ESSAYS ON HOUSE ALLOCATION PROBLEMS

A Dissertation

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ABSTRACT

We study discrete resource allocation problems in which agents have unit demand and strict preferences over a set of indivisible objects. Such problems are known as house allocation problems. We define a new property that we call "balancedness." We characterize the top trading cycles from individual endowments by Pareto efficiency, group strategy-proofness, reallocation-proofness and balancedness. When there are at least four agents or just two agents, we characterize the top trading cycles from individual endowments by Pareto efficiency, group strategy-proofness and balancedness. When there are three agents, an allocation rule is Pareto efficient, group strategy-proof and balanced if and only if it is a top trading cycles rule from individual endowments or a trading cycles rule with three brokers.

We also study house allocation problems with weak preferences. We show that the serial dictatorship with fixed tie-breaking satisfies weak Pareto efficiency, strategy-proofness, non-bossiness, and consistency. Furthermore, the serial dictatorship with fixed tie-breaking is not Pareto dominated by any Pareto efficient and strategy-proof rule. We also show that the random serial dictatorship with fixed (or random) tie-breaking is equivalent to the top trading cycles from random endowments with fixed (or random) tie-breaking.

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1. INTRODUCTION

1.1 House allocation problems

Since the pioneering work of Shapley and Scarf (1974), a large literature has studied discrete resource allocation problems. Such a problem consists of a set of agents and a set of indivisible objects. Each agent consumes one object, and has a privately known preference relation over the set of objects. Each object can only be assigned to one agent, and objects do not care who they are assigned to. These problems are known as one-sided matching problems as opposed to two-sided matching problems. In a two-sided matching problem, there are two disjointed sets of agents, and each agent in one set has a preference relation over the agents in the other set. Real life examples of discrete resource allocation problems include assigning students to university apartments (Abdulkadiroğlu and Sönmez 1999), allocating and exchanging transplant organs, like kidneys (Roth, Sönmez, and Ünver 2004) or lungs (Ergin, Sönmez and Ünver 2015), etc. There are no monetary transfers in these settings.

When Shapley and Scarf (1974) initiated the problem, they used the example of houses and traders. Since then, such problems are now known as house allocation problems. In their problem, each agent initially owns one house and can possibly trade it for a better one in the market. So their model is also known as a housing market model in the literature. They defined the strict core. An allocation is in the strict core if there is no subset of agents such that by reallocating their endowed houses among themselves, at least one of them is strictly better off, and the others in the subset are not worse off. They proposed the top trading cycles algorithm invented by David Gale to find an allocation in the strict core. The top trading cycles algorithm consists of a serial of finite steps. At each step, each remaining agent points to the agent who owns her best choice among the remaining houses. A set of agents forms a cycle whenever their best choices are the houses owned by the agents in this set. A single agent who owns her favorite house also forms a cycle by pointing to herself. Remove all cycles from the market, assign each agent in a cycle her best choice, and proceed to the next step. The algorithm stops when everyone is assigned. Shapley and Scarf (1974) used an example to show that the strict core may disappear if indifferences are allowed. In what follows, we assume that preferences are strict unless otherwise mentioned.

Roth and Postlewaite (1977) showed that the strict core allocation is unique. Roth (1982a) showed that the strict core mechanism is strategy-proof, i.e., truthfully reporting one's preference is a weakly dominant strategy for each agent. Bird (1984) showed that it is group strategy-proof, i.e., there is no subset of agents who can jointly misreport their preferences such that at least one agent in the group is strictly better off while the others are not worse off. Ma (1994) characterized the strict core mechanism by individual rationality (i.e., each agent weakly prefers her assigned object to her endowed object), Pareto efficiency (i.e., no other allocation exists such that all agents are weakly better off and some agent is strictly better off) and strategy-proofness. However, in the two-sided matching problems even with strict preferences, efficiency and strategy-proofness are incompatible. For example, in the marriage problem, it is well-known that there exists no individual rational mechanism that is both Pareto efficient and strategy-proof (Theorem 3 in Roth 1982b; Proposition 1 in Alcalde and Barberà 1994).

Unlike Shapley and Scarf's (1974) housing market model, all objects could be initially owned by a social planner and she determines a priority ranking of agents. Agents choose their best choices sequentially according to a priority ranking. This allocation rule is a serial dictatorship. Svensson (1999) characterized the serial dictatorships by strategy-proofness, non-bossiness (i.e., if an agent cannot change their assignment by misreporting their preference relation, then she cannot change the assignments of other agents) and neutrality (i.e., the "real" outcome is independent of the indexes of objects). Ergin (2002) generalized the serial dictatorships by allowing each distinct object type to have multiple copies (for example, each school has multiple seats for students) and different object types may have different priority rankings of agents. Ergin studied the deferred acceptance (DA) mechanism, defined a property of acyclicity for a priority structure (i.e., for house allocation problems with unit capacity for each object type, there exist no objects o and o' and agents i, j and k such that for object o, i has a higher priority than j and j has a higher priority than k, and for object o', k has a higher priority than i), and showed the equivalence between acyclical priority, Pareto efficient DA mechanism, group strategy-proof DA mechanism, and consistent (i.e., whenever some agents receives their assignments, we can remove these agents and their assignments from the market without changing the assignments of other agents) DA mechanism.

Abdulkadiroğlu and Sönmez (1999) studied a hybrid model known as house allocation with existing tenants. In their model, some agents each initially owns one object and some objects are social endowments. They proposed the so-called "you request my house-I get your turn (YRMH-IGYT)" algorithm to find the allocation. Sönmez and Ünver (2010) characterized the YRMH-IGYT rules by individual rationality, Pareto efficiency, strategy-proofness, weak neutrality (i.e., the labeling of unowned objects has no effect on the outcome of the mechanism), and consistency. Ekici (2013) defined a property reclaim-proofness for a matching (i.e., it is robust to blocking coalitions with respect to any conceivable interim endowments of agents) and showed its relationships between the YRMH-IGYT rules and competitive allocation.

Pápai (2000) extended Gale's top trading cycles algorithm used in the housing market model to more general environments by allowing some agents to initially own more than one object and each object is owned by some agent. She constructed a class of rules known as hierarchical exchange rules and showed that an allocation rule that is Pareto efficient, group strategy-proof, and reallocation-proof (i.e., no pair of agent can jointly misreport their preferences and swap their assignments ex post to make at least one of them strictly better off) if and only if it is a hierarchical exchange rule. A hierarchical exchange rule also utilizes the top trading cycles to find the allocation. Each rule is defined by an inheritance structure which specifies who initially owns and who potentially inherits what. The formal definition of the hierarchical exchange rules is introduced in chapter 2. To distinguish the top trading cycles used in Shapley and Scarf's housing market where each agent initially owns one object and the top trading cycles used in Pápai's (2000) hierarchical exchange rules where some agent may initially own more than one object, we refer to the former as the top trading cycles from individual endowments. For a hierarchical exchange rule, if each agent initially owns one object, then it is a top trading cycles rule from individual endowments; if one agent initially owns all the objects, the second agent inherits all the remaining objects after the first agent chooses, the third agent inherits all the remaining objects after the second agent chooses, and so forth, then it is a serial dictatorship; if different objects have possibly different inheritance orderings of agents, then it is a Ergin's priority rule.

Velez (2014) studied the set of consistent hierarchical exchange rules. He defined the CHE rules, showed the equivalence of the CHE rules and consistent hierarchical exchange rules, and proved that the CHE rules are the only rules that are efficient in two-agent problems, consistent in two-agent problems, and strategy-proof. Tang and Zhang (2016) redefined individual rationality and the strict core for Pápai's hierarchical exchange rules. They characterized these rules by Pareto efficiency, strategy-proofness and their newly defined individual rationality. They also showed that the hierarchical exchange rule selects the unique strict core allocation.

Pycia and Ünver (2016a) generalized Pápai's hierarchical exchange rules by allowing at most one agent to be a broker who brokers only one object, or allowing all the three remaining objects to be brokered by three agents. The assignment is formed by running the same top trading cycles algorithm as a hierarchical exchange rule with the additional requirement that a broker may not be allowed to points to herself if her favorite object is her brokered object. They call such an allocation rule as a trading cycles rule. The formal definition of the trading cycles rules could be found in chapter 3 when the number of agents is equal to or greater than the number of objects. When there are more objects than agents, the definition is available in chapter 2. The trading cycles rules are quite general. They subsume the top trading cycles from individual endowments, the serial dictatorships, and the hierarchical exchange rules as special cases. Surprisingly, Pycia and Ünver showed that the trading cycles rules are the only rules that are Pareto efficient and group strategy-proof.

1.2 Interim fairness

When we design an allocation rule, Pareto efficiency and group strategy-proofness are among the most important properties of our concern. Pareto efficiency should be the minimum requirement when preferences are strict. Group strategy-proofness prevents manipulation among agents, therefore, it minimizes information searching costs. It also does not discriminate agents who do not have access to information and who are less able to play strategically. All the trading cycles rules, including serial dictatorships, are both Pareto efficient and group strategy-proof. But a serial dictatorship seems unfair in the sense that the agent at the top of the priority list is always guaranteed to her best choice, while the agent at the end of the list receives her best choice only if her best choice is not the best choice of any other agent.

A Pareto efficient allocation rule cannot be ex post fair because if all agents have the same preferences, then Pareto efficiency implies that each agent receives a distinct object. Therefore, agents cannot be treated equally. But the random serial dictatorship seems more desirable than a deterministic allocation rule in the sense that all agents have equal chances to be the first to choose, have equal chances to be the second to choose, and so forth. Given a deterministic allocation rule, we can interpret agents in the rule as "roles." A corresponding random allocation rule is defined by assigning agents to the "roles" via a uniform lottery. Surprisingly, Abdulkadiroğlu and Sönmez (1998) showed that the random serial dictatorship is equivalent to the top trading cycles from random endowments as their distributions over assignments are exactly the same for a given preference profile. Pathak and Sethuraman (2011) showed the equivalence between the random serial dictatorship and the multiple lottery mechanism where each object independently draws a lottery to determine agents' priorities. Lee and Sethuraman (2011) showed that this equivalence still holds for all random hierarchical exchange rules. Bade (2014) extended the equivalence result to all random trading cycles rules. Therefore, all random trading cycles rules are equally ex ante fair. We refer to this kind of fairness as ex ante fairness because the expectations of outcomes are based on beliefs before knowing who actually owns or brokers what. After knowing the realizations of the lotteries, it is clear that a serial dictatorship is less equitable than a rule of top trading cycles from individual endowments.

During the course of assigning students to public schools in New York City, policymakers and parents believed that a single lottery used for all schools is less equitable than lotteries at each school. As quoted in Pathak and Sethuraman (2011), a policymaker from the New York City Department of Education said:

> "*I cannot see how the children at the end of the line are not disenfranchised totally if only one run takes place. I believe that one line will not be acceptable to parents. When I answered questions about this at training sessions, (it did come up!) people reacted that the only fair approach was to do multiple runs.*"

Since no mechanism could be ex post fair, and all of the random mechanisms constructed from Pareto efficient and group strategy-proof rules are equally ex ante fair, our chapter 3 tries to formally define a interim fairness property and to characterize the set of rules that satisfies this interim fairness property without compromising Pareto efficiency and group strategy-proofness.

Consider the following timing of the allocation mechanism:

- First: Mechanism designer picks an allocation rule including the realization of the lottery.
- Second: Agents report their preferences.
- Third: Assignment is realized.

After knowing the realization of the lottery, i.e., after knowing which deterministic allocation rule would be used to find the assignment, but before agents submit their preferences, no one knows what are the true preferences. When fixing the allocation rule and considering all possible preference profiles, we can count the number of preference profiles that an agent, say agent i, receives her best choice. We can also count the number of preference profiles that agent i receives her second best choice, and so forth. If for any two agents i and j, the number of preference profiles that agent i receives her best choice is equal to the number of preference profiles that agent j receives her best choice, the number of preference profiles that agent

i receives her second best choice is equal to the number of preference profiles that agent j receives her second best choice, and so forth, then the allocation rule is balanced. Balancedness means that if all possible preference profiles are equally likely to occur, then a balanced deterministic allocation rule assigns all agents to their best choices with equal probabilities, assigns all agents to their second best choices with equal probabilities, and so forth. Unlike the previous equivalence results which consider all possible permutations over the "roles" of a deterministic rule for the fixed preference profile, our definition considers all possible preference profiles for a given deterministic rule.

Our theorem 3.1 states that an allocation rule is Pareto efficient, group strategy-proof, reallocation-proof, and balanced if and only if it is a top trading cycles rule from individual endowments. As stated in theorem 3.2, when there are at least four agents or just two agents, we can drop reallocation-proofness in theorem 3.1; when there are just three agents, a deterministic allocation rule is Pareto efficient, group strategy-proof, and balanced if and only if it is a top trading cycles rule from individual endowments or a trading cycles rule with three brokers. We may also relax the requirements of balancedness by only requiring all agents have equal chances to their worst choices. We show that theorem 3.1 and theorem 3.2 still hold under the relaxed version of balancedness. Our new characterizations of the top trading cycles from individual endowments have important policy implications. Whenever policy makers can freely choose any allocation rule, for the sake of interim fairness, they should randomly assign each agent a distinct object and then use the top trading cycles algorithm to find the allocation.

1.3 Weak preferences

So far, we assume preferences are strict. A large literature on matching theory assumes strict preferences. Without this assumption, many good properties fail to hold. For example, in the marriage problem with strict preferences, stability (that is, a matching is stable if no unmatched pair can be better off by matching each other, and no matched agent can be better off by being single) implies Pareto efficiency (Proposition 2.1 in Abdulkadiroğlu and Sönmez 2013). But with weak preferences, a stable matching might fail to be Pareto efficient. For example, consider a marriage problem with two men and two women. Each man is indifferent between the two women, woman 2 is indifferent between two men, but woman 1 prefers man 1 to man 2. The matching that matches man 1 to woman 2, and man 2 to woman 1 is stable, but it is not efficient. Alternatively, Pápai (2000) showed that group strategy-proofness is equivalent to strategy-proofness and non-bossiness under strict preferences; but this equivalence breaks down on the weak domain (Ehlers 2002).

Due to the undesirable properties and the complexity induced by ties, weak preferences are ignored in most of the existing matching literature. But indifferences prevail in the real world. For example, in the kidney exchange problem (Roth, Sönmez, and Ünver 2004), each patient-donor pair wants to exchange for a compatible kidney from another patient-donor pair. If their preferences are based on checklist criteria such as blood and tissue types, then different kidneys with the same criteria should be regarded as indifferent. Another example is the school choice problem (Erdil and Ergin 2008) which consists of a set of students and a set of public schools with limited numbers of seats. Each school has a priority ranking over students. The ranking is determined by local laws and educational policies. Such priorities are weak orderings and the indifference classes are quite large.

When we design allocation mechanisms for house allocation problems, in addition to the minimum requirement of efficiency or weak efficiency, we also want agents to truthfully reveal their preferences. In the two-sided matching problems even with strict preferences, efficiency and strategy-proofness are incompatible. Fortunately, in the one-sided matching problems, positive results exist. Recently, Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) defined two different classes of rules that are Pareto efficient, strategy-proof, and individual rational for the housing market problem when indifferences are allowed. Ehlers (2014) provided a characterization for top trading cycles with fixed tie-breaking for the housing market problems with indifference by individual rationality, strategy-proofness, weak efficiency, non-bossiness, and consistency.

In chapter 4, we study the serial dictatorship with fixed tie-breaking when indifferences are allowed. We show that it satisfies weak Pareto efficiency, strategy-proofness, non-bossiness, and consistency; moreover, it is not Pareto dominated by any Pareto efficient and strategy-proof rule. As a corollary to Abdulkadiroğlu and Sönmez (1998), the equivalence between the random serial dictatorship and the top trading cycles algorithm from random endowments still holds when we use fixed tie-breaking or random tie-breaking.

The remaining dissertation is organized as follows: chapter 2 surveys axiomatic approaches to house allocation problems; chapter 3 introduces a new property that we call "balancedness," and we characterize the set of allocation rules that satisfies this new fairness property as well as the efficiency and incentive properties; chapter 4 discusses the serial dictatorships with fixed tie-breaking; and chapter 5 concludes.

2. LITERATURE REVIEW

2.1 Relationship between various allocation rules

The seminal work on house allocation problems was written by Shapley and Scarf (1974). They introduced the so-called housing marking model in which each trader initially owns one house and can possibly trade it for a better one in the market. The question they were interested in is whether the strict core allocation defined by weak dominance exists. They showed that the strict core always exists when preferences are strict. They proposed a simple algorithm known as top trading cycles which was invented David Gale to find an allocation in the strict core. Since then, Roth and Postlewaite (1977), Roth (1982a), Bird (1984), Ma (1994), among others, studied properties of the top trading cycles algorithm. Specifically, Ma (1994) characterized top trading cycles rules by Pareto efficiency, individual rationality and strategy-proofness.

Unlike the housing market model, all objects could be initially owned by a social planner and she determines a priority ranking of agents. Agents choose their best choices sequentially according to the priority ranking. This allocation rule is called a serial dictatorship. Svensson (1999) characterized serial dictatorships by strategy-proofness, non-bossiness and neutrality. Ergin (2002) generalized the serial dictatorship by allowing each distinct object type to have multiple copies (for example, each school has multiple seats for students) and different object types may have different priority rankings of agents. Ergin studied deferred acceptance (DA) mechanism, defined a property of acyclicity for priority structure, and showed the equivalence between acyclical priority, Pareto efficient DA mechanism, group strategy-proof DA mechanism, and consistent DA mechanism.

Abdulkadiroğlu and Sönmez (1999) studied a hybrid model in which some agents each initially owns one object and some objects are social endowment. The proposed the so-called you request my house-I get your turn (YRMH-IGYT) algorithm to find the allocation. Sönmez and Ünver (2010) charactized YRMH-IGYT rules by individual rationality, Pareto efficiency, strategy-proofness, weakly neutrality, and consistency. Ekici (2013) defined a property reclaim-proofness and showed its relationship between YRMH-IGYT rules and competitive allocation.

Pápai (2000) extended Gale's top trading cycles algorithm used in housing market model to more general environments by allowing some agents to initially own more than one object and each object is owned by some agent. She constructed a class of rules known as hierarchical exchange rules and characterized these rules by Pareto efficiency, group strategy-proofness, and reallocation-proofness. Velez (2014) studied the set of consistent hierarchical exchange rules. He defined the CHE rules, showed the equivalence of the CHE rules and consistent hierarchical exchange rules, and proved that the CHE rules are only rules that are 2-efficient, 2-consistent, and strategy-proof. Tang and Zhang (2016) redefined individual rationality and the core for Pápai's model. They characterized hierarchical exchange rules by individual rationality, Pareto efficiency, and strategy-proofness. They also showed that the hierarchical exchange rule selects the unique core allocation.

Pycia and Ünver (2016a) modified the top trading cycles algorithm by allowing at most one agent to be a broker who brokers only one object when there are more objects than agents. The assignment is formed by running the same top trading cycles algorithm as a hierarchical exchange rule with the additional requirement that the broker points to her favorite object owned by others. Such allocation rules are called trading cycles rules. Surprisingly, they characterized the trading cycles rules by Pareto efficiency and group strategy-proofness.

The following figure 2.1 shows the relationship between various allocation rules.

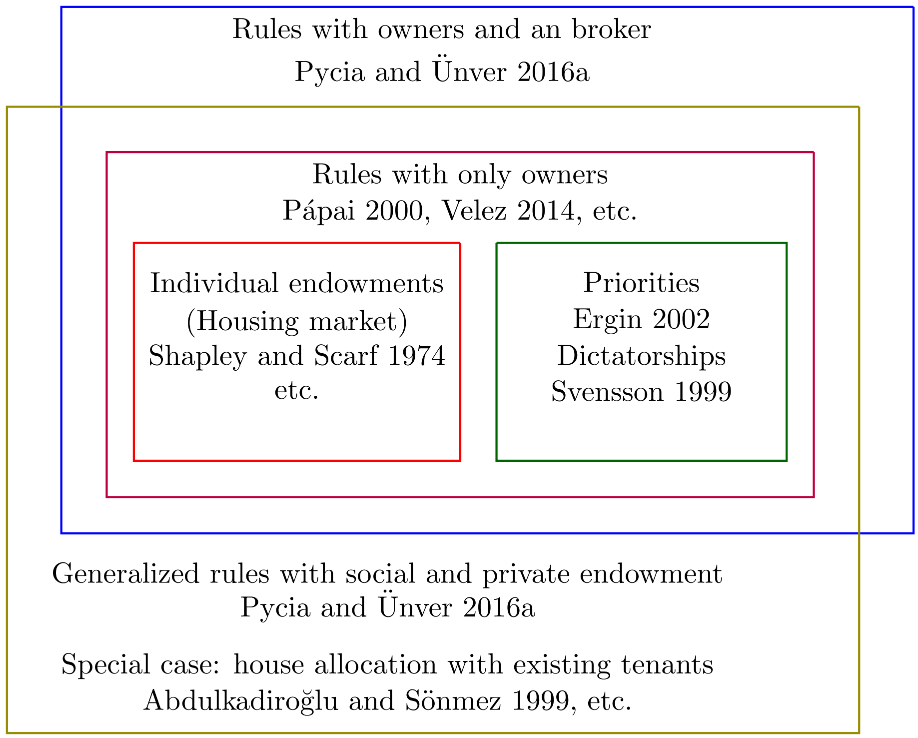

Figure 2.1: Relationship between various allocation rules

The remaining chapter is organized from general models to specific ones. Section 2.2 introduces the notation for general models. Section 2.3 describes the trading cycles. Section 2.4 discusses models without brokers, including models with social and private endowments, the hierarchical exchange, the top trading cycles from individual endowments, and Ergin's priority rules and serial dictatorship.

2.2 The model

Let N be the set of **agents** and O be the set of **objects**. We assume there are more objects than agents unless otherwise mentioned. Each agent i has a **strict**

preference relation over O, denoted by P_i. Let R_i be the weak preference relation associated with P_i, i.e., for all $o, o' \in O$, oR_io' if and only if $o = o'$ or oP_io'. The set of strict preference relations for agent i is $\mathcal{P}_i$. For any $J \subseteq N$, $\mathcal{P}_J$ is the set of preference relations for all agents $j \in J$. A **preference profile** $P = (P_i)_{i \in N}$ is an element from $\mathcal{P} \equiv \mathcal{P}_N$. $P_J = (P_i)_{i \in J}$ is the restriction of P to J. Let P_{-i} denote $P_{N \setminus \{i\}}$. We do not consider outside options, and assume all objects are acceptable to all agents.

A **house allocation problem** (Hylland and Zeckhauser 1979) consists of N, O, and P. The outcome of a house allocation problem is simply a **allocation** (or an **matching**), denoted by μ, such that each agent receives a distinct object, i.e., $\mu : N \to O$ is an injective (one-to-one) function. Let μ_i be the assignment of agent i at matching μ. Let $\mathcal{M}$ denote the set of all matchings. A **submatching** σ is a restriction of a matching to a subset of agents. Let $\mathcal{S}$ be the **set of all submatchings**. For any $\sigma \in \mathcal{S}$, let N_σ be the **set of matched agents** and O_σ be the **set of matched objects** under σ. For each $i \in N_\sigma$, let $\sigma(i)$ denote the assigned object of agent i. For each $o \in O_\sigma$, let $\sigma(o)$ denote the agent that receives o. Let $\overline{N_\sigma}$ be $N \setminus N_\sigma$, $\overline{O_\sigma}$ be $O \setminus O_\sigma$, and $\overline{\mathcal{M}}$ be $\mathcal{S} \setminus \mathcal{M}$. And the set of submatchings that objects o is unmatched is denoted by $\mathcal{S}_{-o}$.

Fixed N and O, the domain of all problems is the set of preference profiles. An **allocation rule** $\varphi : \mathcal{P} \to \mathcal{M}$ is function that assigns a matching for each problem.

2.3 Trading cycles

Pycia and Ünver (2016a) defined trading cycles rules by novelly introducing brokage right.

Definition. A **control rights structure** is a collection of functions

$$\left\{(c_\sigma, b_\sigma) : \overline{O_\sigma} \to \overline{N_\sigma} \times \{ownership, brokerage\}\right\}_{\sigma \in \overline{\mathcal{M}}}.$$

A control rights structure specifies at each submatching σ each unmatched object $o \in \overline{O_\sigma}$ is **controlled** by a unique unmatched agent $c_\sigma(o)$. The type of control $b_\sigma(o) = ownership$ if agent $c_\sigma(o)$ **owns** o at σ, and $b_\sigma(o) = brokerage$ if agent $c_\sigma(o)$ **brokers** o at σ.

A trading cycles rule have to satisfy the following requirements R1 to R6.

Within-round Requirements. For any $\sigma \in \overline{\mathcal{M}}$,

(R1) There exists at most one brokered object at σ.

(R2) If i is the only unmatched agent at σ, then i owns all unmatched objects at σ.

(R3) If agent i brokers an object at σ, then i does not own or broker any other objects at σ.

Across-round Requirements. For any $\sigma \subset \sigma' \in \mathcal{M}$, if agent $i \in \overline{N_{\sigma'}}$ owns object $o \in \overline{O_{\sigma'}}$ at σ, then:

(R4) Agent i owns o at σ'.

(R5) If i' brokers object o' at σ and $i' \in \overline{N_{\sigma'}}$, $o' \in \overline{O_{\sigma'}}$, then i' brokers o' at σ'.

(R6) If agent $i' \in \overline{N_{\sigma'}}$ controls $o' \in \overline{O_{\sigma'}}$ at σ, then i' owns o at $\sigma \cup \{(i, o')\}$.

Requirement R4 postulates that ownership rights are **persistent**: if agent i owns an object at a smaller submatching, and agent i is unmatched at a larger submatching, then agent i still owns the object at the larger submatching. Requirement R5 is a counterpart of R4 for brokage right. It implies that the brokage right **persists** whenever there is at least one owner at the base submatching σ. The loss of brokerage right can only happen when there is no owner at the base submatching σ.

In this case, R2 implies that the unmatched agent owns all unmatched objects. Requirement R6 assumes that when an agent i is matched with an object controlled by i', then i' owns the objects previously owned by i.

A **trading cycles** (**TC**) rule consists of a finite sequence of rounds described below, with a structure of control rights satisfying requirements R1-R6. We set null submatching $\sigma^0 = \varnothing$ and construct submatchings σ^r of matched agents and objects before round $r+1$ for all $r = 1, 2, 3, \cdots$. Each round r consists of three steps:

- *Pointing step.* Each object $o \in \overline{O_{\sigma^{r-1}}}$ points to the agent who controls it at σ^{r-1}. Each owner in $\overline{N_{\sigma^{r-1}}}$ points to her favorite object among $\overline{O_{\sigma^{r-1}}}$. If there is a broker in $\overline{N_{\sigma^{r-1}}}$, she points to her favorite object among objects owned at σ^{r-1}.

- *Trading step.* As the number of agents is finite, each agent points to one object, and each object points to one agent, there exists an integer number n and a trading cycle consisting of a set of agents $\{i^1, i^2, \cdots, i^n\} \subseteq \overline{N_{\sigma^{r-1}}}$ and a set of objets $\{o^1, o^2, \cdots, o^n\} \subseteq \overline{O_{\sigma^{r-1}}}$ such that o^l points to i^l, i^l points to o^{l+1} for all $l = 1, 2, \cdots, n$, and $o^{n+1} = o^1$, i.e., $o^1 \to i^1 \to o^2 \to \cdots \to o^n \to i^n \to o^1$. No two trading cycles intersect. Each agent in a trading cycle is matched with the object she points to.

- *Departure step.* Newly matched agents and objects are removed from $\overline{N_{\sigma^{r-1}}}$ and $\overline{O_{\sigma^{r-1}}}$. Submatching σ^r is the union of σ^{r-1} and newly matched agent-object pairs. The algorithm stops when all agents are matched. The submatching formed at the last round is the outcome of the TC rule.

Example. Consider an economy with three agents 1, 2, 3, four objects a, b, c, d, and a control-right structure that at submatching $\varnothing$, 1 brokers a, 2 owns b and c, and

3 owns d, respectively. Suppose each agent i has the preference relation $aP_ibP_icP_id$ for all $i = 1, 2, 3$. Then in the first round 1 points to b, and 2 and 3 both point to a. A trading cycle "$1 \to b \to 2 \to a \to 1$" exists. 1 is matched with b and 2 is matched with a. In the second round, 3 owns c and d and she is matched with c. The algorithm stops.

Definition. An matching $\mu \in \mathcal{M}$ is **Pareto efficient** if there exists no matching $\mu' \in \mathcal{M}$ such that for all $i \in N$, $\mu'_i R_i \mu_i$, and there exists some agent $j \in N$, $\mu'_j P_j \mu_j$. An allocation rule is **Pareto efficient** if it selects a Pareto efficient matching for each problem.

Definition. An allocation rule φ is **strategy-proof** is truthfully revealing one's preference is a weakly dominant strategy for each agent, i.e., for all $i \in N$, all $P \in \mathcal{P}$, and all $P'_i \in \mathcal{P}_i$, $\varphi_i(P) R_i \varphi_i(P'_i, P_{-i})$. An allocation rule φ is **group strategy-proof** if no group of agent can jointly misreport their preferences to make all agents in the group are weakly better off and some agent in the group are strictly better off, i.e., for all $P \in \mathcal{P}$, there exists no $J \subseteq N$ and $P'_J \in \mathcal{P}_J$ such that $\varphi_i(P) R_i \varphi_i(P'_J, P_{N \setminus J})$ for all $i \in J$, and $\varphi_j(P) P_j \varphi_j(P'_J, P_{N \setminus J})$ for some $j \in J$.

Theorem (Theorem 1 in Pycia and Ünver 2016a): An allocation rule is Pareto efficient and group strategy-proof if and only if it is a trading cycles rule.

For any allocation rule, we call an agent an **owner*** of an object if she obtains the object whenever she ranks it first; we call an agent a **broker*** of an object if she obtains her second best choice among all preference profiles such that all agents rank the brokered object first. To show the only if part of theorem, the authors construct a TC rule that is equivalent to the Pareto efficient and group strategy-proof rule. To prove that this construction finds all candidates, it is necessary to check that each unmatched object is either owned* or brokered*, as shown in the following theorem.

Theorem (Theorem 2 in Pycia and Ünver 2016a): For any Pareto efficient and group strategy-proof rule, for any submatching σ, each unmatched object at σ is either owned* or brokered* by some unique agent.

2.4 Models without brokers

2.4.1 Generalized model with social and private endowment

Let $\mathcal{O} = \{O_i\}_{i\in\{0\}\cup N}$ be a collection of pairwise-disjointed subsets of O such that $\cup_{i\in\{0\}\cup N} O_i = O$. O_0 is social endowment of agents, O_i is private endowment of agent i. Some agents may have empty private endowment.

Definition. A matching is **individually rational** if the assignment of each agent is weakly preferred to any object that she could choose from her private endowment. An allocation rule is **individually rational** if it finds an individually rational matching for each problem.

The following theorem is corollary of theorem 1 in Pycia and Ünver (2016a).

Theorem (Theorem 3 in Pycia and Ünver 2016a): An allocation rule is individually rational, Pareto efficient and group strategy-proof if and only if it is an individually rational trading cycles rule.

The following theorem identifies individually rational trading cycles rule.

Theorem (Theorem 4 in Pycia and Ünver 2016a): An trading cycles rule is individually rational if and only if it may be represented by a persistent structure of control rights in which each agent has the initial ownership rights over all objects from her endowment.

As a corollary, the following theorem characterize the top trading cycles.

Theorem (Theorem 5 in Pycia and Ünver 2016a): Suppose each agent's endowment is nonempty. An allocation rule is individually rational, Pareto efficient, and group strategy-proof if and only if it is a top trading cycles rule that assigns all agents the initial ownership rights over objects from their endowment.

A special case of the generalized model without a broker is a **house allocation problem with existing tenants** (Abdulkadiroğlu and Sönmez 1999) where some agents (existing tenants) each initially owns one object (an occupied house) and unowned objects (vacant houses) are social endowment. They proposed two allocation rules to find the matching. One is a **top trading cycles rule with a priority ordering** of agents over the set of unowned objects (TTC with a priority ordering). It consists of a serial of steps:

- At step 1, the set of **available objects** is the set of unowned objects.
- At step t, the set of available objects is the remaining previously unowned objects at the end of step $t-1$. Each remaining agent points to her best choice among the remaining objects. Each remaining owned object (an occupied house) points to its owner (an existing tenant). And each available object points to the remaining agent with the highest priority. This algorithm is just a special case of trading cycles algorithm without a broker. The definition of cycles and the submatchings formed in this step are similar to that of the trading cycles rule.

Abdulkadiroğlu and Sönmez (1999) showed that the TTC with a priority ordering satisfies good properties.

Theorem (Theorem 1 in Abdulkadiroğlu and Sönmez 1999): Fix a priority ordering, the TTC with the priority ordering is individually rational, Pareto efficient, and strategy-proof.

The other allocation rule that Abdulkadiroğlu and Sönmez (1999) proposed is the **you request my house-I get your turn (YRMH-IGYT) algorithm** described below.

1. For any priority ordering, match the agent with the highest priority her best choice, the agent with the second highest priority her best choice among the remaining one, and so forth, until the object is owned by some owner (an existing tenant).

2. If at some point, the owner whose owned object is demanded is already matched, then step 1 proceeds. Otherwise reorder the priority ordering by adding this owner at the top of the remaining priority ordering and proceed.

3. Similarly, add any owner whose owned object is demanded to the top of the remaining priority ordering and proceed.

4. If at any point a cycle consisting only of owners exists, then match each agent in a cycle her best choice among the remaining ones, remove them, and proceed.

Abdulkadiroğlu and Sönmez (1999) showed the equivalence of the TTC algorithm and YRMH-IGYT algorithm.

Theorem (Theorem 3 in Abdulkadiroğlu and Sönmez 1999): Fix a priority ordering, the TTC with the priority ordering and the YRMH-IGYT with the priority ordering find the same matching.

Sönmez and Ünver (2010) provided a characterization of YRMH-IGYT mechanism. Before introducing their result, we introduce some axioms used in their result.

Definition. An allocation rule is **weakly neutral** if labeling of unowned objects (vacant houses) has no effect on the outcome of the mechanism.

Definition. An allocation rule φ is **consistent** for the house allocation problem with existing tenants if we remove a set of agents J along with their matched objects and some unmatched objects G, provided that the remaining problem is a well-defined reduced problem, then

$$\varphi_i\left(P_{N\setminus J}^{O\setminus\{\varphi_J(P)\cup G\}}\right)=\varphi_i(P),\ \forall i\in N\setminus J,$$

where $P_{N\setminus J}^{O\setminus\{\varphi_J(P)\cup G\}}$ is the restricted preference relations of agents in $N\setminus J$ over the set of objects $O\setminus\{\varphi_J(P)\cup G\}$.

Theorem (Theorem 1 in Sönmez and Ünver 2010): An allocation rule is Pareto-efficient, individually rational, strategy-proof, weakly neutral, and consistent if and only if it is a YRMH-IGYT rule.

Ekici (2013) defined a new property called reclaim-proofness and established a link between reclaim-proof allocations and the class of YRMH-IGYT mechanisms.

Definition. An **interim endowment function** ω of an allocation μ satisfies $\omega(o)\in\{$the agent who initially owns o, the agent who is matched with o at μ, the social planner$\}$ for all o. An allocation is **reclaim-proof** if it is robust to blocking coalitions with respect to every interim endowment function of μ.

Theorem (Theorem 1 and 2 in Ekici 2013): An allocation is reclaim-proof if and only if it is induced by a YRMH-IGYT mechanism and if and only if it is a competitive allocation.

2.4.2 Hierarchical exchange

A hierarchical exchange rule defined by Pápai (2000) is a TC rule without a broker. To define a hierarchical exchange rule, we can simplify the control rights structure to the ownership rights structure.

Definition. A **ownership rights structure** is a collection of functions $\{c_\sigma:\overline{O_\sigma}\to\overline{N_\sigma}\}_{\sigma\in\overline{\mathcal{M}}}$ denoted by $\{c_\sigma\}$ for short. It specifies that at each submatching each unmatched object is owned by a unique unmatched agent. The ownership rights structure is **persistent** if for all submatchings $\sigma\subseteq\sigma'\in\overline{\mathcal{M}}$, if agent i and object o

are unmatched at σ', and i owns o at σ, then i owns o at σ', i.e., if $i \in \overline{N_{\sigma'}}$ owns $o \in \overline{O_{\sigma'}}$ at σ, then i owns o at σ'. For each submatching σ and each unmatched object o, we write $c_\sigma(o) = i$ if agent i owns o at submatching σ.

Definition. Each persistent structure of ownership rights defines a **hierarchical exchange** rule. A **fixed endowment allocation rule** is a hierarchical exchange rule satisfying $c_\sigma(o) = c_{\sigma'}(o)$ for all o, and for all σ and σ such that $N_\sigma = N_{\sigma'}$.

Pápai (2000) defined inheritance trees to describe this persistent TC rule without brokerage right. An inheritance tree of an object specifies who initially owns and who potentially becomes new owner of the object at different submatchings formed when the algorithm proceeds.

An inheritance tree of an object a denoted by Γ_a consists of a set of vertices V and a set of arcs $Q \subseteq V \times V$. A arc $(v_i, v_j) \in Q$ for $v_i, v_j \in V$ if there is an arrow from v_i to v_j. A Q-path from v_1 to v_r is a sequence $\{v_s\}_{s=1}^r$, where $r \geq 2$, such that $(v_s, v_{s+1}) \in Q$. There exists a vertex $v_0 \in V$ which is the unique root of Γ_a, that is, there exists no vertex $v \in V$ such that $(v, v_0) \in Q$.

A well-defined inheritance tree Γ_a satisfies the following requirements.

(A.1) All vertices are labeled by individuals.

(A.2) Every vertex of a Q-path represents a different individual.

(B.1) All arcs are labeled by objects other than a.

(B.2) Every arc of a Q-path represents a different object.

(B.3) Arcs from the same vertex represent different objects.

(C.1) $\max_{v \in V} d(v_0, v) = m - 1$, where $m = \min\{|N|, |O|\}$.

(C.2) The number of arcs starting from v_0 is $|O| - 1$.

(C.3) For all $v \in V$ such that there is a Q-path from v_0 to v, with $d(v_0, v) = r < m - 1$, the number of arcs starting from v is $|O| - r - 1$.

If a path of the tree for an object (say o_1) is $i_1 \overset{o_2}{\to} i_2 \overset{o_3}{\to} i_3 \overset{o_4}{\to} \cdots \overset{o_{|N|}}{\to} i_{|N|}$, then i_1

initially owns o_1; if i_1 is assigned o_2, then i_2 inherits o_1; if i_1 is assigned o_2, and i_2 is assigned o_3, then i_3 inherits o_1, etc.

Definition. An allocation rule φ is **non-bossy** if no agent can misreport her preference to make her allocation unchanged but change the allocation of some other agent, i.e., for all $P \in \mathcal{P}$ and all $P'_i \in \mathcal{P}_i$, $\varphi_i(P'_i, P_{-i}) = \varphi_i(P)$ implies $\varphi(P'_i, P_{-i}) = \varphi(P)$.

Theorem (Lemma 1 in Pápai 2000): An allocation rule is strategy-proof and non-bossy if and only if it is group strategy-proof.

Definition. An allocation rule φ is **reallocation-proof** if no pair of agent can jointly misreport their preferences and swap their assignments ex post to make at least one of them strictly better off, i.e., there exists no $i, j \in N$ such that for some $P \in \mathcal{P}$, $P'_i \in \mathcal{P}_i$ and $P'_j \in \mathcal{P}_j$ with $\varphi(P'_i, P_{-i}) = \varphi(P'_j, P_{-j}) = \varphi(P)$, we have $\varphi_i(P'_i, P'_j, P_{N\setminus\{i,j\}})P_i\varphi_i(P)$ and $\varphi_j(P'_i, P'_j, P_{N\setminus\{i,j\}})R_j\varphi_i(P)$.

The following theorem characterizes the hierarchical exchange rules.

Theorem (Theorem 1 in Pápai 2000): An allocation rule is Pareto efficient, group strategy-proof, and reallocation-proof if and only if it is a hierarchical exchange rule.

Velez (2014) studied the set of consistent hierarchical exchange rules. He defined CHE rules, showed the equivalence of CHE rules and consistent hierarchical exchange rules, and characterized the CHE rules.

Definition. **CHE-1 rules**. Let $\Pi_1 \equiv (\pi_l)_{l=1}^m$ be a partition of agents into sets of at most two agents and $|\pi_m| = 1$. Let $T_1 : O \times \Pi_1 \to N$ be a function such that for $T_i(O, \pi) = \pi$ for all $\pi \in \Pi_1$. For each $o \in O$, the CHE-1 rule satisfies:

C1-1: Between two agents who belong to two different components of Π_1, the one who belongs to the component with the smaller index inherits o first.

C1-2: Between two agents who belong to the same component of Π_1, say π_l, agent $T_1(o, \pi_l)$ inherits o first.

Definition. **CHE-2 rules**. Let $\Pi_2 \equiv (\pi_l)_{l=1}^m$ be a partition of agents into sets of at most two agents and $|\pi_m| = 2$. Let $T_2 : O \times \Pi_2 \setminus \{\pi_m\} \to N$ be a function such that $T_2(O, \pi) = \pi$ for all $\pi \in \Pi_2$. Let $Q : O^2 \setminus \{(o, o) : x \in O\} \to \pi_m$ be an onto function. For each $o \in O$, the CHE-2 rule satisfies:

C2-1 Between two agents who belong to two different components of Π_2 the one who belongs to the component with the smaller index, inherits o first.

C2-2 Between two agents who belong to the same component of $\Pi_2 \setminus \{\pi_m\}$, say π_l, agent $T_2(o, \pi_l)$ inherits o first.

C2-3 If $\pi_m \subseteq N$, then for each $o \in O$ and each leaf edge of $\gamma_{O\setminus\{o\}}$ (i.e. an end arc in a tree), say e, we have that $\Gamma_o(p(e)) = Q(o, \varsigma(e))$, where $p(e)$ is the vertex that connects e and the end vertex, and $\varsigma(e)$ is the labelling of e.

Definition. **CHE-3 rules**. Let $\Pi_3 \equiv (\pi_l)_{l=1}^m$ be a partition of agents into sets of at most two agents and the last three sets in the partition are singletons. Let $T_3 : O \times \Pi_3 \setminus \{\pi_{m-2}, \pi_{m-1}, \pi_m\} \to N$ be a function such that $T_3(O, \pi) = \pi$ for all $\pi \in \Pi_3 \setminus \{\pi_{m-2}, \pi_{m-1}, \pi_m\}$. Let $Q : O^2 \setminus \{(o, o) : x \in O\} \to \pi_m$ be an onto function. Let $\omega \in O$ and $Y \subseteq O \setminus \{\omega\}$ be non-empty. For each $o \in O$, the CHE-3 rule satisfies:

C3-1: Between agent in $\bigcup_{l=1}^{m-3} \pi_l$ who belong to two different components, the one who belongs to the component with the smaller index, inherits o first. Moreover, each agent in $\bigcup_{l=1}^{m-3} \pi_l$ inherits o before each agent in $\bigcup_{l=m-2}^{m} \pi_l$.

C3-2: Between two agents in $\bigcup_{l=1}^{m-3} \pi_l$, who belong to the same component of Π_3, say π_l, agent $T_3(o, \pi_l)$ inherits o first.

C3-3: For that last three agents, i_{m-2} inherits o before i_{m-1} if and only if $o \neq \omega$.

C3-4: i_{m-1} inherits o before i_m.

C3-5: If $\{i_{m-1}, i_m\} \subseteq N$, then for each $o \in O$ and each leaf edge of $\gamma_{O\setminus\{o\}}$, say e, we have that $\Gamma_o(p(e)) = i_m$ if and only if $o \in Y$ and $\varsigma(e) = \omega$.

Definition. A rule is a **CHE rule** if it belongs to one of the three classes defined above.

Theorem (Lemma 1 in Velez 2014): CHE rules are consistent.

Definition. A rule is **2-efficient** if it is efficient for each two-agent problem. A rule is **2-consistent** if it is consistent for each two-agent problem.

Theorem (Theorem 1 in Velez 2014): In a variable population and variable resource environment with at least four objects, a rule is 2-efficient, 2-consistent, and strategyproof if and only if it is a CHE rule.

Theorem (Propositin 1 in Velez 2014): In a variable population and variable resource environment with at least four objects, the following statements of a allocation rule φ are equivalent.

1. φ is a CHE rule.

2. φ is a consistent HE rule.

3. φ is a 2-consistent HE rule.

4. φ is a consistent TC rule.

5. φ is a 2-consistent TC rule.

6. φ is efficient, consistent, and strategy-proof.

Theorem (Theorem 2 in Velez 2014): In a variable population and variable resource environment with at least four objects, A rule is 2-efficient, 2-consistent, and conversely consistent if and only if it is a CHE-1 or a CHE-2 rule.

Tang and Zhang (2016) generalized the definition of individual rationality for the hierarchical exchange rules and provided a new characterization for these rules.

Definition. Given an inheritance structure $\{c_\sigma\}$, the induced set of feasible submatchings denoted by $\mathcal{F}_c$ is a subset of $\mathcal{S}$ such that:

1. $\varnothing \in \mathcal{F}_c$;

2. $\sigma \in \mathcal{F}_c$ if there exists $\sigma' \in \mathcal{F}_c$ with $\sigma' \subsetneq \sigma$ such that σ is minimal in $\{\tilde{\sigma} : \sigma' \subsetneq \tilde{\sigma},\ \mathit{and}\ O_{\tilde{\sigma}} \setminus O_{\sigma'} \subset c_{\sigma'}(N_{\tilde{\sigma}} \setminus N_{\sigma'})\}$.

Condition 2 implies that a submatching is feasible only if it is a minimal enlargement of a feasible submatching that satisfies persistency. The following theorem illustrates the structure of the set of feasible submatchings.

Theorem (Proposition 1 in Tang and Zhang 2016): Let $\sigma, \sigma' \in \mathcal{F}_c$.

1. If $\sigma \subset \sigma'$, then $O_{\sigma'} \setminus O_\sigma \subseteq c_\sigma(N_{\sigma'} \setminus N_\sigma)$.

2. If σ, σ' are both submatchings of a matching, then $\sigma \cap \sigma'$, $\sigma \cup \sigma' \in \mathcal{F}_c$.

Given a matching μ and an agent i, the above result implies that there exists a maximal feasible submatching of μ denoted by $\sigma_{\max}(\mu \setminus i)$ that does not include i.

Definition. The **contingent endowment** of agent i at a matching μ is defined as the set of objects that i would be endowed with a the contingency that the maximal submatching of μ that excludes i has been removed, i.e., $\omega(i|\mu) = \sigma_{\max}(\mu \setminus i)(i)$.

Definition. Agent i is **individually rational** at matching μ if $\mu_i R_i o$ for all $o \in \omega(i|\mu)$. An allocation rule is **individually rational** rational if it always finds an individually rational allocation.

Theorem (Theorem 1 in Tang and Zhang 2016): An allocation rule is Pareto efficient, individually rational, and strategy-proof if and only if it is a hierarchical exchange rule.

Tang and Zhang (2016) also generalized the definition of core defined by Shapley and Scarf (1974) to hierarchical exchange rules.

Definition. A matching μ is in the **core** of house allocation problem with a persistent ownership structure if there do not exist any coalition $B \in N$ and matching ν such that

1. $\nu_i \notin \omega(j|\nu)$, for all $i \in B$ and all $j \in N \setminus B$.

2. $\nu_i R_i \mu_i$ for all $i \in B$, and $\nu_j P_j \mu_j$ for some $j \in B$.

Theorem (Theorem 2 in Tang and Zhang 2016): For any house allocation problem with a persistent structure of ownership, the hierarchical exchange rule selects the unique core allocation.

2.4.3 Top trading cycles from individual endowments

A **housing market proble**m (Shapley and Scarf 1974) is a four-tuple (N, O, P, ω) with $|N| = |O|$, where matching ω is the initial endowment.

Definition. An allocation $\mu \in \mathcal{M}$ belongs to the **weak core** denoted by $\mathcal{C}(P)$ if there is no submarket that could have done strictly better for all its members, i.e., if there does not exist a set of agents $S \subseteq N$ and a matching $\mu' \in \mathcal{M}$ such that

1. For all $i \in S$, $\mu'_i P_i \mu_i$ and
2. $\{\mu'_i | i \in S\} = \{\omega_i | i \in S\}$.

Definition. An allocation $\mu \in \mathcal{M}$ belongs to the **strict core** denoted by $\mathcal{SC}(P)$ if there is no submarket that could make at least one agent in the group strictly better off, while the others in the group are not worse off, i.e., if there does not exist a set of agents $S \subseteq N$ and a matching $\mu' \in \mathcal{M}$ such that

1. For all $i \in S$, $\mu'_i R_i \mu_i$, and for some $j \in S$, $\mu'_j P_j \mu_j$.
2. $\{\mu'_i | i \in S\} = \{\omega_i | i \in S\}$.

Example. The following example shown in table 2.1 illustrates that the weak core and strict core are not equivalent even under strict preferences.

P_1	P_2	P_3
o_3	$\boxed{o_1}$	o_2
$\boxed{o_2}$	o_2	$\boxed{o_3}$
o_1	o_3	o_1

Table 2.1: Weak core $\neq$ strict core.

When $\omega = (o_1, o_2, o_3)$, the boxed allocation is in weak core but not in the strict core.

Shapley and Scarf (1974) utilized Scarf's theorem to show that strict core always exists. They also provided a constructive way by using Gale's top trading cycles algorithm to find an allocation in the strict core. Top trading cycles algorithm from individual endowments is a special case of the hierarchical exchange rules in which each agent is initially endowed with exact one object. A **top trading cycles algorithm from individual endowments** consists of a serial of round. At each round t, each remaining object points to its owner, and each remaining agent points to her favorite object among the remaining ones. A set of agents forms a cycle whenever their favorite objects among the remaining ones are the objects owned by the agents in this set. At least one cycle exists. Assign each agent in a cycle the object she points to and remove the assigned agents along with their assigned objects. The algorithm stops when all agents are assigned.

Theorem (Shapley and Scarf 1974): The strict core exists when preferences are strict. The top trading cycles algorithm finds an allocation in the strict core.

It is clear that the strict core is a subset of the weak core. The weak core always exists, even under weak preferences. However, Shapley and Scarf (1974) also illustrated that strict core may not exist when preferences are not strict, as shown in the following example.

Example. Let $\omega = (o_1, o_2, o_3)$. The preferences are given in table 2.2. For all possible allocations, none of them is in the strict core defined by weak dominance.

P_1	P_2	P_3
o_2	o_1, o_3	o_2
o_1, o_3	o_2	o_1, o_3

Table 2.2: Strict core many not exist

Sönmez (1999) studied a general class of allocation problems that includes housing markets, marriage problems, roommate problems, networks, etc. He showed that the strict core, if it exists, is essentially single-valued.

Theorem (Sönmez 1999): If there exists an individually rational, Pareto efficient, and strategy-proof allocation rule ψ, then we have:

1. For all $R \in \mathcal{R}$, for all $i \in N$, for all $\mu, \mu' \in \mathcal{SC}(R)$, i is indifferent between μ_i and μ_i'.

2. For all $R \in \mathcal{R}$ with $\mathcal{SC}(R) \neq \varnothing$, $\psi(R) \in \mathcal{SC}(R)$.

Roth and Postlewaite (1977) built the relationship between the strict core allocation and competitive allocation.

Definition. The budget set of agent i is defined as $B_i(p) = \{\omega_j | p_j \leq p_i\}$. A matching μ is a **competitive allocation** for a housing market problem if there exists a price vector p, for all $i \in N$, $\mu_i \in \arg\max R_i$, subject to $B_i(p)$.

Theorem (Roth and Postlewaite 1977): In a housing market model with weak preferences, we have:

1. There exists a competitive allocation.

2. If strict core $\mathcal{SC}(R)$ exists, then for all $\mu \in \mathcal{SC}(R)$, μ is a competitive allocation.

When preferences are strict, then:

3. If μ is a competitive allocation, then μ belongs to the strict core.

4. The set of strict core equals the set of competitive allocation and top trading cycles algorithm finds the unique matching in the strict core.

Moreover, the strict core mechanism satisfies some remarkable incentive properties.

Theorem (Roth 1982a): In the top trading cycles procedure, it is a weakly dominant strategy for each player to reveal his true preferences. That is, the strict core mechanism is strategy-proof.

Theorem (Bird 1984): In the top trading cycles procedure, no group of agent can jointly misreport their preferences to make at least one agent in the group strictly better off, and other agents in the group are not worse off. That is, the strict core mechanism is group strategy-proof.

Takamiya (2001) proved the equivalence of group strategy-proofness and Maskin monotonicity.

Definition. Fix an allocation rule φ, a preference profile $P' \in \mathcal{P}$ is a **monotonic transformation** of $P \in \mathcal{P}$ if $\varphi_i(P)P_io$ implies $\varphi_i(P)P'_io$ for all $i \in N$ and all $o \in O$, i.e., for each agent, the set of objects better than the original profile allocation weakly shrinks from the original profile to its transformed profile. An allocation rule φ is **Maskin monotonic** if $\varphi(P') = \varphi(P)$ whenever P' is a monotonic transformation of P.

Theorem (Takamiya 2001): An allocation rule is group strategy-proof if and only if it is Maskin monotonic.

Ma (1994) provided a characterization of top trading cycles from individual endowments.

Theorem (Ma 1994): An allocation rule for the housing market model is Pareto efficient, individually rational, and strategy-proof if and only if it is a top trading cycles rule from individual endowments.

2.4.4 Priority rules

Ergin (2002) studied a indivisible objects model with a finite set of object types and each type has a finite quota of objects. One interesting example is assigning students to public schools. Each schools has a limited number of seats. Students has strict preference over schools, but are indifferent between seats from the same school. In the following section, we assume quota of each object type (or the number of seats of each school) is one, which means this is a house allocation problem. A key component of in Ergin's model is a vector of linear orders known as a priority structure denoted by $\succ = (\succ_o)_{o \in O}$. The priority structure could be viewed as a fixed endowment hierarchical inheritance ownership. The assignment is computed via deferred acceptance algorithm invented by Gale and Shapley.

The **deferred acceptance algorithm** associated with a priority structure $\succ$ denoted by $DA^{\succ}$ consists of a serial of steps as follows:

- At step 1: each agent applies to her favorite object. Each object temporarily accept the applicant with the highest priority.

- In general, at step k: each agent who is rejected in the previous step applies to favorite object among the ones that she has not applied for. Each object temporarily accept the applicant with the highest priority among the new applicants and the applicant that she temporarily accept at step $k-1$, and reject others.

- The algorithm stops when no agent applies.

Unlike the hierarchical exchange rule, the outcome of the deferred acceptance algorithm may not be Pareto efficient. Ergin (2002) defined a new property of a priority structure that is called acyclicity and showed that it is sufficient and necessary for Pareto efficiency, group strategy-proofness, and consistency separately. Acyclicity is defined for a model with object types by a cycle condition and a scarcity condition. In a house allocation problem that each type has one object, the scarcity condition is always satisfied.

Definition. Let $\succ$ be a priority structure. A **cycle** consists of distinct $o, o' \in O$ and $i, j, k \in N$ such that $i \succ_o j \succ_o k \succ_{o'} i$. A priority structure is **acyclical** if it has no cycles.

Theorem (Ergin 2002): The following are equivalent.

1. $DA^{\succ}$ is Pareto efficient.
2. $DA^{\succ}$ is group strategy-proof.
3. $DA^{\succ}$ is consistent.
4. $\succ$ is acyclical.

When all objects use the same priority ordering of agents, then the deferred acceptance algorithm is just a serial dictatorship. In the language of the hierarchical exchange, a **serial dictatorship** is a special case of the hierarchical exchange rules in which one agent owns all unmatched objects at all submatchings.

Definition. Let $\pi : O \to O$ be a bijection, i.e., a permutation (or a change of names) of objects. If μ is an allocation, $\pi\mu$ is defined by $(\pi\mu)(i) = \pi(\mu(i))$ for all $i \in N$. A preference relation $\pi(R_i)$ is defined by $a\pi(R_i)b \Longleftrightarrow \pi(a)R_i\pi(b)$. An allocation rule φ is **neutral** if $\varphi(\pi(R)) = \pi(\varphi(R))$.

Theorem (Svensson 1999): φ is strategy-proof, non-bossy, and neutral if and only if φ is a serially dictatorship.

3. BALANCED HOUSE ALLOCATION

In this chapter, we define a new property that we call "balancedness." It is a fairness property in the sense that if all possible preference profiles are equally likely to happen, an allocation rule is balanced if it assigns all agents to their best choices with equal probabilities, it assigns all agents to their second best choices with equal probabilities, and so forth. We provide new characterizations for the top trading cycles from individual endowments. These rules are the only rules that are Pareto efficient, group strategy-proof, reallocation-proof, and balanced (theorem 3.1). When the number of objects is at least four or just two, they are the only rules that are Pareto efficient, group strategy-proof, and balanced; when there are just three objects, an allocation rule is Pareto efficient, group strategy-proof, and balanced if and only if it is a top trading cycles rule from individual endowments, or a trading cycles rule with three brokers (theorem 3.2).

Previous results imply that all random allocation rules induced by some Pareto efficient and group strategy-proof deterministic allocation rules are equally ex ante fair (Abdulkadiroğlu and Sönmez 1998, Pathak and Sethuraman 2011, Lee and Sethuraman 2011, Bade 2014) because before knowing the realization of the lottery, the distribution over outcomes of a random trading cycles rule is the same as the distribution over outcomes of the random serial dictatorship. However, it is clear that after knowing the realization of the lottery (i.e., knowing which deterministic allocation rule would be used to find the assignment) but before knowing agents' true preferences, not all Pareto efficient and group strategy-proof rules assign all agents to their best choices, to their second best choices, etc, with equal probabilities. Specifically, in a serial dictatorship, the agent with the highest priority is always guaranteed to her best choice, while the agent with the lowest priority has a lower chance to receive her best choice. A Pareto efficient deterministic allocation rule cannot be ex post fair because if all agents have the same preferences, they cannot be treated equally. Balancedness is an interim fairness property since chances that agents receive their best choices, their second best choices, etc., are the expectations calculated after knowing the realization of the lottery, but before knowing agents' preferences.

The intuition of the fact that a top trading cycles rule from individual endowments satisfies balancedness is symmetry: if each agent initially endows with one object, all of them would be treated equally if all possible preference profiles are taken into account with equal weights. We construct a bijection τ from the set of preference profiles to itself, and show that given the top trading cycles from individual endowments, if agent i receives her kth best choice and agent j receives her lth best choice under R, then agent i receives her lth best choice and agent j receives her kth best choice under $\tau(R)$.

The proof of the only if part of theorem 3.1 relies on the persistence property of the hierarchical exchange rules, i.e., once an agent owns an object, she retains it until she is assigned. To prove the only if part of theorem 3.2, we also have to use our proposition 3.1: given a trading cycles rule defined by Pycia and Ünver (2016a), if we change the rule by depriving the control right of one agent and give it to another agent, then some agents' gains mean other agents' losses because the sums of the probabilities that agents receive their best choices are the same. We may also relax the definition of balancedness to the one where all agents have equal chances to their worst choices. Theorem 3.1* and theorem 3.2* state that the top trading cycles from individual endowments can be characterized by the efficiency and the incentive properties and the relaxed version of balancedness.

The remaining chapter is organized as follows: section 3.1 introduces our model and defines the top trading cycles, section 3.2 presents the main result, and section 3.3 concludes.

3.1 The model

Let $N \equiv \{1, 2, \cdots, n\}$ be the set of agents, and $O \equiv \{o_1, o_2, \cdots, o_n\}$ be the set of objects. We assume the number of agents equals the number of objects. Each agent has a strict preference relation over the set of objects. Let $\mathcal{P}$ be the set of strict preference relations on O, and let $\mathcal{P}^N$ be the set of preferences profiles. Note that $|\mathcal{P}| = n!$, and $|\mathcal{P}^N| = (n!)^n$. We use $R \equiv (R_i)_{i \in N} \in \mathcal{P}^N$ to denote a generic preference profile. R_i and P_i denote the weak and strong preference relation for agent i, respectively. For any group of agents $S \subseteq N$, denote the preferences of the agents in S by R_S. Suppose each object is acceptable to all agents and each agent has no use for more than one object. Each object can only be assigned to one agent and objects have no preferences. An assignment μ is a matching such that each agent receives a distinct object. Let μ_i be the object assigned to agent i under the μ. Let $\mathcal{M}$ be the set of all possible matchings. A (deterministic) allocation rule is a mapping from $\mathcal{P}$ to $\mathcal{M}$. We use f to denote a generic allocation rule. $f_i(R)$ is the assignment for agent i under R.

Examples of these problems include allocating and exchanging transplant organs, like kidneys (Roth, Sönmez and Ünver, 2004), assigning students to university apartments (Abdulkadiroğlu and Sönmez, 1999), etc. When Shapley and Scarf (1974) initiate this model, they use an example of houses and traders. Since then, such problems are now known as house allocation problems. The allocation mechanisms in these problems are based on the top trading cycles algorithm invented by David Gale and introduced by Shapley and Scarf (1974). Before describing the algorithm, we first introduce some properties used in our main result.

3.1.1 Axioms

A matching μ is **Pareto efficient** if there does not exist a matching $\mu' \in \mathcal{M}$, such that some agent is strictly better off while others are not worse off. That is, $\forall \mu' \in \mathcal{M}$, $\mu'_i P_i \mu_i$ for some $i \in N$ implies $\mu_j P_j \mu'_j$ for some $j \in N$. An allocation rule f is **Pareto efficient** if it always selects a Pareto efficient matching, i.e., $\forall R \in \mathcal{P}^N$, $f(R)$ is Pareto efficient matching.

An allocation rule f is **strategy-proof** if truthfully revealing her preference is a weakly dominant strategy for each agent, i.e., $\forall i$, $\forall R_i, R'_i \in \mathcal{P}$, $f_i(R) R_i f_i(R'_i, R_{N\setminus\{i\}})$. A stronger version of strategy-proofness is **group strategy-proof** which means no group of agents can be weakly better off by misreporting their preferences, i.e., $\forall$ $R \in \mathcal{P}^N$, there exists no $S \subseteq N$ and $R'_S \in \mathcal{P}^S$, such that $f_i(R'_S, R_{N\setminus S}) R_i f_i(R)$ for all $i \in S$, and $f_j(R'_S, R_{N\setminus S}) P_j\ f_j(R)$ for some $j \in S$.

An allocation rule φ is **reallocation-proof** if no two agents can jointly misreport their preferences and swap their assignments ex post to make at least one of them strictly better off, i.e., there exist no $i, j \in N$ such that for some $R \in \mathcal{P}^N$, $R'_i \in \mathcal{P}$ and $R'_j \in \mathcal{P}$ with $\varphi(R'_i, R_{-i}) = \varphi(R'_j, R_{-j}) = \varphi(R)$, we have $\varphi_i(R'_i, R'_j, R_{N\setminus\{i,j\}}) P_i \varphi_i(P)$ and $\varphi_j(R'_i, R'_j, R_{N\setminus\{i,j\}}) R_j \varphi_i(R)$.

3.1.2 Top trading cycles algorithm

Shapley and Scarf (1974) initiated a model where each agent owns one house and can trade it for a better one in the market. They showed that the strict core (in the sense that there is no submarket such that all agents in a group can be weakly better off and some agent can be strictly better off by reallocating their initial endowments) is nonempty, and the top trading cycles (TTC) algorithm finds an allocation in the strict core. To distinguish the TTC used in Shapley and Scarf's housing market model from the TTC used in Pápai (2000)'s hierarchical exchange rules where some agents may initially own more than one object, we refer to the former as the TTC from individual endowments. It consists of a list of steps described below.

- At the beginning of step 1: each agent owns exactly one object.

- At step 1: each agent points to the agent who owns her favorite object. A list of agents $\{i_1, i_2, \cdots, i_k\}$ forms a **cycle** if agent i_l points to agent i_{l+1} for all $l = 1, 2, \cdots, k$, and $i_{k+1} = i_1$. An agent points to herself also forms a cycle. Because each agent points (as all objects are acceptable) and the number of agents is finite, at least one cycle exists. And no two cycles intersect since preferences are strict. Remove all cycles from the market and assign each agent in a cycle her best choice.

- In general, at step k: each remaining agent points to the agent who owns her best choice among the remaining objects. Again, there exists one cycle. Remove all cycles from the market and assign each agent in a cycle her best choice among the remaining objects.

- The algorithm stops when all agents are removed from the market. Note that the algorithm stops in no more than n steps since at least one agent is removed at each step.

Roth and Postlewaite (1977) proved that the TTC from individual endowments finds the unique matching in the strict core. Roth (1982a) showed that TTC from individual endowments is strategy-proof. Bird (1984) showed that it is group strategy-proof. Ma (1994) characterized the strict core mechanism by Pareto efficiency, individual rationality (one's assignment is at least as good as her endowment), and strategy-proofness.

3.2 The result

Let f be a deterministic allocation rule. For any $i \in N$, define

$$\#(1,i) \equiv \left|\left\{R \in \mathcal{P}^N : f_i(R) \text{ is the best in } R_i\right\}\right|;$$
$$\cdots$$
$$\#(k,i) \equiv \left|\left\{R \in \mathcal{P}^N : f_i(R) \text{ is the } k\text{th best in } R_i\right\}\right|;$$
$$\cdots$$
$$\#(n,i) \equiv \left|\left\{R \in \mathcal{P}^N : f_i(R) \text{ is the worst in } R_i\right\}\right|.$$

Given an allocation rule f, we define $\#(k,i)$ as the number of preference profiles that agent i receives her kth best choice for all $k = 1,\ 2,\ \cdots,\ n$, and all $i \in N$.

Example 3.1: Consider a serial dictatorship with agent t has the tth highest priority for all $t = 1, 2, \cdots, n$. Then, for all $(n!)^n$ possible preference profiles, agent 1 always receives her best choice, so $\#(1,1) = (n!)^n$. Agent 2 has a chance of $\frac{n-1}{n}$ to receive her best choice and a chance of $\frac{1}{n}$ to receive her second best choice, so $\#(1,2) = \frac{n-1}{n} \cdot (n!)^n$ and $\#(2,2) = \frac{1}{n} \cdot (n!)^n$. In general, the chance for agent t to receive her kth choice ($t \leq k$) is given by $\frac{P_{t-1}^{k-1}(n-t+1)(n-k)!}{n!}$, so $\#(k,t) = \frac{(n-t+1)(t-1)!(n-k)!}{(t-k)!n!} \cdot (n!)^n$.

Definition: A deterministic allocation rule f is **balanced** if for any two agents i, $j \in N$, and for all $k = 1, 2, \cdots, n$, we have

$$\#(k,i) = \#(k,j).$$

After knowing the realization of the lottery which determines who controls what, but before agents submit their preferences, the mechanism designer does not know agents' true preferences. If each possible preference profile happens with equal probability, then a rule is balanced if all agents have equal chances to their best choices, have equal chances to their second best choices, and so forth. So we say balancedness is an interim fairness property. Although previous results show that all random trading cycles rules (including random hierarchical exchange rules) are equally ex ante fair in the sense that their distributions over outcomes are exactly the same (Abdulkadiroğlu and Sönmez 1998, Pathak and Sethuraman 2011, Lee and Sethuraman 2011, Bade 2014), our theorem 3.1 shows that any hierarchical exchange rule satisfying balancedness is a TTC from individual endowments, and our theorem 3.2 shows that when there are at least four agents or just two agents, any trading cycles rule satisfying balancedness is a TTC from individual endowments; when there are three agents, a rule is Pareto efficient, group strategy-proof, and balanced if and only if it is a TTC from individual endowments, or a trading cycles rule with three brokers (see section 3.2.2 for its definition). Therefore, our theorem 3.1 and theorem 3.2 provide new characterizations of the TTC from individual endowments.

Theorem 3.1: A deterministic allocation rule is Pareto efficient, group strategy-proof, reallocation-proof, and balanced if and only if it is a TTC from individual endowments.

If we drop reallocation-proofness and restrict the problems to the ones with at least four agents or just two agents, the result of theorem 3.1 still holds, as shown in the following theorem 3.2.

Theorem 3.2: When $|N| = |O| \neq 3$, a deterministic allocation rule is Pareto efficient, group strategy-proof, and balanced if and only if it is a TTC from individual endowments. When $|N| = |O| = 3$, a deterministic allocation rule is Pareto efficient, group strategy-proof, and balanced if and only if it is a TTC from individual endowments or a TC rule with three brokers.

Papai (2000) characterized the hierarchical exchange rules by Pareto efficiency, group strategy-proofness, and reallocation-proofness. Since a TTC from individual endowments belongs to set of the hierarchical exchange rules, to prove theorem 3.1, we just show that among all hierarchical exchange rules, only the rules of TTC from individual endowments are balanced. Similarly, to prove theorem 3.2, we use Pycia and Ünver's (2016a) characterization of trading cycles rules by Pareto efficiency and group strategy-proofness. Trading cycles rules are generalized from Papai's hierarchical exchange rules. In the following subsections, we introduce hierarchical exchange rules first, then introduce trading cycles rules. And finally, we prove theorem 3.2. The proof of theorem 3.1 is similar to part of the proof of theorem 3.2. We also introduce theorem 3.1* and theorem 3.2* in this section.

3.2.1 Hierarchical exchange rules

Pápai (2000) generalized Shapley and Scarf's (1974) model to the case where some agents may initially owns more than one object and each object is initial owned by some agent, but still kept the assumption of unit demand. She defined the hierarchical exchange (HE) rules and characterized them by Pareto efficiency, group strategy-proofness, and reallocation-proofness.

Like in Shapley and Scarf's model, the TTC algorithm finds the allocation for a HE rule. As each agent demands one object but some agent may initially owns more than one object, a HE rule has to specify how the objects are inherited. During the TTC procedure, who inherites what depends only on the assignments that have formed in previous cycles. More specifically, the inheritance rule of a HE rule is defined by an inheritance forest. In the inheritance forest, there is an inheritance tree for each object. Each tree defines who initially owns and who potentially inherits the object under different allocation scenarios. An inheritance tree consists of a set of vertices and a set of arcs. Each vertex is connected to other vertex(es) via arc(s). Each arc connects two distinct vertices. In each tree, there is a unique vertex such that there is no arc from other vertices to the vertex. We call this vertex level 1 vertex. In our model with n agents and n objects, there are $n-1$ arcs pointing from the level 1 vertex to $n-1$ vertices which we call level 2 vertices. For each level 2 vertex, there $n-2$ arcs pointing from the level 2 vertex to $n-2$ vertices which we call level 3 vertices. In general, for each level k vertex $(1 \leq k \leq n-1)$, there are $n-k$ arcs pointing from the level k vertex to $n-k$ level $k+1$ vertices. There is no arc pointing from a level n vertex.

Each vertex is labeled by an agent and each arc is labeled by an object. The labeling of each inheritance tree (say the tree for object o) should satisfy the following conditions to make it well-defined:

- A level 1 vertex can be labeled by any agent (say i). The labeled agent initially owns the object.

- Each arc from vertex i should be labeled by distinct objects other than o; so there are $n-1$ arcs pointing to $n-1$ level 2 vertices.

- There is a unique path from a level 1 vertex to a level n vertex. Each vertex on a path should be labeled by a distinct agent, and each arc on a path should be labeled by a distinct object.

If a path of the tree for an object (say o_1) is $i_1 \overset{o_2}{\to} i_2 \overset{o_3}{\to} i_3 \overset{o_4}{\to} \cdots \overset{o_n}{\to} i_n$, then i_1 initially owns o_1; if i_1 is assigned o_2, then i_2 inherits o_1; if i_1 is assigned o_2, and i_2 is assigned o_3, then i_3 inherits o_1, etc.

Example 3.2: Consider a HE rule with three agents 1, 2, 3, and three objects a, b, c shown in figure 3.1.

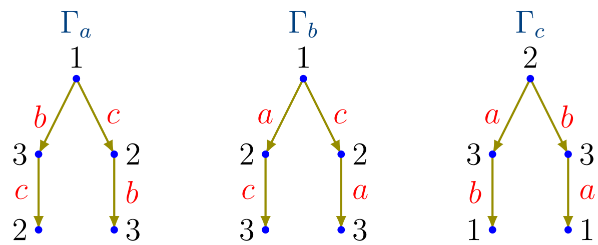

Figure 3.1: A hierarchical exchange rule

The inheritance forest in figure 3.1 implies that agent 1 initially owns objects a and b, agent 2 initially owns c, and agent 3 initially owns nothing. For tree Γ_a, the path $1 \xrightarrow{b} 3 \xrightarrow{c} 2$ means if 1 is assigned b, then 3 inherits a; if 1 is assigned b, and 3 is assigned c, then 2 inherits a. Similarly, for tree Γ_a, the path $1 \xrightarrow{c} 2 \xrightarrow{b} 3$ means if 1 is assigned c, then 2 inherits a; if 1 is assigned c, and 2 is assigned b, then 3 inherits a.

Given a HE rule, the TTC algorithm is applied to find the allocation:

- At the beginning of step 1: each object is owned by an agent. Some agent may initially owns more than one object and some agent may initially owns nothing.

- At step 1: each agent who owns some object(s) points to the agent who owns her best choice. Like the TTC from individual endowments, at least one cycle exists. Remove all cycles from the market and assign each agent in a cycle her best choice. The remaining agents keep their initial endowments, and they inherit the objects left by the assigned agents according to the inheritance forest.

- In general, at the beginning of step k: each remaining object is owned by some agent. Each remaining agent who owns some object(s) points to the agent who owns her favorite object among the remaining ones. Again, at least one cycle exists. Remove all cycles from the market and assign each agent in a cycle her best choice among the remaining objects. Each remaining agent keeps her endowment that she owns at the beginning of step k. Remaining agents also inherit the object(s) left by the assigned agent(s) at step k according to the inheritance forest.

- The algorithm stops when all agents are removed from the market.

A important feature of the HE rules is **persistence** property, i.e., once an agent owns an object (initially endowed or inherited), she retains it until she is assigned. The following example 3.3 illustrates how a HE rule works.

Example 3.3: Suppose the HE rule is determined by the inheritance forest in example 3.2, and the preference profile is given in table 3.1.

R_1	R_2	R_3
b	a	b
a	c	c
c	b	a

Table 3.1: The preferences for a HE rule

Table 3.1 shows agents' preferences over objects (from top to bottom). At step 1, since agent 1 initially owns a and b and her top choice is b, so agent 1 points to herself; agent 2 initially owns c and her top choice is a, so she points to agent 1. There is only one cycle at this step: agent 1 points to herself. So agent 1 is allocated with object b and we remove agent 1 along with her assignment from the market. At the beginning of step 2, agent 2 still owns object c. Object a is left by agent 1. The inheritance tree of object a indicates that given agent 1 receives b, the one who inherits a is agent 3. So at the beginning of step 2, agent 2 owns object c and agent 3 owns object a. In the reduced market, agent 2's best choice among the remaining ones is c, and agent 3's best choice among the remaining ones is a. So agent 2 and agent 3 point to each other and swap their endowments. Figure 3.2 shows the procedure.

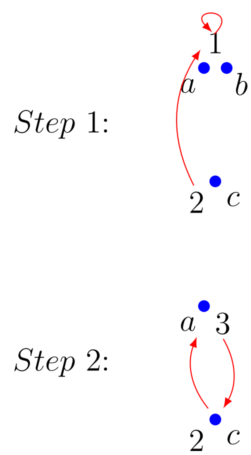

Figure 3.2: The procedure of a HE rule

If each tree in the inheritance forest has a distinct level 1 vertex labeling, then the HE rule is a TTC from individual endowments. Other special cases of HE rules include serial dictatorships and fixed endowment HE rules described below.

3.2.1.1 Serial dictatorships

In a HE rule, if the same level vertices are labeled by the same agent and different levels vertices are labeled by distinct agents for all trees, we call such rule a **serial dictatorship** (Svensson 1999). In a serial dictatorship, the level 1 agent initially owns all objects, the level 2 agent inherits all objects when the level 1 agent chooses, the level 3 agent inherits all objects when the level 2 agent chooses, and so forth.

An agent labeled at a lower level vertex has a higher priority for all objects than an agent labeled at a higher level vertex. Figure 3.3 shows an example of a serial dictatorship in which agent 1 has the highest priority, agent 2 has the second highest priority, and agent 3 has the lowest priority.

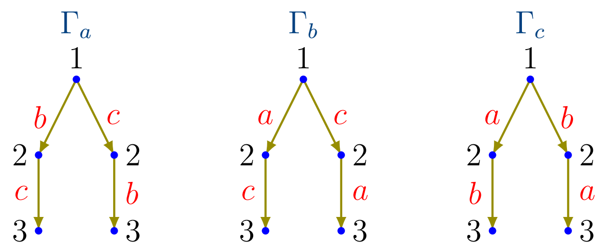

Figure 3.3: A serial dictatorship

The **random serial dictatorship** (or **single lottery mechanism**) proposed by Abdulkadiroğlu and Sönmez (1998) is a random mechanism aims at ex ante fairness. The agent are ordered randomly with any one of the $n!$ orderings being equally likely. For any given priority ordering, the corresponding serial dictatorship finds the allocation. They also define the **TTC from random endowments** as a random mechanism such that each agent random endows with one object with $n!$ possibilities being equally likely. For a given realization of endowments, the TTC from individual endowments is used to find the allocation. Abdulkadiroğlu and Sönmez (1998) showed that the random serial dictatorship is equivalent to the TTC from random endowments as the distributions of assignments are the same for both mechanisms.

3.2.1.2 Fixed endowment HE rules

The inheritance forest of a **fixed endowment HE rule** can be represented by the trees such that different trees may be labeled differently, but the same level vertices in each tree are labeled with the same agent. The following figure 3.4 shows a fixed endowment HE rule.

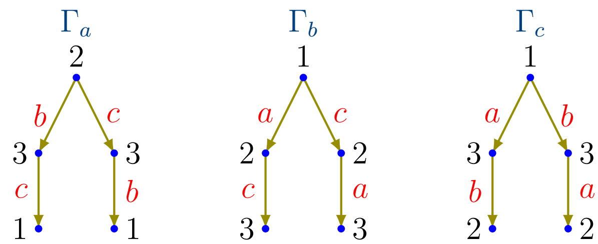

Figure 3.4: A fixed endowment HE rule

The fixed endowment HE rules can be used to define multiple lottery mechanism (Pathak and Sethuraman 2011). In a **multiple lottery mechanism**, each object randomly uses an ordering of agents among $n!$ possibilities, therefore, $(n!)^n$ possible fixed endowment inheritance forests are equally likely to be selected. Given a fixed endowment HE rule, the corresponding allocation is selected as the outcome.

A serial dictatorship is a special case of a fixed endowment HE rule in which the labelings of all trees in the inheritance forest are the same. We can also write a TTC from individual endowments as a fixed endowment HE rule since the level k vertices for $k \geq 2$ will never be reached. Pathak and Sethuraman (2011) generalized Abdulkadiroğlu and Sönmez's (1998) result by showing that the single lottery mechanism (random serial dictatorship) is equivalent to the multiple lottery mechanism as their ex ante distributions over assignments are the same. Given a HE rule, if we view the labeling of vertices as "roles" in the mechanism, then we can define the corresponding **random HE rule** by assigning agents to the "roles" via a uniform lottery over $n!$ possibilities. Lee and Sethuraman (2011) showed that the single lottery mechanism is equivalent to any random HE rule.

3.2.2 Trading cycles rules

Pycia and Ünver (2016a) constructed trading cycles (TC) rules. The TC rules subsume the HE rules as special cases and they are characterized by Pareto efficiency and group strategy-proofness. The major difference between the HE rules and the TC rules is that one agent in a TC rule may not own but, instead, brokers an object, or all the three remaining objects could be brokered by three agents. During the procedure of a TC rule, the broker may not be allowed to point to herself. A TC rule consists of a serial of steps described below.

- In general, at the beginning of step k: each object is either owned or brokered by an agent. There is at most one brokered object, or all three remaining objects are brokered. If an agent brokers an object, then the agent owns nothing.
- At step k, there are two cases.
  - Case 1: there is at most one broker.

    Each owner points to the agent who owns or brokers her best choice. If there exists a broker, then she points to her favorite object owned by other agents. Similar to the TTC from individual endowments, at least one cycle exists. Assign each owner in a cycle her best choice. If a broker is part of a cycle, then the broker is assigned her favorite object owned by other agents. All assigned agents and objects are removed from the market. Each remaining owner keeps her initial endowment, and she may inherit the objects left by the assigned agents according to the inheritance rule. If the broker is still in the market and there exists at least one remaining owner, then the broker retains her brokerage right. If the broker is the only remaining agent, then the brokered object is assigned to her. If there are three remaining agents and none of them owns an object at the beginning of step k, then each one of the three agents or at most one agent brokers an object at the beginning of step $k+1$.

– Case 2: there are three brokers and each brokers one object.

  Each broker points to the agent who brokers her best choice. If these three agents forms a cycle, then assign each broker her best choice. If the cycle of three brokers does not exist, then there exists a cycle such that broker i points to a brokered object and there is another broker points to this brokered object. Then we force broker i points to the agent who brokers her next choice. If now three brokers form a cycle, then assign each broker the object brokered by the agent that she points to. Otherwise, there exists a non-three-agent cycle such that broker j points to a brokered object and there is another broker points to this brokered object. Then we force broker j points to the agent who brokers her next choice. If now three brokers form a cycle, then assign each broker the object brokered by the agent that she points to. Otherwise, iterate the process until three brokers form a cycle.

- The algorithm stops when all agents are assigned.

Note that in a model with at least four agents, there exists at most one broker at the beginning of step 1. As the procedure goes on, we may have three brokers and each of them brokers one object. If there are no brokers throughout the procedure, then the TC rule is a HE rule. Similar to the HE rules, TC rules also satisfy **persistence property**, i.e., once an owner owns an object, she retains it until she receives her assignment, and a broker also keeps her brokerage right until she is assigned. In our model that the number of agents is equal to the number of objects, a broker never has a chance to inherit objects. The following example 3.4 illustrates how a TC rule works.

Example 3.4: Suppose agent 1 initially brokers object a, and agent 2 and 3 initially own objects b and c, respectively. Preference profile is given in table 3.2. Then at step 1, agents 1 points to agent 2, agents 2 and 3 both point to agent 1. So agents 1 and 2 are assigned objects b and a, respectively. At step 2, agent 3 points to herself and is assigned object c.

R_1	R_2	R_3
a	a	a
b	b	b
c	c	c

Table 3.2: The preferences for a TC rule

Fix a TC rule and define the corresponding **random TC rule** by assigning agents to the "roles" in the TC rule via a uniform lottery. Bade (2014) showed that given a preference profile, the distribution over outcomes that arises from a random TC rule is the same as the distribution over outcomes that arises from the random serial dictatorship.

The random serial dictatorship is considered to be **ex ante fair** as all agents have the same chances on the priority list before knowing the realization of the lottery. Therefore, all the random TC rules are equally ex ante fair because their distributions over outcomes are the same. But after knowing its realization, a single lottery ordering seems to be unfair since the agent at the top of the priority list always receives her best choice, while the agent at the end of the list receives her best choice only if her best choice is not the best choice of any other agent. Preferences are private information. Before knowing the preferences, and suppose all possible preference profiles are equally likely to occur, if a deterministic mechanism assigns all agents to their kth best choices with equal probabilities for all $k = 1, 2, \cdots, n$, we say the mechanism is **interim fair**. We show that among all TC rules, an allocation rule that satisfies our new property balancedness if and only if it is a TTC from individual endowments if there are at least three agents or just two agents. Our theorem 3.2 characterizes the TTC from individual endowments by Pareto efficiency, group strategy-proofness, and balancedness when the number of agents is not three.

The fact that a TTC from individual endowments is balanced follows from symmetry. To prove it, we construct a bijection τ from $\mathcal{P}^N$ to $\mathcal{P}^N$ such that for any preference profile $R \in \mathcal{P}^N$, if agent i receives the kth best choice in R_i and agent j receives the lth best choice in R_j, then for the preference profile $R' = \tau(R)$, agent i receives the lth best choice in R'_i and agent j receives the kth best choice in R'_j. We use the following example 3.5 to illustrate the idea.

Example 3.5: In a three agents and three goods economy, the initial endowment is given by $\omega = (o_1, o_2, o_3)$. Under the original preference profile R given in table 3.3, the assignment is shown in boldface where agents 1, 2, and 3 receive their best, second best, and best choices, respectively. We want to construct a new preference profile $R' = \tau(R)$ under which agent 2 and 3 receive their best and second best choices, respectively, while agent 1 still receives her best choice. To this end, we first switch the preferences of R_2 and R_3 to get $\bar{R} = (R_1, R_3, R_2)$. Then for each $\bar{R}_1$, $\bar{R}_2$, and $\bar{R}_3$, we swap the position of o_2 and o_3 to get R' shown in table 3.4. Note that agent 1 is assigned the same ranked object under R' as under R, while the rankings of agents 2 and 3's assignments under R' are switched when compared with their assignments under R.

R_1	R_2	R_3
$\mathbf{o}_2$	o_3	$\mathbf{o}_3$
o_1	$\mathbf{o}_1$	o_2
o_3	o_2	o_1

Table 3.3: Original preference profile R

R'_1	R'_2	R'_3
$\mathbf{o}_3$	$\mathbf{o}_2$	o_2
o_1	o_3	$\mathbf{o}_1$
o_2	o_1	o_3

Table 3.4: Transformed preference profile R'

To prove the only if part of theorem 3.2, we need the following proposition 3.1.

Proposition 3.1: Let f and f' be two different TC rules. Define $\#'(k,i)$ as the number of preference profiles that agent i receive her kth best choice under f', i.e., $\#'(k,i) \equiv \left|\left\{R \in \mathcal{P}^N : f'_i(R) \text{ is the } k\text{th best in } R_i\right\}\right|$. Then

$$\sum_{i \in N} \#(k,i) = \sum_{i \in N} \#'(k,i), \ \forall k = 1, 2, \cdots, n.$$

Proposition 3.1 follows from symmetry and the result of Bade (2014), i.e., random TC rules are equivalent since their distributions over outcomes are the same. We use the following example 3.6 to verify proposition 3.1, and then provide a proof.

Example 3.6: Consider a three agents and three objects economy. Let f be a TC rule such that agent 1 brokers a, agents 2 and 3 own b and c, respectively. Let f' be a TTC from individual endowments such that agent 1, 2, and 3 own objects a, b and c, respectively. Rows 2 to 4, and rows 7 to 9 in table 3.5 show the numbers of preference profiles that agent i receives her kth best choice under rules f and f', respectively. We can verify that $\sum_{i=1}^{3} \#(1,i) = \sum_{i=1}^{3} \#'(1,i) = 432$,

$\sum_{i=1}^{3} \#(2,i) = \sum_{i=1}^{3} \#'(2,i) = 144$, and $\sum_{i=1}^{3} \#(3,i) = \sum_{i=1}^{3} \#'(3,i) = 72$.

f	$\#(1,i)$	$\#(2,i)$	$\#(3,i)$
$i=1$	96	72	48
$i=2$	168	36	12
$i=3$	168	36	12
$\sum_{i=1}^{3} \#(k,i)$	432	144	72
f'	$\#'(1,i)$	$\#'(2,i)$	$\#'(3,i)$
$i=1$	144	48	24
$i=2$	144	48	24
$i=3$	144	48	24
$\sum_{i=1}^{3} \#'(k,i)$	432	144	72

Table 3.5: Example 3.6 for proposition 3.1

PROOF OF PROPOSITION 1. Given a deterministic TC rule f, interpret agents in the rule as "roles" in the rule. By assigning agents to the "roles" via a uniform lottery, we construct a corresponding random allocation rule. A random allocation rule consists of $n!$ deterministic rules. All of them are equally likely to be selected. Denote these deterministic rules as $f(t)$ for all $t = 1, 2, \cdots, n!$. And define

$$\#_t(k,i) \equiv \left| \left\{ R \in \mathcal{P}^N : f(t)_i(R) \text{ is the } k\text{th best in } R_i \right\} \right|,$$

where $f(t)_i(R)$ is the assignment agent i receives under the rule $f(t)$ when preference profile is R. Similarly, for another deterministic TC rule f', we can construct $n!$ deterministic rules by assigning agents to "roles" via a uniform lottery. Denote these rules as $f'(t)$ for all $t = 1, 2, \cdots, n!$. Define

$$\#'_t(k,i) \equiv \left| \left\{ R \in \mathcal{P}^N : f'(t)_i(R) \text{ is the } k\text{th best in } R_i \right\} \right|.$$

Bade (2014) showed that given a preference profile, for any two random TC rules, their distributions over outcomes are the same. When we consider all possible preference profiles, for all $k = 1, 2, \cdots, n$, we have the following result

$$\sum_{t=1}^{n!}\sum_{i=1}^{n} \#_t(k,i) = \sum_{t=1}^{n!}\sum_{i=1}^{n} \#'_t(k,i). \tag{1}$$

It is clear that given a deterministic allocation rule, the assignment depends only on the preferences of "roles". Therefore, for any two t, t', and for all $k = 1, 2, \cdots, n$, we have

$$\sum_{i=1}^{n} \#_t(k,i) = \sum_{i=1}^{n} \#_{t'}(k,i), \tag{2}$$

and

$$\sum_{i=1}^{n} \#'_t(k,i) = \sum_{i=1}^{n} \#'_{t'}(k,i). \tag{3}$$

Equations (1) to (3) imply proposition 3.1. ■

Armed with proposition 3.1, we can prove theorem 3.2.

Proof of theorem 3.2. "$\Leftarrow$" Pycia and Ünver's (2016a, 2016b) results imply that a TTC from individual endowments and a TC rule with three brokers are both Pareto efficient and group strategy-proof. Let f be a TTC from individual endowments. Denote ω_i as the initial endowment of agent i. For any $R \in \mathcal{P}^N$, suppose $f_i(R)$ is the kth in R_i, $f_j(R)$ is the lth in R_j. Construct a mapping $\tau : \mathcal{P}^N \to \mathcal{P}^N$ such that for any $R \in \mathcal{P}^N$, and any $k \notin \{i, j\}$, switch the positions of ω_i and ω_j in R_k to form $\tau(R_k)$, while $\tau(R_i)$ is formed by switching the positions of ω_i and ω_j of R_j, and $\tau(R_j)$ is formed by switching the positions of ω_i and ω_j of R_i. Then $f_i(\tau(R))$ is the lth in $\tau(R_i)$, and $f_j(\tau(R))$ is the kth in $\tau(R_j)$. This is true because during the TTC procedure, any agent points to agent i (or j) under R would point to agent j (or i) under R', while the agent who points to $m \notin \{i, j\}$ under R still points to m under R'. So under the transformed preference profile, agent i plays the "role" as if she was agent j, and agent j plays the "role" as if she was agent i. Hence for the new preference profile R', agent $m \notin \{i, j\}$ still receives the same ranked choice under R' as under R, even though what she receives may not be the same as the one under R, while agent i receives the lth choice in $\tau(R_i)$, and agent j receives the kth choice in $\tau(R_j)$. Also note that τ is a bijection. Therefore, a TTC from individual endowments is balanced. Similarly, we can prove that a TC rule with three brokers is also balanced.

"$\Rightarrow$" When $|N| = |O| \neq 3$, based on Pycia and Ünver's (2016a, 2016b) result, to prove the only if part, we only have to show that among all TC rules, a balanced rule is a TTC from individual endowments. Suppose not. Two cases are considered. Case 1: suppose there exists an agent, say i, who initially owns more than one object. Then by persistence, for any preference profile, agent i will not be assigned her worst choice. But if all agents have the same preferences, then Pareto efficiency implies someone would receive the worst choice, a violation to the last condition of balancedness. Case 2: Suppose no agent initially owns more than one object and the rule is not a TTC from individual endowments, then there exists a broker, say agent i, who brokers one object (say o) while other agents each initially owns one object (we use TC to denote this rule). Now, consider a corresponding TTC from individual endowments where agent i initially not brokers but owns object o (we use TTC to denote this rule), while other agents in the TTC own what they own in the TC. For any preference profile R, two subcases are considered. Subcase 2.1: if TTC assigns agent i an object other than o, then the outcomes of TC and TTC are the same. Subcase 2.2: if TTC assigns agent i object o, then agent i will be assigned o or an object that is ranked below o since she cannot point to herself unless she is the unique agent in the market. For example, if o is the best choice under R_i, then TTC will assign i object o. But if o is also the best choice of all other agents, then TC will assign i her second best choice. Therefore, subcases 2.1 and 2.2 imply that the number preference profiles that agent i receives her best choice under TC is smaller than the number of preference profiles that agent i receives her best choice under TTC. But proposition 2.1 shows that the summation of the numbers of preference profiles that all agents receive their best choices under TTC is equal to the summation of the numbers of preference profiles that all agents receive their best choices under TC. Therefore, the TC rule with one broker is not balanced.

When $|N| = |O| = 3$, a TC rule with a owner who initially owns two or three objects, or a TC rule with two owners and one broker are not balanced; therefore, an Pareto efficient, group strategy-proof, and balanced allocation rule could be a TTC from individual endowments, or a TC rule with three brokers. ■

Three conditions required in Theorem 3.2 are independent. If one of them is violated, we can find an allocation rule that satisfies the others. A serial dictatorship is Pareto efficient and group strategy-proof, but no balanced. A fixed allocation rule where agent i always receives o_i is balanced and group strategy-proof, but not Pareto efficient. The rule in example 3.7 is Pareto efficient and balanced, but not group strategy-proof.

Example 3.7: Let $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, initial endowment $\omega = (a, b, c)$, two preference profiles $\widehat{R}$ and $\widetilde{R}$, and the rule ψ are given by

$$\psi(R) = \begin{cases} (b, c, a), & \text{if } R = \widehat{R}; \\ (c, a, b), & \text{if } R = \widetilde{R}; \\ TTC(R), & \text{otherwise.} \end{cases}$$

$\widehat{R}_1$	$\widehat{R}_2$	$\widehat{R}_3$
$\boxed{b}$	a	$\boxed{a}$
c	$\boxed{c}$	c
a	b	b

$\widetilde{R}_1$	$\widetilde{R}_2$	$\widetilde{R}_3$
$\boxed{c}$	$\boxed{a}$	a
b	b	$\boxed{b}$
a	c	c

The rule ψ is Pareto efficient and balanced, but it is not group strategy-proof as under $\widehat{R}$, if agent 2 reports $\overline{R}_2 = aP_2bP_2c$, then $\psi_2(\widehat{R}_{-2}, \overline{R}_2)\widehat{P}_2\psi_2(\widehat{R})$.

We use the following example 3.8 to verify theorem 3.2.

Example 3.8: Consider an economy with two agents and two objects. Let $N = \{i, j\}$, $O = \{a, b\}$. There are two kinds of preferences and hence four possible preference profiles shown in columns 2 to 5 of table 3.6. And there are eight TC rules including four HE rules. The first column shows these TC rules, where ω_i denotes the initial endowment for agent i, and β_i denotes agent i initially brokers an object. Since there are only two agents, we do not have specify an inheritance structure to define a TC rule. The assignments are shown in boldface. We can see that two TTC rules from individual endowments satisfy balancedness, while the other six rules violate balancedness.

Note that the proof of the only if part in theorem 3.2 implies that if f belongs to the set of the HE rules, then f is a TTC from individual endowment if and only if all agent have equal chances to their worst choice, i.e., for all i, $j \in N$, $\#(n, i) = \#(n, j)$. Based on this observation, we have the following theorem 3.1* and theorem 3.2*.

	R_i	R_j	R_i	R_j	R_i	R_j	R_i	R_j	$\#(k,i)$	$\#(k,j)$
$\omega_i = \{a,b\}$	$\mathbf{a}$	a	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	$\mathbf{b}$	b	$\#(1,i)=4$	$\#(1,j)=2$
$\omega_j = \varnothing$	b	$\mathbf{b}$	b	a	a	b	a	$\mathbf{a}$	$\#(2,i)=0$	$\#(2,j)=2$
$\omega_i = \{a\}$	$\mathbf{a}$	a	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	b	$\mathbf{b}$	$\#(1,i)=3$	$\#(1,j)=3$
$\omega_j = \{b\}$	b	$\mathbf{b}$	b	a	a	b	$\mathbf{a}$	a	$\#(2,i)=1$	$\#(2,j)=1$
$\omega_i = \varnothing$	a	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	b	$\mathbf{b}$	$\#(1,i)=2$	$\#(1,j)=4$
$\omega_j = \{a,b\}$	$\mathbf{b}$	b	b	a	a	b	$\mathbf{a}$	a	$\#(2,i)=2$	$\#(2,j)=0$
$\omega_i = \{b\}$	a	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	$\mathbf{b}$	b	$\#(1,i)=3$	$\#(1,j)=3$
$\omega_j = \{a\}$	$\mathbf{b}$	b	b	a	a	b	a	$\mathbf{a}$	$\#(2,i)=1$	$\#(2,j)=1$
$\beta_i = \{a\}$	a	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	b	$\mathbf{b}$	$\#(1,i)=2$	$\#(1,j)=4$
$\omega_j = \{b\}$	$\mathbf{b}$	b	b	a	a	b	$\mathbf{a}$	a	$\#(2,i)=2$	$\#(2,j)=0$
$\beta_i = \{b\}$	a	$\mathbf{a}$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	b	$\mathbf{b}$	$\#(1,i)=2$	$\#(1,j)=4$
$\omega_j = \{a\}$	$\mathbf{b}$	b	b	a	a	b	$\mathbf{a}$	a	$\#(2,i)=2$	$\#(2,j)=0$
$\omega_i = \{a\}$	$\mathbf{a}$	a	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	$\mathbf{b}$	b	$\#(1,i)=4$	$\#(1,j)=2$
$\beta_j = \{b\}$	b	$\mathbf{b}$	b	a	a	b	a	$\mathbf{a}$	$\#(2,i)=0$	$\#(2,j)=2$
$\omega_i = \{b\}$	$\mathbf{a}$	a	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}$	$\mathbf{a}$	$\mathbf{b}$	b	$\#(1,i)=4$	$\#(1,j)=2$
$\beta_j = \{a\}$	b	$\mathbf{b}$	b	a	a	b	a	$\mathbf{a}$	$\#(2,i)=0$	$\#(2,j)=2$

Table 3.6: Example 3.8 for theorem 3.2

Theorem 3.1*: An allocation rule f is Pareto efficient, group strategy-proof, reallocation-proof and satisfying $\#(n,i) = \#(n,j)$ for all i, $j \in N$ if and only if it is a TTC from individual endowments.

Theorem 3.2*: When $|N| = |O| \neq 3$, an allocation rule f is Pareto efficient, group strategy-proof, and satisfies $\#(n,i) = \#(n,j)$ for all i, $j \in N$ if and only if it is a TTC from individual endowments. When $|N| = |O| = 3$, an allocation rule f is Pareto efficient, group strategy-proof, and satisfies $\#(n,i) = \#(n,j)$ for all i, $j \in N$ if and only if it is a TTC from individual endowments or a TC rule with three brokers.

Proof of theorem 3.2*. We only have the show the only if part. An allocation rule in which some agent initially owns at least two objects violates the condition that $\#(n,i) = \#(n,j)$. Let us pick an agent i in a TTC from individual endowments.

Now change the allocation rule to a TC rule such that i is a broker. Then for any preference profile, i cannot better off. When all agent have the same preferences and rank the brokered object as their $(n-1)$th best choice, then i is strictly worse off under TC by receiving her worst choice. Again, by proposition 3.1, $\#(n,i) = \#(n,j)$ for all i, $j \in N$ is violated. ■

The only if part of theorem 3.1* (or theorem 3.2*) is less restrictive than that of theorem 3.1 (or theorem 3.2). It says that whenever the number of agents is not three, even if we only care the chances that agents would be assigned to their worst choice, only the TTC from individual endowments are desirable among all rules satisfying the efficiency and the incentive properties.

3.3 Conclusion

This chapter is motivated by policymakers and parents' intuition quoted in Pathak and Sethuraman (2011) that the single lottery mechanism seems less equitable than the multiple lottery mechanism in school choice problem. An extreme case of the realization of the multiple lottery is that each agent is initially endowed with one object. But previous results by Abdulkadiroğlu and Sönmez (1998), Pathak and Sethuraman (2011), Lee and Sethuraman (2011), and Bade (2014) showed that given any two Pareto efficient and group strategy-proof deterministic allocation rules, the corresponding random allocation rules, including single lottery mechanism and multiple lottery mechanism, are equivalent. We formally define an interim fairness property that we call "balancedness." It is an interim fairness property in the sense that before knowing agents' true preferences, if all possible preference profiles are equally likely to happen, then a deterministic allocation rule is balanced if all agents have equal chances to their best choices, have equal chances to their second best choices, and so forth. Our main result shows that in models with at least four agents or just two agents, among all Pareto efficient and group strategy-proof rules, only the top trading cycles from individual endowments satisfy balancedness; and when there are only three agents, an allocation rule satisfies Pareto efficiency, group strategy-proofness and balancedness if and only if it is a TTC from individual endowments or a TC rule with three brokers.

In practice, the single lottery mechanism and the multiple lottery mechanism are used in assigning students to public schools. The policy implication of our result is if policymakers can freely choose any allocation rule, for the sake of fairness, then they should randomly assign each agent an object and then use the TTC algorithm to find the assignment.

4. HOUSE ALLOCATION WITH WEAK PREFERENCES

This chapter studies problems of allocating indivisible resources to people. For example, offices have to be assigned to new faculty, public school seats have to be assigned to students, and organs for transplant have to be assigned to patients. Such problems are referred to as one-sided matching problems as opposed to two-sided matching problems in the sense that in the two disjointed sets of agents (offices and faculty members, school seats and students, and organs and patients), only the agents of one side have preferences over the agents of the other side. An important constraint in real-world indivisible goods allocation problems is the lack of monetary transfers: faculty offices, public school seats and organs for transplant cannot be traded for money. This constraint implies that classical competitive market cannot be applied to find efficient outcomes.

The pioneering work on one-sided matching problems was initiated by Shapley and Scarf (1974). They presented a housing market model. In this model, each agent is endowed with one indivisible good and wants to trade it for a better one that might be endowed by another agent in the market. They were interested in whether the core allocation defined by weak dominance exists. They showed that a top trading cycles algorithm invented by David Gale finds an allocation in the core. Roth and Postlewaite (1977) proved that the top trading cycles algorithm finds the unique matching in the core of each problem when preferences are strict. Hylland and Zeckhauser (1979) presented a house allocation model in which no agent initially owns any house. Svensson (1999) showed that for a house allocation problem with strict preferences, an allocation rule is strategy-proof, non-bossy, and neutral if and only if it is serially dictatorial. Abdulkadiroğlu and Sönmez (1998) studied the relationship between the serial dictatorships and the core from assigned endowments, and they also showed that the random serial dictatorship is equivalent to the core from random endowments when indifference is excluded.

A large literature on matching theory assumes that preferences are strict. Without strict preferences, many good properties fail to hold. For example, Shapley and Scarf (1974) provided an example showing that the core may disappear when indifferences are allowed. Alternatively, Pápai (2000) showed that group strategy-proofness is equivalent to strategy-proofness and non-bossiness; but this equivalence breaks down on the weak domain (Ehlers 2002). Due to the undesirable properties and the complexity induced by ties, weak preferences are ignored in most of the existing matching literature. But indifferences are prevailing in the real world. For example, in the kidney exchange problem (Roth, Sönmez, and Ünver 2004), each patient-donor pair wants to exchange for a compatible kidney from another patient-donor pair. If their preferences are based on checklist criteria such as blood and tissue types, then different kidney with the same criteria should be regarded as indifferent. Another example is school choice problem (Erdil and Ergin 2008) which consists of a set of students and a set of public schools with limited numbers of seats. Each school has a priority ranking over students. The ranking is determined by local laws and educational policies. Such priorities are weak orderings and the indifference classes are quite large.

Recently, Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) defined two kinds of allocation rules that are Pareto efficient, strategy-proof, and individual rational for housing market problem when indifferences are allowed. Ehlers (2014) provided a characterization of top trading cycles with fixed tie-breaker for the housing market problem with indifferences by individual rationality, strategy-proofness, weak efficiency, non-bossiness, and consistency.

This chapter studies house allocation problem with weak preferences. By extending the results proved by Abdulkadiroğlu and Sönmez (1998), our corollary 4.1 and corollary 4.2 show that the strong relationship between the serial dictatorships and the top trading cycles, and the equivalence of random serial dictatorship and the top trading cycles with random endowments still hold under weak preferences when we use fixed tie-breaking or random tie-breaking. Our theorem 4.1 and theorem 4.2 show that the serial dictatorship with fixed tie-breaking satisfies weak Pareto efficiency, strategy-proofness, non-bossiness, and consistency; moreover, it is not Pareto dominated by any Pareto efficient and strategy-proof rule.

This chapter is organized as follows: section 4.1 formally presents the model, section 4.2 introduces allocations rules, section 4.3 shows our main result, and section 4.4 concludes.

4.1 The model

Consider the following problem: a department hires n new faculty members and wants to assign each one of them a distinct office. Each faculty member has preferences over n available offices. A **house allocation problem** (Hylland and Zeckhauser 1979) is a triple (A, H, R), where $A = (1, 2, \cdots, n)$ is the set of agents, $H = (h_1, h_2, \cdots, h_n)$ is the set of indivisible goods (hereafter houses), and $R = (R_1, R_2, \cdots, R_n)$ is a preference profile consisting of a list of preference relations over houses. Following the classical assumptions, we assume that the number of agents equals the number of houses.

Let the set of all preference relations be $\mathcal{R}$, and the set of all preference profiles be $\mathcal{R}^A$. Given a preference profile $R \in \mathcal{R}^A$, agent a's preference relation R_a might be weak, and it is complete and transitive. Let R_{-a} be the list of preference relations of all agents but a. For all agents $a \in A$, let P_a and I_a denote the "better than" relation and "indifference" relation induced by the preference relation R_a. For all $S \subseteq A$, let $R_S = (R_a)_{a \in S}$ be the restriction of R to S and $R_{-S} = R_{A \setminus S}$. For all $M \subseteq H$, let $R_a|_M$ be the restriction of R_a to M and $R_S|_M = (R_a|_M)_{a \in S}$.

A binary relation $>_a$ is a **strict order** on a set H if it is irreflexive ($h >_a h$ does not hold for any $h \in H$), asymmetric (if $h >_a h'$, then $h' >_a h$ does not hold), and transitive ($h >_a h'$ and $h' >_a h''$ implies $h >_a h''$). Let a list of strict orders $> = (>_1, >_2, \cdots, >_n)$ denote a profile of **fixed tie-breakers** and $\succ$ be the set of all possible profiles of fixed tie-breakers. Note that the number of all profiles of fixed tie-breakers is $| \succ | = (n!)^n$. The choice of the tie-breakers is arbitrary and it is independent of the preference profile to break ties. The **random tie-breakers** is a uniform distribution over the set of all profiles of fixed tie-breakers, i.e., each profile of fixed tie-breakers is chosen with equal probability.

Given $R_a, R'_a \in \mathcal{R}$, we say R'_a is a **strict transformation** of R_a if (i) R'_a is strict (i.e,. for all $h_i \neq h_j$, we cannot have both $h_i R'_a h_j$ and $h_j R'_a h_i$) and (ii) for all $h_i, h_j \in H$, $h_i P_a h_j$ implies $h_i P'_a h_j$ (where P'_a is the "better than" relation associated with R'_a). For each agent $a \in A$, let $R^>_a$ be the strict transformation of R_a with a profile of fixed tie-breakers $>$ such that for all $h_i, h_j \in H$ with $h_i I_a h_j$ and $h_i >_a h_j$, we have $h_i P^>_a h_j$, where $P^>_a$ is the "better than" relation associated with $R^>_a$. That is, $R^>_a$ is a strict transformation of R_a with a profile of tie-breakers $>$ in the sense that strict preferences are preserved but ties are broken according to tie-breakers $>$. If R_a is a strict preference, then $R^>_a = R_a$. Let $R^> = (R^>_a)_{a \in A}$ be the strict transformation of R with a profile of fixed tie-breakers $>$. Note that only $>_a$ matters to break ties in R_a, so a better way to express $R^>_a$ is $R^{>_a}_a$. But we use the former for notational simplicity.

Given a profile of fixed tie-breakers $>$, the **choice function** $X^>_a(H')$ of an agent $a \in A$ from a set of houses $H' \subseteq H$ is the best house under the transformed strict preference $R_a^{>}$ among H', i.e.,

$$X_a^{>}(H') = h' \iff h' \in H' \text{ and } h' P_a^{>} h \text{ for all } h \in H' \setminus \{h'\}.$$

The outcome of a house allocation problem is an assignment of houses to agents such that each house is assigned to a distinct agent. Formally, the outcome is a **matching** $\mu : A \to H$ which is an one-to-one and onto function from A to H. For all $a \in A$, $\mu(a)$ is the assignment of agent a under matching μ. Let the set of all matchings be $\mathscr{M}$. Note that the number of all matchings is $|\mathscr{M}| = n!$.

Now we consider a slightly different problem: each faculty member is allocated with one office, and they are allowed to trade it for a better one that might be allocated by another agent in the market. A housing market problem (Shapley and Scarf 1974) is simply a house allocation problem with a matching which is referred to as the initial endowment. Formally, a **housing market problem** is a four-tuple (A, H, R, μ), where μ is an initial endowment matching.

A **matching rule** is a mapping from preference profiles to the set of matchings, i.e., a matching rule selects a matching for each problem. We will introduce two matching rules in the next section.

4.2 The matching rules

4.2.1 Serial dictatorships with fixed tie-breaking

We first introduce a matching rule for the house allocation problem. Let an **ordering** $f : \{1, 2, \cdots, n\} \to A$ be a bijective function. We can interpret an ordering as a list of priorities. That is, agent $f(1)$ has the highest priority, agent $f(2)$ has the second highest priority, and so forth. Let $\mathscr{F}$ be the set of all such orderings. Note that the number of all possible orderings is $|\mathscr{F}| = n!$.

Given a preference profile $R \in \mathcal{R}^A$, for any ordering $f \in \mathscr{F}$ and any profile of fixed tie-breakers $>$, defined a **simple serial dictatorship** induced by f with a profile of fixed tie-breaker $>$, $\varphi_f^>$ as

$$(\varphi_f^>)_{f(1)} = X_{f(1)}^>(H),$$

$$(\varphi_f^>)_{f(i)} = X_{f(i)}^> \left(H \setminus \bigcup_{j=1}^{i-1} \{\varphi_f^>(f(j))\} \right) \text{ for } i \subseteq \{2, 3, \cdots, n\}.$$

That is, agent $f(1)$ chooses her top choice among all houses under her transformed preference $R_{f(1)}^>$, agent $f(2)$ chooses her top choice among those remaining under her transformed preference $R_{f(2)}^>$, and so on. When preferences are strict, we can simply write serial dictatorship as φ_f. Now we introduce a matching rule for the housing market problem.

4.2.2 Top trading cycles with fixed tie-breaking

Given a preference profile $R \in \mathcal{R}^A$, for each profile of fixed tie-breakers $>$ and each endowment μ, let $\varphi_\mu^>$ denote the **top trading cycles (TTC) algorithm** with the profile of fixed tie-breakers $>$ from the endowment μ. The algorithm goes as follows:

At step 1: each agent $a \in A$ points to the agent who owns her favorite house under the transformed preferences $R_a^>$ (that is, $X_a^>(H)$). A set of agent $\{a_1, a_2, \cdots, a_m\} \subseteq A$ consists a cycle if agent a_1 points to agent a_2, agent a_2 points to agent a_3, ... , agent a_m points to agent a_1. Since the number of agents is finite and each agent points to one agent, there is at least one cycle. Note that an agent points to herself also forms a cycle. In each cycle, each agent receives the house owned by the agent she points to. All agents with their allocated houses in all cycles are removed from the housing market. At step 1, each agent leaves the market receives (possibly) one of her favorite houses among all houses.

...

At step t: each remaining agent a points to the agent who owns her favorite house among those remaining houses under the transformed preferences $R_a^>$. Again, at least one cycle exists. In each cycle, each agent receives the house owned by the agent she points to and all agents with their houses in the cycles are removed from the housing market. The algorithm stops when no agent is in the market.

Because the number of agents is finite and at lease one agent is removed from the market at each step, the algorithm terminates in finite $k_\mu^>$ steps. We partition agents according to the steps at which they leave the market. Let $A(\mu, >) = \{A^1(\mu, >), A^2(\mu, >), \cdots, A^{k_\mu^>}(\mu, >)\}$ be such partition and we call this the **cycle structures** for μ and $>$.

For all $t \in \{1, 2, \cdots, k_u^>\}$, let $H^t(\mu, >) = \{h \in H : \mu(a) = h \text{ for some } a \in A^t(\mu, >)\}$. That is, $H^t(\mu, >)$ is the set of houses that are initially owned by and allocated to agents who leave the market at the t^{th} step. Given a preference profile $R \in \mathcal{R}^A$, we can write the outcome of the TTC algorithm with the profile of fixed tie-breakers $>$ under endowment μ, $\varphi_\mu^>$ as

$$\forall a \in A^1(\mu, >),\ \varphi_\mu^>(a) = X_a^>(H).$$

$$\forall t \in \{2, 3, \cdots, k_\mu^>\},\ \forall a \in A^t(\mu, >),\ \varphi_\mu^>(a) = X_a^>\left(H \setminus \bigcup_{s=1}^{t-1} H^s(\mu, >)\right).$$

4.3 Main result

A matching ν **Pareto dominates** another matching μ if $\nu(a) R_a \mu(a)$ for all $a \in A$ and $\nu(a) P_a \mu(a)$ for some agent $a \in A$. A matching is **Pareto efficient** if it is not Pareto dominated by any matching. Let $\mathscr{E}$ be the **set of Pareto efficient allocations**. A matching ν **strictly Pareto dominates** another matching μ if $\nu(a) P_a \mu(a)$ for all $a \in A$. A matching is **weakly Pareto efficient** if it is not strictly Pareto dominated by any matching. Let $\mathscr{W}\mathscr{E}$ be the **set of weakly Pareto efficient allocations**.

When preferences are strict, Abdulkadiroğlu and Sönmez (1998) established a strong link between the serial dictatorship and top trading cycles algorithm. They showed that for any ordering f and any matching μ, the serial dictatorship induced by f and the outcome of the TTC algorithm from assigned endowments μ (which is also the unique matching in the core when preferences are strict) both yield Pareto efficient matchings; for any Pareto efficient matching η, there is a serial dictatorship and a outcome of top trading cycles from assigned endowments that yields it. Formally, $\varphi_{\mathscr{M}} = \varphi_{\mathscr{F}} = \mathscr{E}$, where $\varphi_{\mathscr{M}} = \{\eta \in \mathscr{M} : \varphi_{\mu} = \eta \text{ for some } \mu \in \mathscr{M}\}$, $\varphi_{\mathscr{F}} = \{\eta \in \mathscr{M} : \varphi_f = \eta \text{ for some } f \in \mathscr{F}\}$.

But under weak preferences, Ehlers (2014) showed that the outcome of the TTC algorithm with fixed tie-breaking is just weakly Pareto efficient. Similarly, a serial dictatorship with fixed tie-breaking may fail to yield a efficient allocation when preferences are weak. We can illustrate this in an example of two agents and two houses. Preferences are given in the following table 4.1. When the priority ordering satisfies agent 1 chooses first, and the tie-breaker satisfies $h_2 >_1 h_1$, then the outcome of the serial dictatorship with the fixed tie-breaker is (h_2, h_1) which is Pareto dominated by (h_1, h_2).

R_1	R_2
h_1, h_2	h_2
	h_1

Table 4.1: Serial dictatorship with fixed tie-breaking is not Pareto efficient

We have the following corollary that extends Abdulkadiroğlu and Sönmez (1998)'s result when preferences are weak.

Corollary 4.1:

(1) The set of matchings induced by serial dictatorships with fixed tie-breaking coincides with the set of matchings induced by the TTC algorithm with fixed tie-breaking;

(2) the induced matchings are weakly Pareto efficient;

(3) for any Pareto efficient matching, there exists a serial dictatorship with fixed tie-breaking and a TTC from assigned endowment with fixed tie-breaking that yields it. That is,

$$\mathscr{E} \subseteq \varphi^{\succ}_{\mathscr{M}} = \varphi^{\succ}_{\mathscr{F}} \subsetneq \mathscr{W}\mathscr{E},$$

where $\varphi^{\succ}_{\mathscr{M}} = \{\eta \in \mathscr{M} : \varphi^{>}_{\mu} = \eta$ *for some* $\mu \in \mathscr{M}$ *and some* $> \in \succ\}$*, and* $\varphi^{\succ}_{\mathscr{F}} = \{\eta \in \mathscr{M} : \varphi^{>}_{f} = \eta$ *for some* $f \in \mathscr{F}$ *and some* $> \in \succ\}$.

Corollary 4.1 implies that the strong link between serial dictatorship and Gale's TTC algorithm still holds for the full preference domain. Contrary to the scenario of strict preferences, the outcomes are not necessarily Pareto efficient; they are just weakly Pareto efficient. But they are not weak enough in the sense that i) some "extremely" weakly Pareto efficient allocations could not be yielded by any serial dictatorship or any Gale's TTC algorithm (see the following example 4.1); and ii) all the Pareto efficient allocations can be yielded by any one of the two classes of rules. We can illustrate the theorem in the following example 4.1 with three agents and three houses.

Example 4.1: Preferences are given in the following table 4.2.

R_1	R_2	R_3
h_2, h_3	h_1	h_1
h_1	h_3	h_2
	h_2	h_3

Table 4.2: The preferences for example 4.1

There are six serial dictatorships $\mathscr{F} = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$ and six initial endowments $\mathscr{M} = \{(h_1, h_2, h_3),\ (h_1, h_3, h_2),\ (h_2, h_1, h_3),\ (h_2, h_3, h_1),\ (h_3, h_1, h_2),\ (h_3, h_2, h_1)\}$. And there are $(3!)^3 = 216$ profiles of fixed tie-breakers, but only two types of them matter in this problem. Let the type 1 tie-breaker $\bar{>}$ be such that $h_2 \bar{>}_1 h_3$, and the type 2 tie-breaker $\hat{>}$ be satisfying $h_2 \hat{>}_1 h_3$. The following table 4.3 list the outcomes of serial dictatorships and Gale's TTC algorithm.

$\varphi^{>}_{f} \searrow$	$>=\bar{>}$	$>=\hat{>}$	$\varphi^{>}_{\mu} \searrow$	$>=\bar{>}$	$>=\hat{>}$
$f^1 = (1,2,3)$	(h_2, h_1, h_3)	(h_3, h_1, h_2)	$\mu^1 = (h_1, h_2, h_3)$	(h_2, h_1, h_3)	(h_3, h_2, h_1)
$f^2 = (1,3,2)$	(h_2, h_3, h_1)	(h_3, h_2, h_1)	$\mu^2 = (h_1, h_3, h_2)$	(h_2, h_3, h_1)	(h_3, h_1, h_2)
$f^3 = (2,1,3)$	(h_2, h_1, h_3)	(h_3, h_1, h_2)	$\mu^3 = (h_2, h_1, h_3)$	(h_2, h_1, h_3)	(h_3, h_1, h_2)
$f^4 = (2,3,1)$	(h_3, h_1, h_2)	(h_3, h_1, h_2)	$\mu^4 = (h_2, h_3, h_1)$	(h_2, h_3, h_1)	(h_2, h_3, h_1)
$f^5 = (3,1,2)$	(h_2, h_3, h_1)	(h_3, h_2, h_1)	$\mu^5 = (h_3, h_1, h_2)$	(h_3, h_1, h_2)	(h_3, h_1, h_2)
$f^6 = (3,2,1)$	(h_2, h_3, h_1)	(h_2, h_3, h_1)	$\mu^6 = (h_3, h_2, h_1)$	(h_2, h_3, h_1)	(h_3, h_2, h_1)

Table 4.3: The outcomes of example 4.1

From table 4.3 we can see that $\varphi^{\succ}_{\mathscr{F}} = \{(h_2, h_1, h_3), (h_3, h_1, h_2), (h_2, h_3, h_1), (h_3, h_2, h_1)\}$

$= \varphi^{\succ}_{\mathscr{M}}$, and the set of Pareto efficient allocations is $\mathscr{E} = \{(h_2, h_3, h_1), (h_3, h_1, h_2)\}$. Also note that $(h_1, h_3, h_2) \in \mathscr{W}\mathscr{E}$ but $(h_1, h_3, h_2) \notin \varphi^{\succ}_{\mathscr{F}} = \varphi^{\succ}_{\mathscr{M}}$, which verifies $\mathscr{E} \subseteq \varphi^{\succ}_{\mathscr{M}} = \varphi^{\succ}_{\mathscr{F}} \subsetneq \mathscr{W}\mathscr{E}$.

Proof of corollary 4.1. The fact that $\varphi^{\succ}_{\mathscr{M}} = \varphi^{\succ}_{\mathscr{F}}$ follows directly from Abdulkadiroğlu and Sönmez (1998). We first show that $\mathscr{E} \subseteq \varphi^{\succ}_{\mathscr{M}}$. Let $\eta \in \mathscr{E}$. For any fixed tie-breaker $> \in \succ$, we apply Gale's TTC algorithm with the fixed tie-breaker from the endowment η. At each step, no agent would point to a house that is worse than her endowment, i.e., for all $a \in A$, $\varphi^{>}_{\eta}(a) R_a \eta(a)$. Since η is Pareto efficient, we have $\varphi^{>}_{\eta}(a) I_a \eta(a)$ for all $a \in A$. Now we pick up a tie-breaker $\bar{>}$ such that for each agent, her endowment is ordered first, i.e., for each $a \in A$, and each $h \in H \setminus \eta(a)$, $\eta(a) \bar{>}_a h$. With this tie-breaker $\bar{>}$, whenever an agent is indifferent between a house and her endowment and they are in the set of her most preferred houses among the remaining houses, she would point to her endowment under Gale's TTC algorithm. This observation together with $\varphi^{>}_{\eta}(a) I_a \eta(a)$ for all $> \in \succ$ and all $a \in A$ imply $\varphi^{\bar{>}}_{\eta}(a) = \eta(a)$ for all $a \in A$ which in turn implies $\eta = \varphi^{\bar{>}}_{\eta} \in \varphi^{\succ}_{\mathscr{M}}$, completing the proof of $\mathscr{E} \subseteq \varphi^{\succ}_{\mathscr{M}}$. Now we show that $\varphi^{\succ}_{\mathscr{F}} \subsetneq \mathscr{W}\mathscr{E}$. Let $\eta \in \varphi^{\succ}_{\mathscr{F}}$, then there exist $f \in \mathscr{F}$ and $> \in \succ$ such that $\eta = \varphi^{>}_{f}$. Agent $f(1)$ chooses (one of) her most preferred house(s) among all the houses. It is impossible to assign agent $f(1)$ a better house, showing that $\eta \subseteq \mathscr{W}\mathscr{E}$. The example 4.1 shows that $\varphi^{\succ}_{\mathscr{F}} \neq \mathscr{W}\mathscr{E}$. ■

Abdulkadiroğlu and Sönmez (1998) also showed that the random serial dictatorship is equivalent to the TTC from random endowments when preferences are strict. We generalize this result in corollary 4.2 by allowing weak preferences. Before presenting the result, we have to introduce more terminology. A **lottery** m is a probability distribution over matchings, that is, $m = (m_1, m_2, \cdots, m_{n!})$ such that $\sum_{k=1}^{n!} m_k = 1$ and $m_k \geq 0$ for all k. Let m^{μ} be the lottery that assigns probability 1 to matching μ. Let the set of all lotteries be $\Delta\mathscr{M}$. A **lottery rule** selects a lottery for each problem.

Given a preference profile $R \in \mathcal{R}^A$, define a **random serial dictatorship with a profile of fixed tie-breakers** $>$, $\psi^{>}_{rsd}$ as

$$\psi^{>}_{rsd} = \sum_{f \in \mathscr{F}} \frac{1}{n!} m^{\varphi^{>}_{f}},$$

i.e., each serial dictatorship with the fixed tie breakers is chosen with equal probability.

Given a preference profile $R \in \mathcal{R}^A$, define a **random serial dictatorship with random tie-breakers**, $\psi^{\succ}_{rsd}$ as

$$\psi^{\succ}_{rsd} = \sum_{> \in \succ} \sum_{f \in \mathscr{F}} \frac{1}{(n!)^{n+1}} m^{\varphi^{>}_{f}},$$

i.e., each serial dictatorship and each profile of fixed tie breakers are randomly selected with uniform distribution and the induced serial doctorship is applied.

Given a preference profile $R \in \mathcal{R}^A$, define the outcome of the **TTC algorithm with a profile of fixed tie-breakers from random endowments**, $\psi^{>}_{ttcre}$ as

$$\psi^{>}_{ttcre} = \sum_{\mu \in \mathscr{M}} \frac{1}{n!} m^{\varphi^{>}_{\mu}},$$

i.e., endowment is randomly selected with uniform distribution and each outcome of the TTC algorithm with the fixed tie-breakers $>$ is chosen with equal probability.

Given a preference profile $R \in \mathcal{R}^A$, define the outcome of the **TTC algorithm with random tie-breakers from random endowments**, $\psi^{\succ}_{ttcre}$ as

$$\psi^{\succ}_{ttcre} >= \sum_{> \in \succ} \sum_{\mu \in \mathscr{M}} \frac{1}{(n!)^{n+1}} m^{\varphi^{>}_{\mu}},$$

i.e., each endowment and each profile of fixed tie-breakers are randomly chosen with uniform distribution and each outcome produced via the TTC algorithm is chosen with equal probability.

The following corollary 4.2 follows directly from Abdulkadiroğlu and Sönmez (1998).

Corollary 4.2:

(1) For any profile of fixed tie-breakers $>$, the random serial dictatorship with fixed tie-breaking is equal to the TTC from random endowments with fixed tie-breaking, i.e.,

$$\psi_{rsd}^{>} = \psi_{ttcre}^{>}.$$

(2) A random serial dictatorship with random tie-breakers is equivalent to the TTC with random tie-breakers from random endowments, i.e.,

$$\psi_{rsd}^{\succ} = \psi_{ttcre}^{\succ}.$$

In the previous example 4.1,

$$\psi_{rsd}^{\bar{>}} = \left(\frac{2}{6} \otimes (h_2, h_1, h_3), \frac{3}{6} \otimes (h_2, h_3, h_1), \frac{1}{6} \otimes (h_3, h_1, h_2) \right) = \psi_{ttcre}^{\bar{>}}.$$

$$\psi_{rsd}^{\hat{>}} = \left(\frac{3}{6} \otimes (h_3, h_1, h_2), \frac{2}{6} \otimes (h_3, h_2, h_1), \frac{1}{6} \otimes (h_2, h_3, h_1) \right) = \psi_{ttcre}^{\hat{>}}.$$

$$\psi_{rsd}^{\succ} = \left(\frac{2}{12} \otimes (h_2, h_1, h_3), \frac{4}{12} \otimes (h_3, h_1, h_2), \frac{4}{12} \otimes (h_2, h_3, h_1), \frac{2}{12} \otimes (h_3, h_2, h_1) \right) = \psi_{ttcre}^{\succ}.$$

Now we introduce some more definitions to present our theorem 4.1.

A matching rule is **Pareto efficient** if it always selects a Pareto efficient matching for each problem. A matching rule φ **Pareto dominates** another matching rule φ' if for all (A, H, R), $\varphi_a(R) R_a \varphi'_a(R)$ for all $a \in A$, and $\varphi_{\bar{a}}(R) P_{\bar{a}} \varphi'_{\bar{a}}(R)$ for some $\bar{a} \in A$ and some R, where $\varphi_a(R)$ is agent a's allocation under the rule φ with the preference profile R.

A matching rule φ is **strategy-proof** if for all $R \in \mathcal{R}^A$, all $a \in A$, and all $R'_a \in \mathcal{R}$, we have $\varphi_a(R) R_a \varphi_a(R'_a, R_{-a})$. Strategy-proofness requires that no agent can benefit by unilaterally misreporting her preference relation. This incentive-compatible condition ensures that truthfully reveal one's preference is weakly dominant strategy.

A matching rule φ is **non-bossy** if for all $R \in \mathcal{R}^A$, all $a \in A$, and all $R'_a \in \mathcal{R}$, if $\varphi_a(R) = \varphi_a(R'_a, R_{-a})$, then we have $\varphi(R) = \varphi(R'_a, R_{-a})$. Non-bossiness requires that no agent can change the assignments of others by misreporting her preferences without change the her allocation.

A matching rule φ is **consistent** if for all $S \subseteq A$ and all $R \in \mathcal{R}^A$, $\bigcup_{a \in A \setminus S} \{\varphi_a(R)\} = H \setminus M$ (with $M \subseteq H$) implies $\varphi_a(R_S|_M) = \varphi_a(R)$ for all $a \in S$. Consistency requires that whenever some set of agents receives their assigned houses, we can remove these agents and their assignments from the economy without changing the allocation of other agents.

Ehlers (2014) showed that in the housing market problem with indifference, Gale's TTC algorithm with fixed tie-breaking satisfying individual rationality, weak Pareto efficiency, strategy-proofness, non-bossiness and consistency. We obtain a similar result in the house allocation problem with weak preferences.

Theorem 4.1: For any profile of fixed tie-breakers $>$ and any priorities ordering f, the serial dictatorship $\varphi_f^>$ satisfies weak Pareto efficiency, strategy-proofness, non-bossiness and consistency.

Proof of theorem 4.1. Weak Pareto efficiency was proved in corollary 4.1.

Strategy-proofness: for all $R \in \mathcal{R}^A$, all $a \in A$, and all $R'_a \in \mathcal{R}$, since φ_f satisfies strategy-proofness on the strict domain of preferences, so we have

$$(\varphi_f^>)_a(R) R_a^> (\varphi_f^>)_a(R'_a, R_{-a}),$$

where $(\varphi_f^>)_a(R)$ is the allocation for agent a under the matching $\varphi_f^>(R)$. Since $R_a^>$ is the transformed preference that only break ties in R_a, so we have

$$(\varphi_f^>)_a(R) R_a (\varphi_f^>)_a(R'_a, R_{-a}),$$

i.e., strategy-proofness is satisfied.

Non-bossiness: for all $R \in \mathcal{R}^A$, all $a \in A$, and all $R'_a \in \mathcal{R}$, suppose we have $(\varphi_f^>)_a(R) = (\varphi_f^>)_a(R'_a, R_{-a})$; we want to prove $\varphi_f^>(R) = \varphi_f^>(R'_a, R_{-a})$. Since $\varphi_f^>(R) = \varphi_f(R^>)$, so we have $(\varphi_f)_a(R^>) = (\varphi_f)_a({R'_a}^>, {R_{-a}}^>)$. Since φ_f satisfies non-bossiness on the strict domain, we have $(\varphi_f)(R^>) = (\varphi_f)({R'_a}^>, {R_{-a}}^>)$. Again, by $\varphi_f^>(R) = \varphi_f(R^>)$, we have $\varphi_f^>(R) = \varphi_f^>(R'_a, R_{-a})$.

Consistency: for all $S \subseteq A$, all $a \in S$, and all $R \in \mathcal{R}^A$, suppose

$$\bigcup_{a \in A \setminus S} \{(\varphi_f^>)_a(R)\} = H \setminus M;$$

we want to prove $(\varphi_f^>)_a(R_S|_M) = (\varphi_f^>)_a(R)$. Since $\varphi_f^>(R) = \varphi_f(R^>)$, so we have $\bigcup_{a \in A \setminus S} \{(\varphi_f)_a(R^>)\} = H \setminus M$. Since φ_f satisfies consistency on strict domain, so we have $(\varphi_f)_a(R_S^>|_M) = (\varphi_f)_a(R^>)$. Again, by $\varphi_f^>(R) = \varphi_f(R^>)$, we have $(\varphi_f^>)_a(R_S|_M) = (\varphi_f^>)_a(R)$, completing the proof of consistency. ■

Even though fixed tie-breaking fails to achieve strict Pareto efficiency, it is commonly used in matching theory and market design. Abdulkadiroğlu, Pathak and Roth

(2009) and Kesten (2010) independently studied the school choice problem with indifferences in priorities in which ties are broken according to fixed tie-breakers and then apply student-proposing deferred acceptance algorithm to find the stable matching. Kesten (2010) showed that no other strategy-proof and Pareto efficient rules Pareto dominates the DA algorithm with fixed tie-breaking. While Abdulkadiroğlu, Pathak and Roth (2009) defended this practice by showing that no other strategy-proof rules Pareto dominates the DA algorithm with fixed tie-breaking. Ehlers (2014) presented the housing market problem with indifferences and shows that the TTC algorithm with fixed tie-breaking is not Pareto dominated by a rule satisfying strategy-proofness and Pareto efficiency. We prove a similar result as showed in theorem 4.2.

Theorem 4.2: Serial dictatorship with fixed tie-breaking is not Pareto dominated by any strategy-proof and Pareto efficient rule.

Proof of theorem 4.2. Let $A = \{1, 2, 3, 4\}$, $H = \{h_1, h_2, h_3, h_4\}$. Let the preference relations be the same as the ones in Ehlers (2014), as shown in the following table.

R_1	R_2	R_3	R_4	R'_3
h_2, h_3	h_1	h_1	h_3	h_1
h_1	h_2	h_2	h_4	h_2
h_4	h_3	h_4	h_2	h_3
	h_4	h_3	h_1	h_4

Let $f = (1, 2, 3, 4)$ and $>$ be such that $h_2 >^1 h_3$. Then $\varphi_f^>(R) = (h_2, h_1, h_4, h_3)$. Suppose φ dominates $\varphi_f^>$ and φ is Pareto efficient, then we have $\varphi(R) = (h_2, h_1, h_4, h_3)$ $= \varphi_f^>(R)$. We also have $\varphi_f^>(R'_3, R_{-3}) = (h_2, h_1, h_3, h_4)$. Since φ is Pareto efficient and it Pareto dominates $\varphi_f^>$, so $\varphi(R'_3, R_{-3}) = (h_3, h_1, h_2, h_4)$ and thus $\varphi(R'_3, R_{-3}) P_3 \varphi(R)$ which means strategy-proofness is violated. ■

4.4 Conclusion

This chapter studies house allocation problems when indifferences are allowed. It shows that the serial dictatorship with fixed tie-breaking satisfies weak Pareto efficiency, strategy-proofness, non-bossiness, and consistency, and further it is not Pareto dominated by any Pareto efficient and strategy-proof rule. As corollaries to Abdulkadiroğlu and Sönmez (1998), it also shows that the relationship between the serial dictatorships and the TTC algorithm from individual endowments, and the equivalence of random serial dictatorship and the TTC algorithm with random endowments still holds under weak preferences.

5. CONCLUSION

This dissertation studies house allocation problems. The chapter 3 on balance house allocation is motivated by Pathak and Sethuraman's (2011) work. When students are assigned to public schools in New York City, policymakers and parents believe that a single lottery used for all schools is less equitable than lotteries at each school. We formally define a fairness property that we call "balancedness." If all possible preference profiles are equally likely to happen, then a deterministic allocation rule is balanced if all agents have equal chances to their best choices, have equal chances to their second best choices, and so forth. When preferences are strict, we show that in models with non-three agents, an allocation rule satisfies Pareto efficiency, group strategy-proofness, and balancedness if and only if it is a top trading cycles rule from individual endowments; and when there are three agents, an allocation rule is Pareto efficient, group strategy-proof and balanced if and only if it is a TTC from individual endowments or a TC rule with three brokers.

However, this balancedness property cannot hold for school choice problems when capacities of each school is not one. In a school choice problem, agents has a strict preference relation over schools, but they are indifferent between the seats of the same school. The top trading cycles algorithm for school choice problem consists of a serial of steps described as follows:

- At step 1: Each student points to her favorite school. Each school points to the student with the highest priority. At least one cycle exists. Assign each student the school she points to.

- In general, at step k: each remaining student points to her favorite school with non-zero number of available seats. Each school with non-zero number of available seats points to the student with the highest priority among the reaming students. Again, at least one cycle exists. Assign each student the school she points to.

- The algorithm stops when no school or no student points.

Example. Consider a school choice problem with two schools, S_1 and S_2, and four students, i, j, k and l. Each school has two seats. Table 5.1 shows the priority ranking of each school.

S_1	S_2
i	k
j	l

Table 5.1: Two schools with same capacities

The numbers of preference profiles that each student receives her first and her second best choices are given by:

$$\#(1^{st}, i) = 14,\ \#(2^{nd}, i) = 2.\ \#(1^{st}, k) = 14,\ \#(2^{nd}, k) = 2.$$

$$\#(1^{st}, j) = 12,\ \#(2^{nd}, j) = 4.\ \#(1^{st}, l) = 12,\ \#(2^{nd}, l) = 4.$$

We can see that the student at the top of the priority list has a higher chance to her best choice than the student at the bottom of the priority list can.

Example. Consider another example with asymmetric capacities described in table 5.2. School 1 has two seats, but school 2 has one seat.

The numbers of preference profiles that each student receives her first and second best choices are given by:

S_1	S_2
i	k
j	

Table 5.2: Two schools with asymmetric capacities

$$\#(1^{st}, i) = 6,\ \#(2^{nd}, i) = 2.\ \#(1^{st}, k) = 7,\ \#(2^{nd}, k) = 1.$$
$$\#(1^{st}, j) = 5,\ \#(2^{nd}, j) = 3. \qquad .$$

Again, for students who are each initially endowed with one seat at the same school, the student with a higher ranking on the priority list has a higher chance to her best choice than the student at a lower ranking on the priority list. But for students at the same ranking position at distinct schools with different capacities, the student who owns a seat in a school with a smaller capacity is favored.

In geneal, we can partition schools into groups such that schools in the same group have the same capacity, as shown in table 5.3. Let $i_t^k(j)$ be the student on jth position at school S_t^k, where school S_t^k is a school from group G^k with k seats.

G^1			G^2			$\cdots$	G^q		
S_1^1	$\cdots$	$S_{t^1}^1$	S_1^2	$\cdots$	$S_{t^2}^2$	$\cdots$	S_1^q	$\cdots$	$S_{t^q}^q$
$\sqcap$	$\cdots$	$\sqcap$	$\sqcap$	$\cdots$	$\sqcap$	$\cdots$	$\sqcap$	$\cdots$	$\sqcap$
			$\sqcap$	$\cdots$	$\sqcap$	$\cdots$	$\sqcap$	$\cdots$	$\sqcap$
						$\cdots$	$\cdots$	$\cdots$	$\cdots$
							$\sqcap$	$\cdots$	$\sqcap$

Table 5.3: Schools with different capacities

We conjecture that the top trading cycles from individual endowments for school choice problems satisfies:

1. $\#(n^{th}, i_t^k(j)) = \#(n^{th}, i_{t'}^k(j)),\ \forall n,\ \forall k,\ \forall j.$

2. $\#(1^{st}, i_t^k(j)) > \#(1^{st}, i_t^k(j')), \forall j < j'$.

3. $\#(1^{st}, i_t^k(j)) > \#(1^{st}, i_{t'}^{k'}(j)), \forall k < k'$.

That is, two students who are each initially endowed with one seat with the same position at the schools with same capacities have the same chances to their kth best choices. For two students who are each initially endowed with one seat at different positions on the priority lists at the schools with same capacities, the one at a higher ranked position has a higher chance to her best choice. And for two students who are each initially endowed with one seat at the same position on the priority list at the schools with different capacities, the student from the school with smaller capacity has a higher chance to her best choice.

When we talk about the chances or probabilities in the above school choice problem or in the house allocation problem discussed in chapter 3, we basically assume that all possible preferences are equally like to occur. However, some schools or objects might be more popular than others in the real world. How to formalize the definition of fairness in these settings remains an open question.

In chapter 4, we study house allocation problems when indifferences are allowed. We show that the serial dictatorship with fixed tie-breaking satisfies weak Pareto efficiency, strategy-proofness, non-bossiness, and consistency, and further it is not Pareto dominated by any Pareto efficient and strategy-proof rule. As corollaries to Abdulkadiroğlu and Sönmez's (1998) result, the relationship between the serial dictatorships and the top trading cycles algorithm from individual endowments, and the equivalence of random serial dictatorship and the top trading cycles algorithm with random endowments still holds under weak preferences.

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