MEROMORPHIC INNER FUNCTIONS AND THEIR APPLICATIONS

A Dissertation

by

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ABSTRACT

In this dissertation we study an important class of functions in complex analysis, known as Meromorphic Inner Functions (MIF) and we exploit their properties to solve problems from mathematical physics.

In the first part, we answer an old problem, first studied by Louis de Branges about a property of the derivative of MIFs on the real line. We use the theory of Clark measures to solve this problem with the aid of complex and harmonic analysis theory.

In the second part, we study the spectral properties of the Schrödinger operator. Certain inverse spectral problems in this area can be translated into the language of complex analysis and we use the injectivity of Toeplitz operators to solve these problems.

DEDICATION

This dissertation is dedicated to my parents,

Nutan Chandra & Priyaranjan.

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CHAPTER I

INTRODUCTION

A classical question in complex analysis is the following: Given a sequence $\{\lambda_n\}_n$ in $\mathbb{R}$, is the family of exponentials $\{e^{i\lambda_n x}\}_n$ complete in $L^2(-a,a)$? Another way of stating this problem is : Does the sequence $\{\lambda_n\}$ form a zero set in the Paley Wiener space PW_a? This problem was solved by Beurling and Malliavin in a series of papers from the 1960s ([4], [5]). The answers provide metric characterizations of the radius of completeness of a family of exponentials and it was shown to be equal to the *Beurling Malliavin density* of the sequence (times a constant). In the paper [27], Makarov and Poltoratski provided a different proof of the Beurling Malliavin theorem, using model spaces and Toeplitz kernels. By the Paley Wiener theorem, the Hardy space on the upper half plane $\mathcal{H}^2(\mathbb{C}_+)$ can be obtained as the Fourier transform of $L^2(0,\infty)$. Let $S(z) = e^{iz}$, then the Paley Wiener space PW_a can be written in terms of the model spaces as

$$PW_a = S^{-a}[\mathcal{H}^2 \ominus S^{2a}\mathcal{H}^2].$$

The space $K_{S^a} = \mathcal{H}^2 \ominus S^{2a}\mathcal{H}^2$ is called the model space of the inner function S^{2a}. In general, we can define model spaces with respect to any inner function Θ as

$$K_\Theta := \mathcal{H}^2 \ominus \Theta\mathcal{H}^2.$$

We restrict our studies to Θ being a meromorphic inner function. Model spaces play an important role in modern function theory, see [30], [21]. As we saw before, completeness problems for exponentials is the same as the problem of defining uniqueness sets for the model space $K_{S^{2a}}$. The question of uniqueness sets of general model spaces is interesting in its own right, but as we will see, it answers many questions in the inverse problems of 2nd order differential operators.

For $U \in L^\infty(\mathbb{R})$, we define the Toeplitz operator $T_U : \mathcal{H}^2(\mathbb{R}) \to \mathcal{H}^2(\mathbb{R})$ as

$$T_U(f) = P_+(Uf),$$

where P_+ is the orthogonal projection onto $\mathcal{H}^2(\mathbb{R})$. A simple argument shows that $\Lambda \subset \mathbf{C}_+$ is a uniqueness set for K_Θ if and only if the Toeplitz operator $T_{\overline{\Theta} B_\Lambda}$ has a trivial kernel ([27], page 26). Here B_Λ is the Blaschke product with zeroes on Λ. A similar result holds if $\Lambda \subset \mathbb{R}$. The *injectivity* problem for a Toeplitz operator (characterizing symbols U such that ker $T_U = 0$) is useful in the spectral theory of Toeplitz operatos see ([7],[31]). Compared with the invertibility problem, the injectivity problem has received considerably less attention. However, we now understand well the significance of these problems in the context of the spectral theory of 2^{nd} order differential operators ([27],[8]). In this dissertation, we will explore this problem in detail.

The injectivity problem of Toeplitz operators is connected with the following inverse problem. Let us consider the Schrödinger equation

$$-u'' + qu = \lambda u \tag{1}$$

on some interval (a, b) and assume that the potential $q(t)$ is locally integrable and a is a *regular* point i.e., a is finite and q is in L^1 at a. Let us fix the following boundary conditions.

$$\cos(\alpha)u(a) + \sin(\alpha)u'(a) = 0 \tag{2}$$

$$\cos(\beta)u(b) + \sin(\beta)u'(b) = 0. \tag{3}$$

Consider the following problem - Let Λ, a separated sequence on $\mathbb{R}$ be spectral data of the above operator. Can we uniquely reconstruct the operator from this? Borg proved in [6] that in most cases, 2 spectra is sufficient to reconstruct the potential. The condition in Borg's paper was further relaxed by Levinson in [22]. Hochstadt and Liebermann prove in [18] that if half the potential is known, then one spectrum is enough to recover the whole potential. Simon and Gesztezy have proved sufficient conditions for partial potential and one spectrum to determine q uniquely ([15],[16]). Along with Del Rio in [11], they prove a sufficient condition in the case of spectral data coming from 3 spectra. In 2005, Horváth made a breakthrough in [19]. In his result, Horváth allowed all the $\lambda \in \Lambda$ to be in a different spectra. He proved that recovering the potential of a Schrödinger operator is equivalent to the closedness of the family of exponentials $\{e^{\pm 2i\sqrt{\lambda}x}\}_{\lambda\in\Lambda} \cup \{e^{\pm i\mu x}\}$ for some $\mu \notin \Lambda$ in $L^2(a,b)$. More results in this area are to be found in ([32],[9]). In [27], Makarov and Poltoratski establish the relationship of these inverse problems with uniqueness sets in the class of meromorphic inner functions on $\mathbf{C}_+$. In this dissertation, we give necessary and sufficient conditions for a set to be a uniqueness set for a meromorphic inner function. As a consequence, we obtain a metric condition for spectral data to determine the operator uniquely. Moreover, we provide the degree of non-uniqueness of the meromorphic inner function in terms of the dimension of an associated Toeplitz kernel. As a result, one can ask the following question: If some spectral data is not enough to reconstruct the potential uniquely, then can we calculate the number of different potentials that have the same spectral data? We discuss these questions in chapter 8. We also provide some examples of spectral data that do not correspond to any operator, despite being 'close to' spectral data of a given operator in chapter 8.

Another topic presented in this dissertation is an old question that was first studied by Louis de Branges in 1968: Given a *separated* sequence $\{\lambda_n\}$ on $\mathbb{R}$, does there exist an MIF Θ with $\{\lambda_n\}$ as spectrum, such that $|\Theta'|$ is uniformly bounded on $\mathbb{R}$? By spectrum of Θ, we are referring to the level set $\{x \in \mathbb{R} : \Theta(x) = 1\}$ and by separated sequence, we mean that there is a $\delta > 0$ such that $|\lambda_n - \lambda_m| > \delta$, for all $n \neq m$ integers. This question was first studied by Louis de Branges in 1968 in his book 'Hilbert Spaces of Entire Functions' [10]. In the book, De Branges asserts the existence of such a corresponding MIF for *every* separated sequence on $\mathbb{R}$ (lemma 16). In 2011, Anton Baranov discovered a flaw in de Branges proof and, in fact, discovered sequences which served as counterexamples to the assertion. For instance, the one-sided sequence of natural numbers $\mathbb{N}$ does not have a corresponding MIF with bounded derivative.

As we will see, MIFs are ubiquitous in the study of second order differential operators, for instance the Weyl Titchmarsh inner function originating from Schrödinger operators, Dirac systems and similar canonical systems. In particular, MIFs with bounded derivative play a special role in these studies. De Branges himself used this to solve *Beurling's gap problem*: For which sequences Λ does there exists a non-zero measure μ, supported on Λ such that $\hat{\mu}$ vanishes on an interval of positive measure. In [28] Mitkovski and Poltoratski use this property of MIFs to show the relationship between the gap problem and Pólya sequences. In our studies, we often encounter the question of uniqueness sets in model spaces and the related question of invertibility of Toeplitz operators. In [26], [27], Makarov and Poltoratski prove the Beurling Malliavin multiplier theorem, in a general form using Toeplitz kernels. Here, they use MIFs with bounded derivative. We elaborate on these applications in chapter 7.

We discover that in order to bound the derivative of an MIF, the spectrum must have *uniformity*. The exact meaning of this phrase is made clear in our results in chapter 3. We provide characterization of sequences for which there exist corresponding MIFs as well as some counter examples.

CHAPTER II

PRELIMINARIES

2.1 Theory of Hardy Spaces

2.1.1 Fatou's Theorem

In our introduction, we mentioned Dirichlet's problem for functions defined on the boundary of a disk. Let us recall the solution to Dirichlet's problem. Let u be a continuous function defined on $\mathbb{T}$. Then, the following function is harmonic

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} u(t)dt.$$

Moreover, $U(z) \to u(e^{i\phi})$ as $z \to e^{i\phi}$ and this convergence is uniform in ϕ. The function $\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}$ is called the Poisson kernel and is denoted by $P_r(\theta - t)$. The function U is called the Poisson transform of u. A natural question to ask is if the converse is true, i.e. if U is a harmonic function on $\mathbb{D}$, then can it be written as the Poisson transform of a continuous function on the boundary $\mathbb{T}$? The answer is - almost. To be precise, it is true that U is the Poisson transform of some function u defined on the boundary, but this function need not be continuous. This question makes one wonder about the boundary behavior of harmonic functions on the disk. We focus our attention on a particular class of harmonic functions, namely holomorphic functions. A simple question to ask is the following- Given a holomorphic function on the disk, what is the behavior of this function as we approach the boundary of this disk? In general, it is not possible to say anything about this behavior as the function may have singularities on the boundary. Now suppose we impose some additional conditions on the boundedness of the functions as we approach the boundary. Let us say we bound the L^1 norms of this function on consecutive circles, i.e.,

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})| d\theta < \infty. \tag{2.4}$$

In this case Fatou proved that f will have radial boundary limits and the function converges to the boundary function in the L^1 norm.

Lemma 1. *Let f be a holomorphic function on $\mathbb{D}$, satisfying 2.4 above. Then, $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists a.e. and*

$$\lim_{r\uparrow 1} \int_0^{2\pi} |f(re^{i\theta} - f(e^{i\theta})| d\theta = 0$$

Holomorphic functions on the unit disk which satisfy condition 2.4 form a Banach space, denoted by $\mathcal{H}^1$, with norm defined as

$$\|f\|_1 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})| d\theta.$$

Thus,

$$\mathcal{H}^1(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})| d\theta < \infty \right\}.$$

In fact Fatou's theorem is much stronger than that stated above and we refer the reader to [] for further reading. We also mention that $\mathcal{H}^1(\mathbb{D})$ is a *Hardy space.* In general we can define Hardy spaces corresponding to any $1 \leqslant p < \infty$ as follows.

$$\mathcal{H}^p(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}.$$

Fatou's theorem for general Hardy spaces $\mathcal{H}^p(\mathbb{D})$ for $1 \leqslant p < \infty$ is stated as the following.

Theorem 1. *If $f \in \mathcal{H}^p(\mathbb{D})$, then for almost all $e^{i\theta}$, $\lim f(z)$ for $z \to_\angle e^{i\theta}$ exists and is finite. If we call this limit $f(e^{i\theta})$, then*

$$\int_0^\pi |f(e^{i\theta})|^p d\theta = \|f\|_p^p.$$

Moreover, we have that

$$\int_0^\pi |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta \to 0,$$

as $r \to 1$.

By $z \to_\angle e^{i\theta}$, we mean that z approaches $e^{i\theta}$ non tangentially, i.e, there exists a Stolz region, given by $\{w : ||w| - \theta| < c(1 - |w|)\}$ for some fixed $c > 0$, such that z converges to $e^{i\theta}$ in this region. For more information on nontangential limits, we refer the reader to [20].

Let us mention here that analogous spaces exist for the upper half plane $(\mathbb{C}_+)$. We define them as follows:

$$\mathcal{H}^p(\mathbb{C}_+) = \left\{ f \in Hol(\mathbb{C}_+) : \sup_{y>0} |f(x+iy)|^p < \infty \right\}.$$

Given a function f in $\mathcal{H}^p(\mathbb{C}_+)$, where $1 \leqslant p < \infty$, $\lim_{y \to 0} f(x+iy) =: f(x)$ exists and is finite for almost all $x \in \mathbb{R}$. Moreover,

$$f(x+iy) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t) dt.$$

2.1.2 Nevanlinna Class

Let f be an analytic function in $\mathbb{D}$. It is said to be in the **Nevanlinna class** $\mathcal{N}$ if the following is true

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f(re^{i\theta})| d\theta < \infty.$$

Similarly, on $\mathbb{C}_+$, the space $\mathcal{N}(\mathbb{C}_+)$ consists of all those analytic functions f such that

$$\sup_{y>0} \int_{\mathbb{R}} \log^+ |f(x+iy)| dy < \infty.$$

The **Smirnov Nevanlinna class** $\mathcal{N}^+$ consists of all those functions in $\mathcal{N}$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta}| d\theta,$$

in the case of the disk and

$$\lim_{y\to 0} \int_{\mathbb{R}} \log^+ |f(x+iy)| dx = \int_{\mathbb{R}} \log^+ |f(x)| dx,$$

in the case of the upper half plane.

It is easy to see that $H^p \subset \mathcal{N}$ for all $p > 0$. We know that for any $f \in \mathcal{H}^p(\mathbb{D})$, its non-tangential limit to the boundary is in L^p. In fact,

$$\mathcal{H}^p = \mathcal{N}^+ \cap L^p.$$

2.1.3 Inner Outer Factorization

A very useful property of functions that lie in the Hardy spaces is that they can be factorised in terms of certain functions whose behaviour we understand well. This factorization is often called the canonical factorization. We first state them for the unit disk.

A **Blaschke** term B_a is a function of the form

$$B_a(z) = \frac{|a|}{a} \frac{a - z}{1 - \bar{a}z},$$

where $a \in \mathbb{D}$. This function is holomorphic in $\mathbb{D}$, has the property that $|B_a(z)| < 1$ on $\mathbb{D}$ and $|B_a(z)| = 1$ on $\mathbb{T}$. A Blaschke product is a product of Blaschke terms

$$B(z) = e^{ic} \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}z},$$

where we impose the condition that $\sum_n (1 - |z_n|) < \infty$ to ensure that the (possibly) infinite product converges.

A **singular** inner function S has the form

$$S(z) = \exp^{-\frac{1}{2\pi} \int \frac{e^{it}+z}{e^{it}-z} d\sigma(t)},$$

where σ is a measure on the circle $\mathbb{T}$ that is singular with respect to the Lebesgue measure on the circle.

Given a function $f \in \mathcal{H}^p(\mathbb{D})$, the following function is called an outer function O_f corresponding to f.

$$O_f(z) = \exp^{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log |f(e^{it})| dt}.$$

We have the following factorization.

Theorem 2. *Let $f \in \mathcal{H}^p(\mathbb{D})$. Then there exists a Blaschke product B, a singular inner function S and outer function O_f such that the following representation is true in $\mathbb{D}$*

$$f(z) = B(z)S(z)O_f(z).$$

On the upper half plane we have a corresponding factorization. **Blaschke** products are of the form

$$B(z) = \prod_n \frac{z - a_n}{z - \overline{a_n}},$$

where the a_n are in $\mathbb{C}_+$ such that they satisfy the convergence criterion $\sum_n \frac{\Im a_n}{1+|a_n|^2} < \infty$.

A **singular inner function** S is of the form

$$S(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t),$$

where σ is a measure on the real line that is singular with respect to the Lebesgue measure.

An **outer** function O_f is of the form

$$\frac{1}{\pi i} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) f(t) dt,$$

where f is a real valued function in L^1_π, which means that $\int_{\mathbb{R}} \frac{f(t)}{1+t^2} dt < \infty$.

The factorization breaks down in the case of $\mathcal{N}$, as we see from the example $\exp(\frac{1+z}{1-z})$.

2.1.4 Cauchy Transform

As we saw in a previous section, the Poisson kernel is used to recover functions from their boundary values. We study a similar kernel in this section, called the **Herglotz kernel** defined as follows.

$$H_z(\tau) = \frac{\tau + z}{\tau - z},$$

which is analytic in z with $\Re H_z = P_z > 0$, where P_z is the Poisson kernel. The **Herglotz integral** is defined as

$$H\mu(z) := \int H_z(\tau) d\mu(\tau),$$

and is analytic on $\mathbb{D}$ and has positive real part whenever μ is a positive Borel measure. We consider the following result by Herglotz quite crucial to our learning. We refer the reader to [13] for proofs.

Theorem 3. *(Herglotz)*

1. *If $u \geqslant 0$ on $\mathbb{D}$ is harmonic, then $u = P\mu$ for some positive Borel measure μ.*

2. *If f is analytic on $\mathbb{D}$, $\Re f \geqslant 0$ and $f(0) > 0$, then $f = H\mu$ for some positive Borel measure μ.*

A corresponding result holds for the upper half plane as well. The integral corresponding to the Herglotz integral on the upper half plane is the Schwarz integral defined as follows.

$$S\mu(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\mu(t),$$

where μ is a positive measure on $\mathbb{R}$ which is also Poisson finite, i.e.,

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1 + t^2} < \infty.$$

2.2 Meromorphic Inner Functions

2.2.1 Definition and Examples

We have seen in the previous section that an inner function on the upper half plane $\mathbb{C}_+$ is a bounded analytic function on $\mathbb{C}_+$ with unit modulus almost everywhere on the real line $\mathbb{R}$. A **Meromorphic inner function (MIF)** on $\mathbb{C}_+$ is an inner function on $\mathbb{C}_+$ with a meromorphic continuation to $\mathbb{C}$. Examples of MIFs are :

- e^{iaz}, where $a > 0$,
- A Blaschke product $\prod \frac{z-w}{z-\bar{w}}$, where $w \in \mathbb{C}_+$.

It is easy to see that the above examples satisfy the criterion for being MIFs. In fact, the following characterization of MIFs was proved by Riesz and V.I. Smirnov.

Lemma 2. *Given any MIF Θ, it can be represented as*

$$\Theta(z) = B_\Lambda(z)e^{iaz}$$

where $a \geqslant 0$ and B_Λ is the Blaschke product of the zeros of the function given by $\Lambda = \{\lambda_n\}_n$ where $|\lambda_n| \to \infty$ and satisfy the convergence criterion,

$$\sum_{\lambda_n \in \Lambda} \frac{\Im \lambda_n}{1 + |\lambda_n|^2} < \infty.$$

2.2.2 Spectrum and Properties

For a given MIF Θ, we define its **spectrum**, $\sigma(\Theta)$ as follows

$$\sigma(\Theta) = \{x \in \mathbb{R} : \Theta(x) = 1\}.$$

For any MIF theta, $\sigma(\Theta)$ is a discrete set on $\mathbb{R}$.

Lemma 3. *Given any MIF Θ,*

$$\Theta(z) = \frac{1}{\overline{\Theta(\overline{z})}}. \tag{2.5}$$

We can use Weierstrass' factorization to see that the above holds. A proof is given in [17].

Lemma 4. *Given a meromorphic inner function Θ, there exists an increasing, real analytic function $\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$\Theta = e^{i\phi}, \ \ \textit{on } \mathbb{R}$$

We can use the Riesz-Smirnov factorization to verify this property for the functions e^{iaz} and $\frac{z-w}{z-\bar{w}}$ and hence for their products.

2.2.3 Clark Measures

Given a function $f : \mathbb{D} \to \mathbb{D}$, let us consider the following function. For a given $\alpha \in \mathbb{T}$

$$u_\alpha(z) = \Re\left(\frac{\alpha + f(z)}{\alpha - f(z)}\right) = \left(\frac{1 - |f(z)|^2}{|\alpha - f(z)|^2}\right). \tag{2.6}$$

Since u_α is the real part of an analytic function, it must be harmonic on $\mathbb{D}$. Moreover, by Herglotz's theorem, there is a positive Borel measure μ such that

$$u_\alpha(z) = \int_{\mathbb{T}} \left(\frac{\tau + z}{\tau - z}\right) d\mu(\tau).$$

This measure is called the Aleksandrov Clark (AC) or simply the Clark measure for the function f. Conversely if μ is any positive Borel measure on $\mathbb{T}$, then let $H\mu$ be the Herglotz transform of μ, i.e.,

$$H\mu(z) = \int_{\mathbb{T}} \left(\frac{\tau + z}{\tau - z}\right) d\mu(\tau).$$

Then $\Re H\mu > 0$. We can use the fractional linear transformation

$$\lambda \to \frac{\lambda - 1}{\lambda + 1},$$

which maps the right half plane $\Re\lambda > 0$ onto $\mathbb{D}$. Thus, $f(z) := \frac{H\mu(z)-1}{H\mu(z)+1}$ is an analytic self map on $\mathbb{D}$. We notice that

$$H\mu(z) = \frac{1 + f(z)}{1 - f(z)}.$$

Thus, we see that

$$P\mu(z) = \Re(H\mu)(z) = \frac{1 - |f(z)|^2}{|1 - f(z)|^2}.$$

Thus μ is an AC measure for f. If μ was chosen to be singular with respect to the Lebesgue measure, then the corresponding function f in inner. Conversely, if the function f is inner, then its AC measure is singular. As is to be expected, a corresponding theory holds for the upper half plane. Let us focus on MIFs. Let μ be a positive measure defined on the real line such that

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1 + t^2} < \infty.$$

The above property is called being **Poisson finite**. For such a measure, one defines its Cauchy transform as follows.

$$K\mu(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\mu(t).$$

By our discussions in a previous section, we know that $K\mu : \mathbb{C}_+ \to \{\Re z > 0\}$. We can compose this function with the fractional linear transformation $\lambda \to \frac{\lambda-1}{\lambda+1}$ to get a function $\Theta : \mathbb{C}_+ \to \mathbb{D}$. Thus,

$$\Theta = \frac{K\mu - 1}{K\mu + 1}.$$

In particular if the measure has the form

$$\mu(z) = \sum_n w_n \delta_{\lambda_n},$$

where $w_n > 0$ and $\{\lambda_n\}$ is a separated sequence on $\mathbb{R}$ such that

$$\sum_n \frac{w_n}{1+\lambda_n^2} < \infty,$$

then the corresponding function Θ obtained as above is a MIF.

2.3 De Branges Spaces

2.3.1 Definition and Examples

An entire function E with the property that

$$|E(z)| > |E(\bar{z})|, \text{ for } z \in \mathbb{C}_+, \tag{2.7}$$

is called a De Branges function, named after the mathematician Louis de Branges. Given a De Branges function E, the space of functions defined as follows is called a De Branges space of functions, corresponding to E.

$$\mathcal{B}_E := \left\{ f \in Hol(\mathbb{C}) : \frac{f}{E}, \frac{f^\#}{E} \in H^2(\mathbb{C}_+) \right\}. \tag{2.8}$$

We recall that $f^{\#}(z) := \overline{f(\bar{z})}$. The space $\mathcal{B}_E$ is a Hilbert space. In fact, it is a reproducing kernel Hilbert space, with the kernel defined as follows.

$$K_z(w) = \frac{\overline{E(z)}E(w) - E(\overline{z})\overline{E(\overline{w})}}{2i(\overline{z} - w)}.$$

Example: The function $E(z) = e^{-iaz}$ is a De Branges function for $a > 0$. The related De Branges space is the Paley Wiener space PW_a. In this case, the reproduucing kernel is given by the sinc function $\frac{\sin a(\overline{z}-w)}{(\overline{z}-w)}$.

2.3.2 Relationship with MIFs

Let E be a De Branges function. Then, there is a meromorphic inner function Θ that is related to E in the following way.

$$\Theta(z) = \frac{E^{\#}(z)}{E(z)}.$$

Conversely, given any MIF Θ, there is a De Branges function E such that the formula above holds true. Let $\Theta(t) = e^{i\phi(t)}$ on $\mathbb{R}$. Then, the phase function of the related E is defined as $-\frac{1}{2}\phi(t)$. The MIF related to the Paley Wiener space PW_a is e^{2iaz} and the related phase function is $-at$.

CHAPTER III

MODEL SPACES, INVERSE SPECTRAL THEORY AND BEURLING MALLIAVIN THEORY

In this chapter, we reflect on some results by Makarov and Poltoratski in [27] concerning uniqueness sets for model spaces. We will see the connection with the inverse spectral theory of differential operators. The following two sections contain results and proofs from [27].

3.1 Model Spaces

As we observed in the introduction, model spaces are generalized Paley Wiener spaces. Let Θ be an inner function in $\mathcal{H}^2$. Then the **model space** K_Θ is defined as

$$K_\Theta := \mathcal{H}^2 \ominus \Theta\mathcal{H}^2.$$

For any $p \geqslant 1$, the corresponding definition of a model space in $\mathcal{H}^p$ is given by

$$K_\Theta^p = \{f \in \mathcal{H}^p : \overline{\Theta} f \in \overline{\mathcal{H}^p}\}.$$

Given an inner function Θ, we define model spaces in the Smirnov class and general Hardy spaces to be the spaces

$$\begin{aligned} K_\Theta^+ &= \{F \in \mathcal{N}^+ \cap C^\omega(\mathbb{R}) : \Theta\bar{F} \in \mathcal{N}^+\}, \\ K_\Theta^p &= K_\Theta^+ \cap L^p(\mathbb{R}). \end{aligned}$$

We recall that to every $U \in L^\infty(\mathbb{R})$, there corresponds the Toeplitz operator T_U :

$\mathcal{H}^2 \to \mathcal{H}^2$ defined as

$$T_U(f) = P_+(Uf),$$

where P_+ is the projection onto $\mathcal{H}^2$. The *Toeplitz kernel* is defined as

$$\ker_U := \ker T_U.$$

As with model spaces, we can also define the Toeplitz kernels in the Smirnov class and Hardy spaces as

$$\begin{aligned} \ker_U^+ &= \{F \in \mathcal{N}^+ \cap L^1_{loc}(\mathbb{R}) : \overline{UF} \in \mathcal{N}^+\} \\ \ker_U^p &= \ker_U^+ \cap L^p(\mathbb{R}), \quad (0 < p \leqslant \infty). \end{aligned}$$

3.2 Basic Theory in $\mathcal{H}^p$ Spaces

In this section, we see conditions for a Toeplitz operator to be injective. The proofs are reproduced from [27].

Lemma 5. *The Toeplitz kernel* $\ker_U^p \neq \{0\}$ *iff the symbol has the following representation:*

$$U = \bar{\Phi}\frac{\bar{H}}{H},$$

where $H \in \mathcal{H}^p \cap L^1_{\text{loc}}(\mathbb{R})$ *is an outer function and* Φ *is an inner function.*

Proof. Let F and G be the functions in $\mathcal{H}^p$ such that $UF = \overline{G}$. Then, $|F| = |G|$ on $\mathbb{R}$. Consider the following inner-outer factorization: $F = F_i F_e$ and $G = G_i G_e$. We have $F_e = G_e$, and

$$U = (\bar{F}_i \bar{G}_i)\frac{\bar{F}_e}{F_e}.$$

Conversely, if $U = \bar{\Phi}\frac{\bar{H}}{H}$, Then, $UH = \bar{\Phi}\bar{H}$. Since $H \in \mathcal{H}^p$, we have that $\ker_U^p \neq$

$\{0\}$. $\square$

The proposition above also holds for $\ker_U^+$. The following result can be considered an extension of the previous result. We state the condition in terms of the exponents of the symbols involved and provide the same proof as given in [27], as this construction is important for our results ahead.

Proposition 1. *Let $\gamma \in C^\omega(\mathbb{R})$. Then $\ker_{e^{i\gamma}}^+ \neq 0$ iff γ has a representation*

$$\gamma = -\alpha + \tilde{h},$$

where $\alpha \in C^\omega(\mathbb{R})$ is an increasing function and $h \in L^1_\Pi$.

Proof. We first observe that $\ker_U^+ \neq 0$ iff

$$U = \bar{\Phi}\frac{\bar{H}}{H} \qquad \text{on } \mathbb{R} \tag{3.9}$$

or some outer function $H \in C^\omega(\mathbb{R})$ that does not vanish on $\mathbb{R}$, and some meromorphic inner function Φ. Indeed, suppose $\ker_U^+ \neq 0$. Reasoning as in the previous Lemma, we see that

$$U = \bar{I}\frac{\bar{F}}{F},$$

for some meromorphic inner function I and an outer function $F \in C^\omega(\mathbb{R})$. The outer function may have zeros on the real line. Suppose the zeros are simple. Take any meromorphic inner function J such that $\{J = 1\} = \{F = 0\}$. Then the outer function

$$H = \frac{F}{1 - J} \tag{3.10}$$

is zero-free on $\mathbb{R}$ and

$$U = \bar{I} \cdot \frac{1-\bar{J}}{1-J} \cdot \frac{\bar{H}}{H} = -\bar{I}\bar{J}\frac{\bar{H}}{H} := \bar{\Phi}\frac{\bar{H}}{H}.$$

If F has multiple zeros, then we simply repeat this reasoning taking care of the convergence. Next we restate (3.9) in terms of the arguments of the involved functions. Since H is an outer function, it has the following representation:

$$H = e^{-(h+i\tilde{h})/2}, \qquad h \in L^1_\Pi,$$

and since H is zero free we have $\tilde{h} \in C^\omega(\mathbb{R})$. It follows that $\gamma = -\phi + \tilde{h}$, where ϕ is a continuous argument of Φ. Since ϕ is strictly increasing, this gives the "only if" part of the theorem. To prove the "if" part, we observe that given an increasing function α, we can find an inner function with argument ϕ such that

$$\beta := \alpha - \phi \in L^\infty(\mathbb{R}),$$

so

$$\gamma = -\alpha + \tilde{h} = -\phi - \beta + \tilde{h} = -\phi + \tilde{h}_1, \qquad h_1 := h + \tilde{\beta}.$$

□

We can state a similar result for $\mathcal{H}^p$ spaces as follows.

Proposition 2. *Let $U = e^{i\gamma}$ with $\gamma \in C^\omega(\mathbb{R})$. Then $\ker^p_U \neq 0$ iff*

$$U = \bar{\Phi}\frac{\bar{H}}{H},$$

where H is an outer function in $\mathcal{H}^p \cap C^\omega(\mathbb{R})$, $H \neq 0$ on $\mathbb{R}$, and Φ is a meromorphic *inner function. Alternatively,* $\ker^p_U \neq 0$ *iff*

$$\gamma = -\phi + \tilde{h}, \qquad h \in L^1_\Pi, \quad e^{-h} \in L^{p/2}(\mathbb{R}), \tag{3.11}$$

where ϕ *is the argument of some meromorphic inner function.*

Proposition 3. *Let* Θ *be a meromorphic inner function and* $\Lambda \subset \mathbb{R}$*. Then* Λ *is a uniqueness set of* K^p_Θ *if and only if* $\ker^p_{\bar{\Theta}J} = 0$ *for every meromorphic inner function* J *such that* $\{J = 1\} = \Lambda$.

Proof. Suppose we have a non-trivial function $F \in K^p_\Theta$ which is zero on Λ. By Lemma 1, we can find an inner function J with $\{J = 1\} = \Lambda$ such that

$$G = \frac{F}{1 - J} \in \mathcal{H}^p.$$

Then $G \in K^p_\Theta$, and we have

$$\bar{\Theta}JG = \bar{\Theta}(G - F) = \bar{\Theta}F\frac{J}{1 - J} = -\frac{\bar{\Theta}F}{1 - \bar{J}} \in \bar{\mathcal{H}}^p,$$

so the Toeplitz kernel is non-trivial. Conversely, if G is a non-trivial element of $\ker^p_{\bar{\Theta}J}$, then $F = (1 - J)G \in K^p_\Theta$, and so Λ is not a uniqueness set. Indeed, since $G \in \ker^p_{\bar{\Theta}J}$, we have $JG \in K^p_\Theta$, and therefore $G \in K^p_\Theta$ and $G - JG \in K^p_\Theta$. $\square$

3.3 Defining Sets

Let Λ be a separated sequence on $\mathbb{R}$ and let Φ be a MIF such that $\Phi = e^{i\phi}$ on $\mathbb{R}$. Then Λ is said to be **defining** for Φ if for any MIF $\tilde{\Phi}(= e^{i\tilde{\phi}}$ on $\mathbb{R})$,

$$\phi = \tilde{\phi} \text{ on } \Lambda \ \Rightarrow \Phi \equiv \tilde{\Phi}.$$

One would like to characterize defining sets for a given MIF and we see, in the next section that these problems are intimately connected with inverse spectral problems of Schrödinger operator theory.

It is well known that for a given MIF, Φ, its spectrum $\sigma(\Phi)$ is not defining for it. In other words, there are more than one MIFs which have exactly the same spectrum. The details can be found in the following section. We now describe the **two spectra problem**. This corresponds to the case:

$$\Lambda = \{\Phi = 1\} \cup \{\Phi = -1\}.$$

Let Φ be a MIF. Then, $\tilde{\Phi}$ is another MIF with the property that

$$\begin{aligned} \{\tilde{\Phi} = 1\} &= \{\Phi = 1\} \\ \{\tilde{\Phi} = -1\} &= \{\Phi = -1\} \end{aligned}$$

if and only if

$$\frac{1}{\pi i} \log \frac{\tilde{\Phi} + 1}{\tilde{\Phi} - 1} = \mathcal{S}\chi_E + c,$$

where $\mathcal{S}\chi_E$ is the Schwartz transform of the function χ_E, where $E = \{\Im\Phi > 0\}$ and c is a constant. We now describe some results, due to Makarov and Poltoratski [27] that connect the notion of defining sets to that of uniqueness sets of Toeplitz kernels. The following results characterize sufficient conditions for Λ to be a defining set for Φ.

Proposition 4. *Λ is not defining for Φ if there is a non-constant function $G \in K_\Phi^\infty$ such that*

$$G = \bar{G} \quad \text{on} \quad \Lambda. \tag{3.12}$$

Lemma 6. *If* $\tilde{\Phi} = \Phi$ *on* Λ *and* $F = \tilde{\Phi} - \Phi$, *then*

$$F \in K^{\infty}_{\tilde{\Phi}\Phi}, \qquad F = 0 \quad \text{on} \quad \Lambda.$$

In this paper, the authors remark that 'condition 5.31 is very close to the condition that Λ is not a uniqueness set for $K^{\infty}_{\Phi^2}$. The precise relation between the two statements is an interesting question, which we will not discuss here'. We explore this precise relationship in detail in the chapter titled 'Inverse Spectral Theory'.

3.4 Schrödinger Equation

Consider the Schrödinger equation

$$-u'' + qu = \lambda u \tag{3.13}$$

on some interval (a, b) and assume that the potential $q(t)$ is locally integrable and a is a *regular* point i.e., a is finite and q is in L^1 at a. Let us fix the following boundary condition at b.

$$\cos(\beta)u(b) + \sin(\beta)u'(b) = 0. \tag{3.14}$$

Then for each $\lambda \in \mathbb{C}$, there is a solution u_λ to 3.13 that is actually entire [23]. This family of solutions $\{u_\lambda\}_\lambda$ gives rise to a function called the *Weyl-Titchmarsh* m function, defined as

$$m(\lambda) = \frac{u'_\lambda(a)}{u_\lambda(a)}. \tag{3.15}$$

Here we only deal with the compact resolvent case, i.e. when m extends to a meromorphic function. We can then define a meromorphic inner function as follows.

$$\Theta(z) = \frac{m(z) - i}{m(z) + i}. \tag{3.16}$$

This is called the *Weyl* inner function corresponding to the potential q and the boundary condition at b.

Now consider the related Schrödinger operator

$$u \to -u'' + qu,$$

defined on the space $L^2(a,b)$, along with the boundary conditions

$$\cos(\alpha)u(a) + \sin(\alpha)u'(a) = 0 \tag{3.17}$$

$$\cos(\beta)u(b) + \sin(\beta)u'(b) = 0. \tag{3.18}$$

Then, this operator has a discrete spectrum on $\mathbb{R}$, which we denote by $\sigma(q,\alpha,\beta)$. Suppose we fix the Dirichlet boundary condition at a, i.e., $\alpha = 0$ and denote the resulting spectrum thus obtained as $\sigma(q,D,\beta)$. Then,

$$\sigma(\Theta) = \sigma(q,D,\beta), \tag{3.19}$$

where Θ is the Weyl inner function obtained by fixing the Dirichlet boundary condition at a. We can also obtain the spectrum of this operator with varying boundary conditions at a from Θ in the following way. Let us fix the boundary condition α at a, i.e.,

$$\cos(\alpha)u(a) + \sin(\alpha)u'(a) = 0.$$

Denoting the resulting spectrum by $\sigma(q,\alpha,\beta)$, we have the relationship

$$\sigma(e^{-i\alpha}\Theta) = \sigma(q,\alpha,\beta),$$

where Θ is, as before, the Weyl inner function obtained by fixing the Dirichlet boundary condition at a.

3.4.1 Inverse Spectral Theory

Inverse spectral theory concerns the reconstruction of the operator from spectral data. Studies of this nature date back to, at least 1929, when Ambarzumian [1] proved that if you have spectrum at $\{n^2\}_{n\in\mathbb{N}}$ with Neumann boundary condition at both end points, then the corresponding potential must be 0 a.e. In the years 1950-1952, Borg and Marchenko independently proved several results of what are now called the Borg-Marchenko uniqueness type theorems ([6],[24]). Along with Levinson, all three are credited for proving that spectra are enough to recover the potential [22]. In recent times, Simon, Gesztezy and Del Rio have proved results on mixed spectral data (conditions under which data coming from multiple spectra may be enough to recover the potential) ([15], [16],[12]). An important contribution is by Hórvath [19], who proved in 2005 the following result.

Theorem 4. *Let $1 \leqslant p \leqslant \infty$, $q \in L^p(0,\pi)$, $0 \leqslant a < \pi$ and let $\lambda_n \in \sigma(q, \alpha_n, 0)$ be real numbers with $\lambda_n \nrightarrow -\infty$. Then, $\beta = 0$, q on $(0, a)$ and λ_n determine q in L^p if and only if*

$$e(\Lambda) = \{e^{\pm 2i\sqrt{\lambda_n}x}, e^{\pm 2i\mu} : n \geqslant 1\} \tag{3.20}$$

is closed in $L^p(a-\pi, \pi-a)$ for some (any) $\mu \neq \pm\sqrt{\lambda_n}$.

Our approach is to use complex analytic tools, namely Weyl functions, model spaces and Toeplitz kernels to prove our results. Let us describe the fundamental result this theory is based on. Let

$$u \to -u'' + q_i u$$

be two Schrödinger operators on (a, b) with the same boundary condition at b. The related Weyl m functions are given by m_1 and m_2 respectively. Then, Marchenko [24] proved the following

Theorem 5. *Suppose $m_1(z) = m_2(z)$, for all $z \in \mathbb{C}\backslash\mathbb{R}$, then $q_1(x) = q_2(x)$ for almost all $x \in (a, b)$.*

Since the Weyl m functions and the Weyl inner functions are in 1-1 correspondence, one deduces that knowing the Weyl inner function of a Schrödinger operator is enough to recover the potential. As a result, questions about recovery of the potential of a Schrödinger operator from its spectral data can be reduced to the recovery of the related Weyl meromorphic inner function from its spectral data. This question actually has two parts- firstly, given some spectral data, does there exist a MIF corresponding to this data? and secondly, is this MIF unique? We will deal mostly with the second question. We ask and answer questions about uniqueness in the chapter titled 'Inverse Spectral theory'. We will also provide results from a preliminary study on the existence of MIFs corresponding to spectral data in the chapter on future work.

Let us consider some example of questions in this study. We recall the following definition. Let Λ be a separated sequence on $\mathbb{R}$ and let Φ be a MIF such that $\Phi = e^{i\phi}$ on $\mathbb{R}$. Then Λ is said to be **defining** for Φ if for any MIF $\tilde{\Phi}(= e^{i\tilde{\phi}}$ on $\mathbb{R})$,

$$\phi = \tilde{\phi} \text{ on } \Lambda \;\; \Rightarrow \Phi \equiv \tilde{\Phi}.$$

A natural place to start is with the spectrum $\sigma(\Theta)$, of a MIF Θ. Does $\sigma(\Theta)$ define Θ? The answer to this question is - it does not, in general. As we saw in the section with Clark measures that each meromorphic inner function Θ is of the form

$$\Theta = \frac{K\mu - 1}{K\mu + 1},$$

where $K\mu$ is the Cauchy transform of the measure μ, which is defined by

$$\mu = \sum_n w_n \delta_{\lambda_n},$$

where $\{\lambda_n\} = \sigma(\Theta)$. The weights w_n satisfy the conditions- $w_n > 0$ and $\sum_n \frac{w_n}{1+\lambda_n^2} < \infty$. So, there is an infinite choice of $\{w_n\}$, and each choice gives us a different MIF. Hence, there an infinitely many MIFs with spectrum $\sigma(\Theta)$. This indicates that we need more information than just one spectrum, to uniquely identify the related MIF. This leads us to ask if 2 spectra may be sufficient? We use Krein's shift formula to answer this question. Given any two intertwining and separated sequences Λ_+ and Λ_-, there is a one-parameter family of MIFs Φ_c that have the property that $\{\Phi_c = 1\} = \Lambda_+$ and $\{\Phi_c = -1\} = \Lambda_-$. This family is given by the formula

$$\frac{1}{\pi i} \log\left(\frac{\Phi_c + 1}{\Phi_c - 1}\right) = \mathcal{S}\chi_E + c,$$

where $\mathcal{S}\chi_E$ is the Schwartz transform of χ_E and $E = \bigcup_n (\lambda_{n+}, \lambda_{n-})$, where $\lambda_{n+} < \lambda_{n-} < \lambda_{n+1+}$ and $\lambda_{n\pm} \in \Lambda_\pm$. It is easy to see that if in addition to the two spectra we have just one more point, then we can uniquely recover the MIF. In terms of densities, the two spectra case seems to be the ideal case. One can ask that if we have spectral data, where each point may belong to a different spectrum, are we still guaranteed to obtain a unique MIF? We explore this question in detail in the chapter 'Inverse Spectral Theory'.

Let us consider a different problem. We suppose as usual, that $u \to -u'' + qu$

is an operator on $L^1(a,b)$, with $q \in L^1(a,b)$. Let $c \in (a,b)$ and $q_- := q|_{(a,c)}$.

Suppose we know the part potential q_- and the spectrum $\sigma(q,\alpha,\beta)$. Then, can we recover q? Hochstadt and Liebermann proved in [18] that we can recover q provided $c \geqslant (a+b)/2$. This question leads us to many interesting ideas. For instance, consider the operator

$$u \to -u'' + q_- u$$

on (a,c) with boundary condition α at a. Let u_λ be a solution of the corresponding equation. From the family of solutions $\{u_\lambda\}_\lambda$, we construct the Weyl m and inner functions as before: $m_-(\lambda) = -\frac{u'_\lambda(c)}{u_\lambda(c)}$ and $\Theta_- = \frac{m_- - i}{m_- + i}$. In the same vein, , let $q_+ := q|_{(c,b)}$. We consider the operator

$$u \to -u'' + q_+ u,$$

on (c,b) with boundary condition β at b. Let m_+ and Θ_+ be the Weyl m and inner functions for the right part respectively. Then, it is proved in [27] that

Lemma 7. $\sigma(\Theta_-\Theta_+) = \sigma(q,\alpha,\beta)$.

An immediate corollary is

Corollary 1. $\sigma(q_-,\alpha,\sigma(L))$ *determine the operator if and only if* $(\Theta_-,\sigma(\Theta_-\Theta_+))$ *determine* Θ_+, where L is the operator on (a, b). One wonders if it is possible to recover the operator from $\sigma(\Theta_-)$,$\sigma(\Theta_+)$ and $\sigma(\Theta_-\Theta_+)$. We have proved that it indeed is. And in fact, this set of information is equivalent to having 2 spectra (Borg-Marchenko case). The proof and discussions follow in the chapter 'Inverse Spectral Theory'.

3.5 Beurling Malliavin Theory

We recall the problem of completeness of exponentials in some L^2 space, as mentioned in the introduction. Given a separated sequence Λ on $\mathbb{R}$, it's counting funtion n_Λ is the step function that jumps by 1 unit at each point in Λ and is 0 at 0. For $a > 0$, a sequence Λ is said to be *a-regular* if

$$\int_{\mathbb{R}} \frac{|n_\Lambda(x) - ax|}{1 + x^2} dx < \infty.$$

We now define the interior BM density as follows.

$$D_*(\Lambda) = \sup\{a | \exists \text{ an } a - regular \text{ subsequence } \Lambda' \subset \Lambda\}.$$

The exterior Beurling Malliavin density is defined as

$$D^*(\Lambda) = \inf\{a | \exists \text{ an } a - regular \text{ supersequence } \Lambda' \supset \Lambda\}.$$

We can now state the famous Beurling Malliavin description of the radius of completeness. The details can be found in [5].

Theorem 6. *(Beurling & Malliavin) Let Λ be a separated sequence in $\mathbb{R}$. Then the radius of completeness of $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ is equal to $2\pi D^*(\Lambda)$.*

In [28], Mitkovski and Poltoratski proved equivalent descriptions of the Beurling

Malliavin density in terms of Toeplitz kernels.

$$\begin{aligned} D^*(\Lambda) &:= \sup\{a : \ker^2_{\bar{S}^a J} = \{0\}\}, \\ D_*(\Lambda) &:= \inf\{a : \ker^2_{\bar{J} S^a} = \{0\}\}, \end{aligned}$$

where $S(z) = e^{iz}$ and J is any MIF with $\sigma(J) = \Lambda$.

CHAPTER IV

DE BRANGES LEMMA*

In this chapter, we solve a problem that was first studied by Louis de Branges in the 1960s - Given a *separated* sequence $\{a_n\}$ on $\mathbb{R}$, does there exist an MIF Θ with $\{a_n\}$ as spectrum, such that $|\Theta'|$ is uniformly bounded on $\mathbb{R}$? By a separated sequence $\{a_n\}$, we simply mean that there is a $\delta > 0$ such that $|a_n - a_m| > \delta$, for all $n \neq m$ integers.

4.1 Introduction

We recall that an inner function on the upper half plane $\mathbb{C}_+$ is a bounded analytic function on $\mathbb{C}_+$ with unit modulus almost everywhere on the real line $\mathbb{R}$. A meromorphic inner function (MIF) on $\mathbb{C}_+$ is an inner function on $\mathbb{C}_+$ with a meromorphic continuation to $\mathbb{C}$. The spectrum of an MIF Θ is the level set $\{x \in \mathbb{R} : \Theta\ (x) = 1\}$ and we denote it by $\sigma(\Theta)$. Inner functions arise often in the study of complex function theory. A rather well studied object is the Weyl-Titchmarsh inner function that frequently occurs in the study of the spectral theory of differential operators.

In his book 'Hilbert spaces of entire functions' [10], Louis de Branges formulated a result (Lemma 16) that was equivalent to the the statement, 'Given any sequence of separated points $\{a_n\}$ on $\mathbb{R}$, there exists a meromorphic inner function, Θ such that $|\Theta'|$ is uniformly bounded on $\mathbb{R}$ and $\sigma(\Theta) = \{a_n\}$.' In 2011, Anton Baranov discovered this statement to be false and demonstrated this in private communications with mathematicians working in this area [3]. He noticed that any meromorphic inner function having the natural numbers $\mathbb{N}$ as spectrum must indeed have unbounded

*Reprinted with permission from Journal d'Analyse Mathematique: 'Uniform Boundedness of derivatives of Meromorphic Inner Functions on the Real line' by Rishika Rupam, to appear.

derivative on $\mathbb{R}$. In fact, he formulated a more general result which could be loosely stated as – any MIF that has as spectrum- clusters followed by gaps must necessarily have unbounded derivative on $\mathbb{R}$. In this paper, we will characterize sequences for which there do exist corresponding MIFs with bounded derivatives, as well as describe the method used by Baranov to contruct counterexamples.

Before proceeding any further, we must clarify de Branges motivation for his result as well as its application. Lemma 16 that de Branges stated was used to show the existence of a non-zero measure μ that is supported on a sequence Λ, such that its Fourier transform $\hat{\mu}$ vanishes on an interval of positive measure. Readers may recognize this as Beurling's *gap problem* for sequences, wherein he asks the question - under what conditions on the sequence Λ, does there exist a corresponding measure μ with $\hat{\mu}$ vanishing on an interval of positive measure? In [28], Mitkovski and Poltoratski provided a sufficient condition for Beurling's problem for separated sequences, in terms of the Beurling Malliavin density. We notice in hindsight that de Branges was specifically looking at sequences that were *regular* and that these sequences satisfy the requirement as stated in [28]. Thus, despite the erroneous lemma, de Branges application of it still holds. We describe this briefly in the applications below. We remark that even for such special sequences, however, there may not exist any corresponding MIF with a bounded derivative. This and other such counterexamples were constructed by Baranov and we decribe these in the last section.

Apart from this, there have been demands for meromorphic inner functions with a certain spectrum and a bounded derivative in more general contexts. For instance, in the Beurling Malliavin theory for Toeplitz kernels, Makarov and Poltoratski require this to prove the most general form of the BM multiplier theorem [27] (see application 2 below). In [28], Mitkovski and Poltoratski have characterized Pólya sequences and gap conditions using the existence of an MIF with bounded derivative. In his paper [2] on the stability of completeness of a system of exponentials under certain perturbations, Baranov requires the existence of such an MIF with spectrum as the perturbed sequence. He also states a sufficient condition (lemma 5.2) on sequences to possess this special MIF. We will exploit this as well as a new sufficient condition to describe sequences that are spectra for MIFs with bounded derivatives. We also prove a partial converse result.

4.2 Basic Properties of MIFs

It is easy to construct a meromorphic inner function with a given spectrum. We have already demonstrated this in the section with Clark measures. We recall the following main points arising from that construction.

Let $\{a_n\}_{-\infty}^{\infty}$ be a separated sequence on $\mathbb{R}$ (we proceed similarly for one-sided sequences also). Let μ be a Poisson finite, positive measure on $\mathbb{R}$ with point masses at the a_n, i.e.,

$$\mu = \sum_{n=-\infty}^{\infty} w_n \delta_{a_n} \tag{4.21}$$

for some $w_n > 0$ such that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{w_n}{1+a_n^2} < \infty$.The Cauchy transform of a Poisson finite measure ν on $\mathbb{R}$ is given by

$$K\nu(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\nu(t).$$

Applying the Cauchy transform to the measure μ just defined,

$$K\mu(z) = \frac{1}{\pi i} \sum_{n=-\infty}^{\infty} \frac{w_n}{a_n - z} - \frac{w_n a_n}{1+a_n^2},$$

we have that $K\mu$ is an analytic funtion from the upper half plane $\mathbb{C}_+$ to the right half plane. Finally we obtain the MIF Θ :$\mathbb{C}_+ \to \mathbb{D}$ as follows,

$$\Theta(z) = \frac{K\mu(z) - 1}{K\mu(z) + 1}. \tag{4.22}$$

Observe that Θ is a meromorphic inner function on $\mathbb{C}_+$, with spectrum the set, $\{a_n\}_{-\infty}^{\infty}$; For μ is non negative, giving us $\Re K\mu(z) > 0$ on $\mathbb{C}_+$, with $K\mu(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, along with the fact that

$$w \to \frac{w - 1}{w + 1}$$

maps $\{\Re w > 0\}$ onto $\mathbb{D}$, taking $i\mathbb{R}$ onto the unit circle. Morever, we notice that Θ would take the value 1 exactly at the singularities of $K\mu$, i.e. at the a_ns.

We recall the measure μ is known as the Clark measure associated with the function Θ. By a reversal of steps and using Herglotz's theorem one can construct a Clark measure given any meromorphic inner function on $\mathbb{C}_+$. In particular, Clark measures associated with inner functions are singular with respect to the Lebesgue measure, with the spectrum of the inner function as its support except, possibly, the point at infinity. A natural question to ask is if the inner function with spectrum $\{a_n\}$ is unique. A look at (4.21) assures us that that is quite not the case, for the w_n are almost arbitrarily chosen. We can obtain restrictions on the w_n by imposing additional conditions on the function. Here we ask for boundedness of the derivative on $\mathbb{R}$.

4.3 Applications of De Branges Lemma

Let us see how useful it is to have an MIF with a bounded derivative on $\mathbb{R}$ with two applications.

1. We describe the sufficient condition for the gap problem, as described in [28]- If $D_*(\Lambda) > 0$, then there does exist a nonzero measure μ, supported on Λ such that $\hat{\mu}$ vanishes on an interval of positive length. Here $D_*(\Lambda)$ refers to the interior Beurling Malliavin density of Λ. There are several equivalent definitions of D_*. To understand the most relevant definitions here, let us clarify a few things. Given a separated sequence Λ on $\mathbb{R}$, it's counting funtion n_Λ is the step function that jumps by 1 unit at each point in Λ and is 0 at 0. For $a > 0$, a sequence Λ is said to be *a-regular* if

$$\int_{\mathbb{R}} \frac{|n_\Lambda(x) - ax|}{1 + x^2} dx < \infty.$$

We now define the interior BM density as follows.

$$D_*(\Lambda) = \sup\{a | \exists \text{ an } a - regular \text{ subsequence}\, \Lambda' \subset \Lambda\}.$$

Using this definition, it is easy to see that *a-regular* sequences have interior density equal to $a > 0$. Such sequences are also *Pólya*, i.e., it has the following property- if there is an entire function f of zero exponential type that is bounded on this sequence, then f must be a constant. This was proved by de Branges in [10] and it's connection with the gap problem is described in [28]. Thus, for regular sequences, the gap condition holds.

The proof of the gap condition is much simpler in the case the there is an MIF with Λ as spectrum and bounded derivative on $\mathbb{R}$.

Let us explain briefly De Branges' motivation for having an MIF with bounded derivative. We refer the reader to the section on De Branges spaces for definitions and properties. We recall our discussion in the introduction, wherein we mentioned that De Branges required the gap condition to hold for a certain *regular* sequence, i.e., he required the existence of a measure μ that was supported on this regular sequence such that its Fourier transform $\hat{\mu}$ vanished on an interval of positive length. The crucial result that he used was the existence of a certain De Branges space of entire functions on $\mathbb{C}_+$ with some conditions on the mean type of the space. He was able to do this using the fact that $|\phi'|$ is uniformly bounded on $\mathbb{R}$. We refer the reader to 'Theorem 65' in [10] for details.

2. We will refer to [27] for this application. We use the standard notation $\mathcal{H}^p (= \mathcal{H}^p(\mathbb{C}_+))$ to denote Hardy spaces and $\mathcal{N}^+ = \{G/H : G, H \in \mathcal{H}^\infty, H \text{ is outer}\}$ to denote the Smirnov-Nevanlinna class in $\mathbb{C}_+$. We refer the reader to the section on model Spaces for the notation used here. As described in [27], these Toeplitz kernels play a crucial role in answering questions in the inverse spectral theory of differential operators and completeness problems of families of functions, among others. Let us consider one such problem. Let $\Phi = e^{i\phi}$ be a meromorphic inner function and let $\Lambda \subset \mathbb{R}$. We say that Λ is a *defining set* for Φ if for any other meromorphic inner function $\tilde{\Phi} = e^{i\tilde{\phi}}$,

$$\tilde{\phi} = \phi \text{ on } \Lambda \implies \Phi \equiv \tilde{\Phi}.$$

It has been described in [27] that a sufficient condition for Λ to be a defining set for Φ is for Λ to be a uniqueness set for $K^{\infty}_{\Phi^2}$. This condition translates to one about Toeplitz kernels as we saw in 3.

Theorem 7. Λ *is a uniqueness set for* K^{∞}_{Θ} *if and only if for every meromorphic inner function* J *such that* $\sigma(J) = \Lambda$*, we have that* $\ker^{\infty}_{\bar{\Theta}J} = 0$.

Thus, the (non) triviality of Toeplitz kernels $\ker^{\infty}_{\bar{\Theta}J}$ is useful in answering questions about defining sets. In particular, for MIFs that have a bounded derivative, these questions are easier to answer. Let us refer to the following two theorems,

Theorem 8. *Suppose Θ is a tempered inner function. Then for any mermorphic inner function J and any $p > 0$,*

$$\ker^p[\bar{\Theta}J] \neq 0 \implies \exists n, \quad \ker^{\infty}[\bar{b}^n\bar{\Theta}J] \neq 0.$$

By Θ being *tempered* one simply means that Θ' has at most polynomial growth at $\pm\infty$, i.e., $\exists N, \Theta'(x) = O(|x|^N)$, $x \to \infty$. Here, the n we obtain in the theorem is the same as N, regarding the growth of $|\Theta'|$. Thus, in the case of bounded derivative $|\Theta'|$, we have that

$$\ker^p[\bar{\Theta}J] \neq 0 \implies \ker^{\infty}[\bar{\Theta}J] \neq 0.$$

The next theorem comes under the heading of Beurling Malliavin multiplier theorem.

Theorem 9. *Suppose Θ is a meromorphic inner function satisfying $|\Theta'| \leqslant$ const. Then, for any meromorphic inner function J, we have*

$$\ker^{+}[\bar{\Theta}J] \neq 0 \implies \forall \epsilon, \ker^{\infty}[\bar{S}^{\epsilon}\bar{\Theta}J] \neq 0.$$

The space $\ker^{+}$ contains $\ker^{\infty}$ and hence it is easier to construct functions in these spaces. Similarly, it is often easier to work with $\ker^2$, which is a subspace of the Hardy space $\mathcal{H}^2$. Using the above theorems, we can just restrict our attention to these larger spaces in case Θ has a bounded derivative. As mentioned in [27], in the case of a general bounded γ, where $\bar{\Theta}J = e^{i\gamma}$, we cannot multiply down to H^{∞}, elements of $\ker_U (= \ker_U^2)$ even by using factors like $\bar{S}$.

Thus, an MIF having a bounded derivative is an extremely useful object.

4.4 Main Results

We'll denote the gaps between the successive a_ns as,

$$\Delta_n := \begin{cases} a_{n+1} - a_n & \forall n > 0 \\ a_n - a_{n-1} & \forall n \leqslant 0. \end{cases} \tag{4.23}$$

In their paper [26], Makarov and Poltoratski have proved the existence of the required inner function when the Δ_n are uniformly bounded.

Our approach in this paper will be to consider sequences characterized by the growth of their gaps. We start with gaps that are increasing, but very slowly. Formally, the gaps obey the relation

$$\frac{\ln |a_n|}{\ln \ln \Delta_n} \lesssim \Delta_n \lesssim \ln |a_n|.$$

Here, and throughout the paper, $f(n) \asymp g(n)$ will denote the existence of constants $c_1, c_2 > 0$ such that $c_1 f(n) \leqslant g(n) \leqslant c_2 f(n)$ for large enough n. And $f(n) \lesssim g(n)$ will mean $f(n) \leqslant c g(n)$ for some $c \geqslant 0$ and large enough n.

It turns out that this case is a generalization of the result proved in [26].

Lemma 8. *If $\{a_n\}$ is a sequence in $\mathbb{R}$ and the Δ_n, defined by (4.23) are such that*

- $\Delta_{n+1} \asymp \Delta_n$

- $\frac{\ln|a_n|}{\ln\ln\Delta_n} \lesssim \Delta_n \lesssim \ln|a_n|$,

then there is a meromorphic inner function Θ on $\mathbb{C}_+$ such that $\{a_n\}$ is the spectrum of Θ and $|\Theta'|$ is uniformly bounded.

Next, we consider sequences with slightly larger gaps. Baranov's counter example of the one sided sequence $\mathbb{N}$ leads us to ask the natural question- if we have $\mathbb{N}$ on one side, how sparse can the sequence be on the other side? Simple computations tell us that on the other side, the gaps may be at most geometrically increasing, i.e., $|\lambda_n| \lesssim e^{c|n|}$, for some $c > 0$. To put it precisely,

Observation 1. *Let Θ be an MIF on $\mathbb{C}_+$ with uniformly bounded derivative on $\mathbb{R}$ and Λ the spectrum of Θ. If $\Lambda_\pm = \Lambda \cap \mathbb{R}_\pm$ and $\Lambda_+ = \mathbb{N}$, then $\exists$ a $c \geqslant 0$ such that $|\lambda_n| \lesssim e^{c|n|}$ for $\lambda_n \in \Lambda_-$.*

Proof. If $z_n = x_n + iy_n$ are the zeros of $\Theta = e^{i\theta}$, then

$$\theta'(x) = \sum_n \frac{y_n}{(x - x_n)^2 + y_n^2}.$$

We notice that these are sums of Poissons kernels, with the property that $\int_{\mathbb{R}} \frac{y_n}{(x-x_n)^2+y_n^2} dx = \pi$. Let us restrict our attention to the zeroes in the upper right quadrant, i.e. $x_n > 0$ and $y_n > 0$. For any $t < 0$ and fixed x_n, the integral $\int_t^0 \frac{y_n}{(x-x_n)^2+y_n^2} dx$ attains minimum at $y_n^2 = x_n(x_n - t)/t$ and increases for $y_n^2 \geqslant x_n(x_n - t)/t$. But we notice that $\int_t^0 \frac{y_n}{(x-x_n)^2+y_n^2} dx \leqslant \int_t^0 \theta'(x)dx = |\sigma(\Theta) \cap (t, 0)|$. So, when $|t|$ is large enough, then larger the y_ns, the denser Λ_-. Thus, to explore the case when Λ_- is as sparse as possible, we must assume that $y_n \leqslant K$ for all n, for some $K > 0$. On the other hand, the zeros of Θ must be bounded away from the real line in order for $|\Theta'|$ to be bounded. Thus, we can assume, without loss of generality, that $y_n \asymp 1$. Consider the entire function E associated with Θ, i.e. $\Theta(z) = E^{\#}(z)/E(z) = \overline{E(\overline{z})}/E(z)$. These functions are called de Branges functions (associated with an MIF). We refer the reader to [10] and [28] for more on de Branges functions. We know that $E - E^{\#}$ has zeroes on $\mathbb{N}$ (at least), and so must be of exponential type at least π. Thus, the exponential type of E is at least π and so the zeros of E are of the form $z_n \asymp n + iy_n$ where $n \in \mathbb{N}$. Then,

$$|\sigma(\Theta) \cap (t, 0)| \gtrsim \int_t^0 \sum_{\mathbb{N}} \frac{1}{(x-n)^2 + 1} dx \asymp \sum_{n \leqslant |t|} \frac{1}{n} \asymp \ln |t|.$$

□

The extreme case in the above situation, i.e. when $|\lambda_n| \asymp e^{c|n|}$, has the property that the gaps are co-measurable, i.e., $\Delta_n \asymp \Delta_{n+1}$. This gives us motivation for our next result where we analyse a more general case of co-measurable gaps. We recall that the choice of the weights w_n determine the growth of the function. Let us choose $w_n \asymp \Delta_n$.

Lemma 9. *If a_n is a separated sequence on $\mathbb{R}$ and Δ_n, defined as in (4.23) are such that*

- $\Delta_{n+1} \asymp \Delta_n$ *and*

- $\Delta_n \gtrsim (\ln |a_n|)^2$,

then by choosing $w_n \asymp \Delta_n$, Θ defined as in (4.22) is a meromorphic inner function on $\mathbb{C}_+$ with spectrum $\{a_n\}_{-\infty}^{\infty}$ such that $|\Theta'|$ is uniformly bounded on $\mathbb{R}$.

Some examples of such sequences are $a_n = (\operatorname{sgn} n)|n|^k$, where $k > 0$ and $a_n = (\operatorname{sgn} n)r^{|n|}$, where $r > 1$.

Next, we consider sequences that are sparse. By sparse, we mean sequences that are at least geometrically increasing with common ratio bigger than 1, for instance $a_n = (\operatorname{sgn} n)e^{e^{|n|}}$. The following statement may seem technical, but all that is being said is: If we consider finite clusters of points such that consecutive clusters are sparse, then choosing the weights $w_n \asymp 1$, the corresponding inner function will have a bounded derivative on $\mathbb{R}$.

Lemma 10. *For each $n \in \mathbb{N}$, let us consider a finite sequence (cluster) of points $\{a_n^j\}_{1 \leqslant j \leqslant m_n}$ (m_n being uniformly bounded) that is defined by the property $\frac{a_n^{j+1}}{a_n^j} \to 1$ as $n \to \infty$ and $1 \leqslant j \leqslant m_n$. Moreover, the gaps between consecutive clusters is large, in the sense that there is a $d > 0$ such that $\frac{a_{n+1}^j}{a_n^l} - 1 > d > 0$ for $n \geqslant 0$ and $\frac{a_{n-1}^j}{a_n^l} - 1 > d > 0$ for $n < 0$. Then, there is a meromorphic inner function Θ such $\sigma(\Theta) = \{a_n^j\}$ and $|\Theta'|$ is uniformly bounded on $\mathbb{R}$.*

To summarize, we have the following

Theorem 10. *Let $\{a_n\}$ be a separated sequence on $\mathbb{R}$ satisfying one of the conditions below,*

1. $\Delta_{n+1} \asymp \Delta_n$ *and* $\frac{\ln |a_n|}{\ln \ln |\Delta_n|} \lesssim \Delta_n \lesssim \ln |a_n|$ *OR*

2. $\Delta_{n+1} \asymp \Delta_n$ *and* $\Delta_n \gtrsim (\ln |a_n|)^2$ *OR*

3. *there is a $d > 0$ such that the sequence can be partitioned into clusters $\{a_n^j\}_n$, with number of points in each cluster being uniformly bounded, such that for any cluster, $\frac{a_n^j}{a_n^{j+1}} \to 1$ and between successive clusters :$\frac{a_{n+1}^j}{a_n^l} - 1 > d > 0$.*

then there exists a meromorphic inner function with spectrum $\{a_n\}$ with uniformly bounded derivative on $\mathbb{R}$.

The above results cover a wide range of sequences. What happens when the sequence falls in none of these categories? There are several ways ways in which this could happen and we have a partial converse, which is inspried by Baranov's counter example, as described in [3]. Let us first state Baranov's result.

Proposition 5. *Let $\{a_n\}$ be a separated sequence with the following property. Given any $N > 0$, there is a cluster $\{a_n\}_{n=k}^{k+N}$ such that $a_{k+m} = a_k + m$ for $1 \leqslant m \leqslant N$ and $a_{k+N+1} = a_{k+N} + N$. Then, any MIF with spectrum $\{a_n\}$ must have unbounded derivative on $\mathbb{R}$.*

In essence, this sequence has arithmetic clusters followed by unbounded large gaps. We generalise this result as follows. Let $\{a_n\}$ be a sequence of points on $\mathbb{R}$ and a $D > 0$ be a constant such that given any $N > 1$, there is a cluster of points $\{a_n\}_{n=1}^{N}$ such that $a_2 - a_1 \geqslant ND$ and $a_{n+1} - a_n \leqslant D$ for $n \geqslant 2$. Notice that this sequence has clusters whose size grows unboundedly (thus excluding case 3 from above) and $\Delta_1 > N\Delta_2$ (thus the gaps are not co-measurable i.e., $\Delta_n \not\asymp \Delta_{n+1}$). Such sequences serve as counterexamples and we state the result below.

Proposition 6. *Suppose $\{s_n\}$ is a separated sequence on the real line and $D > 0$ is a constant such that given any $N > 0$, there is a subset $\{t_n\}_{n=1}^{N}$ such that $(t_1, t_N) \bigcap \{s_m\} = \{t_n\}_{n=1}^{N}$ for which $t_2 - t_1 > ND$ and $t_{n+1} - t_n < D$ for all $2 \leqslant n \leqslant N-1$ and let Θ be an MIF with this spectrum $\{s_n\}$. Then given any $\delta > 0$, there is a zero $z_n = x_n + iy_n$ of Θ such that $0 < y_n < \delta$. Hence, $|\Theta'|$ is unbounded on $\mathbb{R}$.*

The above result can be used to prove that even in the *regular* case, we are not assured of an MIF with bounded derivative. We describe this in the last section.

4.5 Proofs and Details

As mentioned before, we closely follow the proof of the result in [26] to give a proof of lemma 8

Proof. We use Krein's shift formula to create a meromorphic inner function. Define $b_n := \frac{a_n + a_{n+1}}{2}$ and let $E := \bigcup_n (a_n, b_n)$ to define the function

$$\frac{1}{\pi i} \log \frac{\Theta + 1}{\Theta - 1} = Ku + ic, \;\; u := 1_E - \frac{1}{2}, c \in \mathbb{R} \tag{4.24}$$

Let μ_1 and μ_{-1} be the corresponding Aleksandrov-Clark's measures defined by the Herglotz representation

$$\frac{1+\Theta}{1-\Theta} = K\mu_1 + const., \qquad \frac{1-\Theta}{1+\Theta} = K\mu_{-1} + const.$$

The measures μ_1, μ_{-1} have the following form:

$$\mu_1 = \sum_{n=-\infty}^{\infty} \alpha_n \delta_{a_n}, \qquad \mu_{-1} = \sum_{n=-\infty}^{\infty} \beta_n \delta_{b_n}, \tag{4.25}$$

for some positive numbers α_n, β_n. We claim that

$$\alpha_n \lesssim \Delta_n \ln \Delta_n, \qquad \beta_n \lesssim \Delta_n \ln \Delta_n. \tag{4.26}$$

Since

$$|\Theta'| \asymp |1 - \Theta|^2 |(K\mu_1)'|, \qquad |\Theta'| \asymp |1 + \Theta|^2 |(K\mu_{-1})'|,$$

we have

$$\Theta'(x) \asymp \min\left\{ \sum \frac{\alpha_n}{(x - a_n)^2}, \sum \frac{\beta_n}{(x - b_n)^2} \right\},$$

It follows that if $x \in (a_m, a_{m+1})$, then by (4.26),

$$|\Theta'(x)| \lesssim \int_{|t-x| \geqslant \Delta_m} \frac{\ln|t-x|dt}{(x-t)^2} \lesssim \frac{\ln \Delta_m}{\Delta_m} \lesssim 1.$$

We will prove the estimate for α_ns. The proof for β_ns is similar.

$$\begin{aligned} \alpha_n = Res_{a_n}\left(\sum \frac{\alpha_n}{x-a_n}\right) &= Res_{a_n}(K\mu_1) \\ &= Res_{a_n}\left(\frac{1+\Theta}{1-\Theta}\right) \\ &= const. Res_{a_n} e^{Ku}, \end{aligned}$$

where u is as defined in (4.24).

$$\begin{aligned} e^{Ku} &= \exp\left\{\int_{b_{n-1}}^{b_n} \frac{u(t)dt}{t-z}\right\} \exp\left\{\int_{\mathbb{R}\setminus(b_{n-1},b_n)} \frac{u(t)dt}{t-z}\right\} \\ &= \exp\left\{\int_{b_{n-1}}^{b_n} \frac{u(t)dt}{t-z}\right\} \exp\left\{\int_{\mathbb{R}\setminus(b_{n-1},b_n)} \frac{u(t)dt}{t-z}\right\} \\ &= \frac{\sqrt{(b_n-z)(b_{n-1}-z)}}{a_n-z} \exp\left\{\int_{\mathbb{R}\setminus(b_{n-1},b_n)} \frac{u(t)dt}{t-z}\right\} \end{aligned}$$

Thus,

$$Res_{a_n} e^{Ku} \asymp \Delta_n \exp\left\{\int_{\mathbb{R}\setminus(b_{n-1},b_n)} \frac{u(t)dt}{t-a_n}\right\}$$

Thus, it remains to estimate $\exp\left\{\int_{\mathbb{R}\setminus(b_{n-1},b_n)} \frac{u(t)dt}{t-a_n}\right\}$. This is done as follows.

For $j > n$,

$$\begin{aligned}
\int_{a_j}^{a_{j+1}} \frac{u(t)dt}{t-a_n} &= \ln \frac{b_j - a_n}{a_j - a_n} - \ln \frac{a_{j+1} - a_n}{b_j - a_n} \\
&= \ln\left(1 + \frac{\Delta_j}{a_j - a_n}\right) - \ln\left(1 + \frac{\Delta_j}{b_j - a_n}\right) \\
&= \frac{\Delta_j}{a_j - a_n} - \frac{\Delta_j}{b_j - a_n} + O\biggl(\frac{\Delta_j^2}{(a_j - a_n)^2}\biggr) = O\biggl(\frac{\Delta_j^2}{(a_j - a_n)^2}\biggr)
\end{aligned}$$

Since we are on the positive real line, we can take logs.

We have,

$$\frac{\Delta_j^2}{(a_j - a_n)^2} \lesssim \int_{a_j}^{a_{j+1}} \frac{\ln t}{(t-a_n)^2} dt.$$

Thus,

$$\begin{aligned}
\sum_{j=n+1}^{\infty} \frac{\Delta_j^2}{(a_j - a_n)^2} &\leqslant \int_{b_n}^{\infty} \frac{\ln t dt}{(t-a_n)^2} \\
&\leqslant \int_{b_n}^{\infty} \frac{\ln(t-a_n)dt}{(t-a_n)^2} + \int_{b_n}^{\infty} \frac{\ln a_n dt}{(t-a_n)^2} \\
&\lesssim \frac{\ln \Delta_n}{\Delta_n} + \frac{\ln a_n}{\Delta_n} \\
&\lesssim \ln\ln|\Delta_n|,
\end{aligned}$$

using integration by parts in the second step. Thus,

$$\left| \int_{b_n}^{\infty} \frac{u(t)dt}{t-a_n} \right| \lesssim \ln\ln \Delta_n.$$

Hence, we have obtained the estimate

$$\alpha_n \lesssim \Delta_n e^{\ln\ln|\Delta_n|} = \Delta_n \ln \Delta_n.$$

□

Let us now indulge in a simple observation that will aid us in proving as well as understanding the proofs of lemmas 9 and 10. We recall the construction of the inner function as described in the first section. Using Cauchy's estimate, it is easy to see that if there is a strip around the real axis on which the function is uniformly bounded, then the derivative on the real line is also uniformly bounded. In other words, if there are constants $c, m > 0$ such that for $|\Im z| < c$, $|\Theta(z)| < m$, then

$$|\Theta'(x)| \leqslant \frac{1}{2\pi} \int_{|z-x|=c} \frac{|\Theta(z)|}{|z-x|^2} dz \leqslant \frac{m}{c}.$$

We recall (2.5) and the relationship of Θ with the Cauchy transform (4.22) to formulate a sufficient condition :

Observation 2. *If there exist constansts $c, m > 0$ such that for $0 < \Im z < c$, we have $|K\mu(z) - 1| > m$, then the MIF $\Theta := (K\mu - 1)/(K\mu + 1)$ is such that $|\Theta'|$ is uniformly bounded on $\mathbb{R}$.*

Conversely, however, it is only required that there be a zero free strip for Θ about the real axis. In order to prove lemma 9, we need the following result.

Lemma 11. *If $\Delta_{n+1} \asymp \Delta_n$ then choosing $w_n := \Delta_n$ $\forall n$, we have*

$$\left| \sum_{n \neq k} \left(\frac{w_n}{a_n - a_k} - \frac{w_n a_n}{1 + a_n^2} \right) \right| \lesssim \ln |a_k|.$$

Let us see the effect of this result in the situation when the gaps in the sequence are at least logarithmically increasing.

Proof. (of lemma 9) Let C be a constant such that $\Delta_n \geqslant C \ln^2 |a_n|$. Let us choose and fix a $\delta << C$. We will separate the real line into two disjoint sets:

1. $x \in \left(\frac{a_{k-1}+a_k}{2}, \frac{a_k+a_{k+1}}{2}\right)$ and $|x-a_k| \geqslant \frac{\delta}{2}\frac{\Delta_k}{\ln|a_k|}$,

2. $x \in \left(\frac{a_{k-1}+a_k}{2}, \frac{a_k+a_{k+1}}{2}\right)$ and $|x-a_k| < \frac{\delta}{2}\frac{\Delta_k}{\ln|a_k|}$.

Case 1 We take derivatives in (4.22) to obtain the estimate

$$|\Theta'(z)| \leqslant |1-\Theta|^2 \sum \frac{w_n}{|z-a_n|^2}.$$

For any $x \in \left(\frac{a_{k-1}+a_k}{2}, \frac{a_k+a_{k+1}}{2}\right)$ and $|x-a_k| \geqslant \frac{\delta}{2}\frac{\Delta_k}{\ln|a_k|}$,

$$|\Theta'(x)| \asymp \frac{w_k}{(x-a_k)^2} \leqslant \frac{\Delta_k}{\delta^2\Delta_k^2/4(\ln|a_k|)^2} \leqslant \frac{4(\ln^2|a_k|)}{\delta^2\Delta_k} \lesssim 1. \tag{4.27}$$

Case 2 We first notice that for $z \in D(b,r)$, where

$$b = \frac{1}{2}\left(\frac{a_k+a_{k+1}}{2} + \frac{a_{k-1}+a_k}{2}\right) \text{ and } r = \left(\frac{a_{k+1}-a_{k-1}}{4}\right),$$

we have that

$$|K\mu(z) - K\mu(a_k)| \asymp 1.$$

For,

$$\begin{aligned}
\left|\sum_{n\neq k}\left(\frac{w_n}{a_n-a_k} - \frac{w_n}{a_n-z}\right)\right| \leqslant \left|\sum_{n\neq k}\frac{w_n(a_k-z)}{(a_n-a_k)(a_n-z)}\right| &\lesssim \sum_{n\neq k}\frac{\Delta_n\Delta_k}{|(a_n-a_k)(a_n-z)|} \\
&\asymp \sum_{n\notin\{k-1,k,k+1\}}\frac{\Delta_n\Delta_k}{(a_n-a_k)^2} \\
&\asymp \Delta_k\int_{\mathbb{R}\setminus(a_{k-1},a_{k+1})}\frac{dt}{(t-a_k)^2} \\
&\asymp 1
\end{aligned}$$

Let $x \in (\frac{a_{k-1}+a_k}{2}, \frac{a_k+a_{k+1}}{2})$ and $|x-a_k| \leqslant \frac{\delta\Delta_k}{2\ln|a_k|}$, then for any $z \in D\left(x, \frac{\delta\Delta_k}{2(\ln|a_k|)}\right)$

$$\begin{aligned}
|K\mu(z)| &\geqslant \left|\frac{w_k}{a_k-z} - \frac{w_k a_k}{1+a_k^2}\right| - \left|\sum_{i\neq k}\frac{w_i}{a_i-z} - \frac{w_i a_i}{1+a_i^2}\right| \\
&\geqslant \left|\frac{\Delta_k}{\delta\Delta_k/2\ln|a_k|} - \frac{\Delta_k}{a_k}\right| - C'\ln|a_k| \\
&\geqslant \frac{\ln|a_k|}{2\delta} - C'\ln|a_k| + O(1).
\end{aligned}$$

where C' is such that $\left|\sum_{i\neq k}\frac{w_i}{a_i-z} - \frac{w_i a_i}{1+a_i^2}\right| \leqslant C'\ln|a_k|$, by lemma 11. Thus, by choosing a sufficiently small δ, we have that $K\mu$ is bounded away from 1. We notice that δ is independent of k. Thus $K\mu$ is large on disks centred at points *close to* the a_ns. We recall obsertaion 2 which stated that it is sufficient to have a strip above the real line on which $|K\mu|$ is bounded away from 1. Here, we obtain a slightly weaker configuration - we have disks with centres at a_k and radii $\frac{\delta\Delta_k}{2\ln|a_k|} \gtrsim 1$ such that at each point z in the disk, $|K\mu|$ is bounded away from 1. Thus, $|\Theta'(x)|$ is bounded for $x \in (\frac{a_{k-1}+a_k}{2}, \frac{a_k+a_{k+1}}{2})$ and $|x-a_k| \leqslant \frac{\Delta_k}{2\ln|a_k|}$. Cases 1 and 2 together give us that $|\Theta'|$ is bounded on $\mathbb{R}$.

□

Let's now prove Lemma (11)

Proof. The proof is essentially computation of integrals. The underlying idea is that when $\Delta_{n+1} \asymp \Delta_n$, the singular measure μ, now with weight at w_n equal to the gap Δ_n at a_n, behaves like the Lebesgue measure. Explicitly, we look at the following calculations. Let $n > k$, then

$$\frac{w_n}{a_n-a_k} \lesssim \frac{\Delta_{n-1}}{a_n-a_k} \leqslant \int_{a_{n-1}}^{a_n}\frac{dt}{t-a_k} \qquad \text{and} \qquad \frac{w_n a_n}{1+a_n^2} \geqslant \int_{a_{n-1}}^{a_n}\frac{tdt}{1+t^2}.$$

Thus,

$$
\begin{aligned}
0 \leqslant \left| \sum_{n=k+1}^{\infty} \left(\frac{w_n}{a_n - a_k} - \frac{w_n a_n}{1+a_n^2} \right) \right| &\lesssim \int_{a_k+\epsilon}^{\infty} \left(\frac{1}{t-a_k} - \frac{t}{1+t^2} \right) dt \\
&= \left(\ln|t-a_k| - 1/2 \ln|1+t^2| \right) \Big|_{a_k+\epsilon}^{\infty} \\
&\asymp \ln|a_k|,
\end{aligned}
$$

where ϵ is just some arbitrary positive number that is, say $> 1/2$.

Identical calculations exist for the sum $\sum_{n<k} \left(\frac{w_n}{a_n - a_k} - \frac{w_n a_n}{1+a_n^2} \right)$.

□

We now prove the following result leading to the proof of lemma 10. This lemma considers sparse singletons, which we will generalize to sparse clusters.

Lemma 12. *Let $\{a_n\}$ be a sequence on $\mathbb{R}$ such that, $1 - \frac{a_k}{a_{k+1}} > d > 0 \ \forall k \geqslant 0$ and $1 - \frac{a_k}{a_{k-1}} > d > 0 \ \forall k < 0$, where d is independent of k. Then, there is a meromorphic inner function on $\mathbb{C}_+$ with spectrum $\{a_n\}$ whose derivative is uniformly bounded in $\mathbb{R}$.*

Proof. We will use lemma 5.2 in [2] to prove this result. First note that the ratio test for convergence of a series gives us that $\sum_{n=-\infty}^{\infty} \frac{1}{|a_n|} < \infty$. We also notice that for all $n \neq k$,

$$
\left| \frac{a_k}{a_n} - 1 \right| > \min\left\{ d, \frac{d}{1-d} \right\} =: D. \tag{4.28}
$$

For, if $n > k$, then $1 - \frac{a_k}{a_n} \geqslant 1 - \frac{a_k}{a_{k+1}} > d$ and for $n < k$, $\frac{a_k}{a_n} - 1 > \frac{d}{1-d}$. We notice that this also tells us that $\frac{a_k}{\Delta_k} < \frac{1}{D}$ for all k. Let us choose the weights $w_n = 1$.

We rearrange terms,

$$\begin{aligned}\sum_{n\neq k}\left(\frac{w_n}{a_n-a_k}-\frac{a_nw_n}{1+a_n^2}\right) &= \sum_{n\neq k}\frac{1+a_n^2-a_n^2+a_na_k}{(a_n-a_k)(1+a_n^2)}\\ &= \sum_{n\neq k}\frac{1}{(a_n-a_k)(1+a_n^2)}+\sum_{n\neq k}\frac{a_na_k}{(a_k-a_n)(1+a_n^2)}\\ &= S_1+S_2.\end{aligned}$$

Then,

$$|S_1|\leqslant\sum_{n\neq k}\left|\frac{1}{(a_k-a_n)(1+a_n^2)}\right|\lessapprox\sum_{n\neq k}\left|\frac{1}{a_n^2}\right|<\infty$$

and

$$|S_2|\leqslant\sum_{n\neq k}\left|\frac{a_na_k}{(a_k-a_n)(1+a_n^2)}\right|\leqslant\sum_{n\neq k}\left|\frac{a_k}{\Delta_k}\frac{a_n}{1+a_n^2}\right|\leqslant\frac{1}{D}\sum_{n\neq k}\left|\frac{1}{a_n}\right|<\infty.$$

Thus, we have that

$$\sup_n\left|\sum_{n\neq k}\left(\frac{w_n}{a_n-a_k}-\frac{a_nw_n}{1+a_n^2}\right)\right|<\infty.$$

Hence, by lemma 5.2 in [2], the corresponding MIF, defined by 4.22 has uniformly bounded derivative on $\mathbb{R}$. $\square$

The hypothesis of the above lemma characterizes sequences which are sparse, i.e., at least geometrically increasing with common ratio strictly bigger than 1. Thus, gaps that grow rapidly (but are still finite) do indeed have the required inner function. Notice that we could make this result stronger by allowing sequences that, instead of singletons, have finite bunches that are sparsely distributed. For, each bunch would contribute a (uniformly) bounded weight to the existing sum. We prove lemma 10.

Proof. (of lemma 10) Suppose we choose one point from each cluster and call it $a_{m_0}^{j_0}$, then by the proof of the previous lemma,

$$\sum_{n\neq m_0}\left(\frac{w_n^j}{a_n^j-a_{m_0}^{j_0}}-\frac{a_n^j w_n^j}{1+(a_n^j)^2}\right)=\sum_{n\neq m_0}\left(\frac{1}{a_n^j-a_{m_0}^{j_0}}-\frac{a_n^j}{1+(a_n^j)^2}\right)<B,$$

where B is a bound, independent of k. Consider a point $a_{n_0}^{j_0}$ and let the '*' in the sum denote summation over all points except $a_{m_0}^{j_0}$

$$\begin{aligned}\left|\sum_{*}\frac{1}{a_n^j-a_{m_0}^{j_0}}-\frac{a_n^j}{1+(a_n^j)^2}\right| &= \left|\sum_{j\neq j_0}\frac{1}{a_{m_0}^j-a_{m_0}^{j_0}}-\frac{a_{m_0}^j}{1+(a_{m_0}^j)^2}\right|+\left|\sum_{n\neq i_0,1\leqslant j\leqslant n_m}\frac{1}{a_n^j-a_{m_0}^{j_0}}-\frac{a_n^j}{1+(a_n^j)^2}\right|\\ &\leqslant S+N\left|\sum_{n\neq k}\frac{1}{a_n-a_k}-\frac{a_n}{1+a_n^2}\right|\leqslant S+NB,\end{aligned}$$

where S and B are constants. The maximum size of each cluster N assures that S is independent of m_0 and n_0 and we know from the previous lemma that B is independent of m_0 and n_0.

□

We now proceed to the last part of our discussion. Before we begin our proof of proposition 6, let us elucidate some notations. Let us enumerate the zeroes $z_n(=x_n+iy_n)$ of Θ and let $\Theta(x)=e^{i\phi(x)}$ on $\mathbb{R}$. Let us pick and fix a large N and let $\{t_i\}$ be a set of points on $\mathbb{R}$ as described in the statement of the lemma. Let S be the box $(t_2,t_N)\times(0,\sqrt{ND})$ and T the box $(t_1,t_2)\times(0,\sqrt{ND})$. Let $\tilde{a}$ be the mid point of the interval (t_2,t_N) and let $\tilde{S}$ be the box $(t_2,\tilde{a})\times(0,\sqrt{ND})$. On the adjacent interval, let $\tilde{c}$ be the point in (t_1,t_2) such that $\tilde{a}-t_2=t_2-\tilde{c}$. Since z_n form the zeroes of the Blaschke product of Θ, we can write

$$\phi'(x)=\sum_n\frac{y_n}{(x-x_n)^2+y_n^2}.$$

Proof. Suppose that the zeros are bounded away from the real line, i.e., there is a $\delta > 0$ such that $y_n \delta$ for all all the zeros $z_n = x_n + iy_n$ of Θ. Without loss of generality, let $\delta = 1$. We have that for $t \in (\tilde{c}, t_2)$ and $s \in (t_2, \tilde{a})$,

$$\sum_{z_n \notin S \cup T} \frac{y_n}{(s - x_n)^2 + y_n^2} \leqslant \kappa \sum_{z_n \notin S \cup T} \frac{y_n}{(t - x_n)^2 + y_n^2},$$

where $\kappa > 0$ is a constant independent of N.

Then,

$$\int_{t_2}^{\tilde{a}} \sum_{z_n \notin S \cup T} \frac{y_n}{(s - x_n)^2 + y_n^2} dt \leqslant \kappa \int_{\tilde{c}}^{t_2} \sum_{z_n \notin S \cup T} \frac{y_n}{(t - x_n)^2 + y_n^2} dt \leqslant \kappa\pi,$$

and the zeros in the box $T = (t_1, t_2) \times (0, \sqrt{ND})$ induce the following inequality

$$\int_{t_2}^{\tilde{a}} \sum_{z_n \in T} \frac{y_n}{(s - x_n)^2 + y_n^2} ds \leqslant \int_{\tilde{c}}^{t_2} \sum_{z_n \in T} \frac{y_n}{(t - x_n)^2 + y_n^2} dt \leqslant \pi.$$

Let Z be the number of zeros $\{z_n\}$ in the box S. Then,

$$\int_{t_2}^{\tilde{a}} \sum_{z_n \in S} \frac{y_n}{(s - x_n)^2 + y_n^2} \leqslant Z\pi.$$

Then,

$$\begin{aligned} \int_{t_2}^{\tilde{a}} \sum_{z_n \in S} \frac{y_n dt}{(s - x_n)^2 + y_n^2} &= \int_{t_2}^{\tilde{a}} \left(\sum_{n \in \mathbb{Z}} - \sum_{(S \cup T)^c} - \sum_{T} \right) \frac{y_n}{(s - x_n)^2 + y_n^2} dt \\ &> N\pi - \kappa\pi - \pi. \end{aligned}$$

Thus,

$$\left(N - \kappa - 1 \right) \pi \leqslant Z\pi.$$

This gives us that

$$Z \geqslant N - \kappa - 1.$$

Thus, for a large enough N, $Z \geqslant N/2$. In a similar vein it can be proved that the box $\tilde{S}$ contains at least $N/4$ zeroes. And for any subinterval of the form (t_2, u), containing n_u points from $\sigma(\Theta)$, the box $(t_2, u) \times (0, \sqrt{ND})$ contains at least $n_u/2$ zeros. We know that $t_n \leqslant t_2 + (n-1)D$ for $n \geqslant 2$. Thus, enumerating the zeros inside $\tilde{S}$, we have $x_n \leqslant t_2 + 2(n-1)D$.

Thus,

$$\begin{aligned}
\pi > \int_{\tilde{c}}^{t_2} \phi'(t)dt \geqslant \int_{\tilde{c}}^{t_2} \sum_{\tilde{S}} \frac{y_n}{(t-x_n)^2 + y_n^2} dt \quad &\geqslant \quad \int_{\tilde{c}}^{t_2} \sum_{n=1}^{N/4} \frac{y_n}{(t-(t_2+nD))^2 + y_n^2} dt \\
&\gtrsim \quad \sum_{n=1}^{N/4} \int_{\tilde{c}}^{t_2} \frac{1}{(t-(t_2+nD))^2+1} dt \\
&= \quad \sum_{n=1}^{N/4} \arctan(Dn) - \arctan(Dn + t_2 - \tilde{c}) \\
&= \quad \sum_{n=1}^{N/4} \arctan\left(\frac{t_2 - \tilde{c}}{(Dn + t_2 - \tilde{c})Dn}\right) \\
&\geqslant \quad \sum_{n=1}^{N/4} \arctan\left(\frac{1}{(2Dn/N^2 + 1)Dn}\right) \\
&\gtrsim \quad \frac{1}{D} \sum_{n=1}^{N/4} \frac{1}{n},
\end{aligned}$$

which diverges as $N \to \infty$, which is a contradiction. □

Baranov remarks in [3] that since the placement of 'other points' does not affect the calculations above, we can make such clusters and gaps along a very rare subsequence of $\mathbb{N}$, without affecting the regularity. For example, let us consider the following sequence $\Lambda = \mathbb{N} \backslash A$, where $A = \{2^{n_k} + m\}$ for $m = 1, 2, ..., k$, where n_k is a rare subsequence of $\mathbb{N}$, say the sequence $n_k = 3^k$. Then, we have gaps of length k, which is unbounded, followed by clusters with gaps of size 1, the size of the clusters $\geqslant 2^{n_k+1}$. This sequence is *a-regular*, where $a = 1$. For,

$$\int_{\mathbb{R}} \frac{|n_\Lambda(x) - x|}{1+x^2} dx \asymp \sum_k \frac{k}{1+(2^{n_k})^2} < \infty.$$

Thus, even for regular sequences, there may not exist any MIF with bounded derivative.

CHAPTER V

EXISTENCE AND UNIQUENESS OF POTENTIAL

In this chapter, we find necessary and sufficient conditions for when a separated sequence is defining for a MIF. We also establish the extent of non-uniqueness, in terms of a metric description of the density of the sequence. Our results have consequences on uniquely reconstructing potential from spectral data. In the results that follow, we use the following notations. Let Λ be a separated sequence on $\mathbb{R}$. Let Φ be a meromorphic inner functions on $\mathbb{C}_+$ and let $\mathcal{Z} = \{f \in K^+_{\Phi^2} : f|_\Lambda = 0\}$. We observe that $\mathcal{Z}$ is a subspace of $\mathcal{N}^+$. The results in this section have been proved in collaboration with Mishko Mitkovski [29].

5.1 Uniqueness

5.1.1 Defining Sets for MIFs

The next result follows the proof of proposition 3.

Lemma 13. *Let $f \in \mathcal{Z}$. Let Θ be any MIF with $\sigma(\Theta) = \Lambda$ and define $g = \frac{f}{1-\Theta}$. Then, $g \in \ker^+_{\bar{\Phi}^2\Theta}$.*

Proof. Given that the function $f \in \mathcal{N}^+$ and Θ is an MIF, the function

$$\frac{f}{1-\Theta} \quad \in \mathcal{N}^+.$$

Moreover,

$$\begin{aligned}\overline{\Phi^2}\Theta g = \overline{\Phi^2}(g-f) &= \overline{\Phi^2}\left(\frac{f}{1-\Theta} - f\right)\\ &= \overline{\Phi^2}\left(\frac{f\Theta}{1-\Theta}\right)\\ &= -\frac{\overline{\Phi^2}f}{1-\overline{\Theta}} \;\in \overline{\mathcal{N}^+},\end{aligned}$$

since $f \in K^+_{\Phi^2}$. □

Lemma 14. *Let Λ, Φ and $\mathcal{Z}$ be as described above. Let Θ be any MIF with $\sigma(\Theta) = \Lambda$. Then,*

$$T : \mathcal{Z} \to \ker^+_{\overline{\Phi}^2\Theta} \tag{5.29}$$

defined as

$$T(f) = \frac{f}{1-\Theta}$$

is an isomorphism.

Proof. The previous lemma (lemma 13) tells us that for $f \in \mathcal{Z}$, $\frac{f}{1-\Theta} \in \ker^+_{\overline{\Phi}^2\Theta}$. It is easy to see that T is a homomorphism. Moreover, if $\frac{f}{1-\Theta} \equiv 0$, then, $f \equiv 0$. To see that T is surjective, we simply look at its inverse. Let $g \in \ker^+_{\overline{\Phi}^2\Theta}$. Then, there is an $h \in \mathcal{N}^+$ such that $\overline{\Phi^2}\Theta g = \overline{h}$. We observe that $\Theta g = \Phi^2\overline{h} \in K^+_{\Phi^2}$ and $g = \Phi^2\overline{\Theta h} \in K^+_{\Phi^2}$. Then, $g(1-\Theta) \in K^+_{\Phi^2}$ and $g(1-\Theta)|_\Lambda = 0$. Thus, $g \to g(1-\Theta)$ is T^{-1}. □

Lemma 15. *Let Λ be a separated sequence on $\mathbb{R}$. Let Θ be an MIF with $\sigma(\Theta) = \Lambda$. Then,*

$$\Lambda \textit{ is not defining for } \Phi \Leftrightarrow \ker^+_{\overline{\Phi^2}\Theta} \neq \{0\} \tag{5.30}$$

Proof. Let $\Phi_2 \not\equiv \Phi$ be an MIF such that $\arg(\Phi_2) = \arg(\Phi)$. Then, $F = \Phi - \Phi_2 \in K^+_{\Phi\Phi_2}$,

$F = 0$ on Λ and $F \not\equiv 0$. By the previous lemma, this means that $\ker^{+}_{\overline{\Phi\Phi_2}\Theta} \neq 0$. By proposition 1, this means that

$$\arg(\Theta) - (\arg(\Phi) + \arg(\Phi_2))$$

is almost decreasing. Since $\|\arg(\Phi) - \arg(\Phi_2)\|_\infty \leqslant 2\pi$, we can replace $\arg(\Phi) + \arg(\Phi_2)$ by $2\arg(\Phi)$, while remaining almost decreasing. Thus, $\ker^{+}_{\bar{\Phi}^2\Theta} \neq \{0\}$.

Conversely, if $\ker^{+}_{\overline{\Phi^2}\Theta} \neq \{0\}$, then by the previous lemma, there is a function $G \in K^{+}_{\Phi^2}$, such that $G = 0$ on Λ, $G \not\equiv 0$. Then, there is an $F \in \mathcal{H}^2$ such that $\overline{\Phi^2}G = \overline{F}$. We follow the proof of proposition 4 in [27]. Define a new function $\tilde{\Phi}$ as

$$\tilde{\Phi} = \frac{\Phi^2 + F}{\Phi + \Phi G}.$$

The function $\tilde{\Phi} \not\equiv \Phi$, since if that were true, we would have $G \equiv \bar{G}$, implying that G is a real valued function, thus a constant function, which in this case must be 0. But this would contradict our assumption, hence $\tilde{\Phi} \neq \Phi$. We also have that $\tilde{\Phi}$ is inner because

$$|\Phi^2 + F| = |\Phi^2 + \Phi^2\bar{G}| = |1 + G| = |\Phi + \Phi G|.$$

Moreover,

$$\frac{\Phi^2 + F}{\Phi + \Phi G} = \frac{\Phi^2 + \Phi^2\bar{G}}{\Phi + \Phi G} = \Phi\frac{1 + \bar{G}}{1 + G}.$$

On the set Λ, the function $G = 0$. Thus $\tilde{\Phi} = \Phi$ on Λ. It is an easy geometric proof that $|\arg(1 + \bar{G}) - \arg(1 + G)| = 2g$, where g is the principal value of the argument of the function G. Thus,

$$\|\arg\Phi - \arg\tilde{\Phi}\|_\infty < 2\pi.$$

Thus $\arg\Phi = \arg\tilde{\Phi}$ on Λ. $\square$

The proof above shows: Every new 'dimension' in $\ker^{+}_{\Phi^2\Theta}$ gives us a one-parameter family of MIFs that agree with with Φ on Λ.

Lemma 16. *Let $\langle v\rangle$ be a one-dimensional subspace in $\ker^{+}_{\Phi^2\Theta}$ generated by v. Then, there is a family of MIFs $\{\Phi_{r,v}\}_{r\in\mathbb{R}}$ such that $\arg\Phi = \arg\Phi_{r,v}$ on Λ, $\Phi_{r,v}\not\equiv\Phi$, given by the formula*

$$\Phi_{r,v} := \Phi\left(\frac{1+r\overline{G}}{1+rG}\right),$$

where $G := (1-\Theta)v \in K^{+}_{\Phi^2}$ and $r\in\mathbb{R}$.

We observe another consequence of the proof of lemma 15: If $G\in K^{+}_{\Phi}$ such that $G=\overline{G}$ on Λ, then $\tilde{\Phi} := \Phi\frac{1+\bar{G}}{1+G}$ is a MIF, with $\arg\tilde{\phi} = \arg\phi$ on Λ. Indeed, this is proposition 4 in [27], for the space K^{+}_{Φ}, instead of K^{p}_{Φ}.

Proposition 7. *(Makarov & Poltoratski) Λ is not defining for Φ if there is a non-constant function $G\in K^{+}_{\Phi}$ such that*

$$G=\bar{G} \quad \text{on} \quad \Lambda. \tag{5.31}$$

A natural question to ask is- do all such MIFs $\tilde{\Phi}$ with $\arg\tilde{\Phi} = \arg\Phi$ on Λ have the form $\tilde{\Phi} = \Phi\frac{1+\bar{G}}{1+G}$, with $G\in K^{+}_{\phi}$, $G=\overline{G}$ on Λ? Let us try to construct this G. If such a G were to exist, then

$$\begin{aligned}\arg\frac{\tilde{\Phi}}{\Phi} &= \arg\frac{1+\bar{G}}{1+G}\\ &= \text{Principal Arg}(G).\end{aligned}$$

Let us choose a branch with this principal argument and extend it to $\mathbb{C}_+$ using the

Schwartz transform. Then,

$$\begin{aligned}
\log\left(\frac{\tilde{\Phi}}{\Phi}\right) &= \log\left(\frac{1+\overline{G}}{1+G}\right)+c \\
\frac{\tilde{\Phi}}{\Phi} &= e^c\frac{1+\overline{G}}{1+G}.
\end{aligned}$$

Since $\tilde{\Phi} = \Phi = 1$ on Λ, it must be the case that $e^c = 1$. Since $\arg\tilde{\Phi} = \arg\Phi$ (which also means that $\tilde{\Phi} = \Phi$ on Λ), this forces G to be real on Λ. Moreover, we have the following computation

$$\begin{aligned}
\tilde{\Phi} + \tilde{\Phi}G &= \Phi + \Phi\overline{G} \\
\phi\overline{G} &= \tilde{\phi} + \tilde{\phi}G - \phi \\
\overline{\phi}G &= \overline{\tilde{\phi}} + \overline{\tilde{\phi}G} - \overline{\phi}.
\end{aligned}$$

Thus, $G \in K_\phi^+$. We have proved the following result.

Lemma 17. *Let $\tilde{\Phi}$ be a MIF, different from Φ such that $\arg\Phi = \arg\tilde{\Phi}$ on Λ. Then, there is a $G \in K_\Phi^+$, $G = \overline{G}$ on Λ such that*

$$\tilde{\Phi} = \Phi\left(\frac{1+\overline{G}}{1+G}\right).$$

We now reconnect with the space $K_{\Phi^2}^+$ with the following result.

Lemma 18. *If $G \in K_\phi^+$, $G = \overline{G}$ on Λ, then $\exists\ F \in K_{\phi^2}^+$, $F = 0$ on Λ.*

Proof. Let $H \in K_\phi^+$ be such that $\overline{\phi}G = \overline{H}$. Define a function $F = \phi G - H \in \mathcal{N}^+$.

Then,

$$\begin{aligned}
\overline{\phi}^2 F &= \overline{\phi} G - \overline{\phi}\phi\overline{G} \\
&= \overline{H} - \overline{G} \ \in \overline{\mathcal{N}^+},
\end{aligned}$$

Thus, $F \in K^+_{\phi^2}$. Moreover, $F = \phi G - \phi\overline{G} = 0$ on Λ. □

We can now state a corollary which will serve the purpose of a converse to corollary 16.

Corollary 2. *Let $\tilde{\Phi}$ be a MIF, different from Φ such that* $\arg\Phi = \arg\tilde{\Phi}$ *on* Λ. *Then, there is $v \in \ker^+_{\overline{\Phi}^2\Theta}$ and a one- parameter family of MIFs $\{\Phi_{r,v}\}_{r\in\mathbb{R}}$ with the property* $\arg\Phi_{r,v} = \arg\Phi$ *on* Λ, $\Phi_{r,v} \not\equiv \Phi$ *such that*

$$\Phi_{r,v} = \Phi\left(\frac{1 + r\overline{f}}{1 + rf}\right),$$

where $f := (1 - \Theta)v$ and $r \in \mathbb{R}$.

5.1.2 Applications in the Schrödinger Case

Let us now state some applications of lemma 15. In the following result, we recall the *Beurling Malliavin* densities (BM density), which are defined as follows [28].

$$\begin{aligned}
D_*(\Lambda) &= \sup\{a | \exists \text{ an } a-regular \text{ subsequence}\, \Lambda' \subset \Lambda\}. \\
D^*(\Lambda) &= \inf\{a | \exists \text{ an } a-regular \text{ supersequence}\, \Lambda' \supset \Lambda,
\end{aligned}$$

where a separated sequence Λ' is said to be $a-regular$ is

$$\int_{\mathbb{R}} \frac{|n_{\Lambda'}(x) - ax|}{1 + x^2} dx < \infty,$$

where $n_{\Lambda'}$ is the counting function of Λ'.

Lemma 19. *Let $q_1 \in L^1(a,b)$ be the potential of the operator*

$$u \to -u'' + q_1 u,$$

with boundary conditions as follows. Let α_n be different boundary conditions at a:

$$u(a)cos(\alpha_n) + u'(a)\sin(\alpha_n) = 0,$$

and fixed boundary condition β at b

$$u(b)\cos(\beta) + u'(b)\sin(\beta) = 0.$$

Let $\Lambda = \{\lambda_n\}_n$ be such that $\lambda_n \in \sigma(q_1, \alpha_n, \beta)$. Then the following is true.

1. *Let $q_2 \in L^1(a,b)$ be a potential for another operator with the same boundary conditions and let $\lambda_n \in \sigma(q_2, \alpha_n, \beta)$. If $D^*(\Lambda) > 2D^*(\sigma(\Phi))$, then $q_1 = q_2$ a.e.*

2. *If $D^*(\Lambda) < 2D^*(\sigma(\Phi))$, then there is a $q_2 \not\equiv q_1$ such that $\lambda_n \in \sigma(q_2, \alpha_n, \beta)$.*

Proof. 1. Let Φ_1 be the Weyl inner function corresponding to q_1. Using Marchenko's result, it is enough to show that Λ is a defining set for Φ_1. We know that $\sigma(\Phi_1) = \sigma(H_1, D, \beta)$, where D refers to the Dirichlet boundary condition at a. It is well known that for Schrödinger operators, the spectrum is *regular*, i.e., $D_*(\sigma(\Phi_1)) = D^*(\sigma(\Phi_2))$. By the hypothesis,

$$D^*(\Lambda) > 2D^*(\sigma(\Phi_1)) = 2D_*(\sigma(\Phi_1)). \tag{5.32}$$

Let Θ be any MIF with $\sigma(\Theta) = \Lambda$. By lemma 15, it is enough to show

$$\ker^{+}_{\overline{\Phi_1^2}\Theta} = \{0\}. \tag{5.33}$$

Suppose that $\ker^{+}_{\overline{\Phi_1^2}\Theta} \neq \{0\}$. Let $a \in (D^*(\sigma(\Phi_1), D^*(\Lambda))$. We recall the equivalent definitions of densities, as given in [28].

$$\begin{aligned} D^*(\Lambda) &:= \sup\{a : \ker^2_{\bar{S}^a J} = \{0\}\}, \\ D_*(\Lambda) &:= \inf\{a : \ker^2_{\bar{J} S^a} = \{0\}\}, \end{aligned}$$

where $S(z) = e^{iz}$ and J is any MIF with $\sigma(J) = \Lambda$. Then, by the definition of the densities, we have the following relations.

$$\begin{aligned} \ker^2_{\bar{S}^a \Theta} &= \{0\} && (5.34) \\ \ker^2_{\overline{\Phi_1}^2 S^a} &= \{0\} \\ \ker^2_{\bar{S}^a \Phi^2} &\neq \{0\}(\Rightarrow \ker^{+}_{\bar{S}^a \Phi^2} \neq \{0\}) && (5.35) \\ \ker^{+}_{\overline{\Phi_1}^2 \Theta} &\neq \{0\}. && (5.36) \end{aligned}$$

However, 5.35 and 5.36 combine to give $\ker^{+}[\bar{S}^a\Theta] \neq \{0\}$, which is a contradiction to 5.34. Thus, it must be the case that $\ker^{+}[\overline{\Phi_1}^2\Theta] = \{0\}$.

2. This is an application of Hórvath's result (theorem 4) and the fact that if $R(\Lambda)$ is the radius of completion on the exponentials $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$, then $D^*(\Lambda) = \frac{1}{2\pi}R(\Lambda)$. □

5.1.3 A Special 3-Spectra Case

We also have an extension of Krein's construction to the case of 3 spectra in the following sense.

Lemma 20. *Let* $\Lambda_i = \sigma(\Theta_i)$, $i = 1, 2$ *and* $\Lambda = \sigma(\Theta_1\Theta_2)$. *Given* Λ_1, Λ_2 *and* Λ, *pairwise disjoint, there exist meromorphic inner functions* θ_i *such that* $\sigma(\Theta_i) = \Lambda_i$, *that are unique upto a Möebius transformation.*

Proof. Consider the function

$$f = \frac{\Theta_1\Theta_2 - 1}{(\Theta_1 - 1)(\Theta_2 - 1)}$$

f is a holomorphic function on $\mathbb{C}_+$ and since $\overline{f(x)} = -f(x)$ $\forall x \in \mathbb{R}$, the argument of f only takes the values $\pm\frac{\pi}{2}$. Let

$$\begin{aligned} T_1 &:= \{x \in \mathbb{R} : \Theta_1(x) = 1 \ \textit{or} \ \Theta_2(x) = 1\} \\ T_2 &:= \{x \in \mathbb{R} : \Theta_1\Theta_2 = 1\}. \end{aligned}$$

Simple geometric arguments tell us that

$$\begin{aligned} \{\arg(f) = \pi/2\} &= \{\operatorname{Princ}\arg(\Theta_1) + \operatorname{Princ}\arg(\Theta_2) \ \geqslant 2\pi\}, \\ \{\arg(f) = -\pi/2\} &= \{\operatorname{Princ}\arg(\Theta_1) + \operatorname{Princ}\arg(\Theta_2) \ < 2\pi\}. \end{aligned}$$

Using the above characterization, it is not hard to see that T_1 and T_2 actually interlace i.e. between any two elements of T_1, there is an element of T_2 and between any two of T_2, there is one of T_1.

Consider the function

$$F = \frac{1}{\pi i}\log f.$$

Let $S_{\pi/2} := \{\arg(f) = \pi/2\}$. Then,

$$F(z) = S\chi_{S_{\pi/2}} + ic,$$

where $S\chi_{S_{\pi/2}}$ is the Schwartz transform of $\chi_{S_{\pi/2}}$. We notice that f can be rearranged as

$$f = \frac{1}{\Theta_1 - 1} + \frac{1}{\Theta_2 - 1} + 1 \tag{5.37}$$

If there exist $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ such that they have the same spectrum Λ_1 and Λ_2 and $\tilde{\Theta}_1\tilde{\Theta}_2$ has the spectrum Λ, then

$$\tilde{f} = \frac{1}{\tilde{\Theta}_1 - 1} + \frac{1}{\tilde{\Theta}_2 - 1} + 1 \tag{5.38}$$

differs from f by only a constant, say c. Suppose $a \in \Lambda_1$. Then we take radial limits to the boundary,

$$\begin{aligned} \lim_{z\to a} f(z) &= \lim_{z\to a} \tilde{f}(z) + c && (5.39)\\ \frac{1}{\Theta_1'(a)} &= \frac{1}{\tilde{\Theta}_1'(a)} + c. && (5.40) \end{aligned}$$

Thus, the spectral measure of Θ_1 and $\tilde{\Theta}_1$ are equal, upto a constant. Thus, the two functions are defined upto a constant. The same is true for Θ_2 as well. □

Let us see the application of this result in spectral theory.

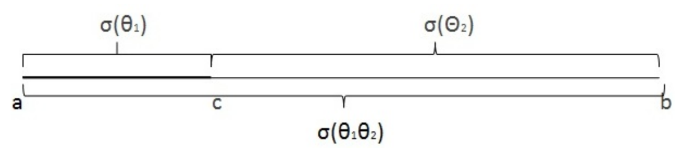
Lemma 21. *Consider the operators*

$$\begin{aligned} H_1 u &:= -u'' + q_1 u, \\ H_2 u &:= -u'' + q_2 u. \end{aligned}$$

on $L^2(a,b)$, with boundary conditions:

$$\begin{aligned} u(a)\cos(\alpha) + u'(a)\sin(\alpha) &= 0 \\ u(b)\cos(\beta) + u'(b)\sin(\beta) &= 0. \end{aligned}$$

Let c be a point in (a,b). Let $\Theta_1 := \Theta^c_{a,\alpha}$ and $\Theta_2 := \Theta^c_{b,\beta}$ be the Weyl inner functions corresponding to H_1 and $\tilde{\Theta}_1 := \Theta^c_{a,\alpha}$ and $\tilde{\Theta}_2 := \Theta^c_{b,\beta}$ be the Weyl inner functions corresponding to H_2. If
$\sigma(\Theta_1) = \sigma(\tilde{\Theta}_1)$, $\sigma(\Theta_2) = \sigma(\tilde{\Theta}_2)$ and $\sigma(\Theta_1\Theta_2) = \sigma(\tilde{\Theta}_1\tilde{\Theta}_2)$,
then $q_1 \equiv q_2$ a.e. Consequently, $q|_{(a,c)}$ is completely determined by Θ_1 and $q|_{(c,b)}$ is completely determined by Θ_2.



Proof. By the previous lemma, Θ_1 and Θ_2 are determined to be unique, upto a Möebius transform by $\sigma(\Theta_1), \sigma(\Theta_2), \sigma(\Theta_1\Theta_2)$. □

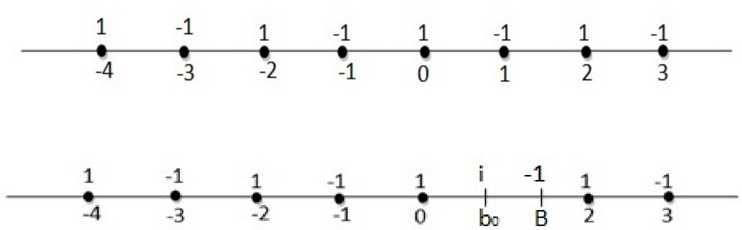
It is interesting to note that in the above case, if any of the spectra agree on all points but one, then we can use Hórvath and Chelkak's result to reconstruct a different potential, also from a Schrödinger operator.

5.2 Existence of Potential

In the previous section, we investigated the uniqueness of potential corresponding to some spectral data. We now ask the existence question: Is a given sequence Λ, is there a potential that corresponds to it? In general, if we have data that corresponds to two spectra of a Schrödinger operator, we can use Borg's result to conclude the existence of the corresponding potential. At the MIF level, this corresponds to 2 spectra of the Weyl inner function. One may ask if the data has the BM density of 2 spectra, but with points from several spectra, will it necessarily correspond to a MIF? The answer is : No! The following result is an example of such a case.

Lemma 22. *Let* $\Lambda_1 = \{a_n\} := \{2n\}_n$ *and* $\Lambda_2 = \{b_n\} := \{2n+1\}_{n\neq 0}$. *There is an* $x \in (0,2)$ *such that if* Θ *is an MIF with* $\sigma(\Theta) = \Lambda_1$, $\Theta(b_n) = -1$ *and* $\Theta(b_0) = i$, *then* $b_0 \leqslant x$, *i.e., the solution to* $\Theta(z) = i$ *cannot lie in the interval* $(x, 2)$ *.*

Proof. Suppose Θ is an MIF with $\sigma(\Theta) = \Lambda_1$, $\Theta(b_n) = -1$ and $\Theta(b_0) = i$. Let B be the point in $(0,2)$ such that $\Theta(B) = -1$. Then, by Krein's construction, we have



that

$$\frac{\Theta+1}{\Theta-1} = \exp(\pi i K u), \tag{5.41}$$

where Ku is the Cauchy transform of $u := \chi_E - \frac{1}{2}$ where $E = \bigcup_{n\neq 0}(a_n, b_n) \cup (a_0, B)$. On $\mathbb{R}$, we have the above equation taking the form

$$\frac{\Theta+1}{\Theta-1} = \exp(\pi i K u) = \exp(\pi i u).\exp(-\pi \tilde{u}), \tag{5.42}$$

where $\tilde{u}$ is the Hilbert transform of u. Thus,

$$\begin{aligned}
-i = \frac{i+1}{i-1} = \frac{\Theta(b_0)+1}{\Theta(b_0)-1} &= i.\exp(-\pi\tilde{u}) \\
&= i\exp\Bigg[-\frac{\pi}{2}\Bigg(\int_0^B \frac{1}{t-b_0}dt + \sum_{n\neq 0}\int_{2n}^{2n+1}\frac{1}{t-b_0}dt \\
&\qquad -\sum_{n\neq 1}\int_{2n-1}^{2n}\frac{1}{t-b_0}dt - \int_B^2 \frac{1}{t-b_0}dt\Bigg)\Bigg]
\end{aligned}$$

The right hand side is a continuous function of B. We will prove that for no $B \in (0,2)$ does the above equation hold.

Notice that since $\Theta(b_0) = i$, $B > b_0$. We evaluate the right hand side

$$\begin{aligned}
& i\exp\Bigg[-\frac{\pi}{2}\Bigg(\int_0^B \frac{1}{t-b_0}dt + \sum_{n\neq 0}\int_{2n}^{2n+1}\frac{1}{t-b_0}dt - \sum_{n\neq 1}\int_{2n-1}^{2n}\frac{1}{t-b_0}dt - \int_B^2 \frac{1}{t-b_0}dt\Bigg)\Bigg] \\
= \;& i\exp\Bigg[-\frac{\pi}{2}\Bigg(\sum_{n=-\infty}^{\infty}\int_{2n}^{2n+1}\frac{1}{t-b_0}dt - \sum_{n=-\infty}^{\infty}\int_{2n-1}^{2n}\frac{1}{t-b_0}dt + 2\int_1^B \frac{1}{t-b_0}dt\Bigg)\Bigg] \\
= \;& -\frac{e^{i\pi b_0}+1}{e^{i\pi b_0}-1}\exp\left(-\pi\int_1^B \frac{1}{t-b_0}dt\right) \\
= \;& -\frac{e^{i\pi b_0}+1}{e^{i\pi b_0}-1}\exp\left(\pi\ln\frac{b_0-1}{B-b_0}\right) \\
=: \;& \alpha i,
\end{aligned}$$

where α is negative. We are interested in the solution to $\alpha = -1$. As B increases, the solution (i.e., the value of b_0) that gives us $\alpha = -1$ also increases. When $B = 2$, the solution is given by $b_0 = 1.5$. Thus, if we let $x > 1.5$, then $\Theta(x) \neq i$, for any value of B in the interval $(0,2)$. In other words, for any $x \in (1.5,2)$, the MIF for which $\sigma(\Theta) = \Lambda_1$, $\Theta(b_n) = -1$ for $n \neq 0$ and $\Theta(x) = i$ does not exist.

$\square$

Will it help if we remove another or a few more $-1s$? The answer is No! Let us remove N $-1s$, situated at $1, 3, 5, .., 2N-1$ and again place an i at 1.5. We will do similar calculations as above to prove that there will not exist an MIF corresponding to this configuration. We denote by S the set $\{0, 1, ..., N-1\}$. Precisely,

Lemma 23. *Let $\Lambda_1 = \{2n\}_n$ and $\Lambda_2 = b_n := \{2n+1\}_{n \notin S}$. Then, there is an $x \in (0, 2)$ such that if Θ is an MIF with $\sigma(\Theta) = \Lambda_1$, $\Theta(b_n) = -1$ for $n \notin S$ and $\Theta(b_0) = i$ for $b_o \in (0, 2)$, then $b_0 \leqslant x$.*

Proof. As before, we will assume, for a contradiction, the existence of such an MIF Θ. For each $i = 0, 1, ..., N-1$, let $B_i \in (2i, 2(i+1))$ such that $\Theta(B_i) = -1$. Then, Θ is determined by Krein's formula and

$$-i = \frac{i+1}{i-1} = \frac{\Theta(b_0)+1}{\Theta(b_0)-1} = i \exp\Bigg[-\frac{\pi}{2}\Bigg(\sum_{i=0}^{N-1} \int_{2i}^{B_i} \frac{1}{t-b_0} dt + \sum_{n \notin S} \int_{2n}^{2n+1} \frac{1}{t-b_0} dt \\ - \sum_{n \notin S} \int_{2n+1}^{2n+2} \frac{1}{t-b_0} dt - \sum_{i=0}^{N-1} \int_{B_i}^{2i+2} \frac{1}{t-b_0} dt \Bigg) \Bigg]. \tag{5.43}$$

We know that $B_0 > 1.5$. For each $i > 0$, suppose that $B_i = 2i+1$, i.e. at $3, 5, 7, ..., 2N-1$. Then, by the previous lemma, i cannot be placed at 1.5- it would have to be in the interval $(0, 1.5)$. It is easy to see that if B_i increases, so does the position of i in $(0, 2)$. So, let us assume that $B_i = i+2$ for all $i > 0$. Then, the right hand side evaluates as

$$\begin{aligned}
& i \quad \exp\left[-\frac{\pi}{2}\left(\sum_{i=0}^{N-1}\int_{2i}^{B_i}\frac{1}{t-b_0}dt+\sum_{n\notin S}\int_{2n}^{2n+1}\frac{1}{t-b_0}dt-\sum_{n\notin S}\int_{2n+1}^{2n+2}\frac{1}{t-b_0}dt-\sum_{i=0}^{N-1}\int_{B_i}^{2i+2}\frac{1}{t-b_0}dt\right)\right] \\
= & -\frac{e^{i\pi b_0}+1}{e^{i\pi b_0}-1}\exp\left(-\pi\int_1^2\frac{1}{t-b_0}dt-\sum_{n=0}^{N-1}\int_{2n+1}^{2n+2}\frac{1}{t-b_0}dt\right) \\
= & -\frac{e^{i\pi b_0}+1}{e^{i\pi b_0}-1}\exp\pi\left(\ln\frac{b_0-1}{2-b_0}+\sum_{n=1}^{N-1}\ln\left(\frac{2n+1-b_0}{2n+2-b_0}\right)\right) \\
= & -\frac{e^{i\pi b_0}+1}{e^{i\pi b_0}-1}\left[\left(\frac{b_0-1}{2-b_0}\right)\left(\frac{3-b_0}{4-b_0}\right)...\left(\frac{2N-1-b_0}{2N-b_0}\right)\right]^{\pi} \\
= & \ \alpha i,
\end{aligned}$$

where α is again a negative real number. Let us call the solution to $\alpha = -1$ as x_N. Then, for $x \in (x_N, 2)$, the MIF Θ for which $\sigma(\Theta) = \Lambda_1$, $\Theta(b_n) = -1$ for $n \notin S$ and $\Theta(x) = i$ does not exist. □

How does x_N approach 2 as N approaches ∞? We notice that when $x \in (1.5, 2)$,

$$(e^{i\pi b_0}+1)(b_0-1)\left[\left(\frac{3-b_0}{4-b_0}\right)...\left(\frac{2N-1-b_0}{2N-b_0}\right)\right] = C\frac{\Gamma(N-y_N)}{\Gamma(N+0.5-y_N)},$$

where C is a constant dependent on N, but lies in $(1, 2)$ and $y_N \in (0, 0.5)$. We recall the asymptotics of the Gamma function: $\Gamma(x) \approx x^x$ as $x \to \infty$. Thus,

$$\frac{\Gamma(N-y_N)}{\Gamma(N+0.5-y_N)} \approx e^{\ln(1/N)}.$$

Thus, the solution x_N of the equation

$$-\frac{e^{i\pi y}+1}{e^{i\pi y}-1}\left[\left(\frac{y-1}{2-y}\right)\left(\frac{3-y}{4-y}\right)...\left(\frac{2N-1-y}{2N-y}\right)\right]^{\pi} = -i$$

obeys the asymptotic relation

$$\frac{1}{(x_N-2)^2}e^{\pi\ln(1/N)}\approx 1.$$

Thus $x_N - 2 = O(\frac{1}{N^{\pi/2}})$.

We generalize lemma 22 to arbitrary intertwining sequences. Let $\Lambda_1 = \{a_n\}$ and $\Lambda_2 = \{b_n\}$ be 2 separated intertwining sequences. Let $\Lambda_2^* = \Lambda_2 \backslash \{b_0\}$.

Theorem 11. *Let Θ be an MIF with $\sigma(\Theta) = \Lambda_1$ and $\Theta(\Lambda_2^*) = -1$, then there is an $x \in (a_0, a_1)$ such that the solution of $\Theta(y) = i$ in the interval (a_0, a_1) must be in (a_0, x).*

Proof. We follow the algorithm in the proof of lemma 22. Let Φ be an MIF with $\sigma(\Phi) = \Lambda_1$ and $\sigma(-\Phi) = \Lambda_2$. Let $B \in (a_0, a_1)$ such that $\Theta(B) = -1$ and let c be the solution of $\Theta(y) = i$ in the interval (a_0, a_1). Then, by Krein's construction again, we have that

$$\begin{aligned} -i &= -\frac{\Phi(c)+1}{\Phi(c)-1}\exp\left(\pi\ln\frac{c-a_0}{B-c}\right) \\ &= -\alpha i, \end{aligned}$$

where α is some positive real number. The position of B varies from c to a_1. Let c_B be the solution of this equation that depends on B. Let $x = c_{a_1}$. Since c_B is an increasing function of B, it follows that $c_B \leqslant c_{a_1} = x$. We know that $x < a_1$ because the RHS blows up at $c = a_1, B = a_1$. □

5.2.1 Application in Mixed Spectra Problem with Incomplete Data

We reflect on the implications of the above results in the theory of differential operators. An immediate consequence of lemma 22 is that since there is no MIF with the configuration as described in the lemma, correspondingly, there cannot be a Schrödinger operator with the same configuration. Thus, it is easy to see that

Lemma 24. *Given any Schrödinger operator L on (a, b), with a fixed boundary condition β at b. If the operator has the following configuration.*

$$\sigma(L, D, \beta) = \{2n\}_n, \qquad \sigma(L, N, \beta) \subset \{2n+1\}_{n \neq 0},$$

then there is no eigenvalue for $(L, e^{i\pi/2}D, \beta)$, in the interval $(1.5, 2)$. Here D and N refer to the Dirichlet and Neumann conditions at a.

Proof. Suppose there is an operator with these properties, then there is a corresponding Weyl inner function Θ, with $\sigma(\Theta) = \{2n\}_n$, $\sigma(-\Theta) \subset \{2n+1\}_{n \neq 0}$ and $\Theta(x) = i$. But this would contradict lemma 22. Thus, the lemma is proved. $\square$

CHAPTER VI

SUMMARY AND FUTURE WORK

6.1 Summary

In this dissertation, we have discussed three problems arising in the study of complex analysis, used in spectral theory.

In the seventh chapter, we discussed an old problem of Louis de Branges that he had first studied in the 1960s: For which separated sequences Λ does there exist a MIF Θ, with $\sigma(\Theta) = \Lambda$ and $|\Theta'|$ uniformly bounded in $\mathbb{R}$. De Brange had claimed this was true for all sequences. But this was not true (lemma 16, [10]). We were able to correct his erroneous lemma and provide a near complete description of sequences for which this holds and also sequences for which it doesn't.

In the eighth chapter, we discussed the recovery of potentials from spectral data. In the first part, we discussed the uniqueness of potential and in the second part, we discussed the existence of potential. In both these studies, we used the Weyl Titchmarsh m and inner functions. Using Makarov and Poltoratski's results in [27], we were able to equate these problems with those about the uniqueness sets in Model spaces as well as kernels of Toeplitz operators. We have characterized exactly when a sequence is defining for an MIF. Moreover, we have also characterized the degree to which a set is not defining for an MIF, and related that with the dimension of the kernel of a related Toeplitz operator. In this chapter, we saw a different problem-that of characterizing the sets which do *not* form spectral data sets of any MIF. This is useful in concluding when a spectral data set does not come from a Schrödinger operator.

6.2 Future Work

6.2.1 De Branges Lemma

We have seen in chapter 7 that for a sequences to have a corresponding MIF with bounded derivative, it sequence itself must be *uniform* in some sense. We also have a counter example in case the sequence is not uniform. But there are some other ways in which the sequences could display non uniformity. Let us describe those here.

1. A way to characterize sparsity would be to have growing clusters that are sparser than any arithmetic progression, with common ratio approaching 1, intertwining with a subsequence having non-comeasurable gaps.

2. Some other cases that we still don't know about is when the gaps are co-measurable, $\Delta_n \asymp \Delta_{n+1}$ and

   (a) The gaps are very small, $\Delta_n \lesssim \frac{\ln |a_n|}{\ln \ln \Delta_n}$.

   (b) The gaps are '*in between*' i.e., $\ln |a_n| \lesssim \Delta_n \lesssim \ln^2 |a_n|$.

   (c) The sequence has clusters and gaps, i.e. $\Delta_n \not\gtrsim (\ln |a_n|)^2$ and $\Delta_n \not\lesssim \ln |a_n|$.

Problem 1: What happens to the derivative of the MIF with spectrum as described above?

6.2.2 Uniqueness of Potential

In the chapter 8, we observed that if $D^*(\Lambda) < 2D^*(\sigma(\Phi))$, then we have a family of MIFs that agree with Φ on Λ. One wonders if each of the MIFs in this family corresponds to a Schrödinger operator. If we can characterize this, then we would have a description of the number of Schrödinger operators which agree on a certain spectral data set.

Problem 2: If there are multiple MIFs that agree on a certain set, and if one of them comes from a Schrödinger operator, do all of them have to come from Schrödinger operators as well?

In general, MIFs have a one-to-one correspondence with canonical systems. Thus, in this wider setting one can ask the more general question

Problem 3: If there are multiple MIFs that agree on a certain set, do all of them have to belong to a certain subclass of canonical systems, eg. Schrödinger operators, Dirac systems.

6.2.3 Existence of Potential

As we saw in the previous chapter, the interpolation of MIFs from a discrete set of $\mathbb{R}$ is a subtle one. We used Krein's formula for two spectra and by altering two spectra slightly, were able to construct sequences that from which we cannot interpolate MIFs. This leads us to ask many more questions.

Problem 4: What if instead of removing finitely many points from two spectra, we remove infinitely many points, and replace it with a point from a third spectrum. Will it still fail to be interpolating for an MIF?

Problem 5: Instead of starting with 2 fixed spectra, suppose we have data coming from multiple spectra, can anything be said in that case?

REFERENCES

[1] Ambarzumian, V., *Über eine Frage der Eigenwerttheorie,* Zeitschrift für Physik 53 (2009).

[2] Baranov, A., *Completeness and Riesz Bases of reproducing kernels in model subspaces*, International Mathematical Research Notices, Volume 2006 (2005), pp. 1-35.

[3] Baranov, A., *De Branges mistake*, Private Communications, 2011.

[4] Beurling, A., Malliavin, P., *On Fourier transforms of measures with compact support,* Acta Math. 107 (1962), pp. 291-302.

[5] Beurling, A., Malliavin, P., *On the closure of characters and the zeros of entire functions,* Acta Math. 118 (1967), pp. 79-93.

[6] Borg, G., *Uniqueness theorems in the spectral theory of* $y'' + (\lambda - q(x))y = 0$, Proc. 11-th Scandinavian Congress of Mathematicians, Johan Grundt Tanums Forlag, Oslo, 1952, pp. 276-287.

[7] Böttcher A., Silbermann B., *Analysis of Toeplitz operators.* Academie-Verlag and Springer-Verlag, Berlin, 1990.

[8] Bourgain, J., *A problem of Douglas and Rudin on factorization,* Pacific J. Math. 121 (1986), pp. 47-50.

[9] Chelkak, D., *An Application of the fixed point theorem to the inverse Sturm-Liouville problem,* Journal of Mathematical Sciences, Vol 66 (1) (2010), pp. 118-126.

[10] De Branges, L., *Hilbert spaces of entire functions,* Prentice - Hall (1968), pp. 281-282.

[11] Del Rio, R., Gesztesy, F., Simon, B., *Inverse Spectral Analysis with partial information on the potential, III. Updating boundary conditions,* International Math. Research Notices, 15 (1997), pp. 751-758.

[12] Del Rio, R., Gesztesy, F., Simon, B., *Corrections and addendums to 'Inverse Spectral Analysis with partial information on the potential, III. Updating boundary conditions',* International Math. Research Notices, 11 (1999), pp. 623-625.

[13] Duren, P., *Theory of H^p spaces* Academic Press, New York (1970), MR 42# 3552.

[14] Everitt, W. N., *On a property of the m-coefficient of a second order linear differential equation,* J. London Math. Soc. 4 (1972), pp. 443-457.

[15] Gesztesy, F., Simon, B., *On the determination of potential from three spectra,* Transactions of the American Mathematical Society, Vol 189(2) (1999), pp. 85-92.

[16] Gesztesy, F., Simon, B., *Inverse Spectral Analysis with partial information on the Potential, II. the case of discrete spectrum,* Transactions of the American Mathematical Society, Vol. 352 (2000), pp. 2765-2787.

[17] Havin, V., Mashreghi, J., *Admissible majorants for model subspaces of H^2, Part I: Slow winding of the generating inner function,* Canad. J. Math. Vol. 55 (6) (2003), pp. 1231-1263.

[18] Hochstadt, H., Lieberman, B., *An inverse Sturm-Liouville problem with mixed given data,* SIAM J. Appl. Math. 34 (1978), pp. 676-680.

[19] Horváth, M., *Inverse Spectral theory and closed exponential systems,* Annals of Mathematics 162 (2005), pp. 885-918.

[20] KOOSIS, P., *An Introduction to H^p spaces,* Cambridge University Press, Cambridge, MA, 1980.

[21] LAX P. D., PHILLIPS R.S., *Scattering theory,* Academic Press, New York (1967).

[22] LEVINSON, N., *The Inverse Sturm Liouville problem,* Mat. Tidsskr. B (1949), pp. 25-30.

[23] LEVITAN, B.M., SARGSJAN, I.S., *Sturm-Liouville and Dirac operators,* Kluwer, Dordrecht, (1991).

[24] MARCHENKO, V., *Some questions in the theory of one-dimensional linear differential operators of the second order, I,* Trudy Mosk. Mat. Obsch. 1 (1952), pp. 327-420.

[25] MARCHENKO, V., *Sturm-Liouville operators and applications,* Birkhauser, Basel, 1986.

[26] MAKAROV, N., POLTORATSKI, A., *Beurling - Malliavin theory for Toeplitz kernels,* Inventiones Mathematicae, Volume 180, Issue 3 (June 2010), pp. 443-480.

[27] MAKAROV, N., POLTORATSKI, A., *Meromorphic inner functions, Toeplitz kernels and the uncertainty principle,* Perspectives in Analysis, Mathematical Physics Studies Volume 27 (2005), pp. 185-252.

[28] MITKOVSKI, M., POLTORATSKI, A., *Pólya sequences, Toeplitz kernels and gap theorems,* Advances in Mathematics, 224 (2010), pp. 1057-1070.

[29] MITKOVSKI, M., RUPAM, R., *Defining sets for MIFs and uniqueness of potentials,* in preparation.

[30] Nikolskii, N. K., *Treatise on the shift operator,* Springer-Verlag, Berlin-New York (1986).

[31] Nikolskii, N.K., *Operators, functions, and systems: an easy reading, Vol. I & II,* AMS, Providence, RI (2002).

[32] Pivovarchik, V.,*An inverse Sturm- Liouville problem by three spectra,* Integr. Equ. Oper. Theory 34 (1999), pp. 234- 243.

[33] Poltoratski, A., Sarason, D., *Aleksandrov Clark measures,* Recent Advances in Operator related Function Theory, Contemp. Math., Vol. 393 (2006), pp. 1-14.

[34] Rupam, R., *Uniform Boundedness of derivatives of Meromorphic Inner functions on the Real line,* to appear in Journal d'Analyse Mathematique.