MODULAR FORMS AND *L*-FUNCTIONS

A Thesis

by

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ABSTRACT

In this thesis, our main aims are expressing some strong relations between modular forms, Hecke operators, and L-functions. We start with background information for modular forms and give some information about the linear space of modular forms. Next, we introduce Hecke operators and their properties. Also we find a basis of Hecke eigenforms. Then, we explain how to construct L-functions from modular forms. Finally, we give a nice functional equation for completed L-function.

Dedicated to my nieces Ahsen Yıldırım, Hüma Yıldırım, Meryem Özkale and my nephew Ömer Özkale.

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TABLE OF CONTENTS

1 INTRODUCTION

A modular form of weight k is a holomorphic function on the upper half plane $\mathbb{H}$ and at $i\infty$ and satisfies the relation

$$f(\gamma(z)) = (cz+d)^k f(z) \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

Modular forms have been one of the main interest area in number theory for almost 200 years. It was first considered by Jacobi in *Fundamenta Nova Theorie Functionum Ellipticarum* which was published in 1829. Then, Riemann, Dedekind, Eisenstein, and Kronecker made great contributions to the area; but, mathematicians have given more consideration to the area after Ramanujan's works, especially his paper about τ-function. Especially, after Ramanujan's death, Hecke extended the area with new theorems which supply new solution methods for old conjectures such as Ramanujan conjectures about τ-function.

The thesis is organized as follows. In Chapter 2, we define modular forms and cusp forms. We provide some examples of the modular forms such as Eisenstein series and Poincare series. Lastly, we prove two main theorems called the valence formula and the dimension formula.

In Chapter 3, we define Hecke operators and show some properties of them such as commutativity. We prove Ramanujan conjectures about τ-function by using Hecke theory. Then, we focus on the Hecke operators for Hecke congruence subgroup. At the end of the chapter, we state the theorems that give us information about eigenfunctions and eigenvalues of the Hecke operators.

In Chapter 4, we introduce ζ-function and *L*-functions and show how to produce an *L*-function from a cusp form. At the end, we present a nice functional equation for the completed *L*-function.

2 MODULAR FORMS

2.1 Modular Group

We start with the definition of the most famous group for number theorists who study the area.

Definition 2.1.1. The set of 2×2 matrices of determinant 1 with integer entries

$$SL_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a,b,c,d \in \mathbb{Z} \ ,ad-bc=1 \right\}$$

forms a group under matrix multiplication. It is called the *(full) modular group.*

Theorem 2.1.2. *[1],[2] $SL_2(\mathbb{Z})$ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.*

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. If $c = 0$, then $a = d = \pm 1$ which implies $\gamma = \pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \pm T^b$. If $\mid c \mid > 0$, without loss of generality, we assume $\mid a \mid > \mid c \mid$ (Otherwise, applying S arranges the desired form since $S\gamma = S\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -c & -d \\ a & b \end{smallmatrix}\right)$). Then, there are $q, r \in \mathbb{Z}$ such that $a = cq + r$ with $0 \leq r < |c|$ by the Division Algorithm. Now $T^{-q}\gamma = \left(\begin{smallmatrix} 1 & -q \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a-cq & b-dq \\ c & d \end{smallmatrix}\right)$ has upper left entry that is smaller then absolute value of lower left entry. Applying S again gives us the desired form again. Finally, we reach the lower left entry equals to 0, which is the $c = 0$ case, by iterating the process finitely many times. $\square$

Remark 2.1.3. $SL_2(\mathbb{Z})$ acts on the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} \mid Im(z) > 0\}$ by the linear fractional transformations

$$\gamma(z) = \frac{az+b}{cz+d} \quad \text{for every} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

Now, we define the principal congruence subgroup of level q.

Definition 2.1.4. [3] Let q be a natural number. The group

$$\Gamma(q) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \;\middle|\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod q \right\}$$

is called *the principal congruence subgroup of level q*. In particular, $\Gamma(1) = SL_2(\mathbb{Z})$.

Remark 2.1.5. $\Gamma(q)$ is a normal subgroup of $SL_2(\mathbb{Z})$. To see this we check $\Gamma(q)$ is the kernel of the homomorphism $\alpha : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/q\mathbb{Z})$ given by $\alpha(g) \equiv g \pmod q$ for every $g \in SL_2(\mathbb{Z})$.

Next, we continue by giving definitions of two special subgroups of $SL_2(\mathbb{Z})$ which contain $\Gamma(q)$.

Definition 2.1.6. [3] The *Hecke congruence group*, denoted by $\Gamma_0(q)$, is defined by the following set

$$\Gamma_0(q) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod q \right\}.$$

Moreover, the map $\beta : \Gamma_0(q) \to (\mathbb{Z}/q\mathbb{Z})^{\times}$ given by $\beta\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv d \pmod q$ is a homomorphism and the set

$$\Gamma_1(q) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod q, d \equiv 1 \pmod q \right\}$$

is the kernel of β.

Remark 2.1.7. Obviously, $T \in \Gamma_0(q)$ for every q; but, $STS^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) \notin \Gamma_0(q)$ for $q > 1$. Hence, $\Gamma_0(q)$ is not a normal subgroup of $SL_2(\mathbb{Z})$. Moreover, the relation

$$\Gamma(q) \subset \Gamma_1(q) \subset \Gamma_0(q) \subset SL_2(\mathbb{Z})$$

can easily be seen.

For simplicity, we use Γ instead of the modular group $SL_2(\mathbb{Z})$.

Now, we introduce fundamental domain for Γ.

Definition 2.1.8. Let $\mathscr{F}$ be a closed subset of the upper half plane $\mathbb{H}$ with connected interior. Then, $\mathscr{F}$ is a *fundamental domain for* Γ if the following conditions are satisfied:

(i) For any $z \in \mathbb{H}$, there exists $w \in \mathscr{F}$ such that z and w are Γ-equivalent, i.e. there exists $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $\gamma(z) = \dfrac{az+b}{cz+d} = w$.

(ii) For any interior points $z, w \in \mathscr{F}$, z and w are not Γ-equivalent.

(iii) The boundary of $\mathscr{F}$ is a finite union of smooth curves.

We need to give a lemma to prove the next theorem.

Lemma 2.1.9. *[3] Let $z = x + iy \in \mathbb{H}$. Then, the set of $(c,d) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ such that $|cz+d| \leq 1$ is finite and nonempty.*

Proof. If $|cz+d| \leq 1$, then $(cx+d)^2 + c^2y^2 \leq 1$ which implies $c^2y^2 \leq 1$ so that $|c| \leq \frac{1}{y}$. That shows there are finitely many possibility for c. Moreover, $(cx+d)^2 \leq 1$ implies $|cx+d| \leq 1$ which means $-1-cx \leq d \leq 1-cx$. Hence there are also finitely many possibility for d. On the other hand, $(c,d) = (0,1)$ satisfies $|cz+1| \leq 1$ that shows the set is nonempty. $\square$

Now, we are ready to introduce a special example of a fundamental domain for Γ.

Theorem 2.1.10. *[2], [4] The set*

$$\mathscr{F} = \{z \in \mathbb{H} \mid |z| \geq 1, |Re(z)| \leq \frac{1}{2}\}$$

is a fundamental domain for Γ.

Proof. Firstly, we observe that for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\operatorname{Im}(\gamma(z)) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2}$. Now, fix $z = x+iy \in \mathbb{H}$. We can push $T^n(z) = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)(z) = z+n$ to be an element of the set $\{z \in \mathbb{H} \mid |\operatorname{Re}(z)| \leq \frac{1}{2}\}$ with an appropriate integer n (Particularly, n equals to either $-\lfloor x \rfloor$, or $-\lfloor x \rfloor - 1$). By Lemma 2.1.9, we may select $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $|cz+d|$ is minimal which implies $\operatorname{Im}(\gamma(z)) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2}$ is maximal. Our claim is $T^n(\gamma(z)) \in \mathscr{F}$. Assume $T^n(\gamma(z)) \notin \mathscr{F}$. Then, $|T^n(\gamma(z)) < 1$. However, the first observation and $|T^n(\gamma(z))| < 1$ imply $\operatorname{Im}(S(T^n(\gamma(z))) = \dfrac{\operatorname{Im}(T^n(\gamma(z))}{|T^n(\gamma(z)|^2} > \operatorname{Im}(T^n(\gamma(z)) = \operatorname{Im}(\gamma(z))$ which contradicts the minimality of $\operatorname{Im}(\gamma(z))$. Hence $T^n(\gamma(z)) \in \mathscr{F}$ and (i) is proven.

On the other hand, suppose $z, w \in \mathscr{F}$ are Γ-equivalent interior points. Then there exists $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $\gamma(z) = w$. Without loss of generality, assume $\operatorname{Im}(z) \leq \operatorname{Im}(w)$. Then, $\operatorname{Im}(z) \leq \operatorname{Im}(\gamma(z)) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2}$ that implies $|cz+d| < 1$ so that $|\operatorname{Im}(cz+d)| = |c|\operatorname{Im}(z) \leq 1$. Therefore, $|c| \leq \dfrac{1}{\operatorname{Im}(z)} < \dfrac{2}{\sqrt{3}}$ and so $c = 0$ or $c = \pm 1$. If $c = 0$, then $ad = 1$ which means $a = d = \pm 1$. Thus, γ is either $-T^n$ or T^n where n is nonzero integer. However, it is not possible unless $z \in \partial\mathscr{F}$; because, $S^2T^n(z) = z+n = T^n(z)$. If $c = 1$, then $1 \geq |cz+d|^2 = (x+d)^2 + y^2 > (x+d)^2 + \dfrac{3}{4}$ that implies $\dfrac{1}{2} > |x+d|$. As $|x| < \dfrac{1}{2}$ and $x+iy$ is fixed, we deduce $d = 0$. Thus, $1 \geq |cz+d| = |z|$ and this derives contradiction because z is taken as an interior point. If $c = -1$, then, by applying S^2 to γ in order to get $c = 1$ case and mimicing the previous case, we prove (ii).

Lastly, it is obvious that the boundary of $\mathscr{F}$ is a finite union of smooth curves. Consequently, $\mathscr{F}$ is a fundamental domain for Γ. $\square$

Definition 2.1.11. $\mathscr{F}$ in the previous theorem is called *the standard fundamental domain* *for* Γ.

2.2 Modular Forms

We now introduce modular forms and cusp forms.

Definition 2.2.1. [5] A *modular form of weight k* is a function which is holomorphic on $\mathbb{H}$ and at $i\infty$ and satisfies the following

$$f(\gamma(z)) = (cz+d)^k f(z) \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

The set of modular forms of weight k is denoted by M_k.

Remark 2.2.2. For odd k, a modular form f vanishes identically since for $\gamma = -I$, $f(\gamma(z)) = f(z) = (-1)^k f(z) = -f(z)$. Hence, there are no modular forms (except identically zero) of weight odd number k.

Remark 2.2.3. By taking $\gamma = T$, we obtain $f(T(z)) = f(z+1) = f(z)$ for every modular form f. Therefore, f is periodic.

Let $q = e^{2\pi iz}$. Since a modular form f is periodic with period 1 by Remark 2.2.3, there is a map from the unit disk to complex numbers, i.e.

$$q \mapsto f(z).$$

Moreover, $f(q)$ is holomorphic on the punctured disk because f is holomorphic. Therefore, $f(q)$ has a Laurent expansion, called q-expansion, around $q = 0$,

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n.$$

Definition 2.2.4. The series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i z n}$$

is called the *Fourier expansion(series) of f at $i\infty$.*

Remark 2.2.5. Note that a Fourier expansion of a modular form starts from $n = 0$ since a modular form is holomorphic at $i\infty$.

Next, we define a special kind of modular forms.

Definition 2.2.6. [5],[6] A *cusp form of weight k* is a modular form of weight k whose Fourier expansion has leading coefficient $a_0 = 0$, i.e. $f(z) = \sum_{n=1}^{\infty} a_n q^n$, $q = e^{2\pi i z}$. The set of cusp forms of weight k is denoted by S_k.

The simplest examples of modular forms are the zero function (of every weight) and constant functions (of weight 0). The first nontrivial example of modular forms of weight $k > 2$ is

$$G_k(z) = \sum_{\substack{(m,n)\in \\ \mathbb{Z}\times\mathbb{Z}\setminus(0,0)}} (mz+n)^{-k} \quad , \quad z \in \mathbb{H}.$$

We give a theorem known as the Lipschitz formula in order to introduce the Eisenstein series.

Theorem 2.2.7 (Lipschitz formula). *[3] For $k \geq 2$ and for $q = e^{2\pi i z}$ where $z \in \mathbb{H}$, we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Proof. By taking logarithmic derivative of the function

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n}\right).$$

When we differentiate both sides $k-1$ times, we complete the proof. $\square$

By using the Lipschitz formula, we deduce the following identity:

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad \text{where} \quad q = e^{2\pi i z}, \sigma_s(n) = \sum_{d|n} d^s. \tag{2.1}$$

Definition 2.2.8. [5] From the identity (2.1), we define the *Eisenstein series* as follows

$$E_k(z) := \frac{G_k(z)}{2\zeta(k)} = \frac{1}{2} \sum_{(m,n)=1} (mz+n)^{-k}.$$

From the identities

$$2\zeta(2k) = -\frac{(2\pi i)^{2k} B_{2k}}{(2k)!}, \tag{2.2}$$

where $k > 0$ and B_k is the k^{th} *Bernoulli number* defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!},$$

and the identity (2.1), we deduce that

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad for \quad k \geq 4 \tag{2.3}$$

is a modular form of weight k that is not a cusp form.

Another famous example of modular forms is the *delta function* defined by

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728}.$$

Moreover, it can easily be checked that the Δ-function is also a cusp form of weight 12.

Ramanujan introduced the following τ-function as being coefficients in the Fourier expansion of the infinite product

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

Jacobi proved that the infinite product equals the Δ-function. Furthermore, the τ-function has properties that are conjectured by Ramanujan [7] , as follows

(i) $\quad \tau(mn) = \tau(m)\tau(n) \quad$ for $\quad (m,n) = 1;$

(ii) $\quad$ if $\quad p \quad$ is prime, then $\quad \tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1}) \quad$ for $\quad \alpha \geq 1;$

(iii) $\quad$ if $\quad p \quad$ is prime, then $\quad |\tau(p)| \leq 2p^{11/2}.$

In the chapter 2, we prove the first and second conjecture. The third conjecture is more complicated. It was proven by Pierre Deligne and he was awarded a field medal in 1978.

We return to the modular forms. Modular forms of weight k and cusp forms of weight k build linear spaces of finite dimensions. By using the identity (2.3), we obtain the following result:

Corollary 2.2.9. *For $k \geq 4$,*

$$dim\, M_k = 1 + dim\, S_k.$$

Proof. For $f \in M_k$, $f(i\infty) = a_0$ which is the constant term in Fourier expansion. Then $f - a_0E_k \in S_k$ because it vanishes at $i\infty$. Hence,

$$M_k = \mathbb{C}E_k + S_k \tag{2.4}$$

as $f = a_0E_k + (f - a_0E_k)$. The conclusion is immediate. □

2.3 The Valence and Dimension Formulas

We now define the order of a meromorphic function and state two famous theorems known as the valence formula and the dimension formula.

Definition 2.3.1. Let f be a not identically zero meromorphic function from $\mathbb{H}$ to $\mathbb{C}$. For every $z_0 \in \mathbb{H}$, there is a unique integer n such that $\frac{f(z)}{(z-z_0)^n}$ is holomorphic and nonzero at z_0. Then, n is called *the order of f at z_0* and denoted by $v_{z_0}(f)$. In particular, $v_{i\infty}(f)$ is the smallest value of n such that a_n is the first nonzero Fourier coefficient.

Theorem 2.3.2 (The valence formula). *[2], [8] Let f be a not identically zero modular form of weight k. Then*

$$v_{i\infty}(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{\substack{z \neq i,\rho \\ z \in \mathscr{F}'}} v_z(f) = \frac{k}{12}$$

where $\rho = e^{2\pi i/3}$ and $\mathscr{F}'$ is the standard fundamental domain except its boundary in first quadrant.

Proof. [2], [3], [9], [10] The main tool for the proof is the residue theorem. A complete proof can be found in ([2] , part 1 chapter 4). □

By the valence formula, we deduce the following corollary:

Corollary 2.3.3. *[6]* $(i)\quad M_0 = \mathbb{C}$,

$(ii)\quad M_2 = \{0\}$,

$(iii)\quad M_4 = \mathbb{C}E_4(z)$,

$(iv)\quad M_6 = \mathbb{C}E_6(z)$,

$(v)\quad M_8 = \mathbb{C}E_8(z) = \mathbb{C}E_4^2(z)$,

$(vi)\quad M_{10} = \mathbb{C}E_{10}(z) = \mathbb{C}E_4(z)E_6(z)$.

Proof. When we plug in $k = 0$ in the valence formula, the right hand size is 0. Then, obviously, $f(z) - f(i) \in M_0$ for any $f \in M_0$ which implies f is constant so that $M_0 = \mathbb{C}$. For $k = 2$, the right hand side of the valence formula is $\frac{1}{6}$; hence, f is identically zero and so $M_2 = \{0\}$. If $k \leq 10$, than the right hand side of the valence formula is less then 1. This implies there is no cusp form f of weight k since, otherwise, $v_{i\infty}(f)$ would be 1 which derives contradiction. Therefore, by identity (2.4), $M_k = \mathbb{C}E_k$ for $k = 4, 6, 8, 10$. On the other hand, particularly, M_8 and M_{10} are one dimensional linear spaces. Furthermore, E_8 and E_{10} are the modular forms of weight 8 and 10, respectively, as well as E_4^2 and E_4E_6. This implies that E_8 and E_{10} are constant times E_4^2 and E_4E_6, respectively. Since $E_8(i\infty) = E_{10}(i\infty) = E_4^2(i\infty) = E_4(i\infty)E_6(i\infty) = 1$, the constant should be 1. Therefore, $E_8 = E_4^2$ and $E_{10} = E_4E_6$ that completes the proof. $\square$

Theorem 2.3.4 (The dimension formula). *[3],[6] For* $k \geq 0$,

$$\dim M_k = \left\{ \begin{array}{cc} [k/12], & \text{if } \; k \equiv 2 \pmod{12} \\ 1 + [k/12], & \text{if } \; k \not\equiv 2 \pmod{12} \end{array} \right\}.$$

Proof. Corollary 2.3.3 implies $M_0, M_4, M_6, M_8, \text{and}\, M_{10}$ are linear spaces of dimension 1 and M_2 is a linear space of dimension 0. On the other hand, for any $f \in M_k$ where $k \geq 12$, $f\Delta \in S_{k+12}$ since the Δ-function is a cusp form of weight 12. Conversely, if $g \in S_{k+12}$, then $\frac{g}{\Delta}$ is analytic because Δ does not vanish on $\mathbb{H}$; hence, $\frac{g}{\Delta} \in M_k$. Consequently, the operator defined by $f \mapsto \Delta f$ is an isomorphism from M_k to S_{k+12}. Lastly, by using Corollary 2.2.9, we deduce $dim M_k = dim S_{k+12} = dim M_{k+12} - 1$. The conclusion follows by induction. $\square$

3 HECKE THEORY

The τ-function has exciting multiplicativity

$$\tau(m)\tau(n) = \sum_{d|(m,n)} d^{11}\tau(mnd^{-2})$$

which was first established by Mordell in 1917; but, E. Hecke systemized the idea in 1936. That is why, although Mordell introduced the operator which is used to prove the multiplicativity, we call it as Hecke operator. Moreover, the theory of Hecke operators gives an explanation to many other identities.

3.1 Hecke Operators

We first mention the slash operators to go further in Hecke theory.

Definition 3.1.1. [6]

The *slash operator* which is applied on functions $f : \mathbb{H} \to \mathbb{C}$ as

$$f|_A(z) = (\det A)^{k/2} j_A^k(z) f(Az) \;\; for \;\; A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R}),$$

where $j_A(z) = (cz+d)$ and $GL_2^+(\mathbb{R})$ is the set of elements in $GL_2(\mathbb{R})$ with positive determinant.

We need the following lemma in order to prove the next theorem.

Lemma 3.1.2. *[3] For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), B = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$,*

$$j_{AB}(z) = j_A(Bz) j_B(z).$$

Proof. By the definition of the j function, we get

$$\begin{aligned} j_{AB}(z) &= (ce+dg)z+(cf+dh) \\ &= \left(c\left(\frac{ez+f}{gz+h}\right)+d\right)(gz+h) \\ &= j_A(Bz)j_B(z). \end{aligned}$$

□

Theorem 3.1.3. *[6] For $A,B \in GL_2^+(\mathbb{R})$, the slash operator is associative, i.e.*

$$f|_{AB} = (f|_A)|_B.$$

Proof. By the definition of slash operator, we have

$$\begin{aligned} (f|_A)|_B)(z) &= (\det B)^{k/2} j_B^{-k}(z)(f|_A)(Bz) \\ &= (\det B)^{k/2} j_B^{-k}(z)\big((\det A)^{k/2} j_A^{-k}(Bz)f(A(B(z))\big) \\ &= (\det AB)^{k/2}\big(j_A(Bz)j_B(z)\big)^{-k} f(ABz). \end{aligned}$$

By Lemma 3.1.2, the equation becomes

$$(\det AB)^{k/2}(j_{AB}(z))^{-k} f(ABz)$$

which is equal to $f|_{AB}(z)$. □

Definition 3.1.4. For a positive integer n, the n^{th} *Hecke operator* T_n on function $f \in M_k$ is defined by

$$T_n f := n^{k/2-1} \sum_{ad=n} \sum_{0 \le b < d} f\big|\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = n^{k/2-1} \sum_{\rho \in \Gamma \backslash G_n} f|_\rho$$

where $G_n := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \quad a,b,c,d \in \mathbb{Z}, \quad ad-bc = n \right\}$.

Lemma 3.1.5. *[6] The set*

$$\Delta_n = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : \quad ad = 0, \quad 0 \leq b < d \right\}$$

is the set of right coset representatives of G_n modulo Γ, i.e.

$$G_n = \bigsqcup_{\rho \in \Delta_n} \Gamma \rho.$$

where $\bigsqcup$ denotes disjoint union.

Proof. Let $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in G_n$. We need to find matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) \in \Delta_n$ such that

$$\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} ar & as+bt \\ cr & cs+dt \end{smallmatrix}\right).$$

It is clear that we need to take $r = (x,z), a = x/r, c = z/a$. Moreover,

$$\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$$

as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. This implies

$$s = dy - bw,$$

$$t = aw - cy.$$

Taking the determinant of both sides implies $rt = n$ and for suitable k, the set

$$\left(\begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right)\left(\begin{smallmatrix} r & s-kt \\ 0 & t \end{smallmatrix}\right)$$

is also a set of representatives with $0 \le s < t$.

On the other hand, assume $\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right), \left(\begin{smallmatrix} r' & s' \\ 0 & t' \end{smallmatrix}\right)$ are two elements of Δ_n that are Γ-equivalent. Then, there is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} ar & as+bt \\ cr & cs+dt \end{smallmatrix}\right) = \left(\begin{smallmatrix} r' & s' \\ 0 & t' \end{smallmatrix}\right).$$

Hence, $c = 0$ which implies $ad = 1$ and so $a = d = \pm 1$. Since $a = r/r'$ and $r, r' > 0$, then $a = d = 1$ which implies $r = r'$ and so $t = t'$. Lastly, in the equation $as + bt = s + bt = s'$, b must be zero since $0 \le s, s' < t$. Therefore, $\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} r' & s' \\ 0 & t' \end{smallmatrix}\right)$. □

By Lemma 3.5, we write T_n as

$$T_n f = n^{k/2-1} \sum_{\rho \in \Delta_n} f|_\rho.$$

3.2 Properties of Hecke Operators

Now, we present some nice properties of Hecke operators.

Theorem 3.2.1. *[3],[6] Let $f \in M_k$ such that the Fourier expansion of f at $i\infty$ is*

$$f = \sum_{n=0}^{\infty} a_n q^n.$$

Then, the Fourier expansion of $T_m f$ at $i\infty$ is

$$T_m f = \sum_{n=0}^{\infty} \left(\sum_{d|(n,m)} d^{k-1} a_{mn/d^2} \right) q^n.$$

Proof. Firstly, by the definitions of slash operator and $T_m f$, we write

$$\begin{aligned} T_m f(z) &= m^{k/2-1} \sum_{ad=m} \sum_{0 \le b < d} \left(f \middle| \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \right)(z) \\ &= \frac{1}{m} \sum_{ad=m} \left(\frac{m}{d}\right)^k \sum_{b=0}^{d-1} \sum_{n=0}^{\infty} a_n e^{2\pi i n (az+b)/d} \qquad . \end{aligned}$$

Then, switching the two inner sums and using the identity

$$\sum_{b=0}^{d-1} e^{2\pi inb/d} = \left\{ \begin{array}{ll} d, & \text{if } \ d|n \\ \\ 0, & otherwise \end{array} \right\},$$

we have

$$T_m f(z) = \sum_{\substack{ad=m \\ d>0}} \left(\frac{m}{d}\right)^{k-1} \sum_{\substack{n=0 \\ d|n}} a_n e^{2\pi inaz/d} \qquad .$$

When we write $n = dr$ and take $ar = s$, we obtain

$$\begin{aligned} T_m f(z) &= \sum_{s=0}^{\infty} \left(\sum_{\substack{ad=m \\ ar=s}} \left(\frac{m}{d}\right)^{k-1} a_{dr} \right) q^n \\ &= \sum_{s=0}^{\infty} \left(\sum_{a|(m,s)} a^{k-1} a_{ms/a^2} \right) q^n, \end{aligned}$$

that proves the theorem. □

Theorem 3.2.2. *[3],[8] For* $(m,n) = 1$,

$$T_m T_n = T_{mn} = T_n T_m.$$

Proof. Let $f \in M_k$ and the Fourier expansion of f be

$$f = \sum_{l=0}^{\infty} a_l q^l.$$

Then, the Fourier expansion of $T_n f$ is

$$T_n f = \sum_{l=0}^{\infty} \left(\sum_{b|(l,n)} b^{k-1} a_{nl/b^2} \right) q^l$$

by Theorem 3.2.1. Hence,

$$\begin{aligned}
T_m(T_n f) &= \sum_{l=0}^{\infty}\left(\sum_{c|(m,l)} c^{k-1} \sum_{b|(n,ml/c^2)} b^{k-1} a_{mnl/b^2c^2}\right) q^l \\
&= \sum_{l=0}^{\infty}\left(\sum_{c|(m,l)} c^{k-1} \sum_{b|(n,l/c)} b^{k-1} a_{mnl/b^2c^2}\right) q^l \\
&= \sum_{l=0}^{\infty}\left(\sum_{c|(m,l)} \sum_{b|(n,l)} (cb)^{k-1} a_{mnl/b^2c^2}\right) q^l \\
&= \sum_{l=0}^{\infty}\left(\sum_{d|(mn,l)} d^{k-1} a_{mnl/d^2}\right) q^l.
\end{aligned}$$

Therefore, for $(m,n) = 1$,

$$T_m T_n = T_{mn} = T_{nm} = T_n T_m.$$

□

Theorem 3.2.3. *[3],[8] For a prime p and $s \geq 1$,*

$$T_{p^s} T_p = T_{p^{s+1}} + p^{k-1} T_{p^{s-1}}.$$

Proof. See ([2], part 1 chapter 7). □

By using Theorem 3.2.3 and induction, we deduce

Theorem 3.2.4. *[3]For $r,s \geq 1$,*

$$T_{p^r} T_{p^s} = T_{p^s} T_{p^r}.$$

We now say the Hecke operators commute.

Theorem 3.2.5. *[3], [6] For all $m,n \geq 1$,*

$$T_n T_m = T_m T_n.$$

Proof. Let $m, n \geq 1$ with the prime decompositions $m = p_1^{r_1} \ldots p_\alpha^{r_\alpha}$, and $n = q_1^{s_1} \ldots q_\beta^{s_\beta}$. Thus, we obtain

$$T_m T_n = T_{p_1^{r_1}} \ldots T_{p_\alpha^{r_\alpha}} T_{q_1^{s_1}} \ldots T_{q_\beta^{s_\beta}}$$

from Theorem 3.2.2. For the distinct primes powers, the Hecke operators commute by Theorem 3.2.2 and for the same primes powers, the Hecke operators also commute by Theorem 3.2.5. Consequently, we reach

$$T_m T_n = T_{p_1^{r_1}} \ldots T_{p_\alpha^{r_\alpha}} T_{q_1^{s_1}} \ldots T_{q_\beta^{s_\beta}} = T_{q_1^{s_1}} \ldots T_{q_\alpha^{s_\alpha}} T_{p_1^{r_1}} \ldots T_{p_\beta^{r_\beta}} = T_n T_m$$

as desired. $\square$

We now give a significant property of Hecke operators.

Theorem 3.2.6. *[3], [6] The Hecke operator T_n maps a modular form to a modular form and a cusp form to a cusp form, i.e.*

$$T_n : M_k \to M_k,$$

$$T_n : S_k \to S_k.$$

Proof. One can show that there is a one-to-one correspondence between $\Delta_n \times \Gamma$ and $\Gamma \times \Delta_n$, i.e. for any $\rho \in \Delta_n, \gamma \in \Gamma$, there are unique $\rho' \in \Delta_n, \gamma' \in \Gamma$ such that $\rho\gamma = \gamma'\rho'$. Therefore, for any $\gamma \in \Gamma$ and $f \in M_k$,

$$
\begin{aligned}
(T_n f)|_\gamma &= n^{k/2-1} \sum_{\rho \in \Delta_n} f|_{\rho\gamma} \\
&= n^{k/2-1} \sum_{\rho' \in \Delta_n} f|_{\gamma'\rho'} \\
&= n^{k/2-1} \sum_{\rho' \in \Delta_n} (f|_{\gamma'})|_{\rho'} \\
&= n^{k/2-1} \sum_{\rho' \in \Delta_n} f|_{\rho'} \\
&= T_n f.
\end{aligned}
$$

By Theorem 3.2.2, we clearly see that $T_n f$ is holomorphic at $i\infty$ for a modular form f and also vanishes at $i\infty$ for a cusp form f. Consequently, T_n maps a modular form to a modular form and a cusp form to a cusp form. □

3.3 Ramanujan τ-function

It is time to prove Ramanujan conjectures about τ-function. Recall that τ-function [7] is defined as

$$q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Theorem 3.3.1. *[3] For $(m,n) = 1$,*

$$\tau(mn) = \tau(m)\tau(n).$$

Proof. We know that Δ-function is a cusp form of weight 12 and so $T_n\Delta \in S_{12}$. On the other hand, S_{12} has dimension one so that Δ spans S_{12} which means there is a constant λ_n

such that $T_n\Delta = \lambda_n\Delta$. Moreover, by Theorem 3.2.1, the m^{th} Fourier coefficient of $T_n\Delta$ is

$$\sum_{d|(m,n)} d^{11}\tau(mn/d^2).$$

Hence, for $(m,n) = 1$, the m^{th} Fourier coefficient of $T_n\Delta$ equals to $\tau(mn)$. However, $\tau(mn) = \lambda_n\tau(m)$ as $T_n\Delta = \lambda_n\Delta$. For $m = 1$, $\tau(n) = \lambda_n$; therefore, $\tau(mn) = \tau(m)\tau(n)$. $\square$

Theorem 3.3.2. *[3] If p is prime,then, for* $\alpha \geq 1$*,*

$$\tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1}).$$

Proof. If we compare the m^{th} coefficients of $\lambda_n\Delta = \tau(n)\Delta$ and $T_n\Delta$, for any m,n, we obtain

$$\tau(n)\tau(m) = \sum_{d|(m,n)} d^{11}\tau(mn/d^2).$$

By taking $n = p$ and $m = p^{\alpha}$, we reach the conclusion that

$$\tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1}),$$

that gives desired result. $\square$

3.4 Hecke Operators for $\Gamma_0(q)$

Hecke theory gives more significant results for the Hecke congruence subgroup

$$\Gamma_0(q) = \left\{ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \quad | \quad c \equiv 0 \pmod{q} \right\}.$$

with the Dirichlet character χ modulo q.

Definition 3.4.1. [6] Let $M_k(\Gamma_1(q))$ be the linear space of modular functions for $\Gamma_1(q)$ and χ be a Dirichlet character modulo q. Then $M_k(\Gamma_0(q),\chi)$ is a linear subspace of $M_k(\Gamma_1(q))$ defined by

$$M_k(\Gamma_0(q),\chi) = \left\{ f \in M_k(\Gamma_1(q)) \quad | \quad f|_{\gamma} = \chi(d) f \quad \text{for all} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q) \right\}.$$

Theorem 3.4.2. *[3] We have the decomposition*

$$M_k(\Gamma_1(q)) = \bigoplus_{\chi} M_k(\Gamma_0(q), \chi).$$

Proof. See ([2], part 1 chapter 8) □

We write coset representatives of G_n modulo $\Gamma_0(q)$ as

$$\Delta_n^q = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_n \quad | \quad (a,q) = 1 \right\}.$$

Likewise the Hecke operators, the Hecke operators with character χ has the following property:

Theorem 3.4.3. *[3], [6] The Hecke operator T_n^{χ} maps a modular form to a modular form and a cusp form to a cusp form, i.e.*

$$T_n^{\chi} : M_k(\Gamma_0(q), \chi) \to M_k(\Gamma_0(q), \chi),$$

$$T_n^{\chi} : S_k(\Gamma_0(q), \chi) \to S_k(\Gamma_0(q), \chi).$$

Proof. In the proof of Theorem 3.2.6, we see that there is one-to-one correspondence between $\Delta_n \times \Gamma$ and $\Gamma \times \Delta_n$. We see that the correspondence is also valid for $\Delta_n^q \times \Gamma_0(q)$ and $\Gamma_0(q) \times \Delta_n^q$, i.e. for any $\rho \in \Delta_n^q, \gamma \in \Gamma_0(q)$, there are unique $\rho' \in \Delta_n^q, \gamma' \in \Gamma_0(q)$ such that $\rho\gamma = \gamma'\rho'$. Therefore, for any $\gamma \in \Gamma_0(q)$ and $f \in M_k(\Gamma_0(q), \chi)$,

$$\bar{\chi}(\rho) f|_{\rho\gamma} = \bar{\chi}(\rho) f|_{\gamma'\rho'} = \bar{\chi}(\rho)(f|_{\gamma'})|\rho' = \bar{\chi}(\rho)\chi(\gamma') f|_{\rho'} = \chi(\gamma)\bar{\chi}(\gamma') f|_{\rho'}$$

which implies $(T_n f)|\rho = \chi(\gamma) T_n f$ for $\gamma \in \Gamma_0(q)$. The rest of the proof goes as the proof of Theorem 3.2.6. □

3.5 Hecke Eigenforms

We start the section with the definition of the Petersson inner product.

Definition 3.5.1. [8] Let f,g be two cusp forms and $z = x+iy$. Then we define the *Petersson inner product* as

$$(f,g) = \int\int_{\mathscr{F}} y^k f(z)\overline{g(z)}\frac{dxdy}{y^2}.$$

Remark 3.5.2. From the definition of the Petersson inner product, we easily deduce

(i) (f,g) is bilinear ;

(ii) (f,g) is conjugate symmetric, i.e. $(f,g) = \overline{(f,g)}$;

(iii) $(f,f) > 0$ for $f \neq 0$.

Hence, the Petersson inner product is a Hermitian inner product on S_k.

Now we give a lemma that help us to show that Hecke operators are Hermitian.

Lemma 3.5.3. *[3] For a prime number p, there is a polynomial $P_s(x) \in \mathbb{Z}[x]$ such that*

$$T_{p^s} = P_s(T_p).$$

Proof. We prove the lemma by induction on s.

For $s = 0$ and $s = 1$, $P_0(x) = 1$ and $P_1(x) = x$ are the desired polynomials.

Now, assume we have a desired polynomial P_s for $t \leq s$. Then, by Theorem 3.2.3, we have

$$\begin{aligned} T_{p^{s+1}} &= T_{p^s}T_p - p^{k-1}T_{p^{s-1}} \\ &= P_s(T_p) - p^{k-1}P_{s-1}T_p \\ &=: P_{s+1} \end{aligned}$$

that is a desired polynomial for $t = s+1$. $\square$

Theorem 3.5.4. *[3] For cusp forms f,g and any $n > 0$,*

$$(T_n(f), g) = (f, T_n(g)).$$

Proof. The algebra of Hecke operators called *Hecke algebra* is generated by T_p's where p is prime by Theorem 3.2.3 and Lemma 3.5.3. Hence it is enough to see

$$(T_p(f), g) = (f, T_p(g)).$$

We write

$$T_p(f) = p^{k/2-1}\left\{ f\big|\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{b=0}^{p-1} f\big|\left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right) \right\}$$

from the definition of T_n. Since the Petersson inner product is bilinear, we obtain the identity

$$(T_p(f), g) = p^{k/2-1}\left\{ \left(f\big|\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), g \right) + \sum_{b=0}^{p-1} \left(f\big|\left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right), g \right) \right\}.$$

One can show that if $\delta = A\gamma$ where $A = \det\gamma$ and $\gamma \in GL_2^+(\mathbb{Q})$, then $(f|_\gamma, g) = (f, g|_\delta)$ for cusp forms f, g. Therefore,

$$\left(f\big|\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), g \right) = \left(f, g\big|\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \right)$$

and

$$\left(f\big|\left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right), g \right) = \left(f, g\big|\left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix}\right) \right).$$

Furthermore, for $1 \le b \le p-1$, $\alpha = \left(\begin{smallmatrix} 0 & -1 \\ 1 & b \end{smallmatrix}\right)$,$\beta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -b \end{smallmatrix}\right) \in \Gamma$ one shows

$$\left(f, g\big|\left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix}\right) \right) = \left(f|\beta, g\big|\left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix}\right)\beta \right).$$

Lastly, we say

$$\left(f|\beta, g|\left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix}\right)\beta\right) = \left(f, g|\alpha\left(\begin{smallmatrix} p & -b \\ 0 & 1 \end{smallmatrix}\right)\beta\right) = \left(f, g|\left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right)\right)$$

because $f|_\beta = f$ and $g|_\alpha = g$. Collecting all of these together, we derive the conclusion that

$$(T_p(f), g) = (f, T_p(g))$$

which implies

$$(T_p(f), g) = (f, T_p(g)).$$

□

Theorem 3.5.5. *The eigenvalues of Hecke operator T_n are real numbers.*

Proof. Let λ_n be an eigenvalue of T_n; then, $T_n(f) = \lambda_n f$. Since $(T_n(f), f) = (f, T_n(f))$ by Theorem 3.5.4, we deduce $(\lambda_n f, f) = (f, \lambda_n f)$ which implies $\lambda_n(f, f) = \overline{\lambda_n}(f, f)$. Therefore, λ_n is a real number. □

Lemma 3.5.6. *Assume $f = \sum_{n=1}^{\infty} a_n q^n \in S_k$ is an eigenfunction for all T_m. Then $a_1 \neq 0$ and the eigenvalue of T_m is a_m/a_1.*

Proof. Let λ_m be an eigenvalue of T_m; then, $T_m(f) = \lambda_m f$. We know the coefficient of q in the Fourier expansion of $T_m(f)$ is a_m by Theorem 3.2.1. Therefore, $a_m = \lambda_m a_1$ which means that $\lambda_m = a_m/a_1$ and $a_m = 0$ for all m when $a_1 = 0$. However, this implies f is zero function and so f is not a eigenvector. Hence $a_1 \neq 0$. □

We now define Hecke eigenforms.

Definition 3.5.7. The cusp form $f = \sum_{n=1}^{\infty} a_n q^n \in S_k$ in Lemma 3.5.6 is called *Hecke eigenform*. If $a_1 = 1$ then it is called *normalized Hecke eigenform*.

Thus, we say that the Fourier coefficients of a normalized Hecke eigenform are eigenvalues of T_m's.

Next, we give the following theorem to go further.

Theorem 3.5.8. *Let R be a commutative ring of Hermitian operators on a finite dimensional Hilbert space V. Then V has an orthogonal basis $f_1, f_2, \ldots, f_r$ of eigenvectors of R.*

Proof. See ([2], part 1 chapter 7). □

We state a theorem which satisfies the relation between cusp forms and Hecke eigenforms.

Theorem 3.5.9. *[6] The space of cusp forms of weight k S_k has an orthogonal basis of Hecke eigenforms.*

Proof. Put the finite dimensional Hilbert space S_k instead of V and the commutative ring of Hecke eigenforms instead of R in Theorem 3.5.8. □

We end this chapter by giving some information about eigenforms of $S_k(\Gamma_0(q), \chi)$.

Theorem 3.5.10. *[3] Let $f \in M_k(\Gamma_1(q))$, and $g \in S_k(\Gamma_1(q))$. For each $a \in (\mathbb{Z}/q\mathbb{Z})^{\times}$, fix $\gamma_a \in \Gamma$ such that*

$$\gamma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \pmod{q},$$

where a^{-1} is the inverse of $a(\mod q)$. If (n,q)=1, then

$$(T_n(f), g) = (f|_{\gamma_n}, T_n(g)).$$

In particular, if $f \in M_k(\Gamma_0(q), \chi)$, then

$$(T_n(f), g) = \chi(n)(f, T_n(g)).$$

From Theorem 3.5.10, we deduce

Corollary 3.5.11. *[3] Let $(n,q) = 1$, let χ be a Dirichlet character modulo q, and let c_n be a square root of $\overline{\chi(n)}$. Then c_nT_n is a Hermitian operator on $S_k(\Gamma_0(q), \chi)$, i.e.*

$$(c_nT_n(f), g) = (f, c_nT_n(g)).$$

Proof. Since c_n is a square root of $\overline{\chi(n)}$, it is a root of unity. By using Theorem 3.5.10, we obtain

$$\begin{aligned}
(c_nT_n(f), g) &= c_n(T_n(f), g) \\
&= c_n\chi(n)(f, T_n(g)) \\
&= c_n\bar{c_n}^2(f, T_n(g)) \\
&= \bar{c_n}(f, T_n(g)) \\
&= (f, c_nT_n(g)),
\end{aligned}$$

as desired. □

Lastly, we refer where to find a basis of eigenforms for T_n.

Theorem 3.5.12. *[6] The space of cusp forms $S_k(\Gamma_0(q), \chi)$ contains a basis of eigenforms for all the T_n with (n,q)=1.*

Proof. First of all, we have $S_k(\Gamma_0(q),\chi)$ is a finite dimensional Hilbert space. Moreover, by Corollary 3.5.11, we say the Hecke operators T_n's with $(n,q)=1$ forms a ring of Hermitian operators. The result is immediate from Theorem 3.5.8. $\square$

4 *L*-FUNCTIONS

In this chapter we discuss L-functions and its relation with modular forms.

4.1 Riemann ζ-function

We first take the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

which is called the *Dirichlet series* [11]. This series was first stated by Dirichlet in 1839 and he was able to proof that there are infinitely many primes by using the series. Nevertheless, Dirichlet did not pay much attention to the case that s is complex number.

Riemann was the first mathematician who discovered the importance of the Dirichlet series for the case that s is a complex number. He especially focused on the case when $a_n = 1$ for all n and define the zeta function which is the Dirichlet series for $a_n = 1$ for all n, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

Although Euler was aware of the zeta function, it is known as Riemann zeta function [11] as Riemann studied it extensively.

By integrating the gamma function

$$\Gamma(s) = \int_0^{\infty} e^{-y} y^{s-1} dy,$$

Riemann reached the conclusion [6] that

$$(2\pi)^{-s}\Gamma(s)\zeta(2s) = \frac{1}{2}\int_0^\infty (\theta(iy)-1)y^{s-1}dy$$

where the theta function is

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}.$$

From that conclusion, we deduce the functional equation [11]

$$\pi^{s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

For the *Dirichlet L-function*

$$L(\chi,s) = \sum_{n=1}^{\infty} \chi(n)n^{-s},$$

the functional equation given above is also supplied. In the chapter we focus contribution of Hecke in L-function theory.

4.2 Introduction to L-functions

We start with a cusp form and derive an L-function from that cusp form. Let $f \in S_k(\Gamma_0(q),\chi)$ with the Fourier expansion at the cusps 0 and $i\infty$

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz),$$

and for $w = \left(\begin{smallmatrix} 0 & -1 \\ q & 0 \end{smallmatrix}\right)$, $g = f|_w$ with the Fourier expansion at the cusps 0 and $i\infty$

$$g(z) = \sum_{n=1}^{\infty} b_n e(nz).$$

Then, $g \in S_k(\Gamma_0(q),\chi)$.

Now we have enough tools to define Hecke L-functions.

Definition 4.2.1. [6] Let f,g be as above. Then the *Hecke L-functions associated to f,g*

are

$$L(f,s)=\sum_{n=1}^{\infty}a_n n^{-s},$$

and

$$L(g,s)=\sum_{n=1}^{\infty}b_n n^{-s}.$$

Remark 4.2.2. From their Fourier expansion, we easily obtain the identity

$$g(z)=q^{k/2}(qz)^{-k}f\left(-\frac{1}{qz}\right),$$

and by taking $z=iy$ for $y>0$, we derive

$$g(iy)=i^{-k}q^{-k/2}y^{-k}f\left(\frac{i}{qy}\right).$$

Now we give a significant theorem about Euler product expressions of L-functions introduced by Hecke.

Theorem 4.2.3 (Hecke). *[6] Let f be a Hecke eigenform with Fourier expansion*

$$f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi inz},$$

and let its associated L-function be

$$L(f,s)=\sum_{n=1}^{\infty}a_n n^{-s}.$$

Then for $\mathrm{Re}(s)>(k+1)/2$, $L(f,s)$ *has the Euler product*

$$L(f,s)=\prod_{p}(1-a_p p^{-s}+p^{k-1-2s})^{-1}.$$

Proof. Since the Fourier coefficients of a Hecke eigenform are eigenvalues of T_m's, we obtain

$$a_n a_m = a_{nm} \quad \text{for} \quad (m,n)=1$$

by commutativity of Hecke operators.Moreover, Theorem 3.2.3 implies

$$a_{p^r} = a_p a_{p^{r-1}} - p^{k-1} a_{p^{r-2}}.$$

and this relation allows us to write

$$\begin{aligned}
\sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} &= 1 + \sum_{r=0}^{\infty} \frac{a_{p^{r+1}}}{p^{(r+1)s}} \\
&= 1 + \sum_{r=0}^{\infty} \frac{a_p a_{p^r} - p^{k-1} a_{p^{r-1}}}{p^{(r+1)s}} \\
&= 1 + \frac{a_p}{p^s} \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} - \frac{p^{k-1}}{p^s} \sum_{r=0}^{\infty} \frac{a_{p^{r-1}}}{p^{rs}} \\
&= 1 + \frac{a_p}{p^s} \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} - \frac{p^{k-1}}{p^s} \sum_{r=1}^{\infty} \frac{a_{p^{r-1}}}{p^{rs}} \\
&= 1 + \frac{a_p}{p^s} \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} - \frac{p^{k-1}}{p^s} \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{(r+1)s}} \\
&= 1 + \frac{a_p}{p^s} \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} - \frac{p^{k-1}}{p^{2s}} \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}}.
\end{aligned}$$

Here, for the first line we use $a_1 = 1$, for the forth line we use $a_{p^{-1}} = 0$. Therefore, we obtain

$$\sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} = \left(1 - \frac{a_p}{p^s} + \frac{p^{k-1}}{p^{2s}}\right)^{-1}.$$

Hence, we deduce

$$\begin{aligned}
L(f,s) &= \sum_{n=1}^{\infty} a_n n^{-s} \\
&= \prod_p \left(\sum_{r=0}^{\infty} a_{p^r} p^{-rs} \right) \\
&= \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}
\end{aligned}$$

$\square$

4.3 Completed L-function

In this section, we introduce the completed L-function which satisfies a nice functional equation.

Theorem 4.3.1. *[6],[3] Let $f,g \in S_k(\Gamma_0(q),\chi)$ with the Fourier expansion at the cusps 0 and $i\infty$*

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz),$$

$$g(z) = \sum_{n=1}^{\infty} b_n e(nz),$$

and

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

$$L(g,s) = \sum_{n=1}^{\infty} b_n n^{-s}.$$

Then $L(f,s)$ and $L(g,s)$ extend analytically to the following entire functions with poles at s=0 and s=k

$$\Lambda(f,s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L(f,s)$$

and

$$\Lambda(g,s) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L(g,s),$$

which satisfy the functional equation

$$\Lambda(f,s) = i^k \Lambda(g,k-s).$$

Proof. First, we say that

$$(2\pi)^{-s}\Gamma(s)L(g,s) = \int_0^{\infty} g(iy) y^{s-1} dy.$$

Then, by plugging in

$$g(iy) = i^{-k} q^{-k/2} y^{-k} f\left(\frac{i}{qy}\right),$$

the integral becomes

$$i^{-k} q^{-k/2} \int_0^\infty f\left(\frac{i}{q}\right) y^{-k+s-1} dy.$$

Next, by changing variables $t = 1/qy$, the integral becomes

$$i^{-k} q^{-k/2} \int_0^\infty f(it) q^{k-s} t^{k-s-1} dt = i^{-k} q^{k/2-s} \int_0^\infty f(it) t^{k-s-1} dt. \tag{4.1}$$

Lastly, taking termwise integration of $L(f,s)$ gives the identity

$$(2\pi)^{-s}\Gamma(s)L(f,s) = \int_0^\infty f(iy) y^{s-1} dy.$$

and by putting this identity into (4.1), the integral equals to

$$i^{-k} q^{k/2-s} (2\pi)^{-(k-s)} \Gamma(k-s) L(f,k-s)$$

which gives the desired functional equation. □

Definition 4.3.2. The entire function $\Lambda(f,s)$ in previous theorem is called $the\ completed\ L-function$.

Although we force f, g to be cusp forms in the previous theorem, it is not essential. However, $\Lambda(f,s)$ and $\Lambda(g,s)$ may have simple poles at $s = 0$ and $s = k$ if we relax the assumption.

Next, we give a theorem stated by Hecke that shows under suitable hypotheses the functional equation

$$\Lambda(f,s) = i^k \Lambda(g,k-s)$$

implies $g = f|_w$. However, we need the following theorem known as Phragmén-Lindelöf theorem to prove Hecke's theorem.

Theorem 4.3.3 (Phragmén-Lindelöf). *[9],[10] Suppose that f(s) is analytic in the vertical strip $a \leq \mathrm{Re}(s) \leq b$, and that for some $\alpha \geq 1$,*

$$|f(s)| = O(e^{|t|^{\alpha}})$$

as $|t| \to \infty$. If f(s) is bounded on the two vertical lines Re*(s)=a and* Re*(s)=b, then f(s) is bounded in the entire vertical strip.*

Theorem 4.3.4 (Hecke). *[6] Assume f, g are functions with Fourier expansions*

$$f(z) = \sum_{n=0}^{\infty} a_n e(nz),$$

$$g(z) = \sum_{n=0}^{\infty} b_n e(nz)$$

with a_n, b_n bounded by $O(n^{\alpha})$ for some positive constant α. Put

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

$$L(g,s) = \sum_{n=1}^{\infty} b_n n^{-s},$$

$$\Lambda(f,s) = \left(\frac{\sqrt{q}}{2\pi}\right)\Gamma(s)L(f,s),$$

$$\Lambda(g,s) = \left(\frac{\sqrt{q}}{2\pi}\right)\Gamma(s)L(g,s)$$

where q is a positive number and for $w = \left(\begin{smallmatrix} 0 & -1 \\ q & 0 \end{smallmatrix}\right)$, $(f|_w)(z) = q^{k/2}(qz)^{-k} f\left(-\frac{1}{qz}\right)$. Then the following conditions are equivalent:

(i) $\quad g = f|_w$,

(ii) $\Lambda(f,s), \Lambda(g,s)$ *extend to the entire complex plane and both*

$$\Lambda(f,s)+\frac{a_0}{s}+\frac{b_0 i^k}{k-s}$$

and

$$\Lambda(g,s)+\frac{b_0}{s}+\frac{a_0 i^{-k}}{k-s}$$

are entire and bounded in every vertical strip and satisfy the functional equation

$$\Lambda(f,s)=i^k\Lambda(g,k-s).$$

Proof. When we assume (i), we have the integral representations

$$\Lambda(f,s)+\frac{a_0}{s}+\frac{b_0 i^k}{k-s}=\int_1^\infty\left(f\left(\frac{it}{\sqrt{q}}\right)-a_0\right)t^{s-1}dt+i^k\int_1^\infty\left(g\left(\frac{it}{\sqrt{q}}\right)-b_0\right)t^{k-s-1}dt,$$

and

$$\Lambda(g,s)+\frac{b_0}{s}+\frac{a_0 i^{-k}}{k-s}=\int_1^\infty\left(g\left(\frac{it}{\sqrt{q}}\right)-b_0\right)t^{s-1}dt+i^{-k}\int_1^\infty\left(f\left(\frac{it}{\sqrt{q}}\right)-a_0\right)t^{k-s-1}dt.$$

Both sides of identities converge for all $s\in\mathbb{C}$ since a_n's and b_n's have polynomial growth. If we put s instead of $k-s$ in the identities, we derive the functional equation

$$\Lambda(f,s)=i^k\Lambda(g,k-s)$$

because the polar part also satisfies the functional equation.

For the converse implication, one uses Mellin inversion, the Phragmén-Lindelöf convexity principal and Stirling's estimate for gamma function.

Firstly, we obtain $\Gamma(s+1)=s\Gamma(s)$ from integration by parts of gamma function. Therefore, we extend $\Gamma(s)$ to the entire plane with only poles at $s=0,-1,-2,\ldots$.

On the other hand, Stirling's estimate for gamma function implies that if σ is fixed, and $|t| \to \infty$, then

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} e^{(-\pi/2)|t|} |t|^{\sigma - 1/2}.$$

Hence, we deduce that, for any vertical strip $\sigma_1 \leq \mathrm{Re}(z) \leq \sigma_2$, $L(f,s)$ and $L(g,s)$ are $O(|t|^{1/2-\sigma} e^{\pi|t|/2})$ as $|t| \to \infty$ if $\Lambda(f,s), \Lambda(g,s)$ are bounded in every vertical strip.

Furthermore, $L(f,s)$ and $L(g,s)$ are bounded in some fixed half plane by the Phragmén-Lindelöf convexity principal. Furthermore, by the functional equation in (ii) and Stirling's estimate, for some positive A,

$$s(s-k)L(f,s) = O(|t|^A),$$

$$s(s-k)L(g,s) = O(|t|^A)$$

in any vertical strip.

On the other hand,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Lambda(f,s) y^{-s} ds &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{\sqrt{q}}{2\pi} \right)^s \Gamma(s) L(f,s) y^{-s} ds \\
&= \sum_1^\infty a_n \left(\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s) \left(\frac{2\pi n y}{\sqrt{q}} \right)^{-s} ds \right) \\
&= \sum_1^\infty a_n e^{-2\pi n y/\sqrt{q}} \\
&= f\left(\frac{iy}{\sqrt{q}} \right) - a_0.
\end{aligned}$$

Then, by moving the line integration to $\mathrm{Re}(s) = -U$ for positive U,

$$\begin{aligned} f\left(\frac{iy}{\sqrt{q}}\right) - a_0 &= \frac{1}{2\pi i}\int_{-U-i\infty}^{-U+i\infty} \Lambda(f,s)y^{-s}ds - a_0 - i^k b_0 y^{-k} \\ &= \frac{1}{2\pi i}\int_{-U-i\infty}^{-U+i\infty} i^k\Lambda(g,k-s)y^{-s}ds - i^k b_0 y^{-k} - a_0 \\ &= \frac{1}{2\pi i}\int_{k+U-i\infty}^{k+U+i\infty} i^k\Lambda(g,s)y^{s-k}ds - i^k b_0 y^{-k} - a_0 \\ &= i^k\left(g\left(\frac{iy}{\sqrt{q}}\right) - b_0\right)y^{-k} - a_0. \end{aligned}$$

Here, for first line we used Cauchy's Theorem, for second line we use the functional equation, for third line we change $k-s$ to s, and for the last line we use the above integral representation for g. Therefore, we reach

$$f\left(\frac{iy}{\sqrt{q}}\right) = i^k\left(g\left(\frac{iy}{\sqrt{q}}\right) - b_0\right)y^{-k}.$$

Lastly, we replace y by $1/\sqrt{q}t$ and we get

$$f\left(\frac{i}{qt}\right) = i^k q^{k/2} t^k g(it),$$

which means, for $z = it$,

$$f\left(-\frac{1}{qz}\right) = q^{k/2} z^k g(z).$$

Thus $f|_w = g$ and that proves the other direction. $\square$

5 SUMMARY

In this thesis, we have discussed three main subject arising in the study of modular forms.

In the second chapter, we defined and explained full modular group and some of its subgroups, standard fundamental domain, modular forms and cusp forms. We also mentioned the Fourier expansion of a modular form and introduced Eisenstein series and delta function as examples of modular forms. At the end of the chapter, we expressed the valence and dimension formulas.

In the third chapter, we discussed Hecke operators and some of their commutativity. Then, we saw that Hecke operators map a modular form to a modular form and a cusp form to a cusp form. Further, we used Hecke operators to prove two conjectures about Ramanujan-τ function. Moreover, we defined Hecke operators for Hecke congruence subgroup. At the end of the chapter, we mentioned Hecke eigenforms and some theorems about that eigenforms.

In the forth chapter, we started with the definition of Riemann-ζ function and some identities. Then, we defined L-function and completed L-function and gave a theorem about its Euler product. At the end of the chapter, we established a nice functional equation about completed L-function.

REFERENCES

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*. Springer-Verlag, 1976.

[2] J.-P. Serre, *A course in arithmetic*. Springer-Verlag, 1973.

[3] M. Murty, M. Dewar, and H. Graves, *Institute of Mathematical Sciences Lecture Notes-1: Problems in the Theory of Modular Forms*. Hindustan Book Agency, 2015.

[4] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*. Springer, 1984.

[5] F. Diamond and J. Shurman, *A First Course in Modular Forms*. Springer-Verlag, 2005.

[6] H. Iwaniec, *Topics in Classical Automorphic Forms*. American Mathematical Society, 1997.

[7] S. Ramanujan, "On certain artihmetical fucntions," *Transactions of the Cambridge Philosophical Society XXII No.9*, 1916.

[8] S. Lang, *Introduction to Modular Forms*. Springer-Verlag, 1976.

[9] J. B. Conway, *Functions of One Complex Variable I*. Springer-Verlag, 1978.

[10] W. Rudin, *Real and Complex Analysis*. McGraw-Hill Book Co., 1987.

[11] H. Davenport, *Multiplicative Number Theory*. Markham, 1967.