

CONTRACTUAL PRICING PROBLEMS FOR RETAIL DISTRIBUTION  
UNDER DIFFERENT CHANNEL STRUCTURES

A Dissertation

by

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## ABSTRACT

In many industries, including the retail industry, the profits of a supply chain primarily come from the revenue determined by pricing decisions, while the costs of a supply chain are mainly determined by production and inventory decisions. Lack of coordination between the involved parties concerning pricing and inventory decisions may cost all parties in the supply chain system. Historically, contracts have been viewed and served as effective mechanisms to achieve supply chain coordination. In particular, a coordination contract is such that the total profit of the entities under the contract is equal to the optimal supply chain profit (a.k.a., system profit) under centralized control. Hence, profit potential of each entity is in fact maximized under a coordination contract. Also, a coordination contract is said to achieve the so-called channel coordination objective.

In this context, we consider supplier-buyer (e.g., manufacturer-retailer) systems and take into account a recent trend shifting the leadership in contract design from the supplier to the buyer. In particular, we are interested in powerful entities (e.g., mass retailers or government) leading contractual efforts in various practical settings. We consider two classes of problems related to such powerful entities.

We first study coordination efforts through contracts in single- and multi-product settings from the supplier- and buyer-driven perspectives by considering supplier- and buyer-driven contracts. Previous literature on the leadership shift focuses on the single-product setting while overlooking general buyer-driven contracts under full information. We propose more general buyer-driven contracts and provide a comparison of supplier- and buyer-driven settings in terms of the realized profit and prices while taking into account for not only the supplier's and buyer's but also the

consumers' perspectives. Our results lead to a new buyer-driven contract called the generic contract: a simple, general, effective, and practical coordination contract which is amenable to generalization for handling multi-product, multi-supplier, and multi-buyer settings. Also, the generic contract offers room for negotiation between the buyer and supplier because even when the supplier is the more powerful entity. Last but not least, the generic contract is advantageous not only for the buyer and the supplier but also for the consumers.

We next study a newsvendor problem for a private retailer where government interventions are implemented to induce the retailer to make socially optimal decisions. Very limited literature has studied the social welfare issue for public interest goods with random price-dependent demand, especially in the multiplicative form. We develop a model and methodology for designing government intervention mechanisms that improve/maximize the expected social welfare and analyze the impact of demand uncertainty on coordination performance. We consider two new government regulatory mechanisms, and a new market intervention along with two existing market interventions. Our results demonstrate that government regularity mechanisms are effective in improving the expected social welfare and using any combination of two market interventions achieves the optimal expected social welfare.

## DEDICATION

To my parents, my husband, and my parents-in-law,

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## 1. INTRODUCTION

In many industries, including the retail industry, the profits of a supply chain primarily come from the revenue determined by pricing decisions, while the costs of a supply chain are mainly determined by production and inventory decisions. Lack of coordination between the involved parties concerning pricing and inventory decisions may lead to inefficiencies in terms of costs and profits. Historically, contracts are viewed as facilitators of *“long-term partnerships by delineating mutual concessions that favor the persistence of the business relationship, as well as specifying penalties for non-cooperative behavior”* (Tsay et al. (1999)). Hence, contracts have served as effective mechanisms, when designed and implemented carefully, for achieving supply chain coordination. In this dissertation, we consider *supplier-buyer* (e.g. manufacturer-retailer) systems and investigate coordination efforts through *contracts in supplier- and buyer-driven channels*. In the supplier-driven channel the supplier moves first to specify a contract and then the buyer makes decisions accordingly. Likewise, in the buyer-driven channel the buyer moves first to specify a contract and then the supplier makes decisions accordingly (Liu and Çetinkaya (2009)).

Liu and Çetinkaya (2009) argue that *“In the context of supply contract design, the more powerful party usually has the ability to assume the leadership position. Traditionally, the supplier (e.g., manufacturer) has been more powerful, and, hence, the existing literature in the area emphasizes supplier-driven contracts”*. They also note that *“in some current markets, such as the B2B grocery channel, the power has shifted to the buyer (e.g., retailer)”*. Other powerful buyers include the government and military. With these current trends in mind, we also focus on supply chain contracts that are of interest for powerful entities leading contractual efforts and

aiming for coordination in the context of four closely related problem settings:

Setting 1. The basic bilateral monopolistic contractual setting under price-sensitive demand (shown in Chapter 3),

Setting 2. Multi-product generalization of Setting 1 (shown in Chapter 4),

Setting 3. The exclusive dealer contractual setting under price-sensitive demand (shown in Chapter 5),

Setting 4. The newsvendor problem setting for a private retailer where contractual government interventions are implemented for social welfare maximization (shown in Chapter 6).

The first three settings consider deterministic demand and full information while the last setting takes into account for stochastic demand. Of particular interest is the case where reservation profits are modeled explicitly for the contractual entities involved. The underlying contractual problems are modeled using the principles of leader-follower games which are also known as Stackelberg games (Tirole (1988) and Fudenberg and Tirole (1991)). Stackelberg game is proposed by von Stackelberg (1934). It represents a sequential leader-follower game, in which, one player, the Stackelberg leader, moves first, and then the other player, the Stackelberg follower, moves sequentially after observing the leader's choice (Vardy (2004)). Also, see Chapter 3, Section 1 in Fudenberg and Tirole (1991) for discussion of Stackelberg game. Since reservation profits are modeled explicitly, the resulting models are presented formally as non-linear programming formulations. Based on a careful account of the existing literature,

- Both supplier- and buyer-driven contracts are investigated in the context of Settings 1, 2, and 3, while

- Alternative government interventions, including regularity interventions and market interventions, are investigated in Setting 4.

Of particular interest in the supplier-driven setting is **the wholesale price contract** (e.g., Bresnahan and Reiss (1985), Choi (1991), Lee and Staelin (1997), and Corbett and Tang (1999)). In the buyer-driven setting, a comprehensive account of existing contracts are summarized and a new contract called **the generic contract** is introduced. Significant advantages of the generic contract are established in Settings 1, 2, and 3 through a careful analysis of the underlying game-theoretic non-linear programming formulations. The new interventions targeting coordination for Setting 4 include **price and quantity regulations** along with a **tax cut mechanism**. The goal in all four settings is to establish methods for achieving contractual coordination and realizing the ideal performance as implied by centralized system-wide profit (Settings 1, 2, and 3) or expected social welfare (Setting 4).

We next proceed with an overview of each one of the four settings introduced above. It is worthwhile to note that the complete analysis for these settings is presented in Chapters 3, 4, 5, and 6.

### 1.1 Setting 1. The basic bilateral monopolistic contractual setting

This setting built on the results developed by Liu and Çetinkaya (2009) who compare the supplier- and buyer-driven channels in the single-product setting under price-sensitive demand. Our eventual goal is to extend Setting 1 to consider multiple products and price competition explicitly. Following Liu and Çetinkaya (2009), an accompanying goal is to provide a comparison of supplier- and buyer-driven settings. To this end, we review the existing results on the supplier-driven wholesale price contract, as well as the buyer-driven margin-only and multiplier-only contracts that appear in the previous literature (e.g., Ingene and Parry (2004), Ertek and Griffin

(2002), and Liu and Çetinkaya (2009)). As noted earlier, of particular interest is the case where reservation profits are modeled explicitly. Based on a detailed account of existing literature, we propose the new generic contract and demonstrate that it is a generalization of both the margin-only and multiplier-only contracts. Hence, it increases the so-called *contract flexibility* for the single-product setting analyzed. While the idea of *contract flexibility* has been investigated in the previous literature by Liu and Çetinkaya (2009) in the context of buyer-driven contracts and by Corbett and Tang (1999) and Corbett et al. (2004) in the context of supplier-driven contracts, the focus of the earlier work on *contract flexibility* is addressing information asymmetry. Our goal is to explore fully the case of complete information by offering a more general contract that is also amenable to generalization so that it is effective in multi-product, multi-supplier, and multi-buyer settings. A careful investigation of Setting 1 considering the single-product case is useful to demonstrate these potential benefits of the generic contract and to justify its value.

In a nutshell, in Setting 1, we demonstrate that the generic contract is a simple, general, effective, and practical **coordination** contract which is amenable to generalization. We also demonstrate that it offers room for negotiation between the buyer and supplier because even when the supplier is the more powerful entity. Last but not least, the generic contract is advantageous not only for the buyer and the supplier but also for the consumers.

## 1.2 Setting 2. Multi-product generalization of Setting 1

This setting is a straightforward generalization of Setting 1 to consider multiple symmetric and asymmetric substitutable products, referred as the multi-product setting. While the multi-product problems of interest here have been investigated in the context of supplier-driven channel under wholesale price contract, there is

no previous work considering the buyer-driven channel. Our results document the conditions under which the generic contract remains to be a simple, yet, effective contract when multiple substitutable products are considered.

### 1.3 Setting 3. The exclusive dealer contractual setting

This setting deals with the exclusive dealer channel with two suppliers (e.g., manufacturers) and two buyers (e.g., dealers), where each supplier produces one product and each buyer sells one supplier’s product exclusively. Here, we are interested in the generic contract under the fully asymmetric assumption with an emphasis on exploring generality and practicality of the generic contract relative to the buyer-driven contracts examined in the prior literature.

It is worthwhile to note that while there is previous work (Lee and Staelin (1997), Trivedi (1998), and Zhang et al. (2012)) examining buyer-driven contracts in this setting, existing studies only consider the margin-only contract under symmetric assumptions and ignore reservation profits for all entities. In this dissertation, we study a more general contract than the margin-only contract under the fully asymmetric assumption where reservation profits for suppliers are considered explicitly.

Though there is also previous work (e.g., McGuire and Staelin (1983), Choi (1996), and Wu and Mallik (2010)) examining this setting under the wholesale price contract from supplier-driven perspective, the prior work considers Bertrand competition (Bertrand (1883)) between buyers. That is, “In the Bertrand model, firms simultaneously choose prices and then must produce enough output to meet demand after the price choices become known” (Fudenberg and Tirole (1991)). Another commonly used competition strategy is Cournot competition (Cournot (1838)). That is, “In the Cournot model, firms simultaneously choose the quantities they will produce, which they then sell at the market-clearing price” (Fudenberg and Tirole (1991)).

Considering these two strategies, in the supplier-driven channel, as the channel follower, the buyers are free of competing on quantities or prices after observing the suppliers' wholesale prices. However, either Cournot or Bertrand competition is not involved in the buyer-driven channel. It is because after observing the suppliers' wholesale prices, the buyers' retail prices and quantities are determined by the contract and committed by the buyers due to the nature of buyer-driven contract.

#### 1.4 Setting 4. The newsvendor problem setting under social welfare objective

A large body of literature exists on the price-setting newsvendor problem (Khouja (1999) and Cachon (2003)). The bulk of existing work takes the viewpoint of a seller who aims to maximize the expected profit. When the product at hand is of public interest, e.g., a safety/health related product and an energy efficient appliance, its *“production and consumption imposes an indirect involuntary benefits or costs on other economic agents who are outside the market place for that good”* (Ovchinnikov and Raz (2014)). Hence, social welfare, the total benefits or costs for all entities involved in the society should be considered explicitly. Setting 4 deals with the question how the government should intervene in the seller's decisions on the retail price and the order quantity to maximize the expected social welfare in the context of the newsvendor problem dealing with a public interest good. The problem at hand is based on the analysis presented by Ovchinnikov and Raz (2014) who consider the same problem with the exception that they focus on the case of stochastic **additive demand** while our focus is on the stochastic **multiplicative demand**. Our goal is also the same in the sense that we are interested in alternative intervention mechanisms achieving contractual coordination.

To this end, extending the results presented by Ovchinnikov and Raz (2014), we propose alternative interventions, including regulatory and market interventions, to



align the seller's decisions with the socially optimal ones. We consider two new regulatory interventions, including the maximum price and the minimum quantity, and a new market intervention called the tax cut along with the two market interventions, i.e., the cost subsidy and the consumer rebate, considered by Ovchinnikov and Raz (2014). We demonstrate that simultaneously applying

- Two regulatory interventions together or
- Any combination of two market interventions

allows the government to achieve coordination. Considering the empirical and theoretical importance of multiplicative demand in the welfare analysis, our results extend the knowledge on contractual coordination under the social welfare objective.

The remainder of this dissertation is organized as follows: In Chapter 3, we study the basic bilateral monopolistic setting and focus on the development of the generic contract. In Chapter 4, we focus on the multiple product generalizations for the basic bilateral monopolistic setting, and we present the relation of optimal contracts in the basic and multi-product bilateral monopolistic settings. In Chapter 5, we study the exclusive dealer setting by examining and comparing supplier- and buyer-driven channels. In Chapter 6, we study a newsvendor setting with social welfare objective and propose alternative intervention mechanisms for channel coordination.

## 2. RELATED LITERATURE

Four streams of closely related work are reviewed in this chapter, and they are organized as follows:

- Literature related to Setting 1, i.e., supplier- and buyer-driven contracts in the basic bilateral monopolistic setting under price-sensitive demand for a single product.
- Literature related to Setting 2, i.e., multiple product generalizations considering the bilateral monopolistic setting under price-sensitive demand.
- Literature related to Setting 3, i.e., multiple product generalizations considering the exclusive dealer setting under price-sensitive demand.
- Literature related to Setting 4, i.e., quantitative work related to inventory pricing models as they relate to newsvendor problem under social welfare objective.

### 2.1 Literature related to Setting 1

In the single-product setting, the buyer's price-sensitive demand function is given by  $q = a - bp$  ( $a, b > 0$ ), where  $q$  and  $p$  denote the demand quantity and retail price, respectively. The decisions of interest to the buyer are  $p$  and  $q$ . Clearly,  $q$  dictates the buyer's order quantity which, in turn, is filled by the supplier at wholesale price, denoted by  $w$ . Hence, the decision of interest to the supplier is  $w$ .

This setting has a long history since Cournot (1838). Machlup and Taber (1960) review the early work. They indicate that if the supplier decides  $w$  and the buyer decides  $p$  and  $q$  under **the wholesale price contract**, then  $p$  would exceed the

retail price under vertical integration. Jeuland and Shugan (1983) emphasize channel coordination between the two entities, i.e., supplier and buyer, through various mechanisms, e.g., joint ownership, transfer pricing schemes, and contracts.

Also, this setting has appeared in recent literature (e.g., Corbett and Tang (1999), Ertek and Griffin (2002), Corbett et al. (2004), and Liu and Çetinkaya (2009)). Of particular interest for us are the results presented by Liu and Çetinkaya (2009) who examine the counterpart supplier- and buyer-driven contracts arising in the single-product setting.

Liu and Çetinkaya (2009) build on Corbett and Tang (1999) and Corbett et al. (2004) who assume the supplier-driven channel where the supplier moves first to specify a contract and then the buyer makes decisions accordingly. In particular, Corbett and Tang (1999) and Corbett et al. (2004) consider three general types of supplier-driven contracts: the one-part linear contract, the two-part linear contract, and the two-part nonlinear contract. We note that the two-part nonlinear contract is introduced to handle the case of asymmetric information which is out the scope of this dissertation. Under the supplier-driven one-part linear contract (also, known as **the wholesale price contract**), the supplier specifies  $w$  independent of  $q$ ; under the supplier-driven two-part linear (nonlinear) contract, the supplier specifies both  $w$  and a fixed lump-sum side payment independent (dependent) of  $q$ .

In contrast, Liu and Çetinkaya (2009) develop **the counterpart buyer-driven contracts** corresponding to these three contracts. While related buyer-driven contracts have been studied (e.g., Ertek and Griffin (2002) and Ingene and Parry (2004)), the counterpart buyer-driven contracts are different and nontrivial as we discuss next. For example, consider the counterpart buyer-driven contract corresponding to the wholesale price contract. As noted by Liu and Çetinkaya (2009), when the buyer moves first and announces  $q$  and  $p$ , the supplier would respond with a very high  $w$

which is equal to  $p$ . Then, the buyer would not gain any profit. If the buyer is at the liberty of choosing  $w$  first, however, the buyer would set  $w$  equal to the supplier's product cost. Then, the supplier would not make any profit. Therefore, designing a meaningful counterpart contracting scheme requires a careful thought process. That is, announcing the values of  $w$ ,  $q$ , or  $p$  does not lead to a meaningful counterpart buyer-driven contract.

Liu and Çetinkaya (2009) demonstrate that a meaningful scheme can be constructed by considering the buyer's optimal response for a given  $w$ . It is easy to verify that, for a given  $w$ , the buyer's optimal  $q$  is given by  $q = a - \theta w - \xi$ , where  $\theta = b/2$ ,  $\xi = a/2 + bc/2$  (see (2) on p. 690 of Liu and Çetinkaya (2009)), and  $c$  is the buyer's unit distribution cost. Then, under the counterpart buyer-driven contract, the buyer moves first and announces the relationship  $q = a - \theta w - \xi$  with sensitivity parameters  $\theta$  and  $\xi$  ( $\theta, \xi \geq 0$ ). Next, the supplier announces  $w$ . For any  $w$  announced by the supplier as the follower, the buyer's optimal  $q$  is uniquely determined by  $q = a - \theta w - \xi$ , and, hence, the buyer has no incentive to deviate.

Under this buyer-driven contract, it is optimal for the buyer to set  $\xi = 0$  (see Liu and Çetinkaya (2009), p. 691, Remark 1), i.e.,  $q = a - \theta w$ . Interestingly, Liu and Çetinkaya (2009) also show that this scheme ( $q = a - \theta w$ ) is equivalent to having the buyer decide a non-negative price multiplier  $k = \theta/b \geq 0$  and commit the market pricing mechanism  $p = kw$ . Hence, while Liu and Çetinkaya (2009) call this contract as the buyer-driven one-part linear contract, we refer to it as **the multiplier-only contract**. It is worthwhile to note that Liu and Çetinkaya (2009) extend this contract to generate buyer-driven two-part linear and two-part nonlinear contracts in the spirit of the supplier-driven counterparts examined by Corbett and Tang (1999) as well as Corbett et al. (2004). Also, it is worthwhile to note that the analysis presented by Liu and Çetinkaya (2009) and Corbett et al. (2004) considers

reservation profits explicitly while Corbett and Tang (1999) ignore this practical consideration.

The pricing scheme  $p = kw$  is also considered by Ertek and Griffin (2002) in the context of designing a buyer-driven contract without a specific focus on a comparative analysis of counterpart supplier- and buyer-driven contracts. While Liu and Çetinkaya (2009) address the credibility issue that the buyer cannot deviate from the contractual retail price after obtaining supply, Ertek and Griffin (2002) do not.

Liu and Çetinkaya (2009) demonstrate that leadership benefits the leader in both supplier- and buyer-driven channels and leadership creates more value for the leader under more general contract types (such as the two-part linear contracts) if information is complete. However, with this finding, Liu and Çetinkaya (2009) move on to examining the case of asymmetric information, and, hence, do not explore other potentially more general contracts which is the focus of this dissertation.

Building on Liu and Çetinkaya (2009)'s results summarized above, in contrast to considering  $p = kw$ , we allow a more general pricing scheme  $p = kw + m$  in the single-product setting. We propose a new contract, called the **generic contract**, under which the buyer decides on the values of  $k$ ,  $k \in \Re$ , and  $m$ ,  $m \in \Re$ , while also committing that the retail price would be set such that  $p = kw + m$  and the order quantity would be set such that  $q = a - bp = a - b(kw + m)$ . Next, the supplier decides  $w$ . Here  $m$  can be positive or negative representing a margin (mark-up) or rebate (mark-down) and  $k$  is allowed to be positive or negative for the sake of generality. However, it is shown later that due to the natural and practical assumptions of the problem setting at hand,  $k$  and  $m$  have upper and lower bounds.

Obviously, the pricing scheme  $p = kw$  considered by Ertek and Griffin (2002) and Liu and Çetinkaya (2009) is a special case of the pricing scheme in the generic contract. It has been called the multiplier-only contract ( $k \geq 1$  is required to gain

a nonnegative profit for the buyer). Another special case with  $p = w + m$  is called the **margin-only contract** ( $m \geq 0$  is required to gain a nonnegative profit for the buyer), which has been studied by Ingene and Parry (2004) and Lau et al. (2007) previously in the setting we analyze here. Ingene and Parry (2004) show that the system profit is the same under the wholesale price and margin-only contracts, and Lau et al. (2007) show that the buyer's profit under the margin-only contract is twice of that under the wholesale price contract.

Table 2.1 provides a classification of the three buyer-driven contracts mentioned so far. The relations between  $q$  and  $w$  for the generic and margin-only contracts are derived by considering the relation between  $p$  and  $w$  as well as the relation between  $q$  and  $p$ . An overview of all the contracts of interest for a comparative analysis is given in Table 2.2.

Table 2.1: Buyer-driven contracts of interest in the single-product setting.

	Relation of $q$ and $p$	Relation of $p$ and $w$	Relation of $q$ and $w$	Contract parameters (Buyer's decision)
Multiplier-only	$q = a - bp$	$p = kw$	$q = a - \theta w$ ( $\theta = bk$ )	$k \geq 1$ or $\theta \geq b$
Generic	$q = a - bp$	$p = kw + m$	$q = a - \theta w - \xi$ ( $\theta = bk, \xi = bm$ )	$k, m \in (-\infty, +\infty)$ or $\theta, \xi \in (-\infty, +\infty)$
Margin-only	$q = a - bp$	$p = w + m$	$q = a - \theta w - \xi$ ( $\theta = b, \xi = bm$ )	$m \geq 0$ or $\xi \geq 0, \theta$ is fixed

Note that both the multiplier-only and generic contracts incorporate the price multiplier decision. This decision plays an important role in supply contracting problems mainly from two aspects:

1. Assigning the multiplier provides the decision maker an opportunity to realize

Table 2.2: Summary of entities' decisions under the basic supplier- and buyer-driven contracts in the single-product setting.

Leader	Contract	Supplier's decision	Buyer's decision
Supplier	Wholesale price	Wholesale price $w$	Quantity $q$
	Margin-only	Wholesale price $w$	Margin $m$
Buyer	Multiplier-only	Wholesale price $w$	Multiplier $k$
	Generic	Wholesale price $w$	Multiplier $k$ and Margin $m$

more profit (Irmén (1997)<sup>a</sup>, Tyagi (2005)<sup>b</sup>, Ertek and Griffin (2002), and Liu and Çetinkaya (2009)<sup>c</sup>); and

2. The multiplier represents a practicable profit-driven measure (Liu and Çetinkaya (2009)) commonly used in the retail industry. In the retail industry, the buyer's multiplier  $p/w$  and its variants have been commonly used as practicable profit-driven measures for buyers, according to surveys and articles on industry applications (e.g., Steiner (1973) and Hall et al. (1997)). The variants of the multiplier include *gross profit margin percentage* (GPMP), which is defined as (unit price - unit purchasing cost)/unit price =  $(p - w)/p = 1 - w/p$  (Liu and Çetinkaya (2009)), and the percentage price margin (Tyagi (2005)).

Overall, we consider the basic bilateral monopolistic setting and propose a new

<sup>a</sup>Irmén (1997) investigates the single-product setting under Nash competition (Fudenberg and Tirole (1991)), under which the supplier and buyer move and make decisions simultaneously. The author proves that the retail price is lower and the buyer's profit is higher if both entities compete on the percentage price margins (i.e., the price multiplier minus one) than if they compete on the price margins.

<sup>b</sup>Tyagi (2005) considers a multi-product channel where multiple suppliers sell multiple products through a common buyer. The author considers a buyer-driven contract under which the buyer decides the percentage price margin, i.e., (unit retail price - unit wholesale price)/unit wholesale price =  $(p - w)/w = p/w - 1$ . The contract is obviously equivalent to the multiplier-only contract under which the buyer decides the multiplier, i.e.,  $p/w$ . Tyagi (2005) shows that it is better for the buyer to decide the percentage price margin than to decide the price margin  $m = p - w$ . However, the paper does not show how to derive the optimal contract.

<sup>c</sup>Ertek and Griffin (2002) and Liu and Çetinkaya (2009) demonstrate that the buyer is better off by assigning the price multiplier than the price margin decision in single-product channels.

contract in the buyer-driven channel called the generic contract. The contract has a more general pricing scheme than the two existing buyer-driven contracts in the literature: the margin-only and the multiplier-only contracts. The generic contract reduces to the margin-only contract when  $k = 1$  and it reduces to the multiplier-only contract when  $m = 0$ . We compare the generic contract with other buyer-driven contracts in the literature and provide evidence that the generic contract has better contractual performance than others from several aspects. We demonstrate that the generic contract is not only optimal for the system and the buyer, it also benefits consumers and even the supplier.

## 2.2 Literature related to Setting 2

In the multi-products setting, the supplier's decisions pertain to the wholesale prices  $w_1$  and  $w_2$ , and the buyer's decisions pertain to the order quantities  $q_1$  and  $q_2$  and the retail prices  $p_1$  and  $p_2$ . The order quantities are dictated by the more general demand function that depends linearly on the retail prices following  $q_i = a - \alpha p_i + \beta p_j$  ( $\alpha > \beta \geq 0$ ,  $i, j = 1, 2$ , and  $i \neq j$ ), where  $a$ ,  $\alpha$ , and  $\beta$  are the parameters of the demand function.

This type of demand function has been frequently used in the literature on price competition (e.g., McGuire and Staelin (1983), Choi (1991), Choi (1996), Trivedi (1998), Pan et al. (2010), and Wu et al. (2012)). In the multi-product setting of interest, price competition between the substitutable products results from cross-price effects, where each product's demand depends on both products' retail prices. Hence, the demand function is known as the "symmetric linear demand function with cross-price effects", which is the special case of the generalized "linear demand function with cross-price effects" (e.g., Pashigian (1961), Ingene and Parry (1995), Tyagi (2005), and Yang and Zhou (2006)). The symmetry assumption for products'



demands has been widely adopted in the literature on channel management to keep the problem formulation simple.

In the supplier-driven multi-product setting, Bresnahan and Reiss (1985) and Yang and Zhou (2006) consider the wholesale price contract. Bresnahan and Reiss (1985) make the first attempt to extend the single-product setting by considering one supplier selling multiple substitutable products to one buyer. Although they show a property that the buyer's profit is one-half the supplier's profit if demand is linear, they do not characterize the optimal contract explicitly as we do. Yang and Zhou (2006) consider a similar setting to ours with the exceptions that the supplier's wholesale price is not differentiated by products and they do not consider the buyer's distribution cost. More importantly, both studies only analyze the channel from the supplier's perspective and do not consider any buyer-driven channel.

In this dissertation we take the wholesale price contract as the benchmark supplier-driven contract when we compare supplier- and buyer-driven contracts, because it has been widely applied in the supplier-driven price competition models (e.g., McGuire and Staelin (1983), Ingene and Parry (1995), Saggi and Vettas (2002), Yang and Zhou (2006), and Adida and DeMiguel (2011)), as well as used as a benchmark to evaluate buyer-driven contracts (e.g., Choi (1991), Trivedi (1998), Ertek and Griffin (2002), Tyagi (2005), Pan et al. (2010), and Wu et al. (2012)). Its prevalence is mainly due to the less cost than other contracts, e.g., the revenue-sharing contract (Pan et al. (2010)) and the quantity discount contract (Jeuland and Shugan (1983)), which require more information exchanged between entities.

To the best of our knowledge, the buyer-driven channel has not been analyzed in the multi-product setting of interest. Several existing papers on buyer-driven channels have considered the margin-only contract as summarized in Table 2.3 that provides an overview of the related work on the contract considering multiple prod-

ucts. However, the papers in Table 2.3 consider different multi-product problem settings than ours, i.e., with either multiple suppliers and/or multiple buyers, and their results cannot be directly applied to our setting. Also, although all the papers in Table 2.3 make an attempt to investigate the benefit of leadership, they do not consider the generic and multiplier-only contracts with the exception of Tyagi (2005)<sup>b</sup>.

Table 2.3: Related work on the margin-only contract considering multiple products.

Our work (one-supplier-one-buyer)	One-supplier-multi-buyer
N/A	Pan et al. (2010) Wu et al. (2012)
Multi-supplier-one-buyer	Multi-supplier-multi-buyer
Choi (1991)	Choi (1996)
Lee and Staelin (1997)	Lee and Staelin (1997)
Tyagi (2005)	Trivedi (1998)
Pan et al. (2010)	

Overall, we consider three different scenarios in the multi-product setting: symmetric two-product, symmetric  $n$ -product ( $n \geq 2$ ), and asymmetric two-product scenarios. We focus on analyzing the generic contract in the multi-product setting with three different scenarios while also consider the wholesale price contract by incorporating the buyer's reservation profit. We show that the optimal generic contract is easy to calculate even in the asymmetric two-product setting. Furthermore, we prove that a contractual problem in a symmetric  $n$ -product ( $n \geq 2$ ) setting can be reduced to a single-product setting. Hence, in the multi-product setting of interest, without solving an  $n$ -product contractual problem, one can directly use the results derived in Setting 1 to identify the optimal contract of interest in Setting 2.

### 2.3 Literature related to Setting 3

In practice, the exclusive dealer setting can be seen in many industries. It is particularly applicable in the automobile industry, where an automobile manufacturer usually distributes products through its own dealer (Bresnahan and Reiss (1985)) and the manufacturer-dealer pairs in different brands compete on substitutable vehicles. This channel structure also represents other numerous diverse markets, e.g., sewing machines, agricultural machinery, and gasoline (Ridgway (1969)).

The comparative analysis of supplier- and buyer-driven contracts on the exclusive dealer setting also has practical importance. It is because in this setting two manufacturer-dealer pairs distribute two products exclusively and compete with each other on retail prices and quantities. Each manufacturer-dealer pair forms a vertical strategic alliance that the manufacturer provides a product to the dealer exclusively (Bresnahan and Reiss (1985)). Due to the exclusiveness, selecting the right partner to become a pair is especially important for both entities, and, hence, leadership and contract settings between the two entities are obviously also important to their profitabilities. Hence, the comparison of the supplier- and buyer-driven channels is important due to the simultaneous existence of vertical strategic alliance and horizontal competition.

Next, we proceed with a detailed discussion of the literature on the exclusive dealer setting and on related contracts in the following two streams:

1. Work related to this setting classified by leadership and entities' decisions, and
2. Literature that supports the use of the wholesale price contract with Cournot competition as the benchmark supplier-driven contract.

In the first stream, Table 2.4 lists the most related work to the exclusive dealer setting classified by leadership and entities' decisions under a contract. The main

Table 2.4: Most closely related work in Setting 3 classified by leadership and entities’ decision.

Leadership	Work	Supplier’s decision	Buyer’s decision
Supplier-driven	McGuire and Staelin (1983)	Wholesale price	Retail price
	Lee and Staelin (1997)	Margin	Margin
	Trivedi (1998)	Wholesale price	Margin
	Wu and Mallik (2010)	Wholesale price	Retail price
Buyer-driven	Choi (1996)	Wholesale price	Margin
	Lee and Staelin (1997)	Margin	Margin
	Trivedi (1998)	Wholesale price	Margin
	Zhang et al. (2012)	Wholesale price	Margin

difference between contracts relies in the different decisions. As we can see, all the studies in Table 2.4 assume decisions of interest for entities are related to prices. That is, suppliers decide either the wholesale prices or the manufacture margins (i.e., difference of the wholesale price and the production cost), and buyers decide either the retail prices or the price margins (i.e., difference of the retail price and the wholesale price). While McGuire and Staelin (1983), Lee and Staelin (1997), Trivedi (1998), and Zhang et al. (2012) study the exclusive dealer setting, Choi (1996) and Wu and Mallik (2010) consider two-supplier-two-buyer settings different than ours: Choi (1996) considers the duopoly common retailer channel, where each supplier sells a product to both buyers with cross sales. Wu and Mallik (2010) consider a setting where one retailer is owned by one manufacturer under vertical integration and the other retailer is privately owned.

In fact, McGuire and Staelin (1983) point out that margin decisions can be easily rescaled to price decisions. Therefore, all the buyer-driven contracts in Table 2.4 are equivalent to the margin-only contract in terms of the equilibrium outcomes. Specifically, Lee and Staelin (1997), Trivedi (1998), and Zhang et al. (2012) study the margin-only contract under symmetric assumptions in this setting. All the supplier-driven contracts in Table 2.4 are equivalent to the wholesale price contract with

*Bertrand competition*, under which the suppliers decide the wholesale prices and then the buyers decide the retail prices, recalling the definition of Bertrand competition in Section 1.3.

Regarding the comparative analysis between leaderships, Lee and Staelin (1997) and Trivedi (1998) demonstrate that each entity is better off to possess leadership. Lee and Staelin (1997) show that the retail prices and system profits under different leaderships are the same, i.e., the system efficiency is independent of whether the suppliers or the buyers play as channel leaders. Lee and Staelin (1997) also claim that the suppliers and buyers' profits are symmetric under different leaderships, i.e., the leaders' profits are the same under both leaderships and so as the followers' profits. Focusing on the duopoly common retailer channel, Choi (1996) shows that each entity is better off to possess leadership.

In the second stream, as a classic economic model, Bertrand competition describes a competition structure in which entities decide prices simultaneously, as we mentioned earlier. Another commonly used economic model, Cournot competition, describes interactions between entities that set quantities simultaneously. Recall that the seminal work of Cournot and Bertrand competition goes back to the nineteenth century by Cournot (1838) and Bertrand (1883), respectively.

A comparison of Cournot and Bertrand competition (i.e., Bertrand-Cournot comparison) appears since Singh and Vives (1984) on a one-tier channel. Singh and Vives (1984) demonstrate the standard conclusion in regard to the comparison. The conclusion is that higher prices, lower quantities, and higher profits are obtained in Cournot than Bertrand and mixed Cournot-Bertrand competition for duopolies if products are substitutes with linear demand functions. Using a geometric approach, Cheng (1985) confirms that it is better for duopoly entities to choose a quantity strategy (Cournot competition) than a price strategy (Bertrand competition) if goods are

substitutes with given costs (i.e., wholesale prices). Vives (1985) extends the standard conclusion to oligopolies with arbitrary numbers of entities and more general demand functions in a symmetric setting. The robustness of the standard conclusion has been intensely investigated in the economic literature by considering variations of problem settings, e.g., cost asymmetries, quality difference (Hackner (2000)), mixed duopolies between private and public firms (Matsumura and Ogawa (2012)).

Although all the literature mentioned above focuses on one-tier channels, the standard conclusion can be applied to the two-tier channel in the following way. In a two-tier supplier-buyer channel, after wholesale prices are determined by the supplier(s), the buyers face the same problem as that in a one-tier channel, and, hence, it is always better for the buyers to compete on quantities (i.e., Cournot competition) according to the standard conclusion assuming that the buyers are rational decision makers. Since the buyers are followers who are at the liberty of choosing a competition strategy after observing wholesale prices, Cournot competition would be always implemented. Limited work explicitly examines two-tier channels. Manasakis and Vlassis (2014) consider the exclusive dealer setting with a more general objective function for the suppliers. Consistent with the results in one-tier channels, they show that Cournot competition is the equilibrium strategy between the buyers while Bertrand competition can never be an equilibrium strategy. This result directly supports the use of the wholesale price contract with Cournot competition as the benchmark contract in the supplier-driven channel.

As we can see, Cournot-Bertrand comparison in one-tier channels has been fully examined in prior literature, and the comparison under the wholesale price contract based on downstream entities' profits in two-tier channels can be also derived accordingly. However, how the competition strategy adopted by downstream entities (i.e., buyers) affects upstream entities' profits (i.e., supplier-tier profit) and system

efficiency has not been paid enough attention.

Overall, we currently focus on the buyer-driven channel. While only the margin-only contract under symmetric assumptions has been studied in the exclusive dealer setting, we examine the more general contract (the generic contract) under the fully asymmetric assumption.

## 2.4 Literature related to Setting 4

We consider a price-setting newsvendor facing stochastic multiplicative demand in the social welfare setting. Two streams of literature provide background for our work:

- The empirical work (e.g., Tellis (1988), Mulhern and Lenone (1991), and Hoch et al. (1995)) supporting the wide applicability of multiplicative demand, and
- The quantitative work on inventory pricing models.

The existing quantitative work on inventory pricing models can be roughly classified by the optimization objective and by the demand, as shown in Table 2.5. In the interest of brevity, our emphasis is on previous work that motivates our problem setting (social welfare and multiplicative demand) by omitting details of less relative work, such as profit-maximization models under deterministic demand, on the bottom left hand side of Table 2.5. We refer readers to review papers in this area, including Cachon (1998) and Vives (2001). In the sequel, we will first review both the empirical and quantitative work that supports the application of multiplicative demand, and then review the exiting work focusing on social welfare problems.

Substantial evidence for the importance of multiplicative demand is provided by the empirical work (e.g., Tellis (1988), Mulhern and Lenone (1991), and Hoch et al. (1995)). Specifically, Tellis (1988) reviews 424 models from 42 studies on estimating

the effect of price on market share using actual observations. The author finds that the number of models using the multiplicative demand function is as twice as the number of models using the additive demand function. Especially, for products that are commonly considered for their social welfare issue, such as pharmaceutical and other health-related products, Tellis (1988) points out that consumers pay more attention to their effectiveness than to the price. Thus, the multiplicative demand function is more appropriate, as the price elasticity intends to be consistent for different prices in this case. The reason that Mulhern and Lenone (1991) prefer to multiplicative demand is that, the additive model, referred as the linear model, presents unacceptable price elasticity when prices are discounted. Furthermore, Hoch et al. (1995) suggest that multiplicative demand is more appropriate in representing the effect of price on demand after analyzing data of 18 product categories from 83 supermarkets.

Table 2.5: Quantitative work related to inventory pricing models.

	Deterministic	Stochastic
Social Welfare	Chick et al. (2008) Deo and Corbett (2009) Cho (2010) Arifolu et al. (2012) Adida et al. (2013) Mamani et al. (2012) Levi et al. (2013)	Bell (2001) Taylor and Yadav (2011) Ovchinnikov and Raz (2014)
Profit	Review papers (Cachon (1998) Vives (2001))	Review papers (Yano and Gilbert (2003) Chan et al. (2004) Chen and Simchi-Levi (2012)) Polatoglu and Sahin (2000) Chen and Simchi-Levi (2004a) Chen and Simchi-Levi (2004b) Song et al. (2009) Cohen et al. (2014) Taylor and Xiao (2014)



On the bottom right hand side of Table 2.5, the literature on inventory pricing models using stochastic demand functions is vast. The work in this area has been reviewed by Yano and Gilbert (2003), Chan et al. (2004), and Chen and Simchi-Levi (2012). The models in this stream can be divided into two groups based on how the uncertainty is modeled in the demand function, using additive demand and multiplicative demand. A large number of studies consider multiplicative demand (e.g., Polatoglu and Sahin (2000), Chen and Simchi-Levi (2004a), Chen and Simchi-Levi (2004b), Song et al. (2009), and Taylor and Xiao (2014)). Though these papers are devoted to profit maximization problems, they definitely support the use of multiplicative demand. In addition, the empirical importance of multiplicative demand has also been noticed by other analytical work, such as Cachon and Kok (2007), Driver and Valletti (2003), and Huang and Van Mieghem (2013). Specifically, Cachon and Kok (2007) argue that the multiplicative function, especially in the forms of  $D(p, \xi) = x(\xi)\alpha p^{-\beta}$  and  $D(p, \xi) = x(\xi)\alpha e^{-\beta p}$ , fits actual data better than the additive demand function. In the continuous discussion on which function is more realistic, Driver and Valletti (2003) prefer to the multiplicative demand, as the price elasticity of demand remains constant to any demand realization. This favor is also supported by Huang and Van Mieghem (2013). Cohen et al. (2014) consider both additive and multiplicative demand in a problem where a retailer sells a public interest good and the government applies the rebate mechanism to stimulate the sale to achieve a given target level. They examine how demand uncertainty (additive and multiplicative) impacts optimal decisions of the government, industry, and consumers. Our work is different with Cohen et al. (2014)'s work in that we consider impacts of demand uncertainty on decisions maximizing social welfare, while they consider the impacts on decisions on achieving a given target sales level.

Besides its applicability, the theoretical importance of the multiplicative demand cannot be ignored as well. Several commonly-used demand functions in stochastic models are multiplicative, e.g., the willingness-to-pay model (e.g., Kocabiyikoglu and Popescu (2011)), referred as the reservation-price model (e.g., Van Ryzin (2005)). The reservation-price model is critical in representing demand for a new product or an existing product using demand forecasts (e.g., Kalish (1985)). Argued by Kocabiyikoglu and Popescu (2011), both the exponential model  $d(p) = e^{(z-bp)}$  and the semilogarithmic function popularly used in the marketing literature, are classified as or can be transformed into the multiplicative specification. In addition, managerial insights are usually different respective of demand function, and some of the insights are even contrasting (e.g., Driver and Valletti (2003), and Salinger and Ampudia (2011)). To complement the existing work on additive demand and for the comparative analysis, it is necessary to study multiplicative demand and investigate the impact of demand uncertainty on decisions in the social welfare setting.

As this dissertation concentrates on the operational issues in the social welfare setting, we proceed with a detailed review on existing work in the operation management area, while the fundamental work on the social welfare in economics (e.g., Arrow (1950) and Andersen (1977)) will be not our emphasis. In the social welfare setting of interest, the newsvendor model is considered by Taylor and Yadav (2011) and Ovchinnikov and Raz (2014). Taylor and Yadav (2011) consider both the price-fixed and price-setting newsvendor problems with the additive demand, and Ovchinnikov and Raz (2014) also consider the additive demand. With different objectives, Ovchinnikov and Raz (2014) aim to maximize the expected social welfare, while Taylor and Yadav (2011) are interested in maximizing both the donor's expected profit and the expected social welfare. Bell (2001) incorporates the demand uncertainty in another way by assuming demand depending on consumers' expected surplus. Prior

work on social welfare issues assuming deterministic demand is comparatively rich, as shown in Table 2.6. Most of the work concentrates in the vaccine market with the random production yield and deterministic demand of vaccine, while assuming different problem settings. For example, Cho (2010) considers a multi-period problem, Arifolu et al. (2012) incorporate the consumption externality, and Adida et al. (2013) consider the effect of network and the consumers' purchase preference. Mamani et al. (2012) assume the deterministic demand depending on both price and coverage of the product.

Table 2.6: Most closely related work considering social welfare classified by demand.

Deterministic demand	Chick et al. (2008) (random production yield)
	Deo and Corbett (2009) (random production yield)
	Cho (2010) (random production yield)
	Arifolu et al. (2012) (random production yield)
	Adida et al. (2013) (random production yield)
	Mamani et al. (2012) (demand depending on price and coverage)
Stochastic demand	Levi et al. (2013) (random production yield)
	Bell (2001) (demand depending on consumers' surplus)
	Taylor and Yadav (2011) (price-fixed/setting and additive)
	Ovchinnikov and Raz (2014) (price-setting and additive)
	Our work (price-setting and multiplicative)

As intervention mechanisms play an important role in coordinating the price and quantity decisions for a public interest good, the work related to social welfare can be also classified based on the type of intervention, as shown in Table 2.7. Specifically, Taylor and Yadav (2011), Adida et al. (2013), and Ovchinnikov and Raz (2014) employ the subsidies (cost subsidies and purchase subsidies), the rebates (consumer rebates and sales subsidies) and their combination. Mamani et al. (2012) adopt the taxes, the subsidies, and their combination. Regards to the effectiveness of a single intervention, Ovchinnikov and Raz (2014) observe that the consumer rebate

is better than the cost subsidy in terms of the less social welfare loss when either price or quantity is coordinated. Taylor and Yadav (2011) have a similar result that the sales subsidy is better for both donors and the whole society under specific conditions. Respect to joint interventions, Adida et al. (2013), Mamani et al. (2012) and Ovchinnikov and Raz (2014) all prove combinations of interventions achieving the system coordination and maximizing the expected social welfare. We summarize the most related work to our work in Table 2.7 by emphasizing our differences on intervention mechanisms of interest.

Table 2.7: Most closely related work classified by intervention mechanism.

	Regulation	Market intervention		
		Taxes	Subsidies	Rebates
Ovchinnikov and Raz (2014)			✓	✓
Taylor and Yadav (2011)			✓	✓
Adida et al. (2013)			✓	✓
Mamani et al. (2012)		✓	✓	
Our work	✓	✓	✓	✓

Realizing the empirical and theoretical importance of the multiplicative demand and the significance of social welfare for marketing a public interest good, to the best of our knowledge, this dissertation is the first to combine the two characteristics and to investigate government intervention mechanisms that maximize the expected social welfare. As mentioned earlier, we prove that the multiplicative demand function is feasible in modeling the social welfare. The proof is built on results by Krishnan (2010) and Mas-Colell et al. (1995).

Overall, we revisit Ovchinnikov and Raz (2014) by considering a social welfare setting, in which a public interest good is distributed by a newsvendor-type seller to

consumers with stochastic demand depending on retail price under the multiplicative demand function. We propose two new government regularity intervention and one new market intervention for channel coordination to maximize the expected social welfare. We investigate contractual performance under various interventions in terms of effectiveness, efficiency and the government cost.

### 3. PRELIMINARIES AND THE GENERIC CONTRACT

#### 3.1 Setting 1. The basic bilateral monopolistic contractual setting (a.k.a. single-product setting)

We consider **the basic bilateral monopolistic setting**, i.e., the supplier-buyer channel, with a single product and price-sensitive deterministic demand illustrated in Figure 3.1 and referred as the single-product setting here. The buyer's price-sensitive demand function is given by

$$q = a - bp, \quad (3.1)$$

where  $q$  and  $p$  denote the demand quantity and retail price, respectively. Naturally,  $0 \leq p \leq a/b$ , so that  $q \geq 0$ . Parameter  $a$ ,  $a > 0$ , represents the market potential which is the demand when price approaches zero (Swartz and Iacobucci (2000)). Hence,  $a$  also represents the part of demand that is not affected by price (Adida and Perakis (2010)). Parameter  $b$ ,  $b > 0$ , represents the sensitivity of demand with respect to price (Ingene and Parry (2004) and Adida and Perakis (2010)). Hence,  $b$  measures how the demand is affected by price. The decisions of interest to the buyer are  $q$  and  $p$ . Clearly,  $q$  dictates the buyer's order quantity, which in turn, is filled by the supplier at wholesale price, denoted by  $w$ , so that the decision of interest to the supplier is  $w$ . The notation introduced so far and used frequently in the remainder of this chapter is summarized in Table 3.1.

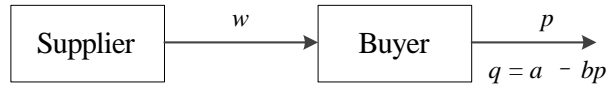


Figure 3.1: The basic bilateral monopolistic contractual setting.

We are interested in studying contractual settings related to the buyer's decisions (i.e.,  $p$  and  $q$ ) as well as the supplier's decision (i.e.,  $w$ ) in this setting. Of particular interest is an explicit comparison of buyer- and supplier-driven contracts as discussed in the next section.

### 3.2 Supplier- and buyer-driven contracts

In *the supplier-driven channel*, the supplier moves first to specify a contract and then the buyer makes decisions accordingly. In contrast, in *the buyer-driven channel*, the buyer moves first to specify a contract and then the supplier makes decisions accordingly (Liu and Çetinkaya (2009)).

For an explicit comparison of buyer- and supplier-driven channels, we consider four specific contracts. Namely, we consider the wholesale price contract, denoted by  $s1$ , in the supplier-driven channel, and the margin-only, multiplier-only, and generic contracts, denoted by  $b1$ ,  $b2$ , and  $b3$ , respectively, in the buyer-driven channel:

$s1$ . Under *the wholesale price contract*, the supplier decides  $w$  and then the buyer decides  $p$ .

$b1$ . Under *the margin-only contract*, the buyer decides the price margin<sup>a</sup>, denoted by  $m$ ,  $m \geq 0$ , representing the difference between the retail and wholesale prices, while also committing that the retail price would be set such that  $p = w + m$  and the order quantity would be set such that  $q = a - bp = a - b(w + m)$ <sup>b</sup>. Next, the supplier decides  $w$ .

$b2$ . Under *the multiplier-only contract*, the buyer decides the price multiplier<sup>c</sup>, de-

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<sup>a</sup>The term price margin is used because  $m \geq 0$  adds a per unit profit to the wholesale price. As shown later, under  $b1$ ,  $m$  satisfies (3.22).

<sup>b</sup>Although we mention that the buyer commits on both relationships for  $p$  and  $q$ , we will show later that it suffices for the buyer to commit only on the latter relationship regarding  $q$  so that there is no credibility issue under this contract.

<sup>c</sup>The term price multiplier is used because  $k \geq 1$  marks up the wholesale price through multi-

noted by  $k$ ,  $k \geq 1$ , representing the ratio of the retail and wholesale prices, while also committing that the retail price would be set such that  $p = kw$  and the order quantity would be set such that  $q = a - bp = a - bkw^b$ . Next, the supplier decides  $w$ .

- b3. Under *the generic contract*, the buyer decides on the values<sup>d</sup> of  $k$ ,  $k \in \mathfrak{R}$ , and  $m$ ,  $m \in \mathfrak{R}$ , while also committing that the retail price would be set such that  $p = kw + m$  and the order quantity would be set such that  $q = a - bp = a - b(kw + m)^b$ . Next, the supplier decides  $w$ .

Contracts  $s1$ ,  $b1$ , and  $b2$  are commonly utilized in practice and analyzed in previous literature<sup>e</sup>. Contract  $b3$  is inspired by  $b1$  and  $b2$  in an attempt to propose a more general pricing scheme and analyzed here for the sake of generality. We are interested in computing the optimal contract parameters under  $s1$ ,  $b1$ ,  $b2$ , and  $b3$ . To this end, we develop basic optimization models, and we utilize the principles of Stackelberg games because the contracting processes are representative of sequential leader-follower games (see Chapter 3 of Fudenberg and Tirole (1991)).

### 3.3 Profit functions

In the single-product setting, using (3.1), the supplier's profit function is given by

$$\pi_s = (w - s)q = (w - s)(a - bp), \quad (3.2)$$

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plication. As shown later, under  $b2$ ,  $k$  also satisfies (3.49).

<sup>d</sup>Observe that, under  $b3$ ,  $m$  can be positive or negative representing a margin (mark-up) or rebate (mark-down). Likewise, under  $b3$ ,  $k$  is allowed to be positive or negative for the sake of generality. However, it is shown later that due to the natural and practical assumptions of the problem setting at hand (e.g., see assumption (3.5)),  $k$  and  $m$  are such that (3.60) and (3.71) hold. Also, the optimal value of  $k$  satisfies  $k \geq 1$ .

<sup>e</sup>For example,  $s1$  has been studied by Corbett et al. (2004) among others;  $b1$  has been studied by Lau et al. (2007) among others; and  $b2$  has been studied by Liu and Çetinkaya (2009) among others. The details of earlier work on these contracts as they apply to our work are given in Sections 3.4 and 3.2.



where  $s$  is the supplier's unit production cost,  $s \geq 0$ . The buyer's profit function is given by

$$\pi_b = (p - w - c)q = (p - w - c)(a - bp), \quad (3.3)$$

where  $c$  is the buyer's unit distribution cost,  $c \geq 0$ . The system profit function is given by

$$\Pi = \pi_s + \pi_b = (p - s - c)q = (p - s - c)(a - bp). \quad (3.4)$$

Recalling (3.1), (3.2), and (3.3), in order to guarantee  $q \geq 0$ ,  $\pi_b \geq 0$ , and  $\pi_s \geq 0$ , we assume  $p \leq a/b$ ,  $w \geq s$ , and  $p \geq w + c$  so that

$$s + c \leq w + c \leq p \leq \frac{a}{b}. \quad (3.5)$$

We pay particular attention to ensure that the contractual problems at hand lead to **nonnegative profits**  $\pi_b$ ,  $\pi_s$ , and  $\Pi$  for the sake of practical realism. We also incorporate the notion of reservation profits, denoted by  $\pi_b^- \geq 0$  for the buyer and  $\pi_s^- \geq 0$  for the supplier, so that not only the profits are **nonnegative** but also they exceed minimum expectations of the entities involved. Then, considering the fact that  $s1$  is a supplier-driven contract, the buyer would not accept  $s1$  unless the buyer's corresponding profit exceeds  $\pi_b^-$ . Likewise, considering the fact that  $b1$  ( $b2$  and  $b3$ ) is a buyer-driven contract, the supplier would not accept  $b1$  ( $b2$  and  $b3$ ) unless the supplier's corresponding profit exceeds  $\pi_s^-$ .

We are primarily interested in the more general case where all external model parameters, i.e.,  $a$ ,  $b$ ,  $s$ ,  $c$ ,  $\pi_b^-$ , and  $\pi_s^-$ , are **positive**. However, when appropriate or necessary, we comment on the cases where  $s = 0$ ,  $\pi_b^- = 0$ , and  $\pi_s^- = 0$  for three specific reasons to include

- These cases have appeared in the literature (e.g., Corbett and Tang (1999))

ignore  $\pi_b^- \geq 0$  under  $s1$  and Lau et al. (2007) assume  $\pi_s^- = 0$  under  $b1$ ),

- They make practical sense (e.g., the case  $s = 0$  is applicable if the supplier is a wholesale distributor, i.e., not a manufacturer. Raju and Zhang (2005) assume  $s = 0$  in a channel with a supplier, a dominant retailer and multiple fringe retailers.), and
- The technical derivations of optimal contract parameters are different (e.g., see the derivations under  $b2$  for  $s = 0$  and  $s > 0$ ).

### 3.3.1 Centralized problem

Under centralized control,  $p$  is decided by the central planner to maximize the system profit  $\Pi$  in (3.4). This is a hypothetical assumption but it is useful to obtain an upper bound on the system profit so that we have a benchmark on the overall performance under the contracts of interest. Hence, using (3.4) and assumption (3.5), the centralized optimization problem can be stated as

$$(Pc) : \quad \max_{s+c \leq p \leq a/b} \Pi = (p - s - c)q = (p - s - c)(a - bp).$$

Clearly,  $w$  is immaterial for  $\Pi$  in (3.4), and, hence, by assumption (3.5), we are only interested in  $p$  values that satisfy

$$s + c \leq p \leq \frac{a}{b}. \quad (3.6)$$

We refer to (3.6) as the main constraint on the decision variable  $p$  of the centralized problem. Now, recalling (3.1) and considering (3.6), we note that

$$0 \leq q \leq a - b(s + c). \quad (3.7)$$

Also, we note that (3.6) as well as (3.7) assure that  $\Pi$  in (3.4) is **nonnegative**. In fact, this is the least<sup>f</sup> the centralized decision maker should target.

Using (3.4), note that

$$\begin{aligned}\frac{d\Pi}{dp} &= a - 2bp + b(s + c) \quad \text{and} \\ \frac{d^2\Pi}{dp^2} &= -2b < 0.\end{aligned}\tag{3.8}$$

It is easy to see that  $\Pi$  is concave in  $p$  and setting  $d\Pi/dp = 0$  in (3.8) leads to

$$p^c = \frac{a + b(s + c)}{2b}.\tag{3.9}$$

Observe that  $p^c$  defined in (3.9) is **the centralized optimal retail price**. This is because by assumption (3.5),

$$\begin{aligned}\frac{a}{b} - p^c &= \frac{a - b(s + c)}{2b} \geq 0 \quad \text{and} \\ p^c - (s + c) &= \frac{a - b(s + c)}{2b} \geq 0,\end{aligned}$$

so that  $p^c$  in (3.9) is realizable over the region (3.6). Using (3.1) and (3.9), **the centralized optimal order quantity** is given by

$$q^c = \frac{a - b(s + c)}{2}.\tag{3.10}$$

Clearly, by assumption (3.5),

$$a - b(s + c) - q^c = \frac{a - b(s + c)}{2} \geq 0,$$

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<sup>f</sup>As we have noted earlier, when we develop optimization models for the contracts of interest, we eventually incorporate the notion of reservation profits so that not only the profits are positive but also they exceed minimum expectations of the entities involved.

so that  $q^c$  defined in (3.10) lies over the region (3.7). Substituting (3.9) in (3.4), the centralized optimal system profit, denoted by  $\Pi^c$ , is given by

$$\Pi^c = \frac{[a - b(s + c)]^2}{4b}. \quad (3.11)$$

It is important to note that  $(Pc)$  is also solved by Lau et al. (2007) leading to  $p^c$  in (3.9) and  $\Pi^c$  in (3.11) (see the expressions in (2) on p. 850 of Lau et al. (2007)). Ingene and Parry (2004) also consider a variant of  $(Pc)$  but allowing a more general cost structure where each entity has a per unit cost as well as a fixed cost. By setting the fixed costs equal zero, their problem (see the problem in (2.3.2) on p. 35 of Ingene and Parry (2004)) is reduced to  $(Pc)$  leading to  $p^c$  in (3.9) and  $\Pi^c$  in (3.11) (see (2.3.4) on p. 35 and (2.3.6) on p. 36 of Ingene and Parry (2004)).

### 3.4 Contract-based optimization problems

#### 3.4.1 Wholesale price contract $s1$

As we have noted in Section 3.2, under  $s1$ , the supplier decides  $w$  first and then the buyer decides  $p$ . By assumption (3.5), under  $s1$ , we are only interested in  $w$  and  $p$  values that satisfy

$$s \leq w \leq \frac{a}{b} - c \quad \text{and} \quad (3.12)$$

$$w + c \leq p \leq \frac{a}{b}. \quad (3.13)$$

Also, recalling (3.1) and considering (3.13), we have

$$0 \leq q \leq a - b(w + c). \quad (3.14)$$

We refer to (3.12) as the main constraint on the decision variable  $w$  of the contract

design problem under  $s1$ . Also, we note that (3.13) as well as (3.14) assure  $\pi_b$  **in** (3.3) **is nonnegative**. In fact, this is the least<sup>f</sup> the supplier should consider in designing the supplier-driven contract  $s1$ .

For a given  $w$  that satisfies (3.12), using (3.3), we have

$$\pi_b = (p - w - c)(a - bp),$$

so that

$$\begin{aligned} \frac{d\pi_b}{dp} &= a - 2bp + b(w + c) \quad \text{and} \\ \frac{d^2\pi_b}{dp^2} &= -2b < 0. \end{aligned} \tag{3.15}$$

Clearly,  $\pi_b$  is concave in  $p$  and setting  $d\pi_b/dp = 0$  in (3.15) leads to

$$p^{s1}(w) = \frac{a + b(w + c)}{2b}. \tag{3.16}$$

Observe that for any  $w$  such that (3.12) is true,  $p^{s1}(w)$  defined in (3.16) is **the buyer's optimal response, i.e., the optimal retail price, under  $s1$** . This is because  $w$  satisfies (3.12) so that

$$\begin{aligned} \frac{a}{b} - p^{s1}(w) &= \frac{a - b(w + c)}{2b} \geq 0 \quad \text{and} \\ p^{s1}(w) - (w + c) &= \frac{a - b(w + c)}{2b} \geq 0. \end{aligned}$$

Hence,  $p^{s1}(w)$  in (3.16) is realizable over the region (3.13). Substituting (3.16) in (3.1), **the buyer's optimal order quantity under  $s1$  for a given  $w$  that satisfies**

(3.12) can be written as

$$q^{s1}(w) = \frac{a - b(w + c)}{2}. \quad (3.17)$$

Again, since  $w$  satisfies (3.12),

$$a - b(w + c) - q^{s1}(w) = \frac{a - b(w + c)}{2} \geq 0,$$

so that  $q^{s1}(w)$  defined in (3.17) lies over the region (3.14).

Also, given the main constraint (3.12) on the decision variable  $w$ , it is easy to verify that assumption (3.5) holds true for  $p = p^{s1}(w)$ , where  $p^{s1}(w)$  is as defined in (3.16). Hence, the buyer's optimal price-quantity response tuple  $(p^{s1}(w), q^{s1}(w))$  does not violate the fundamental assumptions of the problem at hand.

Using (3.16) and (3.17) in (3.2) and (3.3), we have

$$\begin{aligned} \pi_s &= \frac{(w - s)[a - b(w + c)]}{2} \quad \text{and} \\ \pi_b &= \frac{[a - b(w + c)]^2}{4b}. \end{aligned}$$

As noted earlier, considering the fact that  $s1$  is a supplier-driven contract, the buyer would not accept  $s1$  unless the buyer's corresponding profit exceeds  $\pi_b^-$ . Then, recalling (3.12) and considering the two above expressions for  $\pi_s$  and  $\pi_b$ , **the supplier's optimization problem under  $s1$**  can be stated as

$$(Ps1) : \quad \max_{s \leq w \leq a/b - c} \quad \pi_s = \frac{(w - s)[a - b(w + c)]}{2} \quad (3.18)$$

$$s.t. \quad \pi_b = \frac{[a - b(w + c)]^2}{4b} \geq \pi_b^-. \quad (3.19)$$

Clearly,  $(Ps1)$  makes sense only for reasonable values of  $\pi_b^-$ . That is, a natural

upper bound on  $\pi_b^-$  is given by

$$0 \leq \pi_b^- \leq \Pi^c = \frac{[a - b(s + c)]^2}{4b}, \quad (3.20)$$

where  $\Pi^c$  is the optimal centralized system profit in (3.11). Hence, we assume (3.20) holds and, otherwise,  $(Ps1)$  does not have a feasible solution.

Now that we have the complete formulation of  $(Ps1)$ , it is easy to see that the main constraint (3.12) assures that  $\pi_s$  **in** (3.18) **is nonnegative**, i.e., under (3.12), the numerator of (3.18) is nonnegative because  $w - s \geq 0$  and  $a - b(w + c) \geq 0$ .

It is important to note that  $(Ps1)$  is studied by Corbett et al. (2004) (see Case F1 defined in Section 4 on p. 552 of Corbett et al. (2004)). That is, Corbett et al. (2004) consider the formulation of  $(Ps1)$  given by (3.18) and (3.19) and derive  $p^{s1}(w)$  in (3.16) and  $q^{s1}(w)$  in (3.17) (see the expressions in (6) on p. 552 of Corbett et al. (2004)).

#### 3.4.2 Margin-only contract $b1$

As noted in Section 3.2, under  $b1$ , the buyer announces that  $p$  would be set depending on  $w$  according to

$$p = w + m, \quad (3.21)$$

where  $m \geq 0$  is the price margin. Then, the buyer moves first and specifies  $m$  and the supplier selects the optimal  $w$ . By assumption (3.5), under  $b1$ , we are only interested in  $m$ ,  $w$ , and  $p$  values that satisfy

$$c \leq m \leq \frac{a}{b} - s, \quad (3.22)$$

$$s \leq w \leq \frac{a}{b} - m, \quad \text{and} \quad (3.23)$$

$$s + m \leq p \leq \frac{a}{b}. \quad (3.24)$$

Also, recalling (3.1) and considering (3.24), we have

$$0 \leq q \leq a - b(s + m). \quad (3.25)$$

We refer to (3.22) as the main constraint on the decision variable  $m$  of the contract design problem under  $b1$ . Also, we note that (3.23) along with (3.24)–(3.25) assure that  $\pi_s$  **in (3.2) is nonnegative**. In fact, this is the least<sup>f</sup> the buyer should consider in designing the buyer-driven contract  $b1$ .

For a given  $m$  that satisfies (3.22), using (3.21),  $\pi_s$  in (3.2) can be rewritten as

$$\pi_s = (w - s)q = (w - s)(a - bp) = (w - s)[a - b(w + m)]$$

so that

$$\frac{d\pi_s}{dw} = a - 2bw + b(s - m) \quad \text{and} \quad (3.26)$$

$$\frac{d^2\pi_s}{dw^2} = -2b < 0. \quad (3.27)$$

Clearly,  $\pi_s$  is concave in  $w$  and setting  $d\pi_s/dw = 0$  in (3.26) leads to

$$w^{b1}(m) = \frac{a + b(s - m)}{2b}. \quad (3.28)$$

Observe that for any  $m$  such that (3.22) is true,  $w^{b1}(m)$  defined in (3.28) is **the supplier's optimal response, i.e., the optimal wholesale price, under  $b1$** .

This is because  $m$  satisfies (3.22) so that

$$\begin{aligned} \frac{a}{b} - m - w^{b1}(m) &= \frac{a - b(s + m)}{2b} \geq 0 \quad \text{and} \\ w^{b1}(m) - s &= \frac{a - b(s + m)}{2b} \geq 0. \end{aligned}$$



Hence,  $w^{b1}(m)$  in (3.28) is realizable over the region (3.23). Substituting (3.28) in (3.21) and using (3.1), **the corresponding retail price and order quantity for a given  $m$  that satisfies (3.22) are given by**

$$p^{b1}(m) = \frac{a + b(s + m)}{2b} \quad \text{and} \quad (3.29)$$

$$q^{b1}(m) = \frac{a - b(s + m)}{2}, \quad (3.30)$$

respectively. Again, since  $m$  satisfies (3.22),

$$\begin{aligned} \frac{a}{b} - p^{b1}(m) &= \frac{a - b(s + m)}{2b} \geq 0 \quad \text{and} \\ p^{b1}(m) - (s + m) &= \frac{a - b(s + m)}{2b} \geq 0, \end{aligned}$$

so that  $p^{b1}(m)$  defined in (3.29) lies over the region (3.24). Likewise,

$$a - b(s + m) - q^{b1}(m) = \frac{a - b(s + m)}{2} \geq 0,$$

so that  $q^{b1}(m)$  defined in (3.30) lies over the region (3.25).

Also, given the main constraint (3.22) on the decision variable  $m$ , it is easy to verify that assumption (3.5) holds true for  $w = w^{b1}(m)$ , where  $w^{b1}(m)$  is as defined in (3.28). Hence, the supplier's optimal response  $w^{b1}(m)$  does not violate the fundamental assumptions of the problem at hand.

Using (3.28), (3.29), and (3.30) in (3.2) and (3.3), we have

$$\begin{aligned} \pi_s &= \frac{[a - b(s + m)]^2}{4b} \quad \text{and} \\ \pi_b &= \frac{(m - c)[a - b(s + m)]}{2}. \end{aligned}$$

As noted earlier, considering the fact that  $b1$  is a buyer-driven contract, the supplier would not accept  $b1$  unless the supplier's corresponding profit exceeds  $\pi_s^-$ . Hence, we have the constraint  $\pi_s \geq \pi_s^-$ . Then, recalling (3.22) and considering the two above expressions for  $\pi_s$  and  $\pi_b$ , **the buyer's optimization problem under  $b1$**  can be stated as

$$(Pb1) : \quad \max_{c \leq m \leq a/b-s} \quad \pi_b = \frac{(m-c)[a-b(s+m)]}{2} \quad (3.31)$$

$$s.t. \quad \pi_s = \frac{[a-b(s+m)]^2}{4b} \geq \pi_s^-. \quad (3.32)$$

Clearly,  $(Pb1)$  makes sense only for reasonable values of  $\pi_s^-$ . That is, a natural upper bound on  $\pi_s^-$  is given by

$$0 \leq \pi_s^- \leq \Pi^c = \frac{[a-b(s+c)]^2}{4b}, \quad (3.33)$$

where  $\Pi^c$  is the optimal centralized system profit in (3.11). Hence, we assume (3.33) holds and, otherwise,  $(Pb1)$  does not have a feasible solution.

Now that we have the complete formulation of  $(Pb1)$ , it is easy to see that the main constraint (3.22) assures that  $\pi_b$  **in (3.31) is nonnegative**, i.e., under (3.22), the numerator of (3.31) is nonnegative because  $m-c \geq 0$  and  $a-b(s+m) \geq 0$ .

It is important to note that  $(Pb1)$  is studied by Lau et al. (2007) who assume  $\pi_s^- = 0$ . That is, Lau et al. (2007) consider  $\pi_b$  in (3.31) and derive  $w^{b1}(m)$  in (3.28) (see the expression in (5) on p. 852 of Lau et al. (2007)). Ingene and Parry (2004) also consider a variant of  $(Pb1)$  again assuming  $\pi_s^- = 0$  but allowing a more general cost structure, where each entity has a per unit cost as well as a fixed cost. By setting the buyer's fixed cost equal zero, their problem (see the problem in (2.3.29) on p. 39 of Ingene and Parry (2004)) is reduced to  $(Pb1)$  with  $\pi_s^- = 0$ . Hence,  $q^{b1}(m)$  in

(3.30) is also derived by Ingene and Parry (2004) (see the expression in (2.3.28) on p. 39 of Ingene and Parry (2004)).

### 3.4.3 Multiplier-only contract $b2$

As noted in Section 3.2, under  $b2$ , the buyer announces that  $p$  would be set depending on  $w$  according to

$$p = kw, \quad (3.34)$$

where  $k \geq 1$  is the price multiplier. Then, the buyer moves first and specifies  $k$  and the supplier selects the optimal  $w$ . Substituting (3.34) in (3.1), we have

$$q = a - bp = a - bkw \geq 0. \quad (3.35)$$

By assumption (3.5), we have  $w \geq s$ . Hence, using (3.35),

$$a - bsk \geq a - bkw \geq 0.$$

Then, under  $b2$ , we are only interested in  $k$  and  $w$  values that satisfy

$$1 \leq k \leq \frac{a}{bs} \quad \text{and} \quad (3.36)$$

$$s \leq w \leq \frac{a}{bk}. \quad (3.37)$$

Also, using (3.34) in (3.37), we have

$$sk \leq p \leq \frac{a}{b}. \quad (3.38)$$

Then, recalling (3.1) and considering (3.38) leads to

$$0 \leq q \leq a - bsk. \quad (3.39)$$

We refer to (3.36) as the main constraint on the decision variable  $k$  of the contract design problem under  $b2$ . However, unlike in the case of  $b1$ , the main constraint (3.36) along with the accompanying constraints (3.42)–(3.44) do not assure that  $\pi_b$  **in** (3.3) **is nonnegative**<sup>g</sup> under  $b2$ . We address this concern momentarily once we compute the supplier's optimal response.

For a given  $k$  that satisfies (3.36), using (3.34),  $\pi_s$  in (3.2) can be rewritten as

$$\pi_s = (w - s)q = (w - s)(a - bp) = (w - s)(a - bkw),$$

so that

$$\frac{d\pi_s}{dw} = a + bsk - 2bkw \quad \text{and} \quad (3.40)$$

$$\frac{d^2\pi_s}{dw^2} = -2bk < 0. \quad (3.41)$$

Clearly,  $\pi_s$  is concave in  $w$  and setting  $d\pi_s/dw = 0$  in (3.40) leads to

$$w^{b2}(k) = \frac{a + bsk}{2bk}. \quad (3.42)$$

Observe that for any  $k$  such that (3.36) is true,  $w^{b2}(k)$  defined in (3.42) is **the supplier's optimal response, i.e., the optimal wholesale price, under  $b2$ .**

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<sup>g</sup>This is simply because the lower limit of  $k$  in (3.36) is specified as  $k \geq 1$  so that  $p = kw \geq w$ . However, the lower limit  $k \geq 1$  alone does not assure  $p = kw \geq w + c$  which is in fact the condition assuring that  $\pi_b$  **in** (3.3) **is nonnegative**. We momentarily ignore this more strict lower limit on  $k$  but later we establish its equivalent (see (3.46)) and incorporate it in our analysis.

This is because  $k$  satisfies (3.36) so that

$$\begin{aligned}\frac{a}{bs} - w^{b2}(k) &= \frac{a - bsk}{2bk} \geq 0 \quad \text{and} \\ w^{b2}(k) - s &= \frac{a - bsk}{2bk} \geq 0.\end{aligned}$$

Hence,  $w^{b2}(k)$  in (3.42) is realizable over the region (3.37). Substituting (3.42) in (3.34) and using (3.1), **the corresponding retail price and order quantity for a given  $k$  that satisfies (3.36) are given by**

$$p^{b2}(k) = \frac{a + bsk}{2b} \quad \text{and} \quad (3.43)$$

$$q^{b2}(k) = \frac{a - bsk}{2}, \quad (3.44)$$

respectively. Again, since  $k$  satisfies (3.36),

$$\begin{aligned}\frac{a}{b} - p^{b2}(k) &= \frac{a - bsk}{2b} \geq 0 \quad \text{and} \\ p^{b2}(k) - sk &= \frac{a - bsk}{2b} \geq 0,\end{aligned}$$

so that  $p^{b2}(k)$  defined in (3.43) lies over the region (3.38). Likewise,

$$a - bsk - q^{b2}(k) = \frac{a - bsk}{2} \geq 0,$$

so that  $q^{b2}(k)$  defined in (3.44) lies over the region (3.39).

Now, as before, we need to verify that assumption (3.5) holds true for  $w = w^{b2}(k)$ , where  $w^{b2}(k)$  is as defined in (3.42). Unlike in the case of  $b1$ , the main constraint (3.36) on the decision variable  $k$  is not sufficient<sup>h</sup> for this verification under  $b2$ . For

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<sup>h</sup>Also, see footnote <sup>g</sup>.

this reason, rewriting assumption (3.5) while using (3.42), we need to ensure

$$s + c \leq \frac{a + bsk}{2bk} + c \leq p = \frac{a + bsk}{2b} \leq \frac{a}{b} \quad (3.45)$$

holds true. Examining the above inequalities, it is easy to validate that if the lower limit of (3.36) is revised such that

$$k \geq 1 + \frac{2bck}{a + bsk}$$

then (3.45) is ensured. Hence, the main constraint (3.36) needs to be replaced with

$$1 + \frac{2bck}{a + bsk} \leq k \leq \frac{a}{bs}, \quad (3.46)$$

so that assumption (3.5) holds true for  $w = w^{b2}(k)$ .

For obvious reasons, we now refer to (3.46) as the main constraint on the decision variable  $k$  of the contract design problem under  $b2$ . Using the newly established lower limit of  $k$  in (3.46), we note that

$$1 + \frac{2bck}{a + bsk} = k$$

is equivalent to  $f(k) = 0$  where  $f(k)$  is defined as

$$f(k) = bsk^2 + (a - bs - 2bc)k - a. \quad (3.47)$$

Observe that

$$\frac{df(k)}{dk} = 2bsk + a - bs - 2bc \quad \text{and} \quad \frac{d^2f(k)}{dk^2} = 2bs.$$

Hence,  $f(k)$  is convex in  $k$ . Also,  $\lim_{k \rightarrow +\infty} f(k) \rightarrow +\infty$  and  $f(0) = -a < 0$ . Function  $f(k)$  is illustrated in Figures 3.2 and 3.3 when the minimizer  $k_f^*$  of  $f(k)$  is positive and negative, respectively. It is easy to verify that

$$k_f^* = -\frac{a - bs - 2bc}{2bs} \quad (3.48)$$

so that  $k_f^* \geq 0$  if  $a - bs - 2bc \leq 0$  and  $k_f^* < 0$  if  $a - bs - 2bc > 0$ . As illustrated in Figures 3.2 and 3.3, for both cases, there exists a unique positive  $k_l$  such that  $f(k_l) = 0$  so that (3.46) can be rewritten as

$$k_l \leq k \leq \frac{a}{bs}. \quad (3.49)$$

It then follows that, given the final main constraint (3.49) on the decision variable  $k$ , assumption (3.5) holds true for  $w = w^{b2}(k)$ .

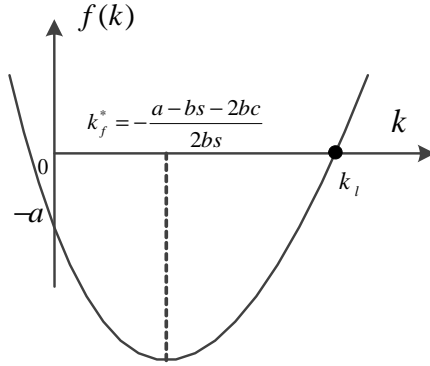


Figure 3.2: An illustration of  $f(k)$  in (3.47) when  $a - bs - 2bc < 0$ .

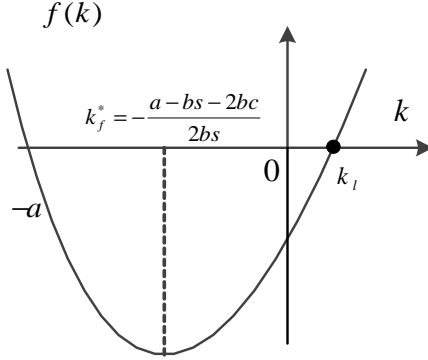


Figure 3.3: An illustration of  $f(k)$  in (3.47) when  $a - bs - 2bc > 0$ .

Using (3.42), (3.43), and (3.44) in (3.2) and (3.3), we have

$$\begin{aligned}\pi_s &= \frac{(a - bsk)^2}{4bk} \quad \text{and} \\ \pi_b &= \frac{(a - bsk)[a(k - 1) + bk(sk - s - 2c)]}{4bk}.\end{aligned}$$

Then, recalling (3.49) and considering the two above expressions for  $\pi_s$  and  $\pi_b$ , **the buyer's optimization problem under b2** can be stated as

$$(Pb2) : \quad \max_{k_l \leq k \leq a/(bs)} \quad \pi_b = \frac{(a - bsk)[a(k - 1) + bk(sk - s - 2c)]}{4bk} \quad (3.50)$$

$$s.t. \quad \pi_s = \frac{(a - bsk)^2}{4bk} \geq \pi_s^-. \quad (3.51)$$

Now that we have the complete formulation of (Pb2), it is easy to see that the main constraint (3.49) assures that  $\pi_b$  **in** (3.50) **is nonnegative**. This is because

- The first term  $a - bsk$  that appears in the numerator of (3.50) is such that  $a - bsk \geq 0$  under (3.49), and
- By definition of  $k_l$  and the properties of  $f(k)$  discussed above, the next term



that appears in the numerator of (3.50) is such that

$$a(k-1) + bk(sk - s - 2c) = bsk^2 + (a - bs - 2bc)k - a \geq 0$$

under (3.49).

It is important to note that (*Pb2*) is formulated and solved by Liu and Çetinkaya (2009) (see (*BP<sub>BF1</sub>*) on p. 692 of Liu and Çetinkaya (2009)). That is, they also derive  $w^{b2}(k)$  in (3.42) (see the expression in (4) on p. 691 of Liu and Çetinkaya (2009)) along with (3.50) and (3.51) (see expressions in (*BP<sub>BF1</sub>*) on p. 692 of Liu and Çetinkaya (2009)).

It is worthwhile to note that the lower limit of  $k$  in the main constraint (3.49) is omitted by Liu and Çetinkaya (2009) as they implicitly assume that  $\pi_b^- = 0$  by arguing that the buyer will not trade if the resulting profit is negative, whereas we provide the region of  $k$  given in (3.49) that guarantees the resulting profit is nonnegative. As we have noted earlier (see footnote <sup>f</sup>), we pay particular attention to ensure that the contractual problems at hand lead to nonnegative profits for both entities for the sake of practical realism.

#### 3.4.4 Generic contract *b3*

As noted in Section 3.2, under *b3*, the buyer announces that  $p$  would be set depending on  $w$  according to

$$p = kw + m, \tag{3.52}$$

where  $k \in \Re$  is the unconstrained multiplier and  $m \in \Re$  is the unconstrained value representing a margin (mark-up) or rebate (mark-down). Then, the buyer moves first and decides  $k$  and  $m$  and the supplier selects the optimal  $w$ .

It is important to note that here  $k \in \Re$  and  $m \in \Re$  are defined as unconstrained

values for the sake of generality because the purpose of this contract is to capture a more general pricing scheme than implied by  $b1$  and  $b2$ . As we demonstrate next,  $k$  and  $m$  are subject to constraints as in the case of the parameters of the other contracts (e.g., see the upper/lower bounds given by (3.22) under  $b1$  and given by (3.49) under  $b2$ .)

Substituting (3.52) in (3.1), we have

$$q = a - bp = a - b(kw + m) \geq 0, \quad (3.53)$$

and, hence,

$$kw + m \leq \frac{a}{b}. \quad (3.54)$$

Also, by assumption (3.5), we have

$$s \leq w \leq \frac{a}{b} - c \quad \text{and} \quad (3.55)$$

$$s + c \leq p \leq \frac{a}{b}. \quad (3.56)$$

Then, recalling (3.1) and considering (3.56),

$$0 \leq q \leq a - b(s + c). \quad (3.57)$$

Given (3.54)–(3.57), for reasons that will become apparent momentarily, let us consider the cases  $k \leq 0$  and  $k > 0$ , separately:

- If  $k \leq 0$  then (3.54) and (3.55) imply that  $k$  and  $m$  should satisfy

$$k \left( \frac{a}{b} - c \right) + m \leq \frac{a}{b}, \quad (3.58)$$

while  $w$ ,  $p$ , and  $q$  should satisfy (3.55), (3.56), and (3.57), respectively.

- If  $k > 0$  then (3.54) and (3.55) imply that

$$sk + m \leq kw + m \leq \frac{a}{b}. \quad (3.59)$$

Using (3.59) along with (3.55) and (3.56), we then conclude that  $k$ ,  $m$ ,  $w$ , and  $p$  should be such that

$$sk + m \leq \frac{a}{b} \quad (3.60)$$

$$s \leq w \leq \frac{a}{bk} - \frac{m}{k}, \quad \text{and} \quad (3.61)$$

$$sk + m \leq p \leq \frac{a}{b}. \quad (3.62)$$

Also, recalling (3.1) and considering (3.62), for the case  $k > 0$ , we have

$$0 \leq q \leq a - b(sk + m). \quad (3.63)$$

We then conclude that the natural **constraints** on the decision variables and parameters of interest are given

- By (3.55), (3.56), (3.57), and (3.58) if  $k < 0$ , and
- By (3.60), (3.61), (3.62), and (3.63) if  $k > 0$ .

Initially, we refer to (3.58) and (3.60) as the main constraints on the decision variables  $k$  and  $m$  of the contract design problem under  $b3$ . Clearly, (3.58) applies to the case  $k \leq 0$  whereas (3.60) applies to the case  $k > 0$ . Similar to the case of  $b2$ , under  $b3$ , (3.58) or (3.60) alone does not assure that  $\pi_b$  in (3.3) is nonnegative.

We proceed with addressing this concern by first computing the supplier's optimal response under  $b3$ .

To this end, let us examine  $\pi_s$  in (3.2) under  $b3$ . For given values of  $k \in \Re$  and  $m \in \Re$ , using (3.52),  $\pi_s$  in (3.2) can be rewritten as

$$\pi_s = (w - s)q = (w - s)(a - bp) = (w - s)[a - b(kw + m)], \quad (3.64)$$

so that

$$\frac{d\pi_s}{dw} = [a - b(kw + m)] - bk(w - s) \quad \text{and} \quad (3.65)$$

$$\frac{d^2\pi_s}{dw^2} = -2bk. \quad (3.66)$$

For reasons that will become apparent shortly, this end, let us again consider the cases  $k \leq 0$  and  $k > 0$ , separately:

*Case 1:  $k \leq 0$*

In this case, considering (3.66) and using (3.53) and (3.55) in (3.65), it can be easily verified that  $\pi_s$  is convex and increasing in  $w$  over the region (3.55). For given values of  $k \leq 0$  and  $m \in \Re$  such that (3.58) holds, the maximizer of  $\pi_s$  in (3.64) is then given by  $a/b - c$  which is the upper limit of (3.55). In turn, using (3.1) and substituting  $w = a/b - c$  in (3.3),

$$\pi_b = (p - w - c)q = \left(p - \frac{a}{b}\right)(a - bp) = -\frac{(a - bp)^2}{b} \leq 0,$$

regardless of the value of  $p$ . Therefore, if  $k \leq 0$  then the buyer does not make any profit. **Hence, we can discard the case  $k \leq 0$  and restrict our attention to the case  $k > 0$ .**

Case 2:  $k > 0$

In this case, it follows from (3.66) that  $\pi_s$  is concave in  $w$ . Setting  $d\pi_s/dw = 0$  in (3.65) leads to

$$w^{b3}(k, m) = \frac{a + b(sk - m)}{2bk}. \quad (3.67)$$

Now, observe that for given values of  $k > 0$  and  $m \in \Re$  such that (3.60) is true, we have

$$\begin{aligned} \left( \frac{a}{bk} - \frac{m}{k} \right) - w^{b3}(k, m) &= \frac{a - b(sk + m)}{2bk} \geq 0 \quad \text{and} \\ w^{b3}(k, m) - s &= \frac{a - b(sk + m)}{2bk} \geq 0. \end{aligned}$$

Then,  $w^{b3}(k, m)$  defined in (3.67) is **the supplier's optimal response, i.e., the optimal wholesale price, under  $b3$**  because it is realizable over the region (3.61). Substituting (3.67) in (3.52) and using (3.1), **the corresponding retail price and order quantity for given values of  $k > 0$  and  $m \in \Re$**  that satisfy (3.60) are given by

$$p^{b3}(k, m) = \frac{a + b(sk + m)}{2b} \quad \text{and} \quad (3.68)$$

$$q^{b3}(k, m) = \frac{a - b(sk + m)}{2}, \quad (3.69)$$

respectively. Again, since we assume (3.60),

$$\begin{aligned} \frac{a}{b} - p^{b3}(k, m) &= \frac{a - b(sk + m)}{2b} \geq 0 \quad \text{and} \\ p^{b3}(k, m) - (sk + m) &= \frac{a - b(sk + m)}{2b} \geq 0, \end{aligned}$$

so that  $p^{b3}(k, m)$  defined in (3.68) lies over the region (3.62). Likewise,

$$a - b(sk + m) - q^{b3}(k, m) = \frac{a - b(sk + m)}{2} \geq 0,$$

so that  $q^{b3}(k, m)$  defined in (3.69) lies over the region (3.63).

Last but not least, we need to verify that assumption (3.5) holds true for  $w = w^{b3}(k, m)$ , where  $w^{b3}(k, m)$  is as defined in (3.67). Similar to the case of  $b2$ , the main constraint (3.60) derived earlier is not sufficient to verify this under  $b3$ . For this reason, recalling assumption (3.5) and using (3.67), we need to ensure

$$s + c \leq \frac{a + b(sk - m)}{2bk} + c \leq p = \frac{a + b(sk - m)}{2b} + m \leq \frac{a}{b} \quad (3.70)$$

holds true. Examining the above inequalities, if

$$\frac{[a + b(sk - m)](k - 1)}{2bk} + m \geq c$$

then (3.70) is ensured. Obviously, the above inequality is equivalent to

$$bsk^2 + [a + b(m - s - 2c)]k - a + bm \geq 0. \quad (3.71)$$

Hence, (3.60) and (3.71) are the main constraints of the problem at hand. Under these two main constraints, assumption (3.5) holds true under  $b3$ , too, as in the cases of  $b1$  and  $b2$ .

Using (3.67), (3.68), and (3.69) in (3.2) and (3.3), we have

$$\begin{aligned} \pi_s &= \frac{[a - b(sk + m)]^2}{4bk} \quad \text{and} \\ \pi_b &= \frac{[a - b(sk + m)][a(k - 1) + b(km + m + sk^2 - sk - 2ck)]}{4bk}. \end{aligned}$$

Finally, recalling (3.60) and (3.71) and considering the above expressions for  $\pi_s$ ,  $\pi_b$  and the constraint  $k > 0$ , **the buyer's optimization problem under b3** can be stated as

$$\begin{aligned}
(Pb3) : \\
\max_{k>0, m \in \mathcal{R}} \pi_b &= \frac{[a - b(sk + m)][a(k - 1) + b(km + m + sk^2 - sk - 2ck)]}{4bk} \quad (3.72) \\
s.t. \quad bsk^2 + [a + b(m - s - 2c)]k - a + bm &\geq 0, \\
sk + m &\leq \frac{a}{b}, \\
\pi_s = \frac{[a - b(sk + m)]^2}{4bk} &\geq \pi_s^-. \quad (3.73)
\end{aligned}$$

Now that we have the complete formulation of (Pb3), it is easy to see that the main constraints (3.60) and (3.71) together also assure that  $\pi_b$  **in** (3.72) **is nonnegative**. That is,

- The first term in square brackets that appears in the numerator of (3.72) is such that  $a - b(sk + m) \geq 0$  under (3.60), and
- The next term in square brackets that appears in the numerator of (3.72) is such that

$$a(k - 1) + b(km + m + sk^2 - sk - 2ck) = bsk^2 + [a + b(m - s - 2c)]k - a + bm \geq 0$$

under (3.71).

Hence,  $\pi_b$  in (3.72) is nonnegative.

A visual investigation of (Pb3) given by (3.72), (3.71), (3.60), and (3.73) reveals that b3 is a generalization of both b1 and b2. Hence, (Pb3) reduces to (Pb1) when  $k = 1$  and to (Pb2) when  $m = 0$ .

### 3.5 Optimal contract parameters

In this section, we present the optimal solutions of  $(Ps1)$ ,  $(Pb1)$ ,  $(Pb2)$ , and  $(Pb3)$  formulated in the previous section. These optimal solutions represent optimal contract parameters under each one of the four contracts of interest.

#### 3.5.1 Optimal solution of $(Ps1)$

Recall  $(Ps1)$  given by (3.18) and (3.19). Using (3.18), observe that

$$\begin{aligned}\frac{d\pi_s}{dw} &= \frac{a - 2bw + b(s - c)}{2} \text{ and} \\ \frac{d^2\pi_s}{dw^2} &= -b < 0.\end{aligned}\tag{3.74}$$

Hence,  $\pi_s$  in (3.18) is concave in  $w$ . Letting  $w^{s1+}$  denote the solution for  $d\pi_s/dw = 0$  in (3.74), we have

$$w^{s1+} = \frac{a + b(s - c)}{2b}.\tag{3.75}$$

Using assumption (3.5) it can be easily verified that

$$\begin{aligned}\frac{a}{b} - c - w^{s1+} &= \frac{a - b(s + c)}{2b} \geq 0 \quad \text{and} \\ w^{s1+} - s &= \frac{a - b(s + c)}{2b} \geq 0.\end{aligned}$$

Hence,  $w^{s1+}$  defined in (3.75) is realizable over the region (3.12) which appears in (3.18). Substituting (3.75) in (3.19), the corresponding buyer's profit is then given by

$$\pi_b^{s1+} = \frac{[a - b(s + c)]^2}{16b} = \frac{\Pi^c}{4},\tag{3.76}$$



where  $\Pi^c$  is the optimal centralized system profit in (3.11). Next, we need to consider  $\pi_b^{s1+}$  given by (3.76) in relation to constraint (3.19) in  $(Ps1)$ .

- If  $\pi_b^{s1+} \geq \pi_b^-$  then (3.19) is satisfied for  $w^{s1+}$  so that **the optimal wholesale price under s1**, denoted by  $w^{s1}$ , is simply given by  $w^{s1+}$  in (3.75).
- Otherwise, i.e.,  $\pi_b^{s1+} < \pi_b^-$ ,  $w^{s1}$  occurs at the boundary of (3.19). That is, it follows from (3.19) that  $w^{s1}$  is dictated by the solution of the polynomial

$$[a - b(w + c)]^2 - 4b\pi_b^- = 0,$$

whose roots are given by

$$w^{s1-} = \frac{a - 2\sqrt{b\pi_b^-}}{b} - c \quad \text{and} \quad \frac{a + 2\sqrt{b\pi_b^-}}{b} - c. \quad (3.77)$$

Now, observe that the latter root is eliminated because it violates (3.12) for  $\pi_b^- > 0$  and it is equal to the former root for  $\pi_b^- = 0$ . Since  $\pi_b^-$  satisfies (3.20), using assumption (3.5) it is easy to verify that

$$\begin{aligned} \frac{a}{b} - c - w^{s1-} &= \frac{2\sqrt{b\pi_b^-}}{b} \geq 0, \quad \text{and} \\ w^{s1-} - s &= \frac{a - b(s + c)}{b} - 2\sqrt{\frac{\pi_b^-}{b}} \\ &\geq \frac{a - b(s + c)}{b} - \sqrt{\frac{[a - b(s + c)]^2}{b^2}} = 0, \end{aligned}$$

so that  $w^{s1-}$  is realizable over the region (3.12) which appears in (3.18). Therefore,  $w^{s1}$  is given by  $w^{s1-}$  in (3.77) and the corresponding buyer's profit is then given by  $\pi_b^-$ .

Using  $\pi_b$  in (3.19), we have

$$\frac{d\pi_b}{dw} = -[a - b(w + c)],$$

and, hence,  $\pi_b$  is decreasing in  $w$  over the region (3.12). That is,

- $\pi_b \geq \pi_b^-$  only for those  $w$  such that  $w \leq w^{s1-}$ , and
- If  $\pi_b^{s1+} < \pi_b^-$  then  $w^{s1-} < w^{s1+}$ .

Consequently, recalling (3.75) and (3.77), we have

$$w^{s1} = \min\{w^{s1-}, w^{s1+}\} = \min\left\{\frac{a - 2\sqrt{b\pi_b^-}}{b} - c, \frac{a + b(s - c)}{2b}\right\}. \quad (3.78)$$

Recalling (3.20), we have  $\pi_b^- \in [0, \Pi^c]$  by assumption. Then, considering (3.76) and (3.78), it is easy to show that the optimal solution of  $(Ps1)$  depends on the value of  $\pi_b^-$ . That is,

- **Case 1:**  $w^{s1} = w^{s1+}$ . If  $\pi_b^- \in [0, \Pi^c/4]$  then  $w^{s1+} \leq w^{s1-}$  so that  $w^{s1} = w^{s1+}$ , and
- **Case 2:**  $w^{s1} = w^{s1-}$ . If  $\pi_b^- \in [\Pi^c/4, \Pi^c]$  then  $w^{s1+} \geq w^{s1-}$  so that  $w^{s1} = w^{s1-}$ .

Note that the above range in terms of  $\pi_b^-$  in Case 1:  $w^{s1} = w^{s1+}$  is equivalent to

$$c \leq \frac{a - 4\sqrt{b\pi_b^-}}{b} - s, \quad (3.79)$$

and the above range in terms of  $\pi_b^-$  in Case 2:  $w^{s1} = w^{s1-}$  is equivalent to

$$\frac{a - 4\sqrt{b\pi_b^-}}{b} - s \leq c \leq \frac{a - 2\sqrt{b\pi_b^-}}{b} - s. \quad (3.80)$$

Cases 1 and 2 discussed above are illustrated in Figures 3.4 and 3.5, respectively. In Case 1, the buyer's reservation profit is relatively small so that the unconstrained maximizer  $w^{s1+}$  of the supplier's profit satisfies the constraint on the buyer's resulting profit. Hence, the optimal wholesale price  $w^{s1}$  is such that the constraint is not binding leading to  $w^{s1} = w^{s1+}$ . However, in Case 2, the buyer's reservation profit is relatively large so that  $w^{s1+}$  violates the constraint. Hence, the optimal wholesale price is dictated by the binding constraint so that  $w^{s1} = w^{s1-}$ .

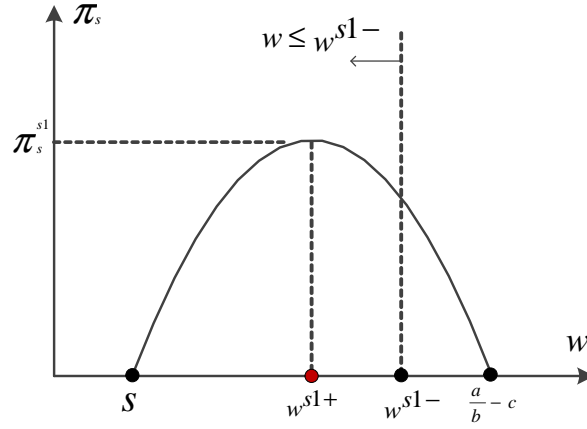


Figure 3.4: An illustration of  $\pi_s^{s1}$  in Case 1:  $w^{s1} = w^{s1+}$ .

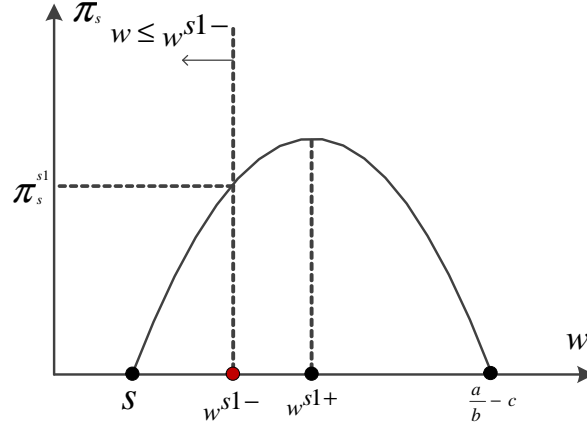


Figure 3.5: An illustration of  $\pi_s^{s1}$  in Case 2:  $w^{s1} = w^{s1-}$ .

Recalling (3.16), (3.17), (3.18), and (3.19) and using (3.78), **the corresponding retail price, order quantity, supplier's, buyer's and system profits under the optimal  $s1$  are given by**

$$\begin{aligned}
 p^{s1} &= \min \left\{ \frac{a - \sqrt{b\pi_b^-}}{b}, \frac{3a + b(s + c)}{4b} \right\}, \\
 q^{s1} &= \max \left\{ \sqrt{b\pi_b^-}, \frac{a - b(s + c)}{4} \right\}, \\
 \pi_s^{s1} &= \begin{cases} \frac{[a - b(s + c) - 2\sqrt{b\pi_b^-}]\sqrt{b\pi_b^-}}{b} & \text{if } \pi_b^- \in [\Pi^c/4, \Pi^c] \\ \frac{[a - b(s + c)]^2}{8b} = \frac{\Pi^c}{2} & \text{if } \pi_b^- \in [0, \Pi^c/4] \end{cases}, \quad (3.81) \\
 \pi_b^{s1} &= \max \left\{ \pi_b^-, \frac{[a - b(s + c)]^2}{16b} \right\}, \quad \text{and} \\
 \Pi^{s1} &= \begin{cases} \frac{[a - b(s + c) - \sqrt{b\pi_b^-}]\sqrt{b\pi_b^-}}{b} & \text{if } \pi_b^- \in [\Pi^c/4, \Pi^c] \\ \frac{3[a - b(s + c)]^2}{16b} & \text{if } \pi_b^- \in [0, \Pi^c/4] \end{cases}.
 \end{aligned}$$

As noted previously,  $(Ps1)$  is solved by Corbett et al. (2004) leading to the optimal wholesale price in (3.78) and the supplier's optimal profit in (3.81) (see the

expressions of Proposition 1 on p. 553 of Corbett et al. (2004)). Corbett et al. (2004) also derive the ranges in terms of  $c$  given by (3.79) and (3.80) (see Proposition 1 on p. 553 of Corbett et al. (2004)).

### 3.5.2 Optimal solution of (Pb1)

Recall (Pb1) given by (3.31) and (3.32). Using (3.31), observe that

$$\begin{aligned}\frac{d\pi_b}{dm} &= \frac{a - 2bm - b(s - c)}{2} \text{ and} \\ \frac{d^2\pi_b}{dm^2} &= -b < 0.\end{aligned}\tag{3.82}$$

Hence,  $\pi_b$  in (3.31) is concave in  $m$ . Letting  $m^{b1+}$  denote the solution for  $d\pi_b/dm = 0$  in (3.82), we have

$$m^{b1+} = \frac{a - b(s - c)}{2b}.\tag{3.83}$$

Using assumption (3.5) it can be easily verified that

$$\begin{aligned}\frac{a}{b} - s - m^{b1+} &= \frac{a - b(s + c)}{2b} \geq 0 \quad \text{and} \\ m^{b1+} - c &= \frac{a - b(s + c)}{2b} \geq 0.\end{aligned}$$

Hence,  $m^{b1+}$  defined in (3.83) is realizable over the region (3.22). Substituting (3.83) in (3.32), the corresponding supplier's profit is given by

$$\pi_s^{b1+} = \frac{[a - b(s + c)]^2}{16b}.\tag{3.84}$$

Next, we need to consider  $\pi_s^{b1+}$  given by (3.84) in relation to constraint (3.32) in (Pb1).

- If  $\pi_s^{b1+} \geq \pi_s^-$  then (3.32) is satisfied for  $m^{b1+}$  so that **the optimal price**

**margin under  $b1$** , denoted by  $m^{b1}$ , is given by  $m^{b1} = m^{b1+}$  defined in (3.83).

- Otherwise, i.e.,  $\pi_s^{b1+} < \pi_s^-$ ,  $m^{b1}$  occurs at the boundary of (3.32). That is, it follows from (3.32) that  $m^{b1}$  is dictated by the solution of the polynomial

$$[a - b(s + m)]^2 - 4b\pi_s^- = 0,$$

whose roots are given by

$$m^{b1-} \equiv \frac{a - 2\sqrt{b\pi_s^-}}{b} - s \quad \text{and} \quad \frac{a + 2\sqrt{b\pi_s^-}}{b} - s. \quad (3.85)$$

Now, observe that the latter root is eliminated because it violates (3.22) for  $\pi_s^- > 0$  and it is equal to the former root for  $\pi_s^- = 0$ . Since  $\pi_s^-$  satisfies (3.33), using assumption (3.5) it is easy to verify that

$$\begin{aligned} \frac{a}{b} - s - m^{b1-} &= \frac{2\sqrt{b\pi_s^-}}{b} \geq 0 \quad \text{and} \\ m^{b1-} - c &= \frac{a - b(s + c)}{b} - 2\sqrt{\frac{\pi_s^-}{b}} \\ &\geq \frac{a - b(s + c)}{b} - \sqrt{\frac{[a - b(s + c)]^2}{b^2}} = 0, \end{aligned}$$

so that  $m^{b1-}$  is realizable over the region (3.22) which appears in (3.31). Therefore,  $m^{b1}$  is given by  $m^{b1-}$  in (3.85) and the corresponding supplier's profit is then given by  $\pi_s^-$ .

Using  $\pi_s$  in (3.32), we have  $d\pi_s/dm = -[a - b(s + m)]$ , and, hence,  $\pi_s$  is decreasing in  $m$  over the region (3.22). That is,

- $\pi_s \geq \pi_s^-$  only for those  $m$  such that  $m \leq m^{b1-}$ , and
- $\pi_s^{b1+} < \pi_s^-$  then  $m^{b1-} < m^{b1+}$ .

Consequently, recalling (3.83) and (3.85), we have

$$m^{b1} = \min\{m^{b1-}, m^{b1+}\} = \min\left\{\frac{a - 2\sqrt{b\pi_s^-}}{b} - s, \frac{a - b(s - c)}{2b}\right\}. \quad (3.86)$$

Recalling (3.33), we have  $\pi_s^- \in [0, \Pi^c]$  by assumption. Then, considering (3.86), it is easy to show that the optimal solution of (Pb1) depends on the value of  $\pi_s^-$ . That is,

- **Case 1:**  $m^{b1} = m^{b1+}$ . If  $\pi_s^- \in [0, \Pi^c/4]$  then  $m^{b1+} \leq m^{b1-}$  so that  $m^{b1} = m^{b1+}$ , and
- **Case 2:**  $m^{b1} = m^{b1-}$ . If  $\pi_s^- \in [\Pi^c/4, \Pi^c]$  then  $m^{b1+} \geq m^{b1-}$  so that  $m^{b1} = m^{b1-}$ .

Cases 1 and 2 discussed above are illustrated in Figures 3.6 and 3.7, respectively. In Case 1, the supplier's reservation profit is relatively small so that the unconstrained maximizer  $m^{b1+}$  of the buyer's profit satisfies the constraint on the supplier's resulting profit. Hence, the optimal price margin  $m^{b1}$  is such that the constraint is not binding leading to  $m^{b1} = m^{b1+}$ . However, in Case 2, the supplier's reservation profit is relatively large so that  $m^{b1+}$  violates the constraint. Hence, the optimal price margin is dictated by the binding constraint so that  $m^{b1} = m^{b1-}$ .

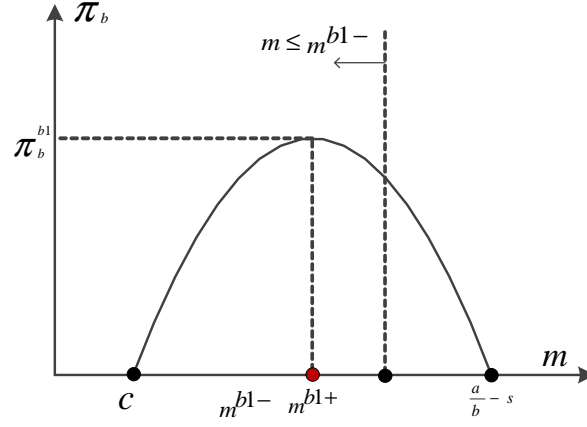


Figure 3.6: An illustration of  $\pi_b^{b1}$  in Case 1:  $m^{b1} = m^{b1+}$ .

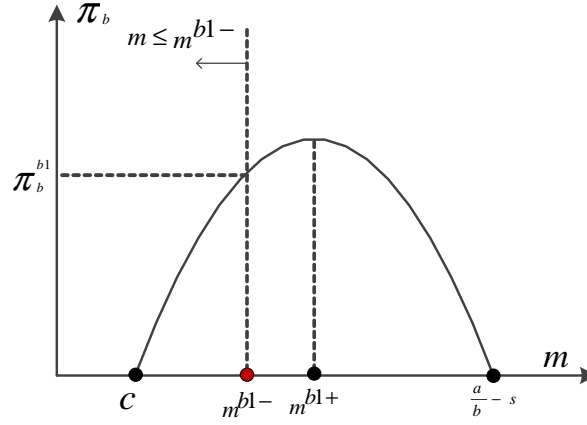


Figure 3.7: An illustration of  $\pi_b^{b1}$  in Case 2:  $m^{b1} = m^{b1-}$ .

Recalling (3.28), (3.29), (3.30), (3.31), and (3.32) and using (3.86), the corresponding wholesale price, retail price, order quantity, supplier's, buyer's, and system



profits under the optimal  $b1$  are given by

$$w^{b1} = \max \left\{ s + \sqrt{\frac{\pi_s^-}{b}}, \frac{a + b(3s - c)}{4b} \right\}, \quad (3.87)$$

$$p^{b1} = \min \left\{ \frac{a - \sqrt{b\pi_s^-}}{b}, \frac{3a + b(s + c)}{4b} \right\}, \quad (3.88)$$

$$q^{b1} = \max \left\{ \sqrt{b\pi_s^-}, \frac{a - b(s + c)}{4} \right\},$$

$$\pi_s^{b1} = \max \left\{ \pi_s^-, \frac{[a - b(s + c)]^2}{16b} \right\}, \quad (3.89)$$

$$\pi_b^{b1} = \begin{cases} \frac{[a - b(s + c) - 2\sqrt{b\pi_s^-}]\sqrt{b\pi_s^-}}{b} & \text{if } \pi_s^- \in [\Pi^c/4, \Pi^c] \\ \frac{[a - b(s + c)]^2}{8b} & \text{if } \pi_s^- \in [0, \Pi^c/4] \end{cases}, \quad \text{and} \quad (3.90)$$

$$\Pi^{b1} = \begin{cases} \frac{[a - b(s + c) - \sqrt{b\pi_s^-}]\sqrt{b\pi_s^-}}{b} & \text{if } \pi_s^- \in [\Pi^c/4, \Pi^c] \\ \frac{3[a - b(s + c)]^2}{16b} & \text{if } \pi_s^- \in [0, \Pi^c/4] \end{cases}. \quad (3.91)$$

As noted previously,  $(Pb1)$  is solved by Lau et al. (2007) by assuming  $\pi_s^- = 0$  leading to the optimal margin in (3.83), retail price in (3.88), and resulting profits in (3.89), (3.90), and (3.91) with  $\pi_s^- = 0$  (see the expressions in Table 1 on p. 851 of Lau et al. (2007)).

### 3.5.3 Optimal solution of $(Pb2)$

Recall  $(Pb2)$  given by (3.50) and (3.51). Using (3.50), observe that

$$\frac{d\pi_b}{dk} = \frac{(a - bsk)^2}{4bk^2} + \frac{s[a - b(sk - c)k]}{2k} \quad \text{and} \quad (3.92)$$

$$\frac{d^2\pi_b}{dk^2} = -\frac{a^2 + k^3s^2b^2}{2bk^3} < 0. \quad (3.93)$$

Hence,  $\pi_b$  in (3.50) is concave over  $k > 0$ . Using (3.92), it is easy to verify that

$$\frac{d\pi_b}{dk} = 0 \quad \Rightarrow \quad g(k) \equiv 2s^2b^2k^3 - s(s+2c)b^2k^2 - a^2 = 0. \quad (3.94)$$

If  $s = 0$  then  $g(k) < 0$  regardless of the value of  $k$ , i.e., there does not exist a solution for  $g(k) = 0$ . For this reason, let us consider the cases  $s = 0$  and  $s > 0$ , separately.

*Case 1:  $s = 0$ .*

Substituting  $s = 0$  in (3.51) and (3.92), we have

$$\pi_s = \frac{a^2}{4bk} \geq \pi_s^- \Rightarrow k \leq \frac{a^2}{4b\pi_s^-} \quad \text{and} \quad (3.95)$$

$$\frac{d\pi_b}{dk} = \frac{a^2}{4bk^2} > 0. \quad (3.96)$$

It then follows that  $\pi_s$  is decreasing while  $\pi_b$  is increasing over  $k > 0$ . Letting  $k^{b2-}$  denote the value of  $k$  such that the reservation profit constraint (3.51) is tight, we have

$$k^{b2-} = \frac{a^2}{4b\pi_s^-}. \quad (3.97)$$

It then follows that if  $k^{b2-}$  is realizable over the region (3.49) (also referred as the main constraint) of (Pb2) then it is also optimal. Using  $s = 0$  in the definition of  $k_l$  (i.e.,  $k_l$  is the root of  $f(k)$  in (3.47)), it is easy to see that (3.49) reduces to

$$k \geq \frac{a}{a - 2bc}. \quad (3.98)$$

Hence, we refer (3.98) as the main constraint of (Pb2) for the case  $s = 0$ . This constraint is sufficient to ensure that the buyer's profit  $\pi_b$  in (3.50) is nonnegative for the case  $s = 0$ .

Using (3.97), it can be easily seen that  $k^{b2-}$  is realizable over the region (3.98) only if

$$\pi_s^- \leq \frac{a^2 - 2abc}{4b}. \quad (3.99)$$

Recalling (3.33), we have  $\pi_s^- \in [0, [a - b(s + c)]^2/(4b)]$  by assumption, so that when  $s = 0$

$$\pi_s^- \in \left[0, \frac{a^2 - 2abc + b^2c^2}{4b}\right].$$

Clearly, the original upper limit of  $\pi_s^-$  in the above equation is higher than the upper limit of  $\pi_s^-$  in (3.99). Hence,  $k^{b2-}$  is realizable over the region (3.98) only if

$$\pi_s^- \in \left[0, \frac{a^2 - 2abc}{4b}\right].$$

As a result, we have the following conclusions.

- **Case 1.1:**  $s = 0$ ,  $k^{b2} = k^{b2-}$ . If  $\pi_s^- \in [0, (a^2 - 2abc)/(4b)]$  then **the buyer's optimal  $k$  under  $b2$** , denoted by  $k^{b2}$ , is given by  $k^{b2} = k^{b2-}$  as defined in (3.97). Furthermore,

- **Case 1.1.a:** If  $\pi_s^- \in (0, (a^2 - 2abc)/(4b)]$  then  $k^{b2}$  is given by (3.97), and

- **Case 1.1.b:** If  $\pi_s^- = 0$  then  $k^{b2}$  is unbounded, i.e.,  $\lim_{\pi_s^- \rightarrow 0} k^{b2} \rightarrow +\infty$ .

Then, we say that  $b2$  does not offer a meaningful solution. Nonetheless, this case (both  $s = 0$  and  $\pi_s^- = 0$ ) does not make practical sense, and, hence, can be safely omitted.

- **Case 1.2:**  $s = 0$ , **infeasible setting**. If  $\pi_s^- \in ((a^2 - 2abc)/(4b), (a^2 - 2abc + b^2c^2)/(4b)]$  then there does not exist a feasible solution for  $(Pb2)$ . While this case makes practical sense ( $\pi_s^-$  is relatively large, say due to fixed costs),  $b2$  does not offer a feasible solution. We revisit this result shortly (see **Case**

**2.3** below) and discuss how the buyer may proceed when  $b_2$  does not offer a practical solution.

Cases 1.1.a and 1.2 discussed above are illustrated in Figures 3.8 and 3.9, respectively. In Case 1.1, the supplier's reservation profit is relatively small so that the optimal price multiplier  $k^{b_2}$  is such that the constraint on the supplier's resulting profit is tight leading to  $k^{b_2} = k^{b_2-}$ . However, in Case 1.2, the supplier's reservation profit is relatively large so that  $k^{b_2-}$  violates the constraint (3.98) that ensures the buyer's resulting profit is nonnegative. Hence, there does not exist a feasible solution for  $(Pb_2)$ .

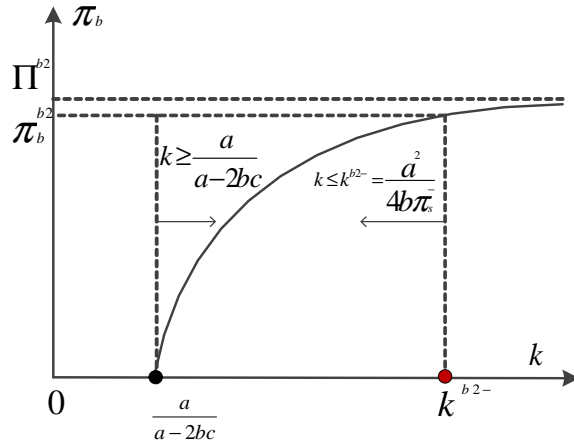


Figure 3.8: An illustration of  $\pi_b^{b_2}$  in Case 1.1.a:  $s = 0$ ,  $k^{b_2} = k^{b_2-}$ .

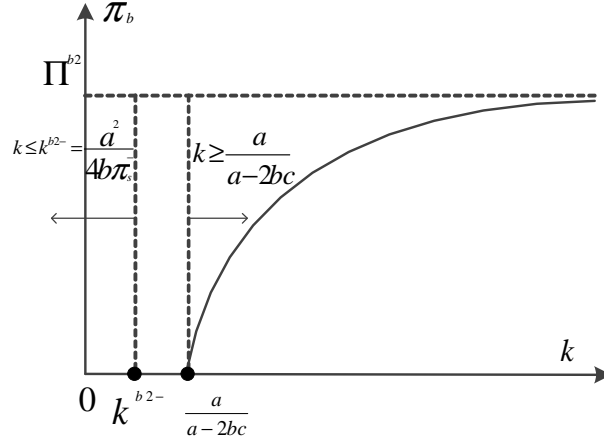


Figure 3.9: An illustration of  $\pi_b$  in Case 1.2:  $s = 0$ , infeasible setting.

Recalling (3.42), (3.43), (3.44), (3.50), and (3.95) and using (3.97), **the corresponding wholesale price, retail price, order quantity, supplier's, buyer's, and system profits under the optimal  $b2$  for  $s = 0$  and  $\pi_s^- \in (0, (a^2 - 2abc)/(4b)]$**  are given by

$$\begin{aligned}
 w^{b2} &= \frac{2\pi_s^-}{a}, \\
 p^{b2} &= \frac{a}{2b}, \\
 q^{b2} &= \frac{a}{2}, \\
 \pi_s^{b2} &= \pi_s^-, \\
 \pi_b^{b2} &= \frac{a^2}{4b} - \frac{ac}{2} - \pi_s^-, \quad \text{and} \\
 \Pi^{b2} &= \frac{a^2}{4b} - \frac{ac}{2}.
 \end{aligned}$$

For the case  $s = 0$ , if  $\pi_s^- \notin (0, (a^2 - 2abc)/(4b)]$  then either the setting (i.e. as in Case 1.1.b) or the contract (i.e., as in Case 1.2) is impractical.

Case 2:  $s > 0$ .

Recalling (3.94), it is easy to verify that

$$g(0) < 0 \quad \text{and} \quad \lim_{k \rightarrow +\infty} g(k) = +\infty,$$

as well as

$$\frac{dg(k)}{dk} = 6s^2b^2k^2 - 2s(s+2c)b^2k. \quad (3.100)$$

It then follows that there are two stationary points  $k_1^0$  and  $k_2^0$  of  $g(k)$  such that  $k_1^0 = 0$  and  $0 < k_2^0 = (s+2c)/(3s) < +\infty$ . Function  $g(k)$  as defined in (3.94) is illustrated in Figure 3.10, where  $k^0$  denotes the reflection point such that

$$\frac{d^2g(k)}{dk^2} = 12s^2b^2k - 2s(s+2c)b^2 = 0 \quad (3.101)$$

for  $k = k^0$ . As we can see from Figure 3.94,  $g(k)$  has a unique positive root, denoted by  $k^{b2+}$ , such that

$$g(k^{b2+}) = 0. \quad (3.102)$$

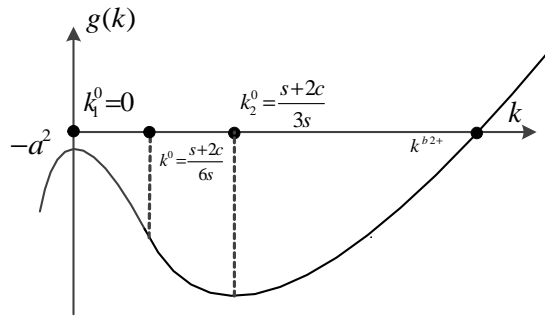


Figure 3.10: An illustration of  $g(k)$  in (3.94).

Let us verify if  $k^{b2+}$  is realizable over the region (3.49) which appears in (3.50). We know from the development of (*Pb2*) that this verification is equivalent to ensuring that  $\pi_b$  in (3.50) is nonnegative for  $k = k^{b2+}$ , i.e., it suffices to verify that

$$\pi_b \big|_{k=k^{b2+}} \geq 0.$$

To this end,

- Recall that  $\pi_b$  is concave (by (3.93)),
- Note that

$$\left. \frac{d\pi_b}{dk} \right|_{k=\frac{a}{bs}} = -\frac{s[a - b(s + c)]}{2} \leq 0 \quad (\text{by (3.92) and assumption (3.5)}), \text{ and}$$

- Note that

$$\pi_b \big|_{k=\frac{a}{bs}} = 0 \quad (\text{by (3.50)}).$$

Hence,  $\pi_b$  is not only decreasing but also it reaches zero at  $k = a/(bs)$ . It then follows from the concavity of  $\pi_b$  and the definition of  $k^{b2+i}$  that  $\pi_b \geq 0$  for  $k^{b2+} \leq k \leq a/(bs)$ . Hence,  $k^{b2+}$  is realizable over the region (3.49).

Substituting  $k = k^{b2+}$  in (3.51), the corresponding supplier's profit is given by

$$\pi_s^{b2+} = \frac{(a - bs k^{b2+})^2}{4b k^{b2+}}. \quad (3.103)$$

Next, we need to consider  $\pi_s^{b2+}$  given by (3.103) in relation to constraint (3.51) in (*Pb2*).

- If  $\pi_s^{b2+} \geq \pi_s^-$  then (3.51) is satisfied for  $k^{b2+}$  so that **the optimal price**

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i.e.,  $k^{b2+}$  is the unique positive stationary point of  $\pi_b$  in (3.50).

**multiplier under  $b_2$** , denoted by  $k^{b_2}$ , is given by  $k^{b_2} = k^{b_2+}$ , which is the unique positive solution for  $g(k^{b_2+}) = 0$  as defined in (3.102).

- Now, consider the case  $\pi_s^{b_2+} < \pi_s^-$  so that  $k^{b_2+}$  violates (3.51). Hence, let us examine the polynomial implied by (3.51)

$$(a - bsk)^2 - 4bk\pi_s^- = 0, \quad (3.104)$$

whose roots are given by

$$k^{b_2-} \equiv \frac{as + 2\pi_s^- - \sqrt{(as + 2\pi_s^-)^2 - a^2s^2}}{bs^2} \quad \text{and} \quad (3.105)$$

$$\frac{as + 2\pi_s^- + \sqrt{(as + 2\pi_s^-)^2 - a^2s^2}}{bs^2}.$$

Now, observe that the latter root is eliminated because it violates the upper limit of  $k$  in (3.49) for  $\pi_s^- > 0$ , and it is equal to the former root for  $\pi_s^- = 0$ .

Next, we examine the conditions under which  $k^{b_2-}$  defined in (3.105) is realizable over the region (3.49) which appears in (3.50). Using (3.105),

$$k^{b_2-} \leq \frac{a}{bs},$$

so that  $k^{b_2-}$  satisfies the upper limit of (3.49). Then, we simply proceed with examining the conditions under which the lower limit of (3.49) is satisfied.

Using (3.105), we have

$$\frac{dk^{b_2-}}{d\pi_s^-} = \frac{2 \left[ \sqrt{(as + 2\pi_s^-)^2 - a^2s^2} - (as + 2\pi_s^-) \right]}{\sqrt{(as + 2\pi_s^-)^2 - a^2s^2}} < 0, \quad (3.106)$$

i.e.,  $k^{b_2-}$  in (3.105) is decreasing in  $\pi_s^-$ . It then follows that there exists a



threshold value of  $\pi_s^-$  such that if  $\pi_s^-$  is greater than the threshold value then  $k^{b2-} \leq k_l$ . Letting  $\pi_s^{k_l}$  denote this threshold value and using (3.104) along with (3.33),

$$\pi_s^{k_l} = \frac{(a - bsk_l)^2}{4bk_l} \leq \Pi^c, \quad (3.107)$$

so that

- If  $\pi_s^- \in [0, \pi_s^{k_l}]$  then  $k^{b2-} \geq k_l$  and
- If  $\pi_s^- \in (\pi_s^{k_l}, \Pi^c]$  then  $k^{b2-} < k_l$ .

Hence,  $k^{b2-}$  lies over the region (3.49) if  $\pi_s^- \in [0, \pi_s^{k_l}]$ . Otherwise, if  $\pi_s^- \in (\pi_s^{k_l}, \Pi^c]$ , there does not exist a feasible solution for (PB2).

Using  $\pi_s$  in (3.51), we have

$$\frac{d\pi_s}{dk} = \frac{-2bs(a - bsk)k - (a - bsk)^2}{4bk^2} = -\frac{(a - bsk)(a + bsk)}{4bk^2}. \quad (3.108)$$

Hence,  $\pi_s$  is decreasing in  $k$  over the region (3.49). That is,

- $\pi_s \geq \pi_s^-$  only for those  $k$  such that  $k \leq k^{b2-}$ , and
- If  $\pi_s^{b2+} < \pi_s^-$  then  $k^{b2-} < k^{b2+}$ .

Consequently, when  $\pi_s^- \in [0, \pi_s^{k_l}]$ , where  $\pi_s^{k_l}$  is defined in (3.107), recalling  $k^{b2+}$  defined in (3.102) and  $k^{b2-}$  defined in (3.105),

$$k^{b2} = \min\{k^{b2-}, k^{b2+}\} = \min\left\{\frac{as + 2\pi_s^- - \sqrt{(as + 2\pi_s^-)^2 - a^2s^2}}{bs^2}, k^{b2+}\right\}. \quad (3.109)$$

Recalling (3.42), (3.43), (3.44), (3.50), and (3.51), **the corresponding whole-sale price, retail price, order quantity, supplier's, buyer's, and system**

profits under the optimal  $b2$  for  $s > 0$  and  $\pi_s^- \in [0, \pi_s^{k_l}]$  are given by

$$\begin{aligned}
w^{b2} &= \frac{a + bsk^{b2}}{2bk^{b2}} = \max \left\{ s + \frac{\sqrt{(as + 2\pi_s^-)^2 - a^2s^2} - 2\pi_s^-}{2(as + 2\pi_s^- - \sqrt{(as + 2\pi_s^-)^2 - a^2s^2})}, \frac{a + bsk^{b2+}}{2bk^{b2+}} \right\}, \\
p^{b2} &= \frac{a + bsk^{b2}}{2b} = \min \left\{ \frac{2as + 2\pi_s^- - \sqrt{(as + 2\pi_s^-)^2 - a^2s^2}}{2bs}, \frac{a + bsk^{b2+}}{2b} \right\}, \quad (3.110) \\
q^{b2} &= \frac{a - bsk^{b2}}{2} = \max \left\{ \frac{\sqrt{(as + 2\pi_s^-)^2 - a^2s^2} - 2\pi_s^-}{2s}, \frac{a - bsk^{b2+}}{2} \right\}, \\
\pi_s^{b2} &= \frac{(a - bsk^{b2})^2}{4bk^{b2}} = \max \left\{ \pi_s^-, \frac{(a - bsk^{b2+})^2}{4bk^{b2+}} \right\}, \\
\pi_b^{b2} &= \frac{(a - bsk^{b2})[a(k^{b2} - 1) + bk^{b2}(sk^{b2} - s - 2c)]}{4bk^{b2}}, \quad \text{and} \\
\Pi^{b2} &= \frac{(a - bsk^{b2})\{a + b[sk^{b2} - 2(s + c)]\}}{4b}.
\end{aligned}$$

Also, recalling (3.33), we have  $\pi_s^- \in [0, \Pi^c]$  by assumption. Then, considering (3.103), (3.109) and the conditions on  $\pi_s^-$  under which  $\pi_b^{b2-}$  defined in (3.105) is realizable, we conclude that the optimal solution of  $(Pb2)$  depends on the value of  $\pi_s^-$ :

- **Case 2.1:**  $s > 0$ ,  $k^{b2} = k^{b2+}$ . If  $\pi_s^- \in [0, \pi_s^{b2+}]$  then  $k^{b2+} \leq k^{b2-}$  so that  $k^{b2} = k^{b2+}$ ,
- **Case 2.2:**  $s > 0$ ,  $k^{b2} = k^{b2-}$ . If  $\pi_s^- \in [\pi_s^{b2+}, \pi_s^{k_l}]$  then  $k^{b2+} \geq k^{b2-}$  so that  $k^{b2} = k^{b2-}$ , and
- **Case 2.3:**  $s > 0$ , **infeasible setting.** If  $\pi_s^- \in (\pi_s^{k_l}, \Pi^c]$  then there does not exist a feasible solution for  $(Pb2)$ . As for **Case 1.2** above, while this case also makes practical sense (perhaps, due to high profit expectation),  $b2$  does not offer a feasible solution. Note that this issue does not arise under  $b1$  which offers a meaningful solution for the buyer for all  $\pi_s^-$  such that  $\pi_s^- \in [0, \Pi^c]$ .

Hence, the buyer can turn to  $b1$  when  $\pi_s^-$  is relatively large. However, there is a clear need for the buyer to pursue a more general contract than  $b1$  and  $b2$ . As we will show in the next section, the more general form of the generic contract  $b3$  does not only provide a practical solution but also it allows the buyer to obtain more profit for  $\pi_s^- \in [0, \Pi^c]$ .

The three cases discussed above are illustrated in Figures 3.11, 3.12, and 3.13 respectively. In Case 1, the supplier's reservation profit is relatively small so that the unconstrained maximizer  $k^{b2+}$  of the buyer's profit satisfies the constraint on the supplier's resulting profit. Hence, the optimal price multiplier  $k^{b2}$  is such that the constraint is not binding leading to  $k^{b2} = k^{b2+}$ . In Case 2, the buyer's reservation profit is relatively large so that  $k^{b2+}$  violates the constraint. Hence, the optimal price multiplier is dictated by the binding constraint so that  $k^{b2} = k^{b2-}$ . However, in Case, the supplier's reservation profit is very high, unlike  $b1$  under which  $(Pb1)$  always has a feasible solution for  $\pi_s^- \in [0, \Pi^c]$ , under  $b2$ ,  $(Pb2)$  does not have a feasible solution for  $\pi_s^- \in (\pi_s^{k_l}, \Pi^c]$ .

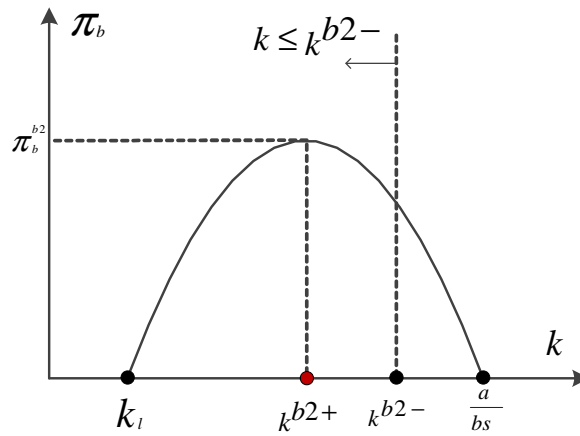


Figure 3.11: An illustration of  $\pi_b^{b2}$  in Case 2.1:  $s > 0$ ,  $k^{b2} = k^{b2+}$ .

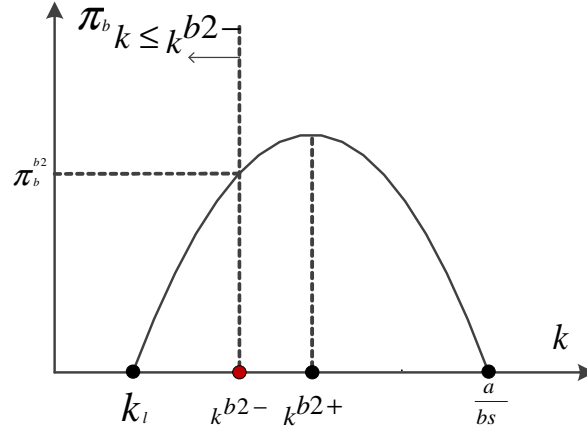


Figure 3.12: An illustration of  $\pi_b^{b2}$  in Case 2.2:  $s > 0$ ,  $k^{b2} = k^{b2-}$ .

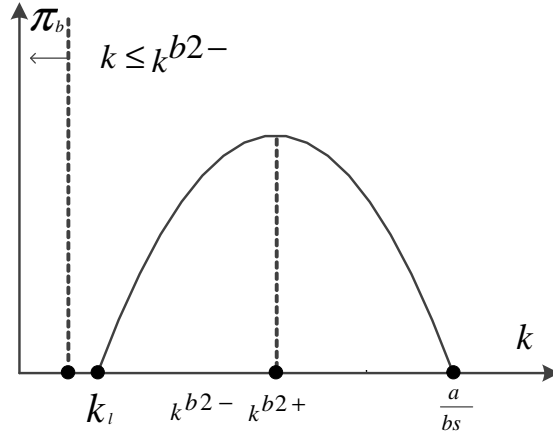


Figure 3.13: An illustration of  $\pi_b$  in Case 2.3:  $s > 0$ , infeasible setting.

As noted previously, (Pb2) is solved by Liu and Çetinkaya (2009) who assume  $s > 0$ . Hence, Liu and Çetinkaya (2009) also derive an equation that is used to solve  $k^{b2}$  for  $s > 0$  (see (6) in Proposition 2 on p. 692 of Liu and Çetinkaya (2009)). They implicitly assume that the buyer will not trade if the resulting profit  $\pi_b$  is

negative, while we provide the region of  $\pi_s^-$  that guarantees  $\pi_b$  is nonnegative under the optimal  $b2$ .

#### 3.5.4 Optimal solution of (Pb3)

We believe that there is a clear need to examine the optimal  $b3$  for three reasons:

- Contract  $b3$ , inspired by  $b1$  and  $b2$ , proposes a more general pricing scheme.
- Contract  $b2$  does not offer a practical solution for the buyer when the supplier's reservation profit is relatively high. We propose  $b3$  in order to overcome this issue while improving on the profit potential for the buyer (i.e., as another alternative contract for the buyer).
- As we show momentarily,  $b3$  is a **coordination contract** (see Cachon (2003) and Tsay et al. (1999) for a formal discussion of coordination contracts). This is because the total profit of the two entities is equal to the optimal centralized system profit  $\Pi^c$  so that profit potential of each party is in fact maximized under  $b3$ . A **coordination contract** is said to achieve the so-called **channel coordination** objective (a phrase coined in the marketing literature as indicated by Tsay et al. (1999)).

Recall (Pb3) given by (3.72) and (3.73). We present two approaches for deriving the optimal solution of (Pb3). In the first approach, we examine the optimal solution for **the unconstrained problem associated with (Pb3)** and then we incorporate the main constraints (3.60) and (3.71) as well as the supplier's reservation profit constraint in (3.73). In the second approach, we establish an upper bound on the objective function given by (3.72) and develop a feasible solution such that the objective function value of this solution achieves the upper bound. Hence, the feasible solution at hand is also optimal.

*Approach 1:*

Using (3.72), for any given value of  $k > 0$ , observe that

$$\begin{aligned}\frac{\partial \pi_b}{\partial m} &= \frac{a - b(km + m + k^2s - kc)}{2k} \quad \text{and} \\ \frac{\partial^2 \pi_b}{\partial m^2} &= -\frac{b(k+1)}{2k} < 0,\end{aligned}\tag{3.111}$$

i.e.,  $\pi_b$  in (3.72) is concave in  $m$  for any  $k > 0$ . Letting  $m(k)$  denote the solution for  $d\pi_b/dm = 0$  in (3.111), we have

$$m(k) = \frac{a + b(kc - k^2s)}{b(k+1)}.\tag{3.112}$$

Substituting (3.112) in (3.71) and (3.60) and using assumption (3.5), it can be easily verified that

$$\begin{aligned}\frac{[a + b(sk - m(k))](k-1)}{2bk} + m(k) &= \frac{a + bc - bs}{2b} \geq c \quad \text{and} \\ sk + m(k) &= sk + \frac{a + b(kc - k^2s)}{b(k+1)} = \frac{a + bk(s+c)}{b(k+1)} \leq \frac{a}{b}.\end{aligned}$$

Hence, for any given  $k > 0$ ,  $k$  and the corresponding  $m(k)$  in (3.112) satisfy the main constraints (3.60) and (3.71) which appear in (Pb3).

Substituting (3.112) in (3.72), the buyer's profit is given by

$$\pi_b = \frac{k[a - b(s+c)]^2}{4b(k+1)} = \frac{[a - b(s+c)]^2}{4b} - \frac{[a - b(s+c)]^2}{4b(k+1)},\tag{3.113}$$

which is increasing over  $k > 0$ . Hence, the maximizer of  $\pi_b$  in (3.113) is unbounded

and  $\pi_b$  reaches its maximum at  $k \rightarrow +\infty$ , i.e.,

$$\lim_{k \rightarrow +\infty} \pi_b = \frac{[a - b(s + c)]^2}{4b}.$$

Substituting (3.112) in (3.73), the corresponding supplier's profit is given by

$$\frac{k[a - b(s + c)]^2}{4b(k + 1)^2},$$

which clearly goes to zero as  $k \rightarrow +\infty$ .

Hence, considering this result in relation to constraint (3.73) in *(Pb3)*, it is clear that (3.73) is violated unless  $\pi_s^- = 0$ . That is, obviously, **the unconstrained solution associated with** *(Pb3)* cannot be the optimal solution for the more general and practical case of interest here, i.e., the case  $\pi_s^- > 0$ .

Now, we consider the case  $\pi_s^- > 0$ . Note that (3.73) is violated so that **the optimal contract parameters under** *b3*, denoted by  $k^{b3}$  and  $m^{b3}$ , occur at the boundary of (3.73). Hence, let us examine the polynomial implied by (3.73)

$$\frac{[a - b(sk + m)]^2}{4bk} - \pi_s^- = 0,$$

which leads to

$$\begin{aligned} sk + m &= \frac{a - 2\sqrt{bk\pi_s^-}}{b} \quad \text{and} \\ sk + m &= \frac{a + 2\sqrt{bk\pi_s^-}}{b}. \end{aligned} \tag{3.114}$$

Now, observe that the latter case is eliminated because it violates the main constraint (3.60) of *(Pb3)*.

Substituting (3.114) in (3.72), the buyer's profit is given by

$$\pi_b = \frac{\sqrt{bk\pi_s^-} [a - \sqrt{bk\pi_s^-} - b(s+c)]}{b} - \pi_s^-. \quad (3.115)$$

Then, we solve the optimal  $k$  that maximizes  $\pi_b$  in the above equation. Letting  $x = \sqrt{bk\pi_s^-}$ ,  $\pi_b$  in the above equation can be rewritten as a function of  $x$  such that

$$\pi_b = \frac{x[a - x - b(s+c)]}{b} - \pi_s^-. \quad (3.116)$$

It is equivalent for the buyer to solving the optimal  $x$  in (3.116) and solving the optimal  $k$  in (3.115). Using (3.116), observe that

$$\begin{aligned} \frac{d\pi_b}{dx} &= \frac{a - 2x - b(s+c)}{b} \text{ and} \\ \frac{d^2\pi_b}{dx^2} &= -\frac{2}{b} < 0. \end{aligned}$$

Hence,  $\pi_b$  in (3.116) is concave and letting  $d\pi_b/dx = 0$  leads to  $x = [a - b(s+c)]/2$ .

Substituting  $x = [a - b(s+c)]/2$  in  $\sqrt{bk\pi_s^-} = x$  and using (3.11) and (3.114), we have

$$k^{b3} = \frac{[a - b(s+c)]^2}{4b\pi_s^-} = \frac{\Pi^c}{\pi_s^-} \text{ and} \quad (3.117)$$

$$m^{b3} = s + c - s \frac{[a - b(s+c)]^2}{4b\pi_s^-} = c - s \left( \frac{\Pi^c}{\pi_s^-} - 1 \right). \quad (3.118)$$

Let us verify if  $k^{b3}$  and  $m^{b3}$  defined in (3.117) and (3.118) satisfy the main constraints (3.60) and (3.71) in ( $Pb3$ ). We know from the development of ( $Pb3$ ) that this verification is equivalent to ensuring that  $\pi_b$  in (3.72) is nonnegative for  $k = k^{b3}$



and  $m = m^{b3}$ , i.e., it suffices to verify that

$$\pi_b \big|_{k=k^{b3}, m=m^{b3}} \geq 0.$$

Using (3.72), we have

$$\pi_b \big|_{k=k^{b3}, m=m^{b3}} = \Pi^c - \pi_s^- \geq 0$$

for  $\pi_s^- \in (0, \Pi^c]$  in (3.33) by assumption. Hence,  $k^{b3}$  and  $m^{b3}$  defined in (3.117) and (3.118) satisfy the main constraints (3.60) and (3.71) in  $(Pb3)$  for  $\pi_s^- \in (0, \Pi^c]$ .

Using (3.72) and (3.73), **the corresponding supplier's, buyer's and system profits under the optimal  $b3$  for  $\pi_s^- \in (0, \Pi^c]$**  are given by

$$\begin{aligned} \pi_b^{b3} &= \Pi^c - \pi_s^-, \\ \pi_s^{b3} &= \pi_s^-, \quad \text{and} \\ \Pi^{b3} &= \Pi^c. \end{aligned} \tag{3.119}$$

The case  $\pi_s^- > 0$  discussed above is illustrated in Figure 3.14. Since the buyer's profit is increasing in  $k$ , the optimal  $k$  for  $(Pb3)$  is such that the constraint on the supplier's resulting profit is tight.

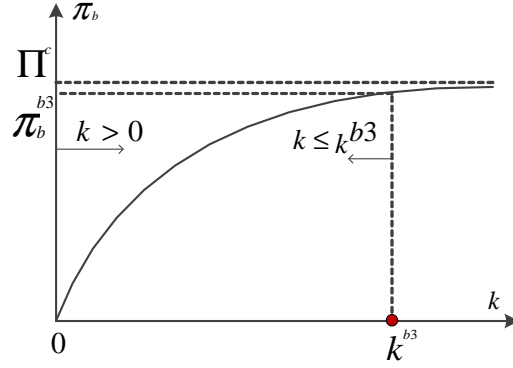


Figure 3.14: An illustration of  $\pi_b$  under  $b3$ .

In summary, the optimal contract parameters for  $b3$  are given by (3.117) and (3.118) leading to a **coordination contract** as indicated by (3.119). It is easy to see from (3.117) and (3.118) that the case  $\pi_s^- = 0$  leads to an unbounded solution as we have also demonstrated through an examination of **the unconstrained solution associated with** ( $Pb3$ ). As noted earlier, unlike the previous literature, we pay particular attention to ensure all resulting profits are positive. Hence, in the remainder of the discussion, we focus on the case  $\pi_s^- > 0$  and say  $b3$  is defined only for this case. Also, unlike  $b2$  under which there does not exist a feasible solution when  $\pi_s^-$  is large (see Cases 1.2 and 2.3 in Section 3.5.3),  $b3$  always provides a meaningful solution for  $\pi_s^- \in (0, \Pi^c]$ .

*Approach 2:*

Using (3.72) and (3.73), the system profit given in (3.4) can be rewritten as

$$\Pi = \pi_s + \pi_b = \frac{[a - b(sk + m)]\{a + b[sk + m - 2(s + c)]\}}{4b} \quad (3.120)$$

for  $k > 0$  and  $m \in \Re$ . Obviously,  $\Pi \leq \Pi^c$  always holds true, where  $\Pi^c$  is the optimal centralized system profit given in (3.11). Therefore, using (3.120), we observe that

$$\pi_b = \Pi - \pi_s = \frac{[a - b(sk + m)]\{a + b[sk + m - 2(s + c)]\}}{4b} - \pi_s \leq \Pi^c - \pi_s^-, \quad (3.121)$$

so that the best profit the buyer can achieve under  $b3$  is given by  $\Pi^c - \pi_s^-$ .

Next, we will provide a feasible solution of  $k$  and  $m$  such that  $\pi_b = \Pi^c - \pi_s^-$  is true. Then, the feasible solution is also optimal under  $b3$ . Letting  $sk + m = x$ , the system profit given in (3.120) can be written as

$$\Pi = \frac{(a - bx)\{a + b[x - 2(s + c)]\}}{4b}, \quad (3.122)$$

so that

$$\begin{aligned} \frac{d\Pi}{dx} &= \frac{b(s + c - x)}{2} \text{ and} \\ \frac{d^2\Pi}{dx^2} &= -\frac{b}{2}. \end{aligned}$$

Hence,  $\Pi$  is concave in  $x$ . The maximizer of  $\Pi$  is given by  $x = s + c$  by solving  $d\Pi/dx = 0$ . It is easy to verify that  $\Pi = \Pi^c$  at  $x = s + c$ . Therefore, the equality in (3.121) holds true if  $sk + m = s + c$  and  $\pi_s = \pi_s^-$ .

Next, we will show a feasible solution for  $k$  and  $m$  that satisfy these two equations above. Substituting  $sk + m = s + c$  in (3.73) and using (3.11), we have

$$\pi_s = \frac{[a - b(sk + m)]^2}{4bk} = \frac{[a - b(s + c)]^2}{4bk} = \frac{\Pi^c}{k}. \quad (3.123)$$

It can be easily verified that  $\pi_s = \pi_s^-$  holds true if  $k = k^{b3}$ , where

$$k^{b3} = \frac{\Pi^c}{\pi_s^-}, \quad (3.124)$$

and  $sk + m = s + c$  holds true if  $m = m^{b3}$ , where

$$m^{b3} = s + c - sk^{b3} = c - s \left( \frac{\Pi^c}{\pi_s^-} - 1 \right). \quad (3.125)$$

Let us verify if  $k^{b3}$  and  $m^{b3}$  satisfy the main constraints (3.60) and (3.71) of (Pb3). We know that the development of (Pb3) that this verification is equivalent to ensuring that  $\pi_b$  in (3.72) is nonnegative for  $k = k^{b3}$  and  $m = m^{b3}$ . Recalling  $\pi_s = \pi_s^-$  and  $sk + m = s + c$  for  $k = k^{b3}$  and  $m = m^{b3}$ , we have

$$\pi_b = \Pi^c - \pi_s^- \geq 0. \quad (3.126)$$

It is because  $\pi_s^- \in [0, \Pi^c]$  by assumption given by (3.33). Hence,  $k^{b3}$  and  $m^{b3}$  satisfy the main constraints (3.60) and (3.71) of (Pb3). Since  $k^{b3}$  and  $m^{b3}$  are feasible, they are also optimal as noted earlier.

Therefore, **the optimal contract parameters under b3** are given by  $k^{b3}$  and  $m^{b3}$  defined in (3.124) and (3.125), respectively. Using (3.67), (3.68), (3.69), (3.72), and (3.73), **the corresponding wholesale price, retail price, order quantity,**

supplier's, buyer's and system profits under the optimal  $b3$  are given by

$$w^{b3} = \frac{[a - b(s + c)]\pi_s^-}{2b\Pi^c} + s \quad (3.127)$$

$$p^{b3} = \frac{a + b(s + c)}{2b}, \quad (3.128)$$

$$q^{b3} = \frac{a - b(s + c)}{2},$$

$$\pi_s^{b3} = \pi_s^-, \quad (3.129)$$

$$\pi_b^{b3} = \Pi^c - \pi_s^-, \quad \text{and} \quad (3.130)$$

$$\Pi^{b3} = \Pi^c. \quad (3.131)$$

Last but not least, using (3.131), we reiterate that  $b3$  is a **coordination contract**. Hence,  $b3$  is optimal from the system perspective.

### 3.6 Discussion and insights regarding $b3$

Now that we have developed *the optimal contracts*<sup>j</sup> under  $s1$ ,  $b1$ ,  $b2$ , and  $b3$ , we provide a general discussion of these contracts while first emphasizing the advantages of  $b3$ .

#### 3.6.1 Advantages of $b3$ relative to $b1$ , $b2$ , $b4$ , and $b5$

First, we note that while  $b3$  has never been studied previously, two other coordination contracts have been investigated in the context of the buyer-driven channel of interest here. These include the following two contracts:

- b4. Under *the buyer-driven two-part linear contract*, the buyer decides the value of  $k$ ,  $k > 0$ , and the value of a lump-sum side payment, denoted by  $L$ ,  $L \in \mathbb{R}$ , while also committing that the retail price would be set such that  $p = kw$  and the order quantity would be set such that  $q = a - bp = a - bkw$ . Next, the

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<sup>j</sup>A summary of all formal results regarding these contracts is shown in Tables 3.2 and 3.3.

supplier decides  $w$ . Note that if  $L > 0$  then a payment is transferred from the supplier to the buyer, and if  $L < 0$  then a payment is transferred from the buyer to the supplier. This contract has been investigated by Liu and Çetinkaya (2009) who demonstrate that  $b4$  is a **coordination contract** for the buyer-driven channel analyzed here.

- b5. Under *the buyer-driven revenue-sharing contract*, the buyer proposes  $\phi$ ,  $\phi \in [0, 1]$ , and commits to share a portion  $(1 - \phi)$  of the selling revenue with the supplier. Next, in return the supplier commits to set a wholesale price lower than the supplier's unit cost, i.e.,  $w = \phi s$ . This contract was studied by Pan et al. (2010). In fact, it was inspired by Cachon and Lariviere (2005) who illustrate some conditions under which it is a **coordination contract**.

Note that while  $b3$  merely specifies a unit wholesale price,  $b4$  and  $b5$  involve both a wholesale and a lump-sum side payment. Hence, we argue that  $b3$  is a simpler, yet, general, effective and practical contract:

- We say that  $b3$  is simpler than  $b4$  and  $b5$  because of the following reason: In a widely cited review paper, Cachon (2003) argues that “*the contract designer may actually prefer to offer a simple contract even if that contract does not optimize the supply chain's performance. A simple contract is particularly desirable if the contract's efficiency is high (the ratio of supply chain profit with the contract to the supply chains optimal profit) and if the contract designer captures the lions share of supply chain profit.*” Under  $b3$ , once the optimal contract parameters are computed and the contract is signed, the only transaction between the supplier and the buyer is based on the wholesale price. Also, the contract's efficiency under the optimal  $b3$  is one. While more sophisticated contracts, e.g., a contract with a side payment such as  $b4$  or a revenue sharing

contract such as  $b5$ , are expected to offer increased flexibility for coordination,  $b3$  achieves **channel coordination** via a single transaction. We refer to Corbett and Tang (1999) for a detailed discussion of progressively sophisticated contracts.

- We say that  $b3$  is more general than not only  $b1$  and  $b2$  but also  $b5$  because of two specific reasons:

- The formulation in  $(Pb3)$  is such that  $b3$  is a generalization of both  $b1$  and  $b2$ , i.e.,  $(Pb3)$  reduces to  $(Pb1)$  when  $k = 1$  and to  $(Pb2)$  when  $m = 0$ .
- The idea of revenue sharing in a contractual setting is first introduced by Cachon and Lariviere (2005) who treat  $\phi$  as an external parameter. They examine both the single-product setting of interest in Figure 3.1 and the more general newsvendor setting while they ignore reservation profits. They are able to prove that  $b5$  is a coordination contract if the distribution cost  $c$  is negligible, i.e.,  $c = 0$ . Since their results do not immediately extend to the case with explicit reservation profits and distribution cost  $c$ , we demonstrate that  $b5$  is not a coordination contract unless  $c = 0$  (see Appendix A). Pan et al. (2010) analyze  $b5$  explicitly in two different settings with two products:

- \* One-supplier-two-buyer setting, and
- \* Two-supplier-one-buyer setting

with linear price-sensitive deterministic demand functions. They also ignore reservation profits and assume that the distribution costs of the buyers are negligible. Since  $b5$  is not a coordination contract neither for the problem settings analyzed by Pan et al. (2010) nor for our problem setting

with explicit reservation profits and distribution cost  $c$ , we argue that  $b3$  is a more promising coordination contract amenable to generalization.

- We say that  $b3$  is effective because of the following two reasons:
  - Only if  $\pi_s^- = \Pi^c$ , the optimal  $b3$  is equivalent to optimal  $b1$  (see footnote <sup>k</sup>).
  - Only if  $\pi_s^- = s\Pi^c/(s+c)$ , the optimal  $b3$  is equivalent to optimal  $b2$  (see footnote <sup>l</sup>).

That is, it is very unlikely for  $b1$  or  $b2$  to be optimal in general while the optimal  $b3$  is system optimal as it is a **coordination contract**. Hence, we argue that  $b3$  is an effective contract, and

- We say that  $b3$  is practical because:  $b3$  offers a practical solution for all realistic levels of the supplier's reservation profit while for example  $b2$  may fail to do so (see Cases 1.2 and 2.3 in Section 3.5.3).

Next, in Section 3.6.2, we also argue that  $b3$  offers flexibility for negotiation between the supplier and the buyer. That is, not only  $b3$  beats  $s1$ ,  $b1$ , and  $b2$  in terms of the system-wide profit and the buyer's profit,  $b3$  can be utilized to create an environment for negotiation when the supplier is the dominant party. Last but not least, in Section 3.6.4, we also prove that  $b3$  benefits consumers in terms of generating a low retail price.

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<sup>k</sup>Suppose  $\pi_s^- = \Pi^c$ . Using (3.86), the optimal  $m$  under  $b1$  is given by  $m^{b1} = c$ . Using (3.124) and (3.125), the optimal  $k$  and  $m$  under  $b3$  are given by  $k^{b3} = 1$  and  $m^{b3} = c$ . Recalling  $k = 1$  under  $b1$ , the optimal contracts under  $b1$  and  $b3$  are equivalent if  $\pi_s^- = \Pi^c$ .

<sup>l</sup>Suppose  $\pi_s^- = s\Pi^c/(s+c)$ . Using (3.109), it can be shown that the optimal  $k$  under  $b2$  is given by  $k^{b2} = (s+c)/s$ . Using (3.124) and (3.125), the optimal  $k$  and  $m$  under  $b3$  are given by  $k^{b3} = (s+c)/s$  and  $m^{b3} = 0$ . Recalling  $m = 0$  under  $b2$ , the optimal contracts under  $b2$  and  $b3$  are equivalent if  $\pi_s^- = s\Pi^c/(s+c)$ .



### 3.6.2 Conditions under which $b3$ is superior to $s1$ for the supplier

We next show that  $b3$  is superior to  $s1$  not only for the buyer but also for the supplier under a specific condition on the supplier's reservation profit. This specific condition is an important finding we wish to elaborate on because Liu and Çetinkaya (2009) argue that leadership is always beneficial for the lead in both supplier and buyer-driven channels when reservation profits are assumed to be equal to zero.

We proceed with comparing  $s1$  and  $b3$  to identify the condition. First, observe that  $b3$  is superior to  $s1$  for the supplier if the supplier has an opportunity to set  $\pi_s^- \geq \pi_s^{s1}$ , where  $\pi_s^- \in [0, \Pi^c]$  is the supplier's reservation profit under  $b3$  and  $\pi_s^{s1}$  is the supplier's optimal profit under  $s1$  given by (3.81). Using  $\pi_s^{s1}$  in (3.81) and Figures 3.4 and 3.5 and considering Cases 1 and 2 discussed in Section 3.5.1, it is easy to see that

- $\pi_s^{s1} = \Pi^c/2$  if  $\pi_b^- \in [0, \Pi^c/4]$  and
- $\pi_s^{s1} \leq \Pi^c/2$  if  $\pi_b^- \in [\Pi^c/4, \Pi^c]$ .

It then follows that

$$\pi_s^{s1} \leq \frac{\Pi^c}{2}$$

for all  $\pi_b^- \in [0, \Pi^c]$ . Then, if the supplier has an opportunity to set

$$\frac{\Pi^c}{2} \leq \pi_s^- \leq \Pi^c \tag{3.132}$$

under  $b3$ ,  $b3$  is superior to  $s1$  from the supplier's perspective, too.

Next, observe that the buyer would hold on to  $b3$  if

$$\pi_b^{b3} \geq \pi_b^{s1} \tag{3.133}$$

for the given level of  $\pi_s^-$  in (3.132), where  $\pi_b^{s1}$  here is given by (3.76) and it is the buyer's corresponding profit when the supplier achieves the maximum profit under  $s1$ , i.e.,  $\pi_b^{s1} = \Pi^c/4$ . Clearly, for the given level of  $\pi_s^-$  in (3.132), we have

$$\pi_b^{b3} = \Pi^c - \pi_s^-.$$

Then, using the above equation in (3.133) and combining (3.133) with (3.132), we have

$$\frac{\Pi^c}{2} \leq \pi_s^- \leq \frac{3\Pi^c}{4}, \quad (3.134)$$

which ensures that  $b3$  is superior to  $s1$  for the supplier. That is, the maximum profit the supplier can achieve under  $s1$  is  $3\Pi^c/4$ . In particular,

- As a rational decision maker, the supplier should set  $\pi_s^- = 3\Pi^c/4$  under  $b3$ . In this case,  $\pi_b^{b3} = \Pi^c/4$ , and, hence, the buyer is indifferent between  $s1$  and  $b3$ .
- If the supplier sets  $\pi_s^- = \Pi^c/2$  then  $\pi_b^{b3} = \Pi^c/2$ . In this case, the supplier is indifferent between  $s1$  and  $b3$ , while the buyer obtains the best profit among all cases when the supplier agrees to switch from  $s1$  to  $b3$ .

Now that we have established the condition in (3.134) under which  $b3$  is superior to  $s1$  for the supplier, we question if  $b4$  (i.e., the other buyer-driven coordination contract) is ever superior to  $s1$  perhaps under a condition similar to (3.134). We do not raise this question for  $b5$  because  $b5$  is not a coordination contract for our problem setting with explicit reservation profits and distribution cost  $c$ .

### 3.6.3 Conditions under which $b4$ is superior to $s1$ for the supplier

Using the results derived by Liu and Çetinkaya (2009) (see expressions in (8) and (9) of Proposition 3 in Liu and Çetinkaya (2009)), the optimal  $b4$  is characterized by

$$\begin{aligned} k^{b4} &= \frac{s+c}{s}, \\ L^{b4} &= -\pi_s^- + \frac{s\Pi^c}{s+c}, \end{aligned} \tag{3.135}$$

$$w^{b4} = \frac{as}{2b(s+c)} + \frac{s}{2}, \tag{3.136}$$

$$\pi_s^{b4} = \pi_s^-, \quad \text{and} \tag{3.137}$$

$$\pi_b^{b4} = \Pi^c - \pi_s^-, \tag{3.138}$$

where superscript for  $b4$  is used in an obvious fashion.

Using (3.129), (3.130), (3.137), and (3.138), we observe that channel performance under  $b3$  and  $b4$  in terms of entities' resulting profits are the same. Therefore, using the same approach discussed above for  $b3$ , we can easily show that  $b4$  is also superior to  $s1$  for the supplier under the condition given by (3.134).

Clearly, the transactions between the supplier and the buyer have different structures under  $b3$  and  $b4$ . Under  $b3$ , the transaction is only based on the wholesale price given by (3.127) which depends on  $\pi_s^-$ . Under  $b4$ , the transaction is

- Based on the wholesale price given by (3.136) which is fixed, as well as,
- Based on the lump-sum side payment given by (3.135) which depends on  $\pi_s^-$ .

In particular, under  $b4$ , using (3.135), we observe that

- If  $\pi_s^- > s\Pi^c/(s+c)$  then  $L^{b4} < 0$  and the buyer makes a payment to the supplier,
- If  $\pi_s^- = s\Pi^c/(s+c)$  then  $L^{b4} = 0$  and there is no lump-sum side payment, and

- If  $\pi_s^- < s\Pi^c/(s+c)$  then  $L^{b4} > 0$  and the supplier makes a payment to the buyer.

#### 3.6.4 Contract $b3$ is beneficial from consumers' perspectives

We prove that  $b3$  generates the lowest wholesale and retail prices among  $s1$ ,  $b1$ ,  $b2$ , and  $b3$ , when  $\pi_s^- = \pi_b^- = 0$  and  $s > 0$ . Hence,  $b3$  is beneficial from both buyer's and consumers' perspectives in terms of the low prices. In particular, we show that when  $\pi_s^- = \pi_b^- = 0$  and  $s > 0$ ,

- $w^{b3} \leq w^{b2} \leq w^{b1} \leq w^{s1}$  and
- $p^{b3} \leq p^{b2} \leq p^{b1} = p^{s1}$ .

Since we do not have closed-form expressions for  $w^{b2}$  and  $p^{b2}$ , we first prove that

- $w^{b3} \leq w^{b1} \leq w^{s1}$  and
- $p^{b3} \leq p^{b1} = p^{s1}$ .

Recalling (3.78) and (3.81), when  $\pi_b^- = 0$ , we have

$$w^{s1} = \frac{a+b(s-c)}{2b} \quad \text{and} \quad p^{s1} = \frac{3a+b(s+c)}{4b},$$

recalling (3.87) and (3.88), when  $\pi_s^- = 0$ , we have

$$w^{b1} = \frac{a+b(3s-c)}{4b} \quad \text{and} \quad p^{b1} = \frac{3a+b(s+c)}{4b},$$

and recalling (3.127) and (3.128), when  $\pi_s^- = 0$ , we have

$$w^{b3} = w \quad \text{and} \quad p^{b3} = \frac{a+b(s+c)}{2b}. \quad (3.139)$$

Using the above equations, we have

$$\begin{aligned} w^{s1} - w^{b1} &= \frac{a - b(s + c)}{4b} \geq 0, & w^{b1} - w^{b3} &= \frac{a - b(s + c)}{4b} \geq 0, \\ p^{s1} - p^{b1} &= 0, & \text{and } p^{b1} - p^{b3} &= \frac{a - b(s + c)}{4b} \geq 0 \end{aligned}$$

using assumption (3.5). Hence we obtain

$$w^{b3} \leq w^{b1} \leq w^{s1} \quad \text{and} \quad p^{b3} \leq p^{b1} = p^{s1}. \quad (3.140)$$

Next, we need to prove that

- $w^{b3} \leq w^{b2} \leq w^{b1}$  and
- $p^{b3} \leq p^{b2} \leq p^{b1}$ .

To this end, recall (3.2) and note that the optimal wholesale price satisfies

$$\frac{d\pi_s}{dw} = \left[ q + (w - s) \frac{dq}{dp} \frac{dp}{dw} \right] = 0 \quad (3.141)$$

under  $b1$  and  $b2$  according to (3.27) and (3.41), respectively. Note that under  $b1$ , deciding on  $w$  is equivalent to deciding on  $p = w + m$  using (3.21) for the supplier given  $m$ . Also, under  $b2$ , deciding on  $w$  is equivalent to deciding on  $p = kw$  using (3.34) given  $k$ . Then, the optimal retail price satisfies

$$\frac{d\pi_s}{dp} = \frac{dw}{dp} q + (w - s) \frac{dq}{dp} + (w - s) \frac{dq}{dp} = 0. \quad (3.142)$$

1. Proof of  $w^{b3} \leq w^{b2} \leq w^{b1}$ .

Under  $b1$ ,  $p = w + m$  implies  $dp/dw = 1$ . Then,  $d\pi_s/dw$  in (3.141) becomes

$$\nabla_w^{b1} = \left[ q + (w - s) \frac{dq}{dp} \right].$$

Under  $b2$ ,  $p/w = k$  implies  $dp/dw = k \geq 1$  and  $d\pi_s/dw$  in (3.141) becomes

$$\nabla_w^{b2} = \left[ q + k(w - s) \frac{dq}{dp} \right].$$

From  $\nabla_w^{b1}$  and  $\nabla_w^{b2}$  we observe that the optimal wholesale price  $w^{b1}$  under  $b1$  that satisfies  $\nabla_w^{b1} = 0$  would result in  $\nabla_w^{b2} \leq 0$ , since  $k \geq 1$  and  $q \geq 0$  by assumption. Recalling that  $\pi_s$  is concave in  $w$  under  $b1$  using (3.27) and under  $b2$  using (3.41), we have  $w^{b1} \geq w^{b2}$ . Also,  $\nabla_w^{b2} = 0$  implies  $w^{b2} \geq s = w^{b3}$ . Then, we conclude  $w^{b3} \leq w^{b2} \leq w^{b1}$ .

## 2. Proof of $p^{b3} \leq p^{b2} \leq p^{b1}$ .

Under  $b1$ ,  $w = p - m$  implies  $dw/dp = 1$  and  $d\pi_s/dp$  in (3.142) becomes

$$\nabla_p^{b1} = \left[ q + (w - s) \frac{dq}{dp} \right].$$

Under  $b2$ ,  $w/p = 1/k$  implies  $dw/dp = 1/k \leq 1$  and  $d\pi_s/dp$  in (3.142) becomes

$$\nabla_p^{b2} = \left[ \frac{q}{k} + (w - s) \frac{dq}{dp} \right].$$

From  $\nabla_p^{b1}$  and  $\nabla_p^{b2}$  we observe that the optimal retail price  $p^{b1}$  under  $b1$  that satisfies  $\nabla_p^{b1} = 0$  would result in  $\nabla_p^{b2} \leq 0$ , since  $k \geq 1$  and  $q \geq 0$  by assumption. Hence,  $p^{b2} \leq p^{b1}$ .

Next, we need to show  $p^{b2} \geq p^{b3}$ . Using (3.110), when  $\pi_s^- = 0$  and  $s > 0$ , we

have

$$p^{b2} = \frac{a + bsk^{b2}}{2b}. \quad (3.143)$$

In order to show  $p^{b2} \geq p^{b3}$ , using (3.139) and (3.143), we need to show that  $k^{b2} \geq (s + c)/s$ . Under  $b2$ , using (3.93),  $\pi_b$  in (3.50) is concave in  $k$  and  $k^{b2}$  satisfies the first-order condition of  $\pi_b$  if  $\pi_s^- = 0$  and  $s > 0$ . Note that  $\pi_b = \Pi - \pi_s$ , where  $\Pi$  is the system profit. Using (3.50) and (3.51), under  $b2$ , we have

$$\begin{aligned} \Pi &= \frac{(a - bsk)\{a + b[sk - 2(s + c)]\}}{4b} \quad \text{and} \\ \frac{d\Pi}{dk} &= \frac{sb(s + c - sk)}{2} \Rightarrow \left. \frac{d\Pi}{dk} \right|_{k=(s+c)/s} = 0. \end{aligned}$$

By (3.108),  $d\pi_s/dk \leq 0$  at  $k = (s + c)/s$  by assumption (3.5). Therefore, we have

$$\left. \frac{d\pi_b}{dk} \right|_{k=(s+c)/s} = \left. \frac{d\Pi}{dk} \right|_{k=(s+c)/s} - \left. \frac{d\pi_s}{dk} \right|_{k=(s+c)/s} \geq 0.$$

Due to the concavity of  $\pi_b$  shown in (3.93),  $k^{b2} \geq (s + c)/s$ . Therefore,  $p^{b2} \geq p^{b3}$ .

Then, we conclude  $p^{b3} \leq p^{b2} \leq p^{b1}$ .

In summary, not only  $b3$  beats  $s1$ ,  $b1$ , and  $b2$  in terms of the system-wide profit, the buyer's profit, and the supplier's profit, it is also beneficial from the consumers' perspectives. Also, the optimal wholesale and retail prices under buyer-driven contracts,  $b1$ ,  $b2$ , and  $b3$ , are not more than those under the supplier-driven contract  $s1$ . The buyer-driven contracts regardless of contract type is more efficient than  $s1$ , since they mitigate the double marginalization problem (Spengler (1950) and Tirole (1988)) which raises the retail price and harms the system profit. The reason why the buyer-driven channel is more efficient is explained by Liu and Çetinkaya (2009): "In the supplier-driven channel, the buyer is the follower and responds to the suppliers

wholesale price. Since the supplier seeks to maximize the profit, the buyer is pushed to select a higher than system optimal retail price which limits the market demand. On the other hand, when the buyer is the leader, she selects the  $k$  value (determines the quantity-price relation first) before the supplier declares the wholesale price, and thus, the buyer has more freedom to warrant a higher market demand by choosing a relatively smaller retail price which is closer to the system optimal retail price.”

### 3.7 Conclusion

In this chapter, we consider the basic bilateral monopolistic setting and propose a new contract in the buyer-driven channel. The new contract referred as the generic contract,  $b3$ , has a more general pricing scheme than the two existing buyer-driven contracts in the literature: the margin-only and the multiplier-only contracts, i.e.,  $b1$  and  $b2$ . Considering the reservation profit for the supplier, we formulate the buyer’s optimization problem ( $Pb3$ ) under  $b3$ . We show that ( $Pb3$ ) reduces to ( $Pb1$ ) (the problem under  $b1$ ) when  $k = 1$  and to ( $Pb2$ ) (the problem under  $b2$ ) when  $m = 0$ . Examining ( $Pb1$ ), ( $Pb2$ ), and ( $Pb3$ ), we study the optimal contracts under  $b1$ ,  $b2$ , and  $b3$ . Our work attempts to contribute the literature regarding the following:

- Contract  $b3$  has never been studied previously;
- We incorporate the supplier’s reservation profit consideration under  $b1$ ;
- We consider both cases  $s = 0$  and  $s > 0$  under  $b2$  and provide the conditions on the supplier’s reservation profit under which  $b2$  is practical.

Finally, we provide an explicit comparison of the buyer- and supplier-driven channels by considering the wholesale price contract  $s1$  in the supplier-driven channel, and  $b1$ ,  $b2$ ,  $b3$ , as well as, existing coordination contracts in the buyer-driven channel: the buyer-driven two-part linear contract,  $b4$ , and the buyer-driven revenue-sharing



contract,  $b5$ . We demonstrate that  $b3$  is not only optimal for the system and the buyer, it also benefits consumers and even the supplier. In summary,

- Contract  $b3$  is a coordination contract and it is effective than  $b1$  and  $b2$ .
- Contract  $b3$  is more general than  $b1$ ,  $b2$ , and  $b5$ .
- Contract  $b3$  is simpler to implement than  $b4$  and  $b5$ .
- Contract  $b3$  is even superior to  $s1$  for the supplier under a specific condition.
- Contract  $b3$  benefits consumers by generating the lowest retail price among the contracts of interest. It also solves the double marginalization problem.

Table 3.1: Summary of notation in the single-product setting.

Symbol	Explanation
$p$	the retail price
$q$	the order quantity, $q = a - bp$
$w$	the wholesale price
$s$	the supplier's unit production cost, $s \geq 0$
$c$	the buyer's unit distribution cost, $c \geq 0$
$k$	the price multiplier, $k > 0$
$m$	the price margin
$\pi_s$	the supplier's profit function
$\pi_b$	the buyer's profit function
$\Pi$	the system profit function, $\Pi = \pi_s + \pi_b$
$s1$	index representing the wholesale price contract in the supplier-driven channel
$b1$	index representing the margin-only contract in the buyer-driven channel
$b2$	index representing the multiplier-only contract in the buyer-driven channel
$b3$	index representing the generic contract in the buyer-driven channel
$\Pi^c$	the centralized optimal system profit
$p^c$	the centralized optimal retail price
$q^c$	the centralized optimal order quantity
$p^{s1}(w)$	the optimal retail price under $s1$ for a given $w$
$q^{s1}(w)$	the optimal order quantity under $s1$ for a given $w$
$w^{b1}(m)$	the optimal wholesale price under $b1$ for a given $m$
$p^{b1}(m)$	the corresponding retail price under $b1$ for a given $m$
$q^{b1}(m)$	the corresponding order quantity under $b1$ for a given $m$
$w^{b2}(k)$	the optimal wholesale price under $b2$ for a given $k$
$p^{b2}(k)$	the corresponding retail price under $b2$ for a given $k$
$q^{b2}(k)$	the corresponding order quantity under $b2$ for a given $k$
$w^{b3}(k, m)$	the optimal wholesale price under $b3$ for given $k$ and $m$
$p^{b3}(k, m)$	the corresponding retail price under $b3$ for given $k$ and $m$
$q^{b3}(k, m)$	the corresponding order quantity under $b3$ for given $k$ and $m$
$k^l$	the optimal $k$ under contract $l = b2, b3$
$m^l$	the optimal $m$ under contract $l = b1, b3$
$w^l$	the optimal wholesale price under contract $l = s1, b1, b2, b3$
$p^l$	the optimal retail price under contract $l = s1, b1, b2, b3$
$\pi_s^l$	the supplier's profit under the optimal contract $l = s1, b1, b2, b3$
$\pi_b^l$	the buyer's profit under the optimal contract $l = s1, b1, b2, b3$
$\Pi^l$	the system profit under the optimal contract $l = s1, b1, b2, b3$
$\pi_s^-$	the supplier's reservation profit, $\pi_s^- \geq 0$
$\pi_b^-$	the buyer's reservation profit, $\pi_b^- \geq 0$

Table 3.2: Summary of results under  $s1$  and  $b1$  in the single-product setting.

Contract	Decisions
Wholesale price Contract ( $\pi_b^- \in [0, \Pi^c]$ )	$w^{s1} = \min\{w^{s1-}, w^{s1+}\} = \min\left\{\frac{a-2\sqrt{b\pi_b^-}}{b} - c, \frac{a+b(s-c)}{2b}\right\}$ $p^{s1} = \min\left\{\frac{a-\sqrt{b\pi_b^-}}{b}, \frac{3a+b(s+c)}{4b}\right\}$ $q^{s1} = \max\left\{\sqrt{b\pi_b^-}, \frac{a-b(s+c)}{4}\right\}$ $\pi_s^{s1} = \begin{cases} \frac{[a-b(s+c)-2\sqrt{b\pi_b^-}]\sqrt{b\pi_b^-}}{b} & \text{if } \pi_b^- \in [\Pi^c/4, \Pi^c] \\ \frac{[a-b(s+c)]^2}{8b} & \text{if } \pi_b^- \in [0, \Pi^c/4] \end{cases}$ $\pi_b^{s1} = \max\left\{\pi_b^-, \frac{[a-b(s+c)]^2}{16b}\right\}$ $\Pi^{s1} = \begin{cases} \frac{[a-b(s+c)-\sqrt{b\pi_b^-}]\sqrt{b\pi_b^-}}{b} & \text{if } \pi_b^- \in [\Pi^c/4, \Pi^c] \\ \frac{3[a-b(s+c)]^2}{16b} & \text{if } \pi_b^- \in [0, \Pi^c/4] \end{cases}$
Margin-only Contract ( $\pi_s^- \in [0, \Pi^c]$ )	$m^{b1} = \min\{m^{b1-}, m^{b1+}\} = \min\left\{\frac{a-2\sqrt{b\pi_s^-}}{b} - s, \frac{a-b(s-c)}{2b}\right\}$ $w^{b1} = \max\left\{s + \sqrt{\frac{\pi_s^-}{b}}, \frac{a+b(3s-c)}{4b}\right\}$ $p^{b1} = \min\left\{\frac{a-\sqrt{b\pi_s^-}}{b}, \frac{3a+b(s+c)}{4b}\right\}$ $q^{b1} = \max\left\{\sqrt{b\pi_s^-}, \frac{a-b(s+c)}{4}\right\}$ $\pi_s^{b1} = \max\left\{\pi_s^-, \frac{[a-b(s+c)]^2}{16b}\right\}$ $\pi_b^{b1} = \begin{cases} \frac{[a-b(s+c)-2\sqrt{b\pi_s^-}]\sqrt{b\pi_s^-}}{b} & \text{if } \pi_s^- \in [\Pi^c/4, \Pi^c] \\ \frac{[a-b(s+c)]^2}{8b} & \text{if } \pi_s^- \in [0, \Pi^c/4] \end{cases}$ $\Pi^{b1} = \begin{cases} \frac{[a-b(s+c)-\sqrt{b\pi_s^-}]\sqrt{b\pi_s^-}}{b} & \text{if } \pi_s^- \in [\Pi^c/4, \Pi^c] \\ \frac{3[a-b(s+c)]^2}{16b} & \text{if } \pi_s^- \in [0, \Pi^c/4] \end{cases}$

Table 3.3: Summary of results under  $b2$  and  $b3$  in the single-product setting.

Contract	Decisions
Multiplier-only Contract ( $s = 0$ , $\pi_s^- \in (0, \frac{a^2-2abc}{4b}]$ )	$k^{b2} = \frac{a^2}{4b\pi_s^-}$
	$w^{b2} = \frac{2\pi_s^-}{a}$
	$p^{b2} = \frac{a}{2b}$
	$\pi_s^{b2} = \pi_s^-$
	$\pi_b^{b2} = \frac{a^2}{4b^2} - \frac{ac}{2} - \pi_s^-$
	$\Pi^{b2} = \frac{a^2}{4b} - \frac{ac}{2}$
Multiplier-only Contract ( $s > 0$ , $\pi_s^- \in [0, \pi_s^{k_l}]$ )	$k^{b2} = \min\{k^{b2-}, k^{b2+}\}$
	$k^{b2-} = \frac{as+2\pi_s^- - \sqrt{(as+2\pi_s^-)^2 - a^2s^2}}{bs^2}$
	$2s^2b^2\{k^{b2+}\}^3 - s(s+2c)b^2\{k^{b2+}\}^2 - a^2 = 0$
	$w^{b2} = \max\left\{s + \frac{\sqrt{(as+2\pi_s^-)^2 - a^2s^2} - 2\pi_s^-}{2(as+2\pi_s^- - \sqrt{(as+2\pi_s^-)^2 - a^2s^2})}, \frac{a+b sk^{b2+}}{2bk^{b2+}}\right\}$
	$p^{b2} = \min\left\{\frac{2as+2\pi_s^- - \sqrt{(as+2\pi_s^-)^2 - a^2s^2}}{2bs}, \frac{a+b sk^{b2+}}{2b}\right\}$
	$q^{b2} = \max\left\{\frac{\sqrt{(as+2\pi_s^-)^2 - a^2s^2} - 2\pi_s^-}{2s}, \frac{a-b sk^{b2+}}{2}\right\}$
	$\pi_s^{b2} = \max\left\{\pi_s^-, \frac{(a-b sk^{b2+})^2}{4bk^{b2+}}\right\}$
	$\pi_b^{b2} = \frac{(a-b sk^{b2})[a(k^{b2}-1)+bk^{b2}(sk^{b2}-s-2c)]}{4bk^{b2}}$
	$\Pi^{b2} = \frac{(a-b sk^{b2})\{a+b[sk^{b2}-2(s+c)]\}}{4b}$
Generic Contract ( $\pi_s^- \in (0, \Pi^c]$ )	$k^{b3} = \frac{\Pi^c}{\pi_s^-}$
	$m^{b3} = c - s\left(\frac{\Pi^c}{\pi_s^-} - 1\right)$
	$w^{b3} = \frac{[a-b(s+c)]\pi_s^-}{2b\Pi^c} + s$
	$p^{b3} = \frac{a+b(s+c)}{2b}$
	$q^{b3} = \frac{a-b(s+c)}{2}$
	$\pi_s^{b3} = \pi_s^-$
	$\pi_b^{b3} = \Pi^c - \pi_s^-$
	$\Pi^{b3} = \Pi^c$

## 4. THE GENERIC CONTRACT IN THE CASE OF MULTIPLE PRODUCTS

### 4.1 Setting 2. The multi-product bilateral monopolistic setting

We extend **the basic bilateral monopolistic setting** (the single-product setting) by considering the more general **multi-product bilateral monopolistic setting** referred as the two-product setting in Figure 4.1. Suppose that the two products are substitutable. The supplier's decisions pertain to the wholesale prices  $w_1$  and  $w_2$ , and the buyer's decisions pertain to the order quantities  $q_1$  and  $q_2$  and the retail prices  $p_1$  and  $p_2$ . The order quantities are dictated by the general demand function that depends linearly on the retail prices following

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} \alpha_1 & -\beta_1 \\ -\beta_2 & \alpha_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \quad (4.1)$$

Price competition occurs between the two substitutable products and it results from cross-price effects (Ingene and Parry (2004)), i.e., each product's demand depends on both products' retail prices. The demand function in (4.1) is the generalized "linear demand function with cross-price effects" (e.g., Pashigian (1961), Ingene and Parry (1995), Tyagi (2005), and Yang and Zhou (2006)). In (4.1),  $a_i$ ,  $\alpha_i$ , and  $\beta_i$  are parameters,  $i = 1, 2$ . Parameter  $a_i$  can be considered as the maximum demand of product  $i$  in market when prices for both products are zero (McGuire and Staelin (1983)). Parameter  $\alpha_i$  represents the sensitivity of demand of product  $i$  to its own retail price and  $\beta_i$  represents the sensitivity of demand of product  $i$  to the substitutable product's retail price (Ingene and Parry (2004)),  $i = 1, 2$ .

Let  $s_i$  and  $c_i$  denote the supplier's unit production cost and the buyer's unit distribution cost, respectively, for product  $i$ ,  $i = 1, 2$ . The notation introduced so far

and used frequently in the remainder of this document is summarized in Table 4.1.

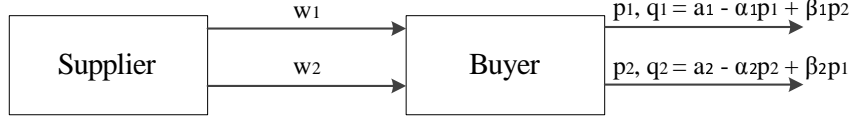


Figure 4.1: The multi-product bilateral monopolistic setting.

Table 4.1: Summary of notation in the multi-product setting.

Symbol	Explanation
$i$	the index of a product, $i = 1, 2$
$p_i$	the retail price for product $i$
$q_i$	the order quantity for product $i$ , $q_i = a_i - \alpha_i p_i + \beta_i p_j$ , $j = 1, 2, j \neq i$
$\alpha_i$	the sensitivity of demand of product $i$ to $p_i$
$\beta_i$	the sensitivity of demand of product $i$ to $p_j$ , $j = 1, 2$ and $j \neq i$
$w_i$	the wholesale price for product $i$
$s_i$	the supplier's unit production cost for product $i$
$c_i$	the buyer's unit distribution cost for product $i$
$k_i$	the price multiplier for product $i$ , $k_i > 0$
$m_i$	the price margin for product $i$ , $m_i \in \mathfrak{R}$

We focus on the examination of the wholesale price contract in the supplier-driven channel and the generic contract in the buyer-driven channel, denoted by *s1* and *b3*, respectively:

- s1. Under *the wholesale price contract*, the supplier decides  $w_1$  and  $w_2$  and then the buyer decides  $p_1$  and  $p_2$ .
- b3. Under *the generic contract*, the buyer decides on the values of  $k_i$ ,  $k_i \in \mathfrak{R}$  and  $m_i$ ,  $m_i \in \mathfrak{R}$ , while also committing that the retail price for product  $i$  would be

set such that  $p_i = k_i w_i + m_i$  and the order quantity for product  $i$  would be set such that  $q_i = a_i - \alpha_i(k_i w_i + m_i) + \beta_i(k_j w_j + m_j)$ ,  $i, j = 1, 2$  and  $j \neq i$ . Next, the supplier decides  $w_i$ ,  $i = 1, 2$ .

## 4.2 Modeling the case of symmetric two products

We start the analysis by considering the so-called symmetric setting (Choi (1991)) for the two products. In the symmetric setting,  $a_i = a$ ,  $\alpha_i = \alpha$ ,  $\beta_i = \beta$ ,  $s_i = s$ , and  $c_i = c$ ,  $i = 1, 2$ . Then, the demand function in (4.1) for product  $i$  can be rewritten as

$$q_i = a - \alpha p_i + \beta p_j, \quad (4.2)$$

$i, j = 1, 2$  and  $j \neq i$ . Hence, (4.2) is known as the “symmetric linear demand function with cross-price effects”, which is the special case of the generalized “linear demand function with cross-price effects” as shown in (4.1). The symmetry assumption for products’ demands as applied in (4.2) has been widely adopted in the literature on channel management to keep the problem formulation simple (e.g., McGuire and Staelin (1983), Choi (1991), Choi (1996), Trivedi (1998), Pan et al. (2010), and Wu et al. (2012)).

Due to the nature of price competition between substitutable products,  $\alpha > 0$  and  $\beta \geq 0$  are required, i.e., each product’s demand decreases in its own price and increases in its competitor’s price. Assuming a product’s demand impacted by its own price more heavily than by its competitor’s price, we set  $\alpha > \beta$  (e.g., McGuire and Staelin (1983), Choi (1991), Choi (1996), Trivedi (1998), Pan et al. (2010), and Wu et al. (2012)). Hence, we assume  $\alpha$  and  $\beta$  such that

$$\alpha > \beta \geq 0. \quad (4.3)$$

In the following, we first examine the symmetric two-product setting. Next, we generalize the symmetric two-product setting to the symmetric  $n$ -product setting,  $n \geq 2$ , and the asymmetric two-product setting. We analyze  $b3$  in these two settings in Sections 4.3 and 4.5. We conclude by summarizing the relations of these settings to **the single-product setting** in Section 4.4.

#### 4.2.1 Profit functions

In the symmetric two-product setting, using (4.2), the supplier's profit function is given by

$$\pi_s = \sum_{i=1,2} (w_i - s)q_i = \sum_{i,j=1,2,j \neq i} (w_i - s)(a - \alpha p_i + \beta p_j). \quad (4.4)$$

The buyer's profit function is given by

$$\pi_b = \sum_{i=1,2} (p_i - w_i - c)q_i = \sum_{i,j=1,2,j \neq i} (p_i - w_i - c)(a - \alpha p_i + \beta p_j). \quad (4.5)$$

The centralized profit function is given by

$$\Pi = \sum_{i=1,2} (p_i - s - c)q_i = \sum_{i,j=1,2,j \neq i} (p_i - s - c)(a - \alpha p_i + \beta p_j). \quad (4.6)$$

Note that using (4.2),  $p_i$  can be written as a function of order quantities such that

$$p_i = \frac{a}{\alpha - \beta} - \frac{\alpha q_i}{\alpha^2 - \beta^2} - \frac{\beta q_j}{\alpha^2 - \beta^2}, \quad (4.7)$$

$i, j = 1, 2$  and  $j \neq i$ .

Recalling (4.4), (4.5), (4.6), and (4.7), in order to guarantee  $q_i \geq 0$  and the supplier's and buyer's profits on product  $i$  are **nonnegative**, we assume  $p_i \leq a/(\alpha -$



$\beta$ ),  $w_i \geq s$ , and  $p_i \geq w_i + c$  so that

$$s + c \leq w_i + c \leq p_i \leq \frac{a}{\alpha - \beta}, \quad (4.8)$$

$i = 1, 2$ . We pay particular attention to ensure that the contractual problems at hand lead to nonnegative profits  $\pi_s$ ,  $\pi_b$ , and  $\Pi$  for the sake of practical realism. The inequalities in (4.8) are proposed to guarantee the nonnegative profits.

#### 4.2.2 Centralized problem

Using (4.6) and assumption (4.8), the centralized optimization problem in the symmetric two-product setting can be stated as

$$(Pc - 2s) : \quad \max_{p_1, p_2 \in [s+c, a/(\alpha-\beta)]} \Pi = \sum_{i,j=1,2, j \neq i} (p_i - s - c)(a - \alpha p_i + \beta p_j).$$

Clearly,  $w_i$  is immaterial for  $\Pi$  in (4.6), and, hence, by assumption (4.8), we are only interested in  $p_i$  values that satisfy

$$s + c \leq p_i \leq \frac{a}{\alpha - \beta}, \quad (4.9)$$

$i = 1, 2$ .

Using (4.6), note that

$$\frac{\partial \Pi}{\partial p_i} = a - 2\alpha p_i + 2\beta p_j + (\alpha - \beta)(s + c), \quad (4.10)$$

$i, j = 1, 2$  and  $j \neq i$ , and the Hessian matrix of  $\Pi$  is given by

$$\begin{bmatrix} \frac{\partial^2 \Pi}{\partial p_1^2} & \frac{\partial^2 \Pi}{\partial p_1 \partial p_2} \\ \frac{\partial^2 \Pi}{\partial p_2 \partial p_1} & \frac{\partial^2 \Pi}{\partial p_2^2} \end{bmatrix} = \begin{bmatrix} -2\alpha & 2\beta \\ 2\beta & -2\alpha \end{bmatrix}.$$

The determinant of the Hessian matrix is given by  $4\alpha^2 - 4\beta^2 > 0$  using (4.3). Hence,  $\Pi$  in (4.6) is negative-definite. Setting  $\partial\Pi/\partial p_i = 0$  in (4.10) for  $i = 1, 2$  leads to

$$p_i^c = \frac{a + (\alpha - \beta)(s + c)}{2(\alpha - \beta)}. \quad (4.11)$$

Observe that  $p_i^c$  defined in (4.11) is **the centralized optimal retail price** for product  $i$ ,  $i = 1, 2$ . This is because by assumption (4.8),

$$\begin{aligned} \frac{a}{\alpha - \beta} - p_i^c &= \frac{a - (\alpha - \beta)(s + c)}{2(\alpha - \beta)} \geq 0 \quad \text{and} \\ p_i^c - (s + c) &= \frac{a - (\alpha - \beta)(s + c)}{2(\alpha - \beta)} \geq 0, \end{aligned}$$

so that  $p_i^c$  defined in (4.11) is realizable over the region (4.9),  $i = 1, 2$ . Using (4.11) in (4.2), **the centralized optimal order quantity** for product  $i$  is given by

$$q_i^c = \frac{a - (\alpha - \beta)(s + c)}{2}, \quad (4.12)$$

$i = 1, 2$ . Substituting (4.11) in (4.6), **the centralized optimal system profit** is given by

$$\Pi^c = \frac{[a - (\alpha - \beta)(s + c)]^2}{2(\alpha - \beta)}. \quad (4.13)$$

It is important to note that the optimal retail prices and order quantities given by (4.11) and (4.12) for both products are the same, i.e., free of the index  $i$ ,  $i = 1, 2$ . It is because the two products are symmetric and it does not make sense to make different decisions for them.

Ingene and Parry (2004) consider a variant of  $(Pc - 2s)$  by allowing a more general cost structure where each entity has a per unit cost as well as a fixed cost for each product and using a more general demand function where  $a_1 \neq a_2$  is possible. By

setting the fixed costs equal zero and letting  $a_1 = a_2$ , their problem (see the problem in (5.3.1) on p. 199 of Ingene and Parry (2004)) is reduced to  $(Pc - 2s)$  leading to  $p^c$  in (4.11) (see (5.3.2) on p. 199 of Ingene and Parry (2004)).

#### 4.2.3 Wholesale price contract s1

Under s1, the supplier decides  $w_1$  and  $w_2$  first and then the buyer decides  $p_1$  and  $p_2$ . By assumption (4.8), we are only interested in  $w_i$  and  $p_i$  values that satisfy

$$s \leq w_i \leq \frac{a}{\alpha - \beta} - c \quad \text{and} \quad (4.14)$$

$$w_i + c \leq p_i \leq \frac{a}{\alpha - \beta}, \quad (4.15)$$

$i = 1, 2$ . We refer to (4.14) as the main constraint on the decision variables of the contract design problem under s1.

##### 4.2.3.1 Formulation of $(Ps1 - 2s)$

For a given  $w_i$ ,  $i = 1, 2$ , such that (4.14) is true, using (4.5), we have

$$\frac{\partial \pi_b}{\partial p_i} = a - 2\alpha p_i + 2\beta p_j + \alpha w_i - \beta w_j + (\alpha - \beta)c, \quad (4.16)$$

$i, j = 1, 2$  and  $j \neq i$ , and the Hessian matrix of  $\pi_b$  in (4.5) is given by

$$\begin{bmatrix} \frac{\partial^2 \pi_b}{\partial p_1^2} & \frac{\partial^2 \pi_b}{\partial p_1 \partial p_2} \\ \frac{\partial^2 \pi_b}{\partial p_2 \partial p_1} & \frac{\partial^2 \pi_b}{\partial p_2^2} \end{bmatrix} = \begin{bmatrix} -2\alpha & 2\beta \\ 2\beta & -2\alpha \end{bmatrix}.$$

The determinant of the Hessian matrix is given by  $4\alpha^2 - 4\beta^2 > 0$  using (4.3). Hence,  $\pi_b$  in (4.5) is negative-definite. Setting  $\partial \pi_b / \partial p_i = 0$  in (4.16) for  $i = 1, 2$  leads to

$$p_i^{s1-2s}(w_i) = \frac{a + (\alpha - \beta)(w_i + c)}{2(\alpha - \beta)}. \quad (4.17)$$

Observe that for any  $w_i$  such that (4.14) is true,  $p_i^{s1-2s}(w_i)$  defined in (4.17) is **the buyer's optimal response, i.e., the optimal retail price** for product  $i$ ,  $i = 1, 2$ . This is because  $w_i$  satisfies (4.14) so that

$$\begin{aligned}\frac{a}{\alpha - \beta} - p_i^{s1-2s}(w_i) &= \frac{a - (\alpha - \beta)(w_i + c)}{2(\alpha - \beta)} \geq 0 \quad \text{and} \\ p_i^{s1-2s}(w_i) - (w_i + c) &= \frac{a - (\alpha - \beta)(w_i + c)}{2(\alpha - \beta)} \geq 0,\end{aligned}$$

$i = 1, 2$ . Hence,  $p_i^{s1-2s}(w_i)$  in (4.17) is realizable over the region (4.15),  $i = 1, 2$ .

Using (4.2), **the buyer's optimal order quantity** for product  $i$  is given by

$$q_i^{s1-2s}(w_i, w_j) = \frac{a - \alpha w_i + \beta w_j - (\alpha - \beta)c}{2}, \quad (4.18)$$

$i, j = 1, 2$  and  $j \neq i$ .

Using (4.17) in (4.4) and (4.5), we have

$$\begin{aligned}\pi_s &= \sum_{i,j=1,2,j \neq i} \frac{(w_i - s)[a - \alpha w_i + \beta w_j - (\alpha - \beta)c]}{2} \quad \text{and} \\ \pi_b &= \sum_{i,j=1,2,j \neq i} \frac{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i + \beta w_j - (\alpha - \beta)c]}{4(\alpha - \beta)}.\end{aligned}$$

Considering the main constraint (4.14) and the buyer's reservation profit  $\pi_b^-$  and using the two above expressions for  $\pi_s$  and  $\pi_b$ , **the supplier's optimization**

**problem under  $s1$**  in the symmetric two-product setting can be stated as

$$(Ps1 - 2s) : \quad \max_{s \leq w_1, w_2 \leq a/(\alpha-\beta)-c} \pi_s = \sum_{i,j=1,2,j \neq i} \frac{(w_i - s)[a - \alpha w_i + \beta w_j - (\alpha - \beta)c]}{2} \quad (4.19)$$

$$\begin{aligned} s.t. \quad \pi_b &= \sum_{i,j=1,2,j \neq i} \frac{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i + \beta w_j - (\alpha - \beta)c]}{4(\alpha - \beta)} \\ &\geq \pi_b^-. \end{aligned} \quad (4.20)$$

Clearly,  $(Ps1 - 2s)$  makes sense only for reasonable values of  $\pi_b^-$ . That is, a natural upper bound on  $\pi_b^-$  is given by

$$0 \leq \pi_b^- \leq \Pi^c = \frac{[a - (\alpha - \beta)(s + c)]^2}{2(\alpha - \beta)}, \quad (4.21)$$

where  $\Pi^c$  is the optimal centralized system profit in (4.13).

It is important to note that a variant of  $(Ps1 - 2s)$  is studied by Yang and Zhou (2006). They consider a channel where a supplier sells a single product with cross-price effects to two buyers who collude to make retail price decisions. The supplier's problem in their work can be considered as a variant of  $(Ps1 - 2s)$ , where the supplier makes only one wholesale price decision. Also, they assume  $\pi_b^- = 0$  and  $c = 0$  (see Section 3.2 on p. 108 of Yang and Zhou (2006)). That is, setting  $w_1 = w_2 = w$ ,  $\pi_b$  in (4.20) and  $p_i^{s1-2s}(w_i)$  in (4.17) also appear in their work (see expressions (11), (12), and (13) on p. 108 of Yang and Zhou (2006)) with  $c = 0$ ,  $i = 1, 2$ .

#### 4.2.3.2 Optimal solution of $(Ps1 - 2s)$

Next, we present two approaches to identify the optimal solution for  $(Ps1 - 2s)$  given by (4.19) and (4.20). In the first approach, we prove that the optimal solution is the same as that for  $(Ps1 - 2s)$  in the single-product setting. In the second approach,

we apply the method of Lagrange Multiplier to directly solve this problem.

*Approach 1*

Using (4.19), we have

$$\pi_s = \frac{1}{2} \sum_{i=1,2} \{(w_i - s)[a - \alpha w_i - (\alpha - \beta)c] - \beta s w_i\} + \beta w_1 w_2. \quad (4.22)$$

Note that

$$2w_1 w_2 \leq w_1^2 + w_2^2 \quad (4.23)$$

for  $w_1, w_2 \in \mathfrak{R}$ , where the equality holds true if  $w_1 = w_2$ . Using (4.22), we have

$$\begin{aligned} \pi_s &\leq \frac{1}{2} \sum_{i=1,2} \{(w_i - s)[a - \alpha w_i - (\alpha - \beta)c] - \beta s w_i\} + \frac{\beta}{2}(w_1^2 + w_2^2) \\ &= \sum_{i=1,2} \frac{(w_i - s)[a - (\alpha - \beta)(w_i + c)]}{2} = \sum_{i=1,2} f(w_i), \end{aligned} \quad (4.24)$$

where the equality holds true if  $w_1 = w_2$  and

$$f(x) = \frac{(x - s)[a - (\alpha - \beta)(x + c)]}{2}. \quad (4.25)$$

Using (4.20) and (4.23), we have

$$\begin{aligned}
\pi_b &= \frac{1}{4(\alpha - \beta)} \cdot \\
&\quad \sum_{i=1,2} \{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i - (\alpha - \beta)c] + [a - (\alpha - \beta)c]\beta w_i\} \\
&\quad - \frac{(\alpha - \beta)\beta w_1 w_2}{2(\alpha - \beta)} \\
&\geq \frac{1}{4(\alpha - \beta)} \cdot \\
&\quad \sum_{i=1,2} \{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i - (\alpha - \beta)c] + [a - (\alpha - \beta)c]\beta w_i\} \\
&\quad - \frac{(\alpha - \beta)\beta(w_1^2 + w_2^2)}{4(\alpha - \beta)} \\
&= \sum_{i=1,2} \frac{[a - (\alpha - \beta)(w_i + c)]^2}{4(\alpha - \beta)} = \sum_{i=1,2} g(w_i), \tag{4.26}
\end{aligned}$$

where the equality holds true if  $w_1 = w_2$  and

$$g(x) = \frac{[a - (\alpha - \beta)(x + c)]^2}{4(\alpha - \beta)}. \tag{4.27}$$

Clearly, if

$$g(w_i) \geq \pi_b^- / 2$$

for  $i = 1, 2$  then using (4.26)

$$\pi_b \geq \sum_{i=1,2} g(w_i) \geq \pi_b^-.$$

Recalling (4.24), we have

$$\pi_s \leq \sum_{i=1,2} f(w_i) \leq \max_{w_1, w_2} \sum_{i=1,2} f(w_i) = \sum_{i=1,2} \max_{w_i} f(w_i), \tag{4.28}$$

where  $\pi_s = \sum_{i=1,2} f(w_i)$  if  $w_1 = w_2$ . Hence, the upper bound of  $\pi_s$  in (4.19) when  $\pi_b \geq \pi_b^-$  is satisfied is given by

$$\begin{aligned} & \sum_{i=1,2} \max_{w_i} f(w_i) \\ \text{s.t. } & g(w_i) \geq \frac{\pi_b^-}{2}, \quad i = 1, 2. \end{aligned}$$

Let  $w_i^*$  denote the optimal solution to the above optimization problem,  $i = 1, 2$ . Since this problem is separable based on  $w_i$ , we have  $w_i^* = w^*$ ,  $i = 1, 2$ , where  $w^*$  is the optimal solution to the following optimization problem

$$\begin{aligned} \max_w f(w) &= \frac{(w-s)[a-(\alpha-\beta)(w+c)]}{2} \\ \text{s.t. } g(w) &= \frac{[a-(\alpha-\beta)(w+c)]^2}{4(\alpha-\beta)} \geq \frac{\pi_b^-}{2}, \end{aligned}$$

using (4.25) and (4.27). It is important to note that this optimization problem is the same as (Ps1) in **the single-product setting**, when  $b = \alpha - \beta$  and the buyer's reservation profit is given by  $\pi_b^-/2$ , where  $\pi_b^-$  is the buyer's reservation profit in the symmetric two-product setting.

Since  $w_1 = w_2 = w^*$ , using (4.28),  $\pi_s = f(w^*) + f(w^*) = \sum_{i=1,2} \max_{w_i} f(w_i) = 2 \max_w f(w)$ . In this case,  $\pi_s$  achieves its upper bound. Hence, **the optimal whole-sale price for product  $i$  under s1** in the symmetric two-product setting is given by  $w_i^{s1-2s} = w^*$ ,  $i = 1, 2$ , where  $w^*$  is the optimal solution to (Ps1) in the single-product setting with  $b = \alpha - \beta$  and the buyer's reservation profit is given by  $\pi_b^-/2$ .



*Approach 2*

Using (4.19), observe that

$$\frac{\partial \pi_s}{\partial w_i} = \frac{(a - 2\alpha w_i + 2\beta w_j + (\alpha - \beta)(s - c))}{2}, \quad (4.29)$$

$i, j = 1, 2$  and  $j \neq i$ , and the Hessian matrix of  $\pi_s$  in (4.19) is given by

$$\begin{bmatrix} \frac{\partial^2 \pi_s}{\partial w_1^2} & \frac{\partial^2 \pi_s}{\partial w_1 \partial w_2} \\ \frac{\partial^2 \pi_s}{\partial w_2 \partial w_1} & \frac{\partial^2 \pi_s}{\partial w_2^2} \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \beta & -\alpha \end{bmatrix}.$$

Hence, the Hessian matrix is negative-definite using (4.3). Letting  $w_i^{s1+}$  denote the solution for  $\partial \pi_s / \partial w_i = 0$  in (4.29), we have

$$w_i^{s1+} = w^{s1+} \equiv \frac{a + (\alpha - \beta)(s - c)}{2(\alpha - \beta)}, \quad (4.30)$$

$i = 1, 2$ . Note that  $w_i^{s1+}$  is free of the index  $i$ ,  $i = 1, 2$ . Hence, the subscript can be omitted. Using assumption (4.8), it can be easily verified that  $w_i^{s1+}$  defined in (4.30) is realizable over the region (4.14) which appears in (4.19),  $i = 1, 2$ . Substituting (4.30) in (4.20), the corresponding buyer's profit is the given by

$$\pi_b^{s1+} = \frac{[a - (\alpha - \beta)(s + c)]^2}{8(\alpha - \beta)}. \quad (4.31)$$

Next, we need to consider  $\pi_b^{s1+}$  given by (4.31) in relation to constraint (4.20) in  $(Ps1 - 2s)$ .

- If  $\pi_b^{s1+} \geq \pi_b^-$  then (4.20) is satisfied for  $w_i = w^{s1+}$  so that **the optimal wholesale price for product  $i$  under  $s1$** , denoted by  $w_i^{s1-2s}$ , is simply given by  $w^{s1+}$  in (4.30),  $i = 1, 2$ .

- Otherwise, i.e.,  $\pi_b^{s1+} < \pi_b^-$ ,  $w_i^{s1-2s}$  occurs at the boundary of (4.20),  $i = 1, 2$ . Then, we solve for  $w_i^{s1-2s}$  using the method of Lagrange multiplier,  $i = 1, 2$ . Let

$$f(\cdot) = \pi_s + \lambda(\pi_b - \pi_b^-),$$

where  $\pi_s$  and  $\pi_b$  are given by (4.19) and (4.20),  $\lambda \geq 0$  is the Lagrange multiplier, and  $f(\cdot)$  is a simplified notation for  $f(w_1, w_2, \lambda)$ . Setting  $\Delta_{w_i, \lambda} f(\cdot) = 0$  leads to

$$\begin{aligned} \frac{\partial f(\cdot)}{\partial w_i} &= \frac{\partial \pi_s}{\partial w_i} + \frac{\lambda \partial \pi_b}{\partial w_i} \\ &= \frac{(1 - \lambda)[a - (\alpha - \beta)c] + (\lambda - 2)(\alpha w_i - \beta w_j) + (\alpha - \beta)s}{2} = 0, \end{aligned} \quad (4.32)$$

$$\frac{\partial f(\cdot)}{\partial \lambda} = \pi_b - \pi_b^- = 0, \quad (4.33)$$

$i, j = 1, 2$  and  $j \neq i$ . Then, (4.32) leads to

$$w_i = \frac{(\lambda - 1)[a - (\alpha - \beta)c] - (\alpha - \beta)s}{(\lambda - 2)(\alpha - \beta)},$$

$i = 1, 2$ . It is obvious that  $w_1 = w_2$  by the above equation. Hence,  $w_1^{s1-2s} = w_2^{s1-2s}$  and let  $w_i^{s1-2s} = w^{s1-2s}$ ,  $i = 1, 2$ . Letting  $w_i = w$ ,  $i = 1, 2$ , in (4.20) and (4.33), we have

$$[a - (\alpha - \beta)(w + c)]^2 - 2(\alpha - \beta)\pi_b^- = 0.$$

Therefore,  $w^{s1-2s}$  is dictated by the solution of the above polynomial. The root of the polynomial are given by

$$w^{s1-} = \frac{a - \sqrt{2(\alpha - \beta)\pi_b^-}}{\alpha - \beta} - c \quad \text{and} \quad \frac{a + \sqrt{2(\alpha - \beta)\pi_b^-}}{\alpha - \beta} - c. \quad (4.34)$$

Now, observe that the latter root is eliminated because it violates (4.14) for  $\pi_b^- > 0$  and it is equal to the former root for  $\pi_b^- = 0$ . Since  $\pi_b^-$  satisfies (4.21), using assumption (4.8), it is easy to verify that

$$\begin{aligned}\frac{a}{\alpha - \beta} - c - w^{s1-} &= \frac{2\sqrt{(\alpha - \beta)\pi_b^-}}{\alpha - \beta} \geq 0, \quad \text{and} \\ w^{s1-} - s &= \frac{a - (\alpha - \beta)(s + c)}{\alpha - \beta} - 2\sqrt{\frac{\pi_b^-}{\alpha - \beta}} \\ &\geq \frac{a - (\alpha - \beta)(s + c)}{\alpha - \beta} - \sqrt{\frac{[a - (\alpha - \beta)(s + c)]^2}{(\alpha - \beta)^2}} = 0,\end{aligned}$$

so that  $w_i^{s1-}$  for  $w_i^{s1-} = w^{s1-}$  defined in (4.34) is realizable over the region (4.14),  $i = 1, 2$ . Therefore,  $w_i^{s1-} = w^{s1-}$ ,  $i = 1, 2$ , and the corresponding buyer's profit is then given by  $\pi_b^-$ .

Substituting  $w_i = w$ ,  $i = 1, 2$ , in (4.20), we have

$$\pi_b = \frac{[a - (\alpha - \beta)(w + c)]^2}{2(\alpha - \beta)}$$

and

$$\frac{d\pi_b}{dw} = -[a - (\alpha - \beta)(w + c)].$$

Hence,  $\pi_b$  is decreasing in  $w$  over the region (4.14). That is,

- $\pi_b \geq \pi_b^-$  only for those  $w \leq w^{s1-}$  and
- If  $\pi_b^{s1+} < \pi_b^-$  then  $w^{s1-} < w^{s1+}$ .

Consequently, recalling (4.30) and (4.34), we have

$$\begin{aligned} w_i^{s1-2s} &= \min\{w^{s1-}, w^{s1+}\} \\ &= \min\left\{\frac{a - \sqrt{2(\alpha - \beta)\pi_b^-}}{\alpha - \beta} - c, \frac{a + (\alpha - \beta)(s - c)}{2(\alpha - \beta)}\right\}. \end{aligned} \quad (4.35)$$

Recalling (4.21), we have  $\pi_b^- \in [0, \Pi^c]$  by assumption. Then, considering (4.31) and (4.35), it is easy to show that the optimal solution of  $(Ps1 - 2s)$  depends on the value of  $\pi_b^-$ . That is,

- **Case 1:**  $w_i^{s1-2s} = w^{s1+}$ . If  $\pi_b^- \in [0, \Pi^c/4]$  then  $w^{s1+} \leq w^{s1-}$  so that  $w_i^{s1-2s} = w^{s1+}$ , and
- **Case 2:**  $w_i^{s1-2s} = w^{s1-}$ . If  $\pi_b^- \in [\Pi^c/4, \Pi^c]$  then  $w^{s1+} \geq w^{s1-}$  so that  $w_i^{s1-2s} = w^{s1-}$ ,  $i = 1, 2$ .

Recalling (4.17), (4.18), (4.19), and (4.20) and using (4.35), **the corresponding retail price, order quantity, supplier's, buyer's and system profits under**

the optimal contract  $s1$  are given by

$$p_i^{s1-2s} = \min \left\{ \frac{2a - \sqrt{2(\alpha - \beta)\pi_b^-}}{2(\alpha - \beta)}, \frac{3a + (\alpha - \beta)(s + c)}{4(\alpha - \beta)} \right\}, \quad (4.36)$$

$$q_i^{s1-2s} = \max \left\{ \sqrt{\frac{(\alpha - \beta)\pi_b^-}{2}}, \frac{a - (\alpha - \beta)(s + c)}{4} \right\},$$

$$\pi_s^{s1-2s} = \begin{cases} \frac{[a - (\alpha - \beta)(s + c) - \sqrt{2(\alpha - \beta)\pi_b^-}] \sqrt{2(\alpha - \beta)\pi_b^-}}{\alpha - \beta} & \text{if } \pi_b^- \in [\Pi^c/4, \Pi^c] \\ \frac{[a - (\alpha - \beta)(s + c)]^2}{4(\alpha - \beta)} = \frac{\Pi^c}{2} & \text{if } \pi_b^- \in [0, \Pi^c/4] \end{cases}, \quad (4.37)$$

$$\pi_b^{s1-2s} = \max \left\{ \pi_b^-, \frac{[a - (\alpha - \beta)(s + c)]^2}{8(\alpha - \beta)} \right\}, \quad \text{and} \quad (4.38)$$

$$\Pi^{s1-2s} = \begin{cases} \frac{[a - (\alpha - \beta)(s + c) - \sqrt{(\alpha - \beta)\pi_b^-/2}] \sqrt{2(\alpha - \beta)\pi_b^-}}{\alpha - \beta} & \text{if } \pi_b^- \in [\Pi^c/4, \Pi^c] \\ \frac{3[a - (\alpha - \beta)(s + c)]^2}{8(\alpha - \beta)} & \text{if } \pi_b^- \in [0, \Pi^c/4] \end{cases} \quad (4.39)$$

As noted previously,  $(Ps1 - 2s)$  is solved by Yang and Zhou (2006) by assuming  $\pi_b^- = 0$  and  $c = 0$  leading to the optimal wholesale price in (4.35), retail price in (4.36), resulting profits in (4.37), (4.38), and (4.39) with  $\pi_b^- = 0$  and  $c = 0$  (see the expressions in Table 1 on p. 110 of Yang and Zhou (2006)).

#### 4.2.4 Generic contract $b3$

Under  $b3$ , the buyer announces that  $p_i$  would be set depending on  $w_i$  according to

$$p_i = k_i w_i + m_i, \quad (4.40)$$

where  $k_i \in \Re$  is the unconstrained multiplier and  $m_i \in \Re$  is the unconstrained value representing a margin (mark-up) or rebate (mark-down),  $i = 1, 2$ . Then, the buyer moves first and decides  $k_i$  and  $m_i$  and the supplier selects the optimal  $w_i$ ,  $i = 1, 2$ .

In **the single-product setting**, we show that if the unconstrained multiplier for the single product is non-positive, i.e.,  $k \leq 0$ , then the buyer does not make any

profit. Hence, we restrict our attention to the case  $k > 0$  under  $b3$ . It is natural to argue the same result in the symmetric/assymmetric two-product setting as well as the symmetric  $n$ -product setting. Hence, we focus on  $k_i > 0$  under  $b3$ ,  $i = 1, 2$ .

Substituting (4.40) in assumption (4.8), we have

$$s + c \leq w_i + c \leq p_i = k_i w_i + m_i \leq \frac{a}{\alpha - \beta},$$

$i = 1, 2$ . Since  $w_i \geq s$  and  $k_i > 0$ ,

$$k_i s + m_i \leq k_i w_i + m_i,$$

$i = 1, 2$ . Hence, using the above inequalities, we conclude that  $k_i$ ,  $m_i$ ,  $w_i$ , and  $p_i$  should be such that

$$k_i s + m_i \leq \frac{a}{\alpha - \beta}, \quad (4.41)$$

$$s \leq w_i \leq \frac{a}{\alpha - \beta} - c, \quad \text{and} \quad (4.42)$$

$$s + c \leq p_i \leq \frac{a}{\alpha - \beta}, \quad (4.43)$$

$i = 1, 2$ .

Let us examine  $\pi_s$  in (4.4) under  $b3$ . For given  $k_i > 0$  and  $m_i \in \Re$ ,  $i = 1, 2$ , using (4.40),  $\pi_s$  in (4.4) can be rewritten as

$$\pi_s = \sum_{i,j=1,2,i \neq j} (w_i - s)[a - \alpha k_i w_i + \beta k_j w_j - (\alpha m_i - \beta m_j)]. \quad (4.44)$$

Note that

$$\frac{\partial \pi_s}{\partial w_i} = a - 2\alpha k_i w_i + \beta(k_i + k_j)w_j + (\alpha - \beta)k_i s - (\alpha m_i - \beta m_j), \quad (4.45)$$

$i, j = 1, 2$  and  $j \neq i$ , and the Hessian matrix of  $\pi_s$  in (4.44) is given by

$$\begin{bmatrix} \frac{\partial^2 \pi_s}{\partial w_1^2} & \frac{\partial^2 \pi_s}{\partial w_1 \partial w_2} \\ \frac{\partial^2 \pi_s}{\partial w_2 \partial w_1} & \frac{\partial^2 \pi_s}{\partial w_2^2} \end{bmatrix} = \begin{bmatrix} -2\alpha k_1 & \beta(k_1 + k_2) \\ \beta(k_1 + k_2) & -2\alpha k_2 \end{bmatrix}.$$

The determinant of the Hessian matrix is given by

$$4\alpha^2 k_1 k_2 - \beta^2 (k_1 + k_2)^2.$$

Here, we momentarily assume this quantity is greater than zero, and, hence, the Hessian matrix is negative-definite. Later, we will show that under the optimal contract the buyer would set  $k_1 = k_2$ , so that determinant of the Hessian matrix reduces to

$$4(\alpha^2 - \beta^2)k_1^2 > 0,$$

since  $k_1 > 0$  by definition and (4.3).

Setting  $\partial \pi_s / \partial w_i = 0$  for  $i = 1, 2$  in (4.45) leads to

$$\begin{aligned} w_i^{b3-2s}(\cdot) &= [(k_i m_i + k_j m_i - k_j^2 s - k_j k_i s)\beta^2 + (k_j m_j - k_i m_j - k_i k_j s + k_j^2 s)\alpha\beta \\ &\quad + (2k_i k_j s - 2m_i k_j)\alpha^2 + (k_i + k_j)a\beta + 2ak_j\alpha] \cdot \\ &\quad \frac{1}{4k_i k_j \alpha^2 - (k_i + k_j)^2 \beta^2}, \end{aligned} \tag{4.46}$$

where  $w_i^{b3-2s}(\cdot)$  is a simplified notation for  $w_i^{b3-2s}(k_1, m_1, k_2, m_2)$ ,  $i, j = 1, 2$  and  $j \neq i$ .

Let us verify if  $w_i^{b3-2s}(\cdot)$  is realizable over the region (4.42),  $i = 1, 2$ . We know from the development of (4.42) that this verification is equivalent to ensuring that  $\pi_s$  in (4.44) is **nonnegative** for  $w_i = w_i^{b3-2s}(\cdot)$ ,  $i = 1, 2$ . Since  $w_i = s$ ,  $i = 1, 2$ , is obviously a feasible solution under which  $\pi_s = 0$ . Therefore, using the definition of

$w_i^{b3-2s}$ ,  $\pi_s \geq 0$  is true for  $w_i = w_i^{b3-2s}(\cdot)$ ,  $i = 1, 2$ . Then,  $w_i^{b3-2s}$  defined in (4.46) is **the supplier's optimal response, i.e., the optimal wholesale price for product  $i$ , under  $b3$**  because it is realizable over the region (4.42). Substituting (4.46) in (4.40) and using (4.2), **the corresponding retail price and order quantity for product  $i$  for given values of  $k_i > 0$  and  $m_i \in \Re$  that satisfy (4.41)** are given by

$$\begin{aligned} p_i^{b3-2s}(\cdot) = & k_i[(k_i m_i + k_j m_i - k_j^2 s - k_j k_i s)\beta^2 + (k_j m_j - k_i m_j - k_i k_j s + k_j^2 s)\alpha\beta \\ & + (2k_i k_j s - 2m_i k_j)\alpha^2 + (k_i + k_j)a\beta + 2ak_j\alpha]/[4k_i k_j \alpha^2 - (k_i + k_j)^2 \beta^2] \\ & + m_i \quad \text{and} \end{aligned} \quad (4.47)$$

$$q_i^{b3-2s}(\cdot) = a - \alpha p_i^{b3-2s}(\cdot) + \beta p_j^{b3-2s}(\cdot), \quad (4.48)$$

$i, j = 1, 2$  and  $j \neq i$ .

As noted in the single-product setting, (4.41), (4.42), and (4.43) are not sufficient to guarantee  $\pi_b \geq 0$  in (4.5). In order to guarantee  $\pi_b \geq 0$ , recalling assumption (4.8) and using (4.46) and (4.47), we need to ensure

$$w_i^{b3-2s}(\cdot) + c \leq p_i^{b3-2s}(\cdot), \quad (4.49)$$

$i = 1, 2$ . Hence, we refer to (4.41) and (4.49) as the main constraints for the problem at hand.

Using (4.4) and (4.5), we have

$$\begin{aligned} \pi_s &= \sum_{i=1,2} [w_i^{b3-2s}(\cdot) - s] q_i^{b3-2s}(\cdot) \quad \text{and} \\ \pi_b &= \sum_{i=1,2} [p_i^{b3-2s}(\cdot) - w_i^{b3-2s}(\cdot) - c] q_i^{b3-2s}(\cdot), \end{aligned}$$

where  $w_i^{b3-2s}(\cdot)$ ,  $p_i^{b3-2s}(\cdot)$ , and  $q_i^{b3-2s}(\cdot)$  are given by (4.46), (4.47), and (4.48),  $i = 1, 2$ .



Finally, considering (4.41), (4.49), the above expressions for  $\pi_s$  and  $\pi_b$ , and the constraint  $k_i > 0$ , **the buyer's optimization problem under b3 for the symmetric two-product setting** can be stated as

$$\begin{aligned}
(Pb3 - 2s) : \max_{k_i > 0, m_i \in \mathbb{R}} \pi_b &= \sum_{i=1,2} [p_i^{b3-2s}(\cdot) - w_i^{b3-2s}(\cdot) - c] q_i^{b3-2s}(\cdot) \quad (4.50) \\
s.t. \quad p_i^{b3-2s}(\cdot) - w_i^{b3-2s}(\cdot) - c &\geq 0 \\
sk_i + m_i &\leq \frac{a}{\alpha - \beta} \\
\pi_s = \sum_{i=1,2} [w_i^{b3-2s}(\cdot) - s] q_i^{b3-2s}(\cdot) &\geq \pi_s^-, \quad (4.51)
\end{aligned}$$

where  $w_i^{b3-2s}(\cdot)$ ,  $p_i^{b3-2s}(\cdot)$ , and  $q_i^{b3-2s}(\cdot)$  are given by (4.46), (4.47), and (4.48), respectively,  $i = 1, 2$ .

Clearly,  $(Pb3 - 2s)$  makes sense only for reasonable values of  $\pi_s^-$ . That is, a natural upper bound on  $\pi_s^-$  is given by

$$0 \leq \pi_s^- \leq \Pi^c = \frac{[a - (\alpha - \beta)(s + c)]^2}{2(\alpha - \beta)}, \quad (4.52)$$

where  $\Pi^c$  is the optimal centralized system profit in (4.13).

Next, we identify the optimal solution to  $(Pb3 - 2s)$  given by (4.50) and (4.51). To this end, we establish an upper bound on the objective function given by (4.50) and develop a feasible solution such that the objective function value of this solution achieves the upper bound. Hence, the feasible solution at hand is also optimal.

Using (4.50) and (4.51), the system profit under b3 is given by  $\Pi = \pi_s + \pi_b$ . Obviously,  $\Pi \leq \Pi^c$  always holds true, where  $\Pi^c$  is the optimal centralized system profit given in (4.13). Therefore, under b3, we have

$$\pi_b = \Pi - \pi_s \leq \Pi^c - \pi_s^-,$$

so that the best profit the buyer can achieve under  $b3$  is given by  $\Pi^c - \pi_s^-$ .

Next, we provide a feasible solution of  $k_i$  and  $m_i$  such that  $\pi_b = \Pi^c - \pi_s$  is true,  $i = 1, 2$ . To this end, inspired by the optimal contract under  $b3$  in the single-product setting, we consider the tuple  $(k_i^{b3-2s}, m_i^{b3-2s})$  such that

$$k_i^{b3-2s} = \frac{\Pi^c}{\pi_s^-} \quad \text{and} \quad (4.53)$$

$$m_i^{b3-2s} = s + c - \frac{s\Pi^c}{\pi_s^-}, \quad (4.54)$$

$i = 1, 2$ .

Next, we need to show that the tuple is a feasible solution that satisfies the main constraints (4.41) and (4.49). We know from the development of  $(Pb3-2s)$  that this verification is equivalent to ensuring that  $\pi_b$  in (4.50) is nonnegative for  $k_i = k_i^{b3-2s}$  and  $m_i = m_i^{b3-2s}$ ,  $i = 1, 2$ . In order to calculate  $\pi_b$ , first, substituting (4.53) and (4.54) in (4.46), (4.47) and (4.48), the corresponding wholesale price, retail price, and order quantity are given by

$$\begin{aligned} w_i^{b3-2s} &= \frac{[a - (\alpha - \beta)(s + c)] \pi_s^-}{2(\alpha - \beta)\Pi^c} + s, \\ p_i^{b3-2s} &= \frac{a + (\alpha - \beta)(s + c)}{2(\alpha - \beta)}, \quad \text{and} \\ q_i^{b3-2s} &= \frac{a - (\alpha - \beta)(s + c)}{2}, \end{aligned}$$

$i = 1, 2$ . Then, using (4.50) and (4.51), we have

$$\begin{aligned} \pi_s^{b3-2s} &= \pi_s^-, \\ \pi_b^{b3-2s} &= \frac{[a - (\alpha - \beta)(s + c)]^2}{2(\alpha - \beta)} - \pi_s^- = \Pi^c - \pi_s^-, \quad \text{and} \\ \Pi^{b3-2s} &= \pi_s^{b3-2s} + \pi_b^{b3-2s} = \frac{[a - (\alpha - \beta)(s + c)]^2}{2(\alpha - \beta)} = \Pi^c. \end{aligned} \quad (4.55)$$

Using (4.55), it is easy to verify that  $\pi_b^{b3-2s} \geq 0$  since  $\pi_s^- \leq \Pi^c$  in (4.52). Hence the tuple  $(k_i^{b3-2s}, m_i^{b3-2s})$  defined in (4.53) and (4.54),  $i = 1, 2$ , is a feasible solution, under which the upper bound of the objective function in (4.50) is achieved, i.e.,  $\pi_b = \Pi^c - \pi_s^-$ . Hence, it is also optimal.

It is important to note that the optimal contract parameters given by (4.53) and (4.54) for the two products are the same, i.e., free of the index  $i$ ,  $i = 1, 2$ . It is because the two products are symmetric and it does not make sense to make different decisions for them.

### 4.3 Modeling the case of symmetric $n$ products

Now that we have illustrated specific approaches simplifying the symmetric two-product problems  $(Ps1-2s)$  and  $(Pb3-2s)$  in Section 4.2, we utilize the approaches to derive the symmetric  $n$ -product problems  $(Ps1-ns)$  and  $(Pb3-ns)$ ,  $n \geq 2$ .

In the symmetric  $n$ -product setting, we assume the order quantity for product  $i$  is given by

$$q_i = a - \alpha p_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} p_j, \quad (4.56)$$

$i = 1, \dots, n$  (Bresnahan and Reiss (1985)). Note that when  $n = 2$ , the demand function in (4.56) is reduced to the demand function in (4.2) in the symmetric two-product setting.

#### *Profit functions:*

In the symmetric  $n$ -product setting, the supplier's profit function is given by

$$\pi_s = \sum_{i=1, \dots, n} (w_i - s)q_i = \sum_{i=1, \dots, n} (w_i - s) \left( a - \alpha p_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} p_j \right). \quad (4.57)$$

The buyer's profit function is given by

$$\begin{aligned}\pi_b &= \sum_{i=1, \dots, n} (p_i - w_i - c)q_i \\ &= \sum_{i=1, \dots, n} (p_i - w_i - c) \left( a - \alpha p_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} p_j \right).\end{aligned}\quad (4.58)$$

Following the symmetric two-product setting, we also assume  $\alpha > \beta \geq 0$ .

#### 4.3.1 Wholesale price contract s1

Under s1, the supplier decides  $w_i$ ,  $i = 1, \dots, n$ , first and then the buyer decides  $p_i$ ,  $i = 1, \dots, n$ . Similar to the symmetric two-product setting, we are only interested in  $w_i$  and  $p_i$  values that satisfy

$$s \leq w_i \leq \frac{a}{\alpha - \beta} - c \quad \text{and} \quad (4.59)$$

$$w_i + c \leq p_i \leq \frac{a}{\alpha - \beta}, \quad (4.60)$$

$i = 1, \dots, n$ . We refer to (4.59) as the main constraint on the decision variables of the contract design problem under s1 in the symmetric  $n$ -product setting.

##### 4.3.1.1 Formulation of (Ps1 - ns)

For a given  $w_i$ ,  $i = 1, \dots, n$ , such that (4.59) is true, using (4.58), we have

$$\begin{aligned}\frac{\partial \pi_b}{\partial p_i} &= a - 2\alpha p_i + \frac{2\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} p_j \\ &\quad + \alpha w_i - \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} w_j + (\alpha - \beta)c,\end{aligned}\quad (4.61)$$

$i, j = 1, \dots, n$  and  $j \neq i$ , and the Hessian matrix of  $\pi_b$  in (4.58) is given by

$$\begin{bmatrix} \frac{\partial^2 \pi_b}{\partial p_1^2} & \cdots & \frac{\partial^2 \pi_b}{\partial p_1 \partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi_b}{\partial p_n \partial p_1} & \cdots & \frac{\partial^2 \pi_b}{\partial p_n^2} \end{bmatrix} = \begin{bmatrix} -2\alpha & \cdots & 2\beta \\ \vdots & \ddots & \vdots \\ 2\beta & \cdots & -2\alpha \end{bmatrix}.$$

The determinant of the Hessian matrix is given by  $(2\alpha)^n - (2\beta)^n > 0$  by assumption. Hence,  $\pi_b$  in (4.58) is negative-definite. Setting  $\partial \pi_b / \partial p_i = 0$  in (4.61) for  $i = 1, \dots, n$  leads to

$$p_i^{s1-ns}(w_i) = \frac{a + (\alpha - \beta)(w_i + c)}{2(\alpha - \beta)}. \quad (4.62)$$

Observe that for any  $w_i$  such that (4.59) is true,  $p_i^{s1-ns}(w_i)$  defined in (4.62) is **the buyer's optimal response, i.e., the optimal retail price** for product  $i$ ,  $i = 1, \dots, n$ . This is because  $w_i$  satisfies (4.59) so that

$$\begin{aligned} \frac{a}{\alpha - \beta} - p_i^{s1-ns}(w_i) &= \frac{a - (\alpha - \beta)(w_i + c)}{2(\alpha - \beta)} \geq 0 \quad \text{and} \\ p_i^{s1-ns}(w_i) - (w_i + c) &= \frac{a - (\alpha - \beta)(w_i + c)}{2(\alpha - \beta)} \geq 0, \end{aligned}$$

$i = 1, \dots, n$ . Hence,  $p_i^{s1-ns}(w_i)$  in (4.62) is realizable over the region (4.60),  $i = 1, \dots, n$ . Using (4.56), **the buyer's optimal order quantity** for product  $i$  is given by

$$q_i^{s1-ns}(w_1, \dots, w_n) = \frac{a - \alpha w_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} w_j - (\alpha - \beta)c}{2}, \quad (4.63)$$

$i, j = 1, \dots, n$  and  $j \neq i$ .

Using (4.62) in (4.57) and (4.58), we have

$$\begin{aligned}\pi_s &= \sum_{i=1, \dots, n} \frac{(w_i - s)[a - \alpha w_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} w_j - (\alpha - \beta)c]}{2} \quad \text{and} \\ \pi_b &= \sum_{i=1, \dots, n} \frac{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} w_j - (\alpha - \beta)c]}{4(\alpha - \beta)}.\end{aligned}$$

Considering the main constraint (4.59) and the buyer's reservation profit  $\pi_b^-$  and using the two above expressions for  $\pi_s$  and  $\pi_b$ , **the supplier's optimization problem under s1** in the symmetric  $n$ -product setting can be stated as

(Ps1 - ns) :

$$\begin{aligned}\max_{s \leq w_i \leq a/(\alpha - \beta) - c, i=1, \dots, n} \\ \pi_s &= \sum_{i=1, \dots, n} \frac{(w_i - s)[a - \alpha w_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} w_j - (\alpha - \beta)c]}{2}\end{aligned}\tag{4.64}$$

$$\begin{aligned}s.t. \quad \pi_b &= \sum_{i=1, \dots, n} \frac{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} w_j - (\alpha - \beta)c]}{4(\alpha - \beta)} \\ &\geq \pi_b^-.\end{aligned}\tag{4.65}$$

Clearly, (Ps1 - ns) makes sense only for reasonable values of  $\pi_b^-$ . That is, a natural upper bound on  $\pi_b^-$  is given by

$$0 \leq \pi_b^- \leq \Pi^{c-n}\tag{4.66}$$

where  $\Pi^{c-n}$  is the centralized optimal system profit, which can be calculated by

$$\begin{aligned}\Pi^{c-n} &= \max_{p_i \in [s+c, a/(\alpha-\beta)], i=1, \dots, n} \Pi \\ &= \sum_{i=1, \dots, n} (p_i - s - c) \left( a - \alpha p_i + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} p_j \right).\end{aligned}\quad (4.67)$$

#### 4.3.1.2 Optimal solution of $(Ps1 - ns)$

Using (4.64), we have

$$\pi_s = \frac{1}{2} \sum_{i=1, \dots, n} \left\{ (w_i - s)[a - \alpha w_i - (\alpha - \beta)c] - \beta s w_i + \frac{\beta}{n-1} w_i \sum_{j=1, \dots, n, j \neq i} w_j \right\}.\quad (4.68)$$

Note that

$$2w_i w_j \leq w_i^2 + w_j^2 \quad (4.69)$$

for  $w_i, w_j \in \mathfrak{R}$ , where the equality holds true if  $w_i = w_j$ ,  $i, j = 1, \dots, n$ ,  $i \neq j$ . Using (4.68), we have

$$\begin{aligned}&\pi_s \\ &\leq \frac{1}{2} \sum_{i=1, \dots, n} \left\{ (w_i - s)[a - \alpha w_i - (\alpha - \beta)c] - \beta s w_i + \frac{\beta}{2} w_i^2 + \frac{\beta}{2(n-1)} \sum_{j=1, \dots, n, j \neq i} w_j^2 \right\} \\ &= \sum_{i=1, \dots, n} \frac{(w_i - s)[a - (\alpha - \beta)(w_i + c)]}{2} = \sum_{i=1, \dots, n} f(w_i),\end{aligned}\quad (4.70)$$

where the equality holds true if  $w_1 = \dots = w_n$  and

$$f(x) = \frac{(x - s)[a - (\alpha - \beta)(x + c)]}{2}.\quad (4.71)$$

Using (4.65) and (4.69), we have

$$\begin{aligned}
& \pi_b \\
&= \frac{1}{4(\alpha - \beta)} \sum_{i=1, \dots, n} \{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i - (\alpha - \beta)c] + [a - (\alpha - \beta)c]\beta w_i\} \\
&\quad - \sum_{i=1, \dots, n} \frac{(\alpha - \beta)\beta w_i \sum_{j=1, \dots, n, j \neq i} w_j}{4(n-1)(\alpha - \beta)} \\
&\geq \frac{1}{4(\alpha - \beta)} \sum_{i=1, \dots, n} \{[a - (\alpha - \beta)(w_i + c)][a - \alpha w_i - (\alpha - \beta)c] + [a - (\alpha - \beta)c]\beta w_i\} \\
&\quad - \sum_{i=1, \dots, n} \frac{(\alpha - \beta)\beta w_i^2}{4(\alpha - \beta)} \\
&= \sum_{i=1, \dots, n} \frac{[a - (\alpha - \beta)(w_i + c)]^2}{4(\alpha - \beta)} = \sum_{i=1, \dots, n} g(w_i), \tag{4.72}
\end{aligned}$$

where the equality holds true if  $w_1 = \dots = w_n$  and

$$g(x) = \frac{[a - (\alpha - \beta)(x + c)]^2}{4(\alpha - \beta)}. \tag{4.73}$$

Clearly, if

$$g(w_i) \geq \pi_b^- / n$$

for  $i = 1, \dots, n$  then using (4.72), we have

$$\pi_b \geq \sum_{i=1, \dots, n} g(w_i) \geq \pi_b^-.$$

Recalling (4.70), we have

$$\pi_s \leq \sum_{i=1, \dots, n} f(w_i) \leq \max_{w_1, \dots, w_n} \sum_{i=1, \dots, n} f(w_i) = \sum_{i=1, \dots, n} \max_{w_i} f(w_i), \tag{4.74}$$

where  $\pi_s = \sum_{i=1, \dots, n} f(w_i)$  if  $w_1 = \dots = w_n$ . Hence, the upper bound of  $\pi_s$  in (4.64)



when  $\pi_b \geq \pi_b^-$  is satisfied is given by

$$\begin{aligned} & \sum_{i=1, \dots, n} \max_{w_i} f(w_i) \\ \text{s.t. } & g(w_i) \geq \frac{\pi_b^-}{n}, \quad i = 1, \dots, n. \end{aligned}$$

Let  $w_i^*$  denote the optimal solution to the above optimization problem,  $i = 1, \dots, n$ . Since this problem is separable based on  $w_i$ , we have  $w_i^* = w^*$ ,  $i = 1, \dots, n$ , where  $w^*$  is the optimal solution to the following optimization problem

$$\max_w f(w) = \frac{(w-s)[a - (\alpha - \beta)(w+c)]}{2} \quad (4.75)$$

$$\text{s.t. } g(w) = \frac{[a - (\alpha - \beta)(w+c)]^2}{4(\alpha - \beta)} \geq \frac{\pi_b^-}{n}, \quad (4.76)$$

using (4.71) and (4.73). It is important to note that this optimization problem is the same as (Ps1) in **the single-product setting**, when  $b = \alpha - \beta$  and the buyer's reservation profit is  $\pi_b^-/n$ .

Since  $w_i = w^*$ ,  $i = 1, \dots, n$ , using (4.74),  $\pi_s = nf(w^*) = \sum_{i=1, \dots, n} \max_{w_i} f(w_i) = n \max_w f(w)$ . In this case,  $\pi_s$  achieves its upper bound. Hence, **the optimal whole-sale price for product  $i$  under  $s1$**  in the symmetric  $n$ -product setting is given by

$$w_i^{s1-ns} = w^*, \quad (4.77)$$

$i = 1, \dots, n$ , where  $w^*$  is the optimal solution to (Ps1) in the single-product setting with  $b = \alpha - \beta$  and the buyer's reservation profit is  $\pi_b^-/n$ .

#### 4.3.2 Generic contract b3

Since the symmetric  $n$ -product setting is a generalization of the symmetric two-product setting, the procedure of derivation for the optimal b3 should be the same.

One of the main differences in the derivation is to extend the ranges of indices  $i, j = 1, 2$  and  $j \neq i$  to  $i, j = 1, \dots, n$  and  $j \neq i$ . To avoid repetition, we summarize the main results without presenting the details.

*The expression of  $\pi_s$ :*

For given  $k_i > 0$  and  $m_i \in \mathfrak{R}$ ,  $i = 1, \dots, n$ , using (4.40), the supplier's profit  $\pi_s$  can be written as

$$\pi_s = \sum_{i=1, \dots, n} (w_i - s) \left[ a - \alpha(k_i w_i + m_i) + \frac{\beta}{n-1} \sum_{j=1, \dots, n, j \neq i} (k_j w_j + m_j) \right]. \quad (4.78)$$

*The buyer's optimization problem:*

**The buyer's optimization problem under  $b3$  for the symmetric  $n$ -product setting** can be stated as

$$\begin{aligned} (Pb3 - ns) : \max_{k_i > 0, m_i \in \mathfrak{R}} \pi_b &= \sum_{i=1, \dots, n} [p_i^{b3-ns}(\cdot) - w_i^{b3-ns}(\cdot) - c] q_i^{b3-ns}(\cdot) \quad (4.79) \\ s.t. \quad p_i^{b3-ns}(\cdot) - w_i^{b3-ns}(\cdot) - c &\geq 0, \\ s_i k_i + m_i &\leq \frac{a}{\alpha - \beta}, \\ \pi_s = \sum_{i=1, \dots, n} [w_i^{b3-ns}(\cdot) - s] q_i^{b3-ns}(\cdot) &\geq \pi_s^-, \quad (4.80) \end{aligned}$$

where  $w_i^{b3-ns}(\cdot)$ ,  $p_i^{b3-ns}(\cdot)$ , and  $q_i^{b3-ns}(\cdot)$  are **the supplier's optimal response, i.e., the optimal wholesale price and the corresponding retail price and order quantity** for product  $i$ ,  $i = 1, \dots, n$  and they are parameterized on  $k_i$  and  $m_i$ ,  $i = 1, \dots, n$ . In particular,  $w_i^{b3-ns}(\cdot)$  can be computed by setting  $\partial \pi_s / \partial w_i = 0$  for  $i = 1, \dots, n$  using (4.78), and  $p_i^{b3-ns}(\cdot)$  and  $q_i^{b3-ns}(\cdot)$  can be then computed using (4.40) and (4.56).

*The optimal b3:*

Inspired by the symmetric two-product setting and using the same approach, we can show that **the optimal contract parameters under b3 in the symmetric  $n$ -product setting** is such that

$$k_i^{b3-ns} = \frac{\Pi^{c-n}}{\pi_s^-} \quad \text{and} \quad (4.81)$$

$$m_i^{b3-ns} = s + c - \frac{s\Pi^{c-n}}{\pi_s^-} \quad (4.82)$$

for  $i = 1, \dots, n$ . Note that  $\Pi^{c-n}$  is the centralized optimal system profit defined in (4.67).

Then, substituting (4.81) and (4.82) in (4.79) and (4.80), we have

$$\begin{aligned} \pi_s^{b3-ns} &= \pi_s^-, \\ \pi_b^{b3-ns} &= \Pi^{c-n} - \pi_s^-, \quad \text{and} \\ \Pi^{b3-ns} &= \Pi^{c-n}. \end{aligned}$$

Note that in the symmetric  $n$ -product setting, the optimal contracts parameters are also free of the index  $i$  for  $i = 1, \dots, n$ . Once the centralized optimal system profit  $\Pi^{c-n}$  and the supplier's reservation profit  $\pi_s^-$  are computed, the optimal contract parameters can be decided using (4.81) and (4.82).

#### 4.4 Relation between the multi-product and single-product contractual settings

In Section (4.3.1), we show that in the symmetric  $n$ -product setting,  $n \geq 2$ , the optimal wholesale prices for  $n$  products under  $s1$  are the same as shown in (4.77). The optimal wholesale price for each product can be solved using the optimization problem given by (4.75) and (4.76). Recalling ( $Ps1$ ) (the supplier's problem under

s1) in the single-product setting, we find that the optimization problem given by (4.75) and (4.76) is the same as (*Ps1*) when

- $b = \alpha - \beta$ , where  $b$  is the price sensitivity parameter in the demand function  $q = a - bp$  in the single-product setting, and
- The buyer's reservation profit in the single-product setting is given by  $\pi_b^-/n$ , where  $\pi_b^-$  is the buyer's reservation profit in the symmetric  $n$ -product setting.

Therefore, instead of solving a symmetric  $n$ -product problem, we can solve the corresponding single-product problem and then the optimal decisions and resulting profits for the former one can be easily obtained.

This fact is due to the symmetry of decisions for different products in the symmetric  $n$ -product setting. Letting  $w_i = w$ ,  $p_i = p$  and  $q_i = q$ ,  $i = 1, \dots, n$ , the demand function in (4.56) and the supplier's and buyer's profits in (4.57) and (4.58) in the symmetric  $n$ -product setting can be reduced to

$$\begin{aligned} q &= a - (\alpha - \beta)p, \\ \pi_s &= n(w - s)[a - (\alpha - \beta)p], \quad \text{and} \\ \pi_b &= n(p - w - c)[a - (\alpha - \beta)p]. \end{aligned}$$

From the above equations, we can see that the demand function in (4.56) becomes the single-product demand function, and both entities' profits are  $n$  times their profit on one product. Hence, the symmetric  $n$ -product problem becomes the single-product problem.

Note that the demand function and the entities' profits are only dependent on the product substitution given by  $\alpha - \beta$  and they are independent of individual values of  $\alpha$  and  $\beta$ . Recall that  $\alpha$  represents the price sensitivity of demand,  $\beta$  represents

the price sensitivity of demand to a substitutable product in (4.56). Therefore, in the symmetric  $n$ -product setting, we should focus on the difference of products' price sensitivities rather than the price sensitivity for each product individually.

In sum, using the results in the single-product setting under  $s1$ , we can solve a symmetric  $n$ -product problem with the demand function in (4.56) and the buyer's reservation profit  $\pi_b^-$  in the following way:

- Consider a single-product setting with the demand function  $q = a - bp$  and the buyer's reservation profit  $\pi_b^{-0}$ . Set  $b = \alpha - \beta$  and set  $\pi_b^{-0} = \pi_b^-/n$ ,  $n \geq 2$ .
- Derive the optimal wholesale price under  $s1$  in the single-product setting and calculating the resulting profits for the supplier and the buyer.
- Then, in the symmetric  $n$ -product setting, the optimal wholesale price for each product is the same as that in the single-product setting, while the supplier (buyer)'s profit is  $n$  times that in the single-product setting,  $n \geq 2$ .

In Section (4.3.2), we also show that the optimal decisions under  $b3$  are same for  $n$  products in the symmetric  $n$ -product setting. Hence, the symmetric  $n$ -product problem under  $b3$  can be also reduced to the single-product problem, and, hence, the existing results in the single-product setting can be utilized in the symmetric  $n$ -product setting under  $b3$ .

#### 4.5 Modeling the case of asymmetric two products

We next examine  $b3$  in the asymmetric two-product setting. The procedure of derivation for the optimal  $b3$  is the same as that in the symmetric two-product setting discussed previously. To avoid repetition, we summarize the main steps and the main results in asymmetric two-product setting.

*Profit functions:*

In the asymmetric two-product setting, the supplier's profit function is given by

$$\pi_s = \sum_{i=1,2} (w_i - s_i)q_i = \sum_{i,j=1,2,i \neq j} (w_i - s_i)(a_i - \alpha_i p_i + \beta_i p_j). \quad (4.83)$$

The buyer's profit function is given by

$$\pi_b = \sum_{i=1,2} (p_i - w_i - c_i)q_i = \sum_{i,j=1,2,j \neq i} (p_i - w_i - c_i)(a_i - \alpha_i p_i + \beta_i p_j). \quad (4.84)$$

We assume  $\alpha_i$  and  $\beta_i$  are such that

$$\alpha_i > \beta_i \geq 0 \quad \text{and} \quad 4\alpha_1\alpha_2 - (\beta_1 + \beta_2)^2 > 0, \quad (4.85)$$

$i = 1, 2$ . The first inequality in (4.85) is assumed based on the nature of price competition, i.e., a product's demand impacted by its own price more heavily than by its competitor's price. The second inequality ensures a property of  $\pi_s$  as we demonstrate next.

*The expression of  $\pi_s$ :*

For given  $k_i > 0$  and  $m_i \in \Re$ ,  $i = 1, 2$ , using (4.40),  $\pi_s$  in (4.83) can be rewritten as

$$\pi_s = \sum_{i,j=1,2,i \neq j} (w_i - s_i)[a_i - \alpha_i k_i w_i + \beta_i k_j w_j - (\alpha_i m_i - \beta_i m_j)]. \quad (4.86)$$

*Compute the supplier's optimal response:*

Note that the Hessian matrix of  $\pi_s$  in (4.86) is given by

$$\begin{bmatrix} \frac{\partial^2 \pi_s}{\partial w_1^2} & \frac{\partial^2 \pi_s}{\partial w_1 \partial w_2} \\ \frac{\partial^2 \pi_s}{\partial w_2 \partial w_1} & \frac{\partial^2 \pi_s}{\partial w_2^2} \end{bmatrix} = \begin{bmatrix} -2\alpha_1 k_1 & \beta_1 k_2 + \beta_2 k_1 \\ \beta_2 k_1 + \beta_1 k_2 & -2\alpha_2 k_2 \end{bmatrix}.$$

The determinant of the Hessian matrix is given by

$$4\alpha_1\alpha_2 k_1 k_2 - (\beta_1 k_2 + \beta_2 k_1)^2.$$

Here, we momentarily assume this quantity is greater than zero, and, hence, the Hessian matrix is negative-definite. Later, we will show that under the optimal contract the buyer would set  $k_1 = k_2$ , so that the determinant above reduces to

$$[4\alpha_1\alpha_2 - (\beta_1 + \beta_2)^2] k_1^2 > 0,$$

since  $k_1 > 0$  by definition and (4.85).

Setting  $\partial\pi_s/\partial w_i = 0$  for  $i = 1, 2$ , we obtain **the supplier's optimal optimal response, i.e., the optimal wholesale price for product  $i$  under  $b3$** , denoted by

$$w_i^{b3-2a}(\cdot). \tag{4.87}$$

Using (4.40) and (4.1), **the corresponding retail price and order quantity for product  $i$**  are given by

$$p_i^{b3-2a}(\cdot) = k_i w_i^{b3-2a}(\cdot) + m_i \quad \text{and} \tag{4.88}$$

$$q_i^{b3-2a}(\cdot) = a_i - \alpha_i p_i^{b3-2a}(\cdot) + \beta_i p_j^{b3-2a}(\cdot), \tag{4.89}$$

$i, j = 1, 2$  and  $j \neq i$ .

*The buyer's optimization problem:*

Note that using (4.1),  $p_i$  can be written as a function of order quantities such that

$$p_i = \frac{a_i\alpha_j + a_j\beta_i}{\alpha_1\alpha_2 - \beta_1\beta_2} - \frac{\alpha_j q_i}{\alpha_1\alpha_2 - \beta_1\beta_2} - \frac{\beta_i q_j}{\alpha_1\alpha_2 - \beta_1\beta_2}, \quad (4.90)$$

$i, j = 1, 2$  and  $j \neq i$ .

Using (4.83), (4.84), and (4.90), in order to guarantee  $q_i \geq 0$  and the supplier's and buyer's profits on product  $i$  are **nonnegative**, we assume  $p_i \leq (a_i\alpha_j + a_j\beta_i)/(\alpha_1\alpha_2 - \beta_1\beta_2)$ ,  $w_i \geq s_i$ , and  $p_i \geq w_i + c_i$  so that

$$s_i k_i + m_i \leq k_i w_i + m_i = p_i \leq \frac{a_i\alpha_j + a_j\beta_i}{\alpha_1\alpha_2 - \beta_1\beta_2} \text{ and } p_i^{b3-2a}(\cdot) - w_i^{b3-2a}(\cdot) - c_i \geq 0, \quad (4.91)$$

$i = 1, 2$ . We refer to (4.91) as the main constraint on the decision variables of the contract design problem under  $b3$  in the asymmetric two-product setting.

Then, **the buyer's optimization problem under  $b3$  for the asymmetric two-product setting** can be stated as

$$\begin{aligned} (Pb3 - 2a) : \max_{k_i > 0, m_i \in \mathbb{R}} \pi_b &= \sum_{i=1,2} [p_i^{b3-2a}(\cdot) - w_i^{b3-2a}(\cdot) - c_i] q_i^{b3-2a}(\cdot) \quad (4.92) \\ s.t. \quad p_i^{b3-2a}(\cdot) - w_i^{b3-2a}(\cdot) - c_i &\geq 0 \\ s_i k_i + m_i &\leq \frac{a_i\alpha_j + a_j\beta_i}{\alpha_1\alpha_2 - \beta_1\beta_2} \\ \pi_s &= \sum_{i=1,2} [w_i^{b3-2a}(\cdot) - s_i] q_i^{b3-2a}(\cdot) \geq \pi_s^-, \quad (4.93) \end{aligned}$$

where  $w_i^{b3-2a}(\cdot)$ ,  $p_i^{b3-2a}(\cdot)$ , and  $q_i^{b3-2a}(\cdot)$  are given by (4.87), (4.88), and (4.89),  $i =$



1, 2.

*The optimal b3:*

We apply the same approach as that in the symmetric two-product setting to identify the optimal contract parameters in the asymmetric two-product setting. Inspired by the optimal contract in the symmetric two-product setting, we consider the tuple  $(k_i^{b3-2a}, m_i^{b3-2a})$  such that

$$k_i^{b3-2a} = \frac{\Pi^{c-2a}}{\pi_s^-} \quad \text{and} \quad (4.94)$$

$$m_i^{b3-2a} = s_i + c_i - \frac{s_i \Pi^{c-2a}}{\pi_s^-}, \quad (4.95)$$

$i = 1, 2$ . Note that  $\Pi^{c-2a}$  is the centralized optimal system profit, which can be calculated by

$$\Pi^{c-2a} = \max_{p_i \in [s_i + c_i, \frac{a_i \alpha_j + a_j \beta_i}{\alpha_1 \alpha_2 - \beta_1 \beta_2}], i=1,2} \Pi = \sum_{i,j=1,2, j \neq i} (p_i - s_i - c_i)(a_i - \alpha_i p_i + \beta_i p_j).$$

Then, substituting (4.94) and (4.95) in (4.92) and (4.93), we have

$$\begin{aligned} \pi_s^{b3-2a} &= \pi_s^-, \\ \pi_b^{b3-2a} &= \Pi^{c-2a} - \pi_s^-, \quad \text{and} \\ \Pi^{b3-2a} &= \pi_s^{b3-2a} + \pi_b^{b3-2a} = \Pi^{c-2a}. \end{aligned} \quad (4.96)$$

We have  $\pi_b^{b3-2a} \geq 0$  in (4.96) for  $\pi_s^- \in [0, \Pi^{c-2a}]$ . Hence, the tuple given by (4.94) and (4.95) is a feasible solution under which the buyer obtains the upper bound of the objective function in (4.92). Therefore, the tuple is also optimal.

It is important to note that in the asymmetric two-product setting, we have

$k_1^{b3-2a} = k_2^{b3-2a}$  while  $m_1^{b3-2a} \neq m_2^{b3-2a}$  is possible using (4.94) and (4.95). Although one may expect that the asymmetric two-product problem is difficult to solve, as we can see, the optimal contract parameters under  $b3$  are easy to calculate. Since the transaction between the supplier and the buyer is only based on the wholesale payment, the contract is also easy to implement.

#### 4.6 Conclusion

In this chapter, we generalize the basic single-product setting to consider multiple symmetric and asymmetric substitutable products. Since there is no previous work on the buyer-driven channel in this setting, we focus on examine the buyer-driven generic contract  $b3$  in the two-product symmetric case, n-product symmetric case, and two-product asymmetric case. Our results document the conditions under which the generic contract remains to be a simple, yet, effective contract when multiple substitutable products are considered.

## 5. THE GENERIC CONTRACT IN THE EXCLUSIVE DEALER SETTING

### 5.1 Setting 3. The exclusive dealer contractual setting

This chapter revisits the two-product channel with two suppliers and two buyers, where each supplier (e.g., manufacturer) produces one product and each buyer (e.g., dealer) sells one supplier's product exclusively. This channel, referred as the **exclusive dealer setting** (Choi (1996)) here, is illustrated in Figure 5.1. Supplier  $i$ 's decision pertains to the wholesale price  $w_i$ , and buyer  $i$ 's decisions pertain to the order quantity  $q_i$  and retail price  $p_i$  for product  $i$ ,  $i = 1, 2$ . We assume a *generalized linear demand function* for product  $i$  given by

$$q_i = a_i - \alpha_i p_i + \beta_i p_j, \quad (5.1)$$

$i, j = 1, 2$ , and  $i \neq j$ . As in Chapter 4,  $a_i$  presents the maximum demand of product  $i$  when prices for both products approach zero (McGuire and Staelin (1983)), and  $\alpha_i$  and  $\beta_i$  represent the sensitivity of demand for product  $i$  to its retail price and to the substitutable product's retail price (Ingene and Parry (2004)), respectively,  $i = 1, 2$ . Let  $s_i$  and  $c_i$  denote supplier  $i$ 's unit production cost and buyer  $i$ 's unit distribution cost, respectively,  $i = 1, 2$ . The notation introduced so far and used frequently in the remainder of this chapter is summarized in Table 5.1. Note that we consider *the generalized asymmetric case*. It is the case where all parameters in (5.1) and costs of different products are different, i.e.,  $a_1 \neq a_2$ ,  $\alpha_1 \neq \alpha_2$ ,  $\beta_1 \neq \beta_2$ ,  $s_1 \neq s_2$ , and  $c_1 \neq c_2$ .

### 5.2 Supplier- and buyer-driven contracts

In Chapter 3, we have studied three buyer-driven contracts and a supplier-driven contract in the basic single-product setting. Now, we are interested in the same

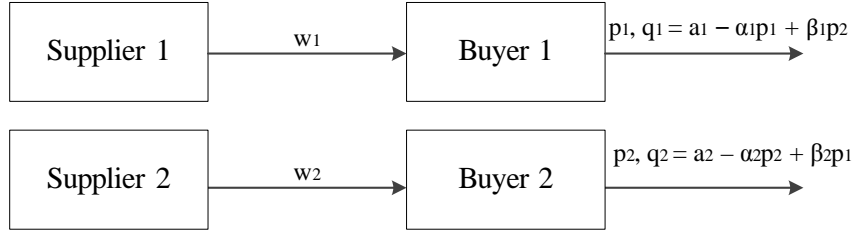


Figure 5.1: The exclusive dealer contractual setting.

Table 5.1: Summary of notation in the exclusive dealer setting.

Symbol	Explanation
$i$	the index of a product, $i = 1, 2$
$p_i$	the retail price for product $i$
$q_i$	the demand for product $i$ , $q_i = a_i - \alpha_i p_i + \beta_i p_j$ , $j = 1, 2, j \neq i$
$\alpha_i$	the sensitivity of $q_i$ to $p_i$
$\beta_i$	the sensitivity of $q_i$ to $p_j$ , $j = 1, 2, j \neq i$
$w_i$	the wholesale price for product $i$
$s_i$	supplier $i$ 's unit production cost
$c_i$	buyer $i$ 's unit distribution cost
$k_i$	the price multiplier for product $i$ , $k_i > 0$
$m_i$	the price margin for product $i$ , $m_i \in \Re$

contracts in the exclusive dealer setting as follows:

- b1. Under *the margin-only contract*, buyer  $i$  decides the price margin of product  $i$ , denoted by  $m_i$ ,  $m_i \geq 0$ , representing the difference between the retail and wholesale prices of product  $i$ ,  $i = 1, 2$ . Also, buyer  $i$  commits that the retail price of product  $i$  would be set such that  $p_i = w_i + m_i$  and its order quantity would be set such that  $q_i = a - \alpha p_i + \beta p_j = a - \alpha(w_i + m_i) + \beta(w_j + m_j)$ ,  $i, j = 1, 2$  and  $j \neq i$ . Next, supplier  $i$  decides  $w_i$ ,  $i = 1, 2$ .
- b2. Under *the multiplier-only contract*, buyer  $i$  decides the price multiplier of product  $i$ , denoted by  $k_i$ ,  $k_i \geq 1$ , representing the ratio of the retail and wholesale

prices of product  $i$ ,  $i = 1, 2$ . Also, buyer  $i$  commits that the retail price of product  $i$  would be set such that  $p_i = k_i w_i$  and its order quantity would be set such that  $q_i = a - \alpha p_i + \beta p_j = a - \alpha k_i w_i + \beta k_j w_j$ ,  $i, j = 1, 2$  and  $j \neq i$ . Next, supplier  $i$  decides  $w_i$ ,  $i = 1, 2$ .

- b3. Under *the generic contract*, buyer  $i$  decides on the values<sup>a</sup> of  $k_i$  and  $m_i$ ,  $k_i > 0$  and  $m_i \in \mathfrak{R}$ , while also committing that the retail price of product  $i$  would be set such that  $p_i = k_i w_i + m_i$  and its order quantity would be set such that  $q_i = a - \alpha p_i + \beta p_j = a - \alpha(k_i w_i + m_i) + \beta(k_j w_j + m_j)$ ,  $i, j = 1, 2$  and  $j \neq i$ . Next, supplier  $i$  decides  $w_i$ ,  $i = 1, 2$ .
- s1. Under *the wholesale price contract with Cournot competition*, supplier  $i$  decides  $w_i$  and then buyer  $i$  decides  $q_i$ ,  $i = 1, 2$ .

We are interested in computing the optimal contract parameters under these contracts. To this end, we develop optimization models and examine the contracts in the buyer-driven channel using the following approach:

- (Vertical Stackelberg game) the buyer tier takes the lead and offers a contract first and then the supplier tier follows. See the definition of Stackelberg game in Chapter 1.
- (Horizontal Nash game) the two entities in a tier have the same negotiation power and make decisions simultaneously. See the discussion of Horizontal Nash game given by Ingene and Parry (2004).

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<sup>a</sup>Observe that, under b3,  $m_i$  can be positive or negative representing a margin (mark-up) or rebate (mark-down). Likewise, under b3,  $k_i$  is allowed to be positive or negative for the sake of generality. However, in the basic single-product setting in Chapter 3, we already show that the optimal multiplier is positive due to the natural and practical assumptions of the contractual setting at hand. Hence, we only consider  $k_i > 0$  in the exclusive dealer setting.

The exclusive dealer setting of interest has appeared in the literature (e.g., McGuire and Staelin (1983), Lee and Staelin (1997), Trivedi (1998), Zhang et al. (2012), Li et al. (2013), and Feng and Lu (2013)). Of particular interest for us are the results presented by Lee and Staelin (1997), Trivedi (1998), and Zhang et al. (2012) who examine the same approach as ours in the buyer-driven channel.

Lee and Staelin (1997) consider a buyer-driven contract under which the buyers decide the price margins (the difference of  $p_i$  and  $w_i$ ), i.e.,  $p_i - w_i$ , simultaneously, first, and then the suppliers decide the manufacture margins (the difference of  $w_i$  and  $s_i$ ), i.e.,  $w_i - s_i$ , simultaneously,  $i = 1, 2$ . Lee and Staelin (1997) assume a symmetric demand function between the two products satisfying that “demand for a product is decreasing in its own retail price and increasing in or independent of other retail prices” (Lee and Staelin (1997)). They also assume symmetric costs for the suppliers and ignore the buyers’ costs, i.e.,  $s_1 = s_2$  and  $c_1 = c_2 = 0$ .

Lee and Staelin (1997) compare the buyer-driven contract with a supplier-driven contract under which the suppliers decide the manufacture margins first and then the buyers decide the price margins. They demonstrate that each entity is better off to possess leadership. They also show that suppliers’ and buyers’ profits are symmetric under different leaderships, i.e., the leaders (followers)’ profits are the same under both leaderships. In addition, the retail prices and the system profits under different leaderships are the same, i.e., the system efficiency is independent of whether the suppliers or the buyers play as channel leaders.

Trivedi (1998) considers a buyer-driven contract under which the buyers decide the price margins, i.e.,  $p_i - w_i$ , simultaneously, first, and then the suppliers decide  $w_i$  simultaneously,  $i = 1, 2$ . The demand function in the paper is a special case of (5.1) where  $a_1 = a_2 = 1$  and  $\alpha_1 = \alpha_2 = 1$ , and all costs are ignored, i.e.,  $s_1 = s_2 = c_1 = c_2 = 0$ . Trivedi (1998) compares the buyer-driven contract with

a supplier-driven contract under which the suppliers decide wholesale prices first and then the buyers decide the price margins. Trivedi (1998) also conclude that leadership is always beneficial.

Zhang et al. (2012) generalize Trivedi (1998) by using a linear demand function with one asymmetric parameter. The demand function is still a special case of (5.1) where  $a_i = (A_i - \theta A_j)/(1 - \theta^2)$ ,  $\alpha_1 = \alpha_2 = 1/(1 - \theta^2)$ , and  $\beta_1 = \beta_2 = \theta/(1 - \theta^2)$ ,  $A_i$ ,  $A_j$ , and  $\theta$  are constants defined in Zhang et al. (2012),  $i, j = 1, 2$  and  $j \neq i$ . They derive the optimal solutions for the supplier- and buyer-driven contracts assuming the asymmetric parameter  $A_i$ ,  $i = 1, 2$ , and compare the two contracts under the symmetric case when  $A_1 = A_2$ . Similar to Trivedi (1998), Zhang et al. (2012) also show that leadership is always beneficial. They also demonstrate that which leadership results in more system profit depends on the product substitutability.

Note that all the three papers that use the same approach as ours consider a buyer-driven contract under which the buyers announce the price margins first and then the suppliers decide  $w_i$ , or equivalently,  $w_i - s_i$ ,  $i = 1, 2$ . This contract is referred to as the *margin-only* contract here. However, the current literature does not concern *whether deciding on margins is optimal* for the buyers and generally overlooks *other buyer-driven contracts*. Also, they either consider *the fully symmetric case* or have *symmetric assumptions*. Moreover, they do not take into account reservation profits for entities. To extend the current literature, in this chapter, we study a more *general* buyer-driven contract (i.e., the generic contract) in *the generalized asymmetric case* based on the following consideration:

- A general contract with more contract flexibility may improve the profit potential for the buyers, and
- The consideration of the generalized asymmetric case provides an extension of

the symmetric case that is commonly assumed in the current literature.

In the following sections, we will first present entities' profit functions, formulate each buyer's optimization problem under the generic contract, and then solve for the optimal decision in the generalized asymmetric case.

### 5.3 Profit functions

In this section, we present entities' profit functions and generate conditions that ensure nonnegative quantities and profits. In the exclusive dealer setting, using (5.1), supplier  $i$ 's profit function is given by

$$\pi_{s_i} = (w_i - s_i)q_i, \quad (5.2)$$

and buyer  $i$ 's profit function is given by

$$\pi_{b_i} = (p_i - w_i - c_i)q_i, \quad (5.3)$$

$i = 1, 2$ . Using (5.1), the inverse demand function is given by

$$p_i = \frac{a_i\alpha_j + a_j\beta_i - \alpha_jq_i - \beta_iq_j}{\alpha_1\alpha_2 - \beta_1\beta_2},$$

$i, j = 1, 2$  and  $j \neq i$ . In order to guarantee  $q_i, \pi_{s_i}, \pi_{b_i} \geq 0$ , we assume  $p_i \leq (a_i\alpha_j + a_j\beta_i - \beta_iq_j)/(\alpha_1\alpha_2 - \beta_1\beta_2)$ ,  $w_i \geq s_i$ , and  $p_i \geq w_i + c_i$  so that

$$s_i + c_i \leq w_i + c_i \leq p_i \leq \frac{a_i\alpha_j + a_j\beta_i - \beta_iq_j}{\alpha_1\alpha_2 - \beta_1\beta_2}, \quad (5.4)$$



$i, j = 1, 2$  and  $j \neq i$ . We assume when retail prices for both products are set at their lower bounds, i.e.,  $p_1 = s_1 + c_1$  and  $p_2 = s_2 + c_2$ , we have  $q_1, q_2 \geq 0$  so that

$$a_i - \alpha_i(s_i + c_i) + \beta_i(s_j + c_j) \geq 0 \quad (5.5)$$

using (5.1),  $i, j = 1, 2$  and  $j \neq i$ . Following Chapter 4, we assume  $\alpha_i > \beta_i \geq 0$ ,  $i = 1, 2$ .

#### 5.4 The generic contract in the generalized asymmetric case

Recall that under *b3*, buyer  $i$  announces that  $p_i$  would be set depending on  $w_i$  according to

$$p_i = k_i w_i + m_i, \quad (5.6)$$

where  $k_i > 0$  and  $m_i \in \mathfrak{R}$  is the unconstrained value representing a margin (mark-up) or rebate (mark-down),  $i = 1, 2$ . Then, the buyers move first and decide  $k_i$  and  $m_i$ , and the suppliers select optimal  $w_i$ ,  $i = 1, 2$ . Next, we proceed with a formal formulation of each buyer's optimization problem given the other buyer's decision under *b3*.

##### 5.4.1 Formulation of (*Pb3* – *ai*)

In the sequential leader-follower game, the buyers make decisions by predicting the suppliers' best responses. In order to formulate each buyer's optimization problem, we need to examine the suppliers' best responses on  $w_1$  and  $w_2$  for given  $k_1, k_2 > 0$  and  $m_1, m_2 \in \mathfrak{R}$ .

For given  $k_1, k_2 > 0$  and  $m_1, m_2 \in \mathfrak{R}$ , supplier  $i$  decides the optimal  $w_i$  corresponding to  $w_j$  and then  $p_i$  and  $q_i$  are determined by  $w_i$ ,  $i, j = 1, 2$  and  $j \neq i$ . We first need to identify the conditions on  $w_i$ ,  $p_i$ , and  $q_i$  that ensure nonnegative profit for supplier  $i$  for given  $w_j$ ,  $k_1, k_2 > 0$  and  $m_1, m_2 \in \mathfrak{R}$ ,  $i, j = 1, 2$  and  $j \neq i$ . After

obtaining the suppliers' best responses and the corresponding  $p_i$  and  $q_i$ , we need to verify the conditions are satisfied and then identify the constraint on  $k_1, k_2, m_1$ , and  $m_2$  for the verification.

Then, considering the suppliers' best responses, buyer  $i$  decides the optimal  $k_i$  and  $m_i$ , for given  $k_j > 0$  and  $m_j \in \mathfrak{R}$ ,  $i, j = 1, 2$  and  $j \neq i$ . To formulate buyer  $i$ 's problem, we also need to identify the constraint on  $k_i$  and  $m_i$  for given  $k_j > 0$  and  $m_j \in \mathfrak{R}$  that ensure nonnegative profit for buyer  $i$ ,  $i, j = 1, 2$  and  $j \neq i$ .

Following this logic, now, we start with identifying the conditions on  $w_i, p_i$ , and  $q_i$  for given  $w_j, k_1, k_2 > 0$  and  $m_1, m_2 \in \mathfrak{R}$ ,  $i, j = 1, 2$  and  $j \neq i$ . Substituting (5.6) in (5.1), we have

$$q_i = a_i - \alpha_i(k_i w_i + m_i) + \beta_i(k_j w_j + m_j) \geq 0,$$

and, hence,

$$k_i w_i + m_i \leq \frac{a_i + \beta_i(k_j w_j + m_j)}{\alpha_i}, \quad (5.7)$$

$i, j = 1, 2$  and  $j \neq i$ .

Since  $k_i > 0$ , by assumption (5.4), (5.7) implies

$$s_i k_i + m_i \leq w_i k_i + m_i \leq \frac{a_i + \beta_i(k_j w_j + m_j)}{\alpha_i}, \quad (5.8)$$

$i, j = 1, 2$  and  $j \neq i$ . Using (5.8) along with (5.7), we then conclude that  $k_i, m_i, w_i$ ,

and  $p_i$  should be such that

$$s_i k_i + m_i \leq \frac{a_i + \beta_i(k_j w_j + m_j)}{\alpha_i}, \quad (5.9)$$

$$s_i \leq w_i \leq \frac{a_i + \beta_i(k_j w_j + m_j)}{\alpha_i k_i} - \frac{m_i}{k_i}, \quad \text{and} \quad (5.10)$$

$$s_i k_i + m_i \leq p_i \leq \frac{a_i + \beta_i(k_j w_j + m_j)}{\alpha_i}, \quad (5.11)$$

$i, j = 1, 2$  and  $j \neq i$ . Note that if (5.11) is true then (5.9) is true. Also, recalling (5.1) and assumption (5.4), we have

$$0 \leq q_i \leq a_i - \alpha_i(s_i + c_i) + \beta_i(k_j w_j + m_j), \quad (5.12)$$

$i, j = 1, 2$  and  $j \neq i$ .

Next, we derive the suppliers' best responses. Let us examine  $\pi_{s_i}$  in (5.2) under b3,  $i = 1, 2$ . For given values of  $k_1, k_2 > 0$  and  $m_1, m_2 \in \Re$ , using (5.1) and (5.6),  $\pi_{s_i}$  in (5.2) can be rewritten as

$$\pi_{s_i} = (w_i - s_i)q_i = (w_i - s_i)[a_i - \alpha_i k_i w_i + \beta_i k_j w_j - (\alpha_i m_i - \beta_i m_j)], \quad (5.13)$$

so that

$$\begin{aligned} \frac{d\pi_{s_i}}{dw_i} &= a_i - 2\alpha_i k_i w_i + \beta_i k_j w_j - (\alpha_i m_i - \beta_i m_j) + k_i \alpha_i s_i \quad \text{and} \\ \frac{d^2 \pi_{s_i}}{dw_i^2} &= -2\alpha_i k_i < 0, \end{aligned} \quad (5.14)$$

$i, j = 1, 2$  and  $j \neq i$ . Hence,  $\pi_{s_i}$  in (5.13) is concave in  $w_i$ ,  $i = 1, 2$ . Setting

$d\pi_{s_1}/dw_1 = d\pi_{s_2}/dw_2 = 0$  in (5.14) leads to

$$w_i^{b3}(\cdot) = \frac{2a_i\alpha_j + a_j\beta_i + 2\alpha_1\alpha_2(s_ik_i - m_i) + \beta_1\beta_2m_i + \alpha_j\beta_i(s_jk_j + m_j)}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i}, \quad (5.15)$$

where  $w_i^{b3}(\cdot)$  which is parameterized on  $k_1, k_2, m_1$ , and  $m_2$  is a simplified notation for  $w_i^{b3}(k_1, k_2, m_1, m_2)$ ,  $i, j = 1, 2$  and  $j \neq i$ .

Now, we verify whether  $w_i^{b3}(\cdot)$  is realizable over the region (5.10). Observe that for given  $k_1, k_2 > 0$ ,  $m_1, m_2 \in \mathfrak{R}$ , we have

$$\frac{a_i + \beta_i(k_j w_j^{b3}(\cdot) + m_j)}{\alpha_i k_i} - \frac{m_i}{k_i} - w_i^{b3}(\cdot) = w_i^{b3}(\cdot) - s_i = \frac{W_i}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i},$$

where

$$W_i = 2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_ik_i + m_i) + \alpha_j\beta_i(s_jk_j + m_j), \quad (5.16)$$

$i, j = 1, 2$  and  $j \neq i$ . Hence, if

$$W_i \geq 0 \quad (5.17)$$

then  $w_i^{b3}(\cdot)$  defined in (5.15) is **supplier  $i$ 's optimal response, i.e., the optimal wholesale price, under  $b3$** , for given values of  $k_1, k_2 > 0$  and  $m_1, m_2 \in \mathfrak{R}$ , because it is realizable over the region (5.10),  $i, j = 1, 2$  and  $j \neq i$ . Substituting (5.15) in (5.6) and using (5.1), **the corresponding retail price and order quantity for**

product  $i$  for given values of  $k_1, k_2 > 0$  and  $m_1, m_2 \in \Re$  are given by

$$\begin{aligned} p_i^{b3}(\cdot) &= k_i w_i^{b3}(\cdot) + m_i \\ &= \frac{2a_i\alpha_j + a_j\beta_i + \alpha_j\beta_i(s_j k_j + m_j) + 2\alpha_1\alpha_2(s_i k_i + m_i)}{4\alpha_1\alpha_2 - \beta_1\beta_2} \quad \text{and} \end{aligned} \quad (5.18)$$

$$\begin{aligned} q_i^{b3}(\cdot) &= a_i - \alpha_i[k_i w_i^{b3}(\cdot) + m_i] + \beta_i[k_j w_j^{b3}(\cdot) + m_j] \\ &= \frac{\alpha_i W_i}{4\alpha_1\alpha_2 - \beta_1\beta_2}, \end{aligned} \quad (5.19)$$

respectively, where  $p_i^{b3}(\cdot)$  and  $q_i^{b3}(\cdot)$  are also parameterized on  $k_1, k_2, m_1$ , and  $m_2$ , and  $W_i$  is given by (5.16),  $i, j = 1, 2$  and  $j \neq i$ . Again, since we assume (5.17),

$$\begin{aligned} \frac{a_i + \beta_i(k_j w_j^{b3}(\cdot) + m_j)}{\alpha_i} - p_i^{b3}(\cdot) &= \frac{W_i}{4\alpha_1\alpha_2 - \beta_1\beta_2} \geq 0 \quad \text{and} \\ p_i^{b3}(\cdot) - s_i k_i - m_i &= \frac{W_i}{4\alpha_1\alpha_2 - \beta_1\beta_2} \geq 0, \end{aligned}$$

so that  $p_i^{b3}(\cdot)$  defined in (5.18) lies over the region (5.11), where  $W_i$  is defined in (5.16),  $i, j = 1, 2$  and  $j \neq i$ . Likewise,

$$a_i - \alpha_i(s_i + c_i) + \beta_i(k_j w_j^{b3}(\cdot) + m_j) - q_i^{b3}(\cdot) = \frac{\alpha_i W_i}{4\alpha_1\alpha_2 - \beta_1\beta_2} \geq 0,$$

so that  $q_i^{b3}(\cdot)$  defined in (5.19) lies over the region (5.12),  $i, j = 1, 2$  and  $j \neq i$ .

Last but not least, we need to verify that assumption (5.4) holds true for  $w_i^{b3}(\cdot)$ , where  $w_i^{b3}(\cdot)$  is as defined in (5.15),  $i = 1, 2$ . Similar to the generic contract in the single-product setting, the constraint (5.17) derived earlier is not sufficient to verify this under  $b3$ . For this reason, recalling assumption (5.4) and using (5.6) and (5.15), we need to ensure

$$s_i + c_i \leq w_i^{b3}(\cdot) + c_i \leq k_i w_i^{b3}(\cdot) + m_i \leq \frac{a_i\alpha_j + a_j\beta_i - \beta_i q_j}{\alpha_1\alpha_2 - \beta_1\beta_2} \quad (5.20)$$

holds true. Examining the above inequalities, if (5.17) is true and

$$(k_i - 1)w_i^{b3}(\cdot) + m_i - c_i \geq 0$$

then (5.20) is ensured,  $i = 1, 2$ . Obviously, the inequality above is equivalent to

$$\frac{(k_i - 1)[2a_i\alpha_j + a_j\beta_i + 2\alpha_1\alpha_2(s_ik_i - m_i) + \beta_1\beta_2m_i + \alpha_j\beta_i(s_jk_j + m_j)]}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i} + m_i - c_i \geq 0, \quad (5.21)$$

$i, j = 1, 2$  and  $j \neq i$ . Hence, we refer to (5.17) and (5.21) as **the main constraints** of the problem at hand.

Using (5.15), (5.18), and (5.19) in (5.2) and (5.3), we have

$$\begin{aligned} \pi_{s_i} &= \frac{\alpha_i[2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_ik_i + m_i) + \alpha_j\beta_i(s_jk_j + m_j)]^2}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i} \quad \text{and} \\ \pi_{b_i} &= \frac{\alpha_i[2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_ik_i + m_i) + \alpha_j\beta_i(s_jk_j + m_j)]}{4\alpha_1\alpha_2 - \beta_1\beta_2} \\ &\quad \cdot \left\{ \frac{(k_i - 1)[2a_i\alpha_j + a_j\beta_i + 2\alpha_1\alpha_2(s_ik_i - m_i) + \beta_1\beta_2m_i + \alpha_j\beta_i(s_jk_j + m_j)]}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i} \right. \\ &\quad \left. + m_i - c_i \right\}, \end{aligned}$$

$i, j = 1, 2$  and  $j \neq i$ .

Although  $k_i > 0$  by definition, in order to apply Karush-Kuhn-Tucker (KKT) conditions (Bazaraa et al. (2006)), under which a strict inequality constraint is not allowed, to solve buyer  $i$ 's optimization problem as shown below, we momentarily assume  $k_i \in \Re$ ,  $i = 1, 2$ . We solve the optimal decision over  $k_i, m_i \in \Re$  and then show that the optimal  $k_i$  is actually positive,  $i = 1, 2$ .

Considering the fact that  $b3$  is a buyer-driven contract, supplier  $i$  would not accept  $b3$  unless supplier  $i$ 's corresponding profit exceeds the reservation profit  $\pi_{s_i}^-$ ,

$i = 1, 2$ . Recalling (5.17) and (5.21) and considering the above expressions for  $\pi_{s_i}$  and  $\pi_{b_i}$  and  $k_i, m_i \in \mathfrak{R}$ , **buyer  $i$ 's optimization problem under  $b3$**  can be stated as

$$\begin{aligned}
& (Pb3 - ai) : \\
& \max_{k_i, m_i \in \mathfrak{R}} \pi_{b_i} \\
& = \frac{\alpha_i[2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_ik_i + m_i) + \alpha_j\beta_i(s_jk_j + m_j)]}{4\alpha_1\alpha_2 - \beta_1\beta_2} \\
& \quad \cdot \left\{ \frac{(k_i - 1)[2a_i\alpha_j + a_j\beta_i + 2\alpha_1\alpha_2(s_ik_i - m_i) + \beta_1\beta_2m_i + \alpha_j\beta_i(s_jk_j + m_j)]}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i} \right. \\
& \quad \left. + m_i - c_i \right\} \tag{5.22} \\
& s.t., \quad 2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_ik_i + m_i) + \alpha_j\beta_i(s_jk_j + m_j) \geq 0, \\
& \quad \frac{(k_i - 1)[2a_i\alpha_j + a_j\beta_i + 2\alpha_1\alpha_2(s_ik_i - m_i) + \beta_1\beta_2m_i + \alpha_j\beta_i(s_jk_j + m_j)]}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i} \\
& \quad + m_i - c_i \geq 0, \\
& \quad \pi_{s_i} = \frac{\alpha_i[2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_ik_i + m_i) + \alpha_j\beta_i(s_jk_j + m_j)]^2}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i} \\
& \quad \geq \pi_{s_i}^-, \tag{5.23}
\end{aligned}$$

given  $k_j, m_j \in \mathfrak{R}$ ,  $i, j = 1, 2$  and  $j \neq i$ .

As discussed earlier, under  $b3$  buyer 1 and buyer 2 make decisions simultaneously, and, hence, their optimal decisions lead to a Nash equilibrium (Fudenberg and Tirole (1991)). That is, each buyer does not have an incentive to deviate if the other buyer's decision is fixed. In the following, we will characterize the Nash equilibrium in three steps:

- First, for given  $k_j, m_j \in \mathfrak{R}$ , we use KKT necessary conditions to identify a KKT point (see the definition of KKT point in Bazaraa et al. (2006)) for  $(Pb3 - ai)$ ,  $i, j = 1, 2$  and  $j \neq i$ . The reason for doing this is the following: According to

Lemma 4.4.1 of Bazaraa et al. (2006), if the Hessian of the Lagrangian function conditioned on Lagrangian multipliers associated with a KKT point is negative semi-definite in the feasible region, then the KKT point is a global maximum.

- Second, using Lemma 4.4.1 of Bazaraa et al. (2006), we show that the KKT point is optimal for  $(Pb3 - ai)$  given  $k_j, m_j \in \mathfrak{R}$ ,  $i, j = 1, 2$  and  $j \neq i$ .
- Finally, using the optimal response of each buyer corresponding to the other buyer's decision, we identify the Nash equilibrium of the two buyers' optimization problems.

#### 5.4.2 Optimal solution of $(Pb3 - ai)$

We start with identifying a KKT point for  $(Pb3 - ai)$ ,  $i = 1, 2$ . Let  $\mu_{i1}$ ,  $\mu_{i2}$ , and  $\mu_{i3}$  denote Lagrangian multipliers corresponding to the three constraints in  $(Pb3 - ai)$ , respectively,  $i = 1, 2$ . The Lagrangian function is given by

$$\mathcal{L}_i = \mathcal{L}_i(k_i, m_i, \mu_{i1}, \mu_{i2}, \mu_{i3}) = \pi_{b_i} + \mu_{i1}g_{i1} + \mu_{i2}g_{i2} + \mu_{i3}g_{i3}, \quad (5.24)$$

where

$$g_{i1} = 2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_ik_i + m_i) + \alpha_j\beta_i(s_jk_j + m_j), \quad (5.25)$$

$$g_{i2} = \frac{(k_i - 1)[2a_i\alpha_j + a_j\beta_i + 2\alpha_1\alpha_2(s_ik_i - m_i) + \beta_1\beta_2m_i + \alpha_j\beta_i(s_jk_j + m_j)]}{(4\alpha_1\alpha_2 - \beta_1\beta_2)k_i} + m_i - c_i, \quad \text{and} \quad (5.26)$$

$$g_{i3} = \pi_{s_i} - \pi_{s_i}^-, \quad (5.27)$$

and  $\pi_{b_i}$  and  $\pi_{s_i}$  are given by (5.22) and (5.23), respectively,  $i, j = 1, 2$  and  $j \neq i$ . Thus, the KKT necessary conditions (see Theorem 4.2.13 in Bazaraa et al. (2006))



are the following:

$$\frac{\partial \mathcal{L}_i}{\partial k_i} = \frac{\partial \pi_{b_i}}{\partial k_i} + \mu_{i1} \frac{\partial g_{i1}}{\partial k_i} + \mu_{i2} \frac{\partial g_{i2}}{\partial k_i} + \mu_{i3} \frac{\partial g_{i3}}{\partial k_i} = 0, \quad (5.28)$$

$$\frac{\partial \mathcal{L}_i}{\partial m_i} = \frac{\partial \pi_{b_i}}{\partial m_i} + \mu_{i1} \frac{\partial g_{i1}}{\partial m_i} + \mu_{i2} \frac{\partial g_{i2}}{\partial m_i} + \mu_{i3} \frac{\partial g_{i3}}{\partial m_i} = 0, \quad (5.29)$$

$$\frac{\partial \mathcal{L}_i}{\partial \mu_{i1}} = g_{i1} \geq 0, \quad (5.30)$$

$$\frac{\partial \mathcal{L}_i}{\partial \mu_{i2}} = g_{i2} \geq 0, \quad (5.31)$$

$$\frac{\partial \mathcal{L}_i}{\partial \mu_{i3}} = g_{i3} \geq 0, \quad (5.32)$$

$$\mu_{i1} g_{i1} = 0, \quad (5.33)$$

$$\mu_{i2} g_{i2} = 0, \quad (5.34)$$

$$\mu_{i3} g_{i3} = 0, \quad (5.35)$$

$$\mu_{i1}, \mu_{i2}, \mu_{i3} \geq 0, \quad i = 1, 2. \quad (5.36)$$

Then, we characterize the Lagrangian multipliers that satisfy (5.28) – (5.36) at optimality, denoted by  $\mu_{i1}^*$ ,  $\mu_{i2}^*$ , and  $\mu_{i3}^*$ ,  $i = 1, 2$ . We first show  $\mu_{i1}^* = \mu_{i2}^* = 0$  is true,  $i = 1, 2$ . Consider all possible cases of  $\mu_{i1}^*$  and  $\mu_{i2}^*$ ,  $i = 1, 2$ :

1. If  $\mu_{i1}^* > 0$  and  $\mu_{i2}^* = 0$  then by (5.33)  $g_{i1} = 0$ . Using (5.25) in (5.22) and (5.23),  $g_{i1} = 0$  implies  $\pi_{b_i} = \pi_{s_i} = 0$ . If  $\pi_{s_i}^- > 0$  then there does not exist a feasible solution. If  $\pi_{s_i}^- = 0$  then obviously the solution for the KKT conditions is not optimal.
2. If  $\mu_{i1}^* = 0$  and  $\mu_{i2}^* > 0$  then by (5.34)  $g_{i2} = 0$ . Using (5.26) in (5.22),  $g_{i2} = 0$  implies  $\pi_{b_i} = 0$ . Obviously, the solution for the KKT conditions is not optimal.
3. If  $\mu_{i1}^* > 0$  and  $\mu_{i2}^* > 0$  then by (5.33) and (5.34)  $g_{i1} = g_{i2} = 0$ . According to

the development of (5.25) and (5.26),

$$g_{i1} = 0 \Leftrightarrow w_i^{b3}(\cdot) = s_i \quad \text{and} \quad g_{i2} = 0 \Leftrightarrow (k_i - 1)w_i^{b3}(\cdot) + m_i - c_i = 0.$$

Hence,

$$g_{i1} = g_{i2} = 0 \Leftrightarrow (k_i - 1)s_i + m_i - c_i = 0 \Leftrightarrow k_i s_i + m_i = s_i + c_i.$$

Using  $k_i s_i + m_i = s_i + c_i$  in (5.25),  $g_{i1} = 0$  is equivalent to

$$A_i = 2a_i\alpha_j + a_j\beta_i - (2\alpha_1\alpha_2 - \beta_1\beta_2)(s_i + c_i) + \alpha_j\beta_i(s_j k_j + m_j) = 0, \quad (5.37)$$

$j = 1, 2, j \neq i$ .

- If  $A_i = 0$  then the solution should satisfy  $k_i s_i + m_i = s_i + c_i$ . Note that  $g_{i1} = g_{i2} = 0$  implies  $\pi_{b_i} = \pi_{s_i} = 0$ . If  $\pi_{s_i}^- > 0$  then there does not exist a feasible solution. If  $\pi_{s_i}^- = 0$  then obviously the solution for the KKT conditions is not optimal.
- Otherwise if  $A_i \neq 0$  then there does not exist a solution for  $g_{i1} = g_{i2} = 0$ .

Therefore, the three cases above are ruled out, and at optimality we have  $\mu_{i1}^* = \mu_{i2}^* = 0$ ,  $i = 1, 2$ . Next, we analyze the solution of (5.28) – (5.36) assuming  $\mu_{i1}^* = \mu_{i2}^* = 0$  for two different cases:

Case 0:  $\mu_{i1}^* = \mu_{i2}^* = 0$  and  $\mu_{i3}^* > 0$ ;

Case 1:  $\mu_{i1}^* = \mu_{i2}^* = 0$  and  $\mu_{i3}^* = 0$ ,  $i = 1, 2$ .

Case 0.

If  $\mu_{i1}^* = \mu_{i2}^* = 0$  and  $\mu_{i3}^* > 0$  then (5.28) and (5.29) reduce to

$$\frac{\partial \mathcal{L}_i}{\partial k_i} = \frac{\partial \pi_{b_i}}{\partial k_i} + \mu_{i3}^* \frac{\partial g_{i3}}{\partial k_i} = \frac{\partial \pi_{b_i}}{\partial k_i} + \mu_{i3}^* \frac{\partial \pi_{s_i}}{\partial k_i} = 0, \quad (5.38)$$

$$\frac{\partial \mathcal{L}_i}{\partial m_i} = \frac{\partial \pi_{b_i}}{\partial m_i} + \mu_{i3}^* \frac{\partial g_{i3}}{\partial m_i} = \frac{\partial \pi_{b_i}}{\partial m_i} + \mu_{i3}^* \frac{\partial \pi_{s_i}}{\partial m_i} = 0, \quad (5.39)$$

and using (5.27), (5.35) implies

$$\pi_{s_i} = \pi_{s_i}^-, \quad (5.40)$$

$i = 1, 2$ .

Then, we solve  $\mu_{i3}^*$  such that (5.38) – (5.40) are satisfied at the point  $(k_i, m_i) = (k_i^*(k_j, m_j), m_i^*(k_j, m_j))$ . Using (5.22) and (5.23), we have

$$\begin{aligned} \frac{\partial \pi_{b_i}}{\partial k_i} &= \frac{\alpha_i}{(4\alpha_1\alpha_2 - \beta_1\beta_2)^2 k_i^2} \{ \alpha_j \beta_i^2 (s_j k_j + m_j) [\alpha_j (s_j k_j + m_j) + \beta_j s_i k_i^2] \\ &\quad + (2a_i \alpha_j + a_j \beta_i) [2a_i \alpha_j + a_j \beta_i + 2\beta_i \alpha_j (s_j k_j + m_j)] \\ &\quad + s_i k_i^2 (\beta_1^2 \beta_2^2 c_i + 8\alpha_1^2 \alpha_2^2 c_i - 6\alpha_1 \alpha_2 \beta_1 \beta_2 c_i + 2a_i \alpha_j \beta_1 \beta_2 + a_j \beta_i^2 \beta_j) \\ &\quad - (2\alpha_1 \alpha_2 - \beta_1 \beta_2) [2\alpha_j \beta_i m_i (s_j k_j + m_j) + 4\alpha_1 \alpha_2 s_i k_i^2 (s_i k_i + m_i) \\ &\quad + 2m_i (2a_i \alpha_j + a_j \beta_i) - m_i^2 (2\alpha_1 \alpha_2 - \beta_1 \beta_2) - 2\alpha_1 \alpha_2 s_i^2 k_i^2] \}, \\ \frac{\partial \pi_{b_i}}{\partial m_i} &= - \frac{\alpha_i}{(4\alpha_1 \alpha_2 - \beta_1 \beta_2)^2 k_i} \{ [k_i (a_j \beta_i^2 \beta_j - 6\alpha_1 \alpha_2 \beta_1 \beta_2 c_i + 8\alpha_1^2 \alpha_2^2 c_i + \beta_1^2 \beta_2^2 c_i \\ &\quad + 2a_i \alpha_j \beta_1 \beta_2)] + \alpha_j \beta_i (s_j k_j + m_j) (4\alpha_1 \alpha_2 - 2\beta_1 \beta_2 + k_i \beta_1 \beta_2) \\ &\quad + 2(2\alpha_1 \alpha_2 - \beta_1 \beta_2) (2a_i \alpha_j + a_j \beta_i) - 4k_i \alpha_1 \alpha_2 (2\alpha_1 \alpha_2 - \beta_1 \beta_2) (k_i s_i + m_i) \\ &\quad + 2\alpha_1 \alpha_2 (\beta_1 \beta_2 s_i k_i - 4\alpha_1 \alpha_2 m_i + 2\beta_1 \beta_2 m_i) \}, \\ \frac{\partial \pi_{s_i}}{\partial k_i} &= - \frac{\alpha_i g_{i1} [2a_i \alpha_j + a_j \beta_i + (2\alpha_1 \alpha_2 - \beta_1 \beta_2) (k_i s_i - m_i) + \alpha_j \beta_i (k_j s_j + m_j)]}{(4\alpha_1 \alpha_2 - \beta_1 \beta_2)^2 k_i^2}, \text{ and} \\ \frac{\partial \pi_{s_i}}{\partial m_i} &= - \frac{2\alpha_i g_{i1} (2\alpha_1 \alpha_2 - \beta_1 \beta_2)}{(4\alpha_1 \alpha_2 - \beta_1 \beta_2)^2 k_i}, \end{aligned} \quad (5.41)$$

$i, j = 1, 2$  and  $j \neq i$ . Substituting the above equations in (5.38) – (5.40) and solving the problem using the software Maple 14, we obtain

$$\begin{aligned}\mu_{i3}^* &= 1, \\ k_i^*(k_j, m_j) &= \frac{A_i^2}{16\alpha_i\alpha_j^2\pi_{s_i}^-}, \quad \text{and}\end{aligned}\tag{5.42}$$

$$\begin{aligned}m_i^*(k_j, m_j) &= \frac{\beta_1\beta_2[2a_i\alpha_j + a_j\beta_i + \alpha_j\beta_i(s_jk_j + m_j)]}{4\alpha_1\alpha_2(2\alpha_1\alpha_2 - \beta_1\beta_2)} \\ &\quad + \frac{(2\alpha_1\alpha_2 - \beta_1\beta_2)(4\alpha_1\alpha_2 - \beta_1\beta_2)(s_i + c_i)}{4\alpha_1\alpha_2(2\alpha_1\alpha_2 - \beta_1\beta_2)} \\ &\quad - s_i k_i^*(k_j, m_j),\end{aligned}\tag{5.43}$$

where  $A_i$  is defined in (5.37),  $i, j = 1, 2$  and  $j \neq i$ .

Next, in order to show that  $(k_i^*(k_j, m_j), m_i^*(k_j, m_j))$  is a KKT point, we need to verify whether (5.30) and (5.31) are satisfied for  $k_i = k_i^*(k_j, m_j)$  and  $m_i = m_i^*(k_j, m_j)$ ,  $i, j = 1, 2$  and  $j \neq i$ . Substituting (5.42) and (5.43) in (5.25) and (5.26), we have

$$\begin{aligned}g_{i1} &= \frac{(4\alpha_1\alpha_2 - \beta_1\beta_2)A_i}{4\alpha_1\alpha_2} \quad \text{and} \\ g_{i2} &= \frac{(4\alpha_1\alpha_2 - \beta_1\beta_2)[A_i^2 - 8\alpha_j(2\alpha_1\alpha_2 - \beta_1\beta_2)\pi_{s_i}^-]A_i}{32\alpha_i\alpha_j^2(2\alpha_1\alpha_2 - \beta_1\beta_2)\pi_{s_i}^-},\end{aligned}$$

where  $A_i$  is defined in (5.37),  $i, j = 1, 2$  and  $j \neq i$ . As we can see,  $g_{i1}, g_{i2} \geq 0$  is true only if

$$A_i \geq 0 \quad \text{and}\tag{5.44}$$

$$\pi_{s_i}^- \in \left[0, \frac{A_i^2}{8\alpha_j(2\alpha_1\alpha_2 - \beta_1\beta_2)}\right],\tag{5.45}$$

$i, j = 1, 2$  and  $j \neq i$ .

Therefore, the KKT necessary conditions in (5.28)–(5.36) are all satisfied for

$(k_i, m_i) = (k_i^*(k_j, m_j), m_i^*(k_j, m_j))$  and  $(\mu_{i1}^*, \mu_{i2}^*, \mu_{i3}^*) = (0, 0, 1)$ , if (5.44) and (5.45) hold true,  $i, j = 1, 2$  and  $j \neq i$ . According to Bazaraa et al. (2006), the solution  $(k_i^*(k_j, m_j), m_i^*(k_j, m_j))$  is a KKT point with Lagrangian multipliers  $(\mu_{i1}^*, \mu_{i2}^*, \mu_{i3}^*)$  corresponding to the three constraints in  $(Pb3 - ai)$ , if (5.44) and (5.45) hold true,  $i, j = 1, 2$  and  $j \neq i$ .

Substituting (5.42) and (5.43) in (5.22), buyer  $i$ 's corresponding profit at the KKT point is given by

$$\pi_{b_i}^* = \frac{A_i^2}{8\alpha_j(2\alpha_1\alpha_2 - \beta_1\beta_2)} - \pi_{s_i}^-, \quad (5.46)$$

$i = 1, 2$ . Since the constraint on supplier  $i$ 's profit is binding as seen in (5.40), supplier  $i$ 's corresponding profit is the reservation profit,  $i = 1, 2$ .

#### Case 1.

If  $\mu_{i1}^* = \mu_{i2}^* = \mu_{i3}^* = 0$  then (5.28) and (5.29) reduce to  $\partial\pi_{b_i}/\partial k_i = 0$  and  $\partial\pi_{b_i}/\partial m_i = 0$ ,  $i = 1, 2$ . Setting  $\partial\pi_{b_i}/\partial m_i = 0$  in (5.41), we obtain

$$\begin{aligned} m_i(k_i) = & \frac{2(2\alpha_1\alpha_2 - \beta_1\beta_2)(2a_i\alpha_j + a_j\beta_i)}{2(2\alpha_1\alpha_2k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)(2\alpha_1\alpha_2 - \beta_1\beta_2)} \\ & + \frac{\alpha_j\beta_i(4\alpha_1\alpha_2 - 2\beta_1\beta_2 + k_i\beta_1\beta_2)(s_jk_j + m_j)}{2(2\alpha_1\alpha_2k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)(2\alpha_1\alpha_2 - \beta_1\beta_2)} \\ & + \frac{(a_j\beta_i^2\beta_j + \beta_1^2\beta_2^2c_i + 2a_i\alpha_j\beta_1\beta_2 + 8\alpha_1^2\alpha_2^2c_i - 6\alpha_1\alpha_2\beta_1\beta_2c_i)k_i}{2(2\alpha_1\alpha_2k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)(2\alpha_1\alpha_2 - \beta_1\beta_2)} \\ & - \frac{(2\alpha_1\alpha_2 - \beta_1\beta_2)(4\alpha_1\alpha_2k_i - \beta_1\beta_2)s_ik_i}{2(2\alpha_1\alpha_2k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)(2\alpha_1\alpha_2 - \beta_1\beta_2)}, \end{aligned}$$

$i, j = 1, 2$  and  $j \neq i$ . Substituting the above equation for  $m_i$  in (5.22), we have

$$\pi_{b_i} = \frac{\alpha_ik_iA_i^2}{4(2\alpha_1\alpha_2k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)(2\alpha_1\alpha_2 - \beta_1\beta_2)}, \quad (5.47)$$

so that

$$\frac{\partial \pi_{b_i}}{\partial k_i} = \frac{\alpha_i A_i^2}{4(2\alpha_1\alpha_2 k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)^2} \geq 0, \quad (5.48)$$

where  $A_i$  is defined in (5.37),  $i, j = 1, 2$  and  $j \neq i$ . Consider the following two cases for  $A_i$ :

- If  $A_i = 0$  then using (5.47)  $\pi_{b_i} = 0$ ,  $i = 1, 2$ . Obviously, it is not optimal.
- Otherwise if  $A_i \neq 0$  then using (5.48),  $\partial \pi_{b_i} / \partial k_i > 0$  over  $-\infty < k_i < +\infty$ ,  $i = 1, 2$ . Hence,  $\partial \pi_{b_i} / \partial k_i = 0$  is not true in general unless  $k_i = \infty$ ,  $i = 1, 2$ . In this case, there is not finite solution for  $\partial \pi_{b_i} / \partial k_i = 0$  such that  $-\infty < k_i < +\infty$ ,  $i = 1, 2$ . By (5.23),  $\pi_{s_i} \geq 0$  is true only if  $k_i > 0$ , and, hence, we can omit  $k_i = -\infty$ ,  $i = 1, 2$ . Now, let us consider  $k_i = +\infty$ ,  $i = 1, 2$ . Using (5.47), we have

$$\lim_{k_i \rightarrow +\infty} \pi_{b_i} = \frac{A_i^2}{8\alpha_j(2\alpha_1\alpha_2 - \beta_1\beta_2)},$$

where  $A_i$  is defined in (5.37),  $i = 1, 2$ . Substituting  $m_i = m_i(k_i)$  in (5.23), we have

$$\pi_{s_i} = \frac{\alpha_i k_i A_i^2}{4(2\alpha_1\alpha_2 k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)^2},$$

so that

$$\lim_{k_i \rightarrow +\infty} \pi_{s_i} = \frac{\alpha_i A_i^2}{16(2\alpha_1\alpha_2 k_i + 2\alpha_1\alpha_2 - \beta_1\beta_2)\alpha_1\alpha_2} = 0,$$

$i = 1, 2$ . Therefore, (5.32) is satisfied only when  $\pi_{s_i}^- = 0$  and (5.32) is binding such that  $\lim_{k_i \rightarrow +\infty} \pi_{s_i} = \pi_{s_i}^- = 0$ ,  $i = 1, 2$ .

Note that this case corresponds to Case 0 when  $\pi_{s_i}^- = 0$ ,  $i = 1, 2$ . If we let  $\pi_{s_i}^- = 0$  in Case 0, we have  $k_i^*(k_j, m_j) = +\infty$  using (5.42), buyer  $i$ 's corresponding

profit in (5.47) is the same as that in (5.46), and supplier  $i$ 's corresponding profit is the reservation profit which is zero. That is, the solution in Case 1, i.e.,  $k_i = +\infty$  for  $\pi_{s_i}^- = 0$ , is included in Case 0. Therefore, it suffices to only consider Case 0 for both products.

In Case 0, we have shown that  $(k_i^*(k_j, m_j), m_i^*(k_j, m_j))$  given by (5.42) and (5.43) is a KKT point with Lagrangian multipliers given by  $\mu_{i1}^* = \mu_{i2}^* = 0$  and  $\mu_{i3}^* = 1$ , if (5.44) and (5.45) hold true,  $i, j = 1, 2$  and  $j \neq i$ . Now, we show that the KKT point is optimal for  $(Pb3 - ai)$ ,  $i = 1, 2$ . For  $\mu_{i1}^* = \mu_{i2}^* = 0$  and  $\mu_{i3}^* = 1$ , the Lagrangian function in (5.24) is given by

$$\mathcal{L}_i = \pi_{b_i} + \mu_{i3}^* g_{i3} = \pi_{b_i} + \mu_{i3}^* (\pi_{s_i} - \pi_{s_i}^-) = \pi_{b_i} + \pi_{s_i} - \pi_{s_i}^-,$$

so that

$$\nabla^2 \mathcal{L}_i = -\frac{4\alpha_i^2 \alpha_j (2\alpha_1 \alpha_2 - \beta_1 \beta_2)}{(4\alpha_1 \alpha_2 - \beta_1 \beta_2)^2} \begin{pmatrix} s_i^2 & s_i \\ s_i & 1 \end{pmatrix},$$

$i = 1, 2$ . Observe that  $|\nabla^2 \mathcal{L}_i| = 0$ , and, hence,  $\nabla^2 \mathcal{L}_i$  is negative semi-definite,  $i = 1, 2$ . Therefore, **the KKT point  $(k_i^*(k_j, m_j), m_i^*(k_j, m_j))$  given by (5.42) and (5.43) is optimal** for  $(Pb3 - ai)$  (see Lemma 4.4.1 in Bazaraa et al. (2006)), for given  $k_j, m_j \in \mathfrak{R}$ , if (5.44) and (5.45) hold true,  $i, j = 1, 2$  and  $j \neq i$ .

#### 5.4.3 Nash equilibrium of $(Pb3 - a1)$ and $(Pb3 - a2)$

Next, using each buyer's optimal solution corresponding to the other buyer's decision given by (5.42) and (5.43), we solve  $(Pb3 - a1)$  and  $(Pb3 - a2)$  simultaneously to identify the Nash equilibrium, denoted by  $(k_1^*, m_1^*)$  and  $(k_2^*, m_2^*)$ . When  $(k_1^*, m_1^*)$

and  $(k_2^*, m_2^*)$  lead to a Nash equilibrium, it should be satisfied that

$$k_1^*(k_2^*, m_2^*) = k_1^*,$$

$$m_1^*(k_2^*, m_2^*) = m_1^*,$$

$$k_2^*(k_1^*, m_1^*) = k_2^*, \quad \text{and}$$

$$m_2^*(k_1^*, m_1^*) = m_2^*,$$

where  $k_i^*(k_j^*, m_j^*)$  and  $m_i^*(k_j^*, m_j^*)$  can be calculated using (5.42) and (5.43), respectively, at  $(k_j, m_j) = (k_j^*, m_j^*)$ ,  $i, j = 1, 2$ ,  $j \neq i$ . Solving the set of equations above, after some algebra we obtain

$$k_i^* = \frac{(2\alpha_1\alpha_2 - \beta_1\beta_2)\Pi_i^*}{2\alpha_1\alpha_2\pi_{s_i}^-} \quad \text{and} \quad (5.49)$$

$$m_i^* = -s_i k_i^* + x_i^*, \quad (5.50)$$

where

$$\begin{aligned} \Pi_i^* &= \frac{2\alpha_j(2\alpha_1\alpha_2 - \beta_1\beta_2)}{(16\alpha_1^2\alpha_2^2 - 12\alpha_1\alpha_2\beta_1\beta_2 + \beta_1^2\beta_2^2)^2} [2a_j\alpha_i\beta_i + a_i(4\alpha_1\alpha_2 - \beta_1\beta_2) \\ &\quad - \alpha_i(4\alpha_1\alpha_2 - 3\beta_1\beta_2)(s_i + c_i) + \beta_i(2\alpha_1\alpha_2 - \beta_1\beta_2)(s_j + c_j)]^2 \quad \text{and} \quad (5.51) \\ x_i^* &= \frac{\beta_1\beta_2[2a_j\alpha_i\beta_i + a_i(4\alpha_1\alpha_2 - \beta_1\beta_2)]}{\alpha_i(16\alpha_1^2\alpha_2^2 - 12\alpha_1\alpha_2\beta_1\beta_2 + \beta_1^2\beta_2^2)} \\ &\quad + \frac{(2\alpha_1\alpha_2 - \beta_1\beta_2)[4\alpha_i(2\alpha_1\alpha_2 - \beta_1\beta_2)(s_i + c_i) + \beta_i^2\beta_j(s_j + c_j)]}{\alpha_i(16\alpha_1^2\alpha_2^2 - 12\alpha_1\alpha_2\beta_1\beta_2 + \beta_1^2\beta_2^2)}, \end{aligned}$$

$i, j = 1, 2$  and  $j \neq i$ . Obviously,  $k_i^* > 0$ ,  $i = 1, 2$ .

Recall that buyer  $i$ 's optimal response is given by (5.42) and (5.43) only if (5.44) and (5.45) hold true,  $i = 1, 2$ . Next, we need to check whether  $A_i \geq 0$  in (5.44) is true for  $(k_j, m_j) = (k_j^*, m_j^*)$ ,  $i, j = 1, 2$  and  $j \neq i$ . Substituting (5.49) and (5.50) in



(5.37), we have

$$A_i = \frac{4\alpha_j(2\alpha_1\alpha_2 - \beta_1\beta_2)B_i}{16\alpha_1^2\alpha_2^2 - 12\alpha_1\alpha_2\beta_1\beta_2 + \beta_1^2\beta_2^2} \geq 0,$$

where  $B_i = (4\alpha_1\alpha_2 - \beta_1\beta_2)[a_i - \alpha_i(s_i + c_i) + \beta_i(s_j + c_j)] + 2\alpha_1\alpha_2[a_j - \alpha_j(s_j + c_j) + \beta_j(s_i + c_i)] \geq 0$  by (5.5),  $i = 1, 2$ .

We also need to identify the range for  $\pi_{s_i}^-$  to ensure (5.45) for  $(k_j, m_j) = (k_j^*, m_j^*)$ ,  $i, j = 1, 2$  and  $j \neq i$ . Substituting (5.49) and (5.50) in (5.37) and using (5.45), we have

$$\pi_{s_i}^- \in \left[0, \frac{A_i^2}{8\alpha_j(2\alpha_1\alpha_2 - \beta_1\beta_2)}\right] \Rightarrow \pi_{s_i}^- \in [0, \Pi_i^*],$$

where  $\Pi_i^*$  is given by (5.51),  $i = 1, 2$ .

Therefore,  $k_i^*$  and  $m_i^*$  given by (5.49) and (5.50) are **the optimal contract parameters under b3** for  $\pi_{s_i}^- \in [0, \Pi_i^*]$ ,  $i = 1, 2$ . Substituting (5.49) and (5.50) in (5.47), **buyer  $i$ 's profit under the optimal b3** for  $\pi_{s_i}^- \in [0, \Pi_i^*]$  is given by

$$\pi_{b_i}^* = \Pi_i^* - \pi_{s_i}^-,$$

and as shown earlier, **supplier  $i$ 's corresponding profit under optimal b3** for  $\pi_{s_i}^- \in [0, \Pi_i^*]$  is given by

$$\pi_{s_i}^* = \pi_{s_i}^-,$$

$i = 1, 2$ . It is easy to see that the total profit of supplier  $i$  and buyer  $i$  is given by  $\Pi_i^*$  as given in (5.51), which is independent of the supplier's reservation profit,  $i = 1, 2$ .

In sum, we conclude that

- If  $\pi_{s_i}^- \in [0, \Pi_i^*]$  for both  $i = 1, 2$  then the tuple  $(k_i^*, m_i^*)$  given by (5.49) and (5.50) is the optimal decision for buyer  $i$  under  $b3$ ,
- If  $\pi_{s_i}^- > \Pi_i^*$  for both  $i = 1, 2$  then there does not exist feasible solutions for  $(Pb3 - a1)$  and  $(Pb3 - a2)$ , and, hence,  $b3$  does not offer a practical solution,
- If  $\pi_{s_i}^- \in [0, \Pi_i^*]$  and  $\pi_{s_j}^- > \Pi_j^*$  then there does not exist a feasible solution for  $(Pb3 - aj)$ , and, hence,  $b3$  also does not offer a practical solution, where  $\Pi_i^*$  is given by (5.51),  $i, j = 1, 2$  and  $j \neq i$ .

## 5.5 Conclusion

In this chapter, we consider the exclusive dealer setting and study the generic contract in a generalized asymmetric case. Considering the suppliers' reservation profits, we formulate each buyer's optimization problem and derive a buyer's optimal decision corresponding to the other buyer's decision. Assuming the two buyers make decisions simultaneously, we characterize the Nash equilibrium of their optimization problems. As a result, under the optimal contract, each supplier's profit is the reservation profit, and each buyer's profit is decreasing in the corresponding supplier's reservation profit. In particular, the total profit of each supplier-buyer pair on a product is constant independent of the supplier's reservation profit under the optimal contract. While the supplier can only obtain the reservation profit, the buyer can obtain the rest of the constant total profit.

## 6. INTERVENTION MECHANISMS IN A NEWSVENDOR PROBLEM FOR PUBLIC INTEREST GOODS

### 6.1 Setting 4. The newsvendor problem setting under social welfare objective

Public interest goods include safety products (e.g., smoke detectors), energy efficient appliances (e.g., water-saving toilets), health-related products (e.g., vaccines) (see e.g., Chick et al. (2008), Deo and Corbett (2009), Cho (2010), Mamani et al. (2012), and Adida et al. (2013)), food in shortage, emission-reduced vehicles (e.g., the electric vehicle) (see e.g., Ovchinnikov and Raz (2014)), and so on. Social welfare, referring to the benefits of all agents involved, is the main concern for marketing a public interest good. Social welfare is composed of benefits for three entities, including the seller's profit, consumers' surplus, and the benefit for the community, net the government cost on implementing interventions, if applicable. This chapter considers a social welfare setting, in which a public interest good is distributed by a newsvendor-type seller to consumers with stochastic demand depending on retail price.

The social welfare setting of interest is illustrated in Figure 6.1. The seller is a newsvendor, who faces a stochastic demand from the market and decides the retail price and the order quantity (i.e., supply quantity). While the traditional newsvendor problem maximizes the seller's expected profit (see e.g., Nahmias (2005)), our model focuses on maximizing the expected social welfare. In this setting, the government plays a significant role in controlling the affordability and availability of a public interest good and leveraging its social welfare. The purpose of the government is to improve/maximize the expected social welfare by intervening in the seller's price and quantity or consumers' purchase decisions through intervention mechanisms.

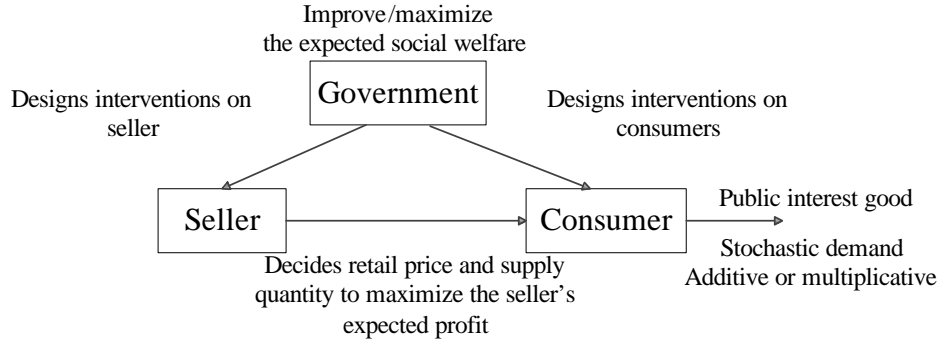


Figure 6.1: The newsvendor problem setting under social welfare objective.

This chapter revisits Ovchinnikov and Raz (2014)'s work by considering the multiplicative demand function. Different than our setting, Ovchinnikov and Raz (2014) apply the additive demand function. The multiplicative demand function is widely applicable and suitable in social welfare analysis for different reasons (see e.g., Tellis (1988), Driver and Valletti (2003), Song et al. (2009), and Huang and Van Mieghem (2013)). As we will demonstrate in this chapter, the demand function also determines the government's decision for choosing a suitable intervention mechanism.

The intervention mechanisms commonly applied by the government can be classified as regulatory interventions (regulations) and market interventions. With regulatory interventions, the government directly imposes restrictions on the seller's behavior, e.g., setting the maximum price (the maximum price regulation) (see e.g., Linhart and Radner (1992)), requiring specific supply, or the combination. With market interventions, the government provides incentives to encourage the seller to make decisions that are socially better. For example, the government adjusts the tax rate (see e.g., Brito et al. (1991), Mas-Colell et al. (1995), Dardan and Stylianou (2000) and Mamani et al. (2012)); the government pays the seller a subsidy for each unit supplied, referred to as the cost subsidy (see e.g., Brito et al. (1991) and

Ovchinnikov and Raz (2014)), or the purchase subsidy (see e.g., Taylor and Yadav (2011), Adida et al. (2013), and Mamani et al. (2012)); and the government pays a consumer a rebate for each unit purchased, referred to as the consumer rebate (see e.g., Ovchinnikov and Raz (2014)), or the sales subsidy (see e.g., Taylor and Yadav (2011) and Adida et al. (2013)).

For additive demand, Ovchinnikov and Raz (2014) investigate two market interventions, the consumer rebate and the cost subsidy, as well as their combination, the joint rebate-subsidy. They conclude that the joint intervention enables the government to coordinate the system and maximize the expected social welfare. For multiplicative demand, our results differ from their work from two aspects:

- Under additive demand, using Ovchinnikov and Raz (2014)'s results, we prove that the socially optimal retail price is less than the seller's production/ordering cost. It is, hence, impossible to align the seller's price with the socially optimal one through a price regulation without providing additional compensations to the seller. Under multiplicative demand, however, we are able to demonstrate that the socially optimal price could be more than the seller's production/ordering cost. Hence, using the maximum price regulation, it is possible to coordinate the price. This is an easily-implemented-and-administrated option for the government.
- While Ovchinnikov and Raz (2014) do not consider the impact of tax on the seller's decisions, we generalize their model by allowing for the sales tax imposed on the seller's revenue. The setting is realistic, as the tax serves as an important tool in leveraging price and quantity decisions in practice (see e.g., Mas-Colell et al. (1995) and Dardan and Stylianou (2000)). With the tax adjustment available, the tax cut and two more joint interventions, the joint tax-rebate

and tax-subsidy, are investigated, besides the rebate-subsidy, which is also considered by Ovchinnikov and Raz (2014). We demonstrate that the joint tax-rebate and tax-subsidy are better options for the government than the rebate-subsidy, as they cost the government less and achieve the same coordination performance.

Based on these results, the main contributions of the chapter are summarized as the following: To the best of our knowledge, this work is the first to investigate the social welfare issue in a newsvendor model with multiplicative uncertainty. With the form of the optimal policy and the performance of intervention mechanisms, our work addresses an important theoretical gap and complements results by Ovchinnikov and Raz (2014). We contribute to the literature by investigating the impact of demand uncertainty on the optimal decisions and intervention mechanisms in a social welfare maximization problem. We employ the joint tax-rebate and tax-subsidy interventions and show that they are better options for the government than the rebate-subsidy used by Ovchinnikov and Raz (2014) for the additive demand. In practice, our work can be used to provide the government/policy maker several ways to control the affordability and availability of a public interest product to improve the expected social welfare.

In the remainder of the chapter, Section 6.2 discusses how the social welfare is modeled using multiplicative demand and Section 6.3 models the problem. Next, Section 6.4 identifies the optimal solutions to the expected profit and social welfare maximization problems, respectively, where the comparison between the two decisions and their economic implications are also provided. Section 6.5 analyzes various government/market interventions and their combinations, and Section 6.6 numerically investigates the intervention performance through a case study. Finally, the

chapter is concluded in Section 6.7 by summarizing the application of the model and avenues of future chapter.

## 6.2 Demand function in welfare analysis

In pricing literature, the stochastic price-dependent demand function  $D(p, \xi)$  is usually assumed to include two components: a deterministic function of price  $d(p)$  and a random variable  $\xi$ . Their mathematical relationships are revealed frequently by the additive form  $D(p, \xi) = \xi + d(p)$  or the multiplicative form  $D(p, \xi) = \xi d(p)$ . For the multiplicative form, we assume that  $\xi$  is a positive random variable with a cumulative distribution function  $F(\cdot)$ , a probability density function  $f(\cdot)$ , and  $E(\xi) = \mu$  and  $var(\xi) = \sigma^2$ .

**Remark 1.** *A multiplicative demand  $D(p, \xi) = \xi d(p)$  is applicable in social welfare analysis, if  $d'(p) \leq 0$  for all  $p$  in the possible range, where  $\xi$  is a positive random variable.*

In microeconomic theory, according to Mas-Colell et al. (1995), a demand function can be applied in welfare analysis only if the demand function is derived based on an underlying model of consumer behavior. That is, consumers should make decisions by choosing from a given set of possible options to maximize their utility. Fortunately, in order to check if the demand function is applicable in welfare analysis, we do not need to figure out the model of consumer behavior behind the demand function. Krishnan (2010) states that if a demand function satisfies certain conditions on the partial derivatives of demand with respect to prices presented in Definition 1, then it is guaranteed that there is an underlying model of consumer behavior that generates the demand function (see e.g., Varian (1992), Mas-Colell et al. (1995), Krishnan (2010)). Definition 1 is taken from Section 2.3 of Krishnan (2010).

**Definition 1.**  $\vec{h}(\vec{p}, u)$  is the Hicksian demand function, and  $\partial \vec{h}(\vec{p}, u)/\partial p_i, \forall i$ , are the price derivatives of the Hicksian demand function, where  $\vec{p}$  is a price vector and  $u$  is a given value of utility. The matrix of partial derivatives has the following properties:

1. The own-price effect is non-positive, i.e.,  $\partial h_i(\vec{p}, u)/\partial p_i \leq 0$ ,
2. The matrix of terms  $\partial h_j(\vec{p}, u)/\partial p_i$  is negative semi-definite, and
3. The matrix of terms  $\partial h_j(\vec{p}, u)/\partial p_i$  is symmetric.

According to Krishnan (2010), the three properties in Definition 1 are not only necessary conditions of the Hicksian demand function, but also sufficient conditions that guarantee that a demand function is generated by utility maximizing consumers. Note that the “Slutsky symmetry” condition (i.e., the third condition in Definition 1) is usually violated by the multiplicative demand functions for a multi-product case. It is because the realizations of the random variables in demand functions for different products might not be the same (see e.g., Krishnan (2010)). However, if there is only a single product, the symmetry condition always holds true. Consider a single-product case with the demand function  $h(p, u) = D(p, \xi) = \xi d(p)$ . The first condition in Definition 1 is satisfied as follows

$$\frac{\partial h(p, u)}{\partial p} = \frac{dD(p, \xi)}{dp} = \xi d'(p) \leq 0,$$

when  $\xi d'(p) \leq 0$  due to Remark 1. The other two conditions are also satisfied, because the matrix of partial derivatives is  $1 \times 1$ , which is naturally symmetric and negative semi-definite if

$$\frac{\partial h(p, u)}{\partial p} \leq 0$$



is true. Therefore, the multiplicative demand function for a single product satisfies all conditions in Definition 1, and the demand is feasible for social welfare analysis.

Most of the commonly-used multiplicative demand functions satisfy the condition  $d''(p) \leq 0$ . Examples include the multiplicative demand functions with the linear deterministic demand  $d(p) = a - bp$  for  $a > 0, b > 0$ , and the power form  $d(p) = ap^{-b}$  (see e.g., Petruzzi and Dada (1999)), and the reservation-price model  $D(p, \xi) = \xi(1 - F(p))$  (see e.g., Ziya et al. (2004)). Furthermore, this condition is intuitively realistic, since it states that the own-price effect is non-positive (see e.g., Varian (1992)) implying that demand decreases in price. In the next section, we will characterize the model in the social welfare setting.

### 6.3 The model

Consider a system comprising a newsvendor-type seller who decides the retail price  $p$  and the order quantity  $q$ , and a government who intervenes in the market to affect the seller's and the consumers' behaviors to maximize the expected social welfare. We follow Ovchinnikov and Raz (2014) by using the same definition of social welfare, which is defined as the summation of all participants' benefits in the channel, net the government cost, given by

$$\begin{aligned} \text{Social welfare} = & \text{Seller's profit} + \text{Consumers' surplus} + \text{Externality benefit} \\ & - \text{Government cost.} \end{aligned}$$

In centralized control, we assume the system is managed by a central planner. That is, the socially optimal decisions are used without any intervention implemented by the government. In this case, the social welfare is the summation of the first three components on the RHS of the above equation without the government cost. In this

section, we focus on modeling the social welfare in centralized control. The impact of the government interventions on the decentralized decisions will be investigated in Section 6.5.

The seller's profit, consumers' surplus, and externality benefit represent the benefits of the seller, the consumers, and the other people in the society, respectively, obtained from the distribution of the product. Let  $D(p, \xi)$  be a generic demand function at price  $p$  and with random component  $\xi$ . Suppose that the production/ordering cost is  $c$  per unit for the seller, any left-over item is salvaged at a value  $s$  per unit, and all unsatisfied demand is lost. A sales tax  $t$  is imposed on the seller's sales revenue. Then, the seller's expected profit function is given by

$$SP(p, q) = (1 - t)p \min \{D(p, \xi), q\} - cq + s (\max \{q - D(p, \xi), 0\}). \quad (6.1)$$

The terms on the RHS of the above equation represent the sales revenue after tax, the production/ordering cost, and the salvage value, respectively.

Consumers' surplus is an important concept in social welfare, which is called Marshallian Consumer Surplus originated from Marshall (1920). It is defined as the difference between a consumer's willing-to-pay price and the market price of a product (Marshall (1920)). Consumers' surplus is regarded as the counterpart of seller's profit, since consumers' surplus decreases and seller's profit increases as the retail price increases. According to Ovchinnikov and Raz (2014), "consumer's surplus is generally defined as the area under the demand curve above the given price." When demand is uncertain, consumers' surplus is not generated for the part of demand that is unmet and lost. While there are several ways to model consumers'

surplus in the case of stockout, following Ovchinnikov and Raz (2014) we obtain:

$$CS(p, q) = \min \left\{ \frac{q}{D(p, \xi)}, 1 \right\} \int_p^\infty D(x, \xi) dx. \quad (6.2)$$

By (6.2), if  $q \geq D(p, \xi)$ , then all demand can be satisfied, and the corresponding consumers' surplus is  $\int_p^\infty D(x, \xi) dx$ , which only depends on the retail price  $p$ . If  $q < D(p, \xi)$ , only  $q/D(p, \xi)$  portion of the total demand is satisfied. Then, the consumers' surplus in this case is given by  $(q/D(p, \xi)) \int_p^\infty D(x, \xi) dx$ .

According to Laffont (2008), “An externality is any indirect effect that either a production or a consumption activity has on a utility function, a consumption set or a production set.” The externality benefit measures the positive influence of owning a product on the other people in the society. Following Ovchinnikov and Raz (2014), the externality benefit is defined as a constant marginal externality,  $\alpha$ , where  $\alpha > 0$ , times the sales amount given as below:

$$EB(p, q) = \alpha \min(D(p, \xi), q). \quad (6.3)$$

Overall, the expected social welfare in centralized control for given  $p$  and  $q$  values is the following:

$$\Pi_C(p, q) = E_{SP}(p, q) + E_{CS}(p, q) + E_{EB}(p, q),$$

where the terms on the RHS are expected values of the seller's profit, consumers' surplus, and the externality benefit, respectively. The centralized problem is to maximize the expected social welfare  $\Pi_C(p, q)$ , and the decentralized problem is to maximize the seller's expected profit  $E_{SP}(p, q)$ .

We state the following assumptions on the demand function and cost parameters.

These assumptions are needed for the analytical tractability and applicability of the models in real-world situations. Assumption 1 on the costs and the retail price is made to guarantee the problem setting is practical. Recall that the multiplicative demand is expressed as  $D(p, \xi) = \xi d(p)$ .

**Assumption 1.** 1. *The production/ordering cost is greater than the salvage value,*

$$c > s,$$

2. *The retail price  $p$  is in the range  $p \in [s/(1-t), \bar{p}]$ , where  $\bar{p}$  satisfies  $d(\bar{p}) = 0$  and  $(1-t)\bar{p} > c$ .*

The seller's revenue per unit sold (i.e., the retail price after tax  $p(1-t)$ ) should be greater than the salvage value; otherwise, the seller would prefer to salvage the product instead of selling it. In addition,  $\bar{p}$  denotes the maximum admissible value of  $p$ , such that  $d(\bar{p}) = 0$ , to avoid negative demand. Therefore, the set of feasible price levels is confined to the finite interval  $[s/(1-t), \bar{p}]$ . Furthermore, it is natural to assume that the product is affordable to consumers when it is profitable to the seller. That is, the production/ordering cost cannot be greater than the highest possible price  $\bar{p}$  after tax, such that  $(1-t)\bar{p} > c$ . The following assumptions for the demand function are needed for analytical tractability to guarantee the existence of the optimal decisions in centralized and decentralized problems.

**Assumption 2.** *For  $p \in [s/(1-t), \bar{p}]$ ,  $d(p)$  satisfies the following conditions:*

1.  *$d(p)$  is positive, strictly decreasing, and continuously differentiable,*
2.  *$d(p)/d'(p)$  is decreasing and concave, and*
3.  *$p + d(p)/d'(p)$  is strictly increasing.*

Assumption 2 is satisfied by common convex functions, such that  $d(p) = ap^{-k}$  for  $a > 0$  and  $k > 0$ ,  $d(p) = (a - bp)^k$  for  $b < 0$  and  $k < -1$ , linear function  $d(p) = a - bp$  for  $a > 0$  and  $b > 0$ , and log-linear function  $ak^{-bp}$  for  $a > 0$ ,  $k > 0$ , and  $b > 0$  (see e.g., Song et al. (2009)). Assumption 3 is from Song et al. (2009), which is used in the proof of Property 6.

**Assumption 3.** *The generalized failure rate of the distribution function of  $\xi$ , defined by  $l(u) = uf(u)/(1 - F(u))$ , is increasing.*

According to Lariviere (2006), “the assumption holds for many common distributions, including uniform, normal, exponential, gamma with shape parameter  $\geq 1$ , and beta with both parameters  $\geq 1$ .”

## 6.4 Optimal decisions in centralized and decentralized controls

In this section, we first characterize the objective functions under the multiplicative demand. After that, we identify the socially optimal retail price and order quantity that maximize the expected social welfare in centralized control. Then, we identify the decentralized optimal decisions that maximize the seller’s expected profit. A comparison of the decisions with the two objectives is also provided.

### 6.4.1 Expressions of objective functions

First, using the multiplicative demand, we derive the expected values of seller’s profit, consumers’ surplus, and the externality benefit from (6.1), (6.2), and (6.3), respectively. Let us define  $z = q/d(p)$ . Recall that  $D(p, \xi) = \xi d(p)$ . The variable  $z$  can be termed as the stocking factor and  $F(z)$  represents the proportion of the demand that is satisfied (see e.g., Petruzzi and Dada (1999)). In the sequel, we will use the following identities. Using  $\max\{x, 0\} = -\min\{x, 0\}$ , the quantity sold can

be written in two forms as given below:

$$\min\{D(p, \xi), q\} = q + \min\{D(p, \xi) - q, 0\} = q - \max\{q - D(p, \xi), 0\} \quad \text{and (6.4)}$$

$$\begin{aligned} \min\{D(p, \xi), q\} &= D(p, \xi) + \min\{q - D(p, \xi), 0\} \\ &= D(p, \xi) - \max\{D(p, \xi) - q, 0\}. \end{aligned} \quad (6.5)$$

Let us define

$$\theta(z) = \int_z^{\bar{r}} (r - z) dF(r) \quad \text{and} \quad (6.6)$$

$$\delta(z) = \int_{\underline{r}}^z (z - r) dF(r). \quad (6.7)$$

Note that the expected unsatisfied demand is given by

$$E[\max\{D(p, \xi) - q, 0\}] = d(p) \int_z^{\bar{r}} (r - z) dF(r) = d(p)\theta(z), \quad (6.8)$$

and the expected leftover inventory is given by

$$E[\max\{q - D(p, \xi), 0\}] = d(p) \int_{\underline{r}}^z (z - r) dF(r) = d(p)\delta(z). \quad (6.9)$$

Using (6.4) and (6.9), the expected sales is written as

$$\begin{aligned} E[\min(D(p, \xi), q)] &= q - E[\max\{q - D(p, \xi), 0\}] \\ &= q - d(p)\delta(z) = d(p)(z - \delta(z)), \end{aligned} \quad (6.10)$$

and using (6.5) and (6.8), the expected sales can also be written as

$$\begin{aligned} E[\min(D(p, \xi), q)] &= E[D(p, \xi)] - E[\max\{D(p, \xi) - q, 0\}] \\ &= d(p)(\mu - \theta(z)). \end{aligned} \quad (6.11)$$

From (6.10) and (6.11), we observe that

$$\mu - \theta(z) = z - \delta(z) = z(1 - F(z)) + \int_{\underline{r}}^z r dF(r). \quad (6.12)$$

Taking the expectation of (6.1), the seller's expected profit is given by

$$E_{SP}(p, q) = (1 - t)pE[\min\{D(p, \xi), q\}] - cq + sE[\max\{q - D(p, \xi), 0\}].$$

Recalling  $z = q/d(p)$  and using (6.9) and (6.11), the seller's expected profit can be written as

$$\begin{aligned} E_{SP}(p, z) &= (1 - t)pd(p)(\mu - \theta(z)) - cd(p)z + sd(p)\delta(z) \\ &= d(p)[(1 - t)p(\mu - \theta(z)) - cz + s\delta(z)]. \end{aligned} \quad (6.13)$$

Let  $H(p) = \int_p^\infty d(x)dx$ . Recalling  $D(p, \xi) = \xi d(p)$  and using (6.2), we obtain

$$CS(p, q) = \begin{cases} \xi H(p), & \text{if } \xi d(p) < q, \\ \frac{q\xi H(p)}{\xi d(p)} = \frac{qH(p)}{d(p)}, & \text{if } \xi d(p) \geq q. \end{cases} \quad (6.14)$$

Then, the expected consumers' surplus is given by

$$\begin{aligned} E_{CS}(p, q) &= \int_{\underline{r}}^{\frac{q}{d(p)}} r H(p) dF(r) + \int_{\frac{q}{d(p)}}^{\bar{r}} \frac{qH(p)}{d(p)} dF(r) \\ &= H(p) \int_{\underline{r}}^{\frac{q}{d(p)}} r dF(r) + \frac{qH(p)}{d(p)} \int_{\frac{q}{d(p)}}^{\bar{r}} dF(r) \\ &= H(p) \int_{\underline{r}}^{\frac{q}{d(p)}} r dF(r) + \frac{qH(p)}{d(p)} \left(1 - F\left(\frac{q}{d(p)}\right)\right). \end{aligned}$$

Substituting  $z = q/d(p)$  into the above equation and using (6.12), we obtain:

$$E_{CS}(p, z) = H(p) \int_{\underline{r}}^z r dF(r) + H(p)z(1 - F(z)) = H(p)(\mu - \theta(z)). \quad (6.15)$$

By (6.3), the expected externality benefit is given by  $E_{EB}(p, q) = \alpha E[\min\{D(p, \xi), q\}]$ .

Using  $z = q/d(p)$  and (6.11), we obtain

$$E_{EB}(p, z) = \alpha d(p) (\mu - \theta(z)). \quad (6.16)$$

Due to mathematical ease, in the sequel, we use variables  $p$  and  $z$  instead of variables  $p$  and  $q$  in the analysis.

#### 6.4.2 Comparison between centralized and decentralized decisions

With the explicit expressions for all components in (6.13), (6.15), and (6.16), the expected social welfare can be written as

$$\begin{aligned} \Pi_C(p, z) &= E_{SP}(p, z) + E_{CS}(p, z) + E_{EB}(p, z) \\ &= [(1 - t)pd(p) + H(p) + \alpha d(p)] [\mu - \theta(z)] + (s\delta(z) - cz) d(p). \end{aligned} \quad (6.17)$$

In centralized control, the government decides  $p$  and  $z$  to maximize the expected social welfare. Hence, the centralized optimization problem is given by

$$\max_{p, z} \Pi_C(p, z).$$

The socially optimal price  $p_C^*$  and stocking factor  $z_C^*$  are presented in Property 1.

**Property 1.** *In centralized control, the optimal retail price  $p_C^*$  and the stocking factor*



$z_C^*$  that maximize the expected social welfare satisfy

$$p - \frac{td(p)}{(1-t)d'(p)} \Big|_{p=p_C^*} = \frac{1}{1-t} \left[ c - \alpha + \frac{(c-s)\delta(z)}{z - \delta(z)} \right] \Big|_{z=z_C^*} \quad (6.18)$$

and

$$F(z_C^*) = \frac{H(p) + d(p)((1-t)p - c + \alpha)}{H(p) + d(p)((1-t)p - s + \alpha)} \Big|_{p=p_C^*}. \quad (6.19)$$

The optimal order quantity  $q_C^*$  is given by  $q_C^* = d(p_C^*)z_C^*$ .

**Proof of Property 1:** Recall (6.6) and (6.7) that define  $\theta(z)$  and  $\delta(z)$ , respectively and  $H(p) = \int_p^\infty d(x)dx$ . We have  $H'(p) = -d(p)$ ,  $\theta'(z) = -(1 - F(z))$  and  $\delta'(z) = F(z)$ . Taking the first and second derivatives of  $\Pi_C(p, z)$  in (6.17) w.r.t  $z$ , we obtain

$$\frac{\partial \Pi_C(p, z)}{\partial z} = [(1-t)pd(p) + H(p) + \alpha d(p)] [1 - F(z)] + sd(p)F(z) - cd(p), \quad (6.20)$$

and

$$\frac{\partial^2 \Pi_C(p, z)}{\partial z^2} = -[((1-t)p + \alpha - s)d(p) + H(p)]f(z) < 0,$$

because  $p(1-t) \geq s$  from Assumption 1 and  $H(p) > 0$  from Assumption 3. Therefore,  $\Pi_C(p, z)$  is concave in  $z$  for any given  $p$  and the optimal  $z$  satisfies the first order condition  $\partial \Pi_C(p, z)/\partial z = 0$ . In addition, we have

$$\begin{aligned} \frac{\partial \Pi_C(p, z)}{\partial p} &= [(1-t)d(p) + (1-t)pd'(p) - d(p) + \alpha d'(p)] (\mu - \theta(z)) \\ &\quad + (s\delta(z) - cz)d'(p), \end{aligned} \quad (6.21)$$

and

$$\begin{aligned} \frac{\partial^2 \Pi_C(p, z)}{\partial p^2} &= ((1-t)d(p) + (1-t)pd'(p) - d(p) + \alpha d'(p))' (\mu - \theta(z)) \\ &\quad + (s\delta(z) - cz)d''(p). \end{aligned} \quad (6.22)$$

Suppose that  $p_0$  satisfies the first order condition  $\partial \Pi_C(p, z)/\partial p|_{p=p_0} = 0$ . Rearranging (6.21), we obtain that  $p_0$  satisfies

$$s\delta(z) - cz = - \left. \frac{[(1-t)d(p) + (1-t)pd'(p) - d(p) + \alpha d'(p)] (\mu - \theta(z))}{d'(p)} \right|_{p=p_0}.$$

Then, using the above equality in (6.22), we have

$$\begin{aligned} &\left. \frac{\partial^2 \Pi_C(p, z)}{\partial p^2} \right|_{p=p_0} \\ &= [(1-t)d(p) + (1-t)pd'(p) - d(p) + \alpha d'(p)]' (\mu - \theta(z)) - \\ &\quad \left. \frac{[(1-t)d(p) + (1-t)pd'(p) - d(p) + \alpha d'(p)] (\mu - \theta(z)) d''(p)}{d'(p)} \right|_{p=p_0}. \end{aligned}$$

By multiplying with  $d'(p)$ , the above equation can be written as

$$\begin{aligned} d'(p) \left. \frac{\partial^2 \Pi_C(p, z)}{\partial p^2} \right|_{p=p_0} &= (1-t) \{ [(d(p) + pd'(p))' d'(p) - (d(p) + pd'(p)) d''(p)] \\ &\quad - [(d'(p))^2 - d(p)d''(p)] \} (\mu - \theta(z)) \Big|_{p=p_0}. \end{aligned}$$

According to Assumption 2, since  $p + d(p)/d'(p)$  is strictly increasing,

$$\left( \frac{d(p) + pd'(p)}{d'(p)} \right)' > 0$$

and we obtain  $(d(p) + pd'(p))' d'(p) - (d(p) + pd'(p)) d''(p) > 0$ . In addition,  $d(p)/d'(p)$  is decreasing, and hence,  $-[(d'(p))^2 - d(p)d''(p)] \geq 0$ . Since  $\mu - \theta(z) > 0$  by (6.12),

we have

$$d'(p) \frac{\partial^2 \Pi_C(p, z)}{\partial p^2} \Big|_{p=p_0} > 0 \quad \text{and then} \quad \frac{\partial^2 \Pi_C(p, z)}{\partial p^2} \Big|_{p=p_0} < 0.$$

Hence,  $\Pi_C(p, z)$  is concave at any stationary point  $p_0$ . Thus, there is a unique stationary point  $p_0$  for a given  $z$  which is the optimal price. Dividing both sides of  $\partial \Pi_C(p, z)/\partial p = 0$  by  $(1-t)d'(p)(\mu - \theta(z))$  (6.21) and using (6.12), we obtain

$$p - \frac{td(p)}{(1-t)d'(p)} - \frac{cz - s\delta(z) - \alpha(z - \delta(z))}{(1-t)(z - \delta(z))} = 0.$$

Rearranging the above equation and  $\partial \Pi_C(p, z)/\partial z = 0$  in (6.20), we obtain the optimal price  $p_C^*$  and the stocking factor  $z_C^*$  satisfy (6.18) and (6.19).  $\square$

Property 1 characterizes the optimal retail price and the order quantity that maximize the expected social welfare. It is interesting to note that the optimal solution to the centralized problem given by (6.19) has an elegant form and can be considered as a generalized solution to the price-setting newsvendor problems with different objectives. The solution to the profit maximization problem that satisfies  $F(z) = (p-c)/(p-s)$  (see e.g., Nahmias (2005)) can be directly obtained by omitting consumers' surplus related term  $H(p)$  and letting  $t = 0$  and  $\alpha = 0$  in (6.19). Note that, in centralized control without the government intervention, the social welfare incorporates three terms: the seller's expected profit, the consumers' surplus, and the externality benefit. Next property represents the relationship between the three terms at the socially optimal decisions.

**Property 2.** *In centralized control,  $E_{SP}(p_C^*, z_C^*) \leq -E_{EB}(p_C^*, z_C^*)$ . More specifically,*

1. *If  $t = 0$ , then  $E_{SP}(p_C^*, z_C^*) = -E_{EB}(p_C^*, z_C^*)$  and  $\Pi_C(p_C^*, z_C^*) = E_{CS}(p_C^*, z_C^*)$ ;*  
*and*

2. If  $t > 0$ , then  $E_{SP}(p_C^*, z_C^*) < -E_{EB}(p_C^*, z_C^*)$ , and  $\Pi_C(p_C^*, z_C^*) < E_{CS}(p_C^*, z_C^*)$ .

**Proof of Property 2:** Multiplying both sides of (6.18) by  $(1-t)d(p)(z-\delta(z))$  and rearranging the equation, we have

$$\begin{aligned} & \{ (1-t)d(p)p(z-\delta(z)) + d(p)(s\delta(z) - cz) \} \Big|_{p=p_C^*, z=z_C^*} \\ = & \left\{ -\alpha d(p)(z-\delta(z)) + \frac{td^2(p)(z-\delta(z))}{d'(p)} \right\} \Big|_{p=p_C^*, z=z_C^*}. \end{aligned}$$

Recall  $E_{SP}(p, z)$  and  $E_{EB}(p, z)$  from (6.13) and (6.16), respectively, and using  $z - \delta(z) = \mu - \theta(z)$ , we observe that

$$E_{SP}(p_C^*, z_C^*) = -E_{EB}(p_C^*, z_C^*) + \frac{td^2(p)(z-\delta(z))}{d'(p)} \Big|_{p=p_C^*, z=z_C^*}.$$

Since  $d'(p) < 0$ ,  $E_{SP}(p_C^*, z_C^*) \leq -E_{EB}(p_C^*, z_C^*)$ . The equality holds true if  $t = 0$ .  $\square$

Property 2 shows that the seller's expected profit is non-positive at the socially optimal decisions. The result directly follows from  $E_{SP}(p_C^*, z_C^*) \leq -E_{EB}(p_C^*, z_C^*)$  and the expected externality benefit  $E_{EB}(p, z) = \alpha d(p)(\mu - \theta(z))$  is always nonnegative due to  $\alpha > 0$ . Hence, the more beneficial the public interest product is to the community, the more loss the seller faces. It is interesting to note that when the tax rate is zero, the absolute values of the seller's expected profit and the expected externality benefit are equal, and hence, the expected social welfare is equal to the expected consumers' surplus.

In decentralized control, the seller maximizes the expected profit without considering the consumers' surplus and the externality benefit. Using (6.13), the decentralized optimization problem is given by

$$\max_{p,z} \Pi_D(p, z) = \max_{p,z} E_{SP}(p, z) = \max_{p,z} d(p) [(1-t)p(\mu - \theta(z)) - cz + s\delta(z)]. \quad (6.23)$$

The optimal price  $p_D^*$  and the stocking factor  $z_D^*$  in decentralized control are presented in the following property.

**Property 3.** *In decentralized control, the optimal retail price  $p_D^*$  and the stocking factor  $z_D^*$  that maximize the seller's expected profit satisfy*

$$p + \frac{d(p)}{d'(p)} \Big|_{p=p_D^*} = \frac{1}{1-t} \left[ c + \frac{(c-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_D^*}$$

and

$$F(z_D^*) = \frac{(1-t)p - c}{(1-t)p - s} \Big|_{p=p_D^*}. \quad (6.24)$$

The optimal order quantity  $q_D^*$  is given by  $q_D^* = d(p_D^*)z_D^*$ .

**Proof of Property 3:** Taking the first and second derivatives of  $\Pi_D(p, z)$  in (6.23) with respect to  $z$  and using  $\theta'(z) = -[1 - F(z)]$  and  $\delta'(z) = F(z)$ , we obtain

$$\frac{\partial \Pi_D(p, z)}{\partial z} = (1-t)pd(p)[1 - F(z)] + sd(p)F(z) - cd(p), \quad (6.25)$$

and

$$\begin{aligned} \frac{\partial^2 \Pi_D(p, z)}{\partial z^2} &= -(1-t)pd(p)f(z) + sd(p)f(z) \\ &= -[(1-t)p - s]d(p)f(z) \leq 0, \end{aligned}$$

because  $(1-t)p \geq s$  from Assumption 1. Next, we analyze two possible cases, respectively, when  $(1-t)p = s$  and  $(1-t)p > s$ . If  $(1-t)p = s$ ,  $\partial^2 \Pi_D(p, z)/\partial z^2 = 0$  and  $\Pi_D(p, z)$  is decreasing in  $z$ , because  $\partial \Pi_D(p, z)/\partial z = (s - c)d(p) < 0$  from Assumption 1. Hence,  $\Pi_D(p, z)$  achieves its optimal value at  $z = 0$ . In this case, the

seller's expected profit is zero such that

$$\Pi_D(p, z)|_{z=0} = d(p) [(1-t)p(\mu - \theta(z)) - cz + s\delta(z)]|_{z=0} = 0.$$

However, it cannot be the optimal solution, because there exists at least one feasible solution that leads to a profit. Since  $(1-t)\bar{p} > c$  from Assumption 1, there exists a feasible retail price  $p^*$ , s.t.  $p^* \in (c/(1-t), \bar{p})$ . Let  $z^* = \underline{z}$ . Then, the expected profit function at the feasible solution  $(p^*, z^*)$  is positive, which is given by

$$\Pi_D(p, z)|_{p=p^*, z=z^*} = d(p) [(1-t)pz - cz]|_{p=p^*, z=z^*} > 0.$$

Therefore, the optimal solution of  $\Pi_D(p, z)$  cannot be achieved when  $(1-t)p = s$ . The optimal retail price should satisfy  $(1-t)p > s$ . For any given  $p$  in the range  $(1-t)p > s$ ,  $\Pi_D(p, z)$  is strictly concave in  $z$  and the optimal  $z$  satisfies the first order condition  $\partial \Pi_D(p, z)/\partial z = 0$ . In addition, we have

$$\frac{\partial \Pi_D(p, z)}{\partial p} = (1-t)(\mu - \theta(z))(d(p) + pd'(p)) + (s\delta(z) - cz)d'(p), \quad (6.26)$$

and

$$\begin{aligned} \frac{\partial^2 \Pi_D(p, z)}{\partial p^2} &= (1-t)(\mu - \theta(z))[d'(p) + d'(p) + pd''(p)] + (s\delta(z) - cz)d''(p). \\ &= (1-t)(\mu - \theta(z))[2d'(p) + pd''(p)] + (s\delta(z) - cz)d''(p). \end{aligned} \quad (6.27)$$

Suppose that  $p_0$  satisfies the first order condition  $\partial \Pi_D(p, z)/\partial p|_{p=p_0} = 0$ . Rearranging (6.26), we obtain that  $p_0$  satisfies

$$s\delta(z) - cz = - \frac{(1-t)(\mu - \theta(z))[d(p) + pd'(p)]}{d'(p)} \Big|_{p=p_0}.$$

Then, using the above equality in (6.27) and multiplying both sides with  $d'(p)$ , we have

$$\begin{aligned} \frac{\partial^2 \Pi_D(p, z)}{\partial p^2} \Big|_{p=p_0} &= (1-t)(\mu - \theta(z)) [2d'(p) + pd''(p)] \\ &\quad - \frac{(1-t)(\mu - \theta(z)) [d(p) + pd'(p)] d''(p)}{d'(p)} \Big|_{p=p_0}. \end{aligned}$$

According to Assumption 2, since  $p + d(p)/d'(p)$  is strictly increasing,

$$\left( \frac{d(p) + pd'(p)}{d'(p)} \right)' > 0$$

and we obtain  $[2d'(p) + pd''(p)] d'(p) - (d(p) + pd'(p)) d''(p) = 2[d'(p)]^2 - d(p)d''(p) > 0$ .

Therefore, we have

$$d'(p) \frac{\partial^2 \Pi_D(p, z)}{\partial p^2} \Big|_{p=p_0} > 0, \quad \text{and} \quad \frac{\partial^2 \Pi_D(p, z)}{\partial p^2} \Big|_{p=p_0} < 0.$$

Hence,  $\Pi_D(p, z)$  is concave at any stationary point  $p_0$ . Thus, there is a unique stationary point of  $\Pi_D(p, z)$  w.r.t  $p$  for a given  $z$  which is the optimal price.

Dividing both sides of  $\partial \Pi_D(p, z)/\partial p = 0$  by  $(1-t)d'(p)(\mu - \theta(z))$  given in (6.26) and using (6.12), we obtain

$$p + \frac{d(p)}{d'(p)} - \frac{cz - s\delta(z)}{(1-t)(\mu - \theta(z))} = p + \frac{d(p)}{d'(p)} - \frac{cz - s\delta(z)}{(1-t)(z - \delta(z))} = 0.$$

Rearranging the above equation and using  $\partial \Pi_D(p, z)/\partial z = 0$  given by (6.25), we obtain that the optimal price  $p_D^*$  and the stocking factor  $z_D^*$  satisfy

$$p + \frac{d(p)}{d'(p)} \Big|_{p=p_D^*} = \frac{1}{1-t} \left[ c + \frac{(c-s)\delta(z)}{z - \delta(z)} \right] \Big|_{z=z_D^*} \quad \text{and} \quad F(z_D^*) = \frac{(1-t)p - c}{(1-t)p - s} \Big|_{p=p_D^*}.$$

□

Due to the impact of consumers' surplus and externality benefit, the optimal decisions are different under the two objectives. Property 4 provides the comparison between the socially optimal decisions and the decentralized optimal decisions.

**Property 4.**  $p_C^* < p_D^*$ ,  $z_C^* > z_D^*$ , and  $q_C^* > q_D^*$ .

**Proof of Property 4:** In Properties 1 and 3, we prove that  $p_C^*$  and  $z_C^*$  satisfy the first order conditions of  $\Pi_C(p, z)$ , and  $p_D^*$  and  $z_D^*$  satisfy the first order conditions of  $\Pi_D(p, z)$ . Using (6.21), we have

$$\begin{aligned} \left. \frac{\partial \Pi_C(p, z)}{\partial p} \right|_{p=p_C^*, z=z_C^*} &= \{[(1-t)d(p) + (1-t)pd'(p) - d(p) + \alpha d'(p)](\mu - \theta(z)) \\ &\quad + (s\delta(z) - cz)d'(p)\} \Big|_{p=p_C^*, z=z_C^*} = 0. \end{aligned} \quad (6.28)$$

From (6.26), we have

$$\begin{aligned} \left. \frac{\partial \Pi_D(p, z)}{\partial p} \right|_{p=p_C^*, z=z_C^*} &= \{d'(p)[(1-t)p(\mu - \theta(z)) - cz + s\delta(z)] \\ &\quad + (1-t)d(p)(\mu - \theta(z))\} \Big|_{p=p_C^*, z=z_C^*} \\ &= \left\{ \frac{\partial \Pi_C(p, z)}{\partial p} + d(p)(\mu - \theta(z)) - \alpha d'(p)(\mu - \theta(z)) \right\} \Big|_{p=p_C^*, z=z_C^*} \\ &= \{d(p)(\mu - \theta(z)) - \alpha d'(p)(\mu - \theta(z))\} \Big|_{p=p_C^*, z=z_C^*}. \end{aligned}$$

According to (6.11) and Assumption 2, the first term  $d(p)(\mu - \theta(z))$  represents the expected sales, which should be positive, as well as  $(\mu - \theta(z)) > 0$  and  $d'(p) < 0$ . We obtain  $\partial \Pi_D(p, z) / \partial p|_{p=p_C^*, z=z_C^*} > 0$ . Hence,  $p_D^* > p_C^*$ .



Using (6.20) and (6.25), and the definitions of  $p_D^*$  and  $z_D^*$ , we obtain

$$\begin{aligned}
& \left. \frac{\partial \Pi_C(p, z)}{\partial z} \right|_{p=p_D^*, z=z_D^*} \\
&= \{ (1-t)pd(p)[1-F(z)] \\
&\quad + sd(p)F(z) - cd(p) + [H(p) + \alpha d(p)][1-F(z)] \}_{p=p_D^*, z=z_D^*} \\
&= \left. \frac{\partial \Pi_D(p, z)}{\partial z} \right|_{p=p_D^*, z=z_D^*} + \{ [H(p) + \alpha d(p)][1-F(z)] \}_{p=p_D^*, z=z_D^*} \\
&= \{ [H(p) + \alpha d(p)][1-F(z)] \}_{p=p_D^*, z=z_D^*} > 0.
\end{aligned}$$

Hence,  $z_C^* > z_D^*$ . Furthermore,  $q_C^* = z_C^* d(p_C^*) > z_D^* d(p_D^*) = q_D^*$ .  $\square$

Property 4 demonstrates that the seller's decentralized optimal retail price is higher and the order quantity is lower than the corresponding socially optimal decisions. In other words, in centralized control, a lower price is charged and more quantity is obtained in order to transfer a part of the seller's profit to consumers to improve the expected social welfare. These results are consistent with our intuition, as consumers always prefer a lower retail price and more quantity available in the market.

## 6.5 Intervention mechanisms

We have already shown in Property 4 that the seller's decisions are not socially optimal in a free market (i.e., in decentralized control). The question arises how the government intervenes in the market to align the seller's decisions with the socially optimal ones. In this section, we examine several government intervention mechanisms and investigate their coordination performance and efficiency. The government intervention mechanisms of interest are illustrated in Figure 6.2. The regulatory mechanisms and market mechanisms will be discussed in Sections 6.1 and 6.2, respectively.

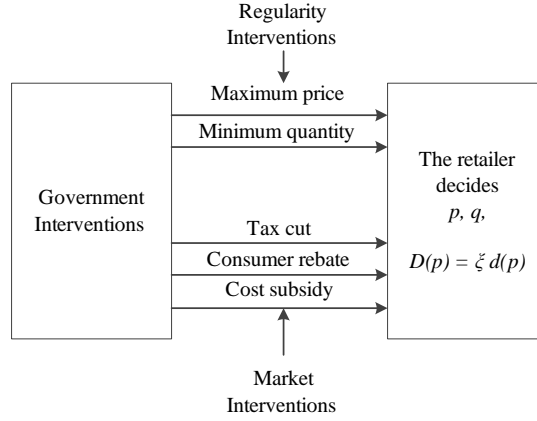


Figure 6.2: The framework of government intervention mechanisms.

We will evaluate and compare these government interventions from three aspects: the effectiveness, coordination efficiency, and the government cost. With the effectiveness of an intervention, we pay attention to whether the socially optimal price, stocking factor, and/or quantity can be achieved through the intervention. With coordination efficiency of an intervention, we are interested in the following questions: a) does the intervention lead to a better expected social welfare, and b) how close is it to the socially optimal expected welfare? Besides coordination efficiency, the government cost is also a critical concern when the government chooses an intervention from multiple options. Hence, the cost should be taken into account for intervention evaluation.

### 6.5.1 Regulatory intervention mechanisms

As discussed previously, regulatory interventions are often implemented by the government to improve social welfare. As shown in Figure 6.2, we consider three government regulations such as maximum price regulation (i.e., the government restricts

the highest retail price to  $p_C^*$ ), minimum quantity regulation (i.e., the government restricts the lowest order quantity to  $q_C^*$ ), and maximum price and minimum quantity regulation (i.e., the government restricts the highest retail price to  $p_C^*$  and lowest order quantity to  $q_C^*$ ). It is worth noting that it is possible for the government to enforce the seller to sell the product at or below a given price under the maximum price regulation only when the seller's expected profit is positive at the price; otherwise, the seller would exit the market. When the demand function is additive (i.e.,  $D(p, \xi) = \xi + d(p)$ ), according to (6) in Ovchinnikov and Raz (2014), the socially optimal price is  $c - \alpha - \Theta(z_C^*)/2$ . This price is less than  $c$ , since the constant marginal benefit  $\alpha > 0$  and the expected shortage  $\Theta(z_C^*) \geq 0$ . In this case, the maximum price regulation will be ruled out, because the seller does not make a profit by selling the product. However, when the demand function is multiplicative, the socially optimal retail price might be more than the production/ordering cost after tax, i.e.,  $p_C^* > c/(1 - t)$ .

The explanation for the different results is the following. Recall that  $E(\xi) = \mu$  and  $var(\xi) = \sigma^2$ . For the additive demand function, the variance is  $var(D(p, \xi)) = var(\xi + d(p)) = \sigma^2$ , and the coefficient of variation is  $CV(D(p, \xi)) = CV(\xi + d(p)) = \sigma d(p)/(d(p) + \mu)$ . Therefore, the variance of the additive demand function is constant in  $p$  while the coefficient of variation increases in  $p$ . On the other hand, for the multiplicative demand function,  $var(D(p, \xi)) = var(\xi d(p)) = d^2(p)\sigma^2$  and  $CV(D(p, \xi)) = CV(\xi d(p)) = \sigma/\mu$ . That is, the variance of the multiplicative demand function decreases in  $p$  and the corresponding coefficient of variation is constant in  $p$ . Note that both variance and coefficient of variation measure extents of variabilities of demands and they are expected to be low. Hence, in the additive case, a lower price is charged to decrease coefficient of variation, while in the multiplicative case, a higher price is charged to decrease variance to reduce the demand uncertainty. The

following property presents a condition that ensures  $p_C^* > c/(1 - t)$ .

**Property 5.** *If*

$$\left. \frac{td(p)}{d'(p)} \right|_{p=p_C^*} + \left. \frac{(c-s)\delta(z)}{z-\delta(z)} \right|_{z=z_C^*} - \alpha > 0, \quad (6.29)$$

*then*  $p_C^* > c/(1 - t)$ .

The proof of Property 5 follows by rearranging (6.18). Note that inequality (6.29) is obviously satisfied when the sales tax  $t$  is zero, and the constant marginal externality  $\alpha$  is less than a threshold value, such as

$$\alpha < \left. \frac{(c-s)\delta(z)}{z-\delta(z)} \right|_{z=z_C^*}.$$

Thus, if  $\alpha$  and  $t$  are small enough, the condition in Property 5 holds true, and hence, the maximum price regulation is effective to coordinate the price. Recall that a higher constant marginal externality  $\alpha$  implies that more benefit is generated to the community from purchasing the product by a consumer. If the benefit is high for a product, then the goal of maximizing the expected social welfare should be achieved through lowering the price, increasing the affordability, and popularizing the product. In this case, the socially optimal price may be very low so that  $p_C^* \leq c/(1 - t)$ . Property 5 reveals scenarios where it is possible for the seller to have a positive expected profit under the mandatory maximum price  $p_C^*$ . Hence, the regulatory intervention on price is effective to coordinate the price for the multiplicative demand function for some cases. It is a significantly different result from the additive case. The following property shows the seller's optimal decisions under the three regulations.

**Property 6.** *1. Under the maximum price regulation, the seller's optimal price*

*$p_{MP}^*$  is the socially optimal price  $p_{MP}^* = p_C^*$ , and the optimal stocking factor*

$z_{MP}^*$  satisfies

$$F(z_{MP}^*) = \frac{(1-t)p_C^* - c}{(1-t)p_C^* - s}. \quad (6.30)$$

We have

$$z_{MP}^* < z_D^* < z_C^*, \text{ and } q_{MP}^* < q_C^*.$$

2. Under the minimum quantity regulation, the seller's optimal quantity  $q_{MQ}^*$  is the socially optimal quantity  $q_{MQ}^* = q_C^*$ , and the optimal retail price  $p_{MQ}^*$  satisfies

$$\begin{aligned} & \left\{ p \left[ \frac{\int_{\underline{r}}^{q/d(p)} r dF(r)}{q/d(p) - \delta(q/d(p))} \right] + \frac{d(p)}{d'(p)} \right\} \Big|_{p=p_{MQ}^*, q=q_C^*} \\ &= \frac{s}{1-t} \left[ \frac{\int_{\underline{r}}^{q/d(p)} r dF(r)}{q/d(p) - \delta(q/d(p))} \right] \Big|_{p=p_{MQ}^*, q=q_C^*}. \end{aligned}$$

We have

$$p_{MQ}^* > p_C^*, \text{ and } z_{MQ}^* = \frac{q_C^*}{d(p_{MQ}^*)} > z_c^* > z_D^*.$$

3. Under the maximum price and minimum quantity regulation, the seller's optimal quantity is  $q_{MPQ}^* = q_C^*$ , and the optimal retail price  $p_{MPQ}^* = p_C^*$ .

### Proof of Property 6:

1. Under the maximum price regulation, the seller's price  $p$  is restricted by the constraint  $p \leq p_C^*$ . From Property 4, we have  $p_D^* > p_C^*$ . Hence, the seller's optimal price,  $p_{MP}^*$ , under the maximum price regulation is binding at the constraint such that  $p_{MP}^* = p_C^*$ . We also know that, from Property 3, for a given  $p$ , the decentralized optimal stocking factor satisfies the first order condition  $\partial \Pi_D(p, z)/\partial z = 0$ . Then, for given  $p = p_C^*$ , using (6.25), the optimal stocking factor  $z_{MP}^*$  under the maximum

price regulation satisfies

$$\left. \frac{\partial \Pi_D(p, z)}{\partial z} \right|_{p=p_C^*, z=z_{MP}^*} = \{[(1-t)p[1-F(z)] + sF(z) - c]d(p)\}_{p=p_C^*, z=z_{MP}^*} = 0.$$

Thus, we have (6.30). Recall  $p_C^* < p_D^*$  and  $z_D^* > z_C^*$  from Property 4. Using (6.24), we have  $F(z_{MP}^*) < F(z_D^*)$ . Then, we have  $z_{MP}^* < z_D^* < z_C^*$  and  $q_{MP}^* < q_C^*$ , since  $q_{MP}^* = d(p_{MP}^*)z_{MP}^* = d(p_C^*)z_{MP}^* < d(p_C^*)z_C^* = q_C^*$  with  $p_{MP}^* = p_C^*$ .

2. Under the minimum quantity regulation, the seller's order quantity  $q$  is restricted by the constraint  $q \geq q_C^*$ . We will first prove that the decentralized optimal price satisfies the first order condition  $\partial \Pi_D(p, q)/\partial q = 0$ , for a given  $q$ . Then, we will show that the centralized optimal price also satisfies the first order condition  $\partial \Pi_C(p, q)/\partial q = 0$  for a given  $q$ , and compare the price decisions under the two situations. The objective function under the decentralized control can be expressed by  $p$  and  $q$  as below

$$\Pi_D(p, q) = [(1-t)pd(p)] [\mu - \theta(q/d(p))] + sd(p)\delta(q/d(p)) - cq, \quad (6.31)$$

which is obtained by substituting  $z = q/d(p)$  into (6.23). The decentralized problem in (6.31) is the typical price-setting newsvendor problem with multiplicative demand excepted that the sales revenue is taxed by  $t$ . The typical newsvendor problem is investigated by Song et al. (2009). They show, in Proposition 1 of their work, that there is a unique price that maximizes the objective function for a given quantity, and the objective function all about quantity is concave under specific assumptions. Incorporating the tax in our model, we are still able to prove the same result following the proof of Proposition 1 of Song et al. (2009). To avoid repetition, we use the result directly and inherit their assumptions, which are Assumptions 2 and 3 in this

chapter. We conclude that, for any given  $q$ , the optimal price  $p$  to  $\Pi_D(p, q)$  in (6.31) satisfies the first order condition  $\partial \Pi_D(p, q)/\partial p = 0$ , and  $\Pi_D(p(q), q)$  is concave in  $q$ , under Assumptions 2 and 3, where  $p(q)$  is the optimal price given  $q$ . Recall  $q_D^* < q_C^*$  from Properties 4. The seller's optimal quantity  $q_{MQ}^*$  under the minimum quantity regulation is binding at the constraint such that  $q_{MQ}^* = q_C^*$ . The optimal price  $p_{MQ}^*$  satisfies  $\Pi_D(p, q)/\partial p|_{p=p_{MQ}^*, q=q_C^*} = 0$  given  $q = q_C^*$ . We will characterize the relation of  $p_{MQ}^*$  and  $q_C^*$  next. Differentiating (6.31) with respect to  $p$ , we have

$$\begin{aligned} \frac{\Pi_D(p, q)}{\partial p} &= [(1-t)d(p) + (1-t)pd'(p)] [\mu - \theta(q/d(p))] + sd'(p)\delta(q/d(p)) \\ &\quad - (1-t)p[1 - F(q/d(p))] \frac{qd'(p)}{d(p)} - sF(q/d(p)) \frac{qd'(p)}{d(p)}. \end{aligned} \quad (6.32)$$

Recall that substituting  $z = q/d(p)$  into (6.12) gives

$$[1 - F(q/d(p))]q/d(p) = q/d(p) - \delta(q/d(p)) - \int_{\underline{r}}^{q/d(p)} rdF(r). \quad (6.33)$$

Dividing both sides of (6.32) by  $(1-t)d'(p)[\mu - \theta(q/d(p))]$  and using (6.33), we have

$$\begin{aligned} \frac{\frac{\Pi_D(p, q)}{\partial p}}{(1-t)d'(p)[\mu - \theta(q/d(p))]} &= p \left[ \frac{\int_{\underline{r}}^{q/d(p)} rdF(r)}{q/d(p) - \delta(q/d(p))} \right] + \frac{d(p)}{d'(p)} \\ &\quad - \frac{s}{1-t} \left[ \frac{\int_{\underline{r}}^{q/d(p)} rdF(r)}{q/d(p) - \delta(q/d(p))} \right]. \end{aligned}$$

Recalling  $\Pi_D(p, q)/\partial p|_{p=p_{MQ}^*, q=q_C^*} = 0$  and using the above equation after some alge-

bra, we have

$$\begin{aligned}
& p \left[ \frac{\int_{\underline{r}}^{q/d(p)} r dF(r)}{q/d(p) - \delta(q/d(p))} \right] + \frac{d(p)}{d'(p)} \Big|_{p=p_{MQ}^*, q=q_C^*} \\
&= \frac{s}{1-t} \left[ \frac{\int_{\underline{r}}^{q/d(p)} r dF(r)}{q/d(p) - \delta(q/d(p))} \right] \Big|_{p=p_{MQ}^*, q=q_C^*}. \tag{6.34}
\end{aligned}$$

Now, we have found the relation of  $p_{MQ}^*$  and  $q_C^*$ . To compare  $p_{MQ}^*$  and  $p_C^*$ , we need to identify the relation of  $p_C^*$  and  $q_C^*$ . In the centralized problem, the expected social welfare function,  $\Pi_C(p, q)$  expressed in  $p$  and  $q$ , can be obtained by substituting  $z = q/d(p)$  into  $\Pi_C(p, z)$  from (6.17). The expected social welfare function is then given by

$$\Pi_C(p, q) = [(1-t)p d(p) + H(p) + \alpha d(p)] [\mu - \theta(q/d(p))] + s d(p) \delta(q/d(p)) - c q.$$

First, we prove that for any given  $q$ , for  $q > 0$ , there exists a  $p(q)$  that satisfies the first order condition:

$$\frac{\partial \Pi_C(p, q)}{\partial p} \Big|_{p=p(q)} = 0.$$

For a given  $q$ , the first derivative of  $\Pi_C(p, q)$  with respect to  $p$  is given by

$$\begin{aligned}
\frac{\partial \Pi_C(p, q)}{\partial p} &= [(1-t)p + \alpha - s] d'(p) \int_{\underline{r}}^{q/d(p)} r dF(r) - t d(p) [\mu - \theta(q/d(p))] \\
&\quad - \frac{H(p)}{d(p)} d'(p) \left[ \mu - \theta(q/d(p)) - \int_{\underline{r}}^{q/d(p)} r dF(r) \right]. \tag{6.35}
\end{aligned}$$

Then, we prove by contradiction that there exists at least one  $p$  that satisfies  $\Pi_C(p, q)/\partial p =$



0 for a given  $q$ . Suppose that there exists a  $q_0$ , such that

$$\frac{\partial \Pi_C(p, q_0)}{\partial p} > 0,$$

for  $\forall p \in [s/(1-t), \bar{p}]$ . Then, the optimal price is achieved at the upper bound  $p = \bar{p}$ , where  $d(\bar{p}) = 0$  defined in Assumption 1. In this case, the expected sales is zero, and the revenue only comes from salvaging the order quantity. The expected profit is  $\Pi_C(\bar{p}, q_0) = (s - c)q_0 < 0$ , since  $s < c$  from Assumption 1. Thus, the solution  $\{\bar{p}, q_0\}$  is not possible to be optimal. Suppose that there exists a  $q_0$ , such that  $\partial \Pi_C(p, q_0)/\partial p < 0$  for  $\forall p \in [s/(1-t), \bar{p}]$ . Then, the optimal price is achieved at the lower bound  $p = \max\{s/(1-t), d^{-1}(q_0/\underline{r})\}$ . It does not make sense to set the price too low that the minimum possible demand is greater than the given quantity. So we need  $\underline{r}d(p) \geq q_0$ , equivalent to  $p \geq d^{-1}(q_0/\underline{r})$ . For the case that  $s/(1-t) \geq d^{-1}(q_0/\underline{r})$ , the lower bound is  $p = s/(1-t)$ . Then, the seller's revenue after tax is the salvage value, and hence, the expected profit is  $(s - c)q_0 < 0$ . So  $p = s/(1-t)$  cannot be the optimal decision. For the other case that  $d^{-1}(q_0/\underline{r}) > s/(1-t)$ , we have the lower bound  $p(q_0) = d^{-1}(q_0/\underline{r})$  and  $q_0/d(p(q_0)) = \underline{r}$ .

Substituting  $z = q/d(p)$  into (6.16), we have  $E_{EB}(p, q_0) = H(p)(\mu - \theta(q_0/d(p)))$ .

Differentiating the sum of (6.15) and the above equation with respect to  $p$  gives

$$\begin{aligned}
& \lim_{p \rightarrow p(q_0)} \frac{\partial[E_{CS}(p, q_0) + E_{EB}(p, q_0)]}{\partial p} \\
&= \lim_{p \rightarrow p(q_0)} \{[-d(p) + \alpha d'(p)](\mu - \theta(q_0/d(p)))\} \\
&\quad + \lim_{p \rightarrow p(q_0)} \left\{ [H(p) + \alpha d(p)] (F(q_0/d(p)) - 1) \frac{q_0/d(p)d'(p)}{d(p)} \right\} \\
&= \lim_{p \rightarrow p(q_0)} \left\{ -q_0/d(p) \frac{d'(p)}{d(p)} H(p) + \alpha d'(p) \int_{\underline{r}}^{q_0/d(p)} r dF(r) - q_0 \right\} \\
&= \lim_{p \rightarrow p(q_0)} -q_0/d(p) \frac{d'(p)}{d(p)} H(p) - q_0 > -\infty.
\end{aligned}$$

Furthermore, substituting  $z = q/d(p)$  into (6.13) and using results in Proposition 1 of Song et al. (2009), we can easily derive that

$$\lim_{p \rightarrow p(q_0)} \frac{\partial E_{SP}(p, q_0)}{\partial p} = \infty.$$

Then,

$$\lim_{p \rightarrow p(q_0)} \frac{\partial \Pi_C(p, q_0)}{\partial p} = \lim_{p \rightarrow p(q_0)} \frac{\partial \Pi_{SP}(p, q) + \Pi_{CS}(p, q) + \Pi_{EB}(p, q)}{\partial p} > 0,$$

which contradicts  $\partial \Pi_C(p, q_0)/\partial p < 0$ . Thus, there exists at least one  $p \in [s/(1-t), \bar{p}]$  satisfying  $\partial \Pi_C(p, q)/\partial p = 0$  for a given  $q$ . As we already prove that the optimal price does not occur at the boundary points, the optimal price  $p_C^*$  should satisfy the first order condition,  $\partial \Pi_C(p, q)/\partial p|_{p=p_C^*, q=q_C^*} = 0$ , at the order quantity  $q = q_C^*$ . Substituting  $\mu - \theta(q/d(p)) = q/d(p) - \delta(q/d(p))$  from (6.33) into (6.35) and using (6.34), when the optimal decision  $\{p = p_{MQ}^*, q = q_C^*\}$  under the minimum quantity

regulation is used, we obtain

$$\begin{aligned} \frac{\partial \Pi_C(p, q)}{\partial p} \Big|_{p=p_{MQ}^*, q=q_C^*} &= \frac{1}{1-t} d'(p) [q/d(p) - \delta(q/d(p))] \cdot \\ &\left\{ \frac{\alpha \int_r^{q/d(p)} r dF(r)}{q/d(p) - \delta(q/d(p))} - \frac{d(p)}{d'(p)} + \frac{H(p)}{d(p)} \left( 1 - \frac{\int_r^{q/d(p)} r dF(r)}{q/d(p) - \delta(q/d(p))} \right) \right\} \Big|_{p_{MQ}^*, q_C^*}. \end{aligned}$$

From (6.12), we have  $q/d(p) - \delta(q/d(p)) - \int_r^{q/d(p)} r dF(r) > 0$ , and hence, the last term of the above equation is positive. Recall  $d'(p) < 0$ . We have  $\partial \Pi_C(p, q)/\partial p|_{p=p_{MQ}^*, q=q_C^*} < 0$ . Recalling  $\partial \Pi_C(p, q)/\partial p|_{p=p_C^*, q=q_C^*} = 0$ , we conclude  $p_C^* < p_{MQ}^*$  and  $z_{MQ}^* = q_C^*/d(p_{MQ}^*) > z_C^* > z_D^*$ .

3. Under both maximum price and minimum quantity regulations, when both constraints  $p \leq p_C^*$  and  $q \geq q_C^*$  are applied, the optimal retail price  $p_B^*$  and quantity  $q_B^*$  are binding at  $q_B^* = q_C^*$  and  $p_B^* = p_C^*$ , because  $q_{MP}^* < q_C^*$  and  $p_{MQ}^* > p_C^*$ .  $\square$

There are several implications about the regularity interventions discussed in Property 6. First, if the government sets the highest retail price to the socially optimal price, then the seller would charge the socially optimal price and order less than the socially optimal quantity. Second, if the government regulates the lowest quantity as the socially optimal quantity, then the seller would order the socially optimal quantity and charge a price higher than the socially optimal price. Third, if the government regulates both the highest price and the lowest quantity, then the seller uses the socially optimal decisions. However, from Property 2, we know that the seller's expected profit is non-positive if the socially optimal price and the order quantity are used. Hence, the third regulation is not applicable in a real-world situation. In addition, note that  $p_C^* > c/(1-t)$  is not a sufficient condition for the seller to have a positive expected profit. If the order quantity is required to be high (i.e., under the minimum quantity regulation) and any leftover is sold at a salvage

value  $s < c$ , then a loss may occur for the seller. Hence, similar to the maximum price regulation, the minimum quantity regulation is also only applicable when the seller have a positive expected profit. Otherwise, if the government intends to practice such regulations that result in a loss, the government has to compensate the seller for the expected profit loss.

### 6.5.2 *Market intervention mechanisms*

Although it is possible for the maximum price and minimum quantity regulations to coordinate the retail price and the order quantity, respectively, the regulations suffer from several limitations. First, a regulation is not effective for coordination when the seller's expected profit is non-positive. Second, an effective regulation, i.e., the maximum price or minimum quantity regulation, cannot align the seller's both decisions simultaneously. Third, there is little flexibility for the seller in choosing the price/quantity decision, under the regulation restricting price/quantity. The inflexibility may eliminate the seller's incentive to collaborate with the government and continue the business to sell a public interest good in long term. To address these problems, in this section, we investigate three market intervention mechanisms, including tax cut, consumer rebate, and cost subsidy mechanisms, and their combinations. We characterize the seller's and consumers' behaviors under the interventions and identify the interventions' effectiveness and efficiency of coordination.

In the sequel, we will first introduce the three market interventions. Then, we derive expressions for the seller's expected profit and the government's expected cost for a generalized case where all three market interventions are applied. Given the general expressions, formulas for each special case, where only one or two of the three market interventions are applied, can be identified easily.

#### 6.5.2.1 Tax cut

Tax cut is reduction in taxes. It usually serves as an important intervention implemented by the government to leverage price and stimulate sales for a product (see e.g., Dardan and Stylianou (2000)). Under the tax cut intervention, the government imposes a lower tax rate  $T$  on the product than the original rate  $t$ . Note that, to increase the flexibility of the mechanism, we do not restrict the new tax rate  $T$  on a positive range, and we allow for the tax rate reduced to a negative value. In this case, the government pays money back instead of charging a tax to the seller to stimulate the seller to make the socially optimal decisions. The tax cut would increase the seller's profit with no raise in the retail price. Hence, the government's expected revenue is decreased, while the seller's revenue from selling a unit product is expected to rise. It gives the seller an incentive to sell more by ordering more quantity and/or charging a lower price. Under the tax cut mechanism, the seller's expected profit function is the same as (6.13) except using the new tax rate  $T$  instead of  $t$ .

#### 6.5.2.2 Consumer rebate

A consumer rebate is a payment transferred from the government to a consumer for each unit that the consumer purchases, with the intention of increasing the affordability of the product. The common types of rebates include cash back, vouchers, and coupons. In this chapter, we assume that cash back is used. Let  $R$  denote the consumer rebate per unit. Let  $p'$  be the retail price the seller charges and  $p$  be the effective price that consumers actually pay for the product after the rebate. Hence, we have  $p = p' - R$ . With rebates, the product is expected to be affordable to more consumers. This intervention provides an incentive for the seller to order more quantity or charge a lower price to satisfy more consumers. For the notational and computational convenience, we model the seller's behavior using the effective price

$p$ , instead of the seller's price  $p'$ , because it is the effective price that determines the amount of demand. In particular, with rebates, the seller's revenue per unit sold is  $p' = p + R$ . Hence, the seller's expected profit with rebates  $E_{SP}(p, z, R)$  can be obtained by (6.13) by replacing  $p$  with  $p + R$  for the seller's revenue per unit sold given by

$$\Pi_R(p, z) = d(p) [(1 - t)(p + R)(\mu - \theta(z)) - cz + s\delta(z)]. \quad (6.36)$$

It is interesting to note that the consumer rebate intervention is equivalent to another intervention, which we call “the seller rebate intervention”. Under the seller rebate intervention, the rebate  $R$  is given to the seller as a cash back for each unit sold, instead of the consumer. Taking  $p$  as the price charged to consumers and  $p + R$  as the seller's revenue per unit sold, the seller's expected profit is the same as the one in (6.36) and the expected government cost also remains unchanged. In this sense, these two intervention mechanisms, the consumer rebate and the seller rebate, are equivalent. The seller rebate intervention is also similar to the sales subsidy intervention considered by Taylor and Yadav (2011), under which the donor pays the sales subsidy to the retailer.

#### 6.5.2.3 Cost subsidy

The cost subsidy is a payment transferred from the government to the seller for each unit the seller purchases or produces, with the intention of inducing the seller to keep more quantity and charge a less retail price. Suppose the cost subsidy is denoted by  $S$ . The cost subsidy given to the seller decreases the effective purchase or production/ordering cost from  $c$  to  $c - S$ . Note that, with subsidies, the seller's optimization problem is equivalent to the decentralized problem given by (6.13) with production/ordering cost  $c - S$ . Thus,  $c - S > s$  is required in practice. Otherwise,

if  $c - S \leq s$ , the seller would order as much as possible because the seller can always make a profit by salvaging the leftovers.

Although the three interventions provide different incentives, all of them can be considered as mechanisms to compensate the seller in different ways. In particular,  $S$  is the compensation paid by the government to the seller for each product ordered,  $R$  is the compensation paid by the government to the seller for each product sold, and tax difference  $t - T$  is the compensation paid by the government for each dollar earned by the seller.

#### 6.5.2.4 Generalized expressions for intervention mechanisms

Next, we investigate a combination of the three interventions and derive the expressions for the seller's expected profit and the government's expected cost when the combination is applied.

Suppose that the government executes tax cut, consumer rebate, and cost subsidy intervention mechanisms at the same time: a new tax rate  $T$  is charged to the seller, a consumer rebate  $R$  is applied on each unit sold to consumers, and a cost subsidy  $S$  is applied on each unit ordered to the seller. Then, the seller's cost per unit ordered is reduced from  $c$  to  $c - S$ , and the seller's revenue per unit sold is increased from  $p$  to  $p + R$ , recalling that,  $p$  refers to the effective price after the rebate. Hence, by replacing  $c$  with  $c - S$  for the seller's unit cost, replacing  $p$  with  $p + R$  for the seller's unit revenue, and replacing the original tax rate  $t$  with the new tax rate  $T$  in the seller's expected profit function given by (6.13), we obtain the seller's expected profit under the combination of the three market interventions as below:

$$E_{SP}(p, z, T, R, S) = d(p) [(1 - T)(p + R)(\mu - \theta(z)) - (c - S)z + s\delta(z)]. \quad (6.37)$$

Meanwhile, the expected revenue of the government is reduced due to the de-

creased tax revenue, the rebates paid to all customers purchasing the product, and the subsidies paid to the seller for ordered units. The government's expected cost,  $E_{GC}(p, z, T, R, S)$ , is measured as the difference between the government's expected revenues without and with the interventions. Without the interventions, the government's expected revenue, the tax rate times the seller's expected revenue (see (6.11)), is given by

$$tpd(p)(\mu - \theta(z)). \quad (6.38)$$

The government pays a subsidy to the seller for each unit the seller orders. The payment is given by

$$Szd(p) \quad (6.39)$$

where  $q = zd(p)$  is the seller's order quantity. The government also pays rebates to all consumers who purchase the product. The expected payment is  $R$  times the seller's expected sales (see (6.11)) given by

$$Rd(p)(\mu - \theta(z)). \quad (6.40)$$

Furthermore, the government's expected tax revenue with the tax rate  $T$  is

$$T(p + R)d(p)(\mu - \theta(z)). \quad (6.41)$$

Thus, the government's expected revenue under the interventions is the expected tax revenue net the expected costs on subsidies and rebates in (6.41), (6.39), and (6.40), respectively, given by

$$T(p + R)d(p)(\mu - \theta(z)) - Rd(p)(\mu - \theta(z)) - Szd(p). \quad (6.42)$$



Therefore, using (6.38) and (6.42), the government's expected cost is given by

$$\begin{aligned}
E_{GC}(p, z, T, R, S) &= tpd(p)(\mu - \theta(z)) \\
&\quad - \{T(p + R)d(p)(\mu - \theta(z)) - Rd(p)(\mu - \theta(z)) - Szd(p)\} \\
&= d(p)(\mu - \theta(z))[(t - T)(p + R) + (1 - t)R] + Szd(p).
\end{aligned}$$

Note that the expressions for the consumers' expected surplus and the expected externality benefit given by (6.15) and (6.16) are not affected by interventions. Table 6.1 shows expressions for expected values of the seller's profit, consumers' surplus, externality benefit and the government cost as  $E_{SP}(p, z, T, R, S)$ ,  $E_{CS}(p, z)$ ,  $E_{EB}(p, z)$  and  $E_{GC}(p, z, T, R, S)$ , respectively. Here, for instance,  $E_{SP}(p, z, T, R, S)$  represents the seller's expected profit when the tax rate  $T$ , the rebate  $R$  and the subsidy  $S$  are applied and the seller's decisions are  $\{p, z\}$ . Using these generalized terms in Table 6.1, it is easy to express one term for any intervention by setting parameters of unapplied interventions as zero, i.e.,  $S = 0$  or the original value, i.e.,  $T = t$ . For example, the seller's expected profit, when only tax cut is applied, is shown as below:

$$E_{SP}(p, z, T, 0, 0) = d(p) [(1 - T)p(\mu - \theta(z)) - cz + s\delta(z)].$$

Table 6.1: Generalized expressions of terms under market intervention mechanisms.

Component	Expressions
$E_{SP}(p, z, T, R, S)$	$d(p) [(1 - T)(p + R)(\mu - \theta(z)) - (c - S)z + s\delta(z)]$
$E_{CS}(p, z)$	$H(p) [\mu - \theta(z)]$
$E_{EB}(p, z)$	$\alpha d(p) [\mu - \theta(z)]$
$E_{GC}(p, z, T, R, S)$	$(t - T)(p + R)d(p)(\mu - \theta(z)) + (1 - t)Rd(p)(\mu - \theta(z)) + Szd(p)$

For presentational convenience, we use  $(T, R, S)$  to represent an intervention when the tax rate  $T$ , the cost subsidy  $S$ , and the consumer rebate  $R$  are applied. Specifically,  $(T, 0, 0)$ ,  $(t, R, 0)$ , and  $(t, 0, S)$  present the tax cut, the rebate, and the subsidy interventions, respectively, where  $t$  is the original tax rate. Similarly,  $(T, R, 0)$ ,  $(T, 0, S)$ , and  $(t, R, S)$  represent the joint tax-rebate, the joint tax-subsidy, and the joint rebate-subsidy mechanisms, respectively.

### 6.5.3 Coordination performance under market intervention mechanisms

As mentioned earlier in Section 6.5.2, intervention parameters  $T$ ,  $R$ , and  $S$  cannot be chosen arbitrarily. That is, the compensation by the government cannot be set so unrewarding or rewarding that the seller's decision is always to order nothing or to order as much as possible under an intervention. We know that, for the cost subsidy,  $c - S > s$  is required. It is also natural to assume that the seller's revenue per unit sold after rebate and tax cut is equal to or greater than the salvage value, so that  $(1 - T)(p + R) \geq s$  holds. The conditions that ensure a meaningful intervention are summarized in Assumption 4.

**Assumption 4.** 1. *The seller's cost after cost subsidy is greater than the salvage value, i.e.,  $c - S > s$ .*

2. *The seller's revenue after rebate and tax cut is greater than the salvage value, i.e.,  $(1 - T)(p + R) > s$  for  $p \leq \bar{p}$ .*

Property 7 characterizes the seller's optimal price and the optimal stocking factor under a combination of interventions, when Assumption 4 is satisfied.

**Property 7.** *Under an intervention mechanism  $L = (T, R, S)$ , where  $T$ ,  $R$ , and  $S$  satisfy Assumption 4, the optimal retail price  $p_L^*$  and the stocking factor  $z_L^*$  that*

maximize the seller's expected profit satisfy

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_L^*} = \frac{1}{1-T} \left[ (c-S) + \frac{(c-S-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_L^*},$$

$$F(z_L^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_L^*},$$

and  $q_L^* = d(p_L^*)z_L^*$ .

**Proof of Property 7:** The proof follows the proof of Property 3. For the notational consistence with  $\Pi_C(p, z)$  and  $\Pi_D(p, z)$ , which have a single-character subscript, denote the seller's expected profit under the government intervention  $(T, R, S)$  as  $\Pi_G(p, z, T, R, S)$ . Obviously  $\Pi_G(p, z, T, R, S) = E_{SP}(p, z, T, R, S)$  given by (6.37). Taking the first and second derivatives of  $E_G(p, z, T, R, S)$  with respect to  $z$  and using  $\theta(z) = -(1 - F(z))$  and  $\delta'(z) = F(z)$ , we obtain

$$\begin{aligned} & \frac{\partial \Pi_G(p, z, T, R, S)}{\partial z} \\ &= [(1-T)(p+R)d(p)] [1 - F(z)] + sd(p)F(z) - (c-S)d(p), \end{aligned} \quad (6.43)$$

and

$$\begin{aligned} \frac{\partial^2 \Pi_G(p, z, T, R, S)}{\partial z^2} &= -[(1-T)(p+R)d(p)] f(z) + sd(p)f(z) \\ &= -[((1-T)(p+R) - s) d(p)] f(z) < 0, \end{aligned}$$

because  $(1-T)(p+R) - s > 0$  from Assumption 4. Therefore,  $\Pi_G(p, z, T, R, S)$  is strictly concave in  $z$  for a given  $p$  and the optimal  $z$  satisfies the first order condition  $\partial \Pi_G(p, z, T, R, S) / \partial z = 0$ . In addition, taking the first and second derivatives of

$E_G(p, z, T, R, S)$  with respect to  $p$ , we have

$$\begin{aligned} \frac{\partial \Pi_G(p, z, T, R, S)}{\partial p} &= [(1 - T)d(p) + (1 - T)(p + R)d'(p)] (\mu - \theta(z)) \\ &\quad + (s\delta(z) - (c - S)z)d'(p), \end{aligned} \quad (6.44)$$

and

$$\begin{aligned} \frac{\partial^2 \Pi_G(p, z, T, R, S)}{\partial p^2} &= [(1 - T)d(p) + (1 - T)(p + R)d'(p)]' (\mu - \theta(z)) \\ &\quad + (s\delta(z) - (c - S)z)d''(p). \end{aligned} \quad (6.45)$$

Next, we will prove that  $\Pi_G(p, z, T, R, S)$  is concave in  $p$  at any stationary point that satisfies the first order condition  $\partial \Pi_G(p, z, T, R, S)/\partial p = 0$  for a given  $z$ . Suppose that  $p_0$  is a stationary point such that  $\partial \Pi_G(p, z, T, R, S)/\partial p|_{p=p_0} = 0$ . Using (6.44), we obtain the relation as below:

$$s\delta(z) - (c - S)z = - \left. \frac{[(1 - T)d(p) + (1 - T)(p + R)d'(p)] (\mu - \theta(z))}{d'(p)} \right|_{p=p_0}.$$

Substituting the above equation into (6.45), we have

$$\begin{aligned} &\left. \frac{\partial^2 \Pi_G(p, z, T, R, S)}{\partial p^2} \right|_{p=p_0} \\ &= [(1 - T)d(p) + (1 - T)(p + R)d'(p)]' (\mu - \theta(z)) - \\ &\quad \left. \frac{[(1 - T)d(p) + (1 - T)(p + R)d'(p)] (\mu - \theta(z))d''(p)}{d'(p)} \right|_{p=p_0}. \end{aligned}$$

By multiplying with  $d'(p)$ , the above equation can be written as

$$\begin{aligned} & d'(p) \frac{\partial^2 \Pi_G(p, z, T, R, S)}{\partial p^2} \Big|_{p=p_0} \\ = & (1-T) [(d(p) + pd'(p))' d'(p) - (d(p) + pd'(p)) d''(p)] (\mu - \theta(z)) \Big|_{p=p_0}. \end{aligned}$$

According to Assumption 2,  $p + d(p)/d'(p)$  is strictly increasing such that

$$\left( \frac{d(p) + pd'(p)}{d'(p)} \right)' > 0.$$

The inequality is equivalent to  $(d(p) + pd'(p))' d'(p) - (d(p) + pd'(p)) d''(p) > 0$ . It is obvious  $\mu - \theta(z) > 0$  by (6.12). Then, we have

$$d'(p) \frac{\partial^2 \Pi_G(p, z, T, R, S)}{\partial p^2} \Big|_{p=p_0} > 0 \quad \text{and} \quad \frac{\partial^2 \Pi_G(p, z, T, R, S)}{\partial p^2} \Big|_{p=p_0} < 0.$$

Hence,  $\Pi_G(p, z, T, R, S)$  is concave at any stationary point  $p_0$ . Thus, there is a unique stationary point  $p_0$  for a given  $z$  which is the optimal price. Dividing both sides of  $\partial \Pi_G(p, z, T, R, S)/\partial p = 0$  given by (6.44) by  $(1-T)d'(p)(\mu - \theta(z))$  and using (6.12), we obtain

$$p + R + \frac{d(p)}{d'(p)} - \frac{(c-S)z - s\delta(z)}{(1-T)(\mu - \theta(z))} = p + R + \frac{d(p)}{d'(p)} - \frac{(c-S)z - s\delta(z)}{(1-T)(z - \delta(z))} = 0.$$

Rearranging the above equation and  $\partial \Pi_G(p, z, T, R, S)/\partial z = 0$  given by (6.43), we observe that the optimal price  $p_L^*$  and the stocking factor  $z_L^*$  satisfy

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_L^*} = \frac{1}{1-T} \left[ (c-S) + \frac{(c-S-s)\delta(z)}{z - \delta(z)} \right] \Big|_{z=z_L^*}, \quad \text{and}$$

$$F(z_L^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_L^*}. \quad \square$$

Using the optimal decisions in Property 7, next, we explore how the government determines the intervention parameters  $T$ ,  $R$ , and  $S$  to coordinate the seller's decisions. Without considering the stocking factor, Ovchinnikov and Raz (2014) also discuss the coordination of the price and order quantity. Since the percentage of satisfied demand (i.e., service level) revealed by  $F(z)$  is also important for consumers when they purchase a public interest good, it is necessary to consider the coordination of stocking factor  $z$  in the analysis.

**Property 8.** *Under an intervention mechanism  $L = (T, R, S)$ , for  $L \in \{(T, 0, 0), (t, R, 0), (t, 0, S), (T, R, 0), (T, 0, S), (t, R, S)\}$ , where  $T$ ,  $R$ , and  $S$  satisfy Assumption 4,*

- *To coordinate the retail price, the corresponding intervention parameters and the stocking factor  $z_L^*$  should satisfy*

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_C^*} = \frac{1}{1-T} \left[ (c-S) + \frac{(c-S-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_L^*} \quad (6.46)$$

and

$$F(z_L^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_C^*};$$

- *To coordinate the stocking factor, the corresponding intervention parameters and the retail price  $p_L^*$  should satisfy*

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_L^*} = \frac{1}{1-T} \left[ (c-S) + \frac{(c-S-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_C^*}$$

and

$$F(z_C^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_L^*};$$

- To coordinate the order quantity, the corresponding intervention parameters, the retail price  $p_L^*$  and the stocking factor  $z_L^*$  should satisfy

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_L^*} = \frac{1}{1-T} \left[ (c-S) + \frac{(c-S-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_L^*},$$

$$F(z_L^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_L^*}, \text{ and } z_L^* = \frac{q_C^*}{d(p_L^*)}; \text{ and}$$

- To coordinate the retail price and the stocking factor, the corresponding parameters should satisfy

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_C^*} = \frac{1}{1-T} \left[ c - S + \frac{(c-S-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_C^*} \quad (6.47)$$

and

$$F(z_C^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_C^*}. \quad (6.48)$$

**Proof of Property 8:** When the price is aligned with the socially optimal price such that  $p_L^* = p_C^*$ , according to Property 7, the optimal stocking factor  $z_L^*$  and the corresponding intervention parameters should satisfy

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_L^*} = \frac{1}{1-T} \left[ c - S + \frac{(c-S-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_C^*}, \text{ and}$$

$$F(z_C^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_L^*}.$$

When the stocking factor is aligned with the socially optimal stocking factor such that  $z_L^* = z_C^*$ , according to Property 7, the optimal price  $p_L^*$  and the corresponding

intervention parameters should satisfy

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_L^*} = \frac{1}{1-T} \left[ c - S + \frac{(c-S-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_C^*}, \text{ and}$$

$$F(z_C^*) = \frac{(1-T)(p+R) - (c-S)}{(1-T)(p+R) - s} \Big|_{p=p_L^*}.$$

When the order quantity is aligned with the socially optimal quantity such that  $q_L^* = z_L^* d(p_L^*) = q_C^*$ , according to Property 7, the optimal price  $p_L^*$ , the optimal stocking factor  $z_L^*$ , and the corresponding intervention parameters should satisfy

$$p + R + \frac{d(p)}{d'(p)} \Big|_{p=p_L^*} = \frac{1}{1-T} \left[ c + \frac{(c-s)\delta(z)}{z-\delta(z)} \right] \Big|_{z=z_L^*},$$

$$F(z_L^*) = \frac{(1-T)(p+R) - c}{(1-T)(p+R) - s} \Big|_{p=p_L^*}, \text{ and}$$

$$z_L^* = \frac{q_C^*}{d(p_L^*)}. \quad \square$$

From Property 8, we observe that in order to coordinate one decision (i.e., the retail price, the stocking factor, or the order quantity), at least one intervention mechanism has to be implemented. This is because when one decision variable is fixed to be the centralized one, one additional degree of freedom on variable  $T$ ,  $R$ , or  $S$  has to be added to satisfy the two equations in each part of Property 8. Furthermore, in order to coordinate both decisions simultaneously, a combination of two market interventions is required for a similar reason. According to Property 8, the system coordination can be achieved through properly setting intervention parameters that make (6.47) and (6.48) hold true.

While applying the tax cut intervention  $(T, 0, 0)$  and the subsidy intervention  $(t, 0, S)$ , several issues should be considered. First, there does not always exist a tax



rate  $T$  that leads the government to coordinate the price in regards to  $(T, 0, 0)$ . Note that the RHS of (6.46) is always positive at  $S = 0$ ,  $R = 0$ , and  $T < 1$ . When the LHS of (6.46) is negative, there does not exist a tax rate  $T < 1$  that satisfies the equation. In addition, there also does not always exist a cost subsidy  $S$  to coordinate the price under  $(t, 0, S)$  for a similar reason. Note that the RHS of (6.46) is positive at  $R = 0$ ,  $S < c - s$ , and  $t < 1$ . When the LHS of (6.46) is negative, there does not exist a cost subsidy  $S < c - s$  that satisfies the equation. Therefore, a single tax cut intervention and a single cost subsidy intervention are not always effective for coordinating the price. This observation will be demonstrated by numerical examples in Section 6.6.

To the best of our knowledge, we are the first to develop implementable structures of joint interventions for a price-setting newsvendor problem with multiplicative demand uncertainty to coordinate the system. The results provide the government several ways to coordinate the price and quantity decisions for a public interest good to achieve the optimal expected social welfare. Even when the system coordination cannot be achieved via a single intervention, the affordability or/and availability of the public interest product can be improved in comparison to the decentralized decision in a free market. With these available mechanisms, to fulfill coordination goals for the public interest good, the government might reasonably combine and tailor the intervention mechanisms according to its political and economical climate.

## 6.6 Application on diversified products

The purpose of this section is to examine coordination performance of the intervention mechanisms of interest on diversified products. We consider six typical public interest goods as presented in Table 6.2. The examples of products 1, 2, 3, and 4 are from Ovchinnikov and Raz (2014). Product 5 represents a typical public interest good with relatively low externality compared to the production cost. Product 6

represents an example for a new launched product, whose demand function is modeled based on the reservation-price model (see the definition of the reservation-price model in Van Ryzin (2005)). Using this model, the demand function can be predicted and obtained through a survey from potential consumers on their reservation prices for the new product, since the historical sales data is not available.

For each product, we simulate 121 scenarios using 11 different values for  $c$  and  $a$  (the 11 sample values in each range are taken with equal intervals starting with the lower bound and ending with the upper bound), respectively. We briefly summarize observations that apply to all these scenarios without presenting all results to avoid repetition:

- Among the **two regularity interventions**, the maximum price regulation leads to a smaller expected profit for the seller than the minimum quantity regulation, and which regulation results in a higher expected social welfare is indeterminant.
- Among the **three single market interventions**, the rebate mechanism leads to the least loss of the expected social welfare regardless of the goal in comparison to the tax cut and the subsidy mechanisms.
- Among the **three joint market interventions** that achieve system coordination, the joint tax-subsidy mechanism is the most cheapest and the joint rebate-subsidy is the most expensive for the government.

## 6.7 Conclusion

This chapter considers a social welfare setting, in which a public interest good is distributed by a newsvendor-type seller to consumers with stochastic demand depending on retail price. We investigate the joint optimal retail price and order quan-

Table 6.2: Characteristics of six different public interest products.

	Example	Cost $c$	$a, d(p) = a - p$	$[r, \bar{r}]$	$\alpha$	t	s
Product 1	Eco-consumable	2 - 4	10 - 15	$[0.95, 1.05]$	0.5	0.1	0
Product 2	Energy-star air conditioner	100 - 200	300 - 500	$[0.5, 1.5]$	20	0.1	30
Product 3	Vaccine	12.5-17.5	30 - 50	$[0.95, 1.05]$	10	0.1	2
Product 4	Emergency power generator	300-500	700 - 900	$[0.5, 1.5]$	200	0.1	50
Product 5	Low externality	180 - 220	240 - 260	$[0.5, 1.5]$	5	0.1	30
Product 6	Reservation-price model	6-7	$d(p) = 1 - bp$ $b \in [1/11, 1/9]$	$[50, 150]$	0.1	0.1	0

tity decisions that maximize the expected social welfare, and maximize the seller's expected profit without government interventions, respectively. We demonstrate that the price and quantity decisions made by the seller in decentralized control without government interventions never reach socially optimal levels. Specifically, the optimal order quantity is lower and the optimal price is higher in decentralized control (maximizing the seller's expected profit) than the centralized control (maximizing the expected social welfare). Motivated by these observations, we investigate intervention mechanisms implemented by the government, including regulative interventions, market interventions, and combinations of market interventions, to align the seller's decisions with the socially optimal ones. Since the socially optimal price can be more than the seller's production/ordering cost considering the effect of the tax, the maximum price regulation enables the government to coordinate the retail price under the multiplicative demand function. In addition, we demonstrate that applying one of the three market interventions, the tax cut, the cost subsidy, and the consumer rebate mechanisms, can lead to socially optimal level of only one decision, the price, the stocking factor or the quantity. Applying a combination of two market interventions can lead to the socially-optimal levels of both price and quantity decisions simultaneously.

We also compare the effectiveness, the efficiency and the government cost of different mechanisms. In terms of the coordination effectiveness, the rebate mechanism

is more effective than the subsidy, which is more effective than the tax cut mechanism. The minimum quantity regulation is more effective than the maximum price regulation. In terms of coordination efficiency, rebate is the best followed by subsidy and tax cut, which perform similarly. The comparison on efficiency between the two regulations is inconclusive. More importantly, the government's cost under the joint tax-rebate mechanism and the joint tax-subsidy mechanism for the system coordination is less than using any single market intervention to coordinate the price or the quantity. Since the joint tax-subsidy leads to a non-positive expected profit for the seller, the joint tax-rebate is the best option for the government in terms of its effectiveness, efficiency and the government cost. We provide several ways to coordinate the channel using appropriate incentive schemes for a public interest good by the government. They provide insights of the role of government involved in public interest good distribution programs.

In a summary, to the best of our knowledge, this chapter makes the first attempt in the literature to analyze social welfare in the price-setting newsvendor model for a public interest good under the multiplicative demand function. We show that the multiplicative demand function is derived based on consumers' choices to maximize their utility, and hence, it can be used to express demand for utility-maximizing consumers. Then, it can be employed in both the profit maximization problem to price private goods and in the social welfare maximization problem to price public interest goods. This observation contrasts with the statement by Ovchinnikov and Raz (2014), who argue that the welfare analysis fails with the multiplicative demand. Furthermore, we have shown the empirical and analytical importance of applying the multiplicative demand in several ways.

As we have mentioned previously, the government's decision on choosing a suitable intervention mechanism is dependent on the demand function. In addition,

Kling (1989) argues that the estimation of consumers' surplus is heavily sensitive to the choice of demand function. Hence, it requires careful investigation on how the uncertain is modeled in the demand function. The error from misusing demand function could be significant for the calculation of social welfare and the decision of interventions. Several ways are optional to decide the form of demand as follows. Based on the different properties of the two functions, given sales data of prices and demands, one method is to calculate variance and coefficient of variation of demand at different prices. As explained in Section 6.5.1, if variance is consistent, the additive demand is appropriate; and if the coefficient of variation is consistent, the multiplicative demand is appropriate. The demand can be also decided based on another observation that the price elasticity of demand remains invariant to any realization of demand variation under the multiplicative form, according to Driver and Valletti (2003). The multiplicative demand is appropriate if the property holds true for the data. Furthermore, Kling (1989) mentions three different ways to choose deterministic demand functions based on intuition, goodness-of-fit tests, and the utility function, respectively. After estimating the deterministic function, we can calculate both the difference error and the ratio error, and then test the randomness of both: if the randomness of the first is significant, then the additive function is appropriate; and if the second one is significantly random, then the multiplicative function is appropriate. Especially, for a new launched product when historical sales data is not available, the reservation-price model with a multiplicative demand function is plausible, thus allowing us to express the relationship between prices and demands through surveying reservation prices of potential consumers.

We believe there are several possible extensions of this topic. For example, this chapter considers a newsvendor problem when a public interest good is distributed by a single seller. It will be interesting to consider competitive situations when there are

multiple sellers selling a product simultaneously. It remains unknown whether these intervention mechanisms also work for channels where pricing competition among sellers exists.

## 7. CONCLUSION AND FUTURE DIRECTIONS

In this dissertation, we investigate contractual pricing problems for retail distribution under different channel structures. In particular, we consider supplier-buyer (e.g., manufacturer-retailer) channels under which powerful entities (e.g., mass retailers or government) take the lead in designing contracts. Characterized by such powerful entities, two classes of contractual problems are studied related to buyer and government, respectively, in this dissertation.

In the first class of problems, we examine contractual coordination efforts with an emphasis on buyer-driven contracts. we propose a new buyer-driven contract, called the generic contract, and examine its performance in different supplier-buyer channels.

First, we consider a basic single-product setting where a supplier sells a product to a buyer. We show that the generic contract is a simple, general, effective, and practical coordination contract that has several advantages relative to the existing buyer-driven contracts. Next, we generalize the basic single-product setting to the multi-product bilateral monopolistic setting where a supplier sells multiple products to a buyer. We show that even in the case of asymmetric two products, the generic contract coordinates the system under which the optimal contract parameters are easy to calculate, and the contract is easy to implement. We also study the generic contract in the exclusive dealer setting in the generalized asymmetric case. In this setting, the contract allows each buyer to extract the system profit on the product less the corresponding supplier's reservation profit, while each supplier can only obtain the reservation profit.

Applying the generic contract in different channel structures, we demonstrate

that the contract is amenable to generalization for handling multi-product, multi-supplier, and multi-buyer settings.

In the second class of problems, we study a newsvendor problem for a private retailer where contractual government interventions are implemented for social welfare maximization.

We study two new government regulatory mechanisms, and a new market intervention along with two existing market interventions. We examine coordination performance of these intervention mechanisms and also investigate the impact of demand uncertainty. Our results demonstrate that the two government regularity mechanisms are effective in improving the expected social welfare and using a combination of any two market interventions achieves the optimal expected social welfare. In particular, using a combination of the new market intervention and one existing market intervention costs the government less than using the combination of the two existing market interventions.

We believe that there are several possible extensions related to buyer-driven contracts studied in this dissertation. One interesting area is the contractual performance of the generic contract under information asymmetry in the single- and multi-product settings. The benefit of leadership and coordination performance under the contract is impacted by incomplete information.

Another extension is to investigate the counterpart supplier-driven contract corresponding to the generic contract and to provide a comparative analysis of the counterpart contracts in the single- and multi-product settings. The concept of “counterpart contract” has been proposed by Liu and Çetinkaya (2009), who develop **the counterpart buyer-driven contracts** corresponding to three general types of supplier-driven contracts that have been studied by Corbett and Tang (1999): the one-part linear contract, the two-part linear contract, and the two-part nonlinear



contract. Another important type of supplier-driven contract is known as the **one-part nonlinear contract**. To the best of our knowledge, the counterpart supplier- and buyer-driven contracts under the one-part nonlinear scheme have never been investigated in current literature. We aim to fill the gap and investigate the relationship between the generic contract and the buyer-driven contract under the one-part nonlinear scheme.

Furthermore, according to Corbett and Tang (1999), more sophisticated contracts, e.g., contracts with more contract parameters, potentially offer increased contract flexibility for negotiation for the channel leader. With the complete results for the counterpart supplier- and buyer-driven contracts under one-part linear, one-part nonlinear, two-part linear, and two-part nonlinear schemes, we are interested in investigating the value of offering more sophisticated contracts under both leaderships.

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## APPENDIX A

### REVENUE-SHARING CONTRACT

We derive the optimal contract under  $b5$  considering the supplier's reservation profit  $\pi_s^- \in [0, \Pi^c]$ . Using the definition of  $b5$  given in Section 3.6.1, (3.1), (3.2), and (3.3), the supplier's and buyer's profits are given by

$$\begin{aligned}\pi_s &= (w - s)q + (1 - \phi)pq = (1 - \phi)(p - s)(a - bp) \quad \text{and} \\ \pi_b &= (p - w - c)q - (1 - \phi)pq = (p - \phi s - c)q - (1 - \phi)pq \\ &= \phi(p - s - c)q - (1 - \phi)cq = \phi(p - s - c)(a - bp) - (1 - \phi)c(a - bp),\end{aligned}$$

respectively. Then, considering  $\pi_s^- \in [0, \Pi^c]$ , assumption (3.5), and the two above expressions for  $\pi_s$  and  $\pi_b$ , **the buyer's optimization problem under  $b5$**  can be stated as

$$(Pb5) : \quad \max_{0 \leq \phi \leq 1, p \geq s+c} \quad \pi_b = \phi(p - s - c)(a - bp) - (1 - \phi)c(a - bp) \quad (\text{A.1})$$

$$s.t. \quad \pi_s = (1 - \phi)(p - s)(a - bp) \geq \pi_s^-. \quad (\text{A.2})$$

Using (A.1), observe that

$$\begin{aligned}\frac{\partial \pi_b}{\partial p} &= \phi(a - 2bp + bs) + bc \quad \text{and} \\ \frac{\partial^2 \pi_b}{\partial p^2} &= -2\phi b \leq 0.\end{aligned}$$

Hence,  $\pi_b$  is concave in  $p$ . Setting  $\partial\pi_b/\partial p = 0$  leads to

$$p^{b5}(\phi) = \frac{a + b(s + c)}{2b} + \frac{(1 - \phi)bc}{2b\phi}. \quad (\text{A.3})$$

Also, using (A.1), observe that

$$\frac{\partial\pi_b}{\partial\phi} = (p - s)(a - bp) > 0.$$

Hence,  $\pi_b$  is increasing in  $\phi$ ,  $\phi \in [0, 1]$ . Therefore,  $\pi_b$  achieves the maximum at the boundary of  $\phi = 1$  or  $\pi_s = \pi_s^-$ .

- When  $\phi = 1$ , by (A.2),  $\pi_s = 0$  and (A.2) are satisfied only if  $\pi_s^- = 0$ . In this case, the supplier does not make any profit. The case is not practical and can be discarded.
- If  $\pi_s^- > 0$  then  $\pi_s = \pi_s^-$  and  $\phi < 1$ . Using (A.3) in (A.2), we have

$$\pi_s = (1 - \phi)b \left[ \left( \frac{a - bs}{2b} \right)^2 - \left( \frac{c}{2\phi} \right)^2 \right] = \pi_s^-. \quad (\text{A.4})$$

Next, we consider  $c = 0$  and  $c > 0$ , separately.

- If  $c = 0$  then using (A.4),

$$\pi_s = (1 - \phi) \frac{(a - bs)^2}{4b} = \pi_s^- \Rightarrow \phi^{b5} = 1 - \frac{\pi_s^-}{\Pi^c},$$

where  $\Pi^c = (a - bs)^2/4b$  given in (3.11). Also, (A.3) implies

$$p^{b5} = \frac{a + bs}{2b} = p^c.$$

Since  $\phi^{b5}$  and  $p^{b5}$  are realizable over the regions in (A.1), they characterize the optimal  $b5$ . Then, using the definition of  $b5$ , (A.1) and (A.2), we have

$$\begin{aligned} L^{b5} &= \frac{(a + bs)(a - bs)\pi_s^-}{4b\Pi^c}, \\ w^{b5} &= \left(1 - \frac{\pi_s^-}{\Pi^c}\right) s, \\ \pi_s^{b5} &= \pi_s^-, \\ \pi_b^{b5} &= \Pi^c - \pi_s^-, \quad \text{and} \\ \Pi^{b5} &= \Pi^c, \end{aligned}$$

where superscript for  $b5$  is used in an obvious fashion. Clearly, the optimal  $b5$  is the coordination contract if  $c = 0$ .

- If  $c > 0$  then using (A.3),  $p^{b5}(\phi) > p^c = \frac{a+b(s+c)}{2b}$  is true for all  $\phi \in [0, 1)$ .

Hence, the contract is not a coordination contract if  $c > 0$ .