_i^{1/2})$ be the diagonal matrix with entries $d_i^{1/2}$. Also let $Dp(x)|_{T_x S^{n-1}}$ be the Jacobian matrix of polynomial system p evaluated at point x restricted to the tangent space of x. Then we define $\kappa(p, x)$ the condition number of the polynomial system p at point x, and $\kappa(p)$ the condition of polynomial system p as follows:

$$\mu(p, x) = \|p\|_W \left\| \Delta(d_i^{1/2}) (Dp(x)|_{T_x S^{n-1}})^{-1} \right\|_2$$

$$\kappa(p, x) = \frac{\|p\|_W}{(\|p\|_W^2 \, \mu(p, x)^{-2} + \|p(x)\|_2^2)^{1/2}}$$

$$\kappa(p) = \max_{x \in S^{n-1}} \kappa(p, x)$$

In ([33], Proposition 3.1) the authors also prove an Eckart-Young type theorem that provides geometric justification for their definition of condition number. In order to state Eckart-Young type theorem of Cucker, Krick, Malajovich, and Wschebor we need to introduce some terminology.

For $x \in S^{n-1}$ we define the set of polynomial systems with singularity at x as

$$\Sigma_R(x) = \{f \in H_D \mid f \text{ has a multiple root at } x\}$$

Then we set Σ_R to be the 'real' disciminant variety

$$\Sigma_R = \{f \in H_D \mid f \text{ has a multiple root in } S^n\} = \bigcup_{x \in S^n} \Sigma_R(x).$$

The main theorem of [33] reads as follows:

Theorem 4.2.1. *[33] For all $p \in H_D$ we have $\kappa(p) = \frac{\|p\|_W}{dist(p,\Sigma_R)}$.* ■

For convenience we define

$$L = L(p) = \min_{x \in S^{n-1}} \{ (\left\| \Delta(d_i^{1/2})(Dp(x)|_{T_x S^{n-1}})^{-1} \right\|_2^{-2} + \|p(x)\|_2^2)^{\frac{1}{2}} \}$$

It directly follows that $\kappa(p) = \frac{\|p\|_W}{L}$. Now we make an important observation, say $M = \Delta(d_i^{1/2})$ and $D_x(p) = Dp(x)|_{T_x S^{n-1}}$ then

$$\left\| M D_x(p)^{-1} \right\|_2^{-1} = \sigma_{min}(M^{-1} D_x(p)) = \min\{ \left\| M^{-1} D_x(p)(y) \right\|_2 : y \perp x, y \in S^{n-1} \}$$

$$L = \min_{x \in S^{n-1}} \{ (\left\| M(D_x(p))^{-1} \right\|_2^{-2} + \|p(x)\|_2^2)^{\frac{1}{2}} \}$$

$$L = \min\{ (\left\| M^{-1} D_x(p)(y) \right\|_2^2 + \|p(x)\|_2^2)^{\frac{1}{2}} : y \perp x, y \in S^{n-1} \}$$

Since the W-norm of a random polynomial system has strong concentration properties for a broad variety of distributions we are mainly interested in behavior of the L quantity. To confuse matters more we set

$$L(x,y) = \sqrt{\|M^{-1} D^{(1)} p(x)(y)\|_2^2 + \|p(x)\|_2^2}$$

It directly follows that $L = \min_{x \perp y} L(x,y)$ and $\kappa(p) = \frac{\|p\|_W}{L}$.

Now we have a correct variant of condition number for square systems, we need to define a correct one for overdetermined systems as well. Newton's method for overdetermined systems was studied at [36]. Inspired by [36] for $m > n-1$ we define

$$\mu(p,x) = \|p\|_W \, \sigma_{min}(\Delta(d_i^{\frac{-1}{2}}) Dp(x)|_{T_x S^{n-1}})^{-1}$$

$$\kappa(p,x) = \frac{\|p\|_W}{(\|p\|_W^2 \, \mu(p,x)^{-2} + \|p(x)\|_2^2)^{1/2}}$$

$$\kappa(p) = \max_{x \in S^{n-1}} \kappa(p,x)$$

where $\sigma_{min}(A)$ is the smallest non-zero singular value of matrix A. Similarly for $m > n-1$ we define

$$L = \min_x \{ \sqrt{\sigma_{min}(M^{-1} D_x(p))^2 + \|p(x)\|_2^2} : x \in S^{n-1} \}$$

Now, we review Kellog's classical Theorem.

Theorem 4.2.2. *Let p be a polynomial with n variables and degree d. Let $\left\| D^{(1)} p \right\|_\infty = \max_{x,u \in S^{n-1}} \left| D^{(1)} p(x)(u) \right|$ and $\|p\|_\infty = \sup_{x \in S^{n-1}} |p(x)|$.*

1. *If p is homogenous we have $\left\| D^{(1)} p \right\|_\infty \leq d \|p\|_\infty$.*
2. *For any polynomial of degree d we have $\left\| D^{(1)} p \right\|_\infty \leq d^2 \|p\|_\infty$.* ■

Let $p = (p_1, \ldots, p_m) \in (\mathbb{C}[x_1, \ldots, x_n])^m$ be a polynomial system with p_i homogeneous of degree d_i for all i, and let $d := \max_i d_i$. We also define $\|p\|_\infty := \sup_{x \in S^{n-1}} \sqrt{\sum_{i=1}^m p_i(x)^2}$, and let $Dp(x)$ denote the Jacobian matrix of the system p at the point x. We also let $Dp(x)(u)$ denote the image of the vector u under the linear operator $Dp(x)$, and set

$$\left\| D^{(1)} p \right\|_\infty := \sup_{x,u \in S^{n-1}} \|Dp(x)(u)\|_2 = \sup_{x,u \in S^{n-1}} \sqrt{\sum_{i=1}^m \langle \nabla p_i(x), u \rangle^2}.$$

Theorem 4.2.3. *Let $p = (p_1, \ldots, p_m)$ be a system of homogenous polynomials p_i with n variables and degrees d_i.*

1. *For the case where $\deg(p_i) = d$ for all $i \in \{1, 2, \ldots, m\}$ we have $\left\|D^{(1)}p\right\|_\infty \leq d\left\|p\right\|_\infty$.*

2. *If $d = \max_i\{d_i\}$ then $\left\|D^{(1)}p\right\|_\infty \leq d^2\left\|p\right\|_\infty$.*

Proof. Assume for (x_0, u_0) we have $\left\|D^{(1)}p\right\|_\infty = \left\|Dp(x_0)(u_0)\right\|_2$, then let $\alpha = (\alpha_1, \ldots, \alpha_m)$ where $\alpha_i = \frac{\langle \nabla p_i(x_0), u_0 \rangle}{\left\|D^{(1)}p\right\|_\infty}$. Note that $\|\alpha\|_2 = 1$. Now we define a polynomial $q(x)$ with n variables and degree d as follows

$$q(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \cdots + \alpha_m p_m(x)$$

$$\nabla q(x) = (\alpha_1 \frac{\partial p_1}{\partial x_1} + \alpha_2 \frac{\partial p_2}{\partial x_1} + \cdots + \alpha_m \frac{\partial p_m}{\partial x_1}, \ldots, \alpha_1 \frac{\partial p_1}{\partial x_n} + \alpha_2 \frac{\partial p_2}{\partial x_n} + \cdots + \alpha_m \frac{\partial p_m}{\partial x_n})$$

$$\langle \nabla q, u \rangle = u_1(\alpha_1 \frac{\partial p_1}{\partial x_1} + \alpha_2 \frac{\partial p_2}{\partial x_1} + \cdots + \alpha_m \frac{\partial p_m}{\partial x_1}) + \cdots + u_n(\alpha_1 \frac{\partial p_1}{\partial x_n} + \alpha_2 \frac{\partial p_2}{\partial x_n} + \cdots + \alpha_m \frac{\partial p_m}{\partial x_n})$$

$$\langle \nabla q(x), u \rangle = \sum_{i=1}^{m} \alpha_i \langle \nabla p_i(x), u \rangle$$

In particular for x_0 and u_0

$$\langle \nabla q(x_0), u_0 \rangle = \sum_{i=1}^{m} \alpha_i \langle \nabla p_i(x_0), u_0 \rangle = \sum_{i=1}^{m} \frac{\langle \nabla p_i(x_0), u_0 \rangle^2}{\|D^{(1)}p\|_\infty} = \left\|D^{(1)}p\right\|_\infty$$

Using the second part of Kellog's Theorem we have

$$\left\|D^{(1)}p\right\|_\infty \leq \sup_{x,u\in S^{n-1}} |\langle \nabla q(x), u\rangle| \leq d^2 \|q\|_\infty$$

Now we observe by the Cauchy-Schwarz Inequality

$$\|q\|_\infty = \sup_{x\in S^{n-1}} \left|\sum_{i=1}^m \alpha_i p_i(x)\right| \leq \sup_{x\in S^{n-1}} \left|(\sum_{i=1}^m p_i(x)^2)^{1/2}\right|.$$

So we conclude that $\left\|D^{(1)}p\right\|_\infty \leq d^2 \|q\|_\infty \leq d^2 \sup_{x\in S^{n-1}}(\sum_{i=1}^m p_i(x)^2)^{1/2} = d^2 \|p\|_\infty$. We also note in the case $\deg(g_i) = d$ for all i, $q(x)$ is a homogenous polynomial of degree d. So for this special case using the first part of Kellog's Theorem we deduce $\left\|D^{(1)}p\right\|_\infty \leq d \|p\|_\infty$. ■

Using our extension of Kellog's Theorem to the polynomial systems we develop useful estimates for $\|p\|_\infty$ and $\left\|D^{(i)}p\right\|_\infty$.

Lemma 4.2.4. *Let $p = (p_1, \ldots, p_m)$ be system of homogenous polynomials p_i with n variables. Let $\mathcal{N}$ be a δ-net on S^{n-1}. Let $\max_{\mathcal{N}}(p) = \sup_{y\in\mathcal{N}} \|p(y)\|_2$ and $\|p\|_\infty = \sup_{x\in S^{n-1}} \|p(x)\|_2$. Similarly let us define,*

$$\max_{\mathcal{N}^{k+1}}(D^{(k)}p) = \sup_{x,u_1,\ldots,u_k\in\mathcal{N}} \left\|D^{(k)}p(x)(u_1,\ldots,u_k)\right\|_2$$

$$\left\|D^{(k)}p\right\|_\infty = \sup_{x,u_1,\ldots,u_k\in S^{n-1}} \left\|D^{(k)}p(x)(u_1,\ldots,u_k)\right\|_2$$

1. *For the case $\deg(p_i) = d$ for all $i \in \{1, 2, \ldots, m\}$ we have*

$$\|p\|_\infty \leq \frac{\max_{\mathcal{N}}(p)}{1 - d\delta}$$

$$\left\|D^{(k)}p\right\|_\infty \leq \frac{\max_{\mathcal{N}^{k+1}}(D^{(k)}p)}{1-\delta d\sqrt{k+1}}$$

2. *For systems of different degree homogenous polynomials with* $max_i\{deg(p_i)\} \leq d$ *we have*

$$\|p\|_\infty \leq \frac{\max_{\mathcal{N}}(p)}{1-d^2\delta}$$

$$\left\|D^{(k)}p\right\|_\infty \leq \frac{\max_{\mathcal{N}^{k+1}}(D^{(k)}p)}{1-\delta d^2\sqrt{k+1}}$$

Proof. For p a system of homogenous polynomials, Lipschitz constant of p on S^{n-1} is bounded by $\left\|D^{(1)}p\right\|_\infty$. A direct way to observe this fact is the following; let $x, y \in S^{n-1}$ and consider the integral

$$p(x) - p(y) = \int_0^1 (Dp(y + t(x-y)).(x-y))\ dt$$

Since $\|y + t.(x-y)\|_2 \leq 1$ for all $t \in [0,1]$ by homogeneity of the system f we have

$$\|Dp(y + t(x-y)).(x-y)\|_2 \leq \left\|D^{(1)}p\right\|_\infty \|x-y\|_2$$

Using the integral formula above we conclude

$$\|p(x) - p(y)\|_2 \leq \left\|D^{(1)}p\right\|_\infty \|x-y\|_2$$

Now, for an unmixed polynomial system p, let the Lipschitz constant of p be L. By Theorem 4.2.3 we have

$$L \leq \left\| D^{(1)} p \right\|_{\infty} \leq d \left\| p \right\|_{\infty}$$

Now let $x_0 \in S^{n-1}$ be such that $\|p(x_0)\|_2 = \|p\|_{\infty}$ and let $y \in \mathcal{N}$ which satisfies $|x_0 - y| \leq \delta$.

$$\|p\|_{\infty} = \|p(x_0)\|_2 \leq \|p(y)\|_2 + \|x_0 - y\|_2 L \leq \max_{\mathcal{N}}(p) + \delta d \|p\|_{\infty}$$

$$\|p\|_{\infty} (1 - d\delta) \leq \max_{\mathcal{N}}(p)$$

For $D^{(k)} p(x)(u_1, \ldots, u_k)$ let us consider the net $\mathcal{N} \times \cdots \times \mathcal{N} = \mathcal{N}^{k+1}$ on $S^{n-1} \times \cdots \times S^{n-1}$. Now let $x = (x_1, \ldots, x_{k+1}) \in S^{n-1} \times \cdots \times S^{n-1}$ and let $y = (y_1, \ldots, y_{k+1}) \in \mathcal{N}^{k+1}$ such that $\|x_i - y_i\|_2 \leq \varepsilon$ for all i. We observe that, $\|x - y\|_2 \leq \sqrt{k+1}\varepsilon$. Since x was an arbitrary point, this argument proves $\mathcal{N}^{k+1}$ is a $\sqrt{k+1}\varepsilon$ net. Also we observe that, $D^{(k)} p(x)(u_1, \ldots, u_k)$ is a homogenous polynomial system with $(k+1)n$ variables and degree d. The desired bound follows from the inequality obtained above.

For the mixed case, the preceding proof carries over verbatim, simply employing the second case of Theorem 4.2.3. ■

4.3 Condition Number of Random Polynomial Systems

4.3.1 Introducing the Randomness

Let $\mathbf{p} = (p_1, \ldots, p_m)$ be a random polynomial system where $p_j = \sum_{|\alpha| = d_j} c_{\alpha}^{(i)} \sqrt{\binom{d_j}{\alpha}} x^{\alpha}$. Let $C_j = (c_{\alpha}^{(j)})_{|\alpha| = d_j} \in \mathbb{R}^{N_j}$ ($N_j := \binom{n+d_j-1}{d_j}$) and $\mathcal{X}_j = (\sqrt{\binom{d_j}{\alpha}} x^{\alpha})_{|\alpha| = d_j}$ then $p_j(x) =< C_j, \mathcal{X}_j >$. We consider random polynomial systems with independent subgaussian centered random vectors C_j that satisfy a small ball probability condition. This assumption allows the coefficients $C_\alpha^{(j)}$ to have dependencies and also covers a broad variety of distributions. More precisely our assumptions on random vectors $C_j \in \mathbb{R}^{N_j}$ are the following:

For every $\theta \in S^{N_j-1}$, $1 \leq j \leq m$,

(1) (Centered vectors) $\quad \mathbb{E}\langle C_j, \theta\rangle = 0$.

(2) (Independent Rows) $\quad C_j, 1 \leq j \leq m$ are independent random vectors.

There exists $K > 0$ such that for every $\theta \in S^{N_j-1}$, $1 \leq j \leq m$,

(3) (Subgaussian Assumption) $\quad \mathbb{P}\left(|\langle C_j, \theta\rangle| \geq t\right) \leq 2e^{-\frac{t^2}{K^2}}$, for all $t > 0$.

There exists $c_0 > 0$ such that for every vector $a \in \mathbb{R}^{N_i}$, $1 \leq j \leq m$, we have

(4) (Small Ball Assumption) $\quad \mathbb{P}\left(|\langle a, C_j\rangle| \leq \varepsilon \left\|a\right\|_2\right) \leq c_0\varepsilon$, for all $\varepsilon > 0$

From this point on, we will always assume that the assumptions (1) and (2) are satisfied. Moreover, we keep the letters K, c_0 to express the constants on the subaussian and Small Ball assumption respectively.

Let us give some examples of random vectors that satisfy our assumptions with some universal constants $c_0, K > 0$. The standard Gaussian measure in $\mathbb{R}^{N_i}$ satisfy the above since the 1-dimensional marginals are again Gaussians. It is a direct computation to check that the uniform measure in the Euclidean ball and in the Euclidean sphere in $\mathbb{R}^{N_i}$ satisfy the above assumptions. A much less trivial example is the case where $C_i := (c_\alpha^{(i)})_{|\alpha|=d_i}$, where c_α are independent subgaussian random variables with bounded densities. Our assumptions are satisfied in this case due to Theorem 4.3.1 and the main result of [102] or [77]. Other non-trivial examples of random vectors that satisfy our assumptions are the uniform measure on $B_p^{N_i}$, $p > 2$, where $B_p^{N_i} := \{x \in R^{N_i} : \sum_{j=1}^{N_i} x_j^p \leq 1\}$. In this case the subgaussian assumption follows from ([12], Section 6) and the Small Ball Assumption is a direct consequence of the well-known fact that $B_p^{N_i}$ satisfies the Hyperplane Conjecture.

Let ξ be a random variable. We denote by $M(\xi)$ a median of ξ, i.e. a number that satisfies

$$\mathbb{P}\left(\xi \leq M(\xi)\right) \geq \frac{1}{2} \text{ and } \mathbb{P}\left(\xi \geq M(\xi)\right) \geq \frac{1}{2}.$$

Let $\xi := |\langle C_j, \theta\rangle|$ and assume that the subgaussian assumption and the Small Ball assumption holds for C_j. Then for $t := 2K$ we have

$$\mathbb{P}(\xi \geq 2K) \leq \frac{1}{2}$$

hence, $M(\xi) \leq 2K$. On the other hand, if we set $\varepsilon = \frac{1}{2c_0}$, our small estimate gives the following

$$\mathbb{P}(\xi \leq \frac{1}{2c_0}) \leq \frac{1}{2}$$

that is; $M(\xi) \geq \frac{1}{2c_0}$. This implies that

$$Kc_0 \geq \frac{1}{4}.$$

In what follows we will use that above inequality several times.

4.3.2 The Subgaussian Assumption and "Operator Norm" Bounds

We start by proving an operator norm type estimate for the polynomial system p. Our operator norm type estimate at Lemma 4.3.2 follows from our subgaussian assumption, our extension of Kellog's Theorem and a standard net-type argument. For the proof we are also going to need the following Hoeffding-type Inequality.

Theorem 4.3.1. *[121, Proposition 5.10] Let $X_1, \ldots, X_n$ be subgaussian (with constant K) random variables with mean zero. Then for every $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$*

and every $t \geq 0$,

$$\mathbb{P}\left(\left|\sum_i a_i X_i\right| \geq t\right) \leq 2\exp\left(-\frac{ct^2}{K^2 \|a\|_2^2}\right)$$

where $c > 0$ *is an absolute constant.* ■

Lemma 4.3.2. *Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *be a polynomial system where* $p_j = \sum_{|\alpha|=d_j} c_\alpha^{(j)} \sqrt{\binom{d_j}{\alpha}} x^\alpha$, C_i *are centered sub-gaussian random vectors with constant* K*. Then for* $\mathcal{N}$ *a* δ*-net over* S^{n-1} *and* $t \geq 2$ *we have the following inequalities*

1. *If* $\deg(p_j) = d$ *for all* $j \in \{1, 2, \ldots, m\}$

$$\mathbb{P}\left(\|\mathbf{p}\|_\infty \leq \frac{2tK\sqrt{m}}{1 - d\delta}\right) \geq 1 - 2|\mathcal{N}|\, e^{-c_1 t^2 m}$$

For specific values $\delta = \frac{1}{3d}$, $t = s\log(ed)$ *with* $s \geq 1$ *we have*

$$\mathbb{P}\left(\|\mathbf{p}\|_\infty \leq 3sK\sqrt{m}\log(ed)\right) \geq 1 - e^{-c_2 s^2 m \log(ed)}$$

where $c_2 \geq 1$ *is an absolute constant.*

2. *If* $\deg(p_j) = d_i$ *and* $\max(d_j) = d$

$$\mathbb{P}\left(\|\mathbf{p}\|_\infty \leq \frac{2tK\sqrt{m}}{1 - d^2\delta}\right) \geq 1 - 2|\mathcal{N}|\, e^{-c_1 t m}$$

For specific values $\delta = \frac{1}{3d^2}$, $t = s\log(ed)$ *with* $s \geq 1$ *we have*

$$\mathbb{P}\left(\|\mathbf{p}\|_\infty \leq 3sK\sqrt{m}\log(ed)\right) \geq 1 - e^{-c_2 s^2 m \log(ed)}$$

where $c_2 \geq 1$ *is an absolute constant.*

Proof. We prove the case (2) since the proof of two cases are identical. We observe that the identity $(x_1^2 + \cdots + x_n^d)^d = \sum_{|\alpha|=d} \binom{d}{\alpha} x^{2\alpha}$ implies $\|\mathcal{X}_j\|_2 = 1$ for all $j \leq m$. We use our subgaussian assumption on random vectors C_i and the observation that $p_j(x) = \langle C_j, \mathcal{X}_j \rangle$, then for every $x \in S^{n-1}$ we have

$$\mathbb{P}\left(|p_j(x)| \geq t\right) \leq 2e^{-\frac{t^2}{K}}$$

Now we need to tensorize the above inequality. By Theorem 4.3.1 for all $a \in S^{m-1}$ we have

$$\mathbb{P}\left(|\langle a, \mathbf{p}(x)\rangle| \geq t\right) \leq 2\exp(-\frac{ct^2}{K^2})$$

Let $\mathcal{M}$ be a δ-net on S^{m-1} then we have

$$\mathbb{P}\left(\max_{a \in \mathcal{M}} |\langle a, \mathbf{p}(x)\rangle| \geq t\right) \leq 2\,|\mathcal{M}| \exp(-\frac{ct^2}{K^2})$$

where we used a union bound on $\mathcal{M}$. Since $\|p(x)\|_2 = \max_{\theta \in S^{m-1}} |\langle \theta, p(x)\rangle|$ using Lemma 4.2.4 for linear functional $\langle\ .\ , \mathbf{p}(x)\rangle$ we have

$$\mathbb{P}\left(\|\mathbf{p}(x)\|_2 \geq \frac{t\sqrt{m}K}{1-\delta}\right) \leq 2\,|\mathcal{M}| \exp(-ct^2 m)$$

It is known that there exist a δ-net $\mathcal{M}$ on S^{m-1} such that $|\mathcal{M}| \leq \left(\frac{3}{\delta}\right)^m$ (see, e.g, [121, Lemma 5.2]). So for $t \geq 1$ and $\delta = \frac{1}{2}$ we have

$$\mathbb{P}\left(\|\mathbf{p}(x)\|_2 \geq 2t\sqrt{m}K\right) \leq 2\exp(-c_1 t^2 m)$$

We arrived to a pointwise estimate on $\|p(x)\|_2$, we again do a union bound on a δ-net $\mathcal{N}$ this time on S^{n-1} then we have

$$\mathbb{P}\left(\max_{x\in\mathcal{N}}\|\mathbf{p}(x)\|_2 \geq 2t\sqrt{m}K\right) \leq 2\left|\mathcal{N}\right| \exp(-c_1t^2m)$$

Using Lemma 4.2.4 again completes the proof.

■

Theorem 4.2.3 and Lemma 4.3.2 imply the following:

Corollary 4.3.3. *Let* $\mathbf{p}$ *be a polynomial system as in Lemma 4.3.2. We have the following inequalities for* $s \geq 1$

1. *If* $\deg(p_j) = d$ *for all* $j \in \{1, 2, \ldots, m\}$

$$\mathbb{P}\left(\left\|D^{(1)}\mathbf{p}\right\|_\infty \leq 3sK\sqrt{m}d\log(ed)\right) \geq 1 - 2e^{-c_2s^2m\log(ed)}$$

$$\mathbb{P}\left(\left\|D^{(2)}\mathbf{p}\right\|_\infty \leq 3sK\sqrt{m}d^2\log(ed)\right) \geq 1 - 2e^{-c_2s^2m\log(ed)}$$

2. *If* $\deg(p_j) = d_j$ *and* $\max(d_j) = d$

$$\mathbb{P}\left(\left\|D^{(1)}\mathbf{p}\right\|_\infty \leq 3sK\sqrt{m}d^2\log(ed)\right) \geq 1 - 2e^{-c_2s^2m\log(ed)}$$

$$\mathbb{P}\left(\left\|D^{(2)}\mathbf{p}\right\|_\infty \leq 3sK\sqrt{m}d^4\log(ed)\right) \geq 1 - 2e^{-c_2s^2m\log(ed)}$$

4.3.3 The Small Ball Assumption and Bounds for the "L" Quantity

We will need the following standard Lemma (see Lemma 2.2 of [101] or [86]).

Lemma 4.3.4. *Let $\xi_1, \cdots, \xi_m$ be independent random variables such that for every $\varepsilon > 0$,*

$$\mathbb{P}\left(|\xi_i| \leq \varepsilon\right) \leq c_0\varepsilon.$$

Then, for every $\varepsilon > 0$

$$\mathbb{P}\left(\sqrt{\xi_1^2 + \cdots + \xi_m^2} \leq \varepsilon\sqrt{m}\right) \leq \left(\tilde{c}c_0\varepsilon\right)^m,$$

where $\tilde{c} > 0$ is an absolute constant.

Lemma 4.3.5. *Let $\mathbf{p}$ be a system of n-variate random homogenous polynomials satisfying the Small Ball assumption (with constant c_0). Then for every $\varepsilon > 0$ and $x \in S^{n-1}$,*

$$\mathbb{P}\{\|\mathbf{p}(x)\|_2 \leq \varepsilon\sqrt{m}\} \leq (\tilde{c}c_0\varepsilon)^m,$$

where $\tilde{c} > 0$ is an absolute constant.

Proof. By the Small Ball assumption on random vectors C_i, and the two observations that $p_i(x) = \langle C_i, \mathcal{X}_i\rangle$, $\|\mathcal{X}_i\|_2 = 1$ for all $x \in S^{n-1}$ we have

$$\mathbb{P}\{|p_i(x)| \leq \varepsilon\} \leq c_0\varepsilon.$$

By Lemma 4.3.4 we get the result. ■

The next Lemma is a variant of a Lemma of H.H. Nguyen ([87], Claim 2.4).

Lemma 4.3.6. *Let $n \geq 3$, $\mathbf{p} = (p_1, \ldots, p_m)$ be a system of n-variate homogenous polynomials, and $\|p\|_\infty \leq \gamma$. Assume for some $x, y \in S^{n-1}$, $x \perp y$ and $L(x, y) \leq \alpha$,then for every w such that $w = x + \beta r y + \beta z$ with $z \in S^{n-1}$, $z \perp x$, $z \perp y$, $|r| \leq 1$, the following inequalities hold*

1. *If* $\deg(p_i) = d$ *for all* $i \in \{1, 2, \ldots, m\}$ *and* $\beta \leq \frac{1}{d}$*, then*

$$\|\mathbf{p}(w)\|_2^2 \leq 5(\alpha^2 + 19\beta^2 d^2 \gamma^2)$$

2. *If* $\deg(p_i) = d_i$*,* $\max_i d_i = d$ *and* $\beta \leq \frac{1}{d^2}$*, then*

$$\|\mathbf{p}(w)\|_2^2 \leq 5(\alpha^2 + 19\beta^2 d^4 \gamma^2)$$

Proof. We prove the case (1) only since the proof of the case (2) is quite similar. We start with some auxiliary observations on $\|\mathbf{p}\|_\infty$. First observation is that by Kellog's inequality $\|\mathbf{p}\|_\infty \leq \gamma$ implies $\left\|D^{(1)}\mathbf{p}\right\|_\infty \leq d\gamma$ and similarly $\left\|D^{(k)}\mathbf{p}\right\|_\infty \leq d^k\gamma$ for every $k \geq 1$. Also for any w and $u_i \in S^{n-1}$ for $i = 1, 2, \ldots, k$ by homogeneity of polynomials in the system p, $\|\mathbf{p}\|_\infty \leq \gamma$ implies

$$\sup_{u_1,\ldots,u_k} \left\|D^{(k)}\mathbf{p}(w)(u_1, \ldots, u_k)\right\|_2 \leq \|w\|_2^{d-k} d^k\gamma$$

These observations above yield the following inequality for $w = x + \beta ry + \beta z$ with $z \in S^{n-1}$, $|r| \leq 1$, $\beta \leq d^{-1}$, $k = 3$ and $u_1, u_2, u_3 \in S^{n-1}$

$$\left\|D^{(3)}\mathbf{p}(w)(u_1, u_2, u_3)\right\|_2 \leq \|w\|_2^{d-3} d^3\gamma \leq (1 + \frac{2}{d})^{d-3} d^3\gamma \leq ed^3\gamma$$

$$p_j(w) = p_j(x) + \langle \nabla p_j(x), \beta ry + \beta z\rangle + 1/2(\beta ry + \beta z)^T D^{(2)}p_j(x)(\beta ry + \beta z) + O(\beta^3)$$

We set $v = \frac{\beta ry+\beta z}{\|\beta ry+\beta z\|_2}$, then we have

$$\begin{aligned}|p_j(w)| \leq |p_j(x)| + \beta|\langle \nabla p_j(x), y\rangle| + \beta|\langle \nabla p_j(x), z\rangle| \\ + 1/2 \|\beta ry + \beta z\|_2^2 |D^{(2)}p_j(x)(v, v)| + |O(\beta^3)|.\end{aligned}$$

Note that $\|\beta ry + \beta z\|_2 \leq 2\beta$, and $|O(\beta^3)| \leq 1/6(2\beta)^3 A_j(x)$ where the integral form of the remainder

$$A_j(x) = \int_{t=0}^{1} D^{(3)} p_j(x + t\,\|v\|_2\, v)(v, v, v)$$

Using Cauchy-Schwarz Inequality (for the coefficients $1, \beta d_j^{\frac{1}{2}}, 1, 1, 1$) implies the following

$$p_j(w)^2 \leq (4 + \beta^2 d_j)(p_j(x)^2 + d_j^{-1} \langle p_j(x), y\rangle^2 + \beta^2 \langle \nabla p_j(x), z\rangle^2 + \frac{1}{4}(2\beta)^4 (D_j^{(2)} p_j(x)(v, v))^2 + \frac{1}{36}(2\beta)^6 A_j(x)^2).$$

Summing up through all $j \leq m$, using the assumption $\|\mathbf{p}\|_\infty \leq \gamma$ and auxiliary observations at the beginning, we have

$$\|p(w)\|_2^2 \leq (4+\beta^2 d)(\|p(x)\|_2^2 + \left\|M^{-1} D^{(1)} p(x)(y)\right\|_2^2 + \beta^2 d^2 \gamma^2 + 4\beta^4 d^4 \gamma^2 + \frac{64}{36}\beta^6 \sum_j A_j(x)^2)$$

Clearly $\sum_{j\leq m} A_j(x)^2 \leq \max_{w \in V_{x,y}} \left\|D^{(3)} \mathbf{p}(w)(u_1, u_2, u_3)\right\|_2^2 \leq e^2 d^6 \gamma^2$, hence we have

$$\|\mathbf{p}(w)\|_2^2 \leq (4 + \beta^2 d)(\alpha^2 + \beta^2 d^2 \gamma^2 + 4\beta^4 d^4 \gamma^2 + \frac{64e^2}{36}\beta^6 d^6 \gamma^2)$$

Since $\beta \leq d^{-1}$ and $\frac{64e^2}{36} \leq 14$

$$\|\mathbf{p}(w)\|_2^2 \leq (4 + \beta^2 d)(\alpha^2 + 19\beta^2 d^4 \gamma^2) \leq 5(\alpha^2 + 19\beta^2 d^4 \gamma^2).$$

The proof is complete. ■

Theorem 4.3.7. *Let $n \geq 3$ and $\mathbf{p} = (p_1, \ldots, p_m)$ be a system of random homogenous n-variate polynomials p_i such that $p_j = \sum_{|\alpha|=d_j} c_\alpha^{(j)} \sqrt{\binom{d_i}{\alpha}} x^\alpha$ where $C_j = (c_\alpha^{(j)})_{|\alpha|=d_j}$ are* *random vectors satisfying the Small Ball assumption (with constant c_0). let $\alpha, \gamma > 0$. Then we have the following inequalities for the $L(\mathbf{p})$ quantity:*

1. *If $\deg(p_j) = d$ for all $1 \leq jm$ and $\alpha \leq \min\{1, \frac{d}{\sqrt{n}}\}\gamma$ then*

$$\mathbb{P}\left(L \leq \alpha\right) \leq C\alpha\sqrt{\frac{n}{m}}\left(\frac{c_0 C d\gamma}{\sqrt{m}}\right)^{n-2}\left(\frac{c_0 C\alpha}{\sqrt{m}}\right)^{m-n+1} + \mathbb{P}\left(\|\mathbf{p}\|_\infty \geq \gamma\right),$$

   where $C > 0$ is an absolute constant.

2. *If $\max_j \deg(p_j) = d$ and $\alpha \leq \min\{1, \frac{d}{\sqrt{n}}\}\gamma$ then*

$$\mathbb{P}\left(L \leq \alpha\right) \leq C\alpha\sqrt{\frac{n}{m}}\left(\frac{c_0 C d^2\gamma}{\sqrt{m}}\right)^{n-2}\left(\frac{c_0 C\alpha}{\sqrt{m}}\right)^{m-n+1} + \mathbb{P}\left(\|\mathbf{p}\|_\infty \geq \gamma\right),$$

   where $C > 0$ is an absolute constant.

Proof. We consider the case where $d = \max_{1\leq j\leq m} d_j$. Let $\alpha > 0, \gamma > 0$ and $\beta \leq \frac{1}{d^2}$. Let $\mathbf{B} := \{\|p\|_\infty \leq \gamma\}$ and let $\mathbf{L} := \{L \leq \alpha\} := \{\exists x \perp y : L(x,y) \leq \alpha\}$. Let $\Gamma := 5(\alpha^2 + 19\beta^2 d^4\gamma^2)$. Lemma 4.3.6 implies that, if the event $B \cap L$ holds, then there exists a set

$$V_{x,y} := \{w \in \mathbb{R}^n : w = x + \beta r y + \beta z, z \in S^{n-1}, |r| \leq 1, x \perp z, y \perp z\} \setminus B_2^n$$

such that for every w in this set, $\|p(w)\|_2^2 \leq \Gamma$. Let $V := |V_{x,y}|$. Note that for $w \in V_{x,y}$, $1 \leq \|w\|_2 \leq 1 + 2\beta^2$. Since $V_{x,y} \subseteq (1 + 2\beta^2)B_2^n \setminus B_2^n$, we have showed that the event $\mathbf{B} \cap \mathbf{L}$ is included in the event

$$\{\left|\{x \in (1 + 2\beta^2)B_2^n \setminus B_2^n : \|\mathbf{p}(x)\|_2 \leq \Gamma\}\right| \geq V\}.$$

Using Markov's inequality, Fubini Theorem and Lemma 4.3.5, we can estimate the probability of this event. Indeed,

$$\begin{aligned}
&\mathbb{P}\left(\left|\{x \in (1+2\beta^2)B_2^n \setminus B_2^n : \|\mathbf{p}(x)\|_2 \le \Gamma\}\right| \ge V\right) \le \\
&\le \frac{1}{V}\mathbb{E}\left|\{x \in (1+2\beta^2)B_2^n \setminus B_2^n : \|\mathbf{p}(x)\|_2^2 \le \Gamma\}\right| \le \frac{1}{V}\int_{(1+2\beta^2)B_2^n \setminus B_2^n} \mathbb{P}\left(\|\mathbf{p}(x)\|_2^2 \le \Gamma\right) dx \\
&\qquad \le \frac{|(1+2\beta^2)B_2^n \setminus B_2^n|}{V} \max_{x \in (1+2\beta^2)B_2^n \setminus B_2^n} \mathbb{P}\left(\|\mathbf{p}(x)\|_2^2 \le \Gamma\right).
\end{aligned}$$

Recall that $|B_2^n| = \frac{\pi^{\frac{n}{2}}}{\Gamma(n/2+1)}$. So, $\frac{|B_2^n|}{|B_2^{n-1}|} \le \frac{C'}{\sqrt{n}}$, where $C' > 0$ is an absolute constant. We also assume that $\beta^2 \le \frac{1}{n}$ which yields $(1+2\beta^2)^n \le e^2$, and we compute that

$$\frac{|(1+2\beta^2)B_2^n \setminus B_2^n|}{V} \le \frac{|B_2^n|\left((1+2\beta^2)^n - 1\right)}{\beta\beta^{n-1}|B_2^{n-1}|} \le C\sqrt{n}\beta^{2-n},$$

where $C > 0$ is an absolute constant. For $x \neq 0$ we write $\tilde{x} := \frac{x}{\|x\|_2}$. For $z \notin B_2^n$, we have that

$$\|\mathbf{p}(z)\|_2^2 = \sum_{j=1}^m |p_j(z)|^2 = \sum_{j=1}^m |p_j(\tilde{z})|^2 \|z\|_2^{2d_j} \ge \sum_{j=1}^m |p_j(\tilde{z})|^2 = \|\mathbf{p}(\tilde{z})\|_2^2,$$

which implies that for every $w \in (1+2\beta^2)B_2^n \setminus B_2^n$,

$$\mathbb{P}\left(\|\mathbf{p}(w)\|_2^2 \le \Gamma\right) \le \mathbb{P}\left(\|p(\tilde{w})\|_2^2 \le \Gamma\right) \le \left(cc_0\sqrt{\frac{\Gamma}{m}}\right)^m,$$

where we have used Lemma 4.3.5. So we conclude that

$$\mathbb{P}(L \le \alpha) \le \mathbb{P}\left(\|\mathbf{p}\|_\infty \ge \gamma\right) + \mathbb{P}\left(\mathbf{B} \cap \mathbf{L}\right) \le \mathbb{P}\left(\|\mathbf{p}\|_\infty \ge \gamma\right) + C\sqrt{n}\beta^{2-n}\left(cc_0\sqrt{\frac{\Gamma}{m}}\right)^m.$$

Recall that $\Gamma = 5(\alpha^2 + 19\beta^2 d^4\gamma^2)$. We choose $\beta := \frac{\alpha}{\gamma d^2}$ and we observe that under our assumption $\alpha \le \gamma$ this is an eligible choice for β. For this choice of β we have that $\Gamma = 100\alpha^2$. So we get that

$$\mathbb{P}(L \le \alpha) \le \mathbb{P}\left(\|\mathbf{p}\|_\infty \ge \gamma\right) + C\sqrt{n}\left(\frac{\alpha}{\gamma d^2}\right)^{2-n}\left(\frac{10cc_0\alpha}{\sqrt{m}}\right)^m.$$

In the case where $d := d_i, 1 \le j \le m$, we have that $\Gamma := 5(\alpha^2 + 19\beta^2 d^2\gamma^2)$. In this case our choice of $\beta = \frac{\alpha}{\gamma d}$ (note that this is eligible in this case) and we conclude that

$$\mathbb{P}(L \le \alpha) \le \mathbb{P}\left(\|\mathbf{p}\|_\infty \ge \gamma\right) + C\sqrt{n}\left(\frac{\alpha}{\gamma d}\right)^{2-n}\left(\frac{10cc_0\alpha}{\sqrt{m}}\right)^m.$$

By adjusting the constants we complete the proof. ∎

4.3.4 The Condition Number Theorem and Consequences

We will need bounds for the Weil-Bombieri norm of the polynomial system. Note that $\|p_j\|_W := \|C_\alpha^{(j)}\|_2$, $1 \le j \le m$. The following Lemma which provides large deviation estimates for the Euclidean norm is standard and follows e.g. from Theorem 4.3.1.

Lemma 4.3.8. *Let $\mathbf{p}$ be a random n-variate polynomial system satisfying our sub-gaussian assumption (with constant K). Let $N_j := \binom{n+d_j-1}{d_j}$, $1 \le j \le m$ and let $N := \sum_{j=1}^m N_j$. Then*

$$\mathbb{P}\left(\|p_j\|_w \ge tcK\sqrt{N_j}\right) \le e^{-t^2 N_i},\ t \ge 1,\ 1 \le j \le m, \tag{4.1}$$

where $c > 0$ is an absolute constant.

$$\mathbb{P}\left(\|\mathbf{p}\|_w \ge tcK\sqrt{N}\right) \le e^{-t^2 N},\ t \ge 1. \tag{4.2}$$

where $c > 0$ is an absolute constant.

We are now ready to prove our main theorem on condition number of random polynomial systems.

Theorem 4.3.9. *There exist universal constants $C, c > 0$ such that if $\mathbf{p} = (p_1, \cdots, p_m)$ is a system of homogenous random polynomials, where $p_j = \sum_{|\alpha|=d_j} c_\alpha^{(i)} \binom{d_j}{\alpha}^{\frac{1}{2}} x^\alpha$, $N = \sum_{j=1}^m \binom{n+d_j-1}{d_j}$ and $C_j = (c_\alpha^{(j)})_{|\alpha|=d_j}$ are random vectors satisfying the subgaussian and Small Ball assumptions (with constants $K, c_0 > 0$ respectively) then the following hold true:*

1. *If $d = \deg(p_j)$ for all $1 \leq j \leq m$ set $M := M(n,d,m,c_0,K)$,*

$$M := cCc_0 \left(\frac{n}{c_0}\right)^{\frac{1}{2(m-n+2)}} \sqrt{N} K \left(3Cc_0 K d \log(ed)\right)^{\frac{n-2}{m-n+2}} m^{-\frac{1}{2}} \max\{1, \frac{\sqrt{n}}{d}\}.$$

Then we denote $\mathbb{P}(\kappa(\mathbf{p}) \geq tM)$ with $\mathbb{P}(t)$, we have

$$P(t) \leq \begin{cases} \frac{3}{t^{m-n+2}} & \text{if } 0 \leq t \leq e^{\frac{m\log(ed)}{m-n+2}} \\ \frac{3}{t^{m-n+2}} \left(\frac{(\log t)(m-n+2)}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{m\log(ed)}{m-n+2}} \leq t \leq e^{\frac{N}{m-n+2}} \\ \frac{3}{t^{m-n+2}} \left(\frac{(\log t)(m-n+2)}{N}\right)^{\frac{m}{2}} \left(\frac{N}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{N}{m-n+2}} \leq t \end{cases}$$

2. *If $\max_{i\leq m} \deg(p_i) = d$ set $M := M(n,d,m,c_0,K)$,*

$$M := cc_0 C \left(\frac{n}{c_0}\right)^{\frac{1}{2(m-n+2)}} \sqrt{N} K \left(3Cc_0 K d^2 \log(ed)\right)^{\frac{n-2}{m-n+2}} m^{-\frac{1}{2}} \max\{1, \frac{\sqrt{n}}{d^2}\}.$$

We consider two cases:

(a) In the case $N \geq m \log(ed)$, we have that

$$\mathbb{P}(t) \leq \begin{cases} \frac{3}{t^{m-n+2}} & \text{if } 1 \leq t \leq e^{\frac{m\log(ed)}{m-n+2}} \\ \frac{3}{t^{m-n+2}} \left(\frac{(\log t)(m-n+2)}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{m\log(ed)}{m-n+2}} \leq t \leq e^{\frac{N}{m-n+2}} \\ \frac{3}{t^{m-n+2}} \left(\frac{(\log t)(m-n+2)}{N}\right)^{\frac{m}{2}} \left(\frac{N}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{N}{m-n+2}} \leq t \end{cases}$$

(b) In the case $N \leq m \log(ed)$, we have that

$$\mathbb{P}(t) \leq \begin{cases} \frac{3}{t^{m-n+2}} & \text{if } 1 \leq t \leq e^{\frac{N}{m-n+2}} \\ \frac{3}{t^{m-n+2}} \left(\frac{(\log t)(m-n+2)}{N}\right)^{\frac{m}{2}} & \text{if } e^{\frac{N}{m-n+2}} \leq t \end{cases}$$

Proof. We consider first the case where $\max_{i\leq N} = d$.

$$\mathbb{P}\left(\kappa(\mathbf{p}) \geq tM\right) \leq \mathbb{P}\left(\|\mathbf{p}\|_W \geq ucK\sqrt{N}\right) + \mathbb{P}\left(L(\mathbf{p}) \leq \frac{c\sqrt{N}uK}{tM}\right).$$

To estimate the above probabilities, we apply Theorem 4.3.7 for $\alpha := \frac{c\sqrt{N}uK}{tM}$ and $\gamma := 3sK\sqrt{m}\log(ed)$, Lemma 4.3.2 and Lemma 4.3.8. First we note our restrictions. We have that $s \geq 1$, $u \geq 1$ and (since $\alpha \leq \min\{1, \frac{d^2}{\sqrt{n}}\}\gamma$),

$$(*)\quad \left(\frac{c_0 m}{n}\right)^{\frac{1}{2(m-n+2)}} \frac{1}{\max\{1, \frac{\sqrt{n}}{d^2}\}} \frac{m^{\frac{1}{2}\frac{m-n+1}{m-n+2}} u}{c_0 C t (c_0 C K d^2 \log(ed))^{\frac{n-2}{m-n+2}}} \leq 3Ks\sqrt{m}\log(ed)\min\{1, \frac{d^2}{\sqrt{n}}\}.$$

Note that, $(*)$ is always true if $u \leq s$ and $t \geq 1$. Under the above restrictions we have that if $P := \mathbb{P}\{\kappa(p) \geq tM\}$,

$$P \leq C\sqrt{\frac{n}{m}}\frac{c\sqrt{N}uK}{tM}\left(\frac{3c_0KCd^2s\sqrt{m}\log(ed)}{\sqrt{m}}\right)^{n-2}\left(\frac{c_0Cc\sqrt{N}uK}{tM\sqrt{m}}\right)^{m-n+1} + e^{-c_2s^2m\log(ed)} + e^{-u^2N}$$

or,

$$P \leq \frac{u^{m-n+2}s^{n-2}}{t^{m-n+2}} + e^{-c_2s^2m\log(ed)} + e^{-u^2N}.$$

We consider first the case where $N \geq m\log(ed)$. If $1 \leq t \leq e^{\frac{c_1 m\log(ed)}{m-n+2}}$, we take $u = s = 1$ (note that $(*)$ is satisfied and that $c_2 \geq 1$) and then we get that

$$P \leq \frac{1}{t^{m-n+2}} + e^{-c_2m\log(ed)} + e^{-N} \leq \frac{3}{t^{m-n+2}},\ t \geq 1,\ u \geq s \geq 1.$$

In the case where $e^{\frac{m\log(ed)}{m-n+2}} \leq t \leq e^{\frac{N}{m-n+2}}$ we choose $u = 1$ and $s := \sqrt{\frac{(\log t)(m-n+2)}{m\log(ed)}} \geq 1$. (Note that $u \leq s$). The above choices give

$$P \leq \frac{1}{t^{m-n+2}}\left(\frac{(\log t)(m-n+2)}{m\log(ed)}\right)^{\frac{n-2}{2}} + \frac{1}{t^{c_2(m-n+2)}} + e^{-N} \leq \frac{3}{t^{m-n+2}}\left(\frac{(\log t)(m-n+2)}{m\log(ed)}\right)^{\frac{n-2}{2}}.$$

In the case where $e^{\frac{N}{m-n+2}} \leq t$, we choose $s := \sqrt{\frac{(\log t)(m-n+2)}{m\log(ed)}}$ and $u := \sqrt{\frac{(\log t)(m-n+2)}{N}}$. (Note that $u \leq s$ also in this case). In this case we get that

$$P \leq \frac{1}{t^{m-n+2}}\left(\frac{(\log t)(m-n+2)}{N}\right)^{\frac{m}{2}}\left(\frac{N}{m\log(ed)}\right)^{\frac{n-2}{2}} + \frac{1}{t^{c_2(m-n+2)}} + \frac{1}{t^{m-n+2}} \leq$$

$$\frac{3}{t^{m-n+2}}\left(\frac{(\log t)(m-n+2)}{N}\right)^{\frac{m}{2}}\left(\frac{N}{m\log(ed)}\right)^{\frac{n-2}{2}}.$$

We consider now the case where $N \leq m \log(ed)$. In the case $1 \leq t \leq e^{\frac{N}{m-n+2}}$, we choose $s = 1$ and $u = 1$ and we get as before

$$P \leq \frac{1}{t^{m-n+2}} + e^{-c_2 m \log(ed)} + e^{-N} \leq \frac{3}{t^{m-n+2}}.$$

In the case $t \geq e^{\frac{N}{m-n+2}}$, we choose $s := u := \sqrt{\frac{(\log t)(m-n+2)}{m \log(ed)}}$. Note that again $(*)$ is satisfied and with these choices we get

$$P \leq \frac{1}{t^{m-n+2}} \left(\frac{(\log t)(m-n+2)}{N} \right)^{\frac{m}{2}} + \frac{1}{t^{\frac{c_2(m-n+2)m \log(ed)}{N}}} + \frac{1}{t^{m-n+2}} \leq$$

$$\frac{3}{t^{m-n+2}} \left(\frac{(\log t)(m-n+2)}{N} \right)^{\frac{m}{2}}.$$

In the case where $d_j = d$ for all $1 \leq j \leq m$, the proof is similar. One has just to observe that in this case it is always true that $N \geq m \log(ed)$. ■

Remark 4.3.1. *Note that the above tail estimates for the probability of the condition number of the random polynomial system imply that*

$$M(\kappa(\mathbf{p})) \leq 6M,$$

where M as defined in the Theorem 4.3.9.

Theorem 4.3.10. *Let $\mathbf{p}$ be a random polynomial system as in Theorem 4.3.9 and let M be as defined in Theorem 4.3.9. Set*

$$\delta_1 := \frac{q\sqrt{\pi n}}{m-n+2} \left(\frac{n-2}{2em \log(ed)} \right)^{\frac{n}{2}-1} \frac{1}{\left(1 - \frac{q}{m-n+2}\right)^{\frac{n}{2}}},$$

$$\delta_2 := \left(\frac{m}{N} \right)^{\frac{m-n+2}{2}} \frac{1}{\left(1 - \frac{q}{m-n+2}\right)^{\frac{m}{2}}} \frac{\sqrt{\pi m} q}{m-n+2-q} e^{-\frac{m}{2}} \frac{1}{(\log(ed))^{\frac{n}{2}-1}}.$$

Then we have the following estimates

1. *If* $d := d_j, 1 \leq j \leq m$, *for all* $0 < q < m - n + 2$ *we have*

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M \left(1 + \frac{q}{m-n-q+2} + \delta_1 + \delta_2\right)^{\frac{1}{q}}.$$

   In particular, if $q \leq (m-n+2)\left(1 - \frac{1}{2\log(ed)}\right)$, *then*

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M \left(\frac{3m\log(ed)}{n}\right)^{\frac{1}{q}},$$

   and if $q \leq \frac{m-n+2}{2}$ *then*

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M4^{\frac{1}{q}}.$$

   Moreover

$$\mathbb{E}\log\kappa(\mathbf{p}) \leq \log M + 1.$$

2. *If* $\max_i \deg(p_i) = d$ *we consider two cases:*

   (a) *In the case* $N \geq m\log(ed)$, *we have for all* $0 < q < m - n + 2$

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M \left(1 + \frac{q}{m-n-q+2} + \delta_1 + \delta_2\right)^{\frac{1}{q}}.$$

   In particular, if $q \leq (m-n+2)\left(1 - \frac{1}{2\log(ed)}\right)$, *then*

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M \left(\frac{3m\log(ed)}{n}\right)^{\frac{1}{q}},$$

   and if $q \leq \frac{m-n+2}{2}$ *then*

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M4^{\frac{1}{q}}.$$

Moreover

$$\mathbb{E}\log\kappa(\mathbf{p}) \leq \log M + 1$$

(b) In the case $N \leq m\log(ed)$

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M\left(1 + \frac{q}{m-n-q+2} + \delta_2\right)^{\frac{1}{q}}$$

In particular, if $q \leq (m-n+2)\left(1 - \frac{m}{eN}\right)$*, then*

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M\left(\frac{3m\log(ed)}{n}\right)^{\frac{1}{q}},$$

and if $q \leq \frac{m-n+2}{2}$ *then*

$$(\mathbb{E}(\kappa(\mathbf{p})^q))^{\frac{1}{q}} \leq M4^{\frac{1}{q}}.$$

Proof. We first consider the case where $d := d_i$, for all $1 \leq i \leq m$. Set

$$\Delta_1 := \left(\frac{m-n+2}{m\log ed}\right)^{\frac{n}{2}-1},\ \Delta_2 := \left(\frac{m-n+2}{N}\right)^{\frac{m}{2}}\left(\frac{N}{m\log ed}\right)^{\frac{n}{2}-1},$$

$$r := m-n-q+3,\ a_1 := \frac{m\log ed}{m-n+2},\ a_2 := \frac{N}{m-n+2}.$$

Note that we have assumed that $r \geq 1$. Using the formula

$$\mathbb{E}\kappa^q(\mathbf{p}) := q\int_0^\infty t^{q-1}\mathbb{P}\left(\{\kappa(p) \geq t\}\right)dt$$

and Theorem 4.3.9, we have that

$$\mathbb{E}\kappa^q(\mathbf{p}) \leq M^q\left(1 + q\int_1^\infty t^{q-1}\mathbb{P}\left(\{\kappa(p) \geq tM\}\right)dt\right) \text{ or}$$

$$\frac{\mathbb{E}\kappa^q(\mathbf{p})}{M^q} \leq 1 + q\int_1^{e^{a_1}} \frac{1}{t^r}dt + q\Delta_1 \int_{e^{a_1}}^{e^{a_2}} \frac{(\log t)^{\frac{n}{2}-1}}{t^r}dt + q\Delta_2 \int_{e^{a_2}}^{\infty} \frac{(\log t)^{\frac{m}{2}}}{t^r}dt.$$

We will give upper estimates for the three integrals appeared in the above inequality. First note that

$$q\int_1^{e^{a_1}} \frac{1}{t^r}dt = \frac{q}{r-1}\left(1 - e^{(r-1)a_1}\right) \leq \frac{q}{r-1}.$$

Also we have that

$$q\Delta_1 \int_{e^{a_1}}^{e^{a_2}} \frac{(\log t)^{\frac{n}{2}-1}}{t^r}dt = q\Delta_1 \int_{a_1}^{a_2} t^{\frac{n}{2}-1} e^{(r-1)t}dt = \frac{q\Delta_1}{(r-1)^{\frac{n}{2}}} \int_{a_1(r-1)}^{a_2(r-1)} t^{\frac{n}{2}-1} e^{-t}dt \leq$$

$$\frac{q\Delta_1}{(r-1)^{\frac{n}{2}}}\Gamma\left(\frac{n}{2}\right) \leq \frac{q\sqrt{\pi n}}{m-n+2}\left(\frac{n-2}{2em\log{(ed)}}\right)^{\frac{n}{2}-1} \frac{1}{\left(1-\frac{q}{m-n+2}\right)^{\frac{n}{2}}}.$$

Finally we check that

$$q\Delta_2 \int_{e^{a_2}}^{\infty} \frac{(\log t)^{\frac{m}{2}}}{t^r}dt = q\Delta_2 \int_{a_2}^{\infty} t^{\frac{m}{2}} e^{(r-1)t}dt = \frac{q\Delta_2}{(r-1)^{\frac{m}{2}+1}} \int_{a_2(r-1)}^{\infty} t^{\frac{m}{2}} e^{-t}dt \leq$$

$$\frac{q\Delta_2}{(r-1)^{\frac{m}{2}+1}}\Gamma\left(\frac{m}{2}+1\right) \leq \frac{\sqrt{\pi m}q}{(m-n-q+2)^{\frac{m}{2}+1}}\left(\frac{m(m-n+2)}{eN}\right)^{\frac{m}{2}}\left(\frac{N}{m\log ed}\right)^{\frac{n}{2}-1} =$$

$$\left(\frac{m}{N}\right)^{\frac{m-n+2}{2}} \frac{1}{\left(1-\frac{q}{m-n+2}\right)^{\frac{m}{2}}} \frac{\sqrt{\pi m}q}{m-n+2-q} e^{-\frac{m}{2}} \frac{1}{(\log{(ed)})^{\frac{n}{2}-1}}.$$

Note that if $q \leq (m-n+2)\left(1-\frac{1}{2\log{(ed)}}\right)$, then $\delta_1, \delta_2 \leq 1$.

The proof for the case $\max_i d_i = d$ and $N \geq m\log(ed)$ is identical. For the case $\max_i d_i = d$ and $N \leq m\log(ed)$, working as before we get that

$$\frac{\mathbb{E}\kappa^q(\mathbf{p})}{M^q} \leq 1 + q\int_1^{e^{a_2}} \frac{1}{t^r}dt + q\Delta_2 \int_{e^{a_2}}^{\infty} \frac{(\log t)^{\frac{m}{2}}}{t^r}dt \leq 1 + \frac{q}{r-1} + \delta_2.$$

In the case where $N \leq m \log(ed)$ we have that

$$\delta_2 \leq \frac{\sqrt{\pi m} q}{m-n+2} \left(\frac{m}{eN}\right)^{\frac{m}{2}} \frac{1}{\left(1 - \frac{q}{m-n+2}\right)^{\frac{m}{2}+1}}.$$

In this case we can show that if $q \leq (m-n+2)\left(1 - \frac{m}{N}\right)$ then $\delta_2 \leq 1$. ■

For the important case $m = n-1$ our main theorems reads as follows: (Note that if $m = n-1$ then $N \geq m \log(ed)$). Moreover, one may check that in that case one do not need to add the $\max\{1, \frac{\sqrt{n}}{d^2}\}$ term on "M", since $(*)$ is satisfied without it.

Corollary 4.3.11. *Let* $\mathbf{p}$ *be a random polynomial system as in Theorem 4.3.9 with* $m = n-1$*. Then we have the following:*

1. *If* $d = \deg(p_i)$, $1 \leq i \leq m$, *set*

$$M := M(n, d, c_0, K) := \sqrt{2} c C \sqrt{c_0} \sqrt{N} K \left(3 C c_0 K d \log(ed)\right)^{n-2}.$$

then

(a)

$$\mathbb{P}\{\kappa(\mathbf{p}) \geq tM\} \leq \begin{cases} \frac{3}{t} & \text{if } 1 \leq t \leq e^{(n-1)\log(ed)} \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{(n-1)\log(ed)} \leq t \leq e^N \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} \left(\frac{\log t}{N}\right)^{\frac{1}{2}} & \text{if } e^N \leq t \end{cases}$$

(b) For all $q \leq 1 - \frac{1}{2\log(ed)}$

$$\left(\mathbb{E}(\kappa(\mathbf{p})^q)\right)^{\frac{1}{q}} \leq M e^{\frac{1}{q}}.$$

Moreover

$$\mathbb{E}\log\kappa(p) \le \log M + 1$$

2. *If* $\max_i \deg(p_i) = d$ *set*

$$M := M(n,d,c_0,K) := \sqrt{2}cC\sqrt{c_0}\sqrt{N}K\left(3Cc_0Kd^2\log(ed)\right)^{n-2}.$$

then

(a)

$$\mathbb{P}\{\kappa(p) \ge tM\} \le \begin{cases} \frac{3}{t} & \text{if } 1 \le t \le e^{(n-1)\log(ed)} \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{(n-1)\log(ed)} \le t \le e^N \\ \frac{3}{t}\left(\frac{\log t}{N}\right)^{\frac{1}{2}}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^N \le t \end{cases}$$

(b) *Then for all* $q \le 1 - \frac{1}{2\log(ed)}$

$$(\mathbb{E}(\kappa(p)^q))^{\frac{1}{q}} \le Me^{\frac{1}{q}}.$$

Moreover

$$\mathbb{E}\log\kappa(p) \le \log M + 1$$

Remark 4.3.2. *The above results has been obtained under the assumptions on the randomness mentioned at the beginning of this section. Even if these assumptions allows us to consider much more general randomness than Gaussian the results are quite similar to the results obtained by Cucker, Malajovich, Krick and Wschebor in the Gaussian case as once can notice by comparing Corollary 4.3.11 with Theorem 4.1.1. In particular if* $d := d_j, 1 \le j \le n-1$, *in the Gaussian case Cucker and all* *proved that*

$$\mathbb{E}\log(\kappa(\mathbf{p})) \leq \frac{1}{2}\log N + \frac{n+1}{2}\log d + \frac{5}{2}\log n + \text{smaller terms}$$

while Corollary 4.3.11 gives

$$\mathbb{E}\log(\kappa(\mathbf{p})) \leq \frac{1}{2}\log N + (n-2)\left(\log d + \log\log(ed) + \log\left(C(c_0, K)\right)\right) + \log\tilde{C},$$

where $C(c_{0,K})$ is a constant depending only on c_0, K and $\tilde{C} \geq 1$ is an absolute constant.

One may notice that the most important difference in the two estimates is a factor "2" missing in the second term in our result.

Unfortunately, we do not know if the estimate of Cucker, Malajovich, Krick and Wschebor is of the right order. The next proposition provide some (most probably) very loose lower bounds for the condition number of a random polynomial system. In order to prove it we will need a slightly different assumption than our Small Ball.

There exists $\tilde{c_0} > 0$ such that for every $1 \leq j \leq m$, we have that
(5) $\mathbb{P}\left(\|C^{(j)}\|_2 \leq \varepsilon\sqrt{N_j}\right) \leq (\tilde{c_0}\varepsilon)^{N_j}$, $\varepsilon > 0$.

Let us comment on the above assumption. If the vectors C_j have independent coordinates then our Small Ball assumption and Lemma 4.3.4 implies that (5) holds true. Moreover, if $C^{(j)}$ are uniformly distributed on a convex body K and satisfy our subgaussian assumption, a result of J. Bourgain [23] (see also [35] or [73] for alternative proofs) implies again that (5) holds true (with a constant $\tilde{c_0}$ depending only on the subgaussian constant K and not the convex body K). In particular, all examples presented at the beginning of this section satisfy (5). For the proof we will need the following extension of Lemma 4.3.4 ([102], Theorem 1.5 and Corollary 8.6).

In order to state Lemma 4.3.12 we need to introduce a bit of terminology.

For a matrix $A := (a_{i,j})_{1\le i,j\le m}$ we write $\|A\|_{HS}$ is the Hilbert-Schmidt norm of A and $\|A\|_{op}$ for the operator norm , i.e.

$$\|A\|_{HS} := \left(\sum_{i,j=1}^{m} a_{i,j}^2\right)^{\frac{1}{2}}, \ \|A\|_{op} := \max_{\theta\in S^{n-1}} \|A\theta\|_2.$$

Lemma 4.3.12. *Let ξ_i, $1\le i\le m$ be independent random variables that satisfy the following small ball assumption:*

$$\mathbb{P}\left(\xi_i \le \varepsilon\right) \le c_0\varepsilon, \ \varepsilon > 0, \ 1\le i\le m.$$

Let $\xi := (\xi_1, \cdots, x_m)$. Then for every A, $m\times m$ matrix we have that

$$\mathbb{P}\left(\|A\xi\|_2 \le \varepsilon\|A\|_{HS}\right) \le (cc_0\varepsilon)^{c\frac{\|A\|_{HS}^2}{\|A\|_{op}^2}}, \ \varepsilon > 0$$

where $c > 0$ is an absolute constant. In the case where $\|A\|_{HS} = \sqrt{m}\|A\|_{op}$ we have that

$$\mathbb{P}\left(\|A\xi\|_2 \le \varepsilon\|A\|_{HS}\right) \le (cc_0\varepsilon)^m, \ \varepsilon > 0$$

We are now ready to prove lower bounds for the condition number of random polynomial systems:

Proposition 4.3.13. *Let $\mathbf{p} = (p_1, \cdots, p_m)$ be a homogeneous n-variate polynomial system with $\deg(p_i) = d_i$. Then we have that*

$$\kappa(\mathbf{p}) \ge \frac{\|p\|_w}{\sqrt{m+1}\|\mathbf{p}\|_\infty}.$$

Moreover if $\mathbf{p} := (p_1, \cdots, p_m)$ is a random polynomial systems satisfying our sub-

gaussian assumption and our assumption (5) (with constants $K, \tilde{c_0}$ respectively) we have that

$$\mathbb{P}\left(\kappa(\mathbf{p}) \leq \varepsilon \frac{\sqrt{N}}{Kmd\log(ed)}\right) \leq (c\tilde{c_0}\varepsilon)^{c'\min\{N\frac{\min_{j\leq m}N_j}{\max_{j\leq m}N_j}, md\log(ed)\}} \text{ and}$$

$$\mathbb{P}\left(\kappa(\mathbf{p}) \leq \varepsilon \frac{\sqrt{N}}{Km\log(ed)}\right) \leq (c\tilde{c_0}\varepsilon)^{c'm\log(ed)}, \text{ if } d = d_j, 1 \leq j \leq m,$$

where $c, c' > 0$ are absolute constants. In particular we have that

$$\exp \mathbb{E}\log(\kappa(\mathbf{p})) \geq c\frac{\sqrt{N}}{m\log(ed)} \text{ when } d = d_j, 1 \leq j \leq m.$$

Proof. First note that Theorem 4.2.2 implies that for every $x, y \in S^{n-1}$,

$$\|d_j^{-1}D^{(1)}p_j(x)y\|_2^2 \leq \|p_j\|_\infty^2.$$

So, we have that

$$\|M^{-1}D^{(1)}\mathbf{p}(x)(y)\|_2^2 \leq \sum_{j=1}^m \|p_j\|_\infty^2 \leq m\|\mathbf{p}\|_\infty^2.$$

Recall that $L^2(x,y) := \|M^{-1}D^{(1)}\mathbf{p}(x)(y)\|_2^2 + \|p(x)\|_2^2$. So, we get that

$$L^2 := \min_{x\perp y} L^2(x,y) \leq (m+1)\|\mathbf{p}\|_\infty^2,$$

which implies that

$$\kappa(\mathbf{p}) \geq \frac{\|p\|_w}{L} \geq \frac{\|p\|_w}{\sqrt{m+1}\|\mathbf{p}\|_\infty}.$$

In case where $d_j = d, 1 \leq j \leq m$ the proof is identical.

We will show that under our assumption (5) the following holds true: for every $\varepsilon > 0$

$$\mathbb{P}\left(\|\mathbf{p}\|_w \leq \varepsilon\sqrt{N}\right) \leq (c\tilde{c_0}\varepsilon)^{cN\frac{\min_{1\leq j\leq m} N_j}{\max_{1\leq j\leq m} N_j}}.$$

Indeed, recall that $\|p_j\|_w = \|C_j\|_{\ell_2^{N_j}}$. Then for any fix $\varepsilon > 0$,

$$\mathbb{P}\left(\|p_j\|_w \leq \varepsilon\sqrt{N_j}\right) \leq (\tilde{c_0}\varepsilon)^{N_j} \leq (\tilde{c_0}\varepsilon)^{N_{j_0}}.$$

where $N_{j_0} := \min_{1\leq j\leq m} N_j$. Let $\xi_j := \frac{\|p_j\|_w}{\sqrt{N_j}}$, $1 \leq j \leq m$. Set $\xi := (\xi_1, \cdots, \xi_m)$ $A := \mathrm{diag}(\sqrt{N_1}, \cdots, \sqrt{N_m})$. Note that $\|\mathbf{p}\|_w = \|A\xi\|_2$, $\|A\|_{HS} = \sqrt{\sum_{j=1}^m N_j} = \sqrt{N}$ and $\|A\|_{op} := \max_{1\leq j\leq m} \sqrt{N_j}$. Then Lemma 4.3.12 implies

$$\mathbb{P}\left(\|p\|_w \leq \varepsilon\sqrt{N}\right) \leq (c\tilde{c_0}\varepsilon)^{cN\frac{\min_{1\leq j\leq m} N_j}{\max_{1\leq j\leq m} N_j}}.$$

Recall that Lemma 4.3.2 implies that for every $t \geq 1$,

$$\mathbb{P}\left(\|p\|_\infty \geq ctK\sqrt{m}\log(ed)\right) \leq e^{-t^2 m \log(ed)}.$$

So, using our lower bound estimate for the condition number we get that

$$\mathbb{P}\left(\frac{\|\mathbf{p}\|_w}{\|p\|_\infty} \geq \frac{c'\varepsilon\sqrt{N}}{tK\sqrt{m}\log(ed)}\right) \leq \mathbb{P}\left(\kappa(\mathbf{p}) \geq \frac{c\varepsilon\sqrt{N}}{tKmd\log(ed)}\right)$$

$$\mathbb{P}\left(\{\|\mathbf{p}\|_w \geq c'\varepsilon\sqrt{N}\} \cap \{\|\mathbf{p}\|_\infty \leq ctK\sqrt{m}\log(ed)\}\right) \leq \mathbb{P}\left(\kappa(\mathbf{p}) \geq \frac{c\varepsilon\sqrt{N}}{tKmd\log(ed)}\right)$$

$$\mathbb{P}\left(\|\mathbf{p}\|_w \geq c'\varepsilon\sqrt{N}\right) + \mathbb{P}\left(\|\mathbf{p}\|_\infty \leq c\sqrt{m}\log(ed)\right) - 1 \geq 1 - (c\tilde{c_0}\varepsilon)^{-cN\frac{\min_{j\leq m} N_j}{\max_{j\leq m} N_j}} - e^{-t^2 m \log(ed)}.$$

We may choose $t := \sqrt{\log\frac{1}{\varepsilon}}$ and by adjusting the constant we get the result. The case where $d_j = d, 1 \leq j \leq m$ is similar. The bounds for the expectation follows by integration. ■

Remark 4.3.3. *The above lemma and Theorem 4.3.10 imply that the following asymptotic formulas holds true* ($n, d \geq 3$, $m = n-1, d_j = d, 1 \leq j \leq n-1$)*:*

$$c_1(n \log d + d \log n) \leq \mathbb{E} \log (\kappa(\mathbf{p})) \leq c_2(n \log d + d \log n)$$

for random polynomial systems $\mathbf{p}$ *satisfying the assumptions at the beginning of the section. This settles the asymptotic order of* $\log(\kappa(\mathbf{p}))$*.*

Remark 4.3.4. *Of course we would like to know the asymptotic behavior of* $\kappa(\mathbf{p})$ *instead of* $\log(\kappa(\mathbf{p}))$*. One must notice that the dominant term in our estimates is* $\sqrt{N}$*, which is the normalization that comes from the Weil-Bombieri norm of the polynomial system. So we would like to know the asymptotic behavior of* $\frac{\kappa(\mathbf{p})}{\sqrt{N}}$*. When* $m = n-1$*, the upper bounds that we get, are exponential with respect to* n*, while the lower bounds are not. But when* $m = 2n-4$ *we have the following estimates (consider the case* $d = d_j, 1 \leq j \leq m$*)*

$$\frac{C_1}{nd \log (ed)} \leq \frac{\mathbb{E}(\kappa(\mathbf{p}))}{\sqrt{N}} \leq \frac{C_2 d \log ed}{\sqrt{n}},$$

where C_1, C_2 *are constants depending on* K, c_0*. This suggest that our estimates tends to be more accurate when* m *is much larger than* n*.*

Remark 4.3.5. *There are similarities on the probability tail estimates of our main result and the estimates in the linear case [103]. In particular our estimates in the quadratic case* $d = 2$ *when* m *is propositional to* n *are quite similar to the optimal result as appeared in [103], which indicates that in the proportional case* ($m \simeq n$) *our result is close to be optimal.*

Remark 4.3.6. *As we mentioned in the introduction H.H. Nguyen proved similar results in [87] for subgaussian random polynomial systems. He worked without the Small Ball assumption that we used that allowed him to state results also in the case of independent Bernoulli's. However he assumed that all the entries in of the random vectors are independent random variables contrary to our setting. It appears that our methods avoids several restrictions that appears in [87] such as the degree of the polynomials must be less than a small power of the number of variables. Since his results are stated for a quantity that resembles the condition number (but it is not) it is not possible to compare the estimates.*

Remark 4.3.7. *Up to our knowledge our results for the condition number of random polynomial systems when $m \geq n$ are the first one to appear, even when the underlying randomness is Gaussian.*

Remark 4.3.8. *As we mention at the beginning of the section, if C_j are uniformly distributed on the unit sphere (of dimension N_j) then our subgaussin and Small Ball assumptions are satisfied. Moreover, in that case, one has that $\|\mathbf{p}\|_w = \sqrt{N}$ with probability 1. So, in that case the estimates on our main results can be slightly improved since one does not need to invoke Lemma 4.3.8. We leave the details to the reader.*

4.4 Random Polynomial Systems With Given Support

In this section we will extend the results of the previous section to the case of random polynomial systems with given support. Let us first fix the notation and provide some basic examples.

4.4.1 Notation and the Basic Quantities

Fix n to be the number of variables, m be the number of equations and d_j, $1 \leq j \leq m$ be the degrees of the polynomials. We will always assume that $n \geq 3$ and $m \geq n-1$. We define

$$S_{n,d} := \{(\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n : \sum_{i=1}^{n} \alpha_i = d\},$$

$$S_{n,d_1,\cdots,d_m} := \{(S_1, \cdots, S_m) : S_j \in S_{n,d_j},\ 1 \leq j \leq m\}.$$

We say that $\mathbf{S} = (S_1, \cdots, S_m) \subseteq S_{n,d_1,\cdots,d_m}$ is eligible (and we write $\mathbf{S} \subseteq S^e_{n,d_1,\cdots,d_m}$) if $S_j \neq \emptyset$ for all $1 \leq j \leq m$, and for every $1 \leq i \leq n$, there exists $\alpha \in \bigcup_{j \leq m} S_j$ such that $\alpha_i \neq 0$.

Given $S \subseteq S^e_{n,d}$ and coefficients $a_\alpha, \alpha \in S$, a polynomial with support S is written as follows:

$$p(x) := \sum_{\alpha \in S} a_\alpha x^\alpha.$$

Given $\mathbf{S} := (S_1, \cdots, S_m) \subseteq S^e_{n,d_1,\cdots,d_m}$ and let $a^{(j)}_\alpha, \alpha \in S_j$, $1 \leq j \leq m$ be coefficients. We write

$$\textbf{(SPS)} \qquad \mathbf{p} := (p_1, \cdots, p_m),\ p_j(x) := \sum_{\alpha \in S_j} a^{(j)}_\alpha x^\alpha,\ 1 \leq j \leq m,$$

for a polynomial system with support $\mathbf{S}$. Let $c^{(j)}_\alpha$, $\alpha_i \in S_j$ be a random vector in $\mathbb{R}^{I_j}$, where I_j is the cardinality of S_j, that satisfies our Sub-Gaussian and Small Ball assumptions (with constants K and c_0) and the vectors $C^{(j)} := c^{(j)}_\alpha$ are independent $1 \leq j \leq m$. Then we write

$$\textbf{(RSPS)} \qquad \mathbf{p} := (p_1, \cdots, p_m),\ p_j(x) := \sum_{\alpha \in S_j} c^{(j)}_\alpha \sqrt{\binom{d}{\alpha}} x^\alpha,\ 1 \leq j \leq m$$

for a random polynomial system with support $\mathbf{S}$. In the case where $\mathbf{S} := S_{n,d_1,\cdot,d_m}$ then the **RSPS** is just the random polynomial system that we have investigated in the previous section.

Of course the same definition of the condition number applies to this case. Our goal is to give bounds for the condition number for the **RSPS** polynomial system. In this case the role of the support $\mathbf{S}$ is very important. It is easy to construct an example of **SPS** that has condition number infinity for any choice of coefficients. Let us give an example where $n = 3, m = 2$ and $d_1 = d_2 = 6$. Consider the system $\mathbf{p} := (p_1, p_2)$, where

$$p_j(x) := a_{j,1}x_1^6 + a_{j,2}x_1^2x_2^4 + a_{j,3}x_1^2x_3^4,\ j = 1, 2.$$

Then every point in the subshpere $\{(y_1, y_2, y_3) \in S^2 : y_1 = 0\}$ is a root of the system of multiplicity 2. But this implies that the condition number of the system is infinity. Under the Small Ball assumption, our random polynomial system will give that $c_\alpha^{(i)}$ are not zero (for all $\alpha \in S_i$, and any $1 \le i \le m$) with probability 1 which imply that the condition number of the system is infinity with probability 1. We will introduce the quantity $H(\mathbf{S})$, which captures the geometry of the support and we will provide bounds for the condition number of **RSPS** with respect to $H(\mathbf{S})$.

In order to introduce the quantity $H(\mathbf{S})$ we will need some additional notation. Let $x \in S^{n-1}$, $S \subseteq S_{n,d}^e$ and let $\mathcal{X}_S : S^{n-1} \to \mathbb{R}^N$, where $N := \binom{n+d+1}{d}$, defined as

$$\mathcal{X}_S(x) = \left(\sqrt{\binom{d}{\alpha}}x^\alpha\right)_{\alpha \in S}.$$

Moreover, we define the function $A_S : S^{n-1} \to \mathbb{R}_+$, as

$$A_S(x) := \|\mathcal{X}_S(x)\|_{\ell_2^N} := \left(\sum_{\alpha \in S} \binom{d}{\alpha} x^{2\alpha} \right)^{\frac{1}{2}}.$$

Note that for every $x \in S^{n-1}$, $A_{S_{n,d}}(x) = \left(\sum_{\alpha \in S_{n,d}} \binom{d}{\alpha} x^{2\alpha} \right)^{\frac{1}{2}} = 1$. Note that, if $S_1, S_2 \subseteq S_{n,d}$ such that $S_1 \cap S_2 = \emptyset$ then for every $x \in S^{n-1}$ we have that

$$A^2_{S_1 \cup S_2}(x) = A^2_{S_1}(x) + A^2_{S_2}(x).$$

The above relation, together with the fact that $A_{S_{n,d}}(x) = 1$, implies that for every $S \subseteq S_{n,d}$,

$$\max_{x \in S^{n-1}} A_S(x) \leq 1 \text{ and } \mathcal{X}_S(x) \in B_2^N.$$

Let $x \in S^{n-1}$, $\mathbf{S} := (S_1, \cdots, S_m) \subseteq S_{n,d_1,\cdots,d_m}$ and let $\mathcal{X}_{\mathbf{S}} : S^{n-1} \to \mathbb{R}^N$, where $N := \sum_{i=1}^m \binom{n+d_i+1}{d_i}$, defined as

$$\mathcal{X}_{\mathbf{S}}(x) = \left(\left(\sqrt{\binom{d}{\alpha}} x^{\alpha} \right)_{\alpha \in S_1}, \cdots, \left(\sqrt{\binom{d}{\alpha}} x^{\alpha} \right)_{\alpha \in S_m} \right).$$

We also define the function $A_{\mathbf{S}} : S^{n-1} \to \mathbb{R}_+$, as

$$A_{\mathbf{S}}(x) := \|\mathcal{X}_{\mathbf{S}}(x)\|_{\ell_2^N} := \left(\sum_{i=1}^m A^2_{S_i}(x) \right)^{\frac{1}{2}}.$$

As before we have that if $\mathbf{S}_1, \mathbf{S}_2 \subseteq S_{n,d_1,\cdots,d_m}$ and $\mathbf{S}_1 \cap \mathbf{S}_2 = \emptyset$, then

$$A^2_{\mathbf{S}_1 \cup \mathbf{S}_2}(x) = A^2_{\mathbf{S}_1}(x) + A^2_{\mathbf{S}_2}(x).$$

Also,

$$\max_{x \in S^{n-1}} A_{\mathbf{S}}(x) \leq \sqrt{m} = A_{S_{n,d_1,\cdots,d_m}}(y),\ y \in S^{n-1}.$$

We can introduce now the quantity $H(\mathbf{S})$. For any eligible support $\mathbf{S} \subseteq S^e_{n,d_1,\cdots,d_m}$, we define

$$H(\mathbf{S}) := \sup_{x \in S^{n-1}} \frac{1}{A_{\mathbf{S}}(x)}.$$

We will also need the following quantity. For any $\mathbf{S} \subseteq S_{n,d_1,\cdots,d_m}$ we define

$$\Delta_S := \begin{cases} c \min_{x \in S^{n-1}} \frac{A_{\mathbf{S}}(x)}{\max_{j \leq m} A_{S_j}(x)\sqrt{m}} & \text{if } \min_{x \in S^{n-1}} \frac{A_{\mathbf{S}}(x)}{\max_{j \leq m} A_{S_j}(x)\sqrt{m}} < 1 \\ 1 & \text{if } \min_{x \in S^{n-1}} \frac{A_{\mathbf{S}}(x)}{\max_{j \leq m} A_{S_j}(x)\sqrt{m}} = 1 \end{cases},$$

where $0 < c < 1$ is an absolute constant that appears in Corollary 4.3.12.

Note that in the special case where $S_j = S$, $1 \leq j \leq m$, we have that $\Delta_S = 1$. In general we have that

$$\frac{c}{\sqrt{m}} \leq \Delta_{\mathbf{S}} \leq 1.$$

Note that

$$H(S_{n,d_1,\cdots,d_m}) = \frac{1}{\sqrt{m}} \text{ and } \Delta_{S_{n,d_1,\cdots,d_m}} = 1.$$

4.4.2 Bounding the Quantity $H(\mathbf{S})$

Before we prove our main result, we will investigate further the quantity $H(\mathbf{S})$. We start with the following

Lemma 4.4.1. *Let $\mathbf{S} \subseteq S^e_{n,d_1,\cdots,d_m}$ and let $\mathbf{p}$ be a* **SPS** *polynomial system with support $\mathbf{S}$. Assume that there exists $y \in S^{n-1}$ such that $A_{\mathbf{S}}(y) = 0$. Then y has to be a common root the polynomial system $\mathbf{p}$ with multiplicity greater than 1.*

Proof. If $A_{\mathbf{S}}(y) = 0$ for some $y \in S^{n-1}$ then for all $\alpha \in \bigcup_{j \leq m} S_j$ we have $y^{2\alpha} = 0$. For

$\alpha \in \mathbb{Z}^n$ we define $supp(\alpha) := \{i \in \{1, 2, \ldots, n\} : \alpha_i \neq 0\}$. $y^\alpha = 0$ implies that there exist $i \in supp(\alpha)$ such that $y_i = 0$. Let $S_y := \{i \in \{1, 2, \ldots, n\} : y_i = 0\}$. Then we get that $A_{\mathbf{S}}(y) = 0$ implies $S_y \cap supp(\alpha) \neq \emptyset$ for all $\alpha \in \bigcup_{j \leq m} S_j$. Since $y \in S^{n-1}$, $S_y \subsetneq \{1, 2, \ldots, n\}$. First we consider the case $|S_y| = n - 1$, wlog say $1 \notin S_y$. Then we dehomogenize the system $\mathbf{p}$ as follows: for all p_j set $\tilde{p_j} = p_j(1, x_2, x_3, \ldots, x_n)$ and $\tilde{\mathbf{p}} = (\tilde{p_1}, \tilde{p_2}, \ldots, \tilde{p}_{n-1})$. We denote the support set of the system $\tilde{\mathbf{p}}$ by $\tilde{S}$. Then for every $\alpha \in \tilde{S}$ we have $S_y \cap supp(\alpha) \neq \emptyset$. Thus, $(0, 0, \ldots, 0)$ is a root of multiplicity greater than 1 for $\tilde{\mathbf{p}}$. Hence $(1, 0, \ldots, 0)$ is a root of multiplicity greater than 1 for $\mathbf{p}$. Second we consider the case $|S_y| \leq n-2$, wlog say $1, 2 \notin S_y$. Then we dehomogenize $\mathbf{p}$ with respect to x_1 as in the first case and define $y_0 = (y_2, y_3, \ldots, y_n)$. We observe that for any j and for any monomial x^α that appears in $\frac{\partial}{\partial x_2}\tilde{p_j}$ we have $supp(\alpha) \cap S_y \neq \emptyset$. That is, for all j we have $\frac{\partial}{\partial x_2}\tilde{p_j}(y_0) = 0$. Therefore the Jacobian matrix of the system $\tilde{\mathbf{p}}$ is not full rank at y_0 i.e y_0 is a root of multiplicity greater than 1 for $\tilde{\mathbf{p}}$. Hence the same conclusion for $\mathbf{p}$ at y. ■

Let $e_i, 1 \leq i \leq n$ be the standard orthonormal basis of $\mathbb{R}^n$. We say that a support $\mathbf{S} \subseteq S^e_{n,d_1,\cdots,d_m}$ is proper if for every $1 \leq i \leq n$, there exists $j \leq m$ such that $d_j e_i \in S_j$. We write $\mathcal{P}_{n,d_1,\cdots,d_m}$ for the set of proper supports.

Proposition 4.4.2. *Let* $\mathbf{S} \notin \mathcal{P}_{n,d_1,\cdots,d_m}$ *be a non-proper eligible support. Then*

$$H(\mathbf{S}) = \infty.$$

Proof. Note that

$$H(\mathbf{S}) = \infty \iff \min_{x \in S^{n-1}} A_{\mathbf{S}}(x) = 0.$$

Since $\mathbf{S}$ is not proper there exists $1 \leq i \leq n$ such that $d_j e_i \notin S_j$ for all $1 \leq j \leq m$. Without loss of generality we may assume that $i = n$. Since $\mathbf{S}$ is eligible we have that

$$\cup_{j=1}^{m}\{\alpha \in S_j : \alpha_n \neq 0\} \neq \emptyset.$$

But the non-proper assumption implies that

$$\cup_{j=1}^{m}\{\alpha \in S_j : \alpha_n \neq 0\} = \cup_{j=1}^{m}\{\alpha \in S_j : \alpha_n \neq 0 \text{ and } \exists\ 1 \leq i < n : \alpha_i \neq 0\}.$$

Let $x_0 = (0, \cdots, 0, 1) \in S^{n-1}$. We clearly have $A_{\mathbf{S}}(x_0) = 0$. ■

Let $p \geq 1$ and $x \in \mathbb{R}^n$. Recall the definition of p-norm of x:

$$\|x\|_p := \|x\|_{\ell_p^n} := \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}.$$

Proposition 4.4.3. *Let* $\mathbf{S} \in \mathcal{P}_{n,d_1,\cdots,d_m}$ *be a proper eligible support. Let* $d := \max_{1\leq j\leq m} d_j$*, then for every* $x \in S^{n-1}$*,*

$$A_{\mathbf{S}}(x) \geq \|x\|_{2d}^{d} \quad \text{and}$$

$$H(\mathbf{S}) \leq \frac{n^{\frac{d}{2}}}{\sqrt{n}}.$$

Moreover, when $m = n - 1$ *and* $d_i = d$ *we have that there exists proper eligible* $\mathbf{T}$ *such that for every* $x \in S^{n-1}$*,*

$$A_{\mathbf{T}}(x) = \|x\|_{2d}^{d} \text{ and } H(\mathbf{T}) = \frac{n^{\frac{d}{2}}}{\sqrt{n}}.$$

Proof. For every $x \in S^{n-1}$, since $d_j \leq d$, for all $i \in \{1, 2, \ldots, n\}$ we have $x_i^{2d_j} \geq x_i^{2d}$. As a direct consequence of S being proper, we get the following inequality

$$A^2_{\mathbf{S}}(x) \geq \sum_{\cup_{j=1}^m \{\alpha \in S_j\}} x^{2\alpha} \geq \sum_{i=1}^n x_i^{2d} = \|x\|_{2d}^{2d}.$$

Note that Hölder inequality implies that for $p \geq 2$ and every $x \in \mathbb{R}^n$,

$$\|x\|_p \geq n^{\frac{1}{p}-\frac{1}{2}} \|x\|_2,$$

with equality when $x = (\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}) \in S^{n-1}$. So we have that

$$H_{\mathbf{S}} = \sup_{x \in S^{n-1}} \frac{1}{A_{\mathbf{S}}(x)} \leq \max_{x \in S^{n-1}} \frac{1}{\|x\|_{2d}^d} \leq \left(n^{\frac{1}{2}-\frac{1}{d}}\right)^d = \frac{n^{\frac{d}{2}}}{\sqrt{n}}.$$

In the case where $m = n-1$ and $d_j = d, 1 \leq j \leq m$, consider $\mathbf{T} := (T_1, \cdots, T_{n-1})$ with $T_j := \{de_j\}$, when $1 \leq j \leq n-2$ and $T_{n-1} := \{de_{n-1}, de_n\}$. It is straightforward to check that $A_{\mathbf{T}}(x) = \|x\|_{2d}^d$. Therefore, $H(\mathbf{T}) = \frac{n^{\frac{d}{2}}}{\sqrt{n}}$. ■

The above two proposition provides a characterization of the supports that $H(\mathbf{S})$ is finite.

Corollary 4.4.4. *Let* $\mathbf{S} \subseteq S^e_{n,d_1,\cdots,d_m}$ *be an eligible support. Then*

$$H(\mathbf{S}) < \infty \iff \mathbf{S} \text{ is proper.}$$

4.4.3 The Main Result

We now turn our attention to the proof of a condition number theorem for random polynomial systems with fixed support. Assume that we have a polynomial system with support $\mathbf{S} := (S_1, \cdots, S_m)$ and random vectors $C^{(j)} \in \mathbb{R}^{I_j}$, $1 \leq j \leq m$ independent with the Subgaussian property (with constant K). Fix $x \in S^{n-1}$. We have that $p_j(x) := \langle C^{(j)}, \mathcal{X}_{S_j}(x)\rangle$. We set $\widetilde{\mathcal{X}_{S_j}(x)} = \frac{\mathcal{X}_{S_j}(x)}{\left\|\mathcal{X}_{S_j}(x)\right\|_2}$. Note that $\left\|\mathcal{X}_{S_j}(x)\right\|_2 = A_{S_j}(x)$.

Then we have the following,

$$\mathbb{P}\left(|p_j(x)| \geq t\right) \leq \mathbb{P}\left(|\langle C^{(j)}, \mathcal{X}_{S_j}(x)\rangle| \geq t\right) \leq$$
$$\mathbb{P}\left(|\langle C^{(j)}, \widetilde{\mathcal{X}_{S_j}(x)}\rangle| \geq \frac{t}{A_{S_j}(x)}\right) \leq 2e^{-\frac{t^2}{A_{S_j}^2(x)K^2}}.$$

Since $A_{S_j}(x) \leq 1$, we have the following

$$\mathbb{P}\left(|p_j(x)| \geq t\right) \leq 2e^{-\frac{t^2}{K^2}}$$

Tensorazing the above inequality, as in Lemma 3.2, for every $x \in S^{n-1}$ we have

$$\mathbb{P}\left(\|p(x)\|_2 \geq t\sqrt{m}\right) \leq 2e^{-\frac{t^2 m}{K^2}}.$$

Working as in section 3 we prove the following

Lemma 4.4.5. *Let* $\mathbf{p}$ *be a* **RSPS** *with the Subgaussian assumption (with constant* K*). Then we have that for every* $s \geq 1$ *and* $d := \max_{i \leq m} d_i$*,*

$$\mathbb{P}\left(\|p\|_\infty \geq 3sK\sqrt{m}\log(ed)\right) \leq e^{-c_2 s^2 m \log(ed)},$$

where $c_2 \geq 1$ *is an absolute constant.*

Corollary 4.4.6. *Let* $\mathbf{p} := (p_1, \cdots, p_m)$ *be a a* **RSPS** *with support* $\mathbf{S}$*, satisfying the Small Ball assumption (with constant* c_0*). Then, for every* $x \in S^{n-1}$*, we have that*

$$\mathbb{P}\left(\|p(x)\|_2 \leq \frac{\varepsilon}{H(\mathbf{S})}\right) \leq (cc_0\varepsilon)^{m\Delta_{\mathbf{S}}^2},$$

where $c > 0$ *is an absolute constant.*

Proof. Fix $x \in S^{n-1}$ and recall that $p_j(x) := \langle C^{(j)}, \mathcal{X}_{S_j}(x)\rangle$. Let $\xi_j := |p_j(x)|$ and $\tilde{\xi} := \frac{\xi_j}{A_j(x)}$. Note that for the random variables $\tilde{x_j}$, our small ball assumption implies that

$$\mathbb{P}\left(\tilde{\xi} \le \varepsilon\right) = \mathbb{P}\left(|\langle C^{(j)}, \widetilde{\mathcal{X}_{S_j}(x)}\rangle| \le \varepsilon\right) \le c_0\varepsilon.$$

Set $\xi := (\xi_1, \cdots, \xi_m)$ and $\tilde{\xi} := (\tilde{\xi_1}, \cdots, \tilde{x_m})$ and $A := \operatorname{diag}(A_{S_1}(x), \cdots, A_{S_m}(x))$. Note that

$$\xi := A\tilde{\xi},\ \|A\|_{HS} := A_{\mathbf{S}}(x),\ \|A\|_{op} := \max_{j\le m} A_{S_j}(x).$$

Assume first that $\min_{x\in S^{n-1}} \frac{A^2_{\mathbf{S}}(x)}{m \max_{j\le m} A^2_{S_j}(x)} < 1$. Since $\|p(x)\|_2 := \|\xi\|_2$, Lemma 4.3.12 implies that

$$\mathbb{P}\left(\|p(x)\|_2 \le \varepsilon A_S(x)\right) \le (cc_0\varepsilon)^{c\frac{A^2_{\mathbf{S}}(x)}{\max_{j\le m} A^2_{S_j}(x)}}.$$

Since for every $x \in S^{n-1}$, $\frac{1}{H(\mathbf{S})} \le A_S(x)$ and $c\frac{A^2_{\mathbf{S}}(x)}{\max_{j\le m} A^2_{S_j}(x)} \ge \Delta_S^2 m$, we conclude that

$$\mathbb{P}\left(\|p(x)\|_2 \le \frac{\varepsilon}{H(\mathbf{S})}\right) \le (cc_0\varepsilon)^{m\Delta^2_{\mathbf{S}}}.$$

We treat the case $\min_{x\in S^{n-1}} \frac{A^2_{\mathbf{S}}(x)}{m \max_{j\le m} A^2_{S_j}(x)} = 1$ similarly. ∎

We are now ready to prove the following

Theorem 4.4.7. *Let $\mathbf{p} := (p_1, \cdots, p_m)$ be a a* **RSPS** *with support* $\mathbf{S}$ *satisfying the Small Ball assumption (with constant c_0). Let $\alpha, \gamma > 0$. Then we have the following*

1. *In the case where $d = \max_{1\le j\le m} d_j$, if $\alpha \le \min\{1, \frac{d^2}{\sqrt{n}}\}\gamma$, we have that*

$$\mathbb{P}(L \le \alpha) \le \mathbb{P}\left(\|\mathbf{p}\|_\infty \ge \gamma\right) + C\sqrt{n}\left(\frac{\alpha}{\gamma d^2}\right)^{2-n} (cc_0\alpha H(\mathbf{S}))^{m\Delta^2_{\mathbf{S}}},$$

where $C, c > 0$ are absolute constants.

2. *In the case where* $d := d_j, 1 \le j \le m$, *if* $\alpha \le \min\{1, \frac{d}{\sqrt{n}}\}\gamma$,

$$\mathbb{P}(L \le \alpha) \le \mathbb{P}\left(\|\mathbf{p}\|_\infty \ge \gamma\right) + C\sqrt{n}\left(\frac{\alpha}{\gamma d}\right)^{2-n} (cc_0\alpha H(\mathbf{S}))^{m\Delta_{\mathbf{S}}^2},$$

where $C, c > 0$ *are absolute constants.*

Proof. The proof is identical to the proof of 4.3.7 with the only difference to be the use of Lemma 4.4.6 instead of Lemma 4.3.5. ■

The proof of the following Lemma is identical to the proof of Lemma 4.3.8.

Lemma 4.4.8. *Let* $\mathbf{p} := (p_1, \cdots, p_m)$ *be a a* **RSPS** *with support* $\mathbf{S} := (S_1, \cdots, S_m)$, *that satisfies our Subgaussian assumption (with constant* K_0*). Let* I_j *be the cardinality of* S_j *and let* $I := \sum_{j=1}^m I_j$. *Then, for every* $t \ge 1$,

$$\mathbb{P}\left(\|p\|_W \ge \tilde{c}tK\sqrt{I}\right) \le e^{-t^2 I},$$

where $\tilde{c} > 0$ *is an absolute constant.*

We are now ready to state the main Theorem of the section. The proof is similar to the proof of Theorem 4.3.9 and will be omitted.

Theorem 4.4.9. *There exist constants* $C, c, \tilde{c} > 0$ *such that the following holds: Let* $\mathbf{p} := (p_1, \cdots, p_m)$ *be a* **RSPS** *with support* $\mathbf{S} := (S_1, \cdots, S_m)$, *that satisfies our subgaussian assumption (with constant* K_0*) and our Small Ball assumption (with constant* c_0*). Let* I_j *be the cardinality of* S_j *and let* $I := \sum_{j=1}^m I_j$. *We also assume that* $m\Delta_{\mathbf{S}}^2 \ge n-1$ *and set* $m\Delta_{\mathbf{S}}^2 - n + 2 = A$. *In the case* $d := \max_{1\le j\le m} d_j$ *we set*

$$M := \sqrt{I}\left(\frac{n}{c_0}\right)^{\frac{1}{2A}} (d^2\log(ed))^{\frac{n-2}{A}} (cc_0K\sqrt{m}H(S))^{\frac{m\Delta_{\mathbf{S}}^2}{A}} m^{-\frac{1}{2}} \max\{1, \frac{\sqrt{n}}{d^2}\}$$

In the case $d := d_j, 1 \le j \le m$, we set

$$M := \sqrt{I}(\frac{n}{c_0})^{\frac{1}{2A}}(d\log(ed))^{\frac{n-2}{A}}(cc_0K\sqrt{m}H(S))^{\frac{m\Delta_{\mathbf{S}}^2}{A}}m^{-\frac{1}{2}}\max\{1, \frac{\sqrt{n}}{d}\}$$

We denote $\mathbb{P}\{\kappa(p) \ge tM\}$ with $\mathbb{P}(t)$ and we consider two cases;

1. *In the case $I \ge m\log(ed)$, we have that*

$$\mathbb{P}(t) \le \begin{cases} \frac{3}{t^A} & \text{if } 1 \le t \le e^{\frac{m\log(ed)}{A}} \\ \frac{3}{t^A}\left(\frac{\log tA}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{m\log(ed)}{A}} \le t \le e^{\frac{I}{A}} \\ \frac{3}{t^A}\left(\frac{\log tA}{I}\right)^{\frac{m\Delta_{\mathbf{S}}^2}{2}}\left(\frac{I}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{I}{A}} \le t. \end{cases}$$

2. *In the case $I \le m\log(ed)$, we have that*

$$\mathbb{P}(t) \le \begin{cases} \frac{3}{t^A} & \text{if } 1 \le t \le e^{\frac{I}{A}} \\ \frac{3}{t^A}\left(\frac{\log tA}{I}\right)^{\frac{m\Delta_{\mathbf{S}}^2}{2}} & \text{if } e^{\frac{I}{A}} \le t. \end{cases}$$

Proof. We give the proof for the all $d_i = d$ case. We first recite the L estimate

$$\mathbb{P}(L \le \alpha) \le \mathbb{P}(\|\mathbf{p}\|_\infty \ge \gamma) + C\sqrt{n}(\gamma d)^{n-2}(\alpha)^{m\Delta_{\mathbf{S}}^2-n+2}(cc_0H(\mathbf{S}))^{m\Delta_{\mathbf{S}}^2}$$

$$\mathbb{P}(\kappa(\mathbf{p}) \ge tM) \le \mathbb{P}\left(\|\mathbf{p}\|_W \ge ucK\sqrt{I}\right) + \mathbb{P}\left(L(\mathbf{p}) \le \frac{c\sqrt{I}uK}{tM}\right).$$

$$\mathbb{P}\left(\kappa(\mathbf{p}) \geq tM\right) \leq e^{-t^2 I} + P\left(\|\mathbf{p}\|_\infty \geq \gamma\right)$$
$$+ C\sqrt{n}\,(\gamma d)^{n-2}\left(\frac{c\sqrt{I}uK}{tM}\right)^{m\Delta_{\mathbf{S}}^2-n+2}(cc_0H(\mathbf{S}))^{m\Delta_{\mathbf{S}}^2}$$

We choose $\gamma = 3sK\sqrt{m}\log(ed)$ and set

$$\left(\frac{M}{\sqrt{I}\max\{1,\frac{\sqrt{n}}{d}\}}\right)^{m\Delta_{\mathbf{S}}^2-n+2} = C\sqrt{n}(d\sqrt{m}\log(ed))^{n-2}\,(cc_0KH(\mathbf{S}))^{m\Delta_{\mathbf{S}}^2}$$

then we have

$$\mathbb{P}\left(\kappa(\mathbf{p}) \geq tM\right) \leq e^{-t^2 I} + e^{-c_2 s^2 m \log(ed)} + \frac{s^{n-2}u^{m\Delta_{\mathbf{S}}^2-n+2}}{t^{m\Delta_{\mathbf{S}}^2-n+2}}$$

The rest of the proof follows as in section 3. ■

As in the previous section we can prove the following

Theorem 4.4.10. *Let* $\mathbf{p}$ *be a random polynomial system as in Theorem 4.4.9. If* $q \leq (m\Delta_{\mathbf{S}}^2 - n + 2)\left(1 - \frac{1}{2\log(ed)}\right)$, *then*

$$\left(\mathbb{E}(\kappa(p)^q)\right)^{\frac{1}{q}} \leq M\left(\frac{3m\log(ed)}{n}\right)^{\frac{1}{q}}.$$

Moreover

$$\mathbb{E}\log\kappa(p) \leq \log M + 1.$$

4.4.4 Consequences of the Main Theorem

As a consequence of Lemma 4.4.1 we have the following

Corollary 4.4.11. *Let* $\mathbf{p}$ *be a* **SPS** *with support* $\mathbf{S} \subseteq S_{n,d_1,\cdots,d_m}$. *Then*

$$\text{If } \kappa(\mathbf{p}) < \infty \text{ then } H(\mathbf{S}) < \infty.$$

Proof. By Lemma 4.4.1 if $H(\mathbf{S}) = \infty$ there exists $y \in S^{n-1}$ such that y is a root of multiplicity 2. But this implies that $\kappa(\mathbf{p}) = \infty$. ∎

Due to standard facts in the theory of A-discriminants such as the Horn-Kapranov uniformization, it is known that for any support S with $H(S) < \infty$, the set of polynomials with support S and a root of multiplicity is nonempty - indeed, in most cases this set is a codimension one variety-. So the reverse of the above corollary is not true. The canonical reference on A-discriminants is [51]. For an accessible review of A-discriminants and the Horn-Kapranov uniformization we refer the reader to [41]. Surpassingly the reverse holds true with probability 1 in the case of a random polynomial system. Theorem 4.4.9 implies the following

Corollary 4.4.12. *Let* $\mathbf{p}$ *be a random polynomial system as in Theorem 4.4.9. Then*

$$\text{If } H(\mathbf{S}) < \infty \text{ then } \kappa(\mathbf{p}) < \infty \quad \text{with probability 1.}$$

Recall that in the special case where the support of the polynomial system $\mathbf{S} := (S, \cdots, S)$, we have that $\Delta_S = 1$. In this special case we have showed the following Theorem, which can been viewed as the generalization of Theorem 4.3.9 in the “sparse" case. The proof follows by Theorem 4.4.9 and Proposition 4.4.3.

Theorem 4.4.13. *There exists* $C, c, \tilde{c} > 0$ *such that for every* $n \geq 3, d \geq 2, m \geq n-1$ *and* $\mathbf{p} := (p_1, \cdots, p_m)$ *be a random polynomial system in* n*-variables with degrees* d_j*, which satisfies the subgaussian and Small Ball assumption with constants* K, c_0 *respectively and has proper eligible support* $\mathbf{S} := (S, \cdots, S)$*, the following holds:*

In the case $d_j = d, 1 \leq j \leq m$ we set

$$M := \sqrt{I}(\frac{n}{c_0})^{\frac{1}{2(m-n+2)}} \left(cc_0 K\sqrt{m}H(S)d\log(ed)\right)^{\frac{n-2}{m-n+2}} cc_0 K H(S)\max\{1, \frac{\sqrt{n}}{d}\}$$

In the case $\max_{1\leq j\leq m} d_j = d$ we set

$$M := \sqrt{I}(\frac{n}{c_0})^{\frac{1}{2(m-n+2)}} \left(cc_0 K\sqrt{m}H(S)d^2\log(ed)\right)^{\frac{n-2}{m-n+2}} cc_0 K H(S)\max\{1, \frac{\sqrt{n}}{d^2}\}$$

We denote $\mathbb{P}\{\kappa(p) \geq tM\}$ by $\mathbb{P}(t)$ and we consider two cases:

1. *In the case $I \geq m\log(ed)$, we have that*

$$\mathbb{P}(t) \leq \begin{cases} \frac{3}{t^{m-n+2}} & \text{if } 1 \leq t \leq e^{\frac{m\log(ed)}{m-n+2}} \\ \frac{3}{t^{m-n+2}}\left(\frac{(\log t)(m-n+2)}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{m\log(ed)}{m-n+2}} \leq t \leq e^{\frac{I}{m-n+2}} \\ \frac{3}{t^{m-n+2}}\left(\frac{(\log t)(m-n+2)}{I}\right)^{\frac{m}{2}}\left(\frac{I}{m\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{\frac{I}{m-n+2}} \leq t. \end{cases}$$

2. *In the case $I \leq m\log(ed)$, we have that*

$$\mathbb{P}(t) \leq \begin{cases} \frac{3}{t^{m-n+2}} & \text{if } 1 \leq t \leq e^{\frac{I}{m-n+2}} \\ \frac{3}{t^{m-n+2}}\left(\frac{(\log t)(m-n+2)}{I}\right)^{\frac{m}{2}} & \text{if } e^{\frac{I}{m-n+2}} \leq t. \end{cases}$$

Moreover,

$$\mathbb{E}\log(\kappa(\mathbf{p})) \leq \log M + 1.$$

Corollary 4.4.14. *There exists $C, c, \tilde{c} > 0$ such that for every $n \geq 3, d \geq 2$ and $\mathbf{p} := (p_1, \cdots, p_{n-1})$ be a random polynomial system in n-variables with degrees d_j, which satisfies the subgaussian and Small Ball assumption with constants K, c_0 respectively and has proper eligible support $\mathbf{S} := (S, \cdots, S)$, the following holds*

In the case $d_j = d, 1 \leq j \leq m$ *we set*

$$M := \sqrt{I} c K H(S)(nc_0)^{\frac{1}{2}} \left(cc_0 K \sqrt{m} H(S) d \log(ed) \right)^{n-2}$$

In the case $\max_{1 \leq j \leq m} d_j = d$ *we set*

$$M := \sqrt{I} c K H(S)(nc_0)^{\frac{1}{2}} \left(cc_0 K \sqrt{m} H(S) d^2 \log(ed) \right)^{n-2}$$

We consider two cases:

1. *In the case* $I \geq (n-1)\log(ed)$*, we have that*

$$\mathbb{P}(t)\{\kappa(p) \geq tM\} \leq \begin{cases} \frac{3}{t} & \text{if } 1 \leq t \leq e^{(n-1)\log(ed)} \\ \frac{3}{t}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^{(n-1)\log(ed)} \leq t \leq e^I \\ \frac{3}{t}\left(\frac{\log t}{I}\right)^{\frac{1}{2}}\left(\frac{\log t}{(n-1)\log(ed)}\right)^{\frac{n-2}{2}} & \text{if } e^I \leq t. \end{cases}$$

2. *In the case* $I \leq (n-1)\log(ed)$*, we have that*

$$\mathbb{P}(t)\{\kappa(p) \geq tM\} \leq \begin{cases} \frac{3}{t} & \text{if } 1 \leq t \leq e^I \\ \frac{3}{t}\left(\frac{\log t}{I}\right)^{\frac{(n-1)}{2}} & \text{if } e^I \leq t. \end{cases}$$

Moreover,

$$\mathbb{E}\log(\kappa(\mathbf{p})) \leq \log M + 1.$$

Remark 4.4.1. *One should compare the bounds that we get in the Theorem 4.4.9, at least in the case where* $\Delta_S = 1$*, with the bounds in Theorem 4.3.9. Let* M_1 *be the* *constant in Theorem 4.3.9 and M_2 be the constant in Theorem 4.4.9. We have that*

$$\frac{M_1}{M_2} = \sqrt{\frac{N}{I}} \left(\frac{1}{\sqrt{m} H(\mathbf{S})} \right)^{\frac{m}{m-n+2}}.$$

It is unclear when the above ration is bigger (or not) to 1. Of course N is possibly much larger than I (possibly by an exponential factor with respect to n or d) but as we have seen $\sqrt{m}H(\mathbf{S})$ can be as large as $n^{d/2}$. So there is a trade-off: The support can be very small but the geometry of the support ($H(\mathbf{S})$) may effect the estimates by a large factor. Nevertheless we have the following proposition.

Proposition 4.4.15. *Let $\mathbf{p}_1 := (p_1, \cdots, p_m)$ be a random polynomial system with eligible and proper support $\mathbf{S} = (S, \cdots, S) \subseteq S_{n,d,\cdots,d}$ and let p_2 be a random polynomial system with support $S_{n,d,\cdots,d}$. Let $M_1 := M(\kappa(\mathbf{p}_1))$ and $M_2 := M(\kappa(\mathbf{p}_2))$. If $m \geq 2n-2$, $d \log ed \leq n$ and I has cardinality at most polynomial with respect to n ($I \leq n^\alpha$, for some $\alpha > 1$), then for n large enough,*

$$M_1 < M_2.$$

In other words if m is at least proportional to n and random polynomial system with at most polynomial with respect to n terms has smaller condition number than a generic random polynomial system (with the same homogeneity degree).

Proof. Note that $m \leq I \leq n^\alpha$. By Lemma 4.3.13 we have that

$$M_2 \geq \frac{c\sqrt{N}}{Kc_0 m \log(ed)} \geq \frac{c\sqrt{n+d}}{\sqrt{nd}\log(ed)} \left(\frac{n+d}{n}\right)^{\frac{n}{2}} \left(\frac{n+d}{d}\right)^{\frac{d}{2}}.$$

By Theorem 4.4.13 we have that

$$M_1 \leq C c_0 K \sqrt{mn} \frac{\sqrt{I}}{\sqrt{n}} n^{\frac{d}{2}}.$$

So,

$$\frac{M_1}{M_2} \leq C'(c_0 K)^2 \frac{\sqrt{dn}\sqrt{m}\log{(ed)}\sqrt{I}}{\sqrt{n+d}} \left(\frac{n}{n+d}\right)^{\frac{n}{2}} \left(\frac{nd}{n+d}\right)^{\frac{d}{2}} \leq$$

$$C'(c_0 K)^2 \log{(ed)} n^{\alpha+\frac{1}{2}} e^{-\frac{n}{2}} d^{\frac{d}{2}} < 1$$

if we assume that n is large enough (larger than a fixed universal constant and larger than $\log c_0 K$). ■

Remark 4.4.2. *One can prove lower bounds for the condition number of random polynomial system with a given support as in Proposition 4.3.13. We omit the details.*

5. SUMMARY

This dissertation consists of three independent sections. Results presented in the section two show the existence of an algorithmically efficient polyhedral approximation for the real part of the zero set of an exponential sum. We derive our approximation using archimedean tropical geometry. Our results capture the quantitative aspects of a certain tropicalization. Our particular tropicalization is computationaly efficient and can be used to aid iterative numerical methods.

In section three, we study nonnegative multihomogenous polynomials and sums of squares. Our results show that for fixed degrees of multihomogeneity, ratio of sums of squares polynomials inside the set of nonnegative polynomials approaches to 0 as number of variables grow. Section three also includes some basic observations on the zonal harmonics. These basic observations indicates a new connection between high dimensional measures and multivariate polynomials.

In section four, we provide probabilistic condition number estimates for a broad family of random polynomial systems. Our techniques are quite different from the existing literature which allows us to work with more general distributions. Despite the difference of our techniques, we proved similar results to the known ones for polynomial systems with Gaussian independent random polynomials. It is not known yet, if our results and the existing results in literature are optimal. Flexibility of our techniques also allow us to work with sparse polynomials; we proved condition number estimates for sparse polynomial systems that involves quantities depending on the geometry of the support. Our results on random sparse polynomial systems indicate the need for a fundamentally different notion of conditioning for sparse polynomial systems.

In this dissertation we always used a blend of algebraic geometry with convex geometric analysis. We believe there are many connections -yet to be discovered- between convex geometric analysis and real algebraic geometry. In particular, two clear questions are the implications of concentration of measure phenomenon and the analogs of the isoperimetric problem in the space real polynomials.

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