PERTURBATIONS OF CERTAIN CROSSED PRODUCT ALGEBRAS BY FREE GROUPS

A Dissertation

by

WAI-KIT CHAN

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Roger Smith
Committee Members,	Kenneth Dykema
	David Kerr
	Frederick Dahm
Head of Department,	Emil Straube

August 2015

Major Subject: Mathematics

ABSTRACT

Given two von Neumann algebras M and N acting on the same Hilbert space, $d(M, N)$ is defined to be the Hausdorff distance between their unit balls. The Kadison-Kastler problem asks whether two sufficiently close von Neumann algebras are spatially isomorphic. In this article, we show that if P_0 is an injective von Neumann algebra with a cyclic tracial vector, G is a free group, α is a free action of G on P_0 and N is a von Neumann algebra such that $d(N, P_0 \rtimes_\alpha G) < 1/7 \cdot 10^{-7}$, then N and $P_0 \rtimes_\alpha G$ are spatially isomorphic. Suitable choices of the actions give the first examples of infinite noninjective factors for which this problem has a positive solution.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Prof. Roger Smith, for his guidance and support in the course of study. I would also like to thank him for suggesting this research project and reading the whole thesis many times.

TABLE OF CONTENTS

1. INTRODUCTION

The study of perturbation of operator algebras was initiated by Kadison and Kastler. In [23], they introduced a metric on all closed *-subalgebras of bounded linear operators on a Hilbert space and asked if two sufficiently close von Neumann algebras are spatially isomorphic. The metric is defined in terms of the Hausdorff distance of the unit balls of the algebras (see Definition 2.1 for the exact details). Examples of close von Neumann algebras can be obtained by conjugating an operator algebra by a unitary close to the identity operator. The Kadison-Kastler problem for injective von Neumann algebras was solved positively by the independent work of Christensen [10], Johnson [22], Raeburn and Taylor [30]. There is also a similar formulation for C*-algebras represented on Hilbert spaces: whether close C*-algebras are isomorphic or spatially isomorphic. Although there are counterexamples where close C^*-algebras are non-isomorphic [7], the problem was solved positively for various classes of C*-algebras (for example, [12], [28], [29]). Recently, Christensen et al. [15] solved it completely for separable and nuclear C*-algebras on separable Hilbert spaces. Other recent progress was obtained in [3], where Cameron et al. provided the first examples of finite non-injective factors for which the Kadison-Kastler problem holds. They considered crossed product algebras associated with trace preserving actions of certain discrete nonamenable groups. Their techniques were specific to the finite case, so it is natural to ask if their results hold for the case of infinite factors. By modifying their methods to apply to the infinite case, we will show that the Kadison-Kastler problem holds for $P_0 \rtimes_\alpha G$ where P_0 is an injective von Neumann algebra with a cyclic tracial vector, G is a free group and α is a free action of G on the von Neumann algebra P_0. We do not assume that G preserves any trace of P_0, so that these crossed products include infinite factors.

When $P_0 \subseteq B(\mathcal{H}_0)$ is an injective von Neumann algebra with a cyclic tracial vector ξ_0 and α is a free action of a free group G on P_0, there is a natural cyclic and separating vector ξ (related to ξ_0) for $M = P_0 \rtimes_\alpha G$ and a group of normalizing unitaries $\{u_g : g \in G\}$ for the algebra $P \subseteq M$ (the image of P_0 in M) and the Hilbert space $\mathcal{H}$, where M is represented, decomposes as a direct sum of closed subspaces spanned by $u_g P\xi$. If we have another von Neumann algebra $N \subseteq B(\mathcal{H})$ close to M, by a theorem of Christensen [12, Theorem 4.3], we may find a unitary w close to the identity so that $P \subseteq N_1 = wNw^*$ and $J_M P J_M \subseteq N_1'$ where J_M is the modular conjugation associated with M and the cyclic and separating vector ξ. Using the methods of Cameron et al. [3], for each $g \in G$, we may find a unitary $v_g \in N_1$ close to u_g so that v_g normalizes P, $v_g P\xi$ and $u_g P\xi$ generate the same closed subpace, and v_g and u_g have the same conjugate action on P. Now, $g \mapsto v_g$ may not be a group homomorphism in general. However, due to the freeness of G, we may find a group homomorphism $g \mapsto \tilde{v}_g$ from G into the group of normalizing unitaries for the inclusion $P \subseteq N_1$ such that $\tilde{v}_g$ and v_g coincide on a generating set of G . Moreover, $\tilde{v}_g$ and u_g have the same conjugate action on P, and $\tilde{v}_g P\xi$ and $u_g P\xi$ generate the same closed subspace. Then there is a natural unitary w_2 such that $w_2 u_g w_2^* = \tilde{v}_g$ for all $g \in G$ and w_2 commutes with P. To show $N_1 = w_2 M w_2^*$, it suffices to show that $\{\tilde{v}_g : g \in G\}$ and P generate N_1. Here we use a different method which does not need the finiteness of N. Let N_2 be the von Neumann algebra generated by $\{\tilde{v}_g : g \in G\}$ and P. We can show that ξ is also a cyclic and separating vector for N_1 and N_2. Let J_{N_1} (resp. J_{N_2}) be the modular conjugation associated with N_1 and ξ (resp. N_2 and ξ). Then $M_1 = w_2^* J_{N_2} N_1' J_{N_2} w_2$ is a von Neumann algebra of M containing P and ξ is cyclic for M_1. We then show that $M_1 = M$. Hence, $N_2 = N_1$ which implies that M and N are spatially isomorphic.

In section 2, we recall some background results in von Neumann algebras and some existing perturbation results of von Neumann algebras.

In section 3, we relax the factor requirement and prove that commutants of a pair of close von Neuman algebras are close when one algebra has a cyclic vector. In [5], the connection between having the similarity property and having close commutants is established. While it is unknown if commutants of a pair of close von Neumann algebras are close, this question is settled when one algebra is properly infinite [11, Corollary 2.5] and when both algebras are II_1 factors with one algebra having a cyclic vector [3, Lemma 4.1(iii)]. This is Corollary 3.7 (see also Proposition 3.6) and the proof is based on a type-decomposition argument and modification of [3, Lemma 4.1(iii)]. This result is needed when we prove the main theorem, Theorem 4.11, of this thesis. It allows us to reduce the problem to the hypothesis of Theorem 4.7.

In section 4, we prove the main result of this thesis, Theorem 4.11, after several propositions. When M and N are close von Neumann algebras with a common injective subalgebra P, we show that several properties can be transferred from the inclusion $P \subseteq M$ to the inclusion $P \subseteq N$.

In section 5, we construct examples of actions of F_∞ (free group of countably infinite many generators) on some measure spaces (Z, μ) which gives non-injective type II_∞ and type III factors $L^\infty(Z) \rtimes F_\infty$. The construction is due to Houdayer and Vaes [21, Corollary B].

The thesis ends with section 6 where, under mild restrictions, we show that the property of being isomorphic to close neighbors is independent of the representation in the following sense: if $M \subseteq B(\mathcal{H})$ is *-isomorphic to any close von Neumann algebra on $\mathcal{H}$ and M_1 is another von Neumann algebra acting on a possibly different Hilbert space $\mathcal{K}$ and is *-isomorphic to M, then M_1 is *-isomorphic to any close von Neumann algebra on $\mathcal{K}$. We do not know if this holds in general, but we establish this for countably decomposable von Neumann algebras in Theorem 6.9. As a consequence, we provide some examples of infinite non-injective von Neumann algebras that are weakly Kadison-Kastler stable and some that are Kadison-Kastler stable (see Definition 6.1 and Corollary 6.12).

Sections 3-6 of this thesis were published in [8].

2. PRELIMINARIES

2.1 Background results in von Neumann algebras theory

We will assume familiarity of the basic theory of von Neumann algebras, as can be found in chapter 5 of [24] and chapters 6-8 of [25]. We will briefly review some of these results in this subsection. Given a Hilbert space $\mathcal{H}$, there are several natural topologies on the set $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$: strong operator topology (SOT), weak operator topology (WOT) and ultraweak operator topology among the others. A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in $B(\mathcal{H})$ converges to some operator $x \in B(\mathcal{H})$ in the strong operator topology if $\|x_\lambda\xi - x\xi\| \to 0$ holds for all vectors ξ in $\mathcal{H}$. A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in $B(\mathcal{H})$ converges to some operator $x \in B(\mathcal{H})$ in the weak operator topology if $\langle x_\lambda\xi, \eta\rangle \to \langle x\xi, \eta\rangle$ holds for all vectors ξ and η in $\mathcal{H}$. A net $\{x_\lambda\}_{i\in\Lambda}$ in $B(\mathcal{H})$ converges to some operator $x \in B(\mathcal{H})$ in the ultraweak operator topology if $\sum_{k=1}^{\infty} \langle x_\lambda\xi_k, \eta_k\rangle \to \sum_{k=1}^{\infty} \langle x\xi_k, \eta_k\rangle$ holds for all sequences $\{\xi_k\}_{k=1}^{\infty}$ and $\{\eta_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that $\sum_{k=1}^{\infty} \|\xi_k\|^2 < \infty$ and $\sum_{k=1}^{\infty} \|\eta_k\|^2 < \infty$.

The identity operator on a Hilbert space $\mathcal{H}$ will be denoted by either $1_{\mathcal{H}}$ or simply 1. Given a subset A of $B(\mathcal{H})$, A' is the commutant of A in $B(H)$, i.e. $\{x \in B(\mathcal{H}) : xy = yx \text{ for all } y \in A\}$. A is a *-subalgebra of $B(\mathcal{H})$ if it is a subalgebra of $B(\mathcal{H})$ such that $x^* \in A$ for all $x \in A$. A von Neumann algebra is a weak-operator-closed *-subalgebra of $B(\mathcal{H})$ containing the identity operator $1_{\mathcal{H}}$. The following theorem gives other equivalent definitions for a *-subalgebra to be a von Neumann algebra. A proof of the theorem can be found in [24, Theorem 5.3.1].

Theorem 2.1 (Double commutant theorem)**.** *Let M be a *-subalgebra of $B(\mathcal{H})$ containing the identity operator* 1*. Then the following are equivalent:*

(i) M is WOT-closed.

(ii) M is SOT-closed.

(iii) $M = M''$, where $M'' = (M')'$.

The crucial part of the following theorem is the norm estimate on the approximating nets $\{x_\lambda\}_{\lambda\in\Lambda}$. A proof of the theorem can be found in [24, Theorem 5.3.5].

Theorem 2.2 (the Kaplansky density theorem)**.** *If A is a *-subalgebra of $B(\mathcal{H})$ and x is an element in the unit ball of $\overline{A}^{WOT}$, then there exists a net $\{x_\lambda\}_{\lambda\in\Lambda}$ in the unit ball of A such that x_λ converges to x in the strong operator topology. If x is a self-adjoint element in the unit ball of $\overline{A}^{WOT}$, then there exists a net $\{x_\lambda\}_{\lambda\in\Lambda}$ of self-adjoint elements in the unit ball of A such that x_λ converges to x in the strong operator topology.*

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. The center of M will be denoted by $Z(M)$ or Z_M. A central projection e of M is a projection in the center of M. For each projection e in M, there exists a smallest projection c_e in $Z(M)$ such that $c_e \geq e$. The projection c_e is called the central support of e relative to M. Murray and von Neumann classified different types of von Neumann algebras by introducing an equvalence relation on the set of projections in M. Given two projections p and q in M, we say that p is equivalent to q (write $p \sim q$) relative to M if there exists $v \in M$ such that $p = v^*v$ and $q = vv^*$. We say that p is subequivalent to q (write $p \preccurlyeq q$) relative to M if there exists a projection q_1 such that $q_1 \leq q$ and $p \sim q$. A projection e in M is finite relative to M if $e = f$ whenever $f \leq e$ and $f \sim e$ relative to M. A projection e in M is called infinite relative to M if it is not finite relative to M. A projection e in M is properly infinite relative to M if pe is an infinite projection relative to M or $pe = 0$ for each projection p in $Z(M)$. These definitions on projections depend on the ambient algebra M. We will drop the words 'relative to M' when the context is clear. We say that M is a finite (properly infinite resp.) von Neumann algebra if 1 is a finite (properly infinite) projection relative to M. For any von Neumann algebra M, there exists a largest projection in $Z(M)$ such that Mp is a finite von Neuman algebra (acting on the Hilbert space $p(\mathcal{H})$). Moreover, $M(1_{\mathcal{H}} - p)$ is a properly infinite von Neumann algebra on $(1_{\mathcal{H}} - p)(\mathcal{H})$ if $1_{\mathcal{H}} - p$ is not zero. A projection e in M is called abelian relative to M if pMp is a commutative algebra. A von Neumann algebra M is type I if there exists an abelian projection e in M such that $c_e = 1$. A von Neumann algebra M is type II_1 if it has no nonzero abelian projections and 1 is a finite projection in M. A von Neumann algebra M is type II_∞ if: it has no nonzero abelian projections, there exists a finite projection e in M such that $c_e = 1$ and 1 is a properly infinite projection in M. A von Neumann algebra is type III if it has no nonzero finite projections.

A proof of the following theorem can be found in [25, Theorem 8.2.8]. The map T is called the center-valued trace of M.

Theorem 2.3 (Center-valued trace)**.** *Let M be a finite von Neumann algebra. Then there exists a unique positive linear map $T : M \to Z(M)$ such that*

(i) $T(ab) = T(ba)$ *for all* a *and* b *in* M.

(ii) $T(c) = c$ *for all* c *in* $Z(M)$.

Morover, for all $a \in A$ *and* $c \in Z(M)$,

(iii) $T(a) > 0$ *if* $a > 0$.

(iv) $T(ca) = cT(a)$.

(v) $\|T(a)\| \leq \|a\|$.

(vi) T *is ultraweakly continuous.*

A proof of the following theorem can be found in [25, Theorem 8.3.5].

Theorem 2.4 (the Diximier approximation theorem)**.** *Let M be a von Neumann algebra and $Z(M)$ be its center. Let x be an element in M and $\overline{conv_M(x)}$ be the norm-closed convex hull of the set $\{uxu^* : u \text{ is a unitary in } M\}$. Then $\overline{conv_M(x)} \cap Z(M) \neq \emptyset$.*

Recall that a linear map between two von Neumann algebras is called normal if it is ultraweakly continuous. Equivalently, a linear map between two von Neumann algebras is normal if and only if it is weak-operator-continuous on the unit ball.

If a von Neumann algebra M acts on a Hilbert space $\mathcal{H}$, then a subset $\mathcal{F}$ of $\mathcal{H}$ is a cyclic set for M if the closed linear span of $\{x\eta : x \in M, \eta \in \mathcal{F}\}$ is $\mathcal{H}$. A set $\mathcal{F}$ is a separating set for M if: $x \in M$ such that $x\eta = 0$ for all $\eta \in \mathcal{F} \Rightarrow x = 0$. A vector η is called cyclic vector (resp. separating vector) for M if the set $\{\eta\}$ is a cyclic set (resp. separating set) for M.

Theorem 2.5. *[25, Theorem 7.2.3] Let $M \subseteq B(\mathcal{H})$ be a von Neumann algebra with a separating vector. Then any normal state τ on M is a vector state, i.e. there exists a vector $\eta \in \mathcal{H}$ such that $\tau(x) = \langle x\eta, \eta \rangle$ for all $x \in M$.*

Proposition 2.6. *[25, Proposition 9.1.2] Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and η be a vector in $\mathcal{H}$. Let e be the orthogonal projection from $\mathcal{H}$ onto $\overline{M'\eta}$ and e' be the orthogonal projection from $\mathcal{H}$ onto $\overline{M\eta}$. Then e is an abelian (resp. finite or infinite) projection in M if and only if e' is an abelian (resp. finite or infinite) projection in M'.*

The following proposition is an immediate and well known consequence of the previous proposition.

Proposition 2.7. *If $M \subseteq B(\mathcal{H})$ is a finite von Neumann algebra and M has a cyclic vector η, then the commutant M' is a finite von Neumann algebra.*

Proof. Let e be the orthogonal projection from $\mathcal{H}$ onto $\overline{M'\eta}$ and e' be the orthogonal projection from $\mathcal{H}$ onto $\overline{M\eta}$. Since M is a finite von Neumann algebra, e is a finite projection in M. Since e' equals 1, by the previous proposition, 1 is a finite projection in M' and hence M' is a finite von Neumann algebra. □

Proposition 2.8. *[25, Excercise 6.9.11] Let M be a finite von Neumann algebra. Let e and f be two projections in M such that $p \sim q$. Then there exists a unitary u in M such that $ueu^* = f$.*

Note that part (ii) of the following proposition follows from part (i) since $M = M''$ by the double commutant theorem.

Proposition 2.9. *[24, Proposition 5.5.11] Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathcal{F}$ be a subset in $\mathcal{H}$. Then*

(i) $\mathcal{F}$ is a cyclic set for M if and only if it is a separating set for M'.

(ii) $\mathcal{F}$ is a separating set for M if and only if it is a cyclic set for M'.

Recall that a vector η is tracial for a von Neumann algebra algebra M if $\langle xy\eta, \eta\rangle = \langle yx\eta, \eta\rangle$ holds for all elements x and y in M.

Proposition 2.10. *[25, Lemma 7.2.14] A cyclic tracial vector for a von Neumann algebra M is a separating vector for M.*

A left ideal I of a von Neumann algebra M is a subspace of M such that $ax \in I$ whenever $a \in M$ and $x \in I$. An ideal I of a von Neumann algebra M is a subspace of M such that $ax \in I$ and $xa \in I$ whenever $a \in M$ and $x \in I$. A *-ideal I of M is an ideal of M such that $x^* \in I$ whenever $x \in I$.

Proposition 2.11. *[25, Proposition 6.8.9] An ideal of a von Neumann algebra is a *-ideal.*

Theorem 2.12. *[25, Theorem 6.8.8] If I is a weak-operator-closed left ideal of a von Neumann algebra M, then there exists a projection p in M such that $I = Mp$. If I is a weak-operator-closed ideal of a von Neumann algebra M, then there exists a central projection p in M such that $I = Mp$.*

Proposition 2.13. *[24, Proposition 5.5.5] Let M be a von Neumann algebra. Let e be a projection in M' and p be the central support of e relative to M'. Let $\Phi : Mp \to Me$ be the map given by $\Phi(xp) = xe$ for all $x \in M$. Then Φ is a *-isomorphism.*

In the following theorem, the maximal condition means: if $\mathcal{A}$ is a family of orthogonal projections in M such that $p \sim g$ for all $p \in \mathcal{A}$ and $\{e_i\}_{i\in\mathcal{F}_1} \subseteq \mathcal{A}$, then $\{e_i\}_{i\in\mathcal{F}_1} = \mathcal{A}$.

Theorem 2.14. *[25, Theorem 6.3.11] Let g be a finite projection in a von Neumann algebra M. Let $\{e_i\}_{i\in\mathcal{F}_1}$ and $\{f_j\}_{j\in\mathcal{F}_2}$ be two orthogonal family of subprojections in M maximal with respect to the property $e_i \sim f_j \sim g$ for all $i \in \mathcal{F}_1$ and $j \in \mathcal{F}_2$. Then $|\mathcal{F}_1| = |\mathcal{F}_2|$.*

Theorem 2.15. *[25, Exercise 10.5.15] Let ϕ be a singular state on a von Neumann algebra M. Let e be a nonzero projection in M. Then there exists a nonzero subprojection f of e such that $\phi(f) = 0$.*

Theorem 2.16. *[31, Theorem IV 5.5] Let $\pi : M \to N$ be a normal *-homomorphism between two von Neumann algebras $M \subseteq B(\mathcal{H})$ and $N \subseteq B(\mathcal{K})$. Then there exists a Hilbert space $\mathcal{L}$, a projection e in $(M\bar{\otimes}\mathbb{C}1_{\mathcal{L}})'$ and a unitary u from $e(\mathcal{H}\otimes\mathcal{L})$ onto $\mathcal{K}$ such that*

$$\pi(x) = u(x \otimes 1)_e u^* \text{ for all } x \in M.$$

2.2 Completely positive maps and conditional expectations

Let A be a C*-algebra and $M_n(A)$ be the set of all $n \times n$ matrix with entries in A. $M_n(A)$ is a vector space under the pointwise scalar multiplication and addition. For $a = (a_{i,j})_{i,j=1,\dots n}$ and $b = (b_{i,j})_{i,j=1,\dots,n}$, define $ab = (c_{i,j})_{i,j=1,\dots n}$ by $c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$ and $a^* = (a^*_{j,i})_{i,j=1\dots n}$. Then $M_n(A)$ is a *-algebra under these operations. Let $\pi : A \to B(\mathcal{H})$ be a faithful *-representation of A. By identifying $M_n(B(\mathcal{H}))$ as $B(\mathcal{H}^n)$, we can define a norm on $M_n(A)$ by $\|(a_{i,j})_{i,j=1,\dots n}\| = \|(\pi(a_{i,j}))_{i,j=1,\dots,n}\|$. Then $M_n(A)$ becomes a C*-algebra under this norm and note that the norm does not depends on the choice of faithful *-representation π.

Let A and B be C*-algebras. Then $M_n(A)$ and $M_n(B)$ are C*-algebras as described in the preceding paragraph. Let $\phi : A \to B$ be a linear map. Define $\phi_n : M_n(A) \to M_n(B)$ by $\phi_n((a_{i,j})_{i,j=1,\dots n}) = (\phi(a_{i,j}))_{i,j=1,\dots n}$. A lineap map ϕ is called completely positive if ϕ_n are positive linear maps for all positive integers n.

Let $B \subseteq A$ be two C*-algebras. A projection from A onto B is a linear map $E : A \to B$ such that $E(b) = b$ for all $b \in B$. A conditional expectation E from A onto B is a completely positive linear map E from A onto B with the following properties:

(i) $\|E(x)\| \leq \|x\|$ for all $x \in A$.

(ii) $E(b) = b$ for all $x \in B$.

(iii) $E(b_1 x b_2) = b_1 E(x) b_2$ for all $b_1, b_2 \in B$ and $x \in A$.

In [33, Theorem 1], Tomiyama showed that a norm-one projection E from a C*-algebra onto a C*-subalgbera is a positive linear map satisfying properties (i)-(iii) of the conditional expectation. The concept of completely positivity was introduced after Tomiyama's paper [33]. A proof of the following theorem can be found in a recent book by Brown and Ozawa [2, Theorem 1.5.10].

Theorem 2.17. *Let $B \subseteq A$ be C*-algebras and E be a projection from A onto B. Then the following are equivalent:*

(i) E is a conditional expectation.

(ii) E is a completely positive linear map and $\|E\| \leq 1$.

(iii) $\|E\| \leq 1$.

Recall that a positive linear map $\phi : A \to B$ between C*-algebras is faithful if $x = 0$ whenever x is a positive element in A and $\phi(x) = 0$.

Proposition 2.18. *[32, Proposition IX 4.3] Let M be a von Neumann algebra and P be a von Neumann subalgebra of M such that $P' \cap M \subseteq P$. Assume that there exists a normal conditional expectation E from M onto P. Then E is faithful and unique.*

2.3 Modular theory

In the following theorem, Δ is called the modular operator associated with M and ξ while J is called the modular conjugation associated with M and ξ.

Theorem 2.19. *[25, Theorem 9.2.9] Suppose that a von Neumann algebra M has a unit cyclic and separating vector ξ in a Hilbert space $\mathcal{H}$ and let $S_0 : M\xi \to \mathcal{H}$ be the map $S_0 x\xi = x^*\xi$ for any x in M. Then S_0 is preclosed and let S be the closure of S_0. Let $S = J\Delta^{1/2}$ be the polar decomposition of S. Then the following hold:*

(i) J is a conjugate-linear isometry onto M such that $J^2 = 1$.

(ii) $S^* x\xi = x^*\xi$ for all $x \in M'$.

(iii) $J\xi = \Delta^{1/2}\xi = \xi$.

(iv) $J\Delta^{it} = \Delta^{it}J$ *for all* $t \in \mathbb{R}$.

(v) $JMJ = M'$.

(vi) $\Delta^{it}M\Delta^{-it} = M$ *for all* $t \in \mathbb{R}$.

Recall that a one-parameter group of *-automorphisms of a von Neumann algebra M is a map $t \mapsto \alpha_t$ from $\mathbb{R}$ to $Aut(M)$ such that $\alpha_{t+s} = \alpha_t\alpha_s$ for all real numbers t and s. A one-parameter group α of *-automorphisms of a von Neumann algebra M satisfies the modular condition relative to a state ω on M if: for any a and b in M, there exists a continuous function f on the region $\{z \in \mathbb{C} : 0 \le Imz \le 1\}$ and is analytic in $\{z \in \mathbb{C} : 0 < Imz < 1\}$ such that

$$f(t) = \omega(\alpha_t(a)b), \qquad f(t+i) = \omega(b\alpha_t(a)) \text{ for all } t \in \mathbb{R}.$$

Assume that a von Neumann algebra M has a unit cyclic and separating vector ξ. Let Δ be the modular operator associated with M and ξ. Let ω be the normal state on M given by $\omega(x) = \langle x\xi, \xi\rangle$ for all $x \in M$. For each $t \in \mathbb{R}$, let α_t be the *-automorphism on M by $\alpha_t(x) = \Delta^{it}x\Delta^{-it}$ for all $x \in M$. Then the map $t \mapsto \alpha_t$ is a one-parameter group of *-automorphisms of M by the Tomita theorem. By [25, Theorem 9.2.13], α satisfies the modular condition relative to ω. Then the following proposition follows from [25, Proposition 9.2.14(iii)].

Proposition 2.20. *Let M be a von Neumann algebra with a unit cyclic and separating vector ξ. Let Δ be the modular operator associated with M and ξ. If x is an element in M such that $\langle ax\xi, \xi\rangle = \langle xa\xi, \xi\rangle$ for all $a \in M$, then $\Delta^{it}x\Delta^{-it} = x$ holds for all real numbers t.*

The first three parts of the next result follow from Stone's Theorem [24, Theorem 5.6.36] and the last part is shown in the proof of [25, Theorem 9.2.16].

Theorem 2.21. *Let $M \subseteq B(\mathcal{H})$ be a von Neumann algebra with a unit cyclic and separating vector ξ. Let Δ be the modular operator associated with M and ξ. Then there exists a (possibly unbounded) self-adjoint operator h defined on a dense subspace of $\mathcal{H}$ such that the following hold:*

(i) $\exp(ith) = \Delta^{it}$.

(ii) *dom*$(h) = \{\xi_1 \in \mathcal{H} : \lim_{t\to 0} \frac{\Delta^{it}\xi_1 - \xi_1}{t}$ *exists in the norm topology*$\}$.

(iii) *For any* $\xi_1 \in$ *dom* (h), $ih\xi_1 = \lim_{t\to 0} \frac{\Delta^{it}\xi_1 - \xi_1}{t}$.

(iv) $\Delta = \exp(h)$.

2.4 Some existing perturbation results

In this subsection, we will recall some related lemmas and theorems known in the literature.

In [23], Kadison and Kastler introduced the following distance on the set of all von Neumann subalgebras of $B(\mathcal{H})$.

Definition 2.22. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$. Given a bounded operator $x \in B(\mathcal{H})$ and a subset F of $B(\mathcal{H})$, define*

$$d(x, F) = \inf\{\|x - y\| : y \in F\} \text{ and}$$

$$d(M, N) = \sup\{d(a, N_1), d(b, M_1) : a \in M_1, b \in N_1\}$$

where M_1 (resp. N_1) denotes the closed unit ball of M (resp. N), $M_1 = \{a \in M : \|a\| \leq 1\}$.

In [12], Christensen introduced the following notion of near inclusion for a pair of von Neumann algebras.

Definition 2.23. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and $\gamma > 0$. $M \subseteq_\gamma N$ means for any $a \in M$ with $\|a\| \leq 1$, there exists $b \in N$ such that $\|a - b\| \leq \gamma$. $M \subset_\gamma N$ means there exists $\gamma_0 < \gamma$ such that $M \subseteq_{\gamma_0} N$.*

The following proposition comes from [9, Lemma 2.7] where the bound for $\|u-1\|$ is sharper than stated here.

Proposition 2.24. *Let x be a bounded operator on a Hilbert space $\mathcal{H}$ and $x = u|x|$ be its polar decomposition. If $\|x - 1\| < 1$, then $\|u - 1\| \leq \sqrt{2}\|x - 1\|$.*

A proof of the following proposition can be found in [27, Lemma 6.2.1].

Proposition 2.25. *Let p and q be two projections in a unital C^*-algebras A with $\|p - q\| < 1$. Then there exists a unitary u in A such that $q = upu^*$ and $\|u - 1\| \leq \sqrt{2}\|p - q\|$.*

The following proposition is [23, Lemma 4].

Proposition 2.26. *Let p be a projection in a von Neumann algebra M. Then p is central if and only if $\|px - xp\| < 1$ for all elements x in the unit ball of M.*

The following proposition is a consequence of [26, Lemma 1.10] since the function $\alpha(t)$ used there satisfies $\alpha(t) \leq \sqrt{2}t$.

Proposition 2.27. *Let A and B be unital C^*-subalgebras of a unital C^*-algebra C (In this case, $1_A = 1_B = 1_C$). Assume that $A \subset_\gamma B$ for some $\gamma < 1$. Then*

(i) If u is a unitary in A, there exists a unitary v in B such that $\|u - v\| < \sqrt{2}\gamma$.

(ii) If p is a projection in A, then there exists a projection q in B such that $\|p-q\| < \frac{\gamma}{\sqrt{2}}$.

The following proposition was used in the proof of [12, Corollary 4.4]. We include a proof for the convenience of the reader.

Proposition 2.28. *Let B be a C^*-subalgebra of a C^*-algebra A. If $A \subset_\gamma B$ for some positive number $\gamma < 1$. Then $A = B$.*

Proof. For each element y in A, we can find recursively a sequence $\{x_n\}_{n=1}^{\infty}$ in B such that $\|y - x_1 - ... - x_n\| \leq \gamma^n \|y\|$. Hence $y = \sum_{n=1}^{\infty} x_n \in B$. □

The following proposition is an immediate consequence of [11, Theorem 2.4]. Note that $\mathcal{D}_x$ in [11, Theorem 2.4] is the map on $B(\mathcal{H})$ such that $\mathcal{D}_x(y) = xy - yx$ for all $y \in B(\mathcal{H})$. Under the hypothesis of (i), any element x in the unit ball of N' statisfies $\|\mathcal{D}_x|_M\| \leq 2\gamma$.

Proposition 2.29. *Let M and N be von Neumann algebras on a Hilbert space $\mathcal{H}$.*

(i) If M is a properly infinite von Neumann algebra and $M \subseteq_\gamma N$, then $N' \subseteq_{3\gamma} M'$.

(ii) If M and N are properly infinite von Neumann algebras, then

$$d(M', N') \leq 6(M, N).$$

The following proposition (with other constant in the conclusion) follows from [13, Corollary 5.4]. The form stated here was obtained recently in [3, Proposition 2.8(ii)] with improved constant.

Proposition 2.30. *Let M and N be von Neumann algebras on a Hilbert space $\mathcal{H}$. Assume that M has a finite cyclic set of m vectors. If $M \subseteq_\gamma N$, then $N' \subseteq_{2(1+\sqrt{2})m\gamma} M'$.*

The following proposition is [14, Lemma 3.4].

Proposition 2.31. *Let M and N be von Neumann algebras on a Hilbert space $\mathcal{H}$. Let Z_M and Z_N be the center of M and N respectively. Assume that $d(M,N) \leq \gamma$ for some $\gamma < 1/6$. Then there exists a unitary $u \in (Z_M \cup Z_N)''$ such that $uZ_Mu^* = Z_N$ and $\|u-1\| \leq 5\gamma$.*

The following proposition can be proved in the same way as the proof of [3, Lemma 4.1(ii)]. We briefly sketch the proof used in [3, Lemma 4.1(ii)]

Proposition 2.32. *Let M and N be von Neumann algebras on a Hilbert space $\mathcal{H}$. Assume that $M \subseteq_\gamma N$ for some positive number γ with $(2+\gamma)2\gamma < 1$. If N is a finite von Neumann algebra, then so is M.*

Proof. Let v be an element in M such that $v^*v = 1$ and $vv^* = e$ for some projection e in M. By hypothesis, there exists an element w in N such that $\|v - w\| \leq \gamma$. Then $\|w^*w - 1\| = \|w^*w - v^*v\| \leq (2+\gamma)\gamma$ and $\|ww^* - vv^*\| \leq (2+\gamma)\gamma$. Since N is a finite von Neumann algebra, there exists a unitary u in N such that $w = u|w|$. Then $ww^* = uw^*wu^*$ and $\|ww^* - 1\| = \|u(w^*w-1)u^*\| \leq (2+\gamma)\gamma$. Hence $\|e-1\| \leq \|vv^* - ww^*\| + \|ww^* - 1\| \leq (2+\gamma)2\gamma < 1$. Since $1-e$ is a projection, the inequality $\|e-1\| < 1$ implies that $e = 1$. This shows that M is a finite von Neumann algebra. □

The following proposition is known (for example, [14, Lemma 3.5]). It is an immediate consequence of the previous proposition.

Proposition 2.33. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with common center Z. Assume that $d(M,N) < \gamma$ for some positive number γ with $(2+\gamma)2\gamma < 1$. Let p (resp. q) be the largest projection in Z such that Mp (resp. Np) is a finite von Neumann algebra. Then $p = q$.*

Proof. We have $Np \subseteq_\gamma Mp$ and Mp is a finite von Neumann algebra on the Hilbert space $p\mathcal{H}$. By previous proposition, Np is a finite von Neumann algebra on $p\mathcal{H}$. Hence $p \leq q$. Similarly, we can show that $q \leq p$. Hence, $p = q$. $\square$

The following proposition is [3, Lemma 4.1 (iii)].

Proposition 2.34. *Let M and N be II_1 factors acting on a Hilbert space $\mathcal{H}$. Suppose that M has a cyclic vector. If $M \subset_\gamma N$ amd $N \subset_\gamma M$ for some $0 < \gamma < 1/47$, then $M' \subset_{4(1+\sqrt{2})\gamma} N'$.*

The following theorem is due to Christensen. Part (i) of the theorem is [12, Theorem 4.3] and part (ii) of the theorem is [12, Corollary 4.4].

Theorem 2.35. *Let M and N be von Neumann algebras on a Hilbert space $\mathcal{H}$. Assume that M is injective.*

(i) If $M \subset_\gamma N$ for some $\gamma < 1/100$, then there exists a unitary $u \in (M \cup N)''$ such that $uMu^* \subseteq N$, $\|u - 1\| \leq 150\gamma$ and $\|uxu^* - x\| \leq 100\gamma\|x\|$ for all $x \in M$.

(ii) If $d(M, N) < \gamma < 1/101$, then there exists a unitary $u \in (M \cup N)''$ such that $uMu^* = N$ and $\|u - 1\| \leq 150\gamma$.

2.5 Crossed product algebras

Let P_0 be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$. Let G be a discrete group with identity element e and let $g \mapsto \alpha_g : G \to Aut(P_0)$ be a free action on P_0. For any $g \in G$, let $\delta_g \in \ell^2(G)$ be the characteristic function of $\{g\}$, let p_g be the orthogonal projection of $\ell^2(G)$ onto $\mathbb{C}\delta_g$ and let $\lambda_g \in B(\ell^2(G))$ be the unitary operator determined by $\lambda_g(\delta_h) = \delta_{gh}$ for any $h \in G$. For any $a \in P_0$, define

$$\pi(a) = \sum_{g \in G} \alpha_{g^{-1}}(a) \otimes p_g.$$

For any $g \in G$, define

$$u_g = 1_{\mathcal{H}_0} \otimes \lambda_g \in B(\mathcal{H}_0 \otimes \ell^2(G)).$$

Then $P_0 \rtimes_\alpha G$ is the von Neumann algebra generated by $\{\pi(a) : a \in P_0\} \cup \{u_g : g \in G\}$.

Recall that the freeness of the action α means for any $g \in G$ with $g \neq e$, if $a \in P_0$ is such that $a\alpha_g(b) = ba$ for every $b \in P_0$, then $a = 0$. It is folklore that the freeness of the action implies $\pi(P_0)' \cap (P_0 \rtimes_\alpha G) \subseteq \pi(P_0)$. Indeed, it follows easily from Proposition 4.9 (i) and (ii).

3. COMMUTANTS OF CLOSE VON NEUMANN ALGEBRAS WITH A CYCLIC VECTOR*

In [23], Kadison and Kastler introduced the following distance on the set of all von Neumann subalgebras of $B(\mathcal{H})$.

Definition 3.1. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$. Given a bounded operator $x \in B(\mathcal{H})$ and a subset F of $B(\mathcal{H})$, define*

$$d(x, F) = \inf\{\|x - y\| : y \in F\} \text{ and}$$

$$d(M, N) = \sup\{d(a, N_1), d(b, M_1) : a \in M_1, b \in N_1\}$$

where M_1 (resp. N_1) denotes the closed unit ball of M (resp. N), $M_1 = \{a \in M : \|a\| \leq 1\}$.

In [12], Christensen introduced the following notion of near inclusion for a pair of von Neumann algebras.

Definition 3.2. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and $\gamma > 0$. $M \subseteq_\gamma N$ means for any $a \in M$ with $\|a\| \leq 1$, there exists $b \in N$ such that $\|a - b\| \leq \gamma$. $M \subset_\gamma N$ means there exists $\gamma_0 < \gamma$ such that $M \subseteq_{\gamma_0} N$.*

In general, it is not known whether close von Neumann algebras have close commutants. Recently, [4] shows that a C*-algebra $\mathcal{A}$ satisfies the Kadison similarly property if and only if commutants are continuous at $\mathcal{A}$. The main result of this section is Corollary 3.7: commutants of close von Neumann algebras are close

*Reprinted with permission from ”Perturbations of certain crossed product algebras by free groups” by Wai-Kit Chan, J. Funct. Anal. 267 (2014), no. 10, 3994-4027, Copyright 2014 by Elsevier.

when one algebra has a cyclic vector. More precisely, there exists $k > 0$ such that $d(M', N') \leq kd(M, N)$ when M has a cyclic vector, where M' denotes the commutant of M. Proposition 2.29 (ii) shows $d(M', N') \leq 6d(M, N)$ when M and N are properly infinite von Neumann algebras. Lemma 6 and Lemma 9 of [23] show that M is a properly infinite von Neumann algebra if and only if N is a properly infinite von Neumann algebra when $d(M, N)$ is sufficiently small. Thus the question is settled when M is properly infinite. By Proposition 2.30, the question is settled when both algebras have a cyclic vector. The question is settled when M and N are type II_1 factors and M has a cyclic vector by Proposition 2.30 and 2.34 . By modifying the proof of Proposition 2.34, we prove Proposition 3.5 which is a version of Proposition 2.34 when M is a finite von Neumann algebra with a cyclic vector. Proposition 3.6 will be used to prove the main result of this work: Theorem 4.11.

The following lemma is analogous to [3, Lemma 2.15] and the proofs are the same. In the following lemma, *conv*(S) refers to the convex hull of a set S.

Lemma 3.3. *Let A and B be von Neumann algebras acting on a Hilbert space $\mathcal{H}$. Let A be a finite von Neumann algebra and suppose that $A \subset_\gamma B$ with $\gamma < 1$. Let T_A be the center-valued trace for A. Let ρ be a state on $B(\mathcal{H})$ such that $\rho|_B$ is tracial. Then, for each $x \in A$,*

$$|\rho(T_A(x)) - \rho(x)| \leq (2 + 2\sqrt{2})\gamma\|x\|.$$

Proof. Let $x \in A$ and let u be a unitary in A. There exists $y \in B$ such that $\|x - y\| \leq \gamma\|x\|$ and there exists a unitary $v \in B$ such that $\|u - v\| < \sqrt{2}\gamma$ by

Proposition 2.27 (i). Then

$$\|uxu^* - vyv^*\| \leq \|(u-v)xu^*\| + \|vx(u^* - v^*)\| + \|v(x-y)v^*\|$$
$$\leq \sqrt{2}\gamma\|x\| + \sqrt{2}\gamma\|x\| + \gamma\|x\|.$$

As $\rho|_B$ is a trace,

$$|\rho(uxu^*) - \rho(x)| \leq |\rho(uxu^* - vyv^*)| + |\rho(vyv^*) - \rho(y)| + |\rho(y-x)|$$
$$\leq (2\sqrt{2}+1)\gamma\|x\| + \gamma\|x\| = (2\sqrt{2}+2)\gamma\|x\|.$$

By the Dixmier approximation theorem (See Theorem 2.4) , $T_A(x)$ lies in the norm closure of $conv\{uxu^* : u \text{ unitary in } A\}$. Thus we have $|\rho(T_A(x)) - \rho(x)| \leq (2\sqrt{2}+2)\gamma\|x\|$. □

As remarked above, it is not known whether close von Neumann algebras have close commutants. Proposition 3.5 establishes this relationship under the additional hypothesis that there is a unit cyclic vector. Proposition 2.31 allows us to reduce to the situation where two algebras have a common center. Proposition 3.5 can be thought of as the generalization of Proposition 2.34 to general finite von Neumann algebras. We extract the main argument of the proof of Proposition 3.5 into the following lemma. Its proof is adapted from the proof of [3, Lemma 4.1.1(iii)] by using a center-valued trace.

Lemma 3.4. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with common center Z. Assume that M is a finite von Neumann algebra and has a unit cyclic vector ξ. Suppose also that $M \subset_\gamma N$ and $N \subset_\gamma M$ for some positive number $\gamma < 1/47$. Then there exists a nonzero projection $\tilde{q} \in Z$ such that $M'\tilde{q} \subset_{4(1+\sqrt{2})\gamma} N'\tilde{q}$.*

Proof. Since M has a cyclic vector and $M \subset_\gamma N$, the near inclusion $N' \subset_{2(1+\sqrt{2})\gamma} M'$ follows from Proposition 2.30. Since M is a finite von Neumann algebra and has a cyclic vector, M' is finite by Proposition 2.7. Note that N' is finite since M' is finite and $N' \subset_{2(1+\sqrt{2})\gamma} M'$ by Proposition 2.32.

The map $x \mapsto \langle T_{M'}(x)\xi, \xi\rangle : M' \to \mathbb{C}$ is a normal state on M' where $T_{M'}$ is the center-valued trace of M'. Now M' has a separating vector, so there exists a unit vector η such that

$$\langle T_{M'}(x)\xi, \xi\rangle = \langle x\eta, \eta\rangle \text{ for any } x \in M' \tag{3.1}$$

by Theorem 2.5. As $J = \{y \in N' : y\eta = 0\}$ is a weak-operator-closed left ideal in N', there exists a projection $p \in N'$ such that

$$J = N'p. \tag{3.2}$$

For any $z \in Z$ such that $\|z\eta\| \neq 0$, define a state ρ_z on $B(\mathcal{H})$ by

$$\rho_z(x) = \langle xz\eta, z\eta\rangle / \|z\eta\|^2,\ x \in B(\mathcal{H}).$$

For $x \in M'$,

$$\begin{aligned}\rho_z(x) &= \langle xz\eta, z\eta\rangle / \|z\eta\|^2 = \langle z^* xz\eta, \eta\rangle / \|z\eta\|^2 \\ &= \langle T_{M'}(z^*xz)\xi, \xi\rangle / \|z\eta\|^2 = \langle T_{M'}(x)z\xi, z\xi\rangle / \|z\eta\|^2,\end{aligned}$$

where the third equality follows from equation (3.1) and the last equality follows from the fact that $T_{M'}$ is a Z-bimodular map Theorem 2.3 (iv). Thus $\rho_z|_{M'}$ is a tracial state on M'. Then, by applying Lemma 3.3 to the pair $N' \subset_{2(1+\sqrt{2})\gamma} M'$, we have for any $y \in N'$,

$$|\rho_z(T_{N'}(y)) - \rho_z(y)| \leq (2 + 2\sqrt{2}) \cdot 2(1 + \sqrt{2})\gamma\|y\|.$$

Now, $\rho_z(p) = \langle pz\eta, z\eta\rangle / |z\eta\|^2 = \langle zp\eta, z\eta\rangle / \|z\eta\|^2 = 0$ since $p\eta = 0$. Hence we have

$$\langle T_{N'}(p)z\eta, z\eta\rangle / \|z\eta\|^2 \leq 4(1 + \sqrt{2})^2\gamma,$$

which is equivalent to

$$\langle T_{N'}(p)z\eta, z\eta\rangle \leq 4(1 + \sqrt{2})^2\gamma \langle z\eta, z\eta\rangle .$$

By density, it follows that for any $\tilde{\eta}$ in the closure of $Z\eta$, we have

$$\langle T_{N'}(p)\tilde{\eta}, \tilde{\eta}\rangle \leq 4(1 + \sqrt{2})^2\gamma \langle \tilde{\eta}, \tilde{\eta}\rangle . \tag{3.3}$$

Let q be the projection onto the closure of $Z\eta$. Then q is a nonzero projection in Z' and the inequality (3.3) implies that

$$T_{N'}(p)q \leq 4(1 + \sqrt{2})^2\gamma q. \tag{3.4}$$

Let $\tilde{q}$ be the central support of q relative to the von Neumann algebra Z'. Then, $\tilde{q} \in Z$. Since the map $a\tilde{q} \mapsto aq$ is a *-isomorphism between $Z\tilde{q}$ and Zq by Proposition 2.13, from (3.4), we have

$$T_{N'}(p)\tilde{q} \leq 4(1 + \sqrt{2})^2\gamma\tilde{q} \leq \frac{1}{2}\tilde{q} \tag{3.5}$$

by the choice of γ. Thus,

$$T_{N'}(p\tilde{q}) \leq T_{N'}((1-p)\tilde{q})$$

and hence

$$p\tilde{q} \preccurlyeq (1-p)\tilde{q} \text{ in } N' \tag{3.6}$$

by Theorem 2.3 (iii), (iv) and [25, Theorem 6.2.7]. As noted in the beginning of the proof, N' is finite and hence (3.6) implies that there exists a unitary $u \in N'$ such that

$$up\tilde{q}u^* \leq (1-p)\tilde{q} \tag{3.7}$$

by Proposition 2.8.

Let $\eta_1 = \tilde{q}\eta$ and $\eta_2 = \tilde{q}u\eta$. If y is an element of $N'\tilde{q}$ such that $y\eta_1 = y\eta_2 = 0$, then

$$y\eta = y\tilde{q}\eta = y\eta_1 = 0.$$

Thus $y \in J$ and $y = yp$ by (3.2). Hence

$$y(1-p) = 0. \tag{3.8}$$

Also, $yu\eta = y\tilde{q}u\eta = y\eta_2 = 0$. Then $yu \in J$ and by (3.2), we have

$$yu = yup.$$

From the last equation, we have

$$y = yupu^* = y\tilde{q}upu^* = yup\tilde{q}u^* = y(1-p)up\tilde{q}u^* = 0$$

by noting that $\tilde{q}$ is in the center Z of N', and that inequality (3.7) and equation (3.8) hold. Thus $\{\eta_1, \eta_2\}$ is separating for $N'\tilde{q}$. The von Neumann algebra $N\tilde{q}$ acting on $\tilde{q}\mathcal{H}$ has a 2-cyclic set $\{\eta_1, \eta_2\}$. Together with $N\tilde{q} \subset_\gamma M\tilde{q}$, we have $M'\tilde{q} \subset_{4(1+\sqrt{2})\gamma} N'\tilde{q}$ by Proposition 2.30. □

Proposition 3.5. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with common center Z. Assume that M is a finite von Neumann algebra and has a unit cyclic vector ξ. Suppose also that $M \subset_\gamma N$ and $N \subset_\gamma M$ for some positive number $\gamma < 1/47$. Then $M' \subset_{4(1+\sqrt{2})\gamma} N'$.*

Proof. Lemma 3.4 shows that for any nonzero projection $p \in Z$, there exists a nonzero subprojection q of p such that $q \in Z$ and $M'q \subset_{4(1+\sqrt{2})\gamma} N'q$. By a maximality argument, there exists an orthogonal family of projections $\{q_i\}_{i\in\Lambda}$ in Z such that $\sum_{i\in\Lambda} q_i = 1$ and $M'q_i \subset_{4(1+\sqrt{2})\gamma} N'q_i$ for each $i \in \Lambda$. The proposition follows. □

Proposition 3.6. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with common center Z. Assume that M has a unit cyclic vector ξ. Suppose also that $M \subset_\gamma N$ and $N \subset_\gamma M$ for some positive number $\gamma < 1/47$. Then $M' \subset_{4(1+\sqrt{2})\gamma} N'$.*

Proof. Let p be the largest projection in Z such that Mp is finite. If $p = 1$, the proposition follows from Proposition 3.5. Otherwise, $1 - p$ is nonzero and $M(1 - p)$ is properly infinite. In this case, p is also the largest projection in Z such that Np is finite and $N(1 - p)$ is properly infinite by Proposition 2.33.

Applying Proposition 3.5 to the pair of von Neumann algebras Mp and Np, we have $M'p \subset_{4(1+\sqrt{2})\gamma} N'p$. Since $N(1-p)$ is properly infinite and $N(1-p) \subset_\gamma M(1-p)$, we have $M'(1 - p) \subset_{3\gamma} N'(1 - p)$ by Proposition 2.29 (i). Thus $M' \subset_{4(1+\sqrt{2})\gamma} N'$. The proposition follows. □

Corollary 3.7. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$. Assume that M has a unit cyclic vector ξ. If $d(M, N) < 1/517$, then $d(M', N') \leq 230d(M, N)$.*

Proof. Let γ be a real number such that $d(M, N) < \gamma < 1/517$. Let $Z(M)$ and $Z(N)$ be the center of M and N respectively. Since $d(M, N) < \gamma < 1/6$, by Proposition 2.31, there exists a unitary $u \in B(\mathcal{H})$ such that $uZ(M)u^* = Z(N)$ and $\|u-1\| \leq 5\gamma$. Let N_1 be u^*Nu. Then $Z(N_1) = Z(M)$ and

$$d(N_1, M) \leq d(N_1, N) + d(N, M) < 2\|u - 1\| + \gamma \leq 11\gamma.$$

Hence $M \subset_{11\gamma} N_1$ and $N_1 \subset_{11\gamma} M$. Since M has a cyclic vector and $M \subset_{11\gamma} N_1$, by Proposition 2.30, we have $N_1' \subset_{2(1+\sqrt{2})\cdot 11\gamma} M'$. Since $11\gamma < 1/47$, by Proposition 3.6, we have $M' \subset_{4(1+\sqrt{2})11\gamma} N_1'$. Hence $d(M', N_1') < 8(1 + \sqrt{2}) \cdot 11\gamma$ and

$$d(M', N') \leq d(M', N_1') + d(N_1', N') < 220\gamma + 2\|u - 1\| < 230\gamma.$$

The corollary follows. □

Remark 3.8. *A similar conclusion can be reached without an upper bound on $d(M, N)$. The inequality $d(M', N') \leq 517d(M, N)$ certainly holds for $d(M, N) \geq 1/517$ since $d(M', N') \leq 1$ generally, while it also holds for $d(M, N) < 1/517$ by Corollary 3.7.*

4. PERTURBATION RESULTS*

In this section, we will prove the main result of this thesis Theorem 4.11: For certain crossed product algebras of the form $P_0 \rtimes_\alpha G$ where P_0 is an injective von Neumann algebra and G is a free group, they are *-isomorphic to any nearby von Neumann algebras. By Corollary B of [21], suitable choices of actions by a free group G give type II_∞ and type III factors of the form $L^\infty(X,\mu) \rtimes_\alpha G$. Propositions 4.2-4.5 and Theorem 4.7 transfer several properties from one algebra to another close algebra, while Propositions 4.8-4.10 are used to identify a subalgebra of the crossed product algebra to be the whole algebra.

Throughout this section, for a subset $\mathcal{F}$ of a Hilbert space, $[\mathcal{F}]$ denotes the closed linear span of $\mathcal{F}$. For a von Neumann subalgebra $P \subseteq B(\mathcal{H})$ and $x \in B(\mathcal{H})$, $\overline{\mathrm{conv}_P(x)}^{WOT}$ is the closure of the set $conv\{uxu^* : u \text{ is a unitary in } P\}$ in the weak operator topology.

In proving the next two propositions, we will need the normal-singular decomposition for a bounded linear functional on a von Neumann algebra and more generally the normal-singular decomposition for a bounded linear map between two von Neumann algebras. We give a brief description here. For details, we refer the reader to the reference [25, section 10.1] or [31, section III.2].

Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let ι be the inclusion map from M into $B(\mathcal{H})$ and let π be the universal representation of M. There exists a unique ultraweakly continuous *-homomorphism $\sigma : \overline{\pi(M)}^{WOT} \to B(\mathcal{H})$ such that $\sigma \circ \pi = \iota$. Then $\ker\sigma$ is an ultraweakly closed two-sided ideal

*Reprinted with permission from "Perturbations of certain crossed product algebras by free groups" by Wai-Kit Chan, J. Funct. Anal. 267 (2014), no. 10, 3994-4027, Copyright 2014 by Elsevier.

of $\overline{\pi(M)}^{WOT}$ and hence there exists a central projection p of $\overline{\pi(M)}^{WOT}$ such that $\ker \sigma = \overline{\pi(M)}^{WOT}(1-p)$. For any bounded linear functional ϕ on M, there exists a unique ultraweakly continuous linear functional $\tilde{\phi}$ on $\overline{\pi(M)}^{WOT}$ such that $\tilde{\phi} \circ \pi = \phi$. Then the normal part ϕ_n of ϕ is defined by $\phi_n(x) = \tilde{\phi}(\pi(x)p)$ for any $x \in M$ and the singular part ϕ_s of ϕ is defined by $\phi_s(x) = \tilde{\phi}(\pi(x)(1-p))$ for any $x \in M$. Then $\phi = \phi_n + \phi_s$ will be called the normal-singular decomposition of ϕ.

Let P be another von Neumann algebra and let $\Phi : M \to P$ be a bounded linear map. There exists a unique ultraweakly continuous linear map $\tilde{\Phi} : \overline{\pi(M)}^{WOT} \to P$ such that $\Phi = \tilde{\Phi} \circ \pi$. The normal part $\Phi_n : M \to P$ of Φ is defined by $\Phi_n(x) = \tilde{\Phi}(\pi(x)p)$ for any $x \in M$ and the singular part $\Phi_s : M \to P$ of Φ is defined by $\Phi_s(x) = \tilde{\Phi}(\pi(x)(1-p))$ for any $x \in M$. Then $\Phi = \Phi_n + \Phi_s$ will be called the normal-singular decomposition of Φ. Note that if Φ is a positive linear map, then so are Φ_n and Φ_s. If $\rho \in P_*$, then $\rho \circ \tilde{\Phi}$ is ultraweakly continuous and $(\rho \circ \tilde{\Phi}) \circ \pi = \rho \circ \Phi$. For any $x \in M$, $(\rho \circ \Phi)_n(x) = \rho \circ \tilde{\Phi}(\pi(x)p) = \rho \circ \Phi_n(x)$. This shows $(\rho \circ \Phi)_n = \rho \circ \Phi_n$. A similar argument shows $(\rho \circ \Phi)_s = \rho \circ \Phi_s$.

Suppose in addition that P is a von Neumann subalgebra of M and $\Phi : M \to P$ is a P-bimodular map. Then for any $a \in P$ and $x \in M$,

$$\tilde{\Phi}(\pi(a)\pi(x)) = \Phi(ax) = a\Phi(x) = a\tilde{\Phi}(\pi(x)).$$

Since $\tilde{\Phi}$ is ultraweakly continuous and $\pi(M)$ is ultraweakly dense in $\overline{\pi(M)}^{WOT}$, we have $\tilde{\Phi}(\pi(a)y) = a\tilde{\Phi}(y)$ for any $y \in \overline{\pi(M)}^{WOT}$. Then for any $a \in P$ and $x \in M$,

$$\Phi_n(ax) = \tilde{\Phi}(\pi(a)\pi(x)p) = a\tilde{\Phi}(\pi(x)p) = a\Phi_n(x).$$

By noting that p is a central projection of $\overline{\pi(M)}^{WOT}$, a similar argument will show that $\Phi_n(xa) = \Phi_n(x)a$ for any $a \in P$ and $x \in M$. Thus Φ_n is a P-bimodular map. A similar argument gives that Φ_s is a P-bimodular map.

Proposition 4.1. *Suppose that M and N are von Neumann algebras acting on a Hilbert space $\mathcal{H}$. Assume that $N \subset_\gamma M$ for some $0 < \gamma < 1$ and that ϕ is a state on $B(\mathcal{H})$ such that $\phi|_M$ is normal. Let $\phi|_N = \phi_n + \phi_s$ be the normal-singular decomposition of $\phi|_N$. Then $\|\phi_s\| \leq \sqrt{2}\gamma$.*

Proof. Since ϕ_s is singular, there is a net of projections $\{p_i\}_{i\in\Lambda}$ in N such that $\phi_s(p_i) = 0$ for all $i \in \Lambda$ and $p_i \to 1$ in the strong operator topology by Theorem 2.15. For each $i \in \Lambda$, there exists a projection $q_i \in M$ such that $\|p_i - q_i\| < 2^{-1/2}\gamma$ by Proposition 2.27 (ii). Using the weak-operator-compactness of the unit ball of M, we may drop to a subnet and assume that $\{q_i\}_{i\in\Lambda}$ converges to some $x \in M$ in the weak operator topology. Since $\|p_i - q_i\| < 2^{-1/2}\gamma$ for all $i \in \Lambda$,

$$\|1 - x\| \leq 2^{-1/2}\gamma. \tag{4.1}$$

Since $0 \leq q_i \leq 1$ for all $i \in \Lambda$,

$$0 \leq x \leq 1. \tag{4.2}$$

For each $i \in \Lambda$,

$$|\phi_n(p_i) - \phi|_M(q_i)| = |\phi(p_i) - \phi(q_i)| \leq \|p_i - q_i\| < 2^{-1/2}\gamma.$$

Since ϕ_n and $\phi|_M$ are normal, by taking limits, we have

$$|\phi_n(1) - \phi|_M(x)| \leq 2^{-1/2}\gamma. \tag{4.3}$$

Now, from (4.1), (4.2) and (4.3), we have

$$\begin{aligned}0 \le \phi_s(1) = \phi(1) - \phi_n(1) &= \phi(1) - \phi(x) - (\phi_n(1) - \phi(x)) \\ &\le \|1 - x\| + |\phi_n(1) - \phi(x)| \le 2 \cdot 2^{-1/2}\gamma = \sqrt{2}\gamma,\end{aligned}$$

showing that $\|\phi_s\| \le \sqrt{2}\gamma$. □

Proposition 4.2. *Suppose that M and N are von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and P is an injective von Neumann subalgebra of $M \cap N$. Assume that $N \subset_\gamma M$ for some $0 < \gamma < 1/\sqrt{2}$. If there exists a normal conditional expectation Φ from M onto P, then there exists a normal conditional expectation θ from N onto P.*

Proof. Since Φ is a unital completely positive map and P is injective, there exists a unital completely positive map Ψ from $B(\mathcal{H})$ onto P such that $\Psi|_M = \Phi$. Then Ψ is a conditional expectation from $B(\mathcal{H})$ onto P and hence Ψ is a P-bimodular map. Let $\Psi|_N = \Psi_n + \Psi_s$ be the normal-singular decomposition of $\Psi|_N$. Then Ψ_n, Ψ_s are positive P-bimodular maps and their ranges lie in P. For each $0 < \lambda < 1$, there exists a normal state ρ on P such that $\rho \circ \Psi_s(1) \ge \lambda\|\Psi_s(1)\| = \lambda\|\Psi_s\|$. Then $\rho \circ \Psi|_N = \rho \circ \Psi_n + \rho \circ \Psi_s$ is the normal-singular decomposition of $\rho \circ \Psi|_N$. By Proposition 4.1, $\|\rho \circ \Psi_s\| \le \sqrt{2}\gamma$. Thus $\lambda\|\Psi_s\| \le \sqrt{2}\gamma < 1$. By letting $\lambda \to 1$, we have $\|\Psi_s\| \le \sqrt{2}\gamma < 1$. Then $\|1 - \Psi_n(1)\| = \|\Psi_s(1)\| < 1$ and $\Psi_n(1)$ is invertible in P. Define $\theta : N \to P$ by

$$\theta(x) = \Psi_n(1)^{-1/2}\Psi_n(x)\Psi_n(1)^{-1/2}.$$

Since $\Psi_n(1)$ lies in the center of P and Ψ_n is a P-bimodular map, $\theta(a) = a$ for any $a \in P$. Also, as θ is a positive linear map, $\|\theta\| = \|\theta(1)\| = 1$. Thus θ is a normal conditional expectation from N onto P. □

In this section, we will need the equivalence between Schwartz Property (P) and injectivity for a general von Neumann algebra. Recall that a von Neumann algebra $M \subseteq B(\mathcal{H})$ has Schwartz Property (P) if $\overline{\mathrm{conv}_M(x)}^{WOT} \cap M' \neq \emptyset$ for all $x \in B(\mathcal{H})$. It can be shown that a von Neumann algebra M having Schwartz Property (P) has an injective commutant M' [25, Exercise 8.7.24] and hence M is injective [1, Theorem IV.2.2.7]. Thus, a von Neumann algebra having Schwartz Property (P) is injective. The backward direction was obtained among the equivalence of other properties for a von Neumann algebra. A von Neumann algebra M is approximately finite dimensional if there is an increasing sequence of finite dimensional *-subalgebras whose union is ultraweakly dense in M. When M is a factor with separable predual, Connes proved that M has Schwartz Property (P) if and only if M is injective [16, Theorem 6] among other properties including approximate finite dimensionality. When M is a von Neumann algebra with separable predual, injectivity of M implies that M is approximately finite dimensional [20, Theorem 2.2, Theorem 5.3 and section 6.3] and hence has Schwartz Property (P). That injectivity implies Schwartz Property (P) for a general von Neumann algebra follows from a result of Elliott [18, Theorem 4]. When the subalgebra $P \subseteq M$ is abelian, this section proceeds without the use of equivalence between Schwartz Property (P) and injectivity.

The following proof is very similar to [3, Lemma 2.16(i)] except that the use of conditional expectations is replaced by the use of the Schwartz Property (P).

Proposition 4.3. *Suppose that M and N are von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and P is an injective von Neumann subalgebra of $M \cap N$. Assume that $N \subset_\gamma M$ for some $0 < \gamma < 1$. If $P' \cap M \subseteq P$, then $P' \cap N = Z(P)$.*

Proof. Note that under the condition $P' \cap M \subseteq P$, we have $P' \cap M = Z(P)$. Fix any $x \in P' \cap N$ with $\|x\| \leq 1$. There exists $y \in M$ such that $\|x - y\| \leq \gamma$. For any unitary $u \in P$,

$$\|x - uyu^*\| = \|uxu^* - uyu^*\| \leq \|x - y\| \leq \gamma.$$

Thus, $\|x - z\| \leq \gamma$ for any $z \in \overline{\text{conv}_P(y)}^{WOT}$. Since P has the Schwartz Property (P), there exists $z_0 \in \overline{\text{conv}_P(y)}^{WOT} \cap P'$, and $z_0 \in M \cap P' = Z(P)$. We have that $\|x - z_0\| \leq \gamma$ with $z_0 \in Z(P)$. This shows that $P' \cap N \subseteq_\gamma Z(P)$. Since $Z(P) \subseteq P' \cap N$ and $P' \cap N \subseteq_\gamma Z(P)$ with $\gamma < 1$, we have $P' \cap N = Z(P)$. $\square$

Suppose that a von Neumann algebra M has a unit cyclic and separating vector ξ in a Hilbert space $\mathcal{H}$ and let $S_0 : M\xi \to \mathcal{H}$ be the map $S_0 x\xi = x^*\xi$ for any x in M. Then S_0 is preclosed by Theorem 2.19 and let S be the closure of S_0. Let $S = J\Delta^{1/2}$ be the polar decomposition of S, where Δ is called the modular operator associated with M and ξ and J is called the modular conjugation associated with M and ξ. Tomita's Theorem (See Theorem 2.19) states that J is a conjugate-linear isometry onto $\mathcal{H}$, $JMJ = M'$ and $\Delta^{it} M \Delta^{-it} = M$ for any $t \in \mathbb{R}$. Note that we also have $J^2 = 1_{\mathcal{H}}$ and $J\xi = \Delta\xi = \xi$ by Theorem 2.19.

The following proposition is known to the experts. We include the proof of it for the convenience of the reader.

Proposition 4.4. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and let P be a von Neumann subalgebra of M. Suppose that M has a unit cyclic and separating vector ξ and that there exists a normal conditional expectation E from M onto P such that $\langle E(x)\xi, \xi \rangle = \langle x\xi, \xi \rangle$ for any x in M. Assume also that ξ is a tracial vector for P, i.e $\langle xy\xi, \xi \rangle = \langle xy\xi, \xi \rangle$ for any $x, y \in P$. Let e_P be the orthogonal projection from $\mathcal{H}$ onto $[P\xi]$. Let Δ (resp. J) be the modular operator (resp. modular conjugation) associated with M and ξ. Then the following hold:*

(i) $E(x)\xi = e_P x\xi$, *for any x in M.*

(ii) $J a\xi = a^*\xi$, *for any* a *in* P *and* $Je_P = e_P J$.

(iii) $P = \{e_P\}' \cap M$.

(iv) $JPJ = (M \cup \{e_P\})'$.

(v) $e_P x e_P = E(x)e_P$ *for any* x *in* M.

Proof. (i) For any $x \in M$ and $a \in P$,

$$\begin{aligned}\langle E(x)\xi, a\xi\rangle &= \langle a^* E(x)\xi, \xi\rangle = \langle E(a^*x)\xi, \xi\rangle \\ &= \langle a^* x\xi, \xi\rangle = \langle x\xi, a\xi\rangle = \langle e_P x\xi, a\xi\rangle .\end{aligned}$$

Since $E(x)\xi$ and $e_P x\xi$ are elements of $[P\xi]$, we have $E(x)\xi = e_P x\xi$.

(ii) For any $x \in M$ and $a \in P$,

$$\langle ax\xi, \xi\rangle = \langle E(ax)\xi, \xi\rangle = \langle aE(x)\xi, \xi\rangle \overset{(*)}{=} \langle E(x)a\xi, \xi\rangle = \langle E(xa)\xi, \xi\rangle = \langle xa\xi, \xi\rangle ,$$

where $(*)$ holds since ξ is a tracial vector for P. By Proposition 2.20, we have $\Delta^{it} a \Delta^{-it} = a$ for any $t \in \mathbb{R}$. By Stone's Theorem [24, Theorem 5.6.36], there exists a closed (possibly unbounded) self-adjoint operator h such that $\exp(ith)=\Delta^{it}$ for any $t \in \mathbb{R}$,

$$\begin{aligned}&\mathrm{dom}(h) = \{\xi_1 \in \mathcal{H} : \lim_{t\to 0} \frac{\Delta^{it}\xi_1 - \xi_1}{t} \text{ exists in the norm topology}\} \text{ and} \\ &ih\xi_1 = \lim_{t\to 0} \frac{\Delta^{it}\xi_1 - \xi_1}{t} \text{ for any } \xi_1 \in \mathrm{dom}(h).\end{aligned}$$

For any $\xi_1 \in \mathrm{dom}(h)$ and $a \in P$,

$$\lim_{t\to 0} \frac{\Delta^{it} a\xi_1 - a\xi_1}{t} = \lim_{t\to 0} \frac{a(\Delta^{it}\xi_1 - \xi_1)}{t} = iah\xi_1.$$

This shows that $a\xi_1 \in \mathrm{dom}(h)$ and $ha\xi_1 = ah\xi_1$. From this, we can see that $ah \subseteq ha$ for any $a \in P$. Thus h is affiliated to P'. By the proof of [25, Theorem 9.2.16], $\exp(h)=\Delta$, hence we have $\Delta^{1/2} = exp(\frac{1}{2}h)$ is affiliated to P' and $a\Delta^{1/2} \subseteq \Delta^{1/2}a$. For any $a \in P$,

$$Ja\xi = Ja\Delta^{1/2}\xi = J\Delta^{1/2}a\xi = a^*\xi.$$

For the second assertion, we now have $Je_P = e_PJe_P$, and by taking adjoints, we have $e_PJ = e_PJe_P = Je_P$.

(iii) $[P\xi]$ is invariant for P, and so e_P lies in P'. Thus $P \subseteq \{e_P\}' \cap M$. Conversely, if $x \in \{e_P\}' \cap M$, then $x\xi = xe_P\xi = e_Px\xi = E(x)\xi$, by (i). Since ξ is a separating vector for M, we have $x = E(x) \in P$.

(iv) By (iii), $JPJ = \{Je_PJ\}' \cap JMJ = \{e_P\}' \cap M'$.

(v) For any $x \in M$ and $a \in P$,

$$e_Pxe_Pa\xi = e_Pxa\xi = E(xa)\xi = E(x)a\xi = E(x)e_Pa\xi. \qquad \square$$

The following proposition is the corresponding version of [3, Lemma 2.10] that we will need for general von Neumann algebras M. The proof is basically the same as the proof of [19, Lemma 3.2] when M is a finite von Neumann algebra.

Proposition 4.5. *Let $P \subseteq M$ be von Neumann algebras satisfying the hypotheses of Proposition 4.4. Further assume that P is injective and $P' \cap M \subseteq P$. Then, for each $y \in P' \cap (M \cup \{e_P\})''$, there exists $a \in P$ such that $y\xi = a\xi$.*

Proof. Let y be an element in $P' \cap (M \cup \{e_P\})''$ and let $\eta = y\xi$. Pick a sequence (x_n) in M such that $\|x_n\xi - \eta\| \to 0$. Then for any unitary $u \in P$,

$$JuJu\eta = JuJuy\xi = yJuJu\xi = y\xi = \eta, \tag{4.4}$$

where the second equality follows from $y \in P'$ and JuJ is in the commutant of $(M \cup \{e_P\})''$ by Proposition 4.4(iv) and the third equality follows from Proposition 4.4(ii). Thus, from (4.4), we have

$$\|JuJux_n\xi - \eta\| = \|JuJu(x_n\xi - \eta)\| = \|x_n\xi - \eta\|. \tag{4.5}$$

Since $JuJux_n\xi = ux_nJuJ\xi = ux_nu^*\xi$ by Proposition 4.4(ii), (iv), from (4.5), for any $w \in \overline{\mathrm{conv}_P(x_n)}^{WOT}$, we have

$$\|w\xi - \eta\| \leq \|x_n\xi - \eta\|.$$

Since P has Schwartz Property (P), for each positive integer n, there exists $y_n \in \overline{\mathrm{conv}_P(x_n)}^{WOT} \cap P' \subseteq M \cap P' \subseteq P$. We have $\|y_n\xi - \eta\| \to 0$ as $n \to \infty$. Then for any $b \in P$,

$$yb\xi = by\xi = \lim_{n\to\infty} by_n\xi.$$

This shows that

$$ye_P = e_Pye_P \in e_P(M \cup \{e_P\})''e_P = Pe_P$$

by Proposition 4.4(v). Thus there exists $a \in P$ such that $ye_P = ae_P$. Then $y\xi = a\xi$. □

Lemma 4.6. *Let P be an injective von Neumann subalgebra of a von Neumann algebra N acting on a Hilbert space $\mathcal{H}$ such that $P' \cap N \subseteq P$. Suppose that there exists a normal conditional expectation E from N onto P. Assume that there exists a vector $\xi \in \mathcal{H}$ such that $\langle uxu^*\xi, \xi\rangle = \langle x\xi, \xi\rangle$ for all unitaries $u \in P$ and elements $x \in N$. Then $\langle E(x)\xi, \xi\rangle = \langle x\xi, \xi\rangle$. Moreover, if ξ is separating for P, then ξ is separating for N.*

Proof. Let u be a unitary in P and let x be an element in N. By the hypothesis, $\langle uxu^*\xi, \xi\rangle = \langle x\xi, \xi\rangle$ and hence we have

$$\langle y\xi, \xi\rangle = \langle x\xi, \xi\rangle \text{ for any } y \in \overline{\mathrm{conv}_P(x)}^{WOT}. \tag{4.6}$$

Also, since E is a P-bimodular map and by the hypothesis

$$\langle E(uxu^*)\xi, \xi\rangle = \langle uE(x)u^*\xi, \xi\rangle = \langle E(x)\xi, \xi\rangle .$$

The normality of E then implies that

$$\langle E(y)\xi, \xi\rangle = \langle E(x)\xi, \xi\rangle \text{ for any } y \in \overline{\mathrm{conv}_P(x)}^{WOT}. \tag{4.7}$$

Since P has Schwartz Property (P), there exists $y_0 \in \overline{\mathrm{conv}_P(x)}^{WOT} \cap P' \subseteq P$. Then combining equations (4.6) and (4.7), we have

$$\langle E(x)\xi, \xi\rangle = \langle E(y_0)\xi, \xi\rangle = \langle y_0\xi, \xi\rangle = \langle x\xi, \xi\rangle . \tag{4.8}$$

This shows the first assertion.

Suppose also that ξ is separating for P. If $x \in N$ is such that $x\xi = 0$, then by (4.8),

$$\|E(x^*x)^{1/2}\xi\|^2 = \langle E(x^*x)\xi, \xi\rangle = \langle x^*x\xi, \xi\rangle = 0.$$

Since ξ is separating for P, $E(x^*x)^{1/2} = 0$ and $E(x^*x) = 0$. Since E is faithful by Proposition 2.18, $x^*x = 0$ and hence $x = 0$. This shows that ξ is separating for N. $\square$

In the following theorem, when P is a von Neumann subalgebra of a von Neumann algebra M, we denote by $\mathcal{N}(P \subseteq M)$ the set $\{u \in M : u \text{ unitary in } M \text{ and } uPu^* = P\}$ of unitary normalizers.

The following theorem can be thought of as a generalization of [3, Lemma 4.4] from a type II_1 factor M to general von Neumann algebras M.

Theorem 4.7. *Let M and N be von Neumann algebras acting on a Hilbert space $\mathcal{H}$. Suppose that M has a unit cyclic and separating vector ξ and let J_M be the modular conjugation associated with M and ξ. Suppose that P is an injective von Neumann subalgebra of M such that $P' \cap M \subseteq P$, $P \subseteq M \cap N$ and $J_M P J_M \subseteq M' \cap N'$. Assume that there exists a normal conditional expectation E_P^M from M onto P such that $\left\langle E_P^M(x)\xi, \xi \right\rangle = \langle x\xi, \xi \rangle$ for any $x \in M$. Suppose also that $\langle xy\xi, \xi\rangle = \langle yx\xi, \xi\rangle$ for any x,$y \in P$. Assume that $d(M, N) < \gamma < 1/7 < 1/(2\sqrt{2}(1 + \sqrt{2}))$. Then*

(i) There exists a normal conditional expectation E_P^N from N onto P and $\left\langle E_P^N(x)\xi, \xi \right\rangle = \langle x\xi, \xi\rangle$ for any $x \in N$.

(ii) ξ is a cyclic and separating vector for N.

(iii) Let J_N be the modular conjugation associated with N and ξ. Let e_P be the orthogonal projection from $\mathcal{H}$ onto $[P\xi]$. Then $J_M x J_M = J_N x J_N$ for any element $x \in P$ and $(M \cup \{e_P\})' = J_M P J_M = J_N P J_N = (N \cup \{e_P\})'$.

(iv) If $u \in \mathcal{N}(P \subseteq M)$, then there exists a normalizing unitary $v \in \mathcal{N}(P \subseteq N)$ such that $uxu^* = vxv^*$ for any $x \in P$, $[uP\xi] = [vP\xi]$ and $\|u - v\| \leq (4 + 2\sqrt{2})\gamma$.

Proof. (i) Applying Proposition 4.2 to the near inclusion $N \subset_\gamma M$ with the common injective subalgebra P, there exists a normal conditional expectation E_P^N from N onto P. Since $P' \cap M \subseteq P$ and $N \subset_\gamma M$, by Proposition 4.3, $P' \cap N \subseteq P$. For any unitary u in P and x in N,

$$\langle uxu^*\xi,\xi\rangle = \langle xu^*\xi,u^*\xi\rangle = \langle xJ_MuJ_M\xi,J_MuJ_M\xi\rangle = \langle x\xi,(J_Mu^*J_M)(J_MuJ_M)\xi\rangle = \langle x\xi,\xi\rangle$$

by Proposition 4.4(ii) and $J_MPJ_M \subseteq N'$. By Lemma 4.6, $\left\langle E_P^N(x)\xi,\xi\right\rangle = \langle x\xi,\xi\rangle$ and ξ is separating for N.

(ii) We showed that ξ is separating in the proof of (i). To show that ξ is cyclic for N, it suffices to show that ξ is separating for N'. Let $Q = J_MPJ_M$. Since ξ is separating for $M' = J_MMJ_M$ and Q is a subalgebra of M', ξ is separating for Q. Note that Q is a subalgebra of N' by the hypothesis. Also, Q is injective since P is injective. Moreover,

$$Q'\cap M' = J_M(P'\cap M)J_M \subseteq J_MPJ_M = Q. \tag{4.9}$$

Since $M \subset_\gamma N$ and M has a cyclic vector, we have $N' \subset_{2(1+\sqrt{2})\gamma} M'$ by Proposition 2.30. Now applying Proposition 4.3 to the near inclusion $N' \subset_{2(1+\sqrt{2})\gamma} M'$ with the common injective subalgebra Q, we conclude that $Q'\cap N' \subseteq Q$ from equation (4.9).

Define $E_Q^{M'} : M' \to Q$ by

$$E_Q^{M'}(x) = J_ME_P^M(J_Mx^*J_M)^*J_M \text{ for any } x \in M'.$$

Then $E_Q^{M'}$ is a normal conditional expectation from M' onto Q. Applying Proposition 4.2 to the near inclusion $N' \subset_{2(1+\sqrt{2})\gamma} M'$ with the common injective subalgebra Q, there exists a normal conditional expectation $E_Q^{N'}$ from N' onto Q. Moreover, for any element y in N' and unitary v in Q, we have $v = J_Mu^*J_M$ for some unitary

$u \in P$ and

$$\langle vyv^*\xi, \xi\rangle = \langle J_M u^* J_M y J_M u J_M \xi, \xi\rangle = \langle y J_M u J_M \xi, J_M u J_M \xi\rangle \overset{(\star)}{=} \langle yu^*\xi, u^*\xi\rangle \overset{(\star\star)}{=} \langle yuu^*\xi, \xi\rangle = \langle y\xi, \xi\rangle \tag{4.10}$$

where equation $(\star)$ holds since $J_M u\xi = u^*\xi$ by Proposition 4.4(ii) and equation $(\star\star)$ holds since $y \in N'$, $u \in P \subseteq N$.

To sum up, we showed that Q is an injective von Neumann subalgebra of N' such that $Q' \cap N' \subseteq Q$ and there exists a normal conditional expectation $E_Q^{N'}$ from N' onto Q. Also, ξ is separating for Q and equation (4.10) holds for any element $y \in N'$ and unitary $v \in Q$. Now apply Lemma 4.6 to the inclusion $Q \subseteq N'$ to conclude that ξ is separating for N'.

(iii) Applying Proposition 4.4(ii) to the inclusion $P \subseteq M$, we have $J_M x\xi = x^*\xi$ for any element $x \in P$. Similarly, we have $J_N x\xi = x^*\xi$ for any element $x \in P$. Then for any element $x \in P$, $J_M x J_M \xi = J_M x\xi = x^*\xi = J_N x\xi = J_N x J_N \xi$. Since $J_M x J_M, J_N x J_N$ are elements of N' and ξ is separating by (ii), $J_M x J_M = J_N x J_N$. This shows the first assertion and the second assertion follows from it and Proposition 4.4(iv).

(iv) By [3, Lemma 3.4(iii)] and its proof, there exists a normalizing unitary $v \in \mathcal{N}(P \subseteq N)$ such that $\|u - v\| \leq (4 + 2\sqrt{2})\gamma$ and $uxu^* = vxv^*$ for any $x \in P$. Thus $u^*v \in P' \cap (M \cup \{e_P\})''$ by (iii). Then by Proposition 4.5, there exists $a \in P$ such that $u^*v\xi = a\xi$ and $[vP\xi] = [uu^*vP\xi] = [uPu^*v\xi] = [uPa\xi] \subseteq [uP\xi]$. By reversing the roles of u and v, the same argument shows that $[uP\xi] \subseteq [vP\xi]$. □

The following notations will be used from Proposition 4.8 to Theorem 4.11.

Let P_0 be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$. Let G be a discrete group with identity element e and let $g \mapsto \alpha_g : G \to Aut(P_0)$ be a free action on P_0. Recall that the freeness of α means that for any $g \in G$ with $g \neq e$, if $a \in P_0$ is such that $a\alpha_g(b) = ba$ for every $b \in P_0$, then $a = 0$. For any $g \in G$, let $\delta_g \in \ell^2(G)$ be the characteristic function of $\{g\}$, let p_g be the orthogonal projection of $\ell^2(G)$ onto $\mathbb{C}\delta_g$ and let $\lambda_g \in B(\ell^2(G))$ be the unitary operator determined by $\lambda_g(\delta_h) = \delta_{gh}$ for any $h \in G$. For any $a \in P_0$, define $\pi(a) = \sum_{g \in G} \alpha_{g^{-1}}(a) \otimes p_g$. For any $g \in G$, define $u_g = 1_{\mathcal{H}_0} \otimes \lambda_g \in B(\mathcal{H}_0 \otimes \ell^2(G))$. Then $P_0 \rtimes_\alpha G$ is the von Neumann algebra generated by $\{\pi(a) : a \in P_0\} \cup \{u_g : g \in G\}$. It is folklore that the freeness of the action implies $\pi(P_0)' \cap (P_0 \rtimes_\alpha G) \subseteq \pi(P_0)$. Indeed, it follows easily from Proposition 4.9 (i) and (ii).

The following proposition is well-known and so we omit the proof.

Proposition 4.8. *Let P_0 be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and ξ_0 is a unit cyclic and separating vector for P_0. Let G be a discrete group and $g \mapsto \alpha_g : G \to Aut(P_0)$ be an action on P_0 and let $\xi = \xi_0 \otimes \delta_e \in \mathcal{H}_0 \otimes \ell^2(G)$. Define $\phi : P_0 \rtimes_\alpha G \to P_0$ by*

$$\langle \phi(x)\eta_1, \eta_2\rangle = \langle x(\eta_1 \otimes \delta_e), \eta_2 \otimes \delta_e\rangle \text{ for any } x \in P_0 \rtimes_\alpha G, \eta_1, \eta_2 \in \mathcal{H}_0.$$

and define $E : P_0 \rtimes_\alpha G \to \pi(P_0)$ by $E = \pi \circ \phi$. Then $\phi(\pi(a)) = a$ and $\phi(\pi(a)u_g) = 0$ for all $a \in P_0$ and $g \in G \setminus \{e\}$. Also, ξ is a unit cyclic and separating vector for $P_0 \rtimes_\alpha G$ and E is a faithful normal conditional expectation from $P_0 \rtimes_\alpha G$ onto $\pi(P_0)$ such that $\langle E(x)\xi, \xi\rangle = \langle x\xi, \xi\rangle$ for any $x \in P_0 \rtimes_\alpha G$.

Note that, in part (v) of the following proposition, we do not assert that N_g is closed in the norm topology or weak operator topology.

Proposition 4.9. *Let P_0 be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and ξ_0 is a unit cyclic and separating vector for P_0. Let G be a discrete group and*

$g \mapsto \alpha_g : G \to Aut(P_0)$ be an action on P_0 and let $\xi = \xi_0 \otimes \delta_e \in \mathcal{H}_0 \otimes \ell^2(G)$. Let $\phi : P_0 \rtimes_\alpha G \to P_0$ be the positive normal linear map constructed in Proposition 4.8. Then the following hold: for any $a \in P_0$, $x, y \in P_0 \rtimes_\alpha G$ and $g \in G$,

(i) $\phi(\pi(a)xu_g^*) = a\phi(xu_g^*)$.

(ii) $\phi(x\pi(a)u_g^*) = \phi(xu_g^*)\alpha_g(a)$.

(iii) $\phi(xu_g^*) = \alpha_g(\phi(u_g^*x))$.

(iv) If $\phi(xu_k) = \phi(yu_k)$ for all $k \in G$, then $x = y$.

*(v) If N is a von Neumann subalgebra of $P_0 \rtimes_\alpha G$ containing $\pi(P_0)$ and $[N\xi] = \mathcal{H}_0 \otimes \ell^2(G)$, then the set $N_g = \{\phi(xu_g^*) : x \in N\}$ is a *-ideal in P_0 and its weak operator closure contains $1_{\mathcal{H}_0}$.*

Proof. (i)-(iv) Parts (i)-(iv) are standard results and so we omit the proof.

(v) From (i) and (ii), N_g is an ideal of P_0. Since any ideal in a von Neumann algebra is a *-ideal [25, Proposition 6.8.9], N_g is a *-ideal. If $\eta \in \mathcal{H}_0$ is orthogonal to the subspace $\{\phi(u_g^*x)\xi_0 : x \in N\}$, then

$$0 = \left\langle \phi(u_g^*x)\xi_0, \eta \right\rangle = \left\langle u_g^*x(\xi_0 \otimes \delta_e), \eta \otimes \delta_e \right\rangle = \left\langle x(\xi_0 \otimes \delta_e), \eta \otimes \delta_g \right\rangle.$$

Since $[N\xi] = \mathcal{H}_0 \otimes \ell^2(G)$, $\eta \otimes \delta_g = 0$ and hence $\eta = 0$. This shows that $\{\phi(u_g^*x)\xi_0 : x \in N\}$ is dense in $\mathcal{H}_0$. From (iii), $\{\phi(u_g^*x) : x \in N\} = {\alpha_g}^{-1}(N_g)$ is a *-subalgebra of P_0. Since $\{\phi(u_g^*x) : x \in N\}$ has a cyclic vector ξ_0, its weak operator closure contains $1_{\mathcal{H}_0}$. Since α_g is a *-automorphism of P_0 and $N_g = \alpha_g(\{\phi(u_g^*x) : x \in N\})$, the weak operator closure of N_g contains $1_{\mathcal{H}_0}$. □

When $N \subseteq M \subseteq B(\mathcal{H})$ are finite von Neumann algebras with the same cyclic and separating vector, N must equal M. The same conclusion may fail when M is not finite [17, Proposition 1.2].

Proposition 4.10. *Let P_0 be an injective von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and ξ_0 is a unit cyclic tracial vector for P_0. Let G be a discrete group, let $g \mapsto \alpha_g : G \to Aut(P_0)$ be a free action on P_0 and let $\xi = \xi_0 \otimes \delta_e \in \mathcal{H}_0 \otimes \ell^2(G)$. Let N be a von Neumann subalgebra of $P_0 \rtimes_\alpha G$ containing $\pi(P_0)$. If $[N\xi] = \mathcal{H}_0 \otimes \ell^2(G)$, then $N = P_0 \rtimes_\alpha G$.*

Proof. Since ξ_0 is a cyclic tracial vector for P_0, ξ_0 is separating for P_0 by [25, Lemma 7.14] and P_0 is a finite von Neumann algebra. Let T be the center-valued trace of P_0 and P be $\pi(P_0)$. Since P_0 is injective, P is injective and hence has Schwartz Property (P).

Let $x \in N$ and $g \in G$. Since P has Schwartz Property (P), there exists $y \in \overline{\mathrm{conv}_P(xu_g^*)}^{WOT} \cap P'$. Then $y \in \overline{\mathrm{conv}_P(xu_g^*)}^{WOT} \cap P' \subseteq (P_0 \rtimes_\alpha G) \cap P' = Z(P)$ where the last equality holds by freeness of the action. Thus

$$y = \pi(y_1) \ \text{ for some element } y_1 \in Z(P_0). \tag{4.11}$$

For any unitary $v \in P$, $vxu_g^*v^*u_g \in N$ since $u_g^*Pu_g = P$, $v \in P$, $x \in N$ and N contains P. Since $y \in \overline{\mathrm{conv}_P(xu_g^*)}^{WOT}$, it follows that

$$yu_g \in N. \tag{4.12}$$

For any unitary $w \in P_0$, $\phi(\pi(w)xu_g^*\pi(w^*)) = w\phi(xu_g^*)w^*$ by Proposition 4.9 (i), (ii) and $u_g^*\pi(w^*)u_g = \pi(\alpha_{g^{-1}}(w^*))$. Thus, by (4.11),

$$y_1 = \phi(y) \in \overline{\mathrm{conv}_{P_0}(\phi(xu_g^*))}^{WOT}. \tag{4.13}$$

Since T is the center-valued trace of P_0, T is a normal linear map and T maps $\overline{\mathrm{conv}_{P_0}(\phi(xu_g^*))}^{WOT}$ onto the singleton set $\{T(\phi(xu_g^*))\}$. Since $y_1 \in Z(P_0)$ and by (4.13), $y_1 = T(y_1) = T(\phi(xu_g^*))$. Hence, by (4.11) and (4.12),

$$\pi(T(\phi(xu_g^*)))u_g = \pi(y_1)u_g = yu_g \in N \text{ for all } x \in N \text{ and } g \in G. \tag{4.14}$$

By Proposition 4.9 (v) and the Kaplansky density theorem, there exists a net $\{x_i\}$ in N such that $\|\phi(x_iu_g^*)\| \leq 1$ and $\phi(x_iu_g^*) \to 1_{\mathcal{H}_0}$ in the weak operator topology. Since T and π are weak-operator continuous on bounded sets, $\pi(T(\phi(x_iu_g^*)))u_g \to \pi(T(1_{\mathcal{H}_0}))u_g = u_g$ in the weak operator topology and hence, by (4.14), $u_g \in N$. Since $u_g \in N$ for all $g \in G$ and N contains P, $N = P_0 \rtimes_\alpha G$. $\square$

In the proof of the next theorem, we will adopt the notations used in the paragraph before Proposition 4.8. The construction of the spatial isomorphism $\mathrm{Ad}(w)$ from the crossed product algebra uses the idea of Choda [6, Lemma 2].

Theorem 4.11. *Let P_0 be an injective von Neumann algebra acting on a Hilbert space $\mathcal{H}_0$ and let ξ_0 be a unit cyclic tracial vector for P_0. Let G be a free group with a free generating set Λ and let $g \mapsto \alpha_g : G \to Aut(P_0)$ be a free action on P_0. Let $M = P_0 \rtimes_\alpha G$ and let N be another von Neumann algebra acting on $\mathcal{H}_0 \otimes \ell^2(G)$ such that $d(M,N) < \gamma < \frac{1}{7} \cdot 10^{-7}$. Then M and N are spatially isomorphic.*

Proof. Let $\xi = \xi_0 \otimes \delta_e \in \mathcal{H}_0 \otimes \ell^2(G)$, $P = \pi(P_0)$ and $\mathcal{H} = \mathcal{H}_0 \otimes \ell^2(G)$. Note that $P' \cap M \subseteq P$ since α is a free action on P_0. Note also that ξ_0 is separating for P_0 since it is a cyclic tracial vector by [25, Lemma 7.2.14].

By Proposition 4.8, ξ is a cyclic and separating vector for M and there exists a faithful normal conditional expectation E_P^M from M onto P such that $\langle E_P^M(x)\xi, \xi\rangle = \langle x\xi, \xi\rangle$ for any $x \in M$. Since ξ_0 is a tracial vector for P_0, ξ is also a tracial vector for

P. Let J_M be the modular conjugation associated with M and ξ. Let Z_M (resp. Z_N) be the center of M (resp. N).

Since $d(M,N) < \gamma < 1/6$, by Proposition 2.31, there exists a unitary $w_1 \in B(\mathcal{H})$ such that $w_1 Z_M w_1^* = Z_N$ and $\|w_1 - 1\| \leq 5\gamma$. Let $N_1 = w_1^* N w_1$. Then M and N_1 have common center and

$$d(M, N_1) \leq 2\|w_1 - 1\| + d(M,N) < 11\gamma.$$

Since P is an injective von Neumann algebra, $P \subseteq M \subset_{11\gamma} N_1$ and $11\gamma < 1/100$, by Theorem 2.35 (i), there exists a unitary $w_2 \in \{P \cup N_1\}''$ such that $w_2 P w_2^* \subseteq N_1$ and $\|w_2 - 1\| \leq 150 \cdot 11\gamma$. Let $N_2 = w_2^* N_1 w_2$. Since $Z_M = Z_{N_1}$, $w_2 \in \{P \cup N_1\}'' \subseteq Z_M{}'$. Thus

$$Z_{N_2} = w_2^* Z_{N_1} w_2 = w_2^* Z_M w_2 = Z_M$$

and $P \subseteq N_2$.

$$d(M, N_2) \leq 2\|w_2 - 1\| + d(M, N_1) < 300 \cdot 11\gamma + 11\gamma = 301 \cdot 11\gamma.$$

Since M and N_2 have common center and M has a cyclic vector and $d(M, N_2) < 301 \cdot 11\gamma < 1/47$, by Proposition 3.6,

$$M' \subset_{4(1+\sqrt{2}) \cdot 301 \cdot 11\gamma} N_2{}'.$$

Since $J_M P J_M \subseteq M'$, we have

$$J_M P J_M \subset_{4(1+\sqrt{2}) \cdot 301 \cdot 11\gamma} N_2{}'.$$

By Theorem 2.35 (i), there exists a unitary $w_3 \in \{J_M P J_M \cup N_2{}'\}''$ such that

$w_3 J_M P J_M w_3^* \subseteq {N_2}'$ and $\|w_3 - 1\| \leq 150 \cdot 4(1+\sqrt{2}) \cdot 301 \cdot 11\gamma$. Let $N_3 = w_3^* N_2 w_3$. Then

$$d(N_3, M) \leq 2\|w_3 - 1\| + d(N_2, M) < 300 \cdot 4(1+\sqrt{2}) \cdot 301 \cdot 11\gamma + 301 \cdot 11\gamma = \gamma_1 < 10^7 \cdot \gamma.$$

Now, $w_3 \in \{J_M P J_M \cup {N_2}'\}'' \subseteq \{M' \cup \ {N_2}'\}'' = (M \cap N_2)' \subseteq P'$, so $P = w_3^* P w_3 \subseteq w_3^* N_2 w_3 = N_3$. By the choice of w_3, $J_M P J_M \subseteq {N_3}'$.

Thus we have $P \subseteq N_3$, $J_M P J_M \subseteq {N_3}'$, $d(M, N_3) < \gamma_1 < 1/7$, N_3 and N are spatially isomorphic. Now we can apply Theorem 4.7 to the pair M and N_3. We have ξ is a cyclic and separating vector for N_3 and there exists a faithful normal conditional expectation $E_P^{N_3}$ from N_3 onto P such that

$$\left\langle E_P^{N_3}(x)\xi, \xi \right\rangle = \langle x\xi, \xi \rangle \text{ for all } x \in N_3. \tag{4.15}$$

For each $g \in G$, by Theorem 4.7 (iv), there exists a unitary $v_g \in \mathcal{N}(P \subseteq N_3)$ such that

$$u_g x u_g^* = v_g x v_g^* \text{ for any } x \in P \text{ and} \tag{4.16}$$

$$[u_g P \xi] = [v_g P \xi]. \tag{4.17}$$

For each $g \in G$, define an automorphism $\theta_g \in Aut(P)$ by

$$\theta_g(x) = u_g x u_g^* \text{ for any } x \in P. \tag{4.18}$$

Then the map $g \mapsto \theta_g : G \to Aut(P)$ is a group homomorphism. Since G is free on the set Λ, there exists a group homomorphism $g \mapsto \tilde{v}_g : G \mapsto \mathcal{N}(P \subseteq N_3)$ such that

$\tilde{v}_g = v_g$ for any $g \in \Lambda$. For each $g \in G$, define $\phi_g \in Aut(P)$ by

$$\phi_g(x) = \tilde{v}_g x \tilde{v}_g^* \text{ for any } x \in P. \tag{4.19}$$

Then $g \mapsto \phi_g : G \to Aut(P)$ is a group homomorphism. Since $\theta_g = \phi_g$ for any g in a generating set Λ for the group G, we have $\theta_g = \phi_g$ for any $g \in G$. So for any $g \in G$ and $x \in P$, by (4.16),

$$v_g x v_g^* = u_g x u_g^* = \tilde{v}_g x \tilde{v}_g^*. \tag{4.20}$$

Then $\tilde{v}_g^* v_g \in P' \cap N_3$. Applying Proposition 4.3 to the near inclusion $N_3 \subseteq_\gamma M$ and $P' \cap M \subseteq P$, we have $P' \cap N_3 \subseteq P$, so $\tilde{v}_g^* v_g \in P' \cap N_3 \subseteq P$. Thus $[\tilde{v}_g P\xi] = [v_g P\xi] = [u_g P\xi]$ by (4.17). Since $\sum_{g \in G} [u_g P\xi] = \mathcal{H}$, there exists a unitary $w \in B(\mathcal{H})$ such that

$$w u_g a\xi = \tilde{v}_g a\xi \text{ for all } a \in P \text{ and } g \in G. \tag{4.21}$$

For all g, $h \in G$, $a \in P$,

$$w u_g u_h a\xi = w u_{gh} a\xi = \tilde{v}_{gh} a\xi = \tilde{v}_g \tilde{v}_h a\xi = \tilde{v}_g w u_h a\xi,$$

hence

$$w u_g = \tilde{v}_g w \text{ for all } g \in G. \tag{4.22}$$

For all $a, b \in P$, $g \in G$, by (4.20) and (4.21),

$$w a u_g b\xi = w u_g u_{g^{-1}} a u_g b\xi = \tilde{v}_g u_{g^{-1}} a u_g b\xi = \tilde{v}_g \tilde{v}_{g^{-1}} a \tilde{v}_g b\xi = a \tilde{v}_g b\xi = a w u_g b\xi,$$

hence

$$wa = aw \text{ for all } a \in P. \tag{4.23}$$

Let N_4 be the von Neumann subalgebra of N_3 generated by $\{\tilde{v}_g : g \in G\} \cup P$. Then, by (4.22) and (4.23),

$$wMw^* = N_4. \tag{4.24}$$

Also,

$$[N_4\xi] = [wMw^*\xi] = w[M\xi] = w\mathcal{H} = \mathcal{H}. \tag{4.25}$$

As noted before (see the sentence preceding (4.15)), ξ is a cyclic and separating vector for N_3. Combining with (4.25), ξ is also a cyclic and separating vector for N_4. Let J_{N_3} (resp. J_{N_4}) be the modular conjugation associated with N_3 and ξ (resp. N_4 and ξ). Using (4.15) and that ξ is a tracial vector for P, we can apply Proposition 4.4(ii) to the inclusion $P \subseteq N_3$ to get

$$J_{N_3}a\xi = a^*\xi \text{ for any } a \in P. \tag{4.26}$$

Similarly, we can apply Proposition 4.4(ii) to the inclusion $P \subseteq N_4$ to get

$$J_{N_4}a\xi = a^*\xi \text{ for any } a \in P. \tag{4.27}$$

By (4.26) and (4.27), $J_{N_3}aJ_{N_3}\xi = a^*\xi = J_{N_4}aJ_{N_4}\xi$ for any $a \in P$. Since ξ is separating for N_4',

$$J_{N_3}aJ_{N_3} = J_{N_4}aJ_{N_4} \text{ for any } a \in P. \tag{4.28}$$

Let $M_1 = w^*J_{N_4}J_{N_3}N_3J_{N_3}J_{N_4}w$. Then Tomita's theorem (See Theorem 2.19 (v)) and (4.24) give

$$M_1 = w^*J_{N_4}N_3'J_{N_4}w \subseteq w^*J_{N_4}N_4'J_{N_4}w = w^*N_4w = M. \tag{4.29}$$

Together with (4.23) and (4.28), M_1 is a von Neumann subalgebra of M containing P. Since $J_{N_3}\xi = J_{N_4}\xi = w\xi = \xi$ and ξ is cyclic for N_3, $[M_1\xi] = w^* J_{N_4} J_{N_3}[N_3\xi] = \mathcal{H}$. By Proposition 4.10, we conclude that $M_1 = M$. Hence $J_{N_4}J_{N_3}N_3J_{N_3}J_{N_4} = N_4$, so $N_3' = J_{N_3}N_3J_{N_3} = J_{N_4}N_4J_{N_4} = N_4'$. Therefore $N_3 = N_4 = wMw^*$ by (4.24). Hence M and N are spatially isomorphic. □

Remark 4.12. *The previous theorem remains true if the condition of G being a free group is replaced by a cohomological condition. The freeness of G is used only when we modify v_g to get $\tilde{v}_g$. For all $g, h \in G$ and $x \in P$, by equation (4.16),*

$$v_g v_h x v_h^* v_g^* = u_g u_h x u_h^* u_g^* = u_{gh} x u_{gh}^* = v_{gh} x v_{gh}^*. \tag{4.30}$$

So $v_{gh}^* v_g v_h \in P' \cap N_3 = Z(P)$. Thus there exists a unitary $\sigma(g,h) \in Z(P)$ such that

$$v_g v_h = \sigma(g,h) v_{gh} \text{ for all } g, h \in G. \tag{4.31}$$

For all $g, h, k \in G$,

$$\sigma(g,hk) = v_g v_{hk} v_{ghk}^{-1} = v_g \sigma(h,k)^{-1} v_h v_k v_{ghk}^{-1} = \theta_g(\sigma(h,k)^{-1}) v_g v_h v_k v_{ghk}^{-1} \tag{4.32}$$

$$= \theta_g(\sigma(h,k)^{-1})\sigma(g,h) v_{gh} v_k v_{ghk}^{-1} = \theta_g(\sigma(h,k)^{-1})\sigma(g,h)\sigma(gh,k). \tag{4.33}$$

Hence, since $\sigma(h,k), \sigma(gh,k), \sigma(g,hk)$ and $\sigma(g,h)$ are unitaries in an abelian algebra $Z(P)$,

$$\theta_g(\sigma(h,k))\sigma(gh,k)^{-1}\sigma(g,hk)\sigma(g,h)^{-1} = 1_{\mathcal{H}}. \tag{4.34}$$

For all $g, h \in G$, define $\sigma_1(g,h) = \pi^{-1}(\sigma(g,h)) \in Z(P_0)$. By equation (4.34) and

$\pi \circ \alpha_g = \theta_g \circ \pi$,

$$\alpha_g(\sigma_1(h,k))\sigma_1(gh,k)^{-1}\sigma_1(g,hk)\sigma_1(g,h)^{-1} = 1_{\mathcal{H}_0}. \tag{4.35}$$

In other words, σ_1 is a 2-cocycle with respect to the action α of G on the unitary group $U(Z(P_0))$ of $Z(P_0)$.

Assume that there exists a function $s : G \to U(Z(P_0))$ such that

$$\sigma_1(g,h) = \alpha_g(s_h)s_{gh}^{-1}s_g \text{ for all } g,h \in G. \tag{4.36}$$

In other words, σ_1 is a 2-coboundary for the action α of G on $U(Z(P_0))$. Let $t_g = \pi(s_g) \in U(Z(P))$ and $\hat{v}_g = t_g^{-1}v_g$. Then, by Eqs. (4.34), (4.36) and $\pi \circ \alpha_g = \theta_g \circ \pi$,

$$\sigma(g,h) = \theta_g(t_h)t_{gh}^{-1}t_g \text{ and} \tag{4.37}$$

$$\hat{v}_{gh} = t_{gh}^{-1}v_{gh} = t_{gh}^{-1}\sigma(g,h)^{-1}v_gv_h \tag{4.38}$$

$$= t_{gh}^{-1}t_g^{-1}t_{gh}\theta_g(t_h)^{-1}v_gv_h \text{ (since } t_{gh}, t_g \text{ are elements in abelian algebra } Z(P)) \tag{4.39}$$

$$= t_g^{-1}\theta_g(t_h)^{-1}v_gv_h = t_g^{-1}v_gt_h^{-1}v_g^{-1}v_gv_h = t_g^{-1}v_gt_h^{-1}v_h = \hat{v}_g\hat{v}_h \tag{4.40}$$

holds for all $g,h \in G$. Also, since t_g is a unitary in P and v_g is a normalizer of P,

$$\hat{v}_gP = t_g^{-1}v_gP = t_g^{-1}Pv_g = Pv_g = v_gP. \tag{4.41}$$

Hence, $[\hat{v}_gP\xi] = [v_gP\xi]$. Replacing $\tilde{v}_g$ by $\hat{v}_g$, the rest of the proof of the previous theorem remains the same. In conclusion, the previous theorem still holds if the *freeness of G is replaced by $H^2(G, U(Z(P_0))) = 0$ for the action α of G on $U(Z(P_0))$.*

Theorem 4.11, Examples 5.6 and 5.7 combine to give the first examples of non-injective type II_∞ and type III von Neumann algebras which are spatially isomorphic to all nearby von Neumann algebras. These results require a representation on a specific Hilbert space. In Section 6, we explore the extent to which they can be made independent of the particular representation.

5. EXAMPLES*

In this section, we would like to construct examples of non-injective type II_∞ and type III factors of the form $L^\infty(Z) \rtimes_\gamma F_\infty$ for some suitable choice of actions of the free group F_∞ of countably infinite many generators. The construction is due to Houdayer and Vaes [21, Corollary B] where $G = F_2$ and $H = \mathbb{Z}$. For background on crossed product algebras, we refer the reader to [25, Section 8.6 and Section 13.1].

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall that $(X, \mathcal{M}, \mu)$ is countably separated if there exist a sequence of $\mathcal{M}$-measurable sets $\{E_n\}_{n=1}^\infty$ such that $\mu(E_n) < \infty$ for all positive integers n and for each pair of distinct points x and y of X, there exists some positive integer j such that $x \in E_j$ and $y \in X \setminus E_j$. When the context is clear, we will simply call an $\mathcal{M}$-measurable set a measurable set.

Let G be a countable discrete group and α be a measurable action of G on $(X, \mathcal{M}, \mu)$, i.e. $g \mapsto \alpha_g$ is a group homomorphism from G into the group of measurable permutations of X. When the context is clear, we write gx for $\alpha_g(x)$. The action α is called

- nonsingular: for any $g \in G$ and $E \subseteq X$, $g(E)$ is measurable if and only if E is measurable. Moreover, if E is measurable, then $\mu(g(E)) = 0$ if and only if $\mu(E) = 0$.
- free: for any $g \in G$ with $g \neq e$, the set $\{x \in X : gx = x\}$ is a μ-null set.
- ergodic: if E is a measurable subset of X such that $\mu(gE \setminus E) = 0$ for all $g \in G$, then $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. In this case, we also say that G acts

ergodically on X.

These definitions depend not only on the map $g \mapsto \alpha_g$ but also on the measure μ. However, if ν is another measure equivalent to μ (for any $E \in \mathcal{M}$, $\mu(E) = 0$ if and only if $\nu(E) = 0$), then $G \curvearrowright^\alpha (X, \mathcal{M}, \mu)$ is nonsingular (resp. free, ergodic) if and only if $G \curvearrowright^\alpha (X, \mathcal{M}, \nu)$ is nonsingular (resp. free, ergodic).

The following proposition comes from [25, Lemma 8.6.6]. It allows us to check the ergodicity of an action by studying the family of bounded "essentially G-invaraiant" measurable functions.

Proposition 5.1. *Let G be a countable discrete group and α be a free non-singular action of G on some countably separated measure space X. The following conditions are equivalent:*

(i) α is ergodic.

(ii) if f is a bounded measurable complex-valued function on X such that for all $g \in G$, $f(gx) = f(x)$ holds for almost all x in X, then there exists $c \in \mathbb{C}$ such that $f(x) = c$ holds for almost all x in X.

Proposition 5.2. *Let $\pi : G \to H$ be a surjective homomorphism between two countable discrete groups. Let α be a free non-singular action of G on some countably separated measure space (X, μ). Let β be a free non-singular action of G on some countably separated measure space (Y, ν). Write gx for $\alpha_g(x)$ and hy for $\beta_h(y)$. Define an action γ of G on $(X \times Y, \mu \times \nu)$ by setting*

$$g(x, y) = (gx, \pi(g)y), \;\; x \in X, y \in Y.$$

Then

(i) $(X \times Y, \mu \times \nu)$ *is countably separated.*

(ii) γ *is a free non-singular action.*

(iii) If $\alpha|_{\ker(\pi)}$ *and* β *are ergodic, then* γ *is also ergodic.*

Proof. (i)-(ii) Parts (i)-(ii) are easy exercises and so we omit the proof.

(iii) Replacing μ (resp. ν) by a probability measure μ_1 (resp. ν_1) equivalent to μ (resp. ν), we may assume that μ and ν are probability measures. Let f be a bounded measurable complex-valued on $X \times Y$ such that for all $g \in G$, we have $f(g(x, y)) = f(x, y)$ for $\mu \times \nu$-almost all (x, y). Let g be an element of $\ker(\pi)$. Then

$$\iint |f(gx, y) - f(x, y)| d\mu(x) d\nu(y) = 0. \tag{5.1}$$

There exists a measurable subset F_g of Y such that $\nu(F_g) = 0$ and

$$\int |f(gx, y) - f(x, y)| d\mu(x) = 0 \text{ for all } y \in Y \setminus F_g. \tag{5.2}$$

Let $F = \cup_{g \in \ker(\pi)} F_g$. Since G is countable, $\nu(F) = 0$, and for all $y \in Y \setminus F$, we have

$$\int |f(gx, y) - f(x, y)| d\mu(x) = 0 \text{ for all } g \in \ker(\pi). \tag{5.3}$$

Hence, for all $y \in Y \setminus F$ and for all $g \in \ker(\pi)$,

$$f(gx, y) = f(x, y) \text{ for almost all } x \text{ in } X. \tag{5.4}$$

Since $\ker(\pi)$ acts ergodically on X, by Proposition 5.1, for each $y \in Y \setminus F$, there exists a complex number c_y such that

$$f(x,y) = c_y \text{ for almost all } x \text{ in } X. \tag{5.5}$$

Since the action α is nonsingular, for all $y \in Y \setminus F$ and all $g \in G$,

$$f(gx,y) = c_y \text{ for almost all } x \text{ in } X. \tag{5.6}$$

Hence, for all $y \in Y \setminus F$ and all $g \in G$,

$$\int |f(gx,y) - f(x,y)| d\mu(x) = 0. \tag{5.7}$$

Also, by Equation (5.5), for all $y \in Y \setminus F$,

$$\iint |f(x,y) - f(x',y)| d\mu(x) d\mu(x') = 0. \tag{5.8}$$

Let h be an element of H. Since π is surjective, there exists an element $g_h \in G$ such that $\pi(g_h) = h$. Then

$$\begin{aligned}
&\iint |f(x,hy) - f(x,y)| d\nu(y) d\mu(x) \\
\leq &\iint |f(x,hy) - f(g_h x, hy)| d\nu(y) d\mu(x) + \iint |f(g_h x, hy) - f(x,y)| d\nu(y) d\mu(x) \\
= &\int_{Y \setminus h^{-1}F} \int |f(x,hy) - f(g_h x, hy)| d\mu(x) d\nu(y) + 0 \\
= &0 \quad \text{(By Equation (5.7))}
\end{aligned}$$

There exists a measurable subset E_h of X such that $\mu(E_h) = 0$, and for all $x \in X \setminus E_h$, we have,

$$\int |f(x,hy) - f(x,y)| d\nu(y) = 0. \tag{5.9}$$

Let $E = \cup_{h \in H} E_h$. Since H is countable, E is measurable, $\mu(E) = 0$, and for all $x \in X \setminus E$, we have

$$\int |f(x,hy) - f(x,y)| d\nu(y) = 0. \tag{5.10}$$

Since H acts ergodically on Y, by Proposition 5.1, for each $x \in X \setminus E$, there exists a complex number d_x such that

$$f(x,y) = d_x \text{ for almost all } y \text{ in } Y. \tag{5.11}$$

Hence, for any $x \in X \setminus E$,

$$\iint |f(x,y) - f(x,y')| d\nu(y) d\nu(y') = 0. \tag{5.12}$$

Let r be $\iint f(x',y') d\mu(x') \nu(y')$. Then

$$\begin{aligned}
&\iint |f(x,y) - r| d\mu(x) d\nu(y) \\
\leq &\iiiint |f(x,y) - f(x',y')| d\mu(x') d\nu(y') d\mu(x) d\nu(y) \\
\leq &\iiiint |f(x,y) - f(x,y')| d\mu(x') d\nu(y') d\mu(x) d\nu(y) + \\
&\iiiint |f(x,y') - f(x',y')| d\mu(x') d\nu(y') d\mu(x) d\nu(y) \\
= &\iiint |f(x,y) - f(x,y')| d\nu(y) d\nu(y') d\mu(x) + \iiint |f(x,y') - f(x',y')| d\mu(x) d\mu(x') d\nu(y') \\
= &0 \quad \text{(By Equations (5.8) and (5.12))}
\end{aligned}$$

Thus, $f(x,y) = r$ for almost all (x,y) in $X \times Y$. By Proposition 5.1, the action γ is ergodic. $\square$

Proposition 5.3. *Let $\pi : G \to H$ be a surjective homomorphism between two countable discrete groups. Let α be a free measure-preserving action of G on a countably separated measure space $(X, \mathcal{M}, \mu)$ such that $\alpha|_{\ker(\pi)}$ is ergodic and $\mu(X) = 1$. Let β be a measurable action of H on some σ-finite measure space. Define an action γ of G on $(X \times Y, \mu \times \nu)$ by setting*

$$g(x, y) = (gx, \pi(g)y), \ \ x \in X, y \in Y.$$

If there is a σ-finite measure ρ on $X \times Y$ such that ρ is G-invariant and ρ is absolutely continuous with respect to $\mu \times \nu$, then there is a σ-finite measure ν_1 on Y such that ν_1 is absolutely continuous with respect to ν and ν_1 is H-invariant.

Proof. Since ρ is absoluely continuous with respect to $\mu \times \nu$, by the Radon-Nikodym Theorem, there exists a non-negative $\mathcal{M} \otimes \mathcal{N}$-measurable function ϕ on $X \times Y$ such that

$$\rho(A) = \int_A \phi(x, y) d(\mu \times \nu)(x, y) \ \text{ for any measurable subsets } A \text{ of } X \times Y. \quad (5.13)$$

For any $g \in \ker(\pi)$ and $A \in \mathcal{M} \otimes \mathcal{N}$,

$$\rho(gA) = \rho(A). \quad (5.14)$$

We have

$$\rho(gA) = \iint \chi_{gA}(x,y)\phi(x,y)d\mu(x)d\nu(y) \tag{5.15}$$
$$= \iint \chi_A(g^{-1}x,y)\phi(x,y)d\mu(x)d\nu(y) \quad \text{(since } g \in \ker(\pi)\text{)} \tag{5.16}$$
$$= \iint \chi_A(x,y)\phi(gx,y)d\mu(x)d\nu(y) \quad \text{(since } \mu \text{ is } G\text{-invariant)} \tag{5.17}$$

and

$$\rho(A) = \iint \chi_A(x,y)\phi(x,y)d\mu(x)d\nu(y). \tag{5.18}$$

By Equations (5.14), (5.17) and (5.18), for any $g \in \ker(\pi)$,

$$\phi(gx,y) = \phi(x,y) \quad \text{for } \mu\times\nu\text{-almost all } (x,y) \text{ in } X \times Y. \tag{5.19}$$

Then

$$\iint |\phi(gx,y) - \phi(x,y)|d\mu(x)d\nu(y) = 0. \tag{5.20}$$

For any $g \in \ker(\pi)$, there exists a measurable subset $T_g \subseteq Y$ such that $\nu(T_g) = 0$ and

$$\int |\phi(gx,y) - \phi(x,y)|d\mu(x) = 0 \quad \text{for any } y \in Y \setminus T_g. \tag{5.21}$$

Let $T = \cup_{g\in\ker(\pi)} T_g$. Then T is a measurable subset of Y and $\nu(T) = 0$. For any $y \in Y \setminus T$ and $g \in \ker(\pi)$,

$$\phi(gx,y) = \phi(x,y) \quad \text{for } \mu\text{-almost all } x \text{ in } X. \tag{5.22}$$

Since $\ker(\pi)$ acts ergodically on X, by Proposition 5.1, there exists a complex number c_y such that

$$\phi(x, y) = c_y \quad \text{for } \mu\text{-almost all } x \text{ in } X. \tag{5.23}$$

Define $\psi : Y \to \mathbb{C}$ by

$$\psi(y) = \begin{cases} \int \phi(x, y) d\mu(x) = c_y & , \text{if } y \in Y \setminus T \\ 0 & , \text{if } y \in T. \end{cases}$$

By the Fubini-Tonelli Theorem, ψ is a measurable function on Y. Define a measure ν_1 on Y by

$$\nu_1(F) = \int_F \psi(y) d\nu(y) \quad \text{for any measurable subset } F \text{ of } Y. \tag{5.24}$$

Then ν_1 is absolutely continuous with respect to ν. We will show that ν_1 is σ-finite and is H-invariant. Since ρ is σ-finite, there exist pairwise disjoint measurable subsets $\{A_n\}_{n=1}^{\infty}$ of $X \times Y$ such that $\cup_{n=1}^{\infty} A_n = X \times Y$ and $\rho(A_n) < \infty$ for all n. Define $\psi_n : Y \to \mathbb{C}$ by

$$\psi_n(y) = \begin{cases} \int \chi_{A_n}(x, y) d\mu(x) & , \text{if } y \in Y \setminus T \\ 0 & , \text{if } y \in T. \end{cases}$$

Then ψ_n is a measurable function on Y and

$$\sum_{n=1}^{\infty} \psi_n(y) = 1 \quad \text{for any } y \in Y \setminus T. \tag{5.25}$$

$$
\begin{aligned}
&\int \psi_n(y)\psi(y)d\nu(y) \\
=&\int_{Y\setminus T}\int c_y\chi_{A_n}(x,y)d\mu(x)d\nu(y) \\
=&\int_{Y\setminus T}\int \chi_{A_n}(x,y)\phi(x,y)d\mu(x)d\nu(y) \\
=&\rho(A_n)
\end{aligned} \tag{5.26}
$$

By Equation (5.25), $Y \setminus T = \cup_{n,k\in\mathbb{N}} F_{n,k}$ where $F_{n,k} = \{y \in Y \setminus T : \psi_n(y) > 1/k\}$.

$$
\begin{aligned}
&\frac{1}{k}\nu_1(F_{n,k}) = \frac{1}{k}\int_{F_{n,k}} \psi(y)d\nu(y) \quad \text{(By Equation (5.24))} \\
\leq &\int_{F_{n,k}} \psi_n(y)\psi(y)d\nu(y) \\
\leq &\int_{F_{n,k}} \psi_n(y)\psi(y)d\nu(y) \\
=&\rho(A_n) < \infty \text{ (By Equation (5.26)).}
\end{aligned} \tag{5.27}
$$

Since $\nu(T) = 0$, $\nu_1(T) = 0$. This shows that ν_1 is σ-finite. For any measurable subset E of X and measurable subset F of Y,

$$\rho(E \times F) = \int_F \int_E \phi(x,y)d\mu(x)d\nu(y) \tag{5.28}$$

$$= \int_F \psi(y)\mu(E)d\nu(y) \tag{5.29}$$

$$= \mu(E)v_1(F). \tag{5.30}$$

Thus $\rho = \mu \times \nu_1$. For any $g \in G$ and measurable subset F of Y, we have

$$\nu_1(\pi(g)F) = \mu(gX)\nu_1(\pi(g)F) = \rho(g(X \times F)) = \rho(X \times F) = \nu_1(F). \tag{5.31}$$

Since π is surjective, ν_1 is H-invariant. $\square$

Proposition 5.4. *[25, Proposition 8.6.10] Let α be a free ergodic nonsingular action of a countable discrete group G on a countably separated measure space (X, μ). Then $L^\infty(X) \rtimes_\alpha G$ is a factor. Moreover,*

(i) $L^\infty(X) \rtimes_\alpha G$ is type $\mathrm{I_n}$ if $\mu(\{x_0\}) > 0$ for some $x_0 \in X$ and n is the cardinality of G.

(ii) $L^\infty(X) \rtimes_\alpha G$ is type II_1 if $\mu(\{x\}) = 0$ for all $x \in X$ and there exists a nonzero finite measure μ_1 on X which is G-invariant and is absolutely continuous with respect to μ.

(iii) $L^\infty(X) \rtimes_\alpha G$ is type II_∞ if $\mu(\{x\}) = 0$ for all $x \in X$ and there exists an infinite measure(σ-finite) μ_1 on X which is G-invariant and is absolutely continuous with respect to μ.

(iv) $L^\infty(X) \rtimes_\alpha G$ is type III if $\mu(\{x\}) = 0$ for all $x \in X$ and there does not exist a nonzero σ-finite measure μ_1 on X which is G-invariant and is absolutely continuous with respect to μ.

The following proposition is due to Houdayer and Vaes [21, Corollary B] where $G = F_2$ and $H = \mathbb{Z}$.

Proposition 5.5. *Let $\pi : G \to H$ be a surjective group homomorphism between two countable discrete groups. Assume that G is nonamenable and H is amenable. Let α be a free ergodic measuring-preserving action on a countably separated measure space (X, μ) with $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for any $x \in X$. Assume also that $\alpha|_{\ker(\pi)}$ is ergodic. Let β be a free ergodic non-singular action of H on a countably separated measure space (Y, ν) with $\nu(\{y\}) = 0$ for any $y \in Y$. Define an action γ of G on*

$(X \times Y, \mu \times \nu)$ by setting

$$g(x, y) = (gx, \pi(g)y), \;\; x \in X, y \in Y.$$

Then

(i) The action γ is a free ergodic nonsingular action.

(ii) $L^\infty(X \times Y, \mu \times \nu) \rtimes_\gamma G$ is a noninjective factor which has the same type (type II_1, II_∞ or III) as $L^\infty(Y) \rtimes_\beta H$.

Proof. (i) This follows from Proposition 5.1.

(ii) By (i), $L^\infty(X \times Y, \mu \times \nu) \rtimes_\gamma$ is a factor. If $L^\infty(Y) \rtimes_\beta H$ is type II_1, then there exists a nonzero finite measure ν_1 on Y which is H-invariant and is absolutely continuous with respesct to ν. Then $\mu \times \nu_1$ is a nonzero finite measure on $X \times Y$ which is G-invariant and is absolutely continuous with respesct to $\mu \times \nu$. Hence, $L^\infty(X \times Y, \mu \times \nu) \rtimes_\gamma G$ is type II_1.

If $L^\infty(Y) \rtimes_\beta H$ is type II_∞, then there exists an infinite measure (σ-finite) ν_1 on Y which is H-invariant and is absolutely continuous with respesct to ν. Then $\mu \times \nu_1$ is an infinite measure (σ-finite) on $X \times Y$ which is G-invariant and is absolutely continuous with respesct to $\mu \times \nu$. Hence, $L^\infty(X \times Y, \mu \times \nu) \rtimes_\gamma G$ is type II_∞.

If $L^\infty(Y) \rtimes_\beta H$ is type III, then there does not exist a nonzero σ-finite measure ν_1 on Y which is H-invariant and is absolutely continuous with respesct to ν. By Proposition 5.3, there does not exist a nonzero σ-finite measure on $X \times Y$ which is G-invariant and is absolutely continuous with respesct to $\mu \times \nu$. Hence, $L^\infty(X \times Y, \mu \times \nu) \rtimes_\gamma G$ is type III.

Noninjectivity of $L^\infty(X \times Y, \mu \times \nu) \rtimes_\gamma G$ can be proved in the same way as [21, Corollary B] where $G = F_2$ and $H = \mathbb{Z}$. $\square$

Proposition 5.5 gives examples of non-injective factors of the form $L^\infty(Z) \rtimes_\gamma F_\infty$.

Example 5.6. (Type II_∞) Take G to be F_∞ (the free group on a countably infinite number of generators $a_1, a_2, ...$), X to be $[0,1]^{F_\infty}$ and μ to be the infinite product measure of Lebesgue measure. Consider the Bernoulli action of G on X: G acts on X by

$$g_1 x(g_2) = x(g_1^{-1} g_2) \text{ for any } g_1, g_2 \in G \text{ and } x \in X.$$

Take H to be the rationals $\mathbb{Q}$, Y to be the real line and ν to be Lebesgue measure. H acts on Y by translation. Let $\pi : F_\infty \to \mathbb{Q}$ be any surjective homomorphism such that $\ker(\pi)$ contains the generator a_1.

Since the Lebesgue measure ν is an infinite measure which is invariant under H, $L^\infty(Y, \nu) \rtimes_\beta H$ is a type II_∞ factor. For details, we refer the reader to [25, Example 8.6.13].

Example 5.7. (Type III) Take G to be F_∞ (free group on countably infinite generators $a_1, a_2, ...$), X to be $[0,1]^{F_\infty}$ and μ to be the infinite product measure of Lebesgue measure on X. G acts on X by

$$g_1 x(g_2) = x(g_1^{-1} g_2) \text{ for any } g_1, g_2 \in G \text{ and } x \in X.$$

Let Y be the real line and let ν be the Lebesgue measure. Let H be the group of maps $x \mapsto cx + d : \mathbb{R} \to \mathbb{R}$ where c,d are rationals and $c \neq 0$. Let $\pi : F_\infty \to \mathbb{Q}$ be any surjective homomorphism such that $\ker(\pi)$ contains the generator a_1.

Since there does not exist a nonzero σ-finite measure ν_1 on Y which is absolutley continuous with respect to ν and is invariant under H, $L^\infty(Y, \nu) \rtimes_\beta H$ is a type III factor. For details, we refer the reader to [25, Example 8.6.14].

6. CHANGING REPRESENTATIONS*

In this section, we will show that if a countably decomposable von Neumann algebra $M \subseteq B(\mathcal{H})$ is *-isomorphic to any close von Neumann algebra and $M_1 \subseteq B(\mathcal{K})$ is *-isomorphic to M, then M_1 is *-isomorphic to any close von Neumann algebra. The question can be solved if for any von Neumann subalgebra N_1 of $B(\mathcal{K})$, we can find a von Neumann algebra $N \subseteq B(\mathcal{H})$ which is *-isomorphic to N_1 and N is close to M. It is a known fact that any *-isomorphism between two von Neumann algebras is a composition of three types of *-isomorphisms (See Theorem 2.16): an amplification, cutting down by a projection in the commutant, a spatial isomorphism. For amplifications and spatial isomorphisms, we can bring two close von Neumann algebras acting on a Hilbert space to two close von Neumann algebras acting on a possibly different Hilbert space. Complications occur since it is not known whether close von Neumann algebras have close commutants. The case of properly infinite von Neumann algebras is easy as it is known that close properly infinite von Neumann algebras have close commutants. The work of [3] solved this question when M is a type II_1 factor. Based on the methods used in [3], we solve this question first for a cyclic representation M_1 of M and then for any faithful normal unital *-representation M_1 of M.

In [3], the following definitions were made:

Definition 6.1. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$.*

(i) We say M is strongly Kadison-Kastler stable if for any positive number $\eta > 0$, there exists a positive number $\epsilon > 0$ such that: for any faithful normal unital

*Reprinted with permission from "Perturbations of certain crossed product algebras by free groups" by Wai-Kit Chan, J. Funct. Anal. 267 (2014), no. 10, 3994-4027, Copyright 2014 by Elsevier.

-representation $\pi : M \to B(\mathcal{K})$ and for any von Neumann algebra $N \subseteq B(\mathcal{K})$ with $d(\pi(M), N) < \epsilon$, then there is a unitary $u \in B(\mathcal{K})$ such that $\|u - 1\| < \eta$ and $uMu^* = N$.

*(ii) We say M is Kadison-Kastler stable if there exists a positive number $\epsilon > 0$ such that: for any faithful normal unital *-representation $\pi : M \to B(\mathcal{K})$ and for any von Neumann algebra $N \subseteq B(\mathcal{K})$ with $d(\pi(M), N) < \epsilon$, then $\pi(M)$ and N are spatially isomorphic.*

*(iii) We say M is weakly Kadison-Kastler stable if there exists a positive number $\epsilon > 0$ such that: for any faithful normal unital *-representation $\pi : M \to B(\mathcal{K})$ and for any von Neumann algebra $N \subseteq B(\mathcal{K})$ with $d(\pi(M), N) < \epsilon$, then $\pi(M)$ and N are *-isomorphic.*

We will need terminology which is slightly different from these. The primary difference is that we consider a von Neumann algebra M in a representation on a specific Hilbert space $\mathcal{H}$.

Definition 6.2. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$.*

(i) We say M is Kadison-Kastler stable on $\mathcal{H}$ if there exists $\epsilon > 0$ such that for any von Neumann algebra $N \subseteq B(\mathcal{H})$ with $d(M, N) < \epsilon$, then M and N are spatially isomorphic.

*(ii) We say M is weakly Kadison-Kastler stable on $\mathcal{H}$ if there exists $\epsilon > 0$ such that for any von Neumann algebra $N \subseteq B(\mathcal{H})$ with $d(M, N) < \epsilon$, then M and N are *-isomorphic.*

A question arises as to whether dependence on $\mathcal{H}$ in definition 4.2 can be removed. We do not know the full answer to this, but in this section we show that for countably decomposable von Neumann algebras $M \subseteq B(\mathcal{H})$, weak Kadison-Kastler stability on $\mathcal{H}$ carries over to any other faithful normal unital *-representation of M on another Hilbert space. We also show the same conclusion for Kadison-Kastler stability on $\mathcal{H}$ when M is a type III factor with separable predual.

Throughout this section, for a projection e in a von Neumann algebra $M \subseteq B(\mathcal{H})$, c_e stands for the central support of e relative to the von Neumann algebra. For a projection $e' \in M'$ and $x \in M$, we denote by $x_{e'}$ the restriction of x on $e'(\mathcal{H})$. Also, we denote by $M_{e'}$ the von Neumann algebra $\{x_{e'} : x \in M\}$.

Lemma 6.3. *Let P be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{L}$ be an infinite dimensional Hilbert space. Let $\{e_i\}_{i\in\Lambda}$ be an orthogonal family of minimal projections in $B(\mathcal{L})$ with sum equal to 1. Then any projection in $P\bar{\otimes}B(\mathcal{L})$ is equivalent to $\sum_{i\in\Lambda} f_i \otimes e_i$ for some projections $f_i \in P$.*

Proof. First, we will show that for any nonzero projection e in $P\bar{\otimes}B(\mathcal{L})$, there exist a nonzero central subprojection p of c_e in $Z(P\bar{\otimes}B(\mathcal{L}))$ and projections f_i in P such that $pe \sim p(\sum_{i\in\Lambda} f_i \otimes e_i)$. Let $\{g_j\}_{j\in\Omega}$ be a maximal orthogonal family of subprojections of e in $P\bar{\otimes}B(\mathcal{L})$ such that $g_j \sim c_e(1\otimes e_i)$ for all j. (If such a family does not exist, take $\Omega = \emptyset$). By maximality, we cannot have $e - \sum_{j\in\Omega} g_j \succcurlyeq c_e(1 \otimes e_i)$ and so there exists a nonzero central projection p' in $Z(P\bar{\otimes}B(\mathcal{L}))$ such that $p'(e - \sum_{j\in\Omega} g_j) \prec p'c_e(1 \otimes e_i)$. Take $p = p'c_e$. Now, $1 \otimes e_i$ is finite in $P\bar{\otimes}B(\mathcal{L})$ and so is $p(1 \otimes e_i)$. Since $p = \sum_{i\in\Lambda} p(1 \otimes e_i)$, by applying generalized invariance of dimension (See Theorem 2.14) to the von Neumann algebra $p(P\bar{\otimes}B(\mathcal{L}))$ with test projection $p(1 \otimes e_i)$, we conclude that $|\Omega| \leqq |\Lambda|$. Fix i_0 in Λ. Since Λ is an infinite set, there exists $\Lambda_1 \subseteq \Lambda \setminus \{i_0\}$ such that $|\Omega| = |\Lambda_1|$. Then

$$\sum_{j\in\Omega} pg_j \sim \sum_{i\in\Lambda_1} p(1\otimes e_i)$$
$$p(e - \sum_{j\in\Omega} g_j) \sim g$$

for some subprojection g of $p(1\otimes e_{i_0})$. Now p is a central projection in $Z(P\bar{\otimes}B(\mathcal{L}))$, so there is a central projection q in $Z(P)$ such that $p = q\otimes 1$. Also, $p(1\otimes e_{i_0})\, P\bar{\otimes}B(\mathcal{L})\, p(1\otimes e_{i_0}) = p(P\bar{\otimes}\mathbb{C}e_{i_0}) \subseteq P\bar{\otimes}\mathbb{C}e_{i_0}$, so there is a projection f_{i_0} in P such that $g = f_{i_0}\otimes e_{i_0}$. So $pe = p(e-\sum_{j\in\Omega} g_j) + \sum_{j\in\Omega} pg_j \sim p(f_{i_0}\otimes e_{i_0} + \sum_{i\in\Lambda_1}(1\otimes e_i))$. This proves our first claim.

Let e be a nonzero projection in $P\bar{\otimes}B(\mathcal{L})$. Let $\{p_\alpha\}_{\alpha\in S}$ be a maximal orthogonal family of nonzero central subprojections of c_e in $P\bar{\otimes}B(\mathcal{L})$ such that each $p_\alpha e \sim p_\alpha(\sum_{i\in\Lambda} f_{i,\alpha}\otimes e_i)$ for some projections $f_{i,\alpha}$ in P. By maximality and the first paragraph, we have $\sum_{\alpha\in S} p_\alpha = c_e$. Since p_α is a central projection in $Z(P\bar{\otimes}B(\mathcal{L}))$, $p_\alpha = q_\alpha\otimes 1$ for some central projection q_α in $Z(P)$. Then $e = \sum_{\alpha\in S} p_\alpha e \sim \sum_{\alpha\in S}\sum_{i\in\Lambda}(q_\alpha f_{i,\alpha})\otimes e_i = \sum_{i\in\Lambda}(\sum_{\alpha\in S} q_\alpha f_{i,\alpha})\otimes e_i$. □

Lemma 6.4. *Let P and Q be von Neumann algebras acting on some Hilbert space $\mathcal{H}$. Assume that $d(P,Q) < \gamma < 1/4$ and e is a projection in $P'\cap Q'$. If the central support of e relative to the von Neumann algebra P' is 1, then the central support of e relative to the von Neumann algebra Q' is 1.*

Proof. Let q be the central support of e relative to the von Neumann algebra Q'. Then there exists a projection p in P such that $\|1-q-p\| \le 2^{-1/2}\gamma$ by Proposition 2.27 (ii). By Proposition 2.26, $p\in Z(P)$. Then

$$\|pe\| = \|(1-q-p)e\| \le 2^{-1/2}\gamma < 1.$$

Since pe is a projection, $pe = 0$. As the central support of e relative to the von Neumann algebra P' is 1, $p = 0$. Thus $\|1 - q\| \leq 2^{-1/2}\gamma < 1$ and $q = 1$. $\square$

Proposition 6.5. *Let $M_0 \subseteq B(\mathcal{H})$ be a finite von Neumann algebra with a cyclic vector and let $\pi\colon M_0 \to M \subseteq B(\mathcal{K})$ be a *-isomorphism of von Neumann algebras. Assume that $N_0 \subseteq B(\mathcal{H})$ is another von Neumann algebra such that $Z(N_0) = Z(M_0)$ and $d(M_0, N_0) < \gamma < 1/84$. Then there exists a von Neumann algebra $N \subseteq B(\mathcal{K})$ with $d(M, N) < 21\gamma$ and a *-isomorphism $\rho : N_0 \to N$.*

Proof. By Theorem 2.16, there exists an infinite dimensional Hilbert space $\mathcal{L}$, a projection $e' \in (M_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})'$ with central support equal to $1_{\mathcal{H}\otimes\mathcal{L}}$ and a unitary $u\colon e'(\mathcal{H} \otimes \mathcal{L}) \to \mathcal{K}$ such that

$$\pi(x) = u(x \otimes 1_{\mathcal{L}})_{e'}u^* \text{ for any } x \in M_0.$$

Fix an orthogonal family of minimal projections $\{e_i\}_{i\in\Lambda}$ in $B(\mathcal{L})$ with sum equals to $1_{\mathcal{L}}$. Since M_0 is a finite von Neumann algebra with a cyclic vector, M_0' is finite by Proposition 2.7. By Lemma 6.3, there exist projections $f_i \in M_0'$ and a partial isometry $v \in M_0' \bar{\otimes} B(\mathcal{L})$ such that

$$v^*v = e'' = \sum_{i\in\Lambda} f_i \otimes e_i, \quad vv^* = e'.$$

Define a unitary $v_1\colon e''(\mathcal{H} \otimes \mathcal{L}) \to e'(\mathcal{H} \otimes \mathcal{L})$ by $v_1\xi = v\xi$ for any $\xi \in e''(\mathcal{H} \otimes \mathcal{L})$. Then for any $x \in M_0$, we have

$$\begin{aligned}(x \otimes 1_{\mathcal{L}})_{e'} &= v_1(x \otimes 1_{\mathcal{L}})_{e''}v_1^*\ ,\\ \pi(x) &= u(x \otimes 1_{\mathcal{L}})_{e'}u^* = uv_1(x \otimes 1_{\mathcal{L}})_{e''}v_1^*u^*.\end{aligned}$$

By Proposition 3.5, $M_0' \subset_{4(1+\sqrt{2})\gamma} N_0'$. By Proposition 2.27 (ii), there exist projections

$g_i \in N_0'$ such that

$$\|f_i - g_i\| < 2^{-1/2} \cdot 4(1+\sqrt{2})\gamma < 1.$$

Then, by Proposition 2.25, there exist unitaries $w_i \in B(\mathcal{H})$ such that

$$\|w_i - 1_{\mathcal{H}}\| \le \sqrt{2}\|f_i - g_i\| < 4(1+\sqrt{2})\gamma,\ w_i^* f_i w_i = g_i.$$

Let $w = \sum_{i\in\Lambda} w_i \otimes e_i$. Then w is a unitary in $B(\mathcal{H}\otimes\mathcal{L})$ such that $w^* e'' w = \sum_{i\in\Lambda} g_i \otimes e_i \in N_0' \bar{\otimes} B(\mathcal{L}) = (N_0 \bar{\otimes} \mathbb{C}1_L)'$, so $e'' \in (M_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}})' \cap w(N_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}})'w^*$.

Define $\rho\colon N_0 \to B(\mathcal{K})$ by

$$\rho(x) = uv_1(w(x\otimes 1_{\mathcal{L}})w^*)_{e''} v_1^* u^*.$$

Since e'' is equivalent to e' in $(M_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}})'$, e'' has central support $1_{\mathcal{H}\otimes\mathcal{L}}$ relative to $(M_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}})'$. Since

$$d(M_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}}, w(N_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}})w^*) \le d(M_0,N_0) + 2\|w - 1_{\mathcal{H}\otimes\mathcal{L}}\| < \gamma + 8(1+\sqrt{2})\gamma < 21\gamma < 1/4\ ,$$

and by Lemma 6.4, e'' has central support $1_{\mathcal{H}\otimes\mathcal{L}}$ relative to $w(N_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}})'w^*$. So ρ is a *-isomorphism onto its image. Let $N = \rho(N_0)$. Then

$$d(M,N) \le d(M_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}}, w(N_0\bar{\otimes}\mathbb{C}1_{\mathcal{L}})w^*) < 21\gamma.$$

□

Proposition 6.6. *Let $M_0 \subseteq B(\mathcal{H})$ be a properly infinite von Neumann algebra and $\pi\colon M_0 \to M \subseteq B(\mathcal{K})$ be a *-isomorphism of von Neumann algebras. Assume that $N_0 \subseteq B(\mathcal{H})$ is another properly infinite von Neumann algebra such that $d(M_0, N_0) <$*

$\gamma < 1/28$. *Then there exists a von Neumann algebra* $N \subseteq B(\mathcal{K})$ *with* $d(M, N) < 7\gamma$ *and a* *-*isomorphism* $\rho : N_0 \to N$.

Proof. By Theorem 2.16, there exist a Hilbert space $\mathcal{L}$, a projection $e' \in (M_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})'$ with central support equal to $1_{\mathcal{H}\otimes\mathcal{L}}$ and a unitary $u\colon e'(\mathcal{H} \otimes \mathcal{L}) \to \mathcal{K}$ such that

$$\pi(x) = u(x \otimes 1_{\mathcal{L}})_{e'} u^* \text{ for any } x \in M_0.$$

Since $N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}}$ is properly infinite and $N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}} \subset_\gamma M_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}}$, by Proposition 2.29 (i), we have

$$(M_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})' \subset_{3\gamma} (N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})'.$$

By Proposition 2.27 (ii), there exists a projection $f' \in (N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})'$ such that $\|f' - e'\| < \frac{1}{\sqrt{2}} \cdot 3\gamma < 1$. By Proposition 2.25, there exists a unitary $w \in B(\mathcal{H} \otimes \mathcal{L})$ such that $w^* e' w = f'$ and $\|w - 1_{\mathcal{H}\otimes\mathcal{L}}\| \leq \sqrt{2}\|e' - f'\| < 3\gamma$. Then $e' \in w(N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})' w^*$ and

$$d(M_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}}, w(N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})w^*) \leq 2\|w - 1_{\mathcal{H}\otimes\mathcal{L}}\| + d(M_0, N_0) < 7\gamma < 1/4.$$

As noted in the beginning of the proof, the central support of e' relative to $(M_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})'$ is $1_{\mathcal{H}\otimes\mathcal{L}}$. By Lemma 6.4, the central support of e' relative to $w(N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})' w^*$ is $1_{\mathcal{H}\otimes\mathcal{L}}$. Define $\rho : N_0 \to B(\mathcal{K})$ by

$$\rho(x) = u(w(x \otimes 1_{\mathcal{L}})w^*)_{e'} u^* \text{ for any } x \in N_0.$$

Then ρ is a *-isomorphism onto its image $N = \rho(N_0)$ and

$$d(M, N) \leq d(M_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}}, w(N_0 \bar{\otimes} \mathbb{C}1_{\mathcal{L}})w^*) < 7\gamma.$$

□

Proposition 6.7. *Let $M_0 \subseteq B(\mathcal{H})$ be a von Neumann algebra with a cyclic vector and $\pi\colon M_0 \to M \subseteq B(\mathcal{K})$ be a *-isomorphism of von Neumann algebras. Assume that $N_0 \subseteq B(\mathcal{H})$ is another von Neumann algebra such that $d(M_0, N_0) < \gamma < 1/924$. Then there exists a von Neumann algebra $N \subseteq B(\mathcal{K})$ with $d(M, N) < 231\gamma$ and a *-isomorphism $\rho : N_0 \to N$.*

Proof. Let $Z(M_0)$ and $Z(N_0)$ be the center of M_0 and N_0 respectively. By Proposition 2.31, there exists a unitary $u \in B(\mathcal{H})$ such that $\|u - 1_{\mathcal{H}}\| \leq 5\gamma$ and $Z(M_0) = uZ(N_0)u^*$. Let $N_1 = uN_0u^*$. Then $Z(N_1) = Z(M_0)$ and $d(N_1, M_0) \leq 2\|u - 1_{\mathcal{H}}\| + d(N_0, M_0) < 11\gamma$. Let p be the largest central projection in $Z(M_0)$ such that M_0p is finite. By [14, Lemma 3.5], p is the largest central projection in $Z(N_1)$ such that N_1p is finite. If p is $1_{\mathcal{H}}$, the proposition follows from Proposition 6.5. Otherwise, $1_{\mathcal{H}} - p$ is nonzero and $M_0(1_{\mathcal{H}} - p)$ is properly infinite. In this case, $N_1(1_{\mathcal{H}} - p)$ is also properly infinite.

Let $q = \pi(p)$. Let $\pi_1 : M_0p \to Mq \subseteq B(q\mathcal{K})$ be the *-isomorphism defined by

$$\pi_1(xp) = \pi(x)q \text{ for any } x \in M_0.$$

Let $\pi_2 : M_0(1_{\mathcal{H}} - p) \to M(1_{\mathcal{K}} - q) \subseteq B((1_{\mathcal{K}} - q)\mathcal{K})$ be the *-isomorphism defined by

$$\pi_2(x(1_{\mathcal{H}} - p)) = \pi(x)(1_{\mathcal{K}} - q) \text{ for any } x \in M_0.$$

Since M_0 has a cyclic vector, M_0p (acting on $p\mathcal{H}$) has a cyclic vector. Applying Proposition 6.5 to M_0p and N_1p, there exists a *-isomorphism $\rho_1 : N_1p \to \rho_1(N_1p) \subseteq B(q\mathcal{K})$ such that $d(\rho_1(N_1p), Mq) < 21 \cdot 11\gamma$.

Applying Proposition 6.6 to $M_0(1_{\mathcal{H}}-p)$ and $N_1(1_{\mathcal{H}}-p)$, there exists a *-isomorphism

$\rho_2 : N_1(1_{\mathcal{H}} - p) \to \rho_2(N_1(1_{\mathcal{H}} - p)) \subseteq B((1_{\mathcal{K}} - q)\mathcal{K})$ such that $d(\rho_2(N_1(1_{\mathcal{H}} - p)), M(1_{\mathcal{K}} - q)) < 7 \cdot 11\gamma$.

Let N be $\rho_1(N_1 p) \oplus \rho_2(N_1(1_{\mathcal{H}} - p))$. Then N is a von Neumann algebra acting on $\mathcal{K}$. Define $\rho : N_0 \to N$ by

$$\rho(x) = \rho_1(uxu^* p) \oplus \rho_2(uxu^*(1_{\mathcal{H}} - p)) \text{ for any } x \in N_0.$$

Then ρ is a *-isomorphism and $d(M, N) < \max\{21 \cdot 11\gamma, 7 \cdot 11\gamma\} = 231\gamma$. □

Proposition 6.8. *Let M_0 and N_0 be von Neumann algebras acting on a Hilbert space $B(\mathcal{H})$. Assume that M_0 is countably decomposable. Suppose that $d(M_0, N_0) < \gamma < 10^{-6}$. Then there exist a Hilbert space $\mathcal{K}$, von Neumann algebras M and N acting on $\mathcal{K}$ such that M has a cyclic vector, M is *-isomorphic to M_0, N is *-isomorphic to N_0 and $d(M, N) < 210\sqrt{\gamma}$.*

Proof. Let $Z(M_0)$ and $Z(N_0)$ be the center of M_0 and N_0 respectively. By Proposition 2.31, there exists a unitary $u \in B(\mathcal{H})$ such that $\|u - 1_{\mathcal{H}}\| \le 5\gamma$ and $Z(M_0) = uZ(N_0)u^*$. Let $N_1 = uN_0u^*$. Then $Z(N_1) = Z(M_0)$ and $d(N_1, M_0) \le 2\|u - 1_{\mathcal{H}}\| + d(N_0, M_0) < 11\gamma$.

By [3, Lemma 4.8] and a maximality argument, there exists an orthogonal family of projections $\{p_i\}_{i\in\Lambda}$ in $Z(M_0)$, cyclic projections $\{e_i\}_{i\in\Lambda}$ in M_0' and projections $\{f_i\}_{i\in\Lambda}$ in N_1' such that

$$\sum_{i\in\Lambda} p_i = 1_{\mathcal{H}} \ ,$$

$$e_i \le p_i, f_i \le p_i, c_{e_i} = p_i \ ,$$

$$\|e_i - f_i\| \le 21\sqrt{11\gamma}.$$

Let $e = \sum_{i\in\Lambda} e_i$ and $f = \sum_{i\in\Lambda} f_i$. Since M_0 is countably decomposable, Λ is countable. Since e is a sum of a countable family of cyclic projections in M_0' and $\{c_{e_i}\}_{i\in\Lambda}$ are pairwise orthogonal, e is a cyclic projection in M_0'. Also, the central support of e relative to M_0' is $1_{\mathcal{H}}$. Since

$$\|e - f\| \leq 21\sqrt{11\gamma} < 1,$$

by Proposition 2.25, there exists a unitary $u \in B(\mathcal{H})$ such that $u^*eu = f$ and

$$\|u - 1_{\mathcal{H}}\| \leq \sqrt{2}\|e - f\| \leq 21\sqrt{2}\sqrt{11\gamma}.$$

Then $e \in uN_1'u^*$. Since

$$d(M_0, uN_1u^*) \leq d(M_0, N_1) + 2\|u - 1_{\mathcal{H}}\| \leq 11\gamma + 42\sqrt{2}\sqrt{11\gamma} < 210\sqrt{\gamma} < 1/4 ,$$

by Lemma 6.4, e has central support 1 relative to $uN_1'u^*$. Let $\mathcal{K}$ be $e\mathcal{H}$, M be $(M_0)_e$ and $N = (uN_1u^*)_e$. Then

$$d(M, N) \leq d(M_0, uN_1u^*) < 210\sqrt{\gamma}$$

and the proposition follows. □

Theorem 6.9. *Let M be a countably decomposable von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If M is weakly Kadison-Kastler stable on $\mathcal{H}$, then M is weakly Kadison-Kastler stable.*

Proof. This follows by combining Propositions 6.7 and 6.8. □

Proposition 6.10. *Let M and N be type* III *factors acting on a Hilbert space $\mathcal{H}$.*

*Assume that M has separable predual and $d(M,N) < \gamma < 1/300$. If $\varphi : M \to N$ is a *-isomorphism, then there exists a unitary $w \in B(\mathcal{H})$ such that $\varphi(x) = wxw^*$ for any $x \in M$.*

Proof. First, observe that for any vector $\xi \in \mathcal{H}$, $[M\xi]$ is separable. Indeed, since M has a separable predual, the closed unit ball of M is compact and metrizable under the weak operator topology. Hence the closed unit ball of M has a countable WOT-dense subset C. Thus $[M\xi] = [C\xi]$ is separable.

Let $\{e_i\}_{i\in\Lambda}$ be an orthogonal family of cyclic projections in M' with sum equal to $1_{\mathcal{H}}$. Let A be the von Neumann algebra generated by $\{e_i : i \in \Lambda\}$. Then A is abelian. Since $N \subset_\gamma M$ and N is properly infinite, by Proposition 2.29 (i), $M' \subset_{3\gamma} N'$. Then $A \subset_{3\gamma} N'$. By Theorem 2.35 (i), there exists a unitary $u \in B(\mathcal{H})$ such that $uAu^* \subseteq N$ and $\|u - 1_{\mathcal{H}}\| < 150 \cdot 3\gamma$. Let $N_1 = u^*Nu$. Then $e_i \in N_1'$ for all $i \in \Lambda$. Define a *-isomorphism $\psi_i : M_{e_i} \to (N_1)_{e_i}$ by

$$\psi_i(x_{e_i}) = (u^*\varphi(x)u)_{e_i} \text{ for any } x \in M.$$

By the first paragraph, $e_i\mathcal{H}$ is separable for all $i \in \Lambda$. Since M_{e_i} and N_{e_i} are type III von Neumann algebras acting on a separable Hilbert space $e_i\mathcal{H}$, by [25, Proposition 9.1.6], both algebras have a cyclic and separating vector. By [25, Theorem 7.2.9], there exists a unitary $v_i \in B(e_i\mathcal{H})$ such that $\psi_i(x_{e_i}) = v_i x_{e_i} v_i^*$ for any $x \in M$. Let $v = \sum_{i\in\Lambda} v_i$. Then v is a unitary in $B(\mathcal{H})$ and for any $x \in M$,

$$vxv^* = \sum_{i\in\Lambda} v_i x_{e_i} v_i^* = \sum_{i\in\Lambda} (u^*\varphi(x)u)_{e_i} = u^*\varphi(x)u.$$

Thus $\varphi(x) = uvx(uv)^*$. □

Theorem 6.11. *Let M be a type* III *factor with a separable predual acting on a*

Hilbert space $\mathcal{H}$. If M is weakly Kadison-Kastler stable on $\mathcal{H}$, then M is Kadison-Kastler stable.

Proof. By Theorem 6.9, M is weakly Kadison-Kastler stable. In other words, there exists $\epsilon > 0$ such that: if $(\rho, \mathcal{L})$ is any faithful normal unital *-representation of M and $N \subseteq B(\mathcal{L})$ is another von Neumann algebra with $d(\rho(M), N) < \epsilon$, then $\rho(M)$ and N are *-isomorphic. By [23, Corollary A and Lemma 11], N is also a type III factor whenever $d(\rho(M), N)$ is sufficiently small (independent of ρ). Then, when $d(\rho(M), N)$ is sufficiently small, any *-isomorphism between $\rho(M)$ and N is spatial by Proposition 6.10. This proves the theorem. $\square$

Corollary 6.12. *The von Neumann algebras of Examples 5.6 and 5.7 are respectively weakly Kadison-Kastler stable and Kadison-Kastler stable.*

Proof. These algebras have separable preduals so Theorems 6.9 and 6.11 apply. $\square$

Lemma 6.13. *Let M and N be von Neumann algebras acting on the same Hilbert space $\mathcal{H}$ and $\mathcal{K}$ be another Hilbert space. Then $d(M, N) \leq d(M \bar{\otimes} B(\mathcal{K}), N \bar{\otimes} B(\mathcal{K}))$.*

Proof. Let $\epsilon = d(M \bar{\otimes} B(\mathcal{K}), N \bar{\otimes} B(\mathcal{K}))$ and let ρ be any normal state on $B(\mathcal{K})$. Then there exists a normal completely positive map $E : B(\mathcal{H}) \bar{\otimes} B(\mathcal{K}) \to B(\mathcal{H}) \bar{\otimes} \mathbb{C}1_{\mathcal{K}}$ such that $E(x \otimes y) = x \otimes \rho(y) 1_{\mathcal{K}}$ for any $x \in B(\mathcal{H})$ and $y \in B(\mathcal{K})$. Note that $E(M \bar{\otimes} \mathbb{C}1_{\mathcal{K}}) = M \bar{\otimes} \mathbb{C}1_{\mathcal{K}}$ and $E(N \bar{\otimes} \mathbb{C}1_{\mathcal{K}}) = N \bar{\otimes} \mathbb{C}1_{\mathcal{K}}$. Let x be any element in the unit ball of M. Then there exists an element z in the unit ball of $N \bar{\otimes} B(\mathcal{K})$ such that $\|x \otimes 1_{\mathcal{K}} - z\| \leq \epsilon$. Note that $E(z) = y \otimes 1_{\mathcal{K}}$ for some element y in the unit ball of N. Then

$$\|x - y\| = \|x \otimes 1_{\mathcal{K}} - y \otimes 1_{\mathcal{K}}\| = \|E(x \otimes 1_{\mathcal{K}} - z)\| \leq \|x \otimes 1_{\mathcal{K}} - z\| \leq \epsilon.$$

By the same argument, we can show that any element in the unit ball of N can be approximated by an element in the unit ball of M within a distance ϵ. This shows that $d(M,N) \leq d(M\bar{\otimes}B(\mathcal{K}), N\bar{\otimes}B(\mathcal{K}))$. $\square$

Proposition 6.14. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\mathcal{K}$ be any Hilbert space. If M is strongly Kadison-Kastler stable, so is $M\bar{\otimes}B(\mathcal{K})$.*

Proof. Let $\eta > 0$. Let $\epsilon > 0$ be a number as in Definition 4.1(i) for M. Let $\pi : M\bar{\otimes}B(\mathcal{K}) \to B(\mathcal{L})$ be any faithful normal unital *-representation. Let ϵ_1 be $\min\{\frac{\eta}{150}, \frac{\epsilon}{301}, \frac{1}{100}\}$, and let $M_1 = \pi(M\bar{\otimes}B(\mathcal{K}))$. Assume that $N \subseteq B(\mathcal{L})$ is any von Neumann algebra such that $d(N, M_1) < \epsilon_1$. Let P be $\pi(\mathbb{C}1_{\mathcal{H}}\bar{\otimes}B(K))$. Then P is injective and $P \subset_{\epsilon_1} N$. By Theorem 2.35 (i), there is a unitary $u \in B(\mathcal{L})$ such that $\|u - 1_{\mathcal{L}}\| \leq 150\epsilon_1$ and $uPu^* \subseteq N$. Let N_1 be u^*Nu. Then $P \subseteq N_1$ and

$$d(M_1, N_1) \leq d(M_1, N) + 2\|u - 1_{\mathcal{L}}\| < \epsilon_1 + 300\epsilon_1 = 301\epsilon_1. \tag{6.1}$$

Let $\{\xi_i\}_{i\in\Lambda}$ be an orthonormal basis for $\mathcal{K}$ and $\{e_{i,j}\}_{i,j\in\Lambda}$ be a system of matrix units for $B(\mathcal{K})$ such that $e_{i,i}$ is a rank-one projection onto the subspace spanned by ξ_i. Fix $i_0 \in \Lambda$. Let $f_{i,j} = \pi(1_{\mathcal{H}} \otimes e_{i,j})$. Let $w : \mathcal{L} \to f_{i_0,i_0}(\mathcal{L})\bar{\otimes}\mathcal{K}$ be the unitary map defined by

$$w(\zeta) = \sum_{i\in\Lambda} f_{i_0,i}\zeta \otimes \xi_i \text{ for any } \zeta \in \mathcal{L}.$$

For any $x \in P'$, x commutes with any $f_{i,j}$ and hence $wxw^* = x_{f_{i_0,i_0}} \otimes 1_{\mathcal{K}}$. This shows that $wM_1'w^* = (M_1')_{f_{i_0,i_0}}\bar{\otimes}\mathbb{C}1_{\mathcal{K}}$ and hence $wM_1w^* = (M_1)_{f_{i_0,i_0}}\bar{\otimes}B(\mathcal{K})$. Similarly, $wN_1w^* = (N_1)_{f_{i_0,i_0}}\bar{\otimes}B(\mathcal{K})$. Then by Lemma 4.13 and inequality (6.1),

$$d((M_1)_{f_{i_0,i_0}}, (N_1)_{f_{i_0,i_0}}) \leq d(wM_1w^*, wN_1w^*) = d(M_1, N_1) < 301\epsilon_1 \leq \epsilon. \tag{6.2}$$

Note that π induces a *-isomorphism between $(M_1)_{f_{i_0,i_0}}$ and $(M\bar{\otimes}B(\mathcal{K}))_{1_{\mathcal{H}}\otimes e_{i_0,i_0}}$. Hence, $(M_1)_{f_{i_0,i_0}}$ is *-isomorphic to M. Since M is strongly Kadison-Kastler stable and by inequality (6.2), there is a unitary $v \in B(f_{i_0,i_0}(\mathcal{L}))$ such that $\|v-1\| < \eta$ and $v(M_1)_{f_{i_0,i_0}}v^* = (N_1)_{f_{i_0,i_0}}$. Let v_2 be $w^*(v\otimes 1_{\mathcal{K}})w$. Then $\|v_2 - 1_{\mathcal{L}}\| = \|v-1\| < \eta$ and $v_2M_1v_2^* = N_1 = u^*Nu$. So $uv_2M_1(uv_2)^* = N$ and $\|uv_2 - 1_{\mathcal{L}}\| \leq \|u-1_{\mathcal{L}}\| + \|v-1_{\mathcal{L}}\| < 150\epsilon_1 + \eta \leq 2\eta$. $\square$

Remark 6.15. *For any integer $n \geq 3$, let $\alpha : SL_n(\mathbb{Z}) \curvearrowright (X,\mu)$ be a free, ergodic and measure preserving action of $SL_n(\mathbb{Z})$ on a standard nonatomic probability space (X,μ). Theorem A of [3] states that $M = (L^\infty(X,\mu) \rtimes_\alpha SL_n(\mathbb{Z}))\bar{\otimes}R$ is a strongly Kadison-Kastler stable* II_1 *factor where R is the hyperfinite* II_1 *factor. Combining this result with Proposition 6.14, $M = (L^\infty(X,\mu)\rtimes_\alpha SL_n(\mathbb{Z}))\bar{\otimes}R\bar{\otimes}B(\mathcal{K})$ is a strongly non-injective Kadison-Kastler stable* II_∞ *factor for any infinite dimensional Hilbert space $\mathcal{K}$.*

7. CONCLUSION

In section 4, we have shown that $P \rtimes_\alpha G$ is *-isomorphic to any nearby algebras when P is an injective von Neumann with a cyclic tracial vector and α is a free action of a free group G of n ($n = 2, 3..., \infty$). As remarked before (See Remark 4.12), the same result holds for more general groups when a cohomological condition is satisfied. It would be interesting to know if this result holds for all countable discrete groups without any cohomological assumptions on the action α.

REFERENCES

[1] Bruce Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.

[2] Nathanial Brown, Narutaka Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, 88, AMS, 2008.

[3] Jan Cameron, Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Alan D. Wiggins, *Kadison-Kastler Stable Factors*, Duke Math. J. 163 (2014), no. 14, 2639-2686.

[4] Jan Cameron, Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Alan D. Wiggins, *Type* II_1 *factors satisfying the spatial isomorphism conjecture*, Proc. Natl. Acad. Sci. USA 109 (2012), 20338-20343.

[5] Jan Cameron, Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Alan D. Wiggins, *A remark on the similarity and perturbation problems*, C. R. Math. Acad. Sci. Soc. R. Can. 35 (2013), 70-76.

[6] Marie Choda, *Normal expectations and crossed products of von Neumann algebras*, Proc. Japan Acad. 50 (1974), 738-742.

[7] Man-Duen Choi and Erik Christensen, *Completely order isomorphic and close C^*-algebras need not be $*$-isomorphic*, Bull. London Math. Soc.15 (1983), no. 6, 604-610.

[8] Wai-Kit Chan, *Perturbations of certain crossed product algebras by free groups*, J. Funct. Anal. 267 (2014), no. 10, 3994-4027.

[9] Erik Christensen, *Perturbations of Type I von Neumann algebras*, J. London Math. Soc.2 (1975), no. 9, 395-405.

[10] Erik Christensen, *Perturbations of Operator Algebras*, Invent. Math. 43 (1977), no. 1, 1-13.

[11] Erik Christensen, *Perturbations of Operator Algebras II*, Indiana Univ. Math. J. 26 (1977), no. 5, 891-904.

[12] Erik Christensen, *Near inclusions of C^*-algebras*, Acta Math. 144 (1980), no. 3-4, 249-265.

[13] Erik Christensen, *Extensions of derivations II*, Math. Scand. 50 (1982), no. 1, 111-122.

[14] Erik Christensen, Allan M. Sinclair, Roger R. Smith, and Stuart White, *Perturbations of C^*-algebraic invariants*, Geom. Funct. Anal. 20 (2010), no. 2, 368-397.

[15] Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Wilhelm Winter, *Perturbations of nuclear C^*-algebras*, Acta. Math. 208 (2012), no. 1, 93-150.

[16] Alain Connes, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. (2) 104 (1976), no. 1, 73-115.

[17] Sergio Doplicher, Roberto Longo, *Standard and split inclusions of von Neumann algebras*, Invent. Math. 75 (1984), no. 3, 493-536.

[18] George Elliott, *On approximately finite-dimensional von Neumann algebras*, II. Canad. Math. Bull. 21 (1978), no. 4, 415-418.

[19] Junsheng Fang, Roger R. Smith, Stuart White, and Alan D. Wiggins, *Groupoid normalizers of tensor products*, J. Funct. Anal. 258 (2010), no. 1, 20-49.

[20] Uffe Haagerup, *A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space*, J. Funct. Anal. 62 (1985), no. 2, 160-201.

[21] Cyril Houdayer and Stefaan Vaes, *Type* III *factors with unique Cartan decomposition*, J. Math. Pures Appl. (9) 100 (2013), no. 4, 564-590.

[22] Barry Johnson, *Perturbations of Banach algebras*, Proc. London Math. Soc. (3) 34 (1977), no. 3, 439-458.

[23] Richard V. Kadison and Daniel Kastler, *Perturbations of von Neumann algebras. I. Stability of type.*, Amer. J. Math. (1972), no 94, 38-54.

[24] Richard V. Kadison and John R. Ringrose, *Fundamentals of the Theory of Operator Algebras Vol.1*, AMS,1997.

[25] Richard V. Kadison and John R. Ringrose, *Fundamentals of the Theory of Operator Algebras Vol.2*, AMS,1997.

[26] Mahmood Khoshkam,*Perturbations of C*-algebras and K-theory*, J. Op. Theory 12 (1984), no. 1, 89-99.

[27] Gerard J. Murphy, *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.

[28] John Phillips and Iain Raeburn, *Perturbations of AF-algebras*, Canad. J. Math.31 (1979), no. 5, 1012-1016.

[29] John Phillips and Iain Raeburn, *Perturbations of C^*-algebras. II*, Proc. London Math. Soc. (3) 43 (1981), no. 1, 46-72.

[30] Iain Raeburn and Joseph Taylor, *Hochschild cohomology and perturbations of Banach algebras*, J. Funct. Anal. 25 (1977), no. 3, 258-266.

[31] Masamichi Takesaki, *Theory of operator algebras. I*, Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin, 2002.

[32] Masamichi Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.

[33] Jun Tomiyama, *On the projection of norm one in W*-algebras*, Proc. Japan Acad. 33, 1957, 608-612.