ON THE INTERPOLATION OF SMOOTH FUNCTIONS VIA RADIAL BASIS FUNCTIONS

A Dissertation

by

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ABSTRACT

We consider the interpolatory theory of bandlimited functions at both the integer lattice and at more general point sets in $\mathbb{R}^d$ by forming interpolants which lie in the linear span of translates of a single Radial Basis Function (RBF). Asymptotic behavior of the interpolants in terms of a given parameter associated with the RBF is considered; in these instances the original bandlimited function can be recovered in L_2 and uniformly by a limiting process.

Additionally, multivariate interpolation of nonuniform data is considered, and sufficient conditions are given on a family of RBFs which allow for recovery of multidimensional bandlimited functions. We also consider the rate of approximation that can be obtained in different cases. Sometimes, we may say something about the rate in terms of the RBF parameter mentioned above, while other times, we achieve rates based on a shrinking mesh size. The latter technique allows us to consider interpolation of Sobolev functions and their associated approximation rates as well.

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It is with no little surprise on my part that I am writing this section of my dissertation in preparation to complete my Ph.D. in Mathematics at Texas A&M. I began this journey nearly five years ago thanks in large part to what must have been quite some recommendations from Professors Raytcho Lazarov, Roger Smith, and Joel Zinn which led the then graduate head Paulo Lima-Filho to take a chance and admit me very late into the program. To them, as well as to the Department of Mathematics at Texas A&M, I give my heartfelt gratitude for providing me with this opportunity.

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I first met Professor Sivakumar when he was my instructor for undergraduate Complex Analysis during my final semester as an undergraduate here. I began his class with the intent to drop it if it was not interesting as I was registered for 18 hours. Unfortunately for my class schedule, nothing could have been further from the truth. Professor Sivakumar took a very interesting subject, and managed to supplement it with his wit and unique flavor of doing mathematics. Though he may never admit it, he has an excellent view of, and approach to, mathematics. His unique perspective has been beneficial in shaping my view of the subject. His precision and thoroughness in computations and writing is something that I have tried to emulate for my own betterment as a mathematician, though admittedly sometimes without success. I am forever grateful for his insight into the problems solved in this dissertation, as well as his wise advice on many matters. It was always a pleasure to sit in his office and talk about nothing in particular, or to see the endless parade of students coming in for Calculus help and seeing his witty interactions with them.

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NOMENCLATURE

$\mathbb{N}$	The Natural numbers (we do not include 0 in this definition)
$\mathbb{N}_0$	The Natural numbers including 0
$\mathbb{Z}$	The Integers
$\mathbb{R}$	The Real numbers
$\mathbb{C}$	The Complex numbers
$C^\infty(\mathbb{R})$	The space of continuous, infinitely differentiable functions on $\mathbb{R}$
$C_0(\mathbb{R})$	The space of continuous functions on $\mathbb{R}$ which decay to 0 at infinity
L_p	$L_p(\mathbb{R})$, the Banach space of p-integrable measurable functions
ℓ_p	$\ell_p(\mathbb{Z})$, the Banach space of (infinite) p-summable sequences
W_p^k	$W_p^k(\mathbb{R})$, the Sobolev space of functions whose first k weak derivatives are in L_p
$\mathscr{S}$	The Schwartz space of rapidly decreasing functions
$\mathscr{S}'$	The space of Tempered Distributions
$\widehat{f}$	The Fourier transform on L_1 or $\mathscr{S}'$
$\mathscr{F}[f]$	The Fourier transform of a function $f \in L_2$
$\lvert \cdot \rvert$	The Euclidean norm on $\mathbb{R}^d$
$\lVert \cdot \rVert_X$	The norm of a linear space X (usually a Banach space)
a.e.	almost everywhere (with respect to Lebesgue measure)
RBF	Radial Basis Function
CIS	Complete Interpolating Sequence
PW_σ	The Paley-Wiener space of bandlimited functions whose Fourier transforms are supported on $[-\sigma, \sigma]$
PW_S	The Paley-Wiener space over the set $S \subset \mathbb{R}^d$

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1. INTRODUCTION

It has long been an endeavour in mathematics to find techniques which allow one to approximate complicated functions by simpler ones. A classical example of this is found in the Weierstrass Approximation Theorem, which states that any continuous function on the interval $[0, 1]$ can be uniformly approximated (as well as one would like) by a polynomial. A constructive proof of Weierstrass's Theorem uses Bernstein polynomials to do the approximation. Let us start with a general formulation of the problem.

Problem 1.0.1 (General Approximation Problem)**.** *Given a class of functions $\mathcal{C}$, form an Approximation Space of functions $\mathcal{A}$ such that given any function f in $\mathcal{C}$, we can find some function g in $\mathcal{A}$ such that g "closely approximates" f.*

Of course the above problem is quite vague. In general, we need some sort of measurement of "closeness" of two functions. Often, the class $\mathcal{C}$ will be a normed linear space, and $\mathcal{A}$ a subspace of $\mathcal{C}$, in which case the notion of distance is provided by the norm. Heuristically, one chooses $\mathcal{A}$ to be a space of functions that is simpler than $\mathcal{C}$ in some manner. Again, the Weierstrass Approximation Theorem is illustrative in that polynomials are generally considered more easily understood than arbitrary continuous functions.

Note that in the above example, there is no guarantee that the value of an approximating polynomial at any point in the interval $[0, 1]$ will actually coincide with the continuous function it is approximating. That is, if p is a polynomial approximating $f \in C[0, 1]$, then there may not exist an $x \in [0, 1]$ such that $p(x) = f(x)$. However, another potential approximation scheme involves *interpolation*, in which one attempts to match the values of the original function at certain prescribed points.

We formulate below a general problem in this vein.

Problem 1.0.2 (General Interpolation Problem)**.** *Given a class, $\mathcal{C}$, of real-valued functions defined on $\mathbb{R}^d$ and a set of distinct points $X := (x_i)_{i\in I} \subset \mathbb{R}^d$, form an Approximation Space of functions $\mathcal{A}$ such that, given any function f in $\mathcal{C}$, we can find some function g in $\mathcal{A}$ such that $g(x_i) = f(x_i)$ for all $i \in I$, and moreover g "closely approximates" f.*

An example of such an interpolation scheme would be to let $\mathcal{C} = C[0,1]$ as in the setting of the Weierstrass Approximation Theorem, let $X = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\right\}$ for some $N \in \mathbb{N}$, and let $\mathcal{A}$ be, say, the set of cubic splines: namely, piecewise cubic polynomials. It is a classic fact that one can uniformly approximate any $f \in C[0,1]$ by a cubic spline p such that $p\left(\frac{i}{N}\right) = f\left(\frac{i}{N}\right)$ for $i = 0, \ldots, N$, as long as N is chosen suitably large. In general, the index set I could be finite or countably infinite. In the sequel, we will focus our attention on the infinite case.

We may further restrict our considerations to forming an approximation space which consists of linear combinations of shifts of a single function. For simplicity, suppose that $\mathcal{C} = C[0,1]$ as before, $X = (x_i)_{i\in I}$ be a set of distinct points in $[0,1]$, and let ϕ be a function on $[0,1]$. We define a potential approximation space $\mathcal{A}_{\phi,X} := \left\{\sum_{i\in I} a_i\phi(\cdot - x_i) : (a_i) \subset \mathbb{R}\right\}$, and ask if we can closely approximate $f \in C[0,1]$ by $g \in \mathcal{A}$.

Problem 1.0.3 (General Approximation by Translates of a Single Function)**.** *Given a class, $\mathcal{C}$, of real-valued functions defined on $\mathbb{R}^d$, a set of distinct points $X := (x_i)_{i\in I} \subset \mathbb{R}^d$, and a function $\phi : \mathbb{R}^d \to \mathbb{R}$, form the Approximation Space*

$$\mathcal{A} := \mathcal{A}_{\phi,X} := \left\{\sum_{i\in I} a_i\phi(\cdot - x_i) : (a_i) \subset \mathbb{R}\right\}.$$

Then for every f in $\mathcal{C}$, find a function g in $\mathcal{A}$ such that g "closely approximates" f.

We can also formulate the corresponding interpolation problem:

Problem 1.0.4 (General Interpolation by Translates of a Single Function)**.** *Given a class of functions $\mathcal{C}$ as before, a set of distinct points $X := (x_i)_{i\in I} \subset \mathbb{R}^d$, and a function $\phi : \mathbb{R}^d \to \mathbb{R}$, form the Approximation Space*

$$\mathcal{A} := \mathcal{A}_{\phi,X} := \left\{ \sum_{i\in I} a_i \phi(\cdot - x_i) : (a_i) \subset \mathbb{R} \right\}.$$

Then for every f in $\mathcal{C}$, can we find a function g in $\mathcal{A}$ such that $g(x_i) = f(x_i)$ for all $i \in I$, and moreover, g "closely approximates" f?

Besides considering different approximation schemes, we will also pay some attention to the rate of approximation of functions; that is, how "fast" the approximation occurs in some sense. Given different approximation schemes, there are typically different ways of measuring the rate of approximation, be it in terms of some limiting parameter or by adding more interpolation points, for example. Our primary technique for examining approximation rates will be in the context of Problem 1.0.4. We will consider a countable sequence of points $X := (x_j)_{j\in\mathbb{Z}}$, and subsequently a parameter $h \in (0, 1]$. Then we pose the following question:

Problem 1.0.5 (General Approximation Rate Problem)**.** *Given a normed space, $(\mathcal{C}, \|\cdot\|)$, of real-valued functions defined on $\mathbb{R}^d$, a set of distinct points $X := (x_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^d$, and an Approximation Space $\mathcal{A}$, find a function $R(h) : (0, 1] \to [0, \infty)$ such that, for every f in $\mathcal{C}$, there exists a function g in $\mathcal{A}$ such that $g(hx_j) = f(hx_j)$ for all $j \in \mathbb{Z}$, and moreover $\|f - g\| \le R(h)\|f\|$. (The function R is usually called the approximation rate).*

Often, the parameter h is called the "mesh size," and the goal is to find approximation schemes such that $R(h) \to 0$ as $h \to 0$. Of course if $R_1(h) \leq R_2(h)$ for all $h \in (0,1]$, then the approximation scheme that yields a rate of R_1 is better than the one with associated rate R_2. In many cases we will consider, we achieve polynomial rates of approximation depending on the smoothness of the target class $\mathcal{C}$; i.e. $R(h) = h^k$ for some $k \in \mathbb{N}$.

Our contribution addresses some of these problems – specifically Problems 1.0.4 and 1.0.5 – for certain classes of functions and approximation spaces. We typically consider $\mathcal{C}$ to be either the class of bandlimited functions on $\mathbb{R}^d$ or of Sobolev functions of a given smoothness and integrability. Additionally, we typically consider Approximation Spaces formed by linear combinations of translates of a single Radial Basis Function (RBF). We will always consider interpolation problems, sometimes by interpolating at the integer lattice, but also more generally at non-uniform sets of points $X \subset \mathbb{R}^d$.

2. BASIC NOTIONS

We begin by introducing some basic tools and definitions which will be used throughout the rest of this work.

Suppose that $d \in \mathbb{N}$ and $1 \le p < \infty$. If $\Omega \subset \mathbb{R}^d$ is a set with positive Lebesgue measure, then define the usual Banach spaces of Lebesgue measurable functions over Ω and their associated norms by

$$L_p(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \|f\|_{L_p(\Omega)} < \infty \right\}, \quad \|f\|_{L_p(\Omega)} := \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}},$$

where dx is the Lebesgue measure on $\mathbb{R}^d$. When $p = \infty$, let

$$L_\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \|f\|_{L_\infty(\Omega)} < \infty \right\},$$

where

$$\|f\|_{L_\infty(\Omega)} := \inf \left\{ C \ge 0 : |f(x)| \le C \text{ for almost every } x \text{ in } \Omega \right\}.$$

Here, almost everywhere is in the sense of the d-dimensional Lebesgue measure. If no set is specified, we mean $L_p(\mathbb{R}^d)$, (or $L_p(\mathbb{R})$ if $d = 1$).

Similarly, for $1 \le p < \infty$, define the sequence spaces indexed by a set I and their norms by

$$\ell_p(I) := \left\{ (a_i)_{i \in I} \subset \mathbb{R} : \|a\|_{\ell_p(I)} < \infty \right\}, \quad \|a\|_{\ell_p(I)} := \left(\sum_{i \in I} |a_i|^p \right)^{\frac{1}{p}}.$$

When $p = \infty$, let $\|a\|_{\ell_\infty(I)} := \sup\limits_{i \in I} |a_i|$, and

$$\ell_\infty(I) := \left\{ (a_i)_{i \in I} \subset \mathbb{R} : \|a\|_{\ell_\infty(I)} < \infty \right\}.$$

If no index set is given, we refer to $\ell_p(\mathbb{Z})$.

Given a function $g \in L_1(\mathbb{R}^d)$, its Fourier transform, $\widehat{g}$, is defined by the following:

$$\widehat{g}(\xi) := \int_{\mathbb{R}^d} g(x)e^{-i\langle \xi, x\rangle}dx, \quad \xi \in \mathbb{R}^d. \tag{2.1}$$

The Fourier transform can be uniquely extended from $L_1 \cap L_2(\mathbb{R}^d)$ to a linear isometry (up to a constant factor) of $L_2(\mathbb{R}^d)$ onto itself. We denote by $\mathscr{F}[g]$, the Fourier transform of a function $g \in L_2(\mathbb{R}^d)$. Moreover, the Parseval-Plancherel Identity states that

$$\|\mathscr{F}[g]\|_{L_2} = (2\pi)^{\frac{d}{2}}\|g\|_{L_2}. \tag{2.2}$$

If g is also continuous and $\mathscr{F}[g] \in L_1$, then the following inversion formula holds:

$$g(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathscr{F}[g](\xi)e^{i\langle \xi, x\rangle}d\xi, \quad x \in \mathbb{R}^d, \tag{2.3}$$

(see, for example, [48]).

Additionally, define $W_2^k := W_2^k(\mathbb{R})$ to be the Sobolev space over $\mathbb{R}$ of functions in L_2 whose first k weak (or distributional) derivatives are in L_2. Recall that if T is a distribution, then its k-th derivative is the distribution defined by

$$\left\langle T^{(k)}, \phi \right\rangle = (-1)^k \left\langle T, \phi^{(k)} \right\rangle, \quad \text{for all } \ \phi \in \mathcal{D},$$

where $\mathcal{D}$ is the set of *test functions* which are in $C^\infty(\mathbb{R})$ and have compact support ($C^\infty(\mathbb{R})$ being the set of all continuous functions on $\mathbb{R}$ which are infinitely differentiable). The seminorm on W_2^k is defined by

$$|g|_{W_2^k} := \left(\int_{\mathbb{R}} |g^{(k)}(x)|^2 dx\right)^{\frac{1}{2}} = \|g^{(k)}\|_{L_2},$$

and the norm on the same space can be defined by

$$\|g\|_{W_2^k} := \left(\|g\|_{L_2}^2 + |g|_{W_2^k}^2 \right)^{\frac{1}{2}}.$$

Another basic property of the Fourier Transform, which will be used frequently, is that for $g \in W_2^k$,

$$\mathscr{F}[g^{(k)}](\xi) = (i\xi)^k \mathscr{F}[g](\xi). \tag{2.4}$$

In light of (2.2) and (2.4), the norms on the spaces L_2 and W_2^k may be expressed as follows:

$$\|g\|_{L_2} = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\mathscr{F}[g](\xi)|^2 \, d\xi \right)^{\frac{1}{2}}, \; \|g\|_{W_2^k} = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} \left(1 + |\xi|^k\right)^2 |\mathscr{F}[g](\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

Similarly, the seminorm becomes

$$|g|_{W_2^k} = \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\xi|^{2k} |\mathscr{F}[g](\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{2.5}$$

This is the seminorm that will be used throughout, since most calculations will be carried out in the Fourier transform domain. Additionally, if $\Omega \subset \mathbb{R}$ is an interval, let $|g|_{W_2^k(\Omega)}$ be as in (2.5) with the integral taken over Ω.

An important class of functions for what follows will be the Paley-Wiener, or bandlimited, functions. For $\sigma > 0$, these spaces are defined as follows:

$$PW_\sigma := \{ f \in L_2 : \mathscr{F}[f] = 0 \text{ almost everywhere outside } [-\sigma, \sigma] \}. \tag{2.6}$$

The Paley-Wiener Theorem asserts that an equivalent definition of this space is that of entire functions of exponential type σ whose restriction to $\mathbb{R}$ is in L_2. Similarly, define the multivariate Paley-Wiener spaces to be

$$PW_\sigma^{(d)} := \left\{ f \in L_2(\mathbb{R}^d) : \mathscr{F}[f] = 0 \text{ almost everywhere outside } [-\sigma,\sigma]^d \right\}. \quad (2.7)$$

Furthermore, let the class PW_σ^k be functions whose k-th derivatives lie in PW_σ. That is

$$PW_\sigma^k := \{f : \mathbb{R} \to \mathbb{R} : f^{(k)} \in PW_\sigma\}.$$

Note that $f \in PW_\sigma^k$ does not imply that f is square-integrable; for example, any polynomial of degree at most $k-1$ is in PW_σ^k for any σ, because its k-th derivative is identically 0, hence bandlimited. Since all one-dimensional Paley-Wiener spaces are isometrically isomorphic (see Theorem A.0.5), we typically consider the canonical case of PW_π. In higher dimensions, let S be a set of positive Lebesegue measure. Then define the Paley-Wiener space of S-bandlimited functions via

$$PW_S := \{f \in L_2(\mathbb{R}^d) : \mathscr{F}[f] = 0 \text{ almost everywhere outside } S\}.$$

As is customary, we use C to denote an absolute constant whose value may vary from line to line, and we use subscripts to denote dependence of C on certain parameters.

Definition 2.0.6. *Let f and g be two functions on $\mathbb{R}$. Then $f(x) = \mathcal{O}(g(x)),\ \ |x| \to \infty$, provided there exists some $x_0 > 0$ and an absolute constant C such that*

$$|f(x)| \leq C|g(x)|, \quad \text{for all } |x| \geq x_0.$$

In this case, it is said that $f(x)$ is "big O" of $g(x)$.

Recall that a function $f : \mathbb{C} \to \mathbb{R}$ is said to be *entire* if it can be represented as a power series which converges on all of the complex plane.

Definition 2.0.7. *An entire function f is said to have* ***exponential type*** $\sigma > 0$ *if for every $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$|f(z)| \leq Ce^{(\sigma+\varepsilon)|z|}, \quad z \in \mathbb{C}. \tag{2.8}$$

The smallest σ for which (2.8) holds is called the ***type*** *of f. One can also calculate the type of f by $\sigma = \limsup_{|z|\to\infty} \frac{|f(z)|}{|z|}$.*

By convention, the constant 0 function is said to have exponential type 0. Any polynomial has exponential type 0 as well. The sinc function, defined by $f(z) := \frac{\sin(\pi z)}{\pi z}$, has exponential type π.

3. ON CARDINAL INTERPOLATION AND THE SAMPLING THEOREM

3.1 Observations on the Sampling Theorem

In this chapter, we study Problem 1.0.4 in the case of interpolation at the (multi) integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ for different RBFs. Specifically, we will consider interpolation of bandlimited functions. Thus Problem 1.0.4 can be more specifically restated as follows.

Problem 3.1.1 (Cardinal Interpolation of Bandlimited Functions). *Let $d \in \mathbb{N}$, $\sigma > 0$, and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a given radial basis function. Define the Approximation Space*

$$\mathcal{A} := \mathcal{A}_{\phi,\mathbb{Z}^d} := \left\{ \sum_{j\in\mathbb{Z}^d} a_j \phi(\cdot - j) : (a_j) \subset \mathbb{R} \right\}.$$

Given an $f \in PW_\sigma^{(d)}$, find a cardinal interpolant, *$\mathscr{I}_\phi f \in \mathcal{A}$, such that $\mathscr{I}_\phi f(j) = f(j)$ for every $j \in \mathbb{Z}^d$, and $\|\mathscr{I}_\phi f - f\|_{L_2(\mathbb{R}^d)}$ is small.*

As mentioned before, there is no loss of generality in reducing to the canonical case $\sigma = \pi$. The function $\mathscr{I}_\phi f$ is called the *cardinal interpolant of f* because it coincides with the function f at the integer lattice $\mathbb{Z}^d$.

This problem may be approached in several ways which are all essentially equivalent. Perhaps the most illustrative approach comes from first considering a classical univariate sampling theorem from the early 1900's, due independently to E. T. Whittaker, Kotelnikov, and Shannon.

Theorem 3.1.2 (WKS Sampling Theorem). *If $f \in PW_\pi$, then*

$$f(x) = \sum_{j\in\mathbb{Z}} f(j) \frac{\sin(\pi(x-j))}{\pi(x-j)}, \quad x \in \mathbb{R}.$$

Many questions arise from this theorem, one of which is the question of summability. The function $\sin(x)/x$ decays like $1/|x|$, which is quite slow. Consequently, the series above does not converge uniformly in general, but locally uniformly. Isaac Schoenberg asked if one can drop the condition of complete recovery, given by the sampling theorem, in exchange for a series that merely interpolates the bandlimited function, but which nonetheless converges uniformly. Schoenberg referred to such techniques as *summability methods*. Before moving on, one more pertinent observation about the series in Theorem 3.1.2 is in order. The function $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ has the following interpolatory property on the integer lattice: $\operatorname{sinc}(0) = 1$, and $\operatorname{sinc}(j) = 0$ for $j \in \mathbb{Z} \setminus \{0\}$. This property is recorded below for future use.

Definition 3.1.3. *A function $L : \mathbb{R}^d \to \mathbb{R}$ is called a* **fundamental function***, or equivalently a* **cardinal function***, provided*

$$L(j) = \delta_{0,j} := \begin{cases} 1, & j = 0 \\ 0, & j \neq 0 \end{cases}, \quad j \in \mathbb{Z}^d.$$

The sinc function gets its name from the abbreviation of the Latin *sinus cardinalis*, meaning "cardinal sine."

Let us now undertake the task of creating more general fundamental functions which are generated by integer translates of a single RBF. Given a function $\phi : \mathbb{R}^d \to \mathbb{R}$, formally define

$$\widehat{L_\phi}(\xi) := \frac{\widehat{\phi}(\xi)}{\sum_{j \in \mathbb{Z}^d} \widehat{\phi}(\xi + 2\pi j)}, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{3.1}$$

Of course, at the moment, this function is not the Fourier transform of anything, despite our suggestive notation. However, under the correct assumptions it will be the Fourier transform of a fundamental function. Assume that $\widehat{\phi}$ is always non-negative; then by definition, $\widehat{L_\phi}$ is non-negative as well. Of course, an equivalent assumption would be that $\widehat{\phi}$ is non-positive, in which case $\widehat{L_\phi}$ would have the same property. If in addition, $\widehat{L_\phi} \in L_1(\mathbb{R}^d)$, then the following holds.

Theorem 3.1.4. *Let $\widehat{\phi} : \mathbb{R}^d \to \mathbb{R}$ be non-negative (or non-positive) such that $\sum_{j\in\mathbb{Z}^d} \widehat{\phi}(\xi + 2\pi j)$ has no zeros, and let $\widehat{L_\phi}$ be defined by (3.1). If $\widehat{L_\phi} \in L_1(\mathbb{R}^d)$, define*

$$L_\phi(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{L_\phi}(\xi) e^{i\langle x,\xi\rangle} d\xi, \quad x \in \mathbb{R}^d. \tag{3.2}$$

The function L_ϕ is continuous, square-integrable, and a fundamental function. Moreover, $\widehat{L_\phi}$ is the Fourier Transform of L_ϕ.

Proof. Continuity and square-integrability of L_ϕ follow from the Fourier inversion theorem, as does the "moreover" statement. To prove that L_ϕ is a fundamental function, we employ a periodization argument that will be used quite frequently. Let $\Omega := [-\pi, \pi]^d$ and $k \in \mathbb{Z}^d$. Then via the substitution $u = \xi + 2\pi\ell$,

$$\begin{aligned}
L_\phi(k) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\widehat{\phi}(\xi)}{\sum_{j\in\mathbb{Z}^d} \widehat{\phi}(\xi + 2\pi j)} e^{i\langle k,\xi\rangle} d\xi \\
&= \frac{1}{(2\pi)^d} \sum_{\ell\in\mathbb{Z}^d} \int_{\Omega+2\pi\ell} \frac{\widehat{\phi}(\xi)}{\sum_{j\in\mathbb{Z}^d} \widehat{\phi}(\xi + 2\pi j)} e^{i\langle k,\xi\rangle} d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\Omega} \sum_{\ell\in\mathbb{Z}^d} \frac{\widehat{\phi}(u - 2\pi\ell) e^{-i\langle k,2\pi\ell\rangle}}{\sum_{j\in\mathbb{Z}^d} \widehat{\phi}(u - 2\pi\ell + 2\pi j)} e^{i\langle k,u\rangle} du
\end{aligned}$$

$$= \frac{1}{(2\pi)^d} \int_\Omega e^{i\langle k,u\rangle} du$$

$$= \delta_{0,k}.$$

The interchange of sum and integral in the third line is justified by Tonelli's Theorem, for example.

$\square$

Armed with a fundamental function, we formally define the cardinal interpolant of a function $f \in PW_\pi^{(d)}$ via

$$\mathscr{I}_\phi f(x) := \sum_{j\in\mathbb{Z}^d} f(j) L_\phi(x-j), \quad x \in \mathbb{R}^d. \tag{3.3}$$

By Theorem 3.1.4,

$$\mathscr{I}_\phi f(k) = \sum_{j\in\mathbb{Z}^d} f(j) L_\phi(k-j) = \sum_{j\in\mathbb{Z}^d} f(j)\delta_{j,k} = f(k), \quad k \in \mathbb{Z}^d.$$

Thus $\mathscr{I}_\phi$ interpolates f on $\mathbb{Z}^d$, and accordingly, we shall call $\mathscr{I}_\phi f$ the *cardinal interpolant of* f, and $\mathscr{I}_\phi$ the *cardinal interpolation operator*.

Typically, there is a parameter associated with the function ϕ that one may exploit to change the decay of the fundamental function, L_ϕ, away from the origin. In the sequel, such parameters will be used to yield asymptotic recovery results for bandlimited functions. Schoenberg pioneered the subject of cardinal interpolation in the 1960's (see for example [41]-[44]). He considered the family of so-called *cardinal B-splines*:

$$\mathcal{M} := \left\{ M_n := \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} \underbrace{*\cdots*}_{n} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} \;:\; n \in \mathbb{N} \right\}.$$

The integer n is the parameter mentioned above, and Schoenberg proved that

$$L_{M_n}(x) = \mathcal{O}(e^{-|x|}), \quad |x| \to \infty,$$

(the big O constant, see Definition 2.0.6, depends on n). Consequently, it is easy to see that the series in (3.3) corresponding to $\mathscr{I}_{M_n}$ converges uniformly as long as the data sequence $(f(j))_{j\in\mathbb{Z}}$ is bounded (which it is for any Paley-Wiener function, f).

The theory of radial basis functions supplied more examples of such fundamental functions, as Schoenberg's work was taken up by a host of followers. In 1992, Baxter [5] discussed the asymptotic behavior of the fundamental function associated with the Hardy multiquadric, $\phi_c(x) := (|x|^2 + c^2)^{\frac{1}{2}}$. This result sparked interest from others, and in the 90's and early 2000's, Baxter, Riemenschneider, and Sivakumar produced a series of fundamental papers [6], [36]-[39], [45] concerning cardinal interpolation via shifts of the Hardy multiquadric and the Gaussian kernel. Namely, they considered the families

$$\mathcal{Q} := \left\{ (|\cdot|^2 + c^2)^{\frac{1}{2}} \ : \ c > 0 \right\}, \quad \text{and}$$

$$\mathcal{G} := \left\{ e^{-\lambda|\cdot|^2} \ : \ \lambda > 0 \right\}.$$

It was shown by Buhmann [7] that the fundamental function associated with the Hardy multiquadric satisfies

$$L_{\phi_c}(x) = \mathcal{O}(|x|^{-5}), \quad |x| \to \infty,$$

while for that associated with the Gaussian, $g_\lambda(x) := e^{-\lambda|x|^2}$,

$$L_{g_\lambda}(x) = \mathcal{O}(e^{-\lambda|x|}), \quad |x| \to \infty.$$

Recent efforts of the author and Ledford [18] have extended the results of Baxter to a broader class of *general multiquadrics*,

$$\mathcal{Q}_\alpha := \left\{ \left(|\cdot|^2 + c^2 \right)^\alpha \;:\; c > 0 \right\}.$$

The following Theorem provides a summary of the decay of the univariate fundamental function associated with the general multiquadric $\phi_{\alpha,c}(x) := (|x|^2 + c^2)^\alpha$, $x \in \mathbb{R}$.

Theorem 3.1.5 (cf. [18] Corollaries 4.4,4.6, and 4.8)**.**

(i) If $\alpha \in (0, \infty)$ *and* $c \geq 1$*, then*

$$L_{\phi_{\alpha,c}}(x) = \mathcal{O}\left(|x|^{-\lfloor 2\alpha+1 \rfloor} \right), \quad |x| \to \infty.$$

(ii) If $\alpha = -1$*, and* $c \geq 1$*, then*

$$L_{\phi_{-1,c}}(x) = \mathcal{O}(|x|^{-k}), \quad |x| \to \infty,$$

for every $k \in \mathbb{N}$*.*

(iii) If $\alpha \in (-\infty, -1)$*, and* $c \geq 1$*, then*

$$L_{\phi_{\alpha,c}}(x) = \mathcal{O}\left(|x|^{-\lceil 2|\alpha|-2 \rceil} \right), \quad |x| \to \infty.$$

Note that the so-called *Poisson kernel*, $\phi_{-1,c}$, exhibits much better behavior - its fundamental function decays faster than any polynomial - owing to the fact that its Fourier transform is purely an exponential function, which is not the case for the other general multiquadrics. Another consideration is that for most values of the exponent α, specifically $\alpha \in (-\infty, -3/2) \cup \{-1\} \cup [1/2, \infty)$, the series in (3.3) will converge uniformly as desired.

The final consideration for Problem 3.1.1 is the desire to make $\|\mathscr{I}_\phi f - f\|_{L_2}$ small in some quantifiable way. To explore the answer to that question, it is germane to consider the role of the parameters of the functions discussed above. In fact, upon considering the asymptotic behavior of the cardinal interpolants in terms of their parameters, an answer presents itself. Intertwined with this conclusion is the answer to another relevant question: that of not only approximating, but recovering, a bandlimited function via its cardinal interpolant.

Let $K \subset (0, \infty)$ be an infinite set, and let $(\phi_\kappa)_{\kappa \in K}$ be a family of functions on $\mathbb{R}^d$, and let $\mathscr{I}_\kappa f$ be defined as in (3.3) with ϕ replaced by ϕ_κ.

Problem 3.1.6 (Recovery of Bandlimited Functions)**.** *Given a sequence of functions $(\phi_\kappa)_{\kappa \in K}$ with associated cardinal interpolation operators $(\mathscr{I}_\kappa)_{\kappa \in K}$, do the following hold for every $f \in PW_\pi^{(d)}$?*

$$\begin{aligned} &1) \quad \lim_{\kappa} \|\mathscr{I}_\kappa f - f\|_{L_2(\mathbb{R}^d)} = 0, \\ &2) \quad \lim_{\kappa} |\mathscr{I}_\kappa f(x) - f(x)| = 0, \text{ uniformly on } \mathbb{R}^d, \end{aligned}$$

where κ approaches some limiting value (typically 0 or ∞).

Schoenberg's investigation of spline interpolants brought forth a positive solution to Problem 3.1.6 for the B-splines; here $\kappa = n$, and recovery is achieved in 1) and 2) as $n \to \infty$. Later, Baxter's work gave a positive solution for the interpolants associated with the Hardy multiquadric when one lets the *shape parameter* $c \to \infty$. In the case of the Gaussian interpolants considered by Baxter, Riemenschneider, and Sivakumar, Problem 3.1.6 once again has a positive solution when the Gaussian parameter $\lambda \to 0^+$.

Recently, Ledford [28] has generalized Problem 3.1.6 in a different way by considering sufficient conditions on the family of functions $(\phi_\kappa)_{\kappa\in K}$ for the problem to have a positive solution. He calls such a family of functions a *regular family of cardinal interpolators.* All of the previously mentioned examples are regular families, as are more general multiquadrics, namely, $\phi_{\alpha,c}(x) := (|x|^2 + c^2)^\alpha$ for $\alpha \in (-\infty, -\frac{2d+1}{2}] \cup [\frac{1}{2}, \infty) \setminus \mathbb{N}_0$. In this case, $\kappa = c \to \infty$.

However, the following section is devoted to showing that, with a more specific analysis, the family

$$\mathcal{Q}_\alpha := \left\{ \left(|\cdot|^2 + c^2\right)^\alpha \ : \ c \geq 1 \right\}$$

provides a positive answer to Problem 3.1.6 for every $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$.

One interesting note on the driving force behind the proof of these facts is the following observation: for every family listed above, the fundamental functions converge pointwise to the sinc function as the parameter κ goes to its limiting value.

3.2 Recovery of Bandlimited Functions Using General Multiquadrics

The purpose of this section is to extend the result of Baxter [5] to the broader family of general multiquadrics, $\mathcal{Q}_\alpha$, discussed in the previous section (the work in this, and only this, section was done jointly with Ledford in [18]). To wit, if $\alpha \in \mathbb{R}$ and $c > 0$, consider the d-dimensional general multiquadric

$$\phi_{\alpha,c}(x) := \left(|x|^2 + c^2\right)^\alpha, \quad x \in \mathbb{R}^d, \tag{3.4}$$

where $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}^d$. One finds from [47, Theorem 8.15] that the distributional Fourier transform of the general multiquadric coincides with a legitimate function when $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$:

$$\widehat{\phi_{\alpha,c}}(\xi) = \frac{2^{1+\alpha}}{\Gamma(-\alpha)} \left(\frac{c}{|\xi|}\right)^{\alpha+\frac{d}{2}} K_{\alpha+\frac{d}{2}}(c|\xi|), \quad \xi \in \mathbb{R}^d \setminus \{0\}, \tag{3.5}$$

where $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is the usual Gamma function, and

$$K_\nu(r) = \frac{1}{2}\int_0^\infty e^{-r\cosh t}e^{\nu t}dt, \quad r > 0,\ \nu \in \mathbb{R}. \tag{3.6}$$

The function K_ν is called the modified Bessel function of the second kind. A few comments concerning these functions are in order. First note that both $\phi_{\alpha,c}$ and its Fourier transform are radial, that is $\phi_{\alpha,c}(x) = \phi_{\alpha,c}(|x|)$. It is also clear from the definition that the modified Bessel function is symmetric in its order; that is, $K_{-\nu} = K_\nu$ for any $\nu \in \mathbb{R}$. Additionally, the Bessel function K_ν comprises a simple pole of order ν at the origin, and a term decaying exponentially away from the origin. Since $\phi_{\alpha,c}$ is integrable and square-integrable whenever $\alpha < -d/2$, traditional Fourier theory shows that its Fourier transform must be continuous and square-integrable as well, a fact that is exhibited by the cancellation of the pole of order $-\alpha - \frac{d}{2}$ resulting from the Bessel function ($K_{\alpha+\frac{d}{2}} = K_{-\alpha-\frac{d}{2}}$) and the power $|\xi|^{-\alpha-\frac{d}{2}}$ in (3.5). This cancellation shows that $\widehat{\phi_{\alpha,c}}$ is bounded at the origin, and the exponential decay of the modified Bessel function provides integrability. A proof of this fact will be given in Proposition 5.2.3, but shall not be needed here. For other values of α, the formula (3.5) coincides with the distributional Fourier transform of the general multiquadric, except in the case that α is a non-negative integer. If $\alpha \in \mathbb{N}_0$, then its distributional Fourier transform exists, but is a differential operator on the Schwartz space, and thus cannot be represented as a function.

For notational convenience, let $\widehat{L_{\alpha,c}}$ be the function in (3.1) with $\widehat{\phi}$ replaced by $\widehat{\phi_{\alpha,c}}$. It is evident from (3.5) and (3.6) that $\widehat{\phi_{\alpha,c}}$ is either non-negative or non-positive throughout the real line (depending on the sign of $\Gamma(-\alpha)$). Consequently, if $\widehat{L_{\alpha,c}}$ is integrable, Theorem 3.1.4 implies that $L_{\alpha,c}$ is a fundamental function associated with the general multiquadric $\phi_{\alpha,c}$. The following lemmas serve as a prelude to the proof of integrability.

Lemma 3.2.1. *Let $R > r > 0$, $c > 0$, and $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$. Then*

$$|\widehat{\phi_{\alpha,c}}(R)| \leq \left(\frac{R}{r}\right)^{-\alpha-\frac{d}{2}} e^{-c(R-r)}|\widehat{\phi_{\alpha,c}}(r)|.$$

Proof. Defining $\lambda := \lambda_{c,\alpha,d} := \frac{2^{1+\alpha}}{\Gamma(-\alpha)}c^{\alpha+\frac{d}{2}}$, equations (3.5) and (3.6) yield the following series of estimates:

$$\begin{aligned}
|\widehat{\phi_{\alpha,c}}(R)| &= |\lambda|R^{-\alpha-\frac{d}{2}}\int_0^\infty e^{-cR\cosh(t)}e^{(\alpha+\frac{d}{2})t}dt \\
&= |\lambda|\left(\frac{R}{r}\right)^{-\alpha-\frac{d}{2}} r^{-\alpha-\frac{d}{2}}\int_0^\infty e^{-c(R-r)\cosh(t)}e^{-cr\cosh(t)}e^{(\alpha+\frac{d}{2})t}dt \\
&\leq |\lambda|\left(\frac{R}{r}\right)^{-\alpha-\frac{d}{2}} r^{-\alpha-\frac{d}{2}}e^{-c(R-r)}\int_0^\infty e^{-cr\cosh(t)}e^{(\alpha+\frac{d}{2})t}dt \\
&= \left(\frac{R}{r}\right)^{-\alpha-\frac{d}{2}} e^{-c(R-r)}|\widehat{\phi_{\alpha,c}}(r)|.
\end{aligned}$$

The inequality above comes from the fact that $\cosh(t) \geq 1$. $\square$

Note that if $\alpha + \frac{d}{2} \geq 0$, then $(R/r)^{-\alpha-\frac{d}{2}} \leq 1$, and so the upper bound is purely exponential, though this will not be expressly needed. The following inequalities are summarized from [47, Section 5.1].

Lemma 3.2.2. *(i) If $\alpha + d/2 \geq 1/2$, then*

$$K_{\alpha+\frac{d}{2}}(r) \geq \sqrt{\frac{\pi}{2}}\, r^{-1/2} e^{-r}, \quad r > 0.$$

(ii) If $\alpha + d/2 < 1/2$ and $r > 1$, then

$$K_{\alpha+\frac{d}{2}}(r) \geq C_{\alpha,d}\, r^{-1/2} e^{-r}, \quad \text{where} \quad C_{\alpha,d} := \frac{\sqrt{\pi}\, 3^{\alpha+\frac{d}{2}-\frac{1}{2}}}{2^{\alpha+\frac{d}{2}+1}\Gamma\left(\alpha+\frac{d}{2}+\frac{1}{2}\right)}.$$

(iii)

$$K_{\alpha+\frac{d}{2}}(r) \leq \sqrt{2\pi}\, r^{-1/2} e^{-r} e^{\frac{|\alpha+\frac{d}{2}|^2}{2r}}, \quad r > 0.$$

(iv)

$$K_{\alpha+\frac{d}{2}}(r) \leq 2^{\alpha+\frac{d}{2}-1}\Gamma\left(\alpha+\frac{d}{2}\right) r^{-\alpha-\frac{d}{2}}, \quad r > 0.$$

Proposition 3.2.3. *Let $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ and $c > 0$. Then $\widehat{L_{\alpha,c}} \in L_1(\mathbb{R}^d)$. Consequently, by Theorem 3.1.4, the function $L_{\alpha,c}$, defined to be the inverse Fourier transform of $\widehat{L_{\alpha,c}}$, is a fundamental function.*

Proof. First, choose a large positive number M. Then, since $|\widehat{L_{\alpha,c}}(\xi)| \leq 1$ for all ξ,

$$\int_{[-M,M]^d} |\widehat{L_{\alpha,c}}(\xi)| d\xi \leq (2M)^d.$$

It remains to estimate

$$I := \int_{\mathbb{R}^d \setminus [-M,M]^d} |\widehat{L_{\alpha,c}}(\xi)| d\xi.$$

To do this, we establish a pointwise estimate for $\widehat{L_{\alpha,c}}(\xi)$. Let $\xi \in \mathbb{R}^d \setminus [-M,M]^d$ be fixed. Since M is large, there exists some $k_\xi \in \mathbb{Z}^d \setminus \{0\}$ such that $2\pi \leq |\xi + 2\pi k_\xi| \leq 4\pi$. Additionally, Lemma 3.2.2(ii) provides a positive constant $\gamma := \gamma_{\alpha,d}$ such that, if $cr \geq 1$, $K_{\alpha+\frac{d}{2}}(cr) \geq \gamma e^{-cr}(cr)^{-\frac{1}{2}}$. Therefore, choose M large enough so that for

$\xi \in \mathbb{R}^d \setminus [-M, M]^d$, we have $c|\xi| \geq 1$. Then if λ is the constant from Lemma 3.2.1, we have via Lemma 3.2.2(ii),

$$\left|\sum_{k \in \mathbb{Z}^d} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k)\right| \geq |\widehat{\phi_{\alpha,c}}(\xi + 2\pi k_\xi)| \geq \gamma|\lambda||\xi + 2\pi k_\xi|^{-\alpha-\frac{d}{2}} e^{-c|\xi+2\pi k_\xi|}(c|\xi + 2\pi k_\xi|)^{-\frac{1}{2}}.$$

Now depending on the sign of $\alpha+\frac{d}{2}$, the expression above is minimized by substituting 2π or 4π for $|\xi + 2\pi k_\xi|$ in the appropriate places. Consequently, there is a positive constant $D := D_{c,\alpha,d}$ such that

$$\left|\sum_{k \in \mathbb{Z}^d} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k)\right| \geq De^{-4\pi c}.$$

Lemma 3.2.2(iii) states that $K_{\alpha+\frac{d}{2}}(r) \leq \sqrt{2\pi}\, r^{-\frac{1}{2}} e^{-r} e^{\frac{|\alpha+\frac{d}{2}|^2}{2r}}$ for every $r > 0$. Consequently, by further adjusting M if need be so that $e^{\frac{|\alpha+\frac{d}{2}|^2}{2c|\xi|}} \leq 2$ for $\xi \in \mathbb{R}^d \setminus [-M, M]^d$, there is a positive constant β such that $K_{\alpha+\frac{d}{2}}(c|\xi|) \leq \beta e^{-c|\xi|}$. We conclude that

$$I \leq D^{-1}e^{4\pi c} \int_{\mathbb{R}^d\setminus[-M,M]^d} |\widehat{\phi_{\alpha,c}}(\xi)|d\xi \leq \beta D^{-1}|\lambda|e^{4\pi c} \int_{\mathbb{R}^d\setminus[-M,M]^d} |\xi|^{-\alpha-\frac{d}{2}} e^{-c|\xi|}d\xi.$$

The integral on the right is convergent, so $\widehat{L_{\alpha,c}} \in L_1(\mathbb{R}^d)$. $\square$

As mentioned above, one of the primary tools in showing convergence of cardinal interpolants is the fact that the fundamental functions converge almost everywhere to the multivariate sinc function, which is equivalent to the Fourier transform of the fundamental functions converging pointwise almost everywhere to the indicator function of the cube $[-\pi, \pi]^d$. The story is no different here. Defining $I(\xi)$ to be the function that takes value 1 whenever $\xi \in [-\pi, \pi]^d$, and 0 elsewhere, the following holds.

Proposition 3.2.4. *Let* $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$. *Then*

$$\lim_{c \to \infty} \widehat{L_{\alpha,c}}(\xi) = I(\xi)$$

for all $\xi \in \mathbb{R}^d$ *such that* $\max\{|\xi_1|, \ldots, |\xi_d|\} \neq \pi$.

Proof. First suppose that $\xi \notin [-\pi, \pi]^d$. Then there exists some $k_0 \in \mathbb{Z}^d$ such that $|\xi + 2\pi k_0| < |\xi|$. Therefore by Lemma 3.2.1,

$$\begin{aligned} |\widehat{\phi_{\alpha,c}}(\xi)| &\leq \left(\frac{|\xi|}{|\xi + 2\pi k_0|}\right)^{-\alpha - \frac{d}{2}} e^{-c(|\xi| - |\xi + 2\pi k_0|)} |\widehat{\phi_{\alpha,c}}(\xi + 2\pi k_0)| \\ &\leq \left(\frac{|\xi|}{|\xi + 2\pi k_0|}\right)^{-\alpha - \frac{d}{2}} e^{-c(|\xi| - |\xi + 2\pi k_0|)} \sum_{k \in \mathbb{Z}^d} |\widehat{\phi_{\alpha,c}}(\xi + 2\pi k)|. \end{aligned}$$

Consequently, since $\widehat{\phi_{\alpha,c}}$ is of one sign, dividing both sides of the above equation by $\sum_{k \in \mathbb{Z}^d} |\widehat{\phi_{\alpha,c}}(\xi + 2\pi k)|$ yields

$$0 \leq \widehat{L_{\alpha,c}}(\xi) \leq \left(\frac{|\xi|}{|\xi + 2\pi k_0|}\right)^{-\alpha - \frac{d}{2}} e^{-c(|\xi| - |\xi + 2\pi k_0|)}.$$

The term on the far right approaches 0 as $c \to \infty$ because the exponent there is negative. Therefore, for $\xi \notin [-\pi, \pi]^d$, $\lim_{c \to \infty} \widehat{L_{\alpha,c}}(\xi) = 0$ by the Squeeze Theorem.

Now suppose that $\xi \in (-\pi, \pi)^d \setminus \{0\}$. Then for all $k \in \mathbb{Z}^d \setminus \{0\}$, $|\xi| < |\xi + 2\pi k|$. By (3.1), we may write

$$\widehat{L_{\alpha,c}}(\xi) = \left(1 + \sum_{k \neq 0} \frac{\widehat{\phi_{\alpha,c}}(\xi + 2\pi k)}{\widehat{\phi_{\alpha,c}}(\xi)}\right)^{-1},$$

so it suffices to show that

$$\lim_{c\to\infty} \sum_{k\neq 0} \frac{\widehat{\phi_{\alpha,c}}(\xi+2\pi k)}{\widehat{\phi_{\alpha,c}}(\xi)} = 0.$$

By Lemma 3.2.1,

$$0 \leq \sum_{k\neq 0} \frac{\widehat{\phi_{\alpha,c}}(\xi+2\pi k)}{\widehat{\phi_{\alpha,c}}(\xi)} \leq \sum_{k\neq 0} \left(\frac{|\xi+2\pi k|}{|\xi|}\right)^{-\alpha-\frac{d}{2}} e^{-c(|\xi+2\pi k|-|\xi|)}.$$

The series on the right is convergent and dominated by the convergent series where c is replaced by 1, so

$$\lim_{c\to\infty} \sum_{k\neq 0} \frac{\widehat{\phi_{\alpha,c}}(\xi+2\pi k)}{\widehat{\phi_{\alpha,c}}(\xi)} = 0$$

as desired. Convergence of the second series stems from the fact that if $\alpha+\frac{d}{2}>0$, then the term $\frac{|\xi+2\pi k|}{|\xi|}$ is less than 1, and so the series is majorized by $\sum_{k\neq 0} e^{-2c\pi|k|}$, which converges. On the other hand, if $\alpha+\frac{d}{2}<0$, the k-th summand is majorized by the k-th summand of the convergent series $\sum_{k\neq 0}(1+\frac{|k|}{|\xi|})^{-\alpha-\frac{d}{2}}e^{-2c\pi|k|}$.

Note that by continuity, we can let $\widehat{L_{\alpha,c}}(0) = \lim_{|\xi|\to 0}\widehat{L_{\alpha,c}}(\xi) = 1$, which concludes the proof. This fact is also shown in [18, Theorem 4.11]. $\square$

We now consider interpolation of bandlimited functions at the lattice $\mathbb{Z}^d$ by translates of the function $L_{\alpha,c}$, beginning the analysis with an L_2 version of the Poisson Summation Formula.

Lemma 3.2.5 (cf. [5] Lemma 3.2)**.** *If $f \in PW_\pi^{(d)}$, then*

$$\sum_{j\in\mathbb{Z}^d} \widehat{f}(\xi+2\pi j) = \sum_{j\in\mathbb{Z}^d} f(j)e^{-i\langle j,\xi\rangle}, \tag{3.7}$$

where the second series is convergent in $L_2(\mathbb{R}^d)$.

Lemma 3.2.6. *Let $f \in PW_\pi^{(d)}$. For $m \in \mathbb{N}$, define*

$$\widehat{\mathscr{I}_{\alpha,c}^m f}(\xi) := \left(\sum_{\|k\|_1 \le m} f(k) e^{-i\langle k, \xi \rangle} \right) \widehat{L_{\alpha,c}}(\xi), \quad \xi \in \mathbb{R}^d,$$

where $\|k\|_1 = \sum_{i=1}^{d} |k_i|$ for $k \in \mathbb{Z}^d$. Then $(\mathscr{I}_{\alpha,c}^m f)_{m\in\mathbb{N}}$ forms a Cauchy sequence in $L_2(\mathbb{R}^d)$.

Proof. Define $Q_m : \mathbb{R}^d \to \mathbb{R}$ via

$$Q_m(\xi) = \sum_{\|k\|_1 \le m} f(k) e^{-i\langle k, \xi \rangle}.$$

Thus, $\widehat{\mathscr{I}_{\alpha,c}^m f}(\xi) = Q_m(\xi)\widehat{L_{\alpha,c}}(\xi)$. From Lemma 3.2.5, it is clear that $(Q_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L_2[-\pi,\pi]^d$. Since $Q_m(\xi + 2\pi k) = Q_m(\xi)$ for every $k \in \mathbb{Z}^d$,

$$\begin{aligned}
\|\widehat{\mathscr{I}_{\alpha,c}^m f} - \widehat{\mathscr{I}_{\alpha,c}^\ell f}\|_{L_2(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d} |Q_m(\xi) - Q_\ell(\xi)|^2 \left(\widehat{L_{\alpha,c}}(\xi)\right)^2 d\xi \\
&= \sum_{k\in\mathbb{Z}^d} \int_{[-\pi,\pi]^d} |Q_m(\xi + 2\pi k) - Q_\ell(\xi + 2\pi k)|^2 \\
&\qquad \times \left(\widehat{L_{\alpha,c}}(\xi + 2\pi k)\right)^2 d\xi \\
&= \int_{[-\pi,\pi]^d} |Q_m(\xi) - Q_\ell(\xi)|^2 \sum_{k\in\mathbb{Z}^d} \left(\widehat{L_{\alpha,c}}(\xi + 2\pi k)\right)^2 d\xi \\
&\le \int_{[-\pi,\pi]^d} |Q_m(\xi) - Q_\ell(\xi)|^2 d\xi\ .
\end{aligned}$$

The interchange of sum and integral is valid by Tonelli's Theorem, and the last inequality follows from the fact that

$$\sum_{k\in\mathbb{Z}^d}\left(\widehat{L_{\alpha,c}}(\xi+2\pi k)\right)^2=\frac{\sum_{k\in\mathbb{Z}^d}\widehat{\phi_{\alpha,c}}^2(\xi+2\pi k)}{\left(\sum_{l\in\mathbb{Z}^d}\widehat{\phi_{\alpha,c}}(\xi+2\pi l)\right)^2}\leq 1. \tag{3.8}$$

Note that because $\widehat{\phi_{\alpha,c}}$ is either non-negative or non-positive, the above inequality is simply the fact that the ℓ_2 norm of a sequence is less that the ℓ_1 norm of the same sequence. In conclusion, $(\widehat{\mathscr{I}^m_{\alpha,c}f})_{m\in\mathbb{N}}$ is a Cauchy sequence in $L_2(\mathbb{R}^d)$ because $\|\widehat{\mathscr{I}^m_{\alpha,c}f}-\widehat{\mathscr{I}^\ell_{\alpha,c}f}\|_{L_2(\mathbb{R}^d)}\leq\|Q_m-Q_\ell\|_{L_2[-\pi,\pi]^d}$, and the latter is Cauchy. $\square$

Lemmas 3.2.5 and 3.2.6 allow us to define

$$\widehat{\mathscr{I}_{\alpha,c}f}(\xi):=\widehat{L_{\alpha,c}}(\xi)\sum_{k\in\mathbb{Z}^d}f(k)e^{-i\langle k,\xi\rangle}, \tag{3.9}$$

where the series is convergent in $L_2(\mathbb{R}^d)$. By a periodization argument similar to that in the proof of Lemma 3.2.6, one can show that $\widehat{\mathscr{I}_{\alpha,c}f}\in L_1(\mathbb{R}^d)$. Indeed,

$$\begin{aligned}
\|\widehat{\mathscr{I}_{\alpha,c}f}\|_{L_1(\mathbb{R}^d)} &= \sum_{l\in\mathbb{Z}^d}\int_{[-\pi,\pi]^d+2\pi l}\widehat{L_{\alpha,c}}(\xi)\left|\sum_{k\in\mathbb{Z}^d}f(k)e^{-i\langle k,\xi\rangle}\right|d\xi\\
&= \int_{[-\pi,\pi]^d}\sum_{l\in\mathbb{Z}^d}\widehat{L_{\alpha,c}}(\xi+2\pi l)\left|\sum_{k\in\mathbb{Z}^d}f(k)e^{-i\langle k,\xi\rangle}\right|d\xi\\
&= \int_{[-\pi,\pi]^d}\left|\sum_{k\in\mathbb{Z}^d}f(k)e^{-i\langle k,\xi\rangle}\right|d\xi
\end{aligned}$$

$$\leq \quad (2\pi)^{\frac{d}{2}} \left\| \sum_{k\in\mathbb{Z}^d} f(k) e^{-i\langle k,\cdot\rangle} \right\|_{L_2[-\pi,\pi]^d}$$

$$= \quad (2\pi)^{\frac{d}{2}} \|\widehat{f}\|_{L_2[-\pi,\pi]^d}.$$

Thus, applying the Fourier inversion formula term by term, we see that

$$\mathscr{I}_{\alpha,c}f(x) = \sum_{k\in\mathbb{Z}^d} f(k) L_{\alpha,c}(x-k), \quad x \in \mathbb{R}^d. \tag{3.10}$$

Since $L_{\alpha,c}$ is a fundamental function, it is evident that $\mathscr{I}_{\alpha,c}f(j) = f(j),\ j \in \mathbb{Z}$.

Theorem 3.2.7. *Let $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$. If $f \in PW_\pi^{(d)}$, then*

$$\lim_{c\to\infty} \|\mathscr{I}_{\alpha,c}f - f\|_{L_2(\mathbb{R}^d)} = 0,$$

and $\lim_{c\to\infty} |\mathscr{I}_{\alpha,c}f(x) - f(x)| = 0$ *uniformly on* $\mathbb{R}^d$.

Proof. We first demonstrate uniform convergence. The proof is the same as in [5]. Again let $I(\xi)$ be the characteristic function of the cube, and let $\Omega := [-\pi,\pi]^d$. Then by the inversion formula, Lemma 3.2.5, and the oft-exploited periodization argument,

$$\begin{aligned} \mathscr{I}_{\alpha,c}f(x) - f(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi + 2\pi k)(\widehat{L_{\alpha,c}}(\xi) - I(\xi)) e^{-i\langle x,\xi\rangle} d\xi \\ &= \frac{1}{(2\pi)^d} \int_\Omega \widehat{f}(\xi) \sum_{k\in\mathbb{Z}^d} (\widehat{L_{\alpha,c}}(\xi + 2\pi k) - I(\xi + 2\pi k)) e^{-i\langle x,\xi+2\pi k\rangle} d\xi. \end{aligned}$$

Therefore, we find that

$$\begin{aligned} |\mathscr{I}_{\alpha,c} f(x) - f(x)| &\leq \frac{1}{(2\pi)^d} \int_{\Omega} |\widehat{f}(\xi)| \sum_{k \in \mathbb{Z}^d} \left| \widehat{L_{\alpha,c}}(\xi + 2\pi k) - I(\xi + 2\pi k) \right| d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\Omega} |\widehat{f}(\xi)| \left(1 - \widehat{L_{\alpha,c}}(\xi) + \sum_{k \neq 0} \widehat{L_{\alpha,c}}(\xi + 2\pi k) \right) d\xi. \end{aligned}$$

But then by definition,

$$\sum_{k \neq 0} \widehat{L_{\alpha,c}}(\xi + 2\pi k) = \frac{\sum_{k \in \mathbb{Z}^d} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k) - \widehat{\phi_{\alpha,c}}(\xi)}{\sum_{l \in \mathbb{Z}^d} \widehat{\phi_{\alpha,c}}(\xi + 2\pi l)} = 1 - \widehat{L_{\alpha,c}}(\xi).$$

Therefore,

$$|\mathscr{I}_{\alpha,c} f(x) - f(x)| \leq 2 \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} |\widehat{f}(\xi)| (1 - \widehat{L_{\alpha,c}}(\xi)) d\xi.$$

As the integrand is non-negative and bounded by $2|\widehat{f}(\xi)| \in L_1[-\pi,\pi]^d$, and $\lim_{c\to\infty}(1 - \widehat{L_{\alpha,c}}(\xi)) = 0$ by Proposition 3.2.4, the Dominated Convergence Theorem implies that

$$\lim_{c \to \infty} |\mathscr{I}_{\alpha,c} f(x) - f(x)| = 0, \quad x \in \mathbb{R}^d.$$

The upper bound is independent of x, hence the convergence is uniform.

Turning to the proof of L_2 convergence, in view of the Plancherel/Parseval Identity, it suffices to show that $\|\widehat{\mathscr{I}_{\alpha,c} f} - \widehat{f}\|_{L_2(\mathbb{R}^d)} \to 0$. We first estimate the norm of $\widehat{\mathscr{I}_{\alpha,c} f} - \widehat{f}$ on the cube $[-\pi,\pi]^d$. Recall that since $(e^{-i\langle k,\cdot\rangle})_{k \in \mathbb{Z}^d}$ is an orthonormal basis for $L_2[-\pi,\pi]^d$, we may write $\widehat{f}(\xi) = \sum_{k\in\mathbb{Z}^d} f(k)e^{-i\langle k,\xi\rangle}$. Moreover,

$$\|\widehat{f}\|_{L_2[-\pi,\pi]^d} = \|f(k)\|_{\ell_2(\mathbb{Z}^d)}.$$

Consequently, (3.9) yields

$$\begin{aligned}
\|\widehat{\mathscr{I}_{\alpha,c}f} - \widehat{f}\|^2_{L_2[-\pi,\pi]^d} &= \int_{[-\pi,\pi]^d} \left| \sum_{k\in\mathbb{Z}^d} f(k)(\widehat{L_{\alpha,c}}(\xi) - 1)e^{-i\langle k,\xi\rangle} \right|^2 d\xi \\
&= \int_{[-\pi,\pi]^d} |\widehat{L_{\alpha,c}}(\xi) - 1|^2 \left| \sum_{k\in\mathbb{Z}^d} f(k)e^{-i\langle k,\xi\rangle} \right|^2 d\xi.
\end{aligned}$$

The right hand side is bounded by $4\|f(k)\|^2_{\ell_2(\mathbb{Z}^d)}$, so by the Dominated Convergence Theorem and Proposition 3.2.4, $\lim_{c\to\infty} \|\widehat{\mathscr{I}_{\alpha,c}f} - \widehat{f}\|_{L_2[-\pi,\pi]^d} = 0$.

Now to estimate the norm outside the cube, if $l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d \setminus \{0\}$, define $Q_l := [-\pi - 2\pi l_1, \pi - 2\pi l_1] \times \cdots \times [-\pi - 2\pi l_d, \pi - 2\pi l_d]$. As f is bandlimited,

$$\|\widehat{\mathscr{I}_{\alpha,c}f} - \widehat{f}\|^2_{L_2(\mathbb{R}^d\setminus[-\pi,\pi]^d)} = \|\widehat{\mathscr{I}_{\alpha,c}f}\|^2_{L_2(\mathbb{R}^d\setminus[-\pi,\pi]^d)} = \sum_{l\neq 0} \|\widehat{\mathscr{I}_{\alpha,c}f}\|^2_{L_2(Q_l)}.$$

Consequently,

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus[-\pi,\pi]^d} |\widehat{\mathscr{I}_{\alpha,c}f}(\xi)|^2 d\xi &= \sum_{l\neq 0} \int_{Q_l} \left| \widehat{L_{\alpha,c}}(\xi) \sum_{k\in\mathbb{Z}^d} f(k)e^{-i\langle k,\xi\rangle} \right|^2 d\xi \\
&= \int_{[-\pi,\pi]^d} \sum_{l\neq 0} |\widehat{L_{\alpha,c}}(\xi + 2\pi l)|^2 \left| \sum_{k\in\mathbb{Z}^d} f(k)e^{-i\langle k,\xi\rangle} \right|^2 d\xi,
\end{aligned}$$

by the Monotone Convergence Theorem.

Recall that $0 \leq \widehat{L_{\alpha,c}}(\xi) \leq 1$, so $|\widehat{L_{\alpha,c}}(\xi+2\pi\ell)|^2 \leq \widehat{L_{\alpha,c}}(\xi+2\pi\ell)$, and as calculated above, $\sum\limits_{\ell\neq 0}\widehat{L_{\alpha,c}}(\xi+2\pi\ell) = 1-\widehat{L_{\alpha,c}}(\xi)$. Consequently, the integral above is majorized by

$$\int_{[-\pi,\pi]^d} |1-\widehat{L_{\alpha,c}}(\xi)| \left|\sum_{k\in\mathbb{Z}^d} f(k)e^{-i\langle k,\xi\rangle}\right|^2 \leq 2\|f(k)\|^2_{\ell_2(\mathbb{Z}^d)}.$$

Therefore, the Dominated Convergence Theorem and Proposition 3.2.4 imply that $\lim\limits_{c\to\infty}\|\widehat{\mathscr{I}_{\alpha,c}f}\|_{L_2(\mathbb{R}^d\setminus[-\pi,\pi]^d)} = 0$, and the proof is complete. □

3.3 Numerical Results

To give some brief numerical results, we focus on univariate cardinal interpolation, first using general multiquadrics and subsequently the Gauss kernel. First note that Proposition 3.2.4 implies that $L_{\alpha,c}$ must converge to the sinc function as $c \to \infty$. Figure 3.1 shows the graph of $L_{\frac{1}{2},c}$ (the fundamental function associated with the Hardy multiquadric) for different values of c. As expected, for the larger value, $c = 10$, the accuracy is much higher. The estimated L_2-error of the difference in $L_{\frac{1}{2},c}$ and the sinc function on the interval $[-10, 10]$ considered in the figure is .0780 when $c = 1$ and .0091 when $c = 10$.

Figure 3.2 shows the multiquadric interpolant (with $\alpha = 1/2$) for the bandlimited function $g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^4 e^{ixt}dt$ on the interval $[-10, 10]$. The estimates for the difference appear in Table 3.1. The numerical entries are the estimated values of the relative error $\|\mathscr{I}_{\alpha,c}g(x) - g(x)\|_{L_\infty[-10,10]}/\|g\|_{L_\infty[-10,10]}$.

To calculate $\mathscr{I}_{\alpha,c}g$, the series in (3.10) was truncated at $k = \pm 100$. From the data in Table 3.1, it seems that for low values of c, it is beneficial to take a large positive value of α. However, for large values of c, there is not as marked a difference when varying α. This could be simply due truncation error in approximating the interpolant. Some similar numerical results for cardinal multiquadric interpolation can be found in [16].

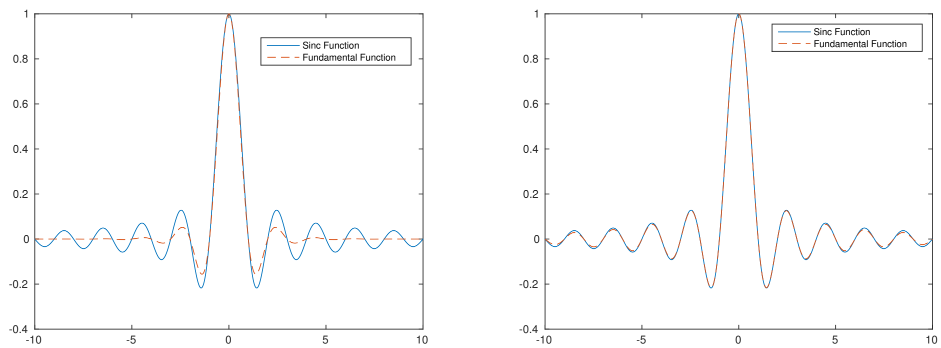

Figure 3.1: Plots of sinc function and Fundamental function for the Hardy multiquadric (with $\alpha = 1/2$) with shape parameters $c = 1$ (left) and $c = 10$ (right).

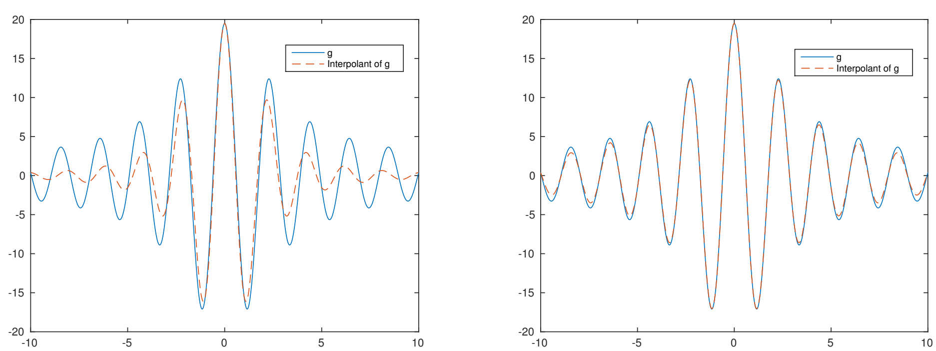

Figure 3.2: Plot of the function g and its multiquadric interpolant for $\alpha = 1/2$ and both $c = 1$ (left) and $c = 10$ (right).

Turning our attention to cardinal interpolation via the Gauss kernel $g_\lambda(x) = e^{-\lambda|x|^2}$, we note that similarly to Proposition 3.2.4, it was shown in [5, Proposition 2.2] that L_λ (the fundamental function associated with the Gaussian) converges to the sinc function as $\lambda \to 0^+$. Again by approximating the fundamental function,

	$c = 1$	$c = 10$	$c = 100$
$\alpha = 7/2$	.1550	.0353	$5.221 \cdot 10^{-4}$
$\alpha = 1/2$	.2470	.0408	$5.318 \cdot 10^{-4}$
$\alpha = -1/2$	.2891	.0428	$5.352 \cdot 10^{-4}$
$\alpha = -1$	.3073	.0439	$5.369 \cdot 10^{-4}$
$\alpha = -7/2$	.4078	.0492	$5.456 \cdot 10^{-4}$

Table 3.1: Estimated relative error of $\mathscr{I}_{\alpha,c}g - g$ for a range of α and c values.

we estimate that the maximum of the difference of L_λ and the sinc function on the interval $[-10, 10]$ was .0747 when $\lambda = 1$ and .0046 when $\lambda = 1/10$. The following table displays the relative L_∞ error of the Gaussian interpolant to the function g used above.

$\lambda = 1$	$\lambda = .1$	$\lambda = .01$
.2401	.0210	$7.60 \cdot 10^{-4}$

Table 3.2: Estimated relative error of $\mathscr{I}_\lambda g - g$ for a range of λ.

We note that the estimated difference of $L_\lambda(x) - \text{sinc}(x)$ was smaller for comparable values of λ than the difference of the fundamental functions associated with the Poisson kernel. Likewise, for smaller values of λ, the cardinal Gaussian interpolant of the bandlimited function g behaved favorably when compared to the multiquadric interpolants for small or negative values of α. The reason for this is likely that the Gaussian fundamental function decays exponentially away from the origin. Additionally, in the instances when we can find approximation rates (such as in Chapters 5 and 6), the Gaussian interpolant of a bandlimited function converges at a faster (or at least equal up to a constant) rate than its multiquadric interpolant.

4. RIESZ BASES OF EXPONENTIALS

For the remainder of this work, our discussion will focus on nonuniform, or scattered-data, interpolation problems. Precisely, we wish to consider Problem 1.0.4 for some of the radial basis functions explored in the previous chapter, where now the interpolation schemes will involve more general point sets $(x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}^d$ rather than the integer lattice. However, some background information is required before making suitable progress toward a solution.

Definition 4.0.1. *Let $\mathcal{H}$ be a separable Hilbert space. A sequence $(h_j)_{j\in\mathbb{N}}$ is said to be a* ***Riesz basis for*** *$\mathcal{H}$ provided there exist an orthonormal basis, $(e_j)_{j\in\mathbb{N}}$, for $\mathcal{H}$ and a bounded, invertible, linear operator, $T : \mathcal{H} \to \mathcal{H}$, such that*

$$h_j = Te_j, \quad j \in \mathbb{N}.$$

The above is merely one way to define a Riesz basis on a Hilbert space as the following theorem demonstrates.

Theorem 4.0.2. *The following are equivalent:*

1. *(h_j) is a Riesz basis for $\mathcal{H}$.*
2. *(h_j) is a bounded, unconditional basis for $\mathcal{H}$ (i.e. $\sup\limits_j \|h_j\| < \infty$ and the series $h = \sum\limits_j \langle h, h_j\rangle\, h_j$ converges unconditionally for every $h \in \mathcal{H}$).*
3. *(h_j) is complete in $\mathcal{H}$ and there exists a number $B \geq 1$ such that for every $(c_j) \in \ell_2$,*

$$\frac{1}{B}\|c\|_{\ell_2} \leq \left\| \sum_j c_j h_j \right\|_{\mathcal{H}} \leq B\|c\|_{\ell_2}. \tag{4.1}$$

The smallest B for which (4.1) *holds is called the* Riesz basis constant *of* (h_j).

4. *(h_j) is a basis for $\mathcal{H}$ and the series $\sum_j c_j h_j$ converges if and only if $(c_j) \in \ell_2$.*

5. *There is an equivalent inner product on $\mathcal{H}$ such that (h_j) is an orthonormal basis for $\mathcal{H}$ equipped with that inner product.*

6. *(h_j) is an exact frame for $\mathcal{H}$ (a sequence (h_j) is a* frame *for $\mathcal{H}$ if there exist constants $A, B > 0$ such that $A\|h\|^2 \leq \sum_j |\langle h, h_j\rangle|^2 \leq B\|h\|^2$ for every $h \in \mathcal{H}$, and a frame is* exact *if removal of any vector h_k makes the new sequence cease to be a frame).*

7. *For every $(c_j) \in \ell_2$, there exists a unique $h \in \mathcal{H}$ such that*

$$\langle h, h_j\rangle = c_j, \quad \text{for every } j.$$

8. *(h_j) is complete, and the* Moment Space *of (h_j) is ℓ_2. (Define $T : \mathcal{H} \to \ell_2$ via $h \mapsto (\langle h, h_j\rangle)_j$. Then $T(\mathcal{H})$ is called the Moment Space).*

There are yet many more equivalences than are stated above, but those mentioned should be sufficient to indicate the nature of Riesz bases of Hilbert spaces. For proof of many of the equivalences in Theorem 4.0.2, refer to [48]. For the purposes of connection with interpolation schemes, the following problem is one of quite natural interest.

Problem 4.0.3. *Let $S \subset \mathbb{R}^d$ be a bounded set with positive Lebesgue measure. Is there a Riesz basis of exponentials, i.e. of the form $\left(e^{-i\langle x_j, \cdot\rangle}\right)_{j\in\mathbb{N}}$, for $L_2(S)$? If there is, we say that $L_2(S)$* admits a Riesz basis of exponentials.

This problem is an old one in abstract harmonic analysis, and there has been much work to characterize both the sets S and the points $(x_j) \subset \mathbb{R}^d$ such that the conclusion is true. As a starting point, the following lemmas provide a simple necessary condition on the sequence (x_j).

Lemma 4.0.4. *Let $S \subset \mathbb{R}^d$, and let $\left(e^{-i\langle x_j,\cdot\rangle}\right)_{j\in\mathbb{N}}$ be a Riesz basis for $L_2(S)$. Let $(e_j^*)_{j\in\mathbb{N}}$ be the associated coordinate (or biorthogonal) functionals (i.e. the functions such that $\left\langle e^{-i\langle x_j,\cdot\rangle}, e_k^*\right\rangle_S = \delta_{j,k}$, where $\langle\cdot,\cdot\rangle_S$ is the usual inner product on $L_2(S)$). Then $(e_j^*)_{j\in\mathbb{N}}$ is also a Riesz basis for $L_2(S)$. Moreover, any function $f \in L_2(S)$ has the following representations*

$$f(x) = \sum_{j\in\mathbb{N}} \left\langle f, e_j^*\right\rangle_S e^{-i\langle x_j,x\rangle}, \qquad f(x) = \sum_{j\in\mathbb{N}} \left\langle f, e^{-i\langle x_j,\cdot\rangle}\right\rangle_S e_j^*(x).$$

In the case that $d = 1$, $S = [-\pi,\pi]$, and $x_j = j$ for $j \in \mathbb{Z}$, then the coordinate functionals are also exponential functions. However, in general, the coordinate functionals may even fail to be continuous. Another important note is that if a Riesz basis satisfies (4.1) with constant B, then its coordinate functionals also satisfy (4.1) with the same constant (see Lemma 6.2.2).

Lemma 4.0.5. *Let $S \subset \mathbb{R}^d$, and let $\left(e^{-i\langle x_j,\cdot\rangle}\right)_{j\in\mathbb{N}}$ be a Riesz basis for $L_2(S)$. Then there exists an $\varepsilon > 0$ such that*

$$|x_k - x_l| \geq \varepsilon, \quad \text{for all } k \neq l,$$

where $|x_k - x_l|$ is the Euclidean distance on $\mathbb{R}^d$.

Proof. By way of contradiction, suppose that there are subsequences $(k_j),(l_j) \subset \mathbb{N}$ such that $\|x_{k_j} - x_{l_j}\| \to 0$ as $j \to \infty$. Then the Dominated Convergence Theorem implies that $\|e^{-i\left\langle x_{k_j},\cdot\right\rangle} - e^{-i\left\langle x_{l_j},\cdot\right\rangle}\|_{L_2(S)} \to 0$ as $j \to \infty$. On the other hand, if $(e_j^*)_{j\in\mathbb{N}}$

are the coordinate functionals to $(e^{-i\langle x_j,\cdot\rangle})$, then for every j,

$$\left\langle e^{-i\left\langle x_{k_j},\cdot\right\rangle} - e^{-i\left\langle x_{l_j},\cdot\right\rangle}, e^*_{k_j}\right\rangle_S = 1.$$

This supplies a contradiction because

$$\left|\left\langle e^{-i\left\langle x_{k_j},\cdot\right\rangle} - e^{-i\left\langle x_{l_j},\cdot\right\rangle}, e^*_{l_j}\right\rangle_S\right| \leq \left\| e^{-i\left\langle x_{k_j},\cdot\right\rangle} - e^{-i\left\langle x_{k_j},\cdot\right\rangle}\right\|_{L_2(S)} \sup_{j\in\mathbb{N}} \|e^*_{k_j}\|_{L_2(S)} \to 0.$$

Note that the supremum on the right hand side above must be finite by Theorem 4.0.2(2). □

4.1 Riesz Bases of Exponentials in One Dimension

Since later our concern will be interpolating bandlimited functions, we first consider the problem of finding Riesz bases of exponentials for intervals in one dimension. In this setting, it is often more natural to index the sequence by the integers rather than the natural numbers. The following beautiful theorem due to M. Kadec shows that such bases are abundant.

Theorem 4.1.1 (Kadec's 1/4-Theorem, [21])**.** *Let $(x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$. If*

$$\sup_{j\in\mathbb{Z}} |x_j - j| < \frac{1}{4},$$

then $\left(e^{-ix_j(\cdot)}\right)_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$.

Consequently, any small enough perturbation of the integer lattice will give a Riesz basis of exponentials. As an aside, the upper bound in Theorem 4.1.1 is sharp:

Proposition 4.1.2 (cf. [48], Theorem 5, p.103)**.** *The set of functions*

$$\left\{e^{\pm i\left(n-\frac{1}{4}\right)(\cdot)} \ : \ n \in \mathbb{N}\right\}$$

is exact, but not a Riesz basis for $L_2[-\pi,\pi]$.

Having so far given separate necessary and sufficient conditions on the sequence (x_j), the following theorem due to B.S. Pavlov gives a necessary and sufficient condition.

Theorem 4.1.3 (Pavlov, [34]). *The set $\left(e^{-ix_j(\cdot)}\right)_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$ if and only if the following three conditions hold.*

(i) (x_j) lies in a horizontal strip along the real axis (i.e. $\sup\limits_j |\Im(x_j)| \leq C$ for some C, with $\Im(x_j)$ being the imaginary part of x_j),

(ii) The function

$$F(z) := \lim_{R\to\infty} \prod_{|x_j|<R} \left(1 - \frac{z}{x_j}\right)$$

is an entire function of exponential type π, and

(iii) F satisfies the following (Muckenhoupt A_2 condition):

$$\sup_{I\subset\mathbb{R}} \left(\frac{1}{|I|}\int_I |F(x)|^2 dx \cdot \frac{1}{|I|}\int_I \frac{1}{|F(x)|^2} dx\right) < \infty,$$

where the supremum is taken over all bounded intervals $I \subset \mathbb{R}$.

The final result on the one dimensional case is a recent one of Kozma and Nitzan.

Theorem 4.1.4 ([24]). *Let $S \subset \mathbb{R}$ be a finite disjoint union of intervals. Then $L_2(S)$ admits a Riesz basis of exponentials. That is, there exists a sequence $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $\left(e^{-ix_j(\cdot)}\right)_{j\in\mathbb{N}}$ is a Riesz basis for $L_2(S)$.*

4.2 Riesz Bases of Exponentials in Higher Dimensions – Cubes

While Problem 4.0.3 has seen some good characterization in one dimension, it remains a significantly more difficult task in higher dimensions due to the fact that the solution is highly dependent on the geometry of the set S, in general. Therefore, we will examine different geometries of the set S and discuss some scenarios when $L_2(S)$ does (or does not) admit a Riesz basis of exponentials. Perhaps the easiest generalization of the one dimensional results comes when considering S to be a cube.

To ease notation, if $X := (x_j)$ is such that the associated exponential functions form a Riesz basis for $L_2(S)$ for a given S, then we call X a *Riesz-basis sequence for* $L_2(S)$.

Theorem 4.2.1. *Let $X_1 := (x_j^{(1)})_{j\in\mathbb{Z}}, \dots, X_d := (x_j^{(d)})_{j\in\mathbb{Z}}$ be Riesz-basis sequences for $L_2[-\pi,\pi]$. Then the Cartesian product, $X_1 \times \cdots \times X_d$, is a Riesz-basis sequence for $L_2[-\pi,\pi]^d$. That is, $\left(e^{-i\left\langle\left(x_{j_1}^{(1)},\dots,x_{j_d}^{(d)}\right),\cdot\right\rangle}\right)_{(j_1,\dots,j_d)\in\mathbb{Z}^d}$ is a Riesz basis for $L_2[-\pi,\pi]^d$.*

Proof. The conclusion follows simply by observing that the tensor product of Riesz bases is again a Riesz basis. More precisely, if $(f_j)_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2(X)$ and $(g_k)_{k\in\mathbb{Z}}$ is a Riesz basis for $L_2(Y)$, then $(f_j\otimes g_k)_{j,k\in\mathbb{Z}}$ (defined by $f_j\otimes g_k(x,y) = f_j(x)g_k(y)$ for $(x,y)\in X\times Y$) is a Riesz basis for $L_2(X)\otimes L_2(Y)$, which is isomorphic to $L_2(X\times Y)$. □

The next theorem, due to Bailey, is a generalization of Kadec's 1/4-Theorem to higher dimensions.

Theorem 4.2.2 ([2], Theorem 2)**.** *If $(x_j)_{j\in\mathbb{Z}^d}\subset\mathbb{R}^d$, and*

$$\sup_{j\in\mathbb{Z}^d}\|x_j - j\|_\infty < \frac{\ln(2)}{\pi d},$$

then $\left(e^{-i\langle x_j,\cdot\rangle}\right)_{j\in\mathbb{Z}^d}$ is a Riesz basis for $L_2[-\pi,\pi]^d$, where $\|x_j - j\|_\infty = \max_{1\le i\le d}|(x_j)_i - j_i|$.

Another similar generalization of the proof of Kadec's 1/4 was given by Sun and Zhou.

Theorem 4.2.3 ([46])**.** *For $d \geq 1$, define*

$$C_d(x) := (1 - \cos(\pi x) + \sin(\pi x) + \operatorname{sinc}(x))^d - (\operatorname{sinc}(x))^d,$$

and let x_d be the unique number such that $0 < x_d \leq 1/4$ and $C_d(x_d) = 1$. If $(x_j)_{j \in \mathbb{Z}^d}$ satisfies

$$\sup_{j \in \mathbb{Z}^d} \|x_j - j\|_\infty < x_d,$$

then $\left(e^{-i\langle x_j, \cdot \rangle}\right)_{j \in \mathbb{Z}^d}$ is a Riesz basis for $L_2[-\pi, \pi]^d$.

It was shown in [2] that Theorems 4.2.2 and 4.2.3 lead asymptotically to the same estimates.

Lastly, similar to the multiband result of Theorem 4.1.4, the following higher dimensional analogue holds.

Theorem 4.2.4 ([25], Theorem 2)**.** *Let $S \subset \mathbb{R}^d$ be a finite disjoint union of rectangles whose edges are parallel to the coordinate axes. Then $L_2(S)$ admits a Riesz basis of exponentials.*

4.3 Riesz Bases of Exponentials in Higher Dimensions – Convex Bodies

Seeing as the interpolation schemes to be considered involve the use of radial basis functions, it is more natural to consider Paley-Wiener spaces (and hence L_2 spaces) over balls or other convex bodies in $\mathbb{R}^d$ rather than cubes. However, finding Riesz bases of exponentials for such spaces can be tricky as the following suggests.

Open Problem 4.3.1. *Does there exist a Riesz basis of exponentials for $L_2(B_2)$, where B_2 is the Euclidean ball in $\mathbb{R}^d$?*

Unfortunately, no solution or counterexample is known for this problem in any dimension larger than 1. Since the Euclidean ball is a very nice and natural convex body, the lack of knowledge here is quite disappointing. Nevertheless, there has been some progress in finding convex bodies that do admit Riesz bases of exponentials. To state the theorems, a definition is in order.

Definition 4.3.2. *A convex body $Z \subset \mathbb{R}^d$ is a* ***zonotope*** *if it is the image of a cube in $\mathbb{R}^m$ $(m \geq d)$ under a projection $P : \mathbb{R}^m \to \mathbb{R}^d$.*

Zonotopes exhibit very special symmetry, and can be equivalently defined to be Minkoswki sums of affine line segments centered at a point. One important fact is that every face of a zonotope is again a zonotope.

The following result is due to Lyubarskii and Rashkovskii.

Theorem 4.3.3 ([31], Proposition 3.2)**.** *If $d \geq 2$ and Z is a zonotope in $\mathbb{R}^d$, then $L_2(Z)$ has a Riesz basis of exponentials.*

The extension to more than 2 dimensions in Theorem 4.3.3 is only alluded to in [31], but the method of proof extends due to the additional symmetry enjoyed by zonotopes (in $d = 2$, zonotopes are simply symmetric convex polytopes). Recent works by Grepstad and Lev [14], and Kolountzakis [23] have extended Theorem 4.3.3 in a different manner.

Theorem 4.3.4 ([14], Corollary 3, and [23])**.** *Let $S \subset \mathbb{R}^d$ be a centrally symmetric polytope whose $(d-1)$-dimensional faces are also centrally symmetric, and whose vertices lie on some lattice, Λ. Then $L_2(S)$ admits a Riesz basis of exponentials.*

Note that Theorem 4.3.4 does not require S to be convex as in Theorem 4.3.3, and so is more inclusive than the zonotope restriction. However, the lattice condition on the vertices is not needed in Theorem 4.3.3.

4.4 Connection With Interpolation

Thus far, the idea of finding Riesz bases of exponentials for domains in higher dimensions has been elucidated, but the ultimate goal is to connect such abstract considerations to concrete interpolation problems. It turns out that there is a highly advantageous characterization of Riesz bases of exponentials which is directly connected to interpolation. To wit, consider the following definition.

Definition 4.4.1. *Let $X := (x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$, and let $S \subset \mathbb{R}^d$ be a bounded set of positive Lebesgue measure. X is said to be a* ***complete interpolating sequence (CIS) for*** *PW_S provided for every $(c_j)_{j\in\mathbb{N}} \in \ell_2$, there exists a unique $f \in PW_S$ such that*

$$f(x_j) = c_j, \quad j \in \mathbb{N}.$$

The following theorem provides a pleasing connection between the two ideas.

Theorem 4.4.2. *$X \subset \mathbb{R}^d$ is a complete interpolating sequence for PW_S if and only if $\left(e^{-i\langle x_j,\cdot\rangle}\right)_{j\in\mathbb{N}}$ is a Riesz basis for $L_2(S)$.*

Proof. Since $\left\langle \widehat{f}, e^{-i\langle x_j,\cdot\rangle}\right\rangle_{L_2(S)} = f(x_j)$ for any $f \in PW_S$ by the Fourier inversion formula, the equivalence is simply a restatement of Theorem 4.0.2(7). $\square$

One consequence of Theorem 4.4.2 is that there is a natural bijection from $PW_S \to \ell_2$ given by the map $f \mapsto (f(x_j))_{j\in\mathbb{Z}}$. In addition, this map is an isomorphism (of Banach spaces) due to the frame inequality found in Theorem 4.0.2(6). That is, if $Tf = (f(x_j))_{j\in\mathbb{Z}}$, then

$$A\|\widehat{f}\|^2_{L_2(S)} \leq \|Tf\|^2_{\ell_2} \leq B\|\widehat{f}\|^2_{L_2(S)},$$

for some positive constants $A, B > 0$ (here, identify PW_S isometrically as $L_2(S)$

via the Fourier transform and Parseval's identity; or equivalently, one could equip PW_S with the norm given by $\|f\|_{PW_S} := \|\widehat{f}\|_{L_2(S)}$). Note that the map T is injective (or one-to-one) because of completeness of the exponential system in $L_2(S)$, and is surjective (or onto) because the moment space is equal to ℓ_2 by Theorem 4.0.2(8).

5. NONUNIFORM SAMPLING IN HIGHER DIMENSIONS

Equipped with the theoretic underpinning of the previous chapter, we now focus our discussion on finding a general framework for interpolation of multivariate bandlimited functions via translates of RBFs. Some intermediate steps in this direction have been taken by Bailey, Schlumprecht, and Sivakumar [4] and Ledford [27]. The former consider Gaussian interpolation of bandlimited functions whose band lies in a ball of small radius β, where the interpolation is done at a Riesz-basis sequence for some larger symmetric convex body (per the discussion of Chapter 4). Ledford worked with squares in two dimensions using Poisson kernels for the interpolation scheme, and mentions an extension to cubes in higher dimensions. Owing to the geometry of the problem, the use of cubes requires a careful analysis when interpolating with radial functions. It seems that with the techniques available, the geometry best suited to bandlimited function interpolation in higher dimensions is that of Paley-Wiener spaces over balls. In fact, the main theorem in [40] (on convergence of the nonuniform Gaussian interpolant to a bandlimited function) holds in higher dimensions for functions whose band lies in the unit ball, but as mentioned above, this may well be vacuous if there is no Riesz-basis sequence for that space. Therefore, we use the fact that one can approximate the Euclidean ball closely by a zonotope which does have an associated Riesz basis of exponentials. That is, for any $\delta < 1$, there exists a zonotope, Z, such that $\delta B_2 \subset Z \subset B_2$, and $L_2(Z)$ has a Riesz basis of exponentials.

Inspired by Ledford's conditions for univariate interpolation in [26] and the higher dimensional Gaussian interpolation results in [4], we give sufficient conditions on a family of functions to form interpolants for the Paley-Wiener space associated with some symmetric convex body in $\mathbb{R}^d$, such as a zonotope, which also provides recovery of bandlimited functions whose Fourier transforms are supported in a ball contained in the convex body. The work of this chapter can also be found in [17].

Now, let Z be a convex set with $(x_j)_{j\in\mathbb{N}}$ being a Riesz-basis sequence for $L_2(Z)$, and we define two operators that will play an important role in our analysis. First, let $(e_j^*)_{j\in\mathbb{N}} \subset L_2(Z)$ be the coordinate functionals for $(e^{-i\langle x_j,\cdot\rangle})_{j\in\mathbb{N}}$. Recall that (e_j^*) is also a Riesz basis for $L_2(Z)$ with the same Riesz basis constant, say R_b. Thus for every $g \in L_2(Z)$, we can write

$$g = \sum_{j\in\mathbb{N}} \left\langle g, e_j^* \right\rangle_Z e^{-i\langle x_j,\cdot\rangle} = \sum_{j\in\mathbb{N}} \left\langle g, e^{-i\langle x_j,\cdot\rangle} \right\rangle_Z e_j^*, \tag{5.1}$$

where $\langle\cdot,\cdot\rangle_Z$ is the usual inner product on $L_2(Z)$. The final expression in (5.1) combined with the Riesz basis constant (see (4.1)) implies that if $f \in PW_Z$, then

$$\|f(x_j)\|_{\ell_2} \le R_b \|\mathscr{F}[f]\|_{L_2(Z)}. \tag{5.2}$$

Notice that for any $g \in L_2(Z)$ and $a \in \mathbb{R}^d$,

$$\begin{aligned}
\left\| \sum_{j\in\mathbb{N}} \left\langle g, e_j^* \right\rangle_Z e^{-i\langle x_j,\cdot\rangle} \right\|_{L_2(a+Z)} &= \left\| \sum_{j\in\mathbb{N}} \left\langle g, e_j^* \right\rangle_Z e^{-i\langle a,x_j\rangle} e^{-i\langle x_j,\cdot\rangle} \right\|_{L_2(Z)} \\
&\le R_b \left\| \left(\left\langle g, e_j^* \right\rangle_Z e^{-i\langle a,x_j\rangle} \right)_j \right\|_{\ell_2} \\
&= R_b \left\| \left(\left\langle g, e_j^* \right\rangle_Z \right)_j \right\|_{\ell_2} \\
&\le R_b^2 \left\| \sum_{j\in\mathbb{Z}} \left\langle g, e_j^* \right\rangle_Z e^{-i\langle x_j,\cdot\rangle} \right\|_{L_2(Z)} \\
&= R_b^2 \|g\|_{L_2(Z)}.
\end{aligned} \tag{5.3}$$

Consequently, the following extension of g is locally square integrable and thus defined almost everywhere on $\mathbb{R}^d$.

$$E(g)(x) := \sum_{j\in\mathbb{N}} \left\langle g, e_j^* \right\rangle_Z e^{-i\langle x_j, x\rangle}, \quad x \in \mathbb{R}^d. \tag{5.4}$$

If $m \in \mathbb{N}$, then we define the *prolongation operator* $A_m : L_2(Z) \to L_2(Z)$ via

$$A_m(g)(\xi) := E(g)(2^m\xi)\chi_{Z\setminus\frac{1}{2}Z}(\xi), \quad \xi \in Z, \tag{5.5}$$

where χ_S is the function taking value 1 on the set S and 0 elsewhere.

It follows from (5.3) that for $g \in L_2(Z)$,

$$\begin{aligned} \|A_m(g)\|^2_{L_2(Z)} = \int_{Z\setminus\frac{1}{2}Z} |E(g)(2^m u)|^2 du = 2^{-dm} \int_{2^m Z\setminus 2^{m-1}Z} |E(g)(v)|^2 dv \\ \leq 2^{-dm}\mathcal{N}^m R_b^4 \|g\|^2_{L_2(Z)}, \end{aligned} \tag{5.6}$$

where $\mathcal{N} = \mathcal{N}(2Z, Z)$ is the minimum number of translates of Z required to cover $2Z$. The constant $\mathcal{N}$ depends only on d, and an induction argument shows that at most $\mathcal{N}^m$ translates of Z are required to cover $2^m Z$.

5.1 Interpolation Scheme

Suppose that $Z \subset \mathbb{R}^d$ is a fixed, convex set. Also assume that $X := (x_j)_{j\in\mathbb{N}}$ is a fixed but arbitrary Riesz-basis sequence for $L_2(Z)$ with basis constant R_b. We explore conditions on interpolation operators formed from translates of a single function that allow for recovery of bandlimited functions through a certain limiting process. The criteria here are inspired by so-called *regular interpolators* developed by Ledford [26]. The results therein are univariate by nature, and our analysis extends to sufficient conditions for interpolation schemes in higher dimensions.

Definition 5.1.1. *We call a function $\phi : \mathbb{R}^d \to \mathbb{R}$ a d-*dimensional interpolator *for PW_Z if the following conditions hold.*

(I1) $\phi \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and $\widehat{\phi} \in L_1(\mathbb{R}^d)$.

(I2) $\widehat{\phi} \geq 0$ and there exists an $\varepsilon > 0$ such that $\widehat{\phi} \geq \varepsilon > 0$ on Z.

(I3) Let $M_j := \sup_{u \in Z \setminus \frac{1}{2}Z} |\widehat{\phi}(2^j u)|$. Then $(2^{-jd}\mathcal{N}^j M_j) \in \ell_1$, where $\mathcal{N}$ is the covering number from (5.6).

It is important to note that for (I2), it is allowable for $\widehat{\phi}$ to be negative everywhere and bounded away from 0 on Z, in which case $-\phi$ satisfies the condition. Condition (I1) allows the use of the Fourier inversion formula (2.3), while (I2) allows one to show existence of an interpolant for a bandlimited function. Finally, (I3) is a technical condition that comes from a periodization argument that is ubiquitous throughout the proofs in the sequel.

Remark 5.1.2. Condition (I1), which is mainly needed to show that an interpolant exists, can also be stated as follows:

(I1') $\phi(x) = \dfrac{1}{(2\pi)^d} \displaystyle\int_{\mathbb{R}^d} \psi(\xi) e^{i\langle x, \xi \rangle} d\xi = \mathscr{F}^{-1}[\psi](x)$ for some $\psi \in L_1 \cap L_2$.

Theorem 5.1.3. *Let $Z \subset \mathbb{R}^d$ be a bounded convex set. Suppose that X is a Riesz-basis sequence for $L_2(Z)$, and that ϕ is a d-dimensional interpolator for PW_Z.*

(i) For every $f \in PW_Z$, there exists a unique sequence $(a_j) \in \ell_2$ such that

$$\sum_{j \in \mathbb{N}} a_j \phi(x_k - x_j) = f(x_k), \quad k \in \mathbb{N}.$$

(ii) The Interpolation Operator $\mathscr{I}_\phi : PW_Z \to L_2(\mathbb{R}^d)$ defined by

$$\mathscr{I}_\phi f(\cdot) = \sum_{j \in \mathbb{N}} a_j \phi(\cdot - x_j),$$

where (a_j) is as in (i), is a well-defined, bounded linear operator from PW_Z to $L_2(\mathbb{R}^d)$. Moreover, $\mathscr{I}_\phi f$ belongs to $C_0(\mathbb{R}^d)$.

The proof of Theorem 5.1.3 will be given in Section 5.3.1. Now we turn to sufficient regularity conditions on a family of d-dimensional interpolators to provide convergence to bandlimited functions both in the L_2 and uniform norms. Our terminology is inspired by that of [26]. Assume that $\delta B_2 \subset Z \subset B_2$, where B_2 is the Euclidean ball in $\mathbb{R}^d$.

Definition 5.1.4. *Let $\beta > 0$. Suppose $A \subset (0, \infty)$ is unbounded, and $(\phi_\alpha)_{\alpha \in A}$ is a family of d-dimensional interpolators for PW_Z. We call this family* regular *for $PW_{\beta B_2}$ if the following hold:*

(R1) If $S_\alpha := \sum_{j \in \mathbb{N}} \mathcal{N}^j M_j(\alpha)$ where $\mathcal{N}$ is the covering number discussed above and $M_j(\alpha)$ is as in (I3), then there is a constant C, independent of α, such that $S_\alpha \leq C M_\alpha$, where $M_\alpha := \sup_{u \in B_2 \setminus \delta B_2} |\widehat{\phi_\alpha}(u)|$.

(R2) Let $m_\alpha(\beta) := \inf_{u \in \beta B_2} |\widehat{\phi_\alpha}(u)|$, and $\gamma_\alpha := \inf_{u \in B_2} |\widehat{\phi_\alpha}(u)|$. Then

$$\frac{M_\alpha^3}{m_\alpha(\beta)\gamma_\alpha^2} \to 0, \text{ as } \alpha \to \infty.$$

Remark 5.1.5. All of the examples considered in Section 5.2 are radial basis functions whose Fourier transforms decrease radially. For such functions, (R2) may be restated as follows:

(R2') $$\frac{\widehat{\phi_\alpha}(\delta)^3}{\widehat{\phi_\alpha}(\beta)\widehat{\phi_\alpha}(1)^2} \to 0, \quad \text{as} \quad \alpha \to \infty.$$

We consider interpolation of bandlimited functions $f \in PW_{\beta B_2}$ for some $\beta \leq \delta$. Hence, $\mathscr{F}[f]$ has support in a subset of Z. The condition (R2) comes from exploiting the geometry of the problem, namely that $\beta B_2 \subset \delta B_2 \subset Z \subset B_2$. This section concludes with the statement of the main result of the chapter, which will be proven in Section 5.3.2.

Theorem 5.1.6. *Let $d \in \mathbb{N}$, $\delta \in (0,1)$, and $\beta \leq \delta$. Suppose that $Z \subset \mathbb{R}^d$ has positive Lebesgue measure, and satisfies $\delta B_2 \subset Z \subset B_2$. Additionally, suppose that X is a Riesz-basis sequence for $L_2(Z)$. Suppose that $(\phi_\alpha)_{\alpha \in A}$ is a family of d-dimensional interpolators for PW_Z that is regular for $PW_{\beta B_2}$, and $\mathscr{I}_\alpha := \mathscr{I}_{\phi_\alpha}$ are the associated interpolation operators. Then for every $f \in PW_{\beta B_2}$,*

$$\lim_{\alpha \to \infty} \|\mathscr{I}_\alpha f - f\|_{L_2(\mathbb{R}^d)} = 0,$$

and

$$\lim_{\alpha \to \infty} |\mathscr{I}_\alpha f(x) - f(x)| = 0, \quad \text{uniformly on } \mathbb{R}^d.$$

The subject of finding Riesz-basis sequences for different geometries in higher dimensions was discussed at length in the previous chapter. In [12, Theorem 4.1.10], it was shown that in any dimension d, there is a zonotope, Z, satisfying the conditions of Theorem 5.1.6. Consequently, by Theorem 4.3.3, Z admits a Riesz basis of exponentials.

5.2 Examples

Since the conditions given above are somewhat abstract, it is prudent to pause and discuss some examples that motivate the general result. Throughout this section, suppose that Z and X satisfy the hypothesis of Theorem 5.1.6. We begin with the Gaussian kernel and show that we recover the main result from [4].

5.2.1 *Gaussians*

To fit the imposed condition of α tending to infinity, we employ a different convention for the Gaussian kernel than [4]:

$$g_\alpha(x) := e^{-\frac{|x|^2}{4\alpha}}, \quad \alpha \geq 1, \quad x \in \mathbb{R}^d.$$

Thus $\widehat{g_\alpha}(\xi) = \frac{1}{(2\alpha)^{\frac{d}{2}}} e^{-\alpha|\xi|^2}$. Conditions (I1)-(I3) are readily verified, and will be discussed in a subsequent example. Evidently, $\widehat{g_\alpha}$ is radially decreasing, so $M_\alpha = (2\alpha)^{-\frac{d}{2}} e^{-\alpha\delta^2}$ and $M_j(\alpha) = (2\alpha)^{-\frac{d}{2}} e^{-\alpha 2^{2(j-1)}}$. Therefore, to check condition (R1), note that

$$S_\alpha \leq (2\alpha)^{-\frac{d}{2}} \sum_{j\in\mathbb{N}} \mathcal{N}^j e^{-\alpha 2^{2(j-1)}} \leq C(2\alpha)^{-\frac{d}{2}} e^{-\alpha} \leq C(2\alpha)^{-\frac{d}{2}} e^{-\alpha\delta^2} = CM_\alpha,$$

where C is some constant depending only on the dimension d. Considering (R2') and noting that $m_\alpha(\beta) = \widehat{g_\alpha}(\beta)$, and $\gamma_\alpha = \widehat{g_\alpha}(1)$, we find that

$$\frac{M_\alpha^3}{m_\alpha(\beta)\gamma_\alpha^2} \leq e^{\alpha(\beta^2+2-3\delta^2)},$$

and the latter tends to 0 as $\alpha \to \infty$ provided $\beta < \sqrt{3\delta^2 - 2}$. This, in turn, requires $\delta > \sqrt{2/3}$ since β must be positive. Consequently, the result of Theorem 5.1.6 coincides with the main theorem in [4], which we restate here in our terminology.

Theorem 5.2.1 (cf. [4], Theorem 3.6)**.** *Let $\delta \in (\sqrt{2/3}, 1)$ and $\beta \in (0, \sqrt{3\delta^2 - 2})$. Then the set of Gaussians $\left(e^{-\frac{|\cdot|^2}{4\alpha}}\right)_{\alpha\in[1,\infty)}$ is a family of d-dimensional interpolators for PW_Z that is regular for $PW_{\beta B_2}$. In particular, for every $f \in PW_{\beta B_2}$, we have $\lim_{\alpha\to\infty} \mathscr{I}_\alpha f = f$ in $L_2(\mathbb{R}^d)$ and uniformly on $\mathbb{R}^d$.*

5.2.2 Inverse Multiquadrics

The next example is a family of *inverse multiquadrics*. For an exponent $\alpha < -d/2$, we define (as in Section 3.2) the general inverse multiquadric with shape parameter $c > 0$ by

$$\phi_{\alpha,c}(x) := (|x|^2 + c^2)^\alpha, \quad x \in \mathbb{R}^d.$$

We will consider the regularity of the family $(\phi_{\alpha,c})_{c\in[1,\infty)}$. This example requires a bit more work up front since the Fourier transform of the inverse multiquadric is somewhat complicated. For now, suppose α is fixed, and we suppress the dependence on α and write ϕ_c for notational ease. Recall that the Fourier transform of ϕ_c is given by (3.5).

One property of the modified Bessel function of the second kind not previously discussed is the following differentiation formula ([1, p. 361]):

$$\frac{d}{dr}\left[r^\nu K_\nu(r)\right] = -r^\nu K_{\nu-1}(r), \tag{5.7}$$

which leads to the following observation.

Proposition 5.2.2. *The function $\widehat{\phi_c}$ is does not change sign, and is radially decreasing. That is, if $|x| \le |y|$, then $|\widehat{\phi_c}(x)| \ge |\widehat{\phi_c}(y)|$.*

Proof. That $\widehat{\phi_c}$ does not change sign is evident from (3.5) and the fact that $K_\nu(r) > 0$ (the sign depends solely on the sign of $\Gamma(-\alpha)$). To see that $\widehat{\phi_c}$ is decreasing, set

$r = |x|$, and note that (3.5) and (5.7) imply

$$\frac{d}{dr}\left[\widehat{\phi}_c(r)\right] = \frac{2^{1+\alpha}}{\Gamma(-\alpha)c^{-\alpha-\frac{d}{2}}}\frac{d}{dr}\left[r^{-\alpha-\frac{d}{2}}K_{-\alpha-\frac{d}{2}}(cr)\right]$$

$$= -\frac{2^{1+\alpha}}{\Gamma(-\alpha)c^{-\alpha-\frac{d}{2}-1}}r^{-\alpha-\frac{d}{2}}K_{-\alpha-\frac{d}{2}-1}(cr),$$

which is negative in the case that $\widehat{\phi}_c$ is positive and positive in the case that $\widehat{\phi}_c$ is negative. Since $\widehat{\phi}_c$ is never 0, the conclusion holds. □

To verify (I1), it is evident from the definition that ϕ_c is integrable, and the following proposition shows that $\widehat{\phi}_c$ is as well.

Proposition 5.2.3. *For $\alpha < -d/2$ and $c > 0$, $\widehat{\phi}_c \in L_1(\mathbb{R}^d)$.*

Proof. According to (3.5), it must be shown that $\int_{\mathbb{R}^d} |\xi|^{-\alpha-\frac{d}{2}}K_{-\alpha-\frac{d}{2}}(c|\xi|)d\xi$ converges. This integral can be split into two pieces, the integral over the Euclidean ball and the integral outside. By Lemma 3.2.2 *(iv)*,

$$I_1 := \int_{B_2} |\xi|^{-\alpha-\frac{d}{2}}K_{-\alpha-\frac{d}{2}}(c|\xi|)d\xi \le C\int_{B_2} |\xi|^{-\alpha-\frac{d}{2}}|\xi|^{\alpha+\frac{d}{2}}d\xi = C\, m(B_2),$$

where C is a finite constant depending on α, d, and c, and $m(B_2)$ is the Lebesgue measure of the Euclidean ball. Furthermore, by Lemma 3.2.2*(iii)*,

$$I_2 := \int_{\mathbb{R}^d\setminus B_2} |\xi|^{-\alpha-\frac{d}{2}}K_{-\alpha-\frac{d}{2}}(c|\xi|)d\xi \le C\int_{\mathbb{R}^d\setminus B_2} |\xi|^{-\alpha-\frac{d}{2}-\frac{1}{2}}e^{-c|\xi|}e^{\frac{|\alpha+\frac{d}{2}|^2}{2c|\xi|}}d\xi$$

$$\le C\int_{\mathbb{R}^d\setminus B_2} |\xi|^{-\alpha-\frac{d}{2}-\frac{1}{2}}e^{-c|\xi|}d\xi,$$

and the right hand side is a convergent integral. Again, C is a finite constant depending on $\alpha, d,$ and c. In the final inequality, we have used the fact that $e^{\frac{|\alpha+\frac{d}{2}|^2}{2c|\xi|}} \le e^{\frac{|\alpha+\frac{d}{2}|^2}{2c}}$. $\square$

Next, notice that (I2) follows from Proposition 5.2.2 and the fact that $\widehat{\phi}_c(1) > 0$. Thus it remains to check (I3) and the regularity conditions. By Proposition 5.2.2 and Lemma 3.2.2(iii),

$$M_j(c) \le |\widehat{\phi}_c(2^{j-1})| \le C_\alpha \left(\frac{2^{j-1}}{c}\right)^{-\alpha-\frac{d}{2}} c^{-\frac{1}{2}} e^{-c2^{j-1}} e^{\frac{|\alpha+\frac{d}{2}|^2}{2^j c}}. \tag{5.8}$$

The right hand side of (5.8) is summable for any fixed c, which yields (I3).

To check (R1), assume, without loss of generality, that c is large enough so that the final exponential term on the right hand side of (5.8) is at most 2, in which case we have (by Lemma 3.2.2(ii))

$$\sum_{j\in\mathbb{N}} \mathcal{N}^j M_j(c) \le C_\alpha \sum_{j\in\mathbb{N}} \mathcal{N}^j 2^{(j-1)(-\alpha-\frac{d}{2})} c^{\frac{d-1}{2}+\alpha} e^{-c2^{j-1}} \le C_{\alpha,d} c^{\frac{d-1}{2}+\alpha} e^{-c} \le C_{\alpha,d} \widehat{\phi}_c(1),$$

which, by Proposition 5.2.2, is at most $C_{\alpha,d}\widehat{\phi}_c(\delta) = C_{\alpha,d} M_c$.

Finally, we check (R2'). By Lemma 3.2.2,

$$\frac{\widehat{\phi}_c(\delta)^3}{\widehat{\phi}_c(\beta)\widehat{\phi}_c(1)^2} \le C_{\alpha,d} \left(\frac{\delta}{\beta}\right)^{-\alpha-\frac{d+1}{2}} c^{\frac{d-1}{2}+\alpha} e^{c(\beta+2-3\delta)}.$$

Consequently, as long as $0 < \beta < 3\delta - 2$ and $\delta > 2/3$, (R2') is satisfied, which leads to the following theorem.

Theorem 5.2.4. *Let $\alpha < -d/2$. Assume $\delta \in (2/3, 1)$ and $\beta \in (0, 3\delta - 2)$, and let Z be as before. Then the set of inverse multiquadrics $((|\cdot|^2 + c^2)^\alpha)_{c\in[1,\infty)}$ is a family of d-dimensional interpolators for PW_Z that is regular for $PW_{\beta B_2}$. In particular, for every $f \in PW_{\beta B_2}$, we have $\lim_{c\to\infty} \mathscr{I}_c f = f$ in $L_2(\mathbb{R}^d)$ and uniformly on $\mathbb{R}^d$.*

5.2.3 A Broad Class of Examples

We end with a large class of examples which includes both the Gaussian and the Poisson kernel as specific cases. These classes provide natural extensions of the results in [4]. For any $p \geq 1$, we define the following function with parameter α:

$$g_\alpha(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\alpha|\xi|^p} e^{i\langle x, \xi\rangle} d\xi, \quad x \in \mathbb{R}^d, \tag{5.9}$$

or in other words, $g_\alpha = \mathscr{F}^{-1}\left[e^{-\alpha|\cdot|^p}\right]$. Note that in the case $d = 1$ and $p \leq 2$, these classes correspond to the so-called p-stable random variables.

By definition, g_α satisfies (I1'). Condition (I2) is evident, and to check (I3), note that since $\widehat{g_\alpha}$ is radially decreasing, $M_j(\alpha) = e^{-\alpha 2^{(j-1)p}}$, and thus $(2^{-jd}\mathcal{N}^j M_j(\alpha))_{j\in\mathbb{N}}$ is summable. We now check the regularity conditions, which will provide bounds on β and δ as in the previous examples. Note that $M_\alpha = e^{-\alpha\delta^p}$. Then as before,

$$S_\alpha = \sum_{j\in\mathbb{N}} \mathcal{N}^j e^{-\alpha 2^{(j-1)p}} \leq Ce^{-\alpha} \leq Ce^{-\alpha\delta^p} = CM_\alpha,$$

where C is some constant independent of α.

Per Remark 5.1.5, consider (R2') as follows:

$$\frac{M_\alpha}{m_\alpha(\beta)\gamma_\alpha^2} = \frac{g_\alpha(\delta)^3}{g_\alpha(\beta)g_\alpha(1)^2} = e^{\alpha(\beta^p + 2 - 3\delta^p)}.$$

Evidently, the right hand side tends to 0 as $\alpha \to \infty$ whenever $\beta < (3\delta^p - 2)^{\frac{1}{p}}$, whence the following.

Theorem 5.2.5. *Let $p \geq 1$. Suppose $\delta \in \left(\left(\frac{2}{3}\right)^{\frac{1}{p}}, 1\right)$, and $\beta \in \left(0, (3\delta^p - 2)^{\frac{1}{p}}\right)$, and Z is as before. Then $(g_\alpha)_{\alpha\in(0,\infty)}$ defined by (5.9) is a family of d-dimensional interpolators for PW_Z that is regular for $PW_{\beta B_2}$. In particular, for every $f \in PW_{\beta B_2}$,* *we have* $\lim_{\alpha\to\infty} \mathscr{I}_\alpha f = f$ *in* $L_2(\mathbb{R}^d)$ *and uniformly on* $\mathbb{R}^d$*.*

Note that in the case $p = 2$, g_α is the Gaussian discussed in the first example, and the condition reads $\beta \in \left(0, \sqrt{3\delta^2 - 2}\right)$; so in this case, Theorems 5.2.1 and 5.2.5 coincide.

5.3 Proofs

5.3.1 Proof of Theorem 5.1.3

Throughout this section, assume that Z is as in the statement of the theorem and that X is a Riesz-basis sequence for $L_2(Z)$. Recall that A_j is the prolongation operator defined by (5.5). The first step in the proof of Theorem 5.1.3 is the following key lemma.

Lemma 5.3.1. *Suppose that* ϕ *is a* d*-dimensional interpolator for* PW_Z*, and let* $A := (\phi(x_m - x_n))_{m,n\in\mathbb{N}}$*. Then* $A : \ell_2 \to \ell_2$ *is a bounded, invertible, linear operator.*

Proof. Linearity is plain, so we will take up boundedness first by looking at $\langle A\mathbf{a}, \mathbf{a}\rangle_{\ell_2}$ for arbitrary $\mathbf{a} := (a_j)_{j\in\mathbb{N}} \in \ell_2$. To show boundedness, use the Dominated Convergence Theorem, (I1), (5.6), and a periodization argument to see that

$$
\begin{aligned}
\sum_{m,n\in\mathbb{N}} a_m \overline{a_n} \phi(x_m - x_n) &= \sum_{m,n\in\mathbb{N}} a_m \overline{a_n} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\phi}(\xi) e^{i\langle x_m - x_n, \xi\rangle} d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\phi}(\xi) \left| \sum_{n\in\mathbb{N}} a_n e^{i\langle x_n, \xi\rangle} \right|^2 d\xi \\
&= \frac{1}{(2\pi)^d} \Bigg[\int_Z \widehat{\phi}(\xi) \left| \sum_{n\in\mathbb{N}} a_n e^{i\langle x_n, \xi\rangle} \right|^2 d\xi \\
&\quad + \sum_{j\in\mathbb{N}} \int_{2^j Z \setminus 2^{j-1} Z} \widehat{\phi}(\xi) \left| \sum_{n\in\mathbb{N}} a_n e^{i\langle x_n, \xi\rangle} \right|^2 d\xi \Bigg]
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{(2\pi)^d}\Bigg[\sup_{u\in Z}|\widehat{\phi}(u)|\ R_b^2\|a\|_{\ell_2}^2 \\
&\quad +\sum_{j\in\mathbb{N}} 2^{-jd}\int_{Z\setminus\frac{1}{2}Z}\widehat{\phi}(2^j\xi)\left|A_j\left(\sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\xi\rangle}\right)\right|^2 d\xi\Bigg] \\
&\leq \frac{1}{(2\pi)^d}\Bigg[\sup_{u\in Z}|\widehat{\phi}(u)|\ R_b^2\|a\|_{\ell_2}^2 \\
&\quad +\sum_{j\in\mathbb{N}} 2^{-jd} M_j 2^{-jd}\mathcal{N}^j R_b^6\|a\|_{\ell_2}^2\Bigg].
\end{aligned}$$

The last term on the right hand side is $R_b^6\|(2^{-2jd}\mathcal{N}^j M_j)_j\|_{\ell_1}$, which by (I3) is finite since $2^{-2jd} < 2^{-jd}$. Equivalently, (I1') could have been used in the first line as we simply needed to write $\phi(x_m - x_n)$ via its Fourier integral.

To show invertibility, we demonstrate a lower bound for the inner product. Indeed, using the Dominated Convergence Theorem again along with (I2),

$$\begin{aligned}
\sum_{m,n\in\mathbb{N}} a_m\overline{a_n}\phi(x_m - x_n) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{\phi}(\xi)\left|\sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\xi\rangle}\right|^2 d\xi \\
&\geq \frac{1}{(2\pi)^d}\int_Z\widehat{\phi}(\xi)\left|\sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\xi\rangle}\right|^2 d\xi \\
&\geq \frac{\varepsilon}{R_b^2(2\pi)^d}\|a\|_{\ell_2}^2.
\end{aligned}$$

□

Proof of Theorem 5.1.3. Note that (i) is a direct consequence of Lemma 5.3.1 and (5.2).

To show (ii), we first prove that the function $\omega := \widehat{\phi}\ \sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\cdot\rangle}$ belongs to

$L_1 \cap L_2$, which follows by a standard periodization argument.

$$\begin{aligned}
\int_{\mathbb{R}^d} |\widehat{\phi}(\xi)| \left| \sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\xi\rangle} \right| d\xi \quad &\leq \quad \sup_{u\in Z} |\widehat{\phi}(u)| \left\| \sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\cdot\rangle} \right\|_{L_1(Z)} \\
&\quad + \sum_{j\in\mathbb{N}} 2^{-jd} M_j \left\| A_j \left(\sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\cdot\rangle} \right) \right\|_{L_1(Z)} \\
&\leq \quad \sup_{u\in Z} |\widehat{\phi}(u)| m(Z)^{\frac{1}{2}} R_b \|a\|_{\ell_2} \\
&\quad + \left\| \left(2^{-\frac{3}{2}jd} \mathcal{N}^{\frac{j}{2}} M_j \right)_j \right\|_{\ell_1} m(Z)^{\frac{1}{2}} R_b \|a\|_{\ell_2}.
\end{aligned}$$

The final step used the Cauchy-Schwarz inequality, and the second term on the right hand side is finite because of (I3) and the fact that $\mathcal{N}^{\frac{j}{2}} \leq \mathcal{N}^j$ since $\mathcal{N} > 1$, and $2^{-\frac{3}{2}jd} \leq 2^{-jd}$. The argument for square-integrability is quite similar:

$$\int_{\mathbb{R}^d} |\widehat{\phi}(\xi)|^2 \left| \sum_{n\in\mathbb{N}} a_n e^{i\langle x_n,\xi\rangle} \right|^2 d\xi \leq \sup_{u\in Z} |\widehat{\phi}(u)|^2 R_b^2 \|a\|_{\ell_2}^2 + \sum_{j\in\mathbb{N}} 2^{-2jd} \mathcal{N}^j M_j^2 R_b^2 \|a\|_{\ell_2}^2.$$

The series on the right is $\|(2^{-jd}\mathcal{N}^{\frac{j}{2}} M_j)_j\|_{\ell_2}^2 \leq \|(2^{-jd}\mathcal{N}^j M_j)_j\|_{\ell_1}^2$, which is finite by (I3). Consequently, $\omega \in L_1 \cap L_2$. It follows from basic techniques and the Riemann-Lebesgue Lemma that the function

$$\mathscr{I}_\phi f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \omega(\xi) e^{i\langle \xi, x\rangle} d\xi = \sum_{j\in\mathbb{N}} a_j \phi(x - x_j)$$

belongs to $C_0(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, and moreover that $\mathscr{F}[\mathscr{I}_\phi f] = \omega$.

Finally, to conclude boundedness, simply notice from the periodization argument above, Lemma 5.3.1, Plancherel's Identity, and (5.2), that

$$\|\mathscr{I}_\phi f\|_{L_2} = \|\mathscr{F}[\mathscr{I}_\phi f]\|_{L_2} \leq C\|a\|_{\ell_2} \leq C\|A^{-1}\|_{\ell_2\to\ell_2} \|f(x_k)\|_{\ell_2}$$

$$\leq C\|A^{-1}\|_{\ell_2\to\ell_2} R_b \|f\|_{L_2}.$$

$\square$

5.3.2 Proof of Theorem 5.1.6

We now embark on the proof of the main result for this chapter. Let Z, X, δ, and β be as in the statement of Theorem 5.1.6, and let $(\phi_\alpha)_{\alpha\in A}$ be a family of d-dimensional interpolators for PW_Z that is regular for $PW_{\beta B_2}$. Recall that γ_α and M_α are as in (R2). The first step is to show that there exists a constant $0 < C < \infty$ so that

$$\|\mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2(Z)} \leq C\frac{M_\alpha}{\gamma_\alpha}\|\mathscr{F}[f]\|_{L_2(Z)}, \quad \alpha \in A,$$

for every $f \in PW_Z$. We proceed in a series of steps following the techniques of [4].

To begin, define the function

$$\Psi_\alpha(u) := \sum_{j\in\mathbb{N}} a_j e^{-i\langle x_j, u\rangle} = \frac{1}{\widehat{\phi_\alpha}(u)}\mathscr{F}[\mathscr{I}_\alpha f](u), \quad u \in \mathbb{R}^d, \tag{5.10}$$

and let ψ_α denote the restriction of Ψ_α to Z.

Remark 5.3.2. It is important to note that by uniqueness of the Riesz basis representation for a function on Z, we have that $\Psi_\alpha(u) = E(\psi_\alpha)(u)$ on $\mathbb{R}^d$. That is, Ψ_α is defined globally by its Riesz basis representation on the body Z. This fact is crucial to the subsequent analysis.

Lemma 5.3.3. *The following holds:*

$$\mathscr{F}[f] = \mathscr{F}[\mathscr{I}_\alpha f] + \sum_{m\in\mathbb{N}} 2^{dm} A_m^*\left(\widehat{\phi}(2^m\cdot)A_m(\psi_\alpha)\right) \text{ a.e. on } Z.$$

Proof. Note that since $\left(e^{-i\langle x_j,\cdot\rangle}\right)$ is a Riesz basis for $L_2(Z)$, it suffices to show that the inner product of both sides above with respect to the basis elements are all equal. First, by (2.3),

$$\left\langle \mathscr{F}[f], e^{-i\langle x_j,\cdot\rangle}\right\rangle_Z = (2\pi)^d f(x_j).$$

On the other hand, the interpolation condition guarantees that

$$\begin{aligned}
(2\pi)^d f(x_j) &= (2\pi)^d \mathscr{I}_\alpha f(x_j) \\
&= \int_{\mathbb{R}^d} \mathscr{F}[\mathscr{I}_\alpha f](u) e^{i\langle x_j,u\rangle} du \\
&= \int_Z \mathscr{F}[\mathscr{I}_\alpha f](u) e^{i\langle x_j,u\rangle} du + \sum_{m\in\mathbb{N}} \int_{2^m Z\setminus 2^{m-1}Z} \widehat{\phi_\alpha}(u)\Psi_\alpha(u) e^{i\langle x_j,u\rangle} du \\
&=: I_1 + I_2.
\end{aligned}$$

Evidently, $I_1 = \left\langle \mathscr{F}[\mathscr{I}_\alpha f], e^{-i\langle x_j,\cdot\rangle}\right\rangle_Z$, whereas

$$\begin{aligned}
I_2 &= \sum_{m\in\mathbb{N}} 2^{dm} \int_{Z\setminus\frac{1}{2}Z} \widehat{\phi_\alpha}(2^m v)\Psi_\alpha(2^m v) e^{i\langle x_j,2^m v\rangle} dv \\
&= \sum_{m\in\mathbb{N}} 2^{dm} \int_{Z\setminus\frac{1}{2}Z} \widehat{\phi_\alpha}(2^m v) A_m(\psi_\alpha)(v) A_m\left(e^{i\langle x_j,\cdot\rangle}\right)(v) dv \\
&= \sum_{m\in\mathbb{N}} 2^{dm} \left\langle \widehat{\phi_\alpha}(2^m\cdot) A_m(\psi_\alpha), A_m\left(e^{-i\langle x_j,\cdot\rangle}\right)\right\rangle_Z \\
&= \sum_{m\in\mathbb{N}} 2^{dm} \left\langle A_m^*\left(\widehat{\phi_\alpha}(2^m\cdot) A_m(\psi_\alpha)\right), e^{-i\langle x_j,\cdot\rangle}\right\rangle_Z,
\end{aligned}$$

whence the identity. $\square$

We now define an operator that is implicit in the previous Lemma:

$$\tau_\alpha : L_2(Z) \to L_2(Z), \quad \text{via} \quad \tau_\alpha(h) := \sum_{m\in\mathbb{N}} A_m^* \left(\widehat{\phi_\alpha}(2^m \cdot) A_m(h) \right). \tag{5.11}$$

Proposition 5.3.4. *The operator τ_α defined by* (5.11) *is a bounded linear operator on $L_2(Z)$ that is positive, (i.e. $\langle \tau_\alpha(h), h\rangle_Z \geq 0$ for all $h \in L_2(Z)$). Moreover, there exists a positive number C, which is independent of α, so that*

$$\|\tau_\alpha\| \leq C M_\alpha. \tag{5.12}$$

Proof. Linearity is plain, and positivity can be seen as follows.

$$\begin{aligned}
\langle \tau_\alpha(h), h\rangle_Z &= \sum_{m\in\mathbb{N}} 2^{dm} \left\langle \widehat{\phi_\alpha}(2^m \cdot) A_m(h), A_m(h) \right\rangle_Z \\
&= \sum_{m\in\mathbb{N}} 2^{dm} \int_{Z\setminus \frac{1}{2}Z} \widehat{\phi_\alpha}(2^m u) |A_m(h)(u)|^2 du \\
&\geq 0,
\end{aligned}$$

the final inequality stemming from the positivity of $\widehat{\phi_\alpha}$.

To prove the upper bound, notice that (5.6) implies that for $h \in L_2(Z)$,

$$\begin{aligned}
\|\tau_\alpha(h)\|_{L_2(Z)} &\leq \sum_{m\in\mathbb{N}} 2^{dm} \left\| A_m^* \left(\widehat{\phi_\alpha}(2^m \cdot) A_m(h) \right) \right\|_{L_2(Z)} \\
&\leq R_b^2 \sum_{m\in\mathbb{N}} 2^{\frac{dm}{2}} \mathcal{N}^{\frac{m}{2}} M_m(\alpha) \|A_m(h)\|_{L_2(Z)}
\end{aligned}$$

$$\begin{aligned} &\le R_b^4 \sum_{m\in\mathbb{N}} \mathcal{N}^m M_m(\alpha)\|h\|_{L_2(Z)} \\ &= R_b^4 S_\alpha \|h\|_{L_2(Z)} \\ &\le C M_\alpha \|h\|_{L_2(Z)}. \end{aligned}$$

The final inequality comes from condition (R1). □

Next, note that the positivity of τ_α and Lemma 5.3.3 imply that

$$\|\mathscr{F}[f]\|_{L_2(Z)}\|\psi_\alpha\|_{L_2(Z)} \ge \langle \mathscr{F}[f], \psi_\alpha\rangle_Z \ge \langle \mathscr{F}[\mathscr{I}_\alpha f], \psi_\alpha\rangle_Z \ge \gamma_\alpha \|\psi_\alpha\|^2_{L_2(Z)}.$$

Therefore,

$$\|\psi_\alpha\|_{L_2(Z)} \le \frac{1}{\gamma_\alpha}\|\mathscr{F}[f]\|_{L_2(Z)}. \tag{5.13}$$

From Lemma 5.3.3, Proposition 5.3.4, and (5.13), we see that

$$\|\mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2(Z)} \le C\frac{M_\alpha}{\gamma_\alpha}\|\mathscr{F}[f]\|_{L_2(Z)}. \tag{5.14}$$

Next we estimate $\|\mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2(\mathbb{R}^d\setminus Z)}$, which is accomplished via a familiar periodization argument.

$$\begin{aligned} \|\mathscr{F}[\mathscr{I}_\alpha f]\|^2_{L_2(\mathbb{R}^d\setminus Z)} &= \sum_{m\in\mathbb{N}} 2^{dm}\int_{Z\setminus\frac{1}{2}Z} |\widehat{\phi_\alpha}(2^m u)|^2 |\Psi_\alpha(2^m u)|^2 du \\ &\le \sum_{m\in\mathbb{N}} 2^{dm} M_m(\alpha)^2 \|A_m(\psi_\alpha)\|^2_{L_2(Z)} \end{aligned}$$

$$\leq \sum_{m\in\mathbb{N}} \mathcal{N}^m M_m(\alpha)^2 R_b^4 \|\psi_\alpha\|^2_{L_2(Z)}$$

$$\leq R_b^4 \frac{1}{\gamma_\alpha^2} \|\mathscr{F}[f]\|^2_{L_2(Z)} \sum_{m\in\mathbb{N}} \mathcal{N}^m M_m(\alpha)^2.$$

Recall again that the covering number $\mathcal{N}$ must be larger than 1, so the series in the final expression above is majorized by

$$\sum_{m\in\mathbb{N}} \mathcal{N}^{2m} M_m(\alpha)^2 \leq \left(\sum_{m\in\mathbb{N}} \mathcal{N}^m M_m(\alpha)\right)^2 = S_\alpha^2 \leq C M_\alpha^2.$$

The first inequality above comes from the fact that the ℓ_2 norm is subordinate to the ℓ_1 norm, and the final inequality is a consequence of (R1). Consequently, the following holds.

Theorem 5.3.5. *There exists a constant C, independent of α, such that*

$$\|\mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2(\mathbb{R}^d)} \leq C \frac{M_\alpha}{\gamma_\alpha} \|\mathscr{F}[f]\|_{L_2(Z)}, \quad f \in PW_Z.$$

The next step toward the proof of Theorem 5.1.6 involves the definition of a multiplication operator $T_\alpha : L_2(Z) \to L_2(Z)$ defined by

$$T_\alpha(h) := \frac{\gamma_\alpha}{\widehat{\phi_\alpha}} h.$$

The definition of γ_α implies that $\|T_\alpha\| \leq 1$. Lemma 5.3.3 can be rewritten as as

$$\mathscr{F}[f] = \mathscr{F}[\mathscr{I}_\alpha f] + \sum_{m\in\mathbb{N}} 2^{dm} A_m^* \left(\frac{\widehat{\phi_\alpha}(2^m\cdot)}{\gamma_\alpha} A_m\left(\gamma_\alpha \psi_\alpha\right)\right),$$

which by (5.10) is

$$\mathscr{F}[f] = \mathscr{F}[\mathscr{I}_\alpha f] + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha \left(\mathscr{F}[\mathscr{I}_\alpha f]\right) = \left(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha\right)\mathscr{F}[\mathscr{I}_\alpha f],$$

where I is the identity operator on $L_2(Z)$.

Proposition 5.3.6. *The map $I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha$ is an invertible operator on $L_2(Z)$, and*

$$\left(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha\right)^{-1}\mathscr{F}[f] = \mathscr{F}[\mathscr{I}_\alpha f], \quad f \in PW_Z.$$

Moreover,

$$\left\| \left(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha\right)^{-1} \right\| \leq \frac{M_\alpha}{\gamma_\alpha}.$$

Proof. Surjectivity of the operator in question follows from the identity in Lemma 5.3.3. To see injectivity, suppose that $(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha)h = 0$. Let $f \in PW_Z$ be the function satisfying $\mathscr{F}[f] = h$. Then by Lemma 5.3.3 and positivity of τ_α, we have

$$0 = \left\langle \left(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha\right)\mathscr{F}[f], T_\alpha \mathscr{F}[f] \right\rangle_Z \geq \langle \mathscr{F}[f], T_\alpha \mathscr{F}[f]\rangle_Z \geq 0.$$

Consequently, $T_\alpha \mathscr{F}[f] = 0$, which implies that $\mathscr{F}[f] = 0$, hence $h = 0$, since $T_\alpha g = 0$ if and only if $g = 0$.

Finally, the norm estimate follows from Theorem 5.3.5 and Lemma 5.3.3.

□

Now all of the necessary ingredients have been assembled to complete the proof.

Proof of Theorem 5.1.6. By Proposition 5.3.6 and Lemma 5.3.3, the following identity holds on Z:

$$\mathscr{F}[f] - \mathscr{F}[\mathscr{I}_\alpha f] = \left[I - \left(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha\right)^{-1}\right](\mathscr{F}[f])$$

$$= \left(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha\right)^{-1} \circ \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha(\mathscr{F}[f]).$$

Therefore, if $f \in PW_{\beta B_2}$, Theorem 5.3.5 and Proposition 5.3.4 imply

$$\|\mathscr{F}[f] - \mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2(Z)} \leq \left\|\left(I + \frac{1}{\gamma_\alpha}\tau_\alpha \circ T_\alpha\right)^{-1}\right\| \frac{1}{\gamma_\alpha}\|\tau_\alpha\|\|T_\alpha \mathscr{F}[f]\|_{L_2(Z)}$$

$$\leq C\frac{M_\alpha}{\gamma_\alpha}M_\alpha \left\|\frac{1}{\widehat{\phi_\alpha}(\cdot)}\mathscr{F}[f]\right\|_{L_2(\beta B_2)}$$

$$\leq C\frac{M_\alpha^2}{\gamma_\alpha m_\alpha(\beta)}\|\mathscr{F}[f]\|_{L_2(\beta B_2)}. \tag{5.15}$$

Next, we estimate $\|\mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2(\mathbb{R}^d\setminus Z)}$ by familiar techniques.

$$\|\mathscr{F}[\mathscr{I}_\alpha f]\|^2_{L_2(\mathbb{R}^d\setminus Z)} \leq \sum_{m\in\mathbb{N}} 2^{dm} M_m(\alpha)^2 \|A_m(\psi_\alpha)\|^2_{L_2(Z)}$$

$$\leq R_b^2 S_\alpha^2 \left\|\frac{1}{\widehat{\phi_\alpha}(\cdot)}\mathscr{F}[\mathscr{I}_\alpha f]\right\|^2_{L_2(Z)}$$

$$\leq C\frac{M_\alpha^2}{\gamma_\alpha^2}\|T_\alpha \mathscr{F}[\mathscr{I}_\alpha f]\|^2_{L_2(Z)}$$

$$\leq C \left[\frac{M_\alpha}{\gamma_\alpha} \| T_\alpha \left(F[\mathscr{I}_\alpha f] - \mathscr{F}[f] \right) \|^2_{L_2(Z)} + \| T_\alpha \mathscr{F}[f] \|_{L_2(\beta B_2)} \right]^2$$

$$\leq C \left[\frac{M_\alpha^3}{\gamma_\alpha^2 m_\alpha(\beta)} + \frac{M_\alpha}{m_\alpha(\beta)} \right]^2 \| \mathscr{F}[f] \|^2_{L_2(\beta B_2)}. \tag{5.16}$$

Convergence of $\|\mathscr{F}[\mathscr{I}_\alpha f] - \mathscr{F}[f]\|_{L_2(\mathbb{R}^d)}$ depends ostensibly on the three ratios in (5.15) and (5.16). However, the largest is $\frac{M_\alpha^3}{\gamma_\alpha^2 m_\alpha(\beta)}$. Indeed one obtains this by multiplying $\frac{M_\alpha}{m_\alpha(\beta)}$ by $\frac{M_\alpha^2}{\gamma_\alpha^2}$ which is at least 1 by definition. Similarly, $\frac{M_\alpha^2}{\gamma_\alpha m_\alpha(\beta)} \leq \frac{M_\alpha^2}{\gamma_\alpha m_\alpha(\beta)} \frac{M_\alpha}{\gamma_\alpha} = \frac{M_\alpha^3}{\gamma_\alpha^2 m_\alpha(\beta)}$. Consequently, if (R2) is satisfied, then $\|\mathscr{F}[\mathscr{I}_\alpha f] - \mathscr{F}[f]\|_{L_2(\mathbb{R}^d)}$ converges to 0 as $\alpha \to \infty$.

To show uniform convergence on $\mathbb{R}^d$, use the Fourier inversion formula and the fact that $\mathscr{F}[f] = 0$ almost everywhere outside of Z to see that

$$|\mathscr{I}_\alpha f(x) - f(x)| \leq \frac{1}{(2\pi)^d} \left(\|\mathscr{F}[\mathscr{I}_\alpha f] - \mathscr{F}[f]\|_{L_1(Z)} + \|\mathscr{F}[\mathscr{I}_\alpha f]\|_{L_1(\mathbb{R}^d \setminus Z)} \right).$$

The Cauchy-Schwarz inequality and convergence in the corresponding L_2 norm imply that $\|\mathscr{F}[\mathscr{I}_\alpha f] - \mathscr{F}[f]\|_{L_1(Z)} \to 0$ as $\alpha \to \infty$. A similar periodization argument to the one above and another appeal to the Cauchy-Schwarz inequality shows that $\|\mathscr{F}[\mathscr{I}_\alpha f]\|_{L_1(\mathbb{R}^d \setminus Z)} \to 0$, which concludes the proof. □

5.4 Remarks

Remark 5.4.1. There are many ways in which one could choose to periodize the integrals over $\mathbb{R}^d \setminus Z$, and consequently, condition (I3) could well be formulated differently. For example, if one periodizes using the annuli $jZ \setminus (j-1)Z$, then the condition would be that $(j^{-d} \mathcal{N}(jZ \setminus (j-1)Z, Z) M_j) \in \ell_1$, once the definition of M_j is modified suitably. However, this modification of M_j essentially counteracts the change in annuli, and so does not give a substantially different condition.

Remark 5.4.2. As a by-product of the proof of Theorem 5.1.6, one can deduce approximation rates in terms of the parameter α for functions $f \in PW_{\beta B_2}$ in terms of the ratio $\frac{M_\alpha^3}{m_\alpha(\beta)\gamma_\alpha^2}$. In fact, all of the examples from Section 5.2 exhibit exponential approximation rates in this case. Precisely, one can see the following.

Theorem 5.4.3. *Given a family $(\phi_\alpha)_{\alpha \in A}$ of d-dimensional interpolators that is regular for $PW_{\beta B_2}$, there exists a constant C, independent of α, so that for any $f \in PW_{\beta B_2}$,*

$$\|\mathscr{I}_\alpha f - f\|_{L_2} \leq C \frac{M_\alpha^3}{m_\alpha(\beta)\gamma_\alpha^2} \|f\|_{L_2}.$$

Remark 5.4.4. It is worth discussing the limiting case briefly. Each of Theorems 5.2.1, 5.2.4, and 5.2.5 hold in the case that $\delta = \beta = 1$, or in other words, $Z = B_2$. Indeed one needs only look at the end of the proof of Theorem 5.1.6 and see that the Dominated Convergence Theorem can be applied to show that $\lim\limits_{\alpha \to \infty} \|T_\alpha g\|_{L_2(B_2)} = 0$ for $g \in L_2(B_2)$. However, as mentioned above, the result may be vacuous, because it is unknown if there is any Riesz-basis sequence for $L_2(B_2)$. This is the primary reason for the analysis we have done here, to exploit the fact that we know there are Riesz-basis sequences for some convex bodies contained in the Euclidean ball.

Remark 5.4.5. For further reading on the interesting problem of finding Riesz-basis sequences, the reader is referred to [21, 24, 34, 48] for results in one dimension, and [2, 3, 31, 46] for higher dimensions.

6. APPROXIMATION RATES FOR NONUNIFORM INTERPOLATION BY GAUSSIANS AND REGULAR INTERPOLATORS*

The final problem that will be considered in this work is perhaps the most interesting. In this chapter, we consider Problem 1.0.5 on finding approximation rates for nonuniform interpolation schemes from the previously discussed approximation spaces formed by translates of the Gauss kernel and Ledford's so-called regular interpolators [26] (these are defined in Section 6.6). Interestingly, the technique of proof allows for interpolation of Sobolev (W_2^k) functions rather than simply band-limited ones, providing a nice generalization. The inspiration for this problem lies in the work of Hangelbroek, Madych, Narcowich, and Ward [19], and the work of this chapter is published as [15]. For clarity, the specific problem to be considered is stated below.

Problem 6.0.6. *Let $W_2^k(\mathbb{R})$ be the Sobolev space of functions whose first k weak derivatives lie in $L_2(\mathbb{R})$. Let $X := (x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ be a Riesz-basis sequence for $L_2[-\pi,\pi]$, and ϕ be either the Gauss kernel or a regular interpolator, and let*

$$\mathcal{A} := \mathcal{A}_{\phi,X} := \left\{ \sum_{j\in\mathbb{Z}} a_j \phi(x - x_j) \ : \ (a_j) \subset \mathbb{R} \right\}$$

be the approximation space. Find a function $R(h) : (0,1] \to [0,\infty)$ such that given any $f \in W_2^k(\mathbb{R})$, there is an interpolant $I^{hX}(f) \in \mathcal{A}$ such that

$$I^{hX}(f)(hx_j) = f(hx_j), \quad j \in \mathbb{Z},$$

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and moreover

$$\|I^{hX}(f) - f\|_{L_2(\mathbb{R})} \leq R(h)|f|_{W_2^k(\mathbb{R})}.$$

The subsequent analysis will provide a positive solution to this problem with $R(h) = h^k$ for all of the interpolators considered. Note that the function R will depend on the choice of the interpolator ϕ, but not on the function f that is being interpolated.

6.1 Main Results

In this section, we will formulate a series of theorems on the way to the main result, and using these we supply the proof of the main theorem of this chapter, Theorem 6.1.5.

We will form two interpolants involving translates of a fixed Gaussian function; one is the original interpolation operator from [40] which interpolates functions in PW_π at a Riesz-basis sequence X, and the second is similar to that of [19], which will be used to interpolate W_2^k functions at hX, given a parameter $0 < h \leq 1$. Let $\lambda > 0$ be fixed, and let $X := (x_j)_{j\in\mathbb{Z}}$ be a Riesz-basis sequence for $L_2[-\pi, \pi]$. It was shown in [40] that given $f \in PW_\pi$, there exists a unique ℓ_2 sequence $(a_j)_{j\in\mathbb{Z}}$ depending on λ, f, and X such that the Gaussian interpolant

$$\mathscr{I}_\lambda^X(f)(x) := \sum_{j\in\mathbb{Z}} a_j e^{-\lambda(x-x_j)^2}, \quad x \in \mathbb{R}, \tag{6.1}$$

is continuous and square-integrable on $\mathbb{R}$, and satisfies the interpolatory condition

$$\mathscr{I}_\lambda^X(f)(x_j) = f(x_j), \quad j \in \mathbb{Z}. \tag{6.2}$$

Where the sequence is clear, we will omit the superscript X. The main result from their paper is the following.

Theorem 6.1.1 ([40], Theorems 4.3 and 4.4)**.** *Let X be a Riesz-basis sequence for $L_2[-\pi,\pi]$, and $0<\lambda\leq 1$. Then for every $f\in PW_\pi$, the Gaussian interpolant $\mathscr{I}_\lambda^X f$ satisfies*

$$\lim_{\lambda\to 0^+}\|\mathscr{I}_\lambda^X f - f\|_{L_2(\mathbb{R})}=0, \quad \text{and} \quad \lim_{\lambda\to 0^+}|\mathscr{I}_\lambda^X f(x)-f(x)|=0,$$

uniformly on $\mathbb{R}$.

Define the second interpolant via the following identity:

$$I^{hX}(f)(x) := \frac{1}{h}\mathscr{I}_{h^2}^X\left(f^h\right)\left(\frac{x}{h}\right), \tag{6.3}$$

where

$$f^h(x) := hf(hx), \quad x\in\mathbb{R}. \tag{6.4}$$

Defined this way, it is evident that this operator satisfies a similar interpolation condition to (6.2). Precisely, due to (6.2), (6.3), and (6.4),

$$I^{hX}(f)(hx_j) = \frac{1}{h}\mathscr{I}_{h^2}^X\left(f^h\right)\left(\frac{hx_j}{h}\right) = \frac{1}{h}f^h(x_j) = f(hx_j). \tag{6.5}$$

An important fact, and indeed the reason for defining I^{hX} in this manner, is that if $f\in PW_{\frac{\pi}{h}}$, then $f^h\in PW_\pi$. Moreover, the relation $\mathscr{F}[f^h](\xi)=\mathscr{F}[f](\xi/h)$ holds. The second interpolation operator enters the analysis in the following way: to interpolate Sobolev functions, we first interpolate them by bandlimited functions whose band size increases depending on h, and then use the original Gaussian interpolant to do the rest of the work. More precisely, the following holds.

Theorem 6.1.2. *Let $k \in \mathbb{N}$, $h > 0$, and let X be a fixed Riesz-basis sequence for $L_2[-\pi, \pi]$. Then for every $g \in W_2^k$, there exists a unique $F \in PW_{\frac{\pi}{h}}$ such that*

$$F(hx_j) = g(hx_j), \quad j \in \mathbb{Z}, \tag{6.6}$$

$$|F|_{W_2^k} \leq C|g|_{W_2^k}, \tag{6.7}$$

and

$$\|g - F\|_{L_2} \leq Ch^k|g|_{W_2^k}, \tag{6.8}$$

where C is a constant depending on k and X.

We note that for the case $X = \mathbb{Z}$ but general p, a similar result was obtained in [19, Lemma 2.2], however, the proof of Theorem 6.1.2 is substantially different.

We then consider stability of the Gaussian interpolant $\mathscr{I}_\lambda$ as an operator from W_2^k to itself. In [40], it was shown that $(\mathscr{I}_\lambda)_{\lambda\in(0,1]}$ is uniformly bounded as a set of operators from PW_π to L_2, as well as being uniformly bounded from PW_π to $C_0(\mathbb{R})$. We adapt the techniques of that paper to show that $(\mathscr{I}_\lambda)_{\lambda\in(0,1]}$ is uniformly bounded as a set of operators from PW_π to W_2^k, which in light of Theorem 6.1.2, means that this family is uniformly bounded on W_2^k. We summarize this in the following theorem:

Theorem 6.1.3. *Let $k \in \mathbb{N}$ and let X be a Riesz-basis sequence for $L_2[-\pi, \pi]$. Then there exists a constant C depending only on k and X such that for every $0 < \lambda \leq 1$,*

$$|\mathscr{I}_\lambda(g)|_{W_2^k} \leq C|g|_{W_2^k}, \quad \text{for all } g \in W_2^k. \tag{6.9}$$

Consequently, $(\mathscr{I}_\lambda)_{\lambda\in(0,1]}$ is uniformly bounded as a set of operators from W_2^k to itself.

Combined with some calculations (see Section 6.4), this theorem yields the following corollary:

Corollary 6.1.4. *Let* $k \in \mathbb{N}$ *and let* X *be a Riesz-basis sequence for* $L_2[-\pi, \pi]$*. Then there exists a constant* C *depending only on* k *and* X *such that for every* $0 < h \leq 1$*,*

$$|I^{hX}(g)|_{W_2^k} \leq C|g|_{W_2^k}, \quad \text{for all } g \in W_2^k. \tag{6.10}$$

Consequently, $\left(I^{hX}\right)_{h\in(0,1]}$ *is uniformly bounded as a set of operators from* W_2^k *to itself.*

A combination of these results yields the main theorem:

Theorem 6.1.5. *Let* $k \in \mathbb{N}$*,* $0 < h \leq 1$*, and let* X *be a Riesz-basis sequence for* $L_2[-\pi, \pi]$*. Then there exists a constant depending only on* k *and* X *such that for every* $g \in W_2^k$*,*

$$\|I^{hX}(g) - g\|_{L_2} \leq Ch^k|g|_{W_2^k}. \tag{6.11}$$

We also obtain derivative convergence:

Corollary 6.1.6. *Let* $k \geq 2$*,* $1 \leq j < k$*,* $0 < h \leq 1$*, and let* X *be a Riesz-basis sequence for* $L_2[-\pi, \pi]$*. Then there exists a constant depending only on* j, k*, and* X *such that for every* $g \in W_2^k$*,*

$$|I^{hX}(g) - g|_{W_2^j} \leq Ch^{k-j}|g|_{W_2^k} . \tag{6.12}$$

As $PW_\sigma \subset W_2^k$ for every k, an easy corollary of Theorem 6.1.5 is the following:

Corollary 6.1.7. *Let* $0 < h \leq 1$*,* $\sigma > 0$*, and let* X *be a Riesz-basis sequence for* $L_2[-\pi, \pi]$*. Then for each* $k \in \mathbb{N}$*, there exists a constant depending only on* k *and* X

such that for every $\phi \in PW_\sigma$,

$$\|I^{hX}(\phi) - \phi\|_{L_2} \leq Ch^k |\phi|_{W_2^k}. \tag{6.13}$$

Note that Corollary 6.1.7 holds for $\phi \in \mathscr{S}(\mathbb{R})$ as well.

Before giving the proof of the main result, we display a theorem (that will be used several times) of Madych and Potter that gives an estimate on the norm of functions with many zeroes.

Theorem 6.1.8 (cf. [33], Corollary 1)**.** *Suppose* $k \in \mathbb{N}$ *and* $f \in W_p^k(\mathbb{R})$. *Let* $Z := \{x \in \mathbb{R} : f(x) = 0\}$ *and suppose that* $h := \max\{dist(x, Z) : x \in \mathbb{R}\} < \infty$. *Then there exists a constant* C *independent of* f, h *and* Z *such that*

$$|f|_{W_p^j} \leq Ch^{k-j} |f|_{W_p^k}$$

for $j = 0, 1, \ldots, k$.

We now provide a proof of Theorem 6.1.5 using the results we have collected thus far in this section.

Proof of Theorem 6.1.5. Theorem 6.1.2 provides a function $F \in PW_{\frac{\pi}{h}}$ which interpolates g at (hx_j). Then

$$\|I^{hX}(g) - g\|_{L_2} \leq \|I^{hX}(g) - F\|_{L_2} + \|F - g\|_{L_2} =: I_1 + I_2.$$

By (6.8),

$$I_2 \leq Ch^k |g|_{W_2^k}.$$

Since $F(hx_j) = g(hx_j)$, uniqueness of the interpolation operator guarantees that $I^{hX}(F) = I^{hX}(g)$, and therefore $I_1 = \|I^{hX}(F) - F\|_{L_2}$. Applying Theorem 6.1.8 to

$I^{hX}(F) - F$ along with Corollary 6.1.4 and equation (6.7) yields

$$I_1 \leq Ch^k |I^{hX}(F) - F|_{W_2^k} \leq Ch^k \left(|I^{hX}(F)|_{W_2^k} + |F|_{W_2^k} \right) \leq Ch^k |F|_{W_2^k} \leq Ch^k |g|_{W_2^k}.$$

□

6.2 Interpolation of W_2^k Functions by Bandlimited Functions

The main goal of this section is to provide the proof of Theorem 6.1.2, which follows from a series of intermediate steps. We begin with a proposition which will lead to the existence and uniqueness of a bandlimited interpolant for a given Sobolev function. The proposition is stated and proved for general p rather than in the specific case $p = 2$ because it is interesting in its own right and gives a sort of reverse inequality of [33, Theorem 1].

Proposition 6.2.1. *Let $(x_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence such that $\inf_{n\in\mathbb{Z}} (x_{n+1} - x_n) =: q > 0$. If $g \in W_p^k(\mathbb{R})$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, then $(g(x_n))_{n\in\mathbb{Z}} \in \ell_p$. Moreover, if $\frac{1}{p} + \frac{1}{p'} = 1$, and $p \neq \infty$, then*

$$\|g(x_n)\|_{\ell_p} \leq 2^{\frac{1}{p'}} \left(\frac{3}{2q} \right)^{\frac{1}{p}} \|g\|_{L_p} + 2^{\frac{1}{p'}} \left(\frac{2q}{3} \right)^{\frac{1}{p'}} \|g'\|_{L_p}. \tag{6.14}$$

Proof. First, in the case $p = \infty$, if $g \in W_\infty^1(\mathbb{R})$, then g is continuous and bounded, and the result follows.

For $1 \leq p < \infty$, let

$$I_n := \left[x_n - \frac{q}{3}, x_n + \frac{q}{3} \right].$$

These intervals are pairwise disjoint and have length $\frac{2}{3}q$. The former condition ensures that

$$\sum_{n\in\mathbb{Z}} \int_{I_n} |g(t)|^p dt \leq \int_{\mathbb{R}} |g(t)|^p dt < \infty.$$

As g admits an absolutely continuous representative (see, for example, [29, Theorem 7.13, p.222]), we may assume, without loss of generality, that g itself is absolutely continuous. Therefore, we may choose $y_n \in I_n$ such that $|g(y_n)|^p = \min\{|g(x)|^p : x \in I_n\}$; then we have that

$$|g(y_n)|^p \frac{2}{3} q \leq \int_{I_n} |g(t)|^p dt,$$

and consequently, $(g(y_n))_{n\in\mathbb{Z}} \in \ell_p$. Moreover,

$$\left(\sum_{n\in\mathbb{Z}} |g(y_n)|^p\right)^{\frac{1}{p}} \leq \left(\frac{3}{2q}\right)^{\frac{1}{p}} \|g\|_{L_p}. \tag{6.15}$$

Additionally, by the Fundamental Theorem of Calculus,

$$g(x_n) = g(y_n) + \int_{y_n}^{x_n} g'(t)dt.$$

Using the inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, we have

$$\begin{aligned} |g(x_n)|^p &\leq 2^{p-1}\left[|g(y_n)|^p + \left|\int_{y_n}^{x_n} g'(t)dt\right|^p\right] \\ &\leq 2^{p-1}\left[|g(y_n)|^p + \left(\tfrac{2q}{3}\right)^{\frac{p}{p'}} \int_{I_n} |g'(t)|^p dt\right]. \end{aligned}$$

The second inequality follows by Hölder's Inequality because

$$\left|\int_{y_n}^{x_n} g'(t)dt\right|^p \leq \left(\int_{y_n}^{x_n} |g'(t)|dt\right)^p \leq \left(\int_{I_n} |g'(t)|dt\right)^p$$

$$\leq \left(\int_{I_n} |g'(t)|^p dt\right)^{p\frac{1}{p}} |I_n|^{\frac{p}{p'}} = \left(\frac{2q}{3}\right)^{\frac{p}{p'}} \int_{I_n} |g'(t)|^p dt.$$

Therefore by (6.15) and the fact that $p/p' = p-1$,

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}}|g(x_n)|^p &\le 2^{p-1}\sum_{n\in\mathbb{Z}}|g(y_n)|^p + 2^{p-1}\left(\tfrac{2q}{3}\right)^{p-1}\sum_{n\in\mathbb{Z}}\int_{I_n}|g'|^p \\
&\le 2^{p-1}\left(\tfrac{2q}{3}\right)^{-1}\|g\|_{L_p}^p + 2^{p-1}\left(\tfrac{2q}{3}\right)^{p-1}\|g'\|_{L_p}^p \\
&= 2^{p-1}\left(\tfrac{2q}{3}\right)^{-1}\left[\|g\|_{L_p}^p + \left(\tfrac{2q}{3}\right)^{p}\|g'\|_{L_p}^p\right].
\end{aligned}
$$

Now since the function $x \mapsto |x|^{\frac{1}{p}}$ is concave,

$$
\begin{aligned}
\left(\sum_{n\in\mathbb{Z}}|g(x_n)|^p\right)^{\frac{1}{p}} &\le 2^{\frac{p-1}{p}}\left(\tfrac{2q}{3}\right)^{-\frac{1}{p}}\left(\|g\|_{L_p} + \tfrac{2q}{3}\|g'\|_{L_p}\right) \\
&= 2^{\frac{1}{p'}}\left(\frac{3}{2q}\right)^{\frac{1}{p}}\|g\|_{L_p} + 2^{\frac{1}{p'}}\left(\tfrac{2q}{3}\right)^{\frac{1}{p'}}\|g'\|_{L_p},
\end{aligned}
$$

which is (6.14). □

From now on, let $X := (x_j)_{j\in\mathbb{Z}}$ be a fixed Riesz-basis sequence for $L_2[-\pi,\pi]$ with Riesz basis constant B. It is a necessary condition for a Riesz-basis sequence to be uniformly separated; i.e. there are $0 < q \le Q$ such that

$$
q \le x_{j+1} - x_j \le Q, \quad j \in \mathbb{Z}, \tag{6.16}
$$

(without loss of generality, assume that $x_j < x_{j+1}$ for every j).

Note that if X is a Riesz-basis sequence for $L_2[-\pi,\pi]$, then hX is a Riesz-basis sequence for $L_2\left[-\frac{\pi}{h},\frac{\pi}{h}\right]$. Indeed, define the map

$$
J_{\pi\sigma} : L_2[-\pi,\pi] \to L_2[-\sigma,\sigma] \quad \text{via} \quad F(x) \mapsto \left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}} F\left(\frac{\pi}{\sigma}x\right).
$$

Note that $J_{\pi\sigma}$ is a norm-one linear isometry, and hence that $\left(J_{\pi\sigma}\left(e^{-ix_j(\cdot)}\right)\right)_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\sigma,\sigma]$ with the same basis constant due to the following lemma.

Lemma 6.2.2. *Let X, Y be Hilbert spaces. If $T : X \to Y$ is a linear isometry with $\|T\| = 1$, and (ϕ_j) is a Riesz basis for X, then $(T\phi_j)$ is a Riesz basis for Y with the same basis constant.*

Proof. Let B be the Riesz basis constant of (ϕ_j) as in Theorem 4.0.2(3). Now every $x \in X$ can be written uniquely as $x = \sum_j a_j\phi_j$, and since T is surjective, if $y \in Y$, let $x \in X$ be such that $Tx = y$, then we have

$$y = Tx = T\left(\sum_j a_j\phi_j\right) = \sum_j a_j(T\phi_j),$$

and since this representation of x is unique, y is uniquely represented in this form by injectivity of T. And for the inequalities in Theorem 4.0.2(3), we have that

$$\frac{1}{B}\left(\sum_j |c_j|^2\right)^{\frac{1}{2}} \leq \left\|\sum_j c_j\phi_j\right\|_X = \left\|\sum_j c_jT(\phi_j)\right\|_Y \leq B\left(\sum_j |c_j|^2\right)^{\frac{1}{2}},$$

which proves the claim. $\square$

Taking $\sigma = \pi/h$, we see that $\left(h^{\frac{1}{2}}e^{-ihx_j(\cdot)}\right)_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$ with basis constant B. This is equivalent, via Theorem 4.0.2(7), to the fact that given any data sequence $(y_j) \in \ell_2$, there exists a unique bandlimited function $f \in PW_{\frac{\pi}{h}}$ such that $f(hx_j) = y_j$ for all $j \in \mathbb{Z}$. Consequently, Proposition 6.2.1 implies that if $g \in W_2^k$, then there exists a unique $F \in PW_{\frac{\pi}{h}}$ such that $F(hx_j) = g(hx_j),\ j \in \mathbb{Z}$.

We now turn toward the proof of Theorem 6.1.2. We have just shown the unique existence of a bandlimited interpolant satisfying (6.6), so it remains to prove (6.7) and (6.8). First, we demonstrate that (6.6) and (6.7) imply (6.8). Indeed, by (6.6), (6.7), and Theorem 6.1.8, we have that

$$\|g - F\|_{L_2} \le Ch^k |g - F|_{W_2^k} \le Ch^k \left(|g|_{W_2^k} + |F|_{W_2^k} \right) \le Ch^k |g|_{W_2^k},$$

which is (6.8).

To prove (6.7), we use the techniques of [32]. Define the sequence of *first forward divided differences* via the following formula:

$$g^{[1]}(hx_j) := \frac{g(hx_{j+1}) - g(hx_j)}{h(x_{j+1} - x_j)}, \quad j \in \mathbb{Z}, \tag{6.17}$$

and for $k \ge 2$, the k-th forward divided difference is defined recursively:

$$g^{[k]}(hx_j) := \frac{g^{[k-1]}(hx_{j+1}) - g^{[k-1]}(hx_j)}{h(x_{j+k} - x_j)} \quad j \in \mathbb{Z}. \tag{6.18}$$

The following pair of lemmas combine to show (6.7).

Lemma 6.2.3. *Let $h > 0$, $k \in \mathbb{N}$, and let X be a Riesz-basis sequence for $L_2[-\pi, \pi]$. Given $g \in W_2^k$, let $F \in PW_{\frac{\pi}{h}}$ be the unique bandlimited interpolant satisfying* (6.6)*. Then there exists a constant depending only on k and X such that*

$$|F|_{W_2^k} \le Ch^{\frac{1}{2}} \left\| \left(g^{[k]}(hx_j) \right)_j \right\|_{\ell_2}.$$

We relegate the proof of this lemma to Section 6.5 as it is somewhat technical.

Lemma 6.2.4. *Let $(x_j)_{j \in \mathbb{Z}}$ be a strictly increasing sequence such that $\inf_{j \in \mathbb{Z}} (x_{j+1} - x_j) =: q > 0$. If $h > 0$, $k \in \mathbb{N}$, and $g \in W_p^k(\mathbb{R})$, then*

$$\left\| \left(g^{[k]}(hx_j) \right) \right\|_{\ell_p} \le \frac{1}{(k-1)!} \left(\frac{1}{hq} \right)^{\frac{1}{p}} \|g^{(k)}\|_{L_p}. \tag{6.19}$$

Proof. We give the arguments for $k = 1$ and general k separately to exhibit better the idea behind the proof. If $k = 1$, then by Hölder's Inequality and the fact that $p/p' = p - 1$,

$$\sum_{j\in\mathbb{Z}} \left| \frac{g(hx_{j+1}) - g(hx_j)}{h(x_{j+1} - x_j)} \right|^p = \sum_{j\in\mathbb{Z}} |h(x_{j+1} - x_j)|^{-p} \left| \int_{hx_j}^{hx_{j+1}} g'(t)dt \right|^p$$

$$\leq \sum_{j\in\mathbb{Z}} |h(x_{j+1} - x_j)|^{-p} |h(x_{j+1} - x_j)|^{p-1} \int_{hx_j}^{hx_{j+1}} |g'(t)|^p dt$$

$$\leq \frac{1}{hq} \|g'\|_{L_p}^p.$$

Thus

$$\left\| \left(g^{[1]}(hx_j) \right) \right\|_{\ell_p} \leq \left(\frac{1}{hq} \right)^{\frac{1}{p}} \|g'\|_{L_p}.$$

If $k \geq 2$ we use the fact (see for example [8, Corollary 3.4.2]) that

$$g^{[k-1]}(hx_j) = \frac{g^{(k-1)}(\xi_j)}{(k-1)!}$$

for some $\xi_j \in [hx_j, hx_{j+k-1}]$.

Therefore,

$$(k-1)! g^{[k]}(hx_j) = (k-1)! \frac{g^{[k-1]}(hx_{j+1}) - g^{[k-1]}(hx_j)}{h(x_{j+k} - x_j)}$$

$$= \frac{g^{(k-1)}(\xi_{j+1}) - g^{(k-1)}(\xi_j)}{h(x_{j+k} - x_j)}$$

$$= \frac{1}{h(x_{j+k} - x_j)} \int_{\xi_j}^{\xi_{j+1}} g^{(k)}(t)dt.$$

Consequently,

$$\begin{aligned}
[(k-1)!]^p \, |g^{[k]}(hx_j)|^p &\leq |h(x_{j+k} - x_j)|^{-p} \left(\int_{\xi_j}^{\xi_{j+1}} |g^{(k)}(t)| dt \right)^p \\
&\leq |h(x_{j+k} - x_j)|^{-p} \left(\int_{hx_j}^{hx_{j+k}} |g^{(k)}(t)| dt \right)^p \\
&\leq |h(x_{j+k} - x_j)|^{-p+p-1} \int_{hx_j}^{hx_{j+k}} |g^{(k)}(t)|^p dt \, .
\end{aligned}$$

Thus,

$$\sum_{j\in\mathbb{Z}} |g^{[k]}(hx_j)|^p \leq \frac{1}{[(k-1)!]^p} (khq)^{-1} \sum_{j\in\mathbb{Z}} \int_{hx_j}^{hx_{j+k}} |g^{(k)}(t)|^p dt$$

$$= \frac{1}{[(k-1)!]^p} (khq)^{-1} k \|g^{(k)}\|_{L_p}^p,$$

where the k in the last term comes from the fact that the integral from hx_j to hx_{j+1} appears k times for each j. We conclude that

$$\left\| \left(g^{[k]}(hx_j) \right)_j \right\|_{\ell_p} \leq \frac{1}{(k-1)!} \left(\frac{1}{hq} \right)^{\frac{1}{p}} \|g^{(k)}\|_{L_p}.$$

$\square$

We conclude with the proof of Theorem 6.1.2 which combines the above results.

Proof of Theorem 6.1.2. Given $g \in W_2^k$, let $F \in PW_{\frac{\pi}{h}}$ be the unique function satisfying (6.6). Then by Lemmas 6.2.3 and 6.2.4, there exists a constant C depending only on k and X such that

$$|F|_{W_2^k} \leq Ch^{\frac{1}{2}} \left\| \left(F^{[k]}(hx_j)\right)_j \right\|_{\ell_2} = Ch^{\frac{1}{2}} \left\| \left(g^{[k]}(hx_j)\right)_j \right\|_{\ell_2} \leq C|g|_{W_2^k},$$

which is (6.7). As commented above, (6.6) and (6.7) imply (6.8), whence Theorem 6.1.2 follows. □

6.3 Stability of Interpolants

This section turns to the proof of Theorem 6.1.3, which will be done in a series of steps reminiscent of the proofs in [40]. The first issue that bears discussing is how Gaussian interpolation of W_2^k functions is even possible. It was shown in [40] that the Gaussian matrix associated with $\lambda > 0$ and the Riesz-basis sequence X,

$$G := G_{\lambda,X} := \left(e^{-\lambda(x_i - x_j)^2}\right)_{i,j\in\mathbb{Z}},$$

is an invertible operator on ℓ_p for every $1 \leq p \leq \infty$. Consequently, for a given $g \in W_2^k$, Proposition 6.2.1 implies the existence of a Gaussian interpolant of the form (6.1), where the sequence (a_j) is given by $a = G^{-1}y$ with $y_j = g(x_j), j \in \mathbb{Z}$. Moreover, invertibility of G provides uniqueness of the Gaussian interpolant for a given data sequence, so if $F \in PW_\pi$ is the function from Theorem 6.1.2 that interpolates g at X, then $\mathscr{I}_\lambda(g) = \mathscr{I}_\lambda(F)$.

Now assume $h \in L_2[-\pi, \pi]$. Then by definition of a Riesz-basis sequence, there exists a unique sequence $(a_j)_{j\in\mathbb{Z}} \in \ell_2$ such that $h(t) = \sum_{j\in\mathbb{Z}} a_j e^{-ix_j t}, t \in [-\pi, \pi]$. Let H be the extension of h to all of $\mathbb{R}$ given by $H(u) := \sum_{j\in\mathbb{Z}} a_j e^{-ix_j u}$, $u \in \mathbb{R}$. On account of the Riesz basis condition, H is locally square-integrable, and hence well-defined almost everywhere. Define the shift operator A_ℓ for $\ell \in \mathbb{Z}$ by

$$A_\ell(h)(t) := H(t + 2\pi\ell), \quad t \in [-\pi, \pi]. \tag{6.20}$$

From the Riesz basis constant inequality in Theorem 4.0.2(3), it follows that $(A_\ell)_{\ell\in\mathbb{Z}}$ is a uniformly bounded set of operators on $L_2[-\pi, \pi]$ with norm at most B^2. Let A_ℓ^* denote the adjoint of A_ℓ. The following is a straightforward adaptation of [40, Theorem 3.3].

Theorem 6.3.1. *Let X be a Riesz-basis sequence, let $\lambda > 0$ be fixed, and let $f \in PW_\pi$. Let ψ_λ denote the restriction, to the interval $[-\pi, \pi]$, of the function*

$$\Psi_\lambda(\xi) := \sqrt{\frac{\lambda}{\pi}} e^{\frac{\xi^2}{4\lambda}} \mathscr{F}[\mathscr{I}_\lambda(f)](\xi).$$

Then

$$\mathscr{F}[f] = \mathscr{F}[\mathscr{I}_\lambda(f)] + \sqrt{\frac{\pi}{\lambda}} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} A_\ell^* \left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}} A_\ell(\psi_\lambda) \right) \quad \text{on } [-\pi, \pi]. \tag{6.21}$$

Consequently, the equation

$$(i\xi)^k \mathscr{F}[f](\xi) = (i\xi)^k \mathscr{F}[\mathscr{I}_\lambda(f)](\xi) + (i\xi)^k \sqrt{\frac{\pi}{\lambda}} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} A_\ell^* \left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}} A_\ell(\psi_\lambda) \right)(\xi) \tag{6.22}$$

holds for every $\xi \in [-\pi, \pi]$.

Lemma 6.3.2. *The following holds for every $\ell \in \mathbb{Z}$ and every $\xi \in [-\pi, \pi]$:*

$$(i\xi)^k A_\ell^* \left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}} A_\ell(\psi_\lambda) \right)(\xi) = A_\ell^* \left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}} A_\ell\left((i\cdot)^k \psi_\lambda\right) \right)(\xi). \tag{6.23}$$

Proof. It suffices to show equality of the inner products of the above expressions with the Riesz basis elements $e_j := e^{-ix_j(\cdot)}$. Here and elsewhere, $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L_2[-\pi,\pi]$.

$$\begin{aligned}
\left\langle (i\cdot)^k A_\ell^*\left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}}A_\ell(\psi_\lambda)\right), e_j\right\rangle &= \left\langle A_\ell^*\left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}}A_\ell(\psi_\lambda)\right), (-i\cdot)^k e_j\right\rangle \\
&= \left\langle e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}}A_\ell(\psi_\lambda), A_l\left((-i\cdot)^k e_j\right)\right\rangle \\
&= \int_{-\pi}^{\pi} e^{-\frac{(\xi+2\pi\ell)^2}{4\lambda}}\psi_\lambda(\xi+2\pi\ell)(i(\xi+2\pi\ell))^k \\
&\quad \times e^{ix_j(\xi+2\pi\ell)}d\xi \\
&= \left\langle e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}}A_\ell\left((i\cdot)^k\psi_\lambda\right), A_\ell(e_j)\right\rangle \\
&= \left\langle A_\ell^*\left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}}A_\ell\left((i\cdot)^k\psi_\lambda\right)\right), e_j\right\rangle.
\end{aligned}$$

□

Lemma 6.3.2 and (6.22) imply the following:

Corollary 6.3.3. *If $\xi\in[-\pi,\pi]$, then*

$$(i\xi)^k\mathscr{F}[f](\xi) = (i\xi)^k\mathscr{F}[\mathscr{I}_\lambda(f)](\xi) + \sqrt{\frac{\pi}{\lambda}}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} A_\ell^*\left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}}A_\ell\left((i\cdot)^k\psi_\lambda\right)\right)(\xi). \tag{6.24}$$

Lemma 6.3.4 (cf. [40], Corollary 3.4)**.** *Suppose that λ, f, ψ_λ, and B are as defined above. Then*

$$\left\|(i\cdot)^k\mathscr{F}\left[\mathscr{I}_\lambda(f)\right]\right\|_{L_2[-\pi,\pi]} \le \sqrt{2\pi}|f|_{W_2^k} + \sqrt{\frac{\pi}{\lambda}}B^4\frac{2e^{-\frac{\pi^2}{4\lambda}}}{1-e^{-\frac{\pi^2}{4\lambda}}}\|(i\cdot)^k\psi_\lambda\|_{L_2[-\pi,\pi]}. \tag{6.25}$$

Proof. Note that (6.24) yields the appropriate bound as long as we can show that

$$\left\| \sum_{\ell \in \mathbb{Z} \setminus \{0\}} A_\ell^* \left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}} A_\ell \left((i\cdot)^k \psi_\lambda \right) \right) \right\|_{L_2[-\pi,\pi]} \leq B^4 \frac{2e^{-\frac{\pi^2}{4\lambda}}}{1-e^{-\frac{\pi^2}{4\lambda}}} \|(i\cdot)^k \psi_\lambda\|_{L_2[-\pi,\pi]}.$$

This is true because the desired term is bounded above by

$$B^2 \sum_{\ell \in \mathbb{Z} \setminus \{0\}} e^{-\frac{(2|\ell|-1)^2\pi^2}{4\lambda}} \|A_\ell \left((i\cdot)^k \psi_\lambda \right) \|_{L_2[-\pi,\pi]},$$

via the triangle inequality and the facts that $\|A_\ell^*\| \leq B^2$ for all ℓ, and that $e^{-\frac{(\xi+2\pi\ell)^2}{4\lambda}} \leq e^{-\frac{(2|\ell|-1)^2\pi^2}{4\lambda}}$ for $\xi \in [-\pi, \pi]$. Similarly, since $\|A_\ell\| \leq B^2$ for all $\ell \in \mathbb{Z}$ and $e^{-\frac{(2|\ell|-1)^2\pi^2}{4\lambda}} \leq e^{-\frac{(2|\ell|-1)\pi^2}{4\lambda}}$, the above term is majorized by

$$B^4 \sum_{\ell \in \mathbb{Z} \setminus \{0\}} e^{-\frac{(2|\ell|-1)\pi^2}{4\lambda}} \|(i\cdot)^k \psi_\lambda\|_{L_2[-\pi,\pi]} \leq B^4 \frac{2e^{-\frac{\pi^2}{4\lambda}}}{1-e^{-\frac{\pi^2}{4\lambda}}} \|(i\cdot)^k \psi_\lambda\|_{L_2[-\pi,\pi]}.$$

The fraction comes from summing the geometric series. □

This brings us to our final proposition before the proof of the main theorem in this section:

Proposition 6.3.5. *Suppose that λ, f, ψ_λ, and B are as defined above. Then*

$$\|(i\cdot)^k \psi_\lambda\|_{L_2[-\pi,\pi]} \leq \sqrt{2\lambda} e^{\frac{\pi^2}{4\lambda}} |f|_{W_2^k}.$$

Proof. We begin by taking the inner product of both sides of (6.24) with $(i\cdot)^k \psi_\lambda$, and noticing that both terms on the right hand side are non-negative. Indeed, it is easily checked that the operator

$$T_\lambda(h) := \sum_{\ell\in\mathbb{Z}\setminus\{0\}} A_\ell^* \left(e^{-\frac{(\cdot+2\pi\ell)^2}{4\lambda}} A_\ell(h) \right)$$

is positive in the sense that $\langle T_\lambda(h), h\rangle \geq 0$ for all $h \in L_2[-\pi,\pi]$. Moreover,

$$\left\langle (i\cdot)^k \mathscr{F}[\mathscr{I}_\lambda(f)], (i\cdot)^k \psi_\lambda \right\rangle = \sqrt{\frac{\pi}{\lambda}} \int_{-\pi}^{\pi} |\xi|^{2k} e^{-\frac{\xi^2}{4\lambda}} |\psi_\lambda(\xi)|^2 d\xi \ \geq \ 0$$

by the definition of ψ_λ in Theorem 6.3.1. Therefore, from (6.24), we have that

$$\left\langle (i\cdot)^k \mathscr{F}[\mathscr{I}_\lambda(f)], (i\cdot)^k \psi_\lambda \right\rangle \leq \left\langle (i\cdot)^k \mathscr{F}[f], (i\cdot)^k \psi_\lambda \right\rangle.$$

Finally,

$$\begin{aligned}
\sqrt{\frac{\pi}{\lambda}} e^{-\frac{\pi^2}{4\lambda}} \|(i\cdot)^k \psi_\lambda\|^2_{L_2[-\pi,\pi]} &= \sqrt{\frac{\pi}{\lambda}} e^{-\frac{\pi^2}{4\lambda}} \int_{-\pi}^{\pi} |(i\xi)^k \psi_\lambda(\xi)|^2 d\xi \\
&\leq \int_{-\pi}^{\pi} \left[\sqrt{\frac{\pi}{\lambda}} e^{-\frac{\xi^2}{4\lambda}} (i\xi)^k \psi_\lambda(\xi) \right] \overline{(i\xi)^k \psi_\lambda(\xi)} d\xi \\
&= \left\langle (i\cdot)^k \mathscr{F}[\mathscr{I}_\lambda(f)], (i\cdot)^k \psi_\lambda \right\rangle \\
&\leq \left\langle (i\cdot)^k \mathscr{F}[f], (i\cdot)^k \psi_\lambda \right\rangle \\
&\leq \|(i\cdot)^k \mathscr{F}[f]\|_{L_2[-\pi,\pi]} \|(i\cdot)^k \psi_\lambda\|_{L_2[-\pi,\pi]},
\end{aligned}$$

where the last step is a consequence of the Cauchy-Schwarz inequality. The required result follows, taking into account (2.5). □

Before finishing the proof of Theorem 6.1.3, we note that in light of Theorem

6.1.2, we need only prove the upper bound for functions $f \in PW_\pi$. That is, if we show that for all $f \in PW_\pi$, $|\mathscr{I}_\lambda(f)|_{W_2^k} \leq C|f|_{W_2^k}$, then if $F \in PW_\pi$ is the function given by Theorem 6.1.2 for a certain $g \in W_2^k$, we have

$$|\mathscr{I}_\lambda(g)|_{W_2^k} = |\mathscr{I}_\lambda(F)|_{W_2^k} \leq C|F|_{W_2^k} \leq C|g|_{W_2^k},$$

which is the conclusion of Theorem 6.1.3.

Proof of Theorem 6.1.3. In light of (2.5), we need to estimate both $\left\|(i\cdot)^k \mathscr{F}\left[\mathscr{I}_\lambda(f)\right]\right\|_{L_2[-\pi,\pi]}$ and $\left\|(i\cdot)^k \mathscr{F}\left[\mathscr{I}_\lambda(f)\right]\right\|_{L_2(\mathbb{R}\setminus[-\pi,\pi])}$.

By Lemma 6.3.4 and Proposition 6.3.5,

$$\begin{aligned}
\left\|(i\cdot)^k \mathscr{F}\left[\mathscr{I}_\lambda(f)\right]\right\|_{L_2[-\pi,\pi]} &\leq \sqrt{2\pi}|f|_{W_2^k} + \sqrt{\frac{\pi}{\lambda}} B^4 \frac{2e^{-\frac{\pi^2}{4\lambda}}}{1-e^{-\frac{\pi^2}{4\lambda}}} \|(i\cdot)^k \psi_\lambda\|_{L_2[-\pi,\pi]} \\
&\leq \sqrt{2\pi}(1+4B^4)|f|_{W_2^k}.
\end{aligned}$$

In the second inequality, we used the fact that $2/(1-e^{-\frac{\pi^2}{4\lambda}}) \leq 4$.

On $\mathbb{R}\setminus[-\pi,\pi]$, we use a periodization argument:

$$\begin{aligned}
\left\|(i\cdot)^k \mathscr{F}\left[\mathscr{I}_\lambda(f)\right]\right\|^2_{L_2(\mathbb{R}\setminus[-\pi,\pi])} &= \frac{\pi}{\lambda} \int_{\mathbb{R}\setminus[-\pi,\pi]} e^{-\frac{\xi^2}{2\lambda}} |(i\xi)^k \Psi_\lambda(\xi)|^2 d\xi \\
&= \frac{\pi}{\lambda} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \int_{(2\ell-1)\pi}^{(2\ell+1)\pi} e^{-\frac{\xi^2}{2\lambda}} |(i\xi)^k \Psi_\lambda(\xi)|^2 d\xi \\
&= \frac{\pi}{\lambda} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \int_{-\pi}^{\pi} e^{-\frac{(t+2\pi\ell)^2}{2\lambda}} |A_\ell((i\cdot)^k \psi_\lambda)(t)|^2 dt
\end{aligned}$$

$$\leq \frac{\pi}{\lambda} \sum_{\ell \in \mathbb{Z}\setminus\{0\}} e^{-\frac{(2|\ell|-1)^2\pi^2}{2\lambda}} \|A_\ell((i\cdot)^k \psi_\lambda)\|^2_{L_2[-\pi,\pi]}$$

$$\leq \frac{B^4\pi}{\lambda} \frac{2e^{-\frac{\pi^2}{2\lambda}}}{1-e^{-\frac{\pi^2}{2\lambda}}} \|(i\cdot)^k \psi_\lambda\|^2_{L_2[-\pi,\pi]}$$

$$\leq 8\pi B^4 |f|^2_{W_2^k}.$$

Consequently, we have that

$$|\mathscr{I}_\lambda(f)|_{W_2^k(\mathbb{R})} \leq \sqrt{2\pi(1+14B^4+16B^8)}\, |f|_{W_2^k(\mathbb{R})}.$$

Uniform boundedness comes from the above inequality as well as the fact that $\|\mathscr{I}_\lambda(f)\|_{L_2} \leq C\|f\|_{L_2}$ for $f \in PW_\pi$ (use the same method of proof taking $k=0$). □

6.4 Proofs of Corollaries 6.1.4 and 6.1.6

We begin with the proof of the Corollary 6.1.4. Arguing along the same lines as in Theorem 6.1.3, it suffices to show the estimate for functions $F \in PW_{\frac{\pi}{h}}$. To do so, we must explore the relationship between the seminorms of the two interpolants, and that between bandlimited functions F and F^h.

Firstly,

$$|F^h|_{W_2^k} = h^{k+\frac{1}{2}} |F|_{W_2^k}, \tag{6.26}$$

because

$$|F^h|^2_{W_2^k} = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} |\xi|^{2k} |\mathscr{F}\left[F^h\right](\xi)|^2 d\xi$$

$$= \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} |\xi|^{2k} |\mathscr{F}[F](\xi/h)|^2 d\xi$$

$$= \frac{1}{(2\pi)^d} h \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} h^{2k} |u|^{2k} |\mathscr{F}[F](u)|^2 du$$

$$= h^{2k+1} |F|^2_{W_2^k}.$$

Now we consider the seminorms of the interpolants. By (6.3) and the identity $\mathscr{F}[g(\frac{\cdot}{h})](\xi) = h\mathscr{F}[g](h\xi)$,

$$\mathscr{F}\left[I^{hX}(F)(\cdot)\right](\xi) = \frac{1}{h}\mathscr{F}\left[\mathscr{I}^X_{h^2}\left(F^h\right)\left(\frac{\cdot}{h}\right)\right](\xi) = \mathscr{F}\left[\mathscr{I}^X_{h^2}\left(F^h\right)(\cdot)\right](h\xi).$$

Putting these together, we have the following relation between the seminorms of the interpolants:

$$\left|I^{hX}(F)\right|_{W_2^k} = \frac{1}{h^{k+\frac{1}{2}}} \left|\mathscr{I}^X_{h^2}\left(F^h\right)\right|_{W_2^k}, \tag{6.27}$$

which is seen as follows.

$$\begin{aligned}
\left|I^{hX}(F)\right|_{W_2^k} &= \frac{1}{\sqrt{2\pi}} \left\|(i\cdot)^k \mathscr{F}\left[\mathscr{I}^X_{h^2}\left(F^h\right)\right](h\cdot)\right\|_{L_2} \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\xi|^{2k} \left|\mathscr{F}\left[\mathscr{I}^X_{h^2}\left(F^h\right)\right](h\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{h}\int_{\mathbb{R}} \left|\frac{u}{h}\right|^{2k} \left|\mathscr{F}\left[\mathscr{I}^X_{h^2}\left(F^h\right)\right](u)\right|^2 du\right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{h^{k+\frac{1}{2}}} \left\|(i\cdot)^k \mathscr{F}\left[\mathscr{I}^X_{h^2}\left(F^h\right)\right]\right\|_{L_2} \\
&= \frac{1}{h^{k+\frac{1}{2}}} \left|\mathscr{I}^X_{h^2}\left(F^h\right)\right|_{W_2^k}.
\end{aligned}$$

Proof of Corollary 6.1.4. Let $F \in PW_{\frac{\pi}{h}}$ be the function given by Theorem 6.1.2. Then $I^{hX}(g) = I^{hX}(F)$, and by (6.26), (6.27) and Theorem 6.1.3, we see that

$$|I^{hX}F|_{W_2^k} = \frac{1}{h^{k+\frac{1}{2}}}|\mathscr{I}^X_{h^2}(F^h)|_{W_2^k} \leq \frac{C}{h^{k+\frac{1}{2}}}|F^h|_{W_2^k} \leq \frac{Ch^{k+\frac{1}{2}}}{h^{k+\frac{1}{2}}}|F|_{W_2^k} = C|F|_{W_2^k},$$

which, on account of (6.7), gives (6.10). □

Proof of Corollary 6.1.6. Let $j < k$, $0 < h \leq 1$, and let F be the function given by Theorem 6.1.2. Then, as in the proof of Theorem 6.1.5, we see via Theorem 6.1.8 that

$$\begin{aligned}|I^{hX}(g) - g|_{W_2^j} &\leq |I^{hX}(F) - F|_{W_2^j} + |F - g|_{W_2^j}\\ &\leq Ch^{k-j}|I^{hX}(F) - F|_{W_2^k} + Ch^{k-j}|F - g|_{W_2^k} =: I_1 + I_2\ .\end{aligned}$$

By Corollary 6.1.4 and (6.7), we have

$$I_1 \leq Ch^{k-j}|F|_{W_2^k} \leq Ch^{k-j}|g|_{W_2^k},$$

and

$$I_2 \leq Ch^{k-j}|g|_{W_2^k},$$

from which the corollary follows. We note that this does give a convergence result under the operative assumption $k > j$. □

6.5 Proof of Lemma 6.2.3

First, recall that if X is a Riesz-basis sequence for $L_2[-\sigma, \sigma]$, then for any sequence $y \in \ell_2$, there exists a unique function $f \in PW_\sigma$ such that $f(x_j) = y_j, j \in \mathbb{Z}$. Similar to (6.17), we define the *k-th forward divided difference* of a data sequence y by identifying y_j as a function on the data sites X; that is, consider $y_j = y(x_j)$. Then we find from [32, Theorem 2], that if y is a sequence such that $y^{[k]} \in \ell_2$, then there exists a unique function $f \in PW_\sigma^k$ such that $f(x_j) = y_j, j \in \mathbb{Z}$. Moreover, there is a constant, C, such that $\|f^{(k)}\|_{L_2} \leq C\|y^{[k]}\|_{\ell_2}$. But as we have been considering increasing values of σ, namely π/h where $h \to 0^+$, the dependence of this constant on h must be recorded carefully. We will come to the final conclusion that the constant is of order $\sqrt{h}$.

We make the preliminary observation that if $y \in \ell_2$, then the function $F \in PW_\sigma$ and the function $G \in PW_\sigma^k$ that satisfy the interpolation conditions are in fact the same (note that $y \in \ell_2$ implies that $y^{[k]} \in \ell_2$). This conclusion follows from the simple fact that if $F \in PW_\sigma$, then $F \in PW_\sigma^k$ since the Paley-Wiener space is closed under differentiation, and since $F - G|_X = 0$ where X is a Riesz-basis sequence for $L_2[-\sigma, \sigma]$, $F = G$.

Therefore, it suffices simply to determine the constant, C, such that $\|f^{(k)}\|_{L_2} \leq C\|y^{[k]}\|_{\ell_2}$. We do this via some intermediate steps involving spline interpolants. Our proofs here rely heavily on the work of Madych in [32]. Essentially, we track the constants through modified proofs of [32, Theorems 1,2]. For the sake of self-containment, we present the proofs here with the necessary modifications.

Theorem 6.5.1 (cf. [32], Theorem 1)**.** *Suppose X is a Riesz-basis sequence for $L_2[-\pi, \pi]$ and $k \in \mathbb{N}$. Then for every sequence y such that $y^{[k]} \in \ell_2$, and $0 < h \leq 1$, there is a function $F \in PW_{\pi/2h}^k$ such that $(y_j - F(hx_j))_j \in \ell_2$ and $|F|_{W_2^k} \leq Ch^{\frac{1}{2}}\|y^{[k]}\|_{\ell_2}$. Here, the constant C depends on k and the Riesz-basis sequence X. Moreover,*

$$\left\| (y_j - F(hx_j))_j \right\|_{\ell_2} \leq Ch^k \|y^{[k]}\|_{\ell_2}.$$

Proof. Let q and Q be as in (6.16), and B the Riesz basis constant of X. By work of Golomb [13], such a data sequence y has a unique minimal piecewise polynomial spline extension (or interpolant), s, of order $2k$ such that $s(hx_j) = y_j$; further, s is a polynomial of degree $2k-1$ on any interval of the form $[hx_m, hx_{m+1}]$. Then by a lovely estimate of de Boor [9], we have the following bound:

$$\|s^{(k)}\|_{L_2} \leq k!k^{1-\frac{1}{2}}k(2k+1)(2k-1)^{k-1} \left\| \left(\left(\frac{h(x_{j+k}-x_j)}{k} \right)^{\frac{1}{2}} y_j^{[k]} \right)_j \right\|_{\ell_2}. \tag{6.28}$$

Thus we see that the right hand side is at most

$$C_k Q^{\frac{1}{2}} h^{\frac{1}{2}} \|y^{[k]}\|_{\ell_2},$$

where

$$C_k = k!(k)(2k+1)(2k-1)^{k-1}.$$

Next we construct $F \in PW_\sigma^k$ via $\widehat{F}(\xi) = \widehat{s}(\xi)\widehat{\phi}(\xi/\sigma)$ (we will specify σ later), where $\widehat{\phi}$ is an infinitely differentiable function with support in $[-1,1]$, with $\widehat{\phi}(\xi) = 1$ for ξ in some neighborhood of the origin, and s is the spline interpolant discussed above. By definition, it is evident that $\widehat{F}$ has support in $[-\sigma, \sigma]$. Moreover, it satisfies the bound

$$\|F^{(k)}\|_{L_2} \leq \|\phi\|_{L_1(\mathbb{R})} \|s^{(k)}\|_{L_2}, \tag{6.29}$$

which by (6.28) is at most

$$C_k Q^{\frac{1}{2}} h^{\frac{1}{2}} \|\phi\|_{L_1(\mathbb{R})} \|y^{[k]}\|_{\ell_2} =: C_{k,Q} h^{\frac{1}{2}} \|y^{[k]}\|_{\ell_2}. \tag{6.30}$$

As a consequence of (6.29), (6.30), and the Paley-Wiener Theorem, $F^{(k)} \in PW_\sigma$, and so $F \in PW_\sigma^k$.

Now let $\widehat{\phi_\sigma}(\xi) := \widehat{\phi}(\xi/\sigma)$. We observe that

$$\widehat{s}(\xi) - \widehat{s}(\xi)\widehat{\phi_\sigma}(\xi) = (i\xi)^k \widehat{s}(\xi) \frac{1 - \widehat{\phi_\sigma}(\xi)}{(i\xi)^k} =: (i\xi)^k \widehat{s}(\xi)\widehat{I_{\sigma,k}}(\xi).$$

Thus,

$$s(hx_j) - F(hx_j) = s^{(k)} * I_{\sigma,k}(hx_j) = \int_{\mathbb{R}} s^{(k)}(hx_j - x) I_{\sigma,k}(x) dx. \tag{6.31}$$

Using Minkowski's Integral Inequality, (6.31), and the fact that $y_j = s(hx_j)$, we find that

$$\begin{aligned}
\left\| (y_j - F(hx_j))_j \right\|_{\ell_2} &= \left(\sum_{j\in\mathbb{Z}} \left| \int_{\mathbb{R}} s^{(k)}(hx_j - x) I_{\sigma,k}(x) dx \right|^2 \right)^{\frac{1}{2}} \\
&\leq \int_{\mathbb{R}} \left(\sum_{j\in\mathbb{Z}} \left| s^{(k)}(hx_j - x) \right|^2 |I_{\sigma,k}(x)|^2 \right)^{\frac{1}{2}} dx \\
&= \int_{\mathbb{R}} \|s^{(k)}(hx_j - x)\|_{\ell_2} |I_{\sigma,k}(x)| dx.
\end{aligned}$$

Our next step is to estimate $\|s^{(k)}(hx_j - x)\|_{\ell_2}$. We will see that

$$\|s^{(k)}(hx_j - x)\|_{\ell_2} \leq \left(\frac{ND_{k-1}}{hq} \right)^{\frac{1}{2}} \|s^{(k)}\|_{L_2},$$

where $N := \lceil \frac{Q}{q} \rceil$. To show this, we require the following:

Lemma 6.5.2 (cf. [32], Lemma 2). *Let $\mathcal{P}_m$ denote the class of algebraic polynomials of degree at most m. If P is a polynomial in $\mathcal{P}_m$ which is not identically zero, then* *for any* $-\infty < a < b < \infty$,

$$1 \leq \frac{\max\limits_{a \leq x \leq b} |P(x)|}{\left(\frac{1}{b-a}\int_a^b |P(x)|^2 dx\right)^{\frac{1}{2}}} \leq D_m,$$

where D_m *is a finite constant which depends on* m *but is independent of* a *and* b.

To apply this to our situation, simply notice that if $hx_j - x \in [hx_m, hx_{m+1}]$ for some m, then $s^{(k)}$ is a polynomial of degree at most $k-1$ on this interval, and consequently

$$|s^{(k)}(hx_j - x)|^2 \leq \frac{D_{k-1}}{h(x_{m+1} - x_m)} \int_{hx_m}^{hx_{m+1}} |s^{(k)}(y)|^2 dy,$$

where D_{k-1} is the constant from Lemma 6.5.2. Since $hq \leq h(x_{m+1} - x_m) \leq hQ$ for all m, there are at most $N = \lceil \frac{Q}{q} \rceil$ terms of the form $hx_j - x$ in any given interval $[hx_m, hx_{m+1}]$. Consequently, we have that for any $x \in \mathbb{R}$,

$$\sum_{j \in \mathbb{Z}} |s^{(k)}(hx_j - x)|^2 = \sum_{m \in \mathbb{Z}} \left(\sum_{hx_j - x \in [hx_m, hx_{m+1}]} |s^{(k)}(hx_j - x)|^2 \right) \leq \frac{ND_{k-1}}{hq} \|s^{(k)}\|_{L_2}^2,$$

which is what we set out to show.

Finally, noting that $\|I_{\sigma,k}\|_{L_1} = \sigma^{-k}\|I_{1,k}\|_{L_1}$, and taking $\sigma = \pi/(2h)$, we obtain

$$\left\| (y_j - F(hx_j))_j \right\|_{\ell_2} \leq C_{k,q,Q} h^k \|y^{[k]}\|_{\ell_2},$$

where

$$C_{k,q,Q} = C_{k,Q} \left(\frac{ND_{k-1}}{q} \right)^{\frac{1}{2}} \frac{2^k}{\pi^k} \|I_{1,k}\|_{L_1}.$$

□

Proof of Lemma 6.2.3. Let $g \in W_2^k$, and let $y_j := g(hx_j)$. Let F_0 be the $PW^k_{\pi/2h}$ function given by Theorem 6.5.1. Since $(y_j - F_0(hx_j))_j \in \ell_2$, and hX is a Riesz-basis sequence for $L_2\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$, there exists a unique $F_1 \in PW_{\frac{\pi}{h}}$ such that $F_1(hx_j) = y_j - F_0(hx_j), j \in \mathbb{Z}$. Moreover, this function satisfies

$$\|F_1\|_{L_2} \leq \sqrt{2\pi} B h^{\frac{1}{2}} \left\| (y_j - F_0(hx_j))_j \right\|_{\ell_2}.$$

Recall from the discussion in Section 6.2 that $\left(h^{\frac{1}{2}} e^{-ihx_j(\cdot)}\right)_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$ with basis constant B, the basis constant for $\left(e^{-ix_j(\cdot)}\right)_{j\in\mathbb{Z}}$. Therefore,

$$\|F_1\|_{L_2} \leq B \left(\sum_{j\in\mathbb{Z}} \left| \left\langle \widehat{F_1}, h^{\frac{1}{2}} e^{-ihx_j(\cdot)} \right\rangle_{L_2\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]} \right|^2 \right)^{\frac{1}{2}} = \sqrt{2\pi} B h^{\frac{1}{2}} \left\| (F_1(hx_j))_j \right\|_{\ell_2},$$

the last step coming from the inversion formula.

Define $F := F_0 + F_1$. Then by construction, $F(hx_j) = y_j = g(hx_j), j \in \mathbb{Z}$. Moreover, we have that

$$\|F_1\|_{L_2} \leq \sqrt{2\pi} B h^{\frac{1}{2}} \left\| (F_1(hx_j))_j \right\|_{\ell_2} = \sqrt{2\pi} B h^{\frac{1}{2}} \left\| (y_j - F_0(hx_j))_j \right\|_{\ell_2}$$

$$\leq C_{k,q,Q} \sqrt{2\pi} B h^{k+\frac{1}{2}} \|y^{[k]}\|_{\ell_2}.$$

Recalling that for $F \in PW_\sigma$, the relation $|F|_{W_2^k} \leq \sigma^k \|F\|_{L_2}$ holds, we see that

$$|F_1|_{W_2^k} \leq C_{k,q,Q,B} h^{\frac{1}{2}} \|y^{[k]}\|_{\ell_2}. \tag{6.32}$$

Therefore (6.30) and (6.32) lead to the conclusion that

$$|F|_{W_2^k} \leq C_{k,q,Q,B} h^k \|y^{[k]}\|_{\ell_2} + C_{k,Q} h^{\frac{1}{2}} \|y^{[k]}\|_{\ell_2} \leq C h^{\frac{1}{2}} \|y^{[k]}\|_{\ell_2},$$

where C depends on k, q, Q, and B (we assumed $h \leq 1$, so we note that $h^{\frac{1}{2}}$ is the biggest term involving h). This concludes the proof. $\square$

6.6 Interpolation by Means of Regular Interpolators

Here we extend the discussion of approximation and convergence rates of W_2^k functions, but rather than limiting ourselves to the case of the Gaussian interpolation operator, we consider families of *regular interpolators*, a notion developed by Ledford [26].

A function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be an *interpolator* for PW_π if the following hold:

(A1) $\phi \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ and $\widehat{\phi} \in L_1(\mathbb{R})$.

(A2) $\widehat{\phi}(\xi) \geq 0, \xi \in \mathbb{R}$, and $\widehat{\phi}(\xi) > 0$ on $[-\pi, \pi]$.

(A3) If $M_j := \sup\limits_{|\xi|\leq\pi} \widehat{\phi}(\xi + 2\pi j)$, then $(M_j)_{j\in\mathbb{Z}} \in \ell_1$.

Next, a family of interpolators $(\phi_\alpha)_{\alpha\in A}$ indexed by an unbounded set $A \subset (0, \infty)$ is called *regular* if

(R1) ϕ_α is an interpolator for PW_π for every $\alpha \in A$.

(R2) Let $M_j(\alpha) := \sup\limits_{|\xi|\leq\pi} \widehat{\phi_\alpha}(\xi + 2\pi j)$, and $m_\alpha := \inf\limits_{|\xi|\leq\pi} \widehat{\phi_\alpha}(\xi)$. Then there exists a constant C independent of α such that for all $\alpha \in A$, $\sum\limits_{j\neq 0} M_j(\alpha) \leq C m_\alpha$.

(R3) For almost every $\xi \in [-\pi, \pi]$, $\lim\limits_{\alpha\to\infty} \dfrac{m_\alpha}{\widehat{\phi_\alpha}(\xi)} = 0$.

Here, α plays the role of $1/h$ in our previous discussions.

It was shown ([26, Corollary 1]) that if X is a Riesz-basis sequence for $L_2[-\pi, \pi]$ and $f \in PW_\pi$, then there is a unique sequence $(a_j)_{j\in\mathbb{Z}} \in \ell_2$ such that the interpolant

$$\mathscr{I}_\phi(f)(x) := \sum_{j\in\mathbb{Z}} a_j \phi(x - x_j) \tag{6.33}$$

is continuous and satisfies $\mathscr{I}_\phi(f)(x_j) = f(x_j),\ j \in \mathbb{Z}$.

The main result of that paper is the following:

Theorem 6.6.1 (cf. [26], Theorems 1 and 2)**.** *If $(\phi_\alpha)_{\alpha\in A}$ is a set of regular interpolators and X a Riesz-basis sequence for $L_2[-\pi,\pi]$, then for every $f \in PW_\pi$,*

$$\lim_{\alpha\to\infty} \mathscr{I}_{\phi_\alpha}(f) = f,$$

both in $L_2(\mathbb{R})$ and uniformly on $\mathbb{R}$.

6.6.1 Results

Given a sequence of regular interpolators, and their associated interpolation operators $\mathscr{I}_\alpha : PW_\pi \to L_2$ (where to ease notation we identify $\mathscr{I}_\alpha = \mathscr{I}_{\phi_\alpha}$), define another set of interpolation operators via

$$I^\alpha(f)(x) := \alpha \mathscr{I}_{\alpha^2}\left(f^{\frac{1}{\alpha}}\right)(\alpha x), \quad \text{where } f^{\frac{1}{\alpha}}(x) := \frac{1}{\alpha} f\left(\frac{x}{\alpha}\right).$$

It is easily verified that if $f \in PW_{\alpha\pi}$, then $f^{\frac{1}{\alpha}} \in PW_\pi$ and $I^\alpha(f)\left(\frac{x_j}{\alpha}\right) = f\left(\frac{x_j}{\alpha}\right),\ j \in \mathbb{Z}$. Indeed

$$I^\alpha(f)\left(\frac{x_j}{\alpha}\right) = \alpha \mathscr{I}_{\alpha^2}\left(f^{\frac{1}{\alpha}}\right)\left(\frac{\alpha x_j}{\alpha}\right) = \alpha \frac{1}{\alpha} f\left(\frac{x_j}{\alpha}\right).$$

Again, the idea here is that to interpolate Sobolev functions $g \in W_2^k$, we first interpolate them by bandlimited functions of increasing band (namely by $F \in PW_{\alpha\pi}$), where the interpolation is done at the shrinking set of points $\left(\frac{x_j}{\alpha}\right)$. We summarize the analogous results to those found in Section 6.1.

Theorem 6.6.2. *Let $(\phi_\alpha)_{\alpha\in A}$ be a set of regular interpolators for PW_π. Let $k \in \mathbb{N}$ and let X be a Riesz-basis sequence for $L_2[-\pi,\pi]$. Then there exists a constant* *depending only on k, X and the functions (ϕ_α) such that for every $g \in W_2^k$,*

$$\|I^\alpha(g) - g\|_{L_2} \leq C\alpha^{-k}|g|_{W_2^k}. \tag{6.34}$$

Corollary 6.6.3. *Let $(\phi_\alpha)_{\alpha\in A}$ be a set of regular interpolators for PW_π. Let $k \geq 2$, $1 \leq j < k$ and let X be a Riesz-basis sequence for $L_2[-\pi,\pi]$. Then there exists a constant depending on j, k, X and the functions (ϕ_α) such that for every $g \in W_2^k$,*

$$|I^\alpha(g) - g|_{W_2^j} \leq C\alpha^{j-k}|g|_{W_2^k}\ . \tag{6.35}$$

Corollary 6.6.4. *Let $(\phi_\alpha)_{\alpha\in A}$ be a set of regular interpolators for PW_π and let X be a Riesz-basis sequence for $L_2[-\pi,\pi]$. Then for each $k \in \mathbb{N}$, there exists a constant depending on X and the functions (ϕ_α) such that for every $\psi \in \mathscr{S}(\mathbb{R})$ and $\psi \in PW_\sigma$ for some $\sigma > 0$,*

$$\|I^\alpha(\psi) - \psi\|_{L_2} \leq C\alpha^{-k}|\psi|_{W_2^k}. \tag{6.36}$$

To conclude this section, we give four examples of families of regular interpolators. The first example is the family of Gaussians we have already discussed at length, namely

$$\mathcal{G} := \left(e^{-\lambda x^2}\right)_{\lambda\in(0,1]}.$$

In Ledford's notation, α here corresponds to $1/\lambda$. The next two examples are given in [26]: the family of Poisson kernels,

$$\mathcal{F} := \left(\sqrt{\frac{2}{\alpha}}\frac{\alpha}{\alpha^2 + x^2}\right)_{\alpha\in[1,\infty)},$$

and the sequence of differenced convolutions of the Hardy multiquadric, $\phi(x) :=$

$\sqrt{1+x^2}$,

$$\mathcal{M} := \left((-1)^k \Delta^k \phi_*^k(x)\right)_{k\in\mathbb{N}},$$

where

$$\Delta^1 f(x) := f(x+1) + f(x-1) - 2f(x),$$

and recursively $\Delta^k f(x) := \Delta^1\left(\Delta^{k-1}f\right)(x)$, and $f_*^k(x) := \left(f \underbrace{*\cdots*}_{k} f\right)(x)$.

Our final example is one which is not listed in [26], but an important one nonetheless.

Theorem 6.6.5. *For each fixed* $\alpha \in (-\infty, -1/2)$, *the class of inverse multiquadrics*

$$\mathcal{Q}_\alpha := \left((x^2+c^2)^\alpha\right)_{c\in[1,\infty)}$$

is a family of regular interpolators.

Proof. Let $\alpha \in (-\infty, -1/2)$ be fixed. To check (R1), consider a fixed value of $c \in [1,\infty)$, and the conditions (A1)-(A3) need to be shown. To see condition (A1), it is evident that $\phi_{\alpha,c}$ is integrable from the definition, and that $\widehat{\phi_{\alpha,c}} \in L_1(\mathbb{R})$ follows from the restriction on α and Proposition 5.2.3. Condition (A2) has already been discussed: the sign of $\widehat{\phi_{\alpha,c}}$ depends only upon the sign of $\Gamma(-\alpha)$, and does not change for any value of ξ; moreover, $K_{\alpha+\frac{1}{2}}$ is strictly positive on $\mathbb{R}$, whence the condition.

For (A3), if $M_j := \sup\limits_{|\xi|\leq\pi}|\widehat{\phi_{\alpha,c}}(\xi+2\pi j)|$, $j\in\mathbb{Z}$, then we need to show that $M_j \in \ell_1$. Recall that $\widehat{\phi_{\alpha,c}}$ is decreasing (Proposition 5.2.2). Consequently, $M_0 = \lim\limits_{\xi\to 0}\widehat{\phi_{\alpha,c}}(\xi)$, which is a finite constant (depending on α and c since $\phi_{\alpha,c} \in L_1 \cap L_2(\mathbb{R})$ implies $\widehat{\phi_{\alpha,c}} \in C_0(\mathbb{R})$ via the Riemann-Lebesgue Lemma). Then for $j \neq 0$, we have

$$M_j = \widehat{\phi_{\alpha,c}}((2|j|-1)\pi) = A_{\alpha,c}((2|j|-1)\pi)^{-\alpha-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(c(2|j|-1)\pi),$$

where $A_{\alpha,c}$ is a constant. By Lemma 3.2.2(iii),

$$\begin{aligned} M_j &\leq A_{\alpha,c}((2|j|-1)\pi)^{-\alpha-\frac{1}{2}}c^{-\frac{1}{2}}((2|j|-1)\pi)^{-\frac{1}{2}}e^{-c(2|j|-1)\pi}e^{\frac{|\alpha+\frac{1}{2}|^2}{2c(2|j|-1)\pi}} \\ &\leq A_{\alpha,c}((2|j|-1)\pi)^{-\alpha-1}e^{-c(2|j|-1)\pi}. \end{aligned}$$

The second inequality follows because the final exponential term in the first inequality may be bounded by a constant involving α and c by taking $|j|=1$. Putting together these estimates, we find that

$$\sum_{j\in\mathbb{Z}} M_j \leq A_{\alpha,c}\left[1+\sum_{j\neq 0}((2|j|-1)\pi)^{-\alpha-\frac{1}{2}}e^{-c(2|j|-1)\pi}\right] < \infty,$$

which is (A3). Consequently, (R1) is satisfied and $\phi_{\alpha,c}$ is an interpolator for PW_π for every $c \in [1,\infty)$.

To show (R2), let $M_j(c) := \sup_{|\xi|\leq\pi} \widehat{\phi_{\alpha,c}}(\xi+2\pi j)$, and $m_c := \inf_{|\xi|\leq\pi} \widehat{\phi_{\alpha,c}}(\xi)$. By Lemma 3.2.2(i) and (ii) and Proposition 5.2.2, we have

$$m_c = \widehat{\phi_{\alpha,c}}(\pi) \geq A_\alpha c^\alpha e^{-c\pi}. \tag{6.37}$$

Additionally, $M_j(c) = \widehat{\phi_{\alpha,c}}(\pi)$ whenever $|j|=1$, so $M_1(c)/m_c = M_{-1}(c)/m_c = 1$ for every c. Following the steps above, but keeping better track of c, it follows from Lemma 3.2.2(iv) that for $|j|>1$,

$$M_j(c) \leq A_\alpha c^{\alpha+\frac{1}{2}}((2|j|-1)\pi)^{-\alpha-\frac{1}{2}}e^{-c(2|j|-1)\pi}. \tag{6.38}$$

Thus by (6.37) and (6.38),

$$\frac{\sum_{j\neq 0} M_j(c)}{m_c} \leq 2 + \sum_{|j|\geq 2} A_\alpha c^{\frac{1}{2}}((2|j|-1)\pi)^{-\alpha-\frac{1}{2}} e^{-c(2|j|-1)\pi} \leq A_\alpha.$$

The series on the right hand side above is convergent for every c, and moreover is dominated by the convergent series obtained by replacing c with 1, whence the final inequality above.

It remains to check (R3), which is to show that $\lim_{c\to\infty} \frac{m_c}{\widehat{\phi_{\alpha,c}}(\xi)} = 0$ for almost every $\xi \in [-\pi,\pi]$. By Lemma 3.2.2(i), (ii), and (iii), we find that up to a constant depending on α,

$$\begin{aligned} \frac{m_c}{\widehat{\phi_{\alpha,c}}(\xi)} &\leq \frac{c^\alpha \pi^{-\alpha-1} e^{-c\pi} e^{\frac{|\alpha+\frac{1}{2}|^2}{2c\pi}}}{c^\alpha |\xi|^{-\alpha-1} e^{-c|\xi|}} \\ &= \left(\frac{\pi}{|\xi|}\right)^{-\alpha-1} e^{-c(\pi-|\xi|)} e^{\frac{|\alpha+\frac{1}{2}|^2}{2c\pi}}. \end{aligned}$$

Therefore, since $0 \leq \frac{m_c}{\widehat{\phi_{\alpha,c}}(\xi)}$, for any $|\xi| < \pi$, we have $\lim_{c\to\infty} \frac{m_c}{\widehat{\phi_{\alpha,c}}(\xi)} = 0$ since $\pi - |\xi|$ is positive. We conclude that $\mathcal{Q}_\alpha$ is a family of regular interpolators for PW_π.

$\square$

6.6.2 Proofs for Regular Interpolators

In this section we turn to the proof of the uniform boundedness of $\mathscr{I}_\alpha f$ in the Sobolev seminorm (i.e. the analogue of Theorem 6.1.3), which will be accomplished in a series of steps reminiscent of the proofs in [40]. Again assume that $X := (x_j)_{j\in\mathbb{Z}}$ is a fixed Riesz-basis sequence with Riesz basis constant B.

From the definition of the interpolation operators, (6.33), and basic Fourier transform properties, we see that

$$\mathscr{F}[\mathscr{I}_\alpha(f)](\xi) = \widehat{\phi_\alpha}(\xi) \sum_{j\in\mathbb{Z}} a_j e^{-ix_j\xi} =: \widehat{\phi_\alpha}(\xi)\Psi_\alpha(\xi), \quad \xi \in \mathbb{R}. \tag{6.39}$$

Again letting ψ_α be the restriction of Ψ_α to $[-\pi,\pi]$, we obtain the following.

Proposition 6.6.6 ([26], Proposition 2)**.** *If $f \in PW_\pi$, then*

$$\mathscr{F}[f](\xi) = \widehat{\phi_\alpha}(\xi)\psi_\alpha(\xi) + \sum_{j\neq 0} A_j^*\left(\widehat{\phi_\alpha}(\cdot+2\pi j)A_j(\psi_\alpha)\right)(\xi), \quad \xi \in [-\pi,\pi]. \tag{6.40}$$

The following proposition will be quite useful in obtaining estimates on the norms of the derivatives.

Proposition 6.6.7. *For every $j \in \mathbb{Z}$,*

$$(i\xi)^k A_j^*\left(\widehat{\phi_\alpha}(\cdot+2\pi j)A_j(\psi_\alpha)\right)(\xi) = A_j^*\left(\widehat{\phi_\alpha}(\cdot+2\pi j)A_j\left((i\cdot)^k\psi_\alpha\right)\right)(\xi),$$

for $\xi \in [-\pi,\pi]$.

Proof. Define $e_j := e^{-ix_j(\cdot)}$ on $[-\pi,\pi]$. Then because (e_j) forms a Riesz basis for $L_2[-\pi,\pi]$, it suffices to show that the inner products of the left and right hand sides with e_j are equal for all $j \in \mathbb{Z}$. So let $j,\ell \in \mathbb{Z}$ be arbitrary, and let $\langle\cdot,\cdot\rangle$ denote the usual inner product on $L_2[-\pi,\pi]$. Then

$$\begin{aligned}\left\langle (i\cdot)^k A_j^*\left(\widehat{\phi_\alpha}(\cdot+2\pi j)A_j(\psi_\alpha)\right), e_\ell\right\rangle &= \left\langle \widehat{\phi_\alpha}(\cdot+2\pi j)A_j(\psi_\alpha), A_j\left((-i\cdot)^k e_\ell\right)\right\rangle \\ &= \int_{-\pi}^{\pi} \widehat{\phi_\alpha}(\xi+2\pi j)\psi_\alpha(\xi+2\pi j) \\ &\quad \times (i(\xi+2\pi j))^k e^{-ix_\ell(\xi+2\pi j)}d\xi \\ &= \left\langle \widehat{\phi_\alpha}(\cdot+2\pi j)A_j\left((i\cdot)^k\psi_\alpha\right), A_j(e_\ell)\right\rangle\end{aligned}$$

$$= \left\langle A_j^* \left(\widehat{\phi_\alpha}(\cdot + 2\pi j) A_j \left((i\cdot)^k \psi_\alpha \right) \right), e_\ell \right\rangle .$$

$\square$

To simplify the above formulae, define

$$B_\alpha g(\xi) := \sum_{j\neq 0} A_j^* \left(\frac{\widehat{\phi_\alpha}(\cdot + 2\pi j)}{m_\alpha} A_j(g) \right)(\xi).$$

As a consequence of Propositions 6.6.7 and 6.6.6, the following holds.

Corollary 6.6.8. *Let $f \in PW\pi$, and $\phi_\alpha, \psi_\alpha, B_\alpha$ be as defined above. Then*

$$(i\xi)^k \widehat{f}(\xi) = (i\xi)^k \mathscr{F}[\mathscr{I}_\alpha(f)](\xi) + B_\alpha \left((i\cdot)^k m_\alpha \psi_\alpha \right)(\xi), \quad \xi \in [-\pi, \pi].$$

In [26], it was shown that $\|B_\alpha g\|_{L_2[-\pi,\pi]} \leq C\|g\|_{L_2[-\pi,\pi]}$ where C is independent of α and g. It follows that

$$\|B_\alpha \left((i\cdot)^k m_\alpha \psi_\alpha \right)\|_{L_2[-\pi,\pi]} \leq C m_\alpha \|(i\cdot)^k \psi_\alpha\|_{L_2[-\pi,\pi]}, \tag{6.41}$$

the right hand side of which is estimated as follows:

Lemma 6.6.9.

$$\|(i\cdot)^k \psi_\alpha\|_{L_2[-\pi,\pi]} \leq \frac{\sqrt{2\pi}}{m_\alpha} |f|_{W_2^k(\mathbb{R})}$$

Proof. Begin by taking the inner product in $L_2[-\pi,\pi]$ of the equation in Corollary 6.6.8 with $(i\xi)^k \psi_\alpha(\xi)$:

$$\begin{aligned}\left\langle (i\cdot)^k \mathscr{F}[f], (i\cdot)^k \psi_\alpha \right\rangle &= \left\langle (i\cdot)^k \widehat{\phi_\alpha} \psi_\alpha, (i\cdot)^k \psi_\alpha \right\rangle \\ &\quad + \sum_{j\neq 0} \left\langle \widehat{\phi_\alpha}(\cdot + 2\pi j) A_j \left((i\cdot)^k \psi_\alpha \right), A_j \left((i\cdot)^k \psi_\alpha \right) \right\rangle .\end{aligned}$$

By property (A2), each term on the right hand side is non-negative. Consequently,

$$\begin{aligned} m_\alpha \|(i\cdot)^k \psi_\alpha\|^2_{L_2[-\pi,\pi]} &= m_\alpha \left\langle (i\cdot)^k \psi_\alpha, (i\cdot)^k \psi_\alpha \right\rangle \\ &\le \left\langle (i\cdot)^k \widehat{\phi_\alpha} \psi_\alpha, (i\cdot)^k \psi_\alpha \right\rangle \\ &\le \left\langle (i\cdot)^k \mathscr{F}[f], (i\cdot)^k \psi_\alpha \right\rangle \\ &\le \|(i\cdot)^k \mathscr{F}[f]\|_{L_2[-\pi,\pi]} \|(i\cdot)^k \psi_\alpha\|_{L_2[-\pi,\pi]}, \end{aligned}$$

from which the result follows recalling that $|f|_{W_2^k} = \frac{1}{\sqrt{2\pi}} \|(i\cdot)^k \mathscr{F}[f]\|_{L_2(\mathbb{R})}$, and the fact that $f \in PW_\pi$. $\square$

Proof of Theorem 6.1.3. By Corollary 6.6.8, (6.41), and Lemma 6.6.9,

$$\begin{aligned} \|(i\cdot)^k \mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2[-\pi,\pi]} &\le \sqrt{2\pi} |f|_{W_2^k} + \|B_\alpha \left(m_\alpha (i\cdot)^k \psi_\alpha \right) \|_{L_2[-\pi,\pi]} \\ &\le \sqrt{2\pi} |f|_{W_2^k} + C m_\alpha \|(i\cdot)^k \psi_\alpha\|_{L_2[-\pi,\pi]} \\ &\le \sqrt{2\pi} |f|_{W_2^k} + C m_\alpha \frac{1}{m_\alpha} |f|_{W_2^k}. \end{aligned}$$

Consequently, $\|(i\cdot)^k \mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2[-\pi,\pi]} \le C |f|_{W_2^k}$, where C is independent of α and f. Now to estimate $\|(i\cdot)^k \mathscr{F}[\mathscr{I}_\alpha f]\|_{L_2(\mathbb{R}\setminus[-\pi,\pi])}$.

$$\|(i\cdot)^k \mathscr{F}[\mathscr{I}_\alpha f]\|^2_{L_2(\mathbb{R}\setminus[-\pi,\pi])} = \int_{\mathbb{R}\setminus[-\pi,\pi]} \left| (i\xi)^k \widehat{\phi_\alpha}(\xi) \Psi_\alpha(\xi) \right|^2 d\xi$$

$$
\begin{aligned}
&= \sum_{j\neq 0} \int_{-\pi}^{\pi} \left| \widehat{\phi_\alpha}(\xi + 2\pi j) A_j \left((i\cdot)^k \psi_\alpha \right)(\xi) \right|^2 d\xi \\
&\leq \sum_{j\neq 0} \sup_{|\xi|\leq\pi} \left| \widehat{\phi_\alpha}(\xi + 2\pi j) \right|^2 \left\| A_j \left((i\cdot)^k \psi_\alpha \right) \right\|^2_{L_2[-\pi,\pi]} \\
&\leq \sum_{j\neq 0} M_j(\alpha)^2 B^2 \|(i\cdot)^k \psi_\alpha\|^2_{L_2[-\pi,\pi]} \\
&\leq B^2 \left(\sum_{j\neq 0} M_j(\alpha) \right)^2 \frac{1}{m_\alpha} |f|_{W_2^k} \\
&\leq C|f|_{W_2^k},
\end{aligned}
$$

where the last two inequalities come from property (R2) and the fact that $\|(M_j(\alpha))\|^2_{\ell_2} \leq \|(M_j(\alpha))\|^2_{\ell_1}$.

Consequently, we find that

$$
|\mathscr{I}_\alpha f|_{W_2^k} \leq C|f|_{W_2^k}
$$

where C is independent of α and f (C depends ϕ and B). $\square$

6.7 Remarks on the Multivariate Case

A simple extension of the main theorem (Theorem 6.1.5) can be made by a tensor product argument. This requires considering Riesz-basis sequences for $L_2[-\pi,\pi]^d$ that form a grid (i.e. Cartesian products of d Riesz-basis sequences for $L_2[-\pi,\pi]$).

However, this is far from the general case of nonuniform data sites in higher dimensions. The reason for the lack of a better multivariate extension is twofold. Firstly, we do not have a proper multidimensional version of Theorem 6.1.2. Secondly, not much is known about Riesz-basis sequences associated with more general sets in higher dimensions as discussed in Chapter 4.

REFERENCES

[1] M. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables.* No. 55, Courier Dover Publications, 1972.

[2] B. A. Bailey, An asymptotic equivalence between two frame perturbation theorems, *Proceedings of Approximation Theory XIII: San Antonio 2010*, Springer New York (2012), 1-7.

[3] B. A. Bailey, Sampling and recovery of multidimensional bandlimited functions via frames, *J. Math. Anal. App.* **370** (2010), 374-388.

[4] B. A. Bailey, Th. Schlumprecht and N. Sivakumar, Nonuniform sampling and recovery of multidimensional bandlimited functions by Gaussian radial-basis functions, *J. Fourier Anal. Appl.* **17** (2011), 519-533.

[5] B. J. C. Baxter, On the asymptotic cardinal function of the multiquadric $\phi(r) = (r^2 + c^2)^{1/2}$ as $c \to \infty$, *Comput. Math. Applic.* **24** (1992), 1-6.

[6] B. J. C. Baxter and N. Sivakumar, On shifted cardinal interpolation by Gaussians and multiquadrics, *J. Approx. Theory* **87** (1996), 36-59.

[7] M. Buhmann, Multivariate cardinal interpolation with radial-basis functions, *Constr. Approx.* **6.3** (1990), 225-255.

[8] P. J. Davis, *Interpolation and Approximation*, Dover (1975).

[9] C. de Boor, How small can one make the derivatives of an interpolating function? *J. Approx. Theory* **13** (1975), 105-116.

[10] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[11] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications, Second Edition*, John Wiley & Sons, 2013.

[12] R. J. Gardner, *Geometric Tomography, Second Edition*, Cambridge University Press, 2006.

[13] M. Golomb, $\mathcal{H}^{m,p}$-extensions by $\mathcal{H}^{m,p}$-splines, *J. Approx. Theory* **5** (1972), 238-275.

[14] S. Grepstad and N. Lev, Multi-tiling and Riesz bases, *Adv. Math.* **252** (2014), 1-6.

[15] K. Hamm, Approximation rates for interpolation of Sobolev functions via Gaussians and allied functions, *J. Approx. Theory*, **189** (2015), 101-122.

[16] K. Hamm, Sampling and recovery using multiquadrics, *11th International Conference on Sampling Theory and Applications (SampTA 2015)*, Washington D.C. 2015, to appear.

[17] K. Hamm, Nonuniform sampling and recovery of bandlimited functions in higher dimensions, preprint, arXiv: 1411.5610, submitted.

[18] K. Hamm and J. Ledford, Cardinal interpolation with general multiquadrics, preprint, arXiv: 1501.01899, submitted.

[19] T. Hangelbroek, W. Madych, F. Narcowich and J. Ward, Cardinal interpolation with Gaussian kernels, *J. Fourier Anal. Appl.* **18** (2012), 67-86.

[20] T. Hangelbroek, and A. Ron, Nonlinear approximation using Gaussian kernels, *J. Funct. Anal.* **259**(1) (2010), 203-219.

[21] M. I. Kadec, The exact value of the Paley-Wiener constant, *Dokl. Adad. Nauk SSSR* **155** (1964), 1243-1254.

[22] Y. Katznelson, *An Introduction to Harmonic Analysis, Third Edition*, Cambridge University Press, 2004.

[23] M. N. Kolountzakis, Multiple lattice tiles and Riesz bases of exponentials, *Proc. Amer. Math. Soc.* **143** (2015), 741-747.

[24] G. Kozma, S. Nitzan, Combining Riesz bases, *Invent. Math.*, to appear.

[25] G. Kozma, S. Nitzan, Combining Riesz bases in $\mathbb{R}^d$, preprint, arXiv: 1501.05257.

[26] J. Ledford, Recovery of Paley-Wiener functions using scattered translates of regular interpolators, *J. Approx. Theory* **173** (2013), 1-13.

[27] J. Ledford, Recovery of bivariate band-limited functions using scattered translates of the Poisson kernel, *J. Approx. Theory* **189** (2015), 170-180.

[28] J. Ledford, On the convergence of regular families of cardinal interpolators, *Adv. Comput. Math.* **41** (2015), 357-371.

[29] G. Leoni, *A First Course in Sobolev Spaces* (Vol. 105), American Mathematical Society, 2009.

[30] Yu. Lyubarskii and W. R. Madych, The recovery of irregularly sampled band limited functions via tempered splines, *J. Funct. Anal.* **125** (1994), 201-222.

[31] Yu. Lyubarskii and A. Rashkovskii, Complete interpolating sequences for Fourier transforms supported by convex symmetric polygons, *Ark. Math.* **38** (2000), 139-170.

[32] W. R. Madych, Summability of Lagrange type interpolation series, *J. Anal. Math.* **84** (2001), 207-229.

[33] W. R. Madych and E. H. Potter, An estimate for multivariate interpolation, *J. Approx. Theory* **43** (1985), 132-139.

[34] B. S. Pavlov, The basis property of a system of exponentials and the condition of Muckenhoupt, *Dokl. Akad. Nauk SSSR* **247** (1979), 37-40.

[35] S. D. Riemenschneider, Multivariate cardinal interpolation, in *Approximation theory VI, Vol. II*, Eds: C.K. Chui, L. L. Schumaker and J. D. Ward, Academic Press, Boston, MA (1989), 561-580.

[36] S. D. Riemenschneider and N. Sivakumar, On the cardinal-interpolation operator associated with the one-dimensional multiquadric, *East J. Approx.* **7**, no. 4 (1999), 485-514.

[37] S. D. Riemenschneider and N. Sivakumar, On cardinal interpolation by Gaussian radial-basis functions: properties of fundamental functions and estimates for Lebesgue constants, *J. Anal. Math.* **79** (1999), 33-61.

[38] S. D. Riemenschneider and N. Sivakumar, Gaussian radial basis functions: Cardinal interpolation of ℓ_p and power growth data, *Adv. Comput. Math.* **11** (1999), 229-251.

[39] S. D. Riemenschneider and N. Sivakumar, Cardinal interpolation by Gaussian functions: A survey, *J. Analysis* **8** (2000), 157-178.

[40] Th. Schlumprecht and N. Sivakumar, On the sampling and recovery of band-limited functions via scattered translates of the Gaussian, *J. Approx. Theory* **159** (2009), 128-153.

[41] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions - Part A, *Quart. Appl. Math.* **IV** (1946), 45-99.

[42] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions - Part B, *Quart. Appl. Math.* **IV** (1946), 112-141.

[43] I. J. Schoenberg, Notes on spline functions III: on the convergence of the interpolating cardinal splines as their degree tends to infinity, *Israel J. Math.* **16** (1973), 87-93.

[44] I. J. Schoenberg, *Cardinal Spline Interpolation*, SIAM, 1973.

[45] N. Sivakumar, A note on the Gaussian cardinal-interpolation operator, *Proc. Edinb. Math. Soc.* **40** (1997), 137-150.

[46] W. Sun, X. Zhou, On the stability of multivariate trigonometric systems, *J. Math. Anal. App.* **235** (1999), 159-167.

[47] H. Wendland, *Scattered Data Approximation.* Vol. 17. Cambridge University Press, 2005.

[48] R. M. Young, *An Introduction to Non-harmonic Fourier Series, Revised Edition*, Academic Press, 2001.

APPENDIX A

GENERAL THEOREMS AND FACTS

In this brief Appendix, we provide some auxiliary theorems that were used without proof in the preceding chapters for the sake of completeness.

Theorem A.0.1 (Monotone Convergence Theorem). *Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of positive, measurable functions on $\mathbb{R}^d$ ($d \in \mathbb{N}$) such that $f_n(x) \leq f_{n+1}(x)$ for every n and every $x \in \mathbb{R}^d$. Let $f(x) := \lim_{n\to\infty} f_n(x) = \sup_n f_n(x)$. Then*

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx.$$

Theorem A.0.2 (Dominated Convergence Theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L_1(\mathbb{R}^d)$ such that $f_n \to f$ almost everywhere on $\mathbb{R}^d$, and suppose there exists a non-negative $g \in L_1(\mathbb{R}^d)$ such that $|f_n(x)| \leq g(x)$ a.e. for every n. Then $f \in L_1(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)dx.$$

The statements of these convergence theorems was taken from Chapter 2 of [11].

Theorem A.0.3 (Riemann-Lebesgue Lemma). *If $f \in L_1(\mathbb{R})$, then*

$$\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0.$$

This leads to an important corollary which we have used throughout this work.

Corollary A.0.4 (cf. [22] p.155, Theorem 1.7). *If $f \in L_1 \cap L_2(\mathbb{R})$, then $\widehat{f} \in C_0 \cap L_2(\mathbb{R})$.*

Proof. Basic Fourier transform theory shows that $\widehat{f} \in L_2(\mathbb{R})$ and moreover is continuous. Then the Riemann-Lebesgue Lemma implies that $\widehat{f} \in C_0(\mathbb{R})$. $\square$

In Chapter 6, it was stated that every Paley-Wiener space over an interval in $\mathbb{R}$ is isometrically isomorphic to PW_π. We supply the proof of this fact here.

Theorem A.0.5. *For every $\sigma > 0$, the spaces PW_π and PW_σ are isometrically isomorphic. That is, there exists a bijection $J_{\sigma\pi} : PW_\sigma \to PW_\pi$ such that for every $f \in PW_\sigma$,*

$$\|f\| = \|J_{\sigma\pi} f\|.$$

Proof. For now, let the norm on the Paley-Wiener spaces be $\|f\|_{PW_\sigma} := \|f\|_{L_2(\mathbb{R})}$ for every $\sigma > 0$. Then define the map

$$J_{\sigma\pi} f(x) = \left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}} f\left(\frac{\pi}{\sigma} x\right), \quad x \in \mathbb{R}.$$

First, note that $J_{\sigma\pi} : PW_\sigma \to PW_\pi$. Indeed, via the substitution $u = \frac{\pi}{\sigma} x$,

$$\begin{aligned} \widehat{J_{\sigma\pi} f}(\xi) &= \left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}} \int_{\mathbb{R}} f\left(\frac{\pi}{\sigma} x\right) e^{-ix\xi} dx \\ &= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} f(u) e^{-iu\frac{\sigma}{\pi}\xi} du \\ &= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \widehat{f}\left(\frac{\sigma}{\pi}\xi\right). \end{aligned}$$

Since $f \in PW_\sigma$, $\widehat{f}\left(\frac{\sigma}{\pi}\xi\right)$ is only nonzero whenever $\frac{\sigma}{\pi}\xi \in [-\sigma, \sigma]$, or in other words, whenever $\xi \in [-\pi, \pi]$. Consequently, $J_{\sigma\pi} f \in PW_\pi$.

To see the isometry, by the same substitution as above, note that

$$\|J_{\sigma\pi}f\|^2_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \frac{\pi}{\sigma} \left| f\left(\frac{\pi}{\sigma}x\right)\right|^2 dx = \int_{\mathbb{R}} |f(u)|^2 du = \|f\|^2_{L_2(\mathbb{R})}.$$

$\square$

Having not defined orthonormal bases previously, we supply the definition here.

Definition A.0.6. *A set of vectors $(\phi_n)_{n\in\mathbb{N}}$ in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is said to be* **complete** *if $\langle h, \phi_n\rangle = 0$ for all n implies that $h = 0$ in $\mathcal{H}$. A set of complete vectors in $\mathcal{H}$ is called an* **orthonormal basis for** $\mathcal{H}$ *if*

$$\langle \phi_n, \phi_m\rangle = \delta_{n,m}, \quad n, m \in \mathbb{N}.$$