STOCHASTIC CLEARING MODELS WITH APPLICATIONS IN SHIPMENT CONSOLIDATION

A Dissertation

by

BO WEI

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Chair of Committee,	Sıla Çetinkaya
Co-Chair of Committee,	Daren B.H. Cline
Committee Members,	Guy L. Curry
	Richard M. Feldman
Head of Department,	César O. Malavé

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ABSTRACT

This dissertation focuses on the average cost and service performance models in the shipment consolidation setting, which is treated as an application of stochastic clearing models. Specifically, we consider generalized control policies, generalized demand pattern, multi-item systems, and alternative performance criteria, where various techniques in stochastic analysis and stochastic optimal control are applied. By using stochastic impulsive control technique, we prove that, in the single item shipment consolidation model with drifted Brownian motion demand, the optimal quantity-based policy achieves the least average cost in the long run, among the admissible policies. In multi-item shipment consolidation model, we propose a $(Q+\tau)$ policy and an instantaneous rate policy. We prove that among all $(Q + \tau)$ policies, either a quantity-based policy or a time-based policy is optimal in terms of average cost. Furthermore, we demonstrate that the optimal instantaneous rate policy would dominate the optimal $(Q + \tau)$ policy in terms of average cost. In terms of service performance criteria, we propose average order delay in the single-item case and average weighted delay rate in the multi-item case. From a martingale point of view, we provide a unified method to calculate the service measures. Moreover, by revealing new properties of truncated random variables, we provide comparative results among different control policies in terms of the service measures. Finally, we provide an analytical integrated inventory/hybrid consolidation model, and give comparative results in the integrated inventory/shipment consolidation models in terms of service measures and average cost.

DEDICATION

To my parents and my sister

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I express my deepest gratitude to my advisor Dr. Sila Çetinkaya for guiding me into the supply chain management area, and teaching me the necessary abilities in academic community. I also express my deepest thanks to my co-advisor Dr. Daren B.H. Cline for leading me into the stochastic processes area, and demonstrating how to think when encountering difficulties in research. Further, I also thank Dr. Guy L. Curry and Dr. Richard M. Feldman for serving as members of my advisory committee and providing their valuable knowledge to me.

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1. INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

"A stochastic clearing system is characterized by a non-decreasing stochastic input process $\{Y(t), t \geq 0\}$*, where* $Y(t)$ *is the cumulative quantity entering the system in* $[0, t]$*, and an output mechanism that intermittently and instantaneously clears the system, that is, removes all the quantity currently present"* (Stidham, 1974). Stidham (1974) considers the case that the system is cleared when the quantity in the system, y, exceeds the threshold q, and derives the explicit expression of the limiting distribution of the quantity in the system. Stidham (1977) studies the optimal level of q, to minimize the average cost, where there are fixed clearing and variable holding costs. For the other work in stochastic clearing systems, see Whitt (1981), Stidham (1986), Boxma et al. (2001), Yang et al. (2002), and Kella et al. (2003).

Shipment consolidation is the strategy of combining small size shipments or customer orders, i.e., input process realizations, into a larger load. The purpose of shipment consolidation is achieving scale economies and increasing resource utilization. The customer orders represent the stochastic input process. The consolidated loads are dispatched at specific times that correspond to clearing instances with random loads and possibly random clearing times. Hence, a shipment consolidation system can be considered as a stochastic clearing system. For practical examples, the reader is referred to Çetinkaya and Bookbinder (2003).

Early work in shipment consolidation model focuses on simulation approaches. For a review of earlier work, see Çetinkaya (2005). More recent work places an emphasis on analytical models. A detailed account of the analytical literature is provided in Çetinkaya (2005) and Mutlu et al. (2010), and existing analytical models can be classified as deterministic models and stochastic models. Our focus here is only on stochastic models.

Çetinkaya et al. (2014) indicate that *"three classes of shipment consolidation policies are common in practice: quantity-based policy (QP), time-based policy (TP), and hybrid policy (HP). The QP is aimed at consolidating a load of q units before releasing a shipment. There are two types of TPs. Under the first, called TP1, a shipment is made every T units of time, and all orders that arrive between the two shipment epochs are consolidated. Under the other, called TP2, the arrival time of the first order after a shipment is recorded, and the next shipment is made T time units after the arrival time of the first order. Likewise, there are two types of HPs. The first is a combination of QP and TP1, called HP1, and the second is a combination of QP and TP2, called HP2. Stated formally, under HP1, the goal is to consolidate a load of size q. However, if the time since the last shipment epoch exceeds T, then a shipment decision is made. Under HP2, the goal is also to consolidate a load of size q; but, if the waiting time of the first order after the last shipment exceeds T, then a shipment decision is made".* Figure 1.1 illustrates the three classes of shipment consolidation policies with continuous input process.

According to Çetinkaya (2005), existing stochastic models for shipment consolidation can be classified into two groups. The first group is pure consolidation models where the shipment consolidation policy is implemented without coordination. The work in this category considers the practical policies introduced above, for the purpose of providing optimization techniques to obtain the optimal parameters and comparing the cost and service performance of the policies (Bookbinder and Higginson, 2002; Çetinkaya and Bookbinder, 2003; Mutlu et al., 2010; Çetinkaya et al., 2014). Another line of research is the analysis of integrated shipment consolida-

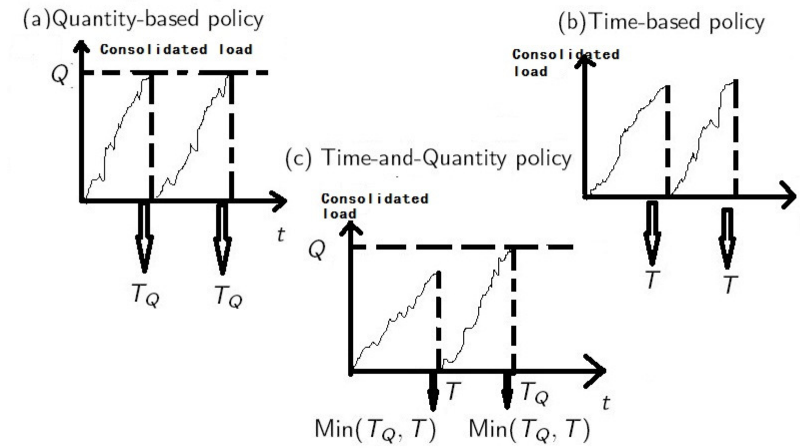

Figure 1.1: Three Classes of Shipment Consolidation Policies with Continuous Input Process.

tion/inventory decisions. We call these integrated inventory/shipment consolidation models. This approach aims at investigating the impact of alternative consolidation policies in the context of joint inventory and transportation decisions (Çetinkaya and Lee, 2000; Axsäter, 2001; Çetinkaya et al., 2006, 2008; Marklund, 2011).

Our work contributes to understanding both pure consolidation models and integrated inventory/shipment consolidation models, in several aspects, by treating them as applications of stochastic clearing systems. For pure consolidation models, we consider:

- generalized control policies;
- generalized demand patterns;
- multi-item systems; and
- alternative performance criteria (cost versus service).

We also study integrated models and build on the existing literature on integrated models through our formal results for pure consolidation models. In particular, we develop cost and service based integrated models under alternative control policies.

More specifically, the proposed research is presented as follows:

- In Chapter 2, we propose to prove the optimality of quantity-based policy among the admissible policies, not limited to alternative practical policies, in terms of average cost, in the single item shipment consolidation model with drifted Brownian motion demand.

- In Chapter 3, we propose a generalized control policy, called a $(Q + \tau)$-policy, in the multi-item shipment consolidation model with drifted Brownian motion demand, and show that the optimal policy among all $(Q + \tau)$-policies is either a quantity-based policy or a time-based policy.

- In Chapter 4, we consider service performance for shipment consolidation model with Poisson process input and provide comparison results among alternative policies.

- In Chapter 5, we propose a new control policy, called an instantaneous rate policy, in the multi-item shipment consolidation model with drifted Brownian motion demand, and show that the optimal instantaneous rate policy achieves the least average cost, among a large class of renewal type clearing policies. We also provide comparison results among alternative policies in terms of service measure.

- In Chapter 6, we develop cost and service performance models in integrated models under alternative control policies, and provide comparison results in terms of average cost and service measures.

Previous analytical work on shipment consolidation models assumes the input process (also referred as demand process or arrival process) is a Poisson process (Çetinkaya and Lee, 2000; Çetinkaya et al., 2006; Mutlu et al., 2010), or a renewal process (Çetinkaya and Bookbinder, 2003; Çetinkaya et al., 2008), or a discrete time Markov chain (Higginson and Bookbinder, 1995; Bookbinder et al., 2011). We model demand instead with a drifted Brownian motion. Also, all previous work in shipment consolidation considers alternative practical shipment consolidation policies for the single-item case. Ours is the first work considering a multi-item joint transportation model and aiming at obtaining an optimal control policy among a large class of admissible policies. Except for Higginson and Bookbinder (1994), Çetinkaya et al. (2006), and Çetinkaya et al. (2014), most previous work investigates the cost criterion under different policies. We provide a unified method to calculate the average order delay, which is an important indicator for the service performance, from a martingale point of view. We also strengthen results in Çetinkaya et al. (2014) by revealing new properties of truncated random variables, that refine comparison results among different policies. Further, we generalize the service performance model for shipment consolidation system with drifted Brownian motion demand instead of Poisson process demand.

1.2 Related Literature

By Theorem 3.3.5 in Ross (1996) (p.108), a renewal demand process with inter-arrival time having a mean of $\frac{1}{\lambda}$ and a standard deviation σ_0 can be approximated by a drifted Brownian motion $\lambda t + \lambda^{3/2}\sigma_0 B(t)$.

One justification for modeling the demand process by Brownian motion has been given as follows: *"The sample paths of Brownian motion have infinite variation and this it cannot represent the difference between a potential input process and a potential* *output process. Nonetheless, a netput process may be well approximated by Brownian motion under certain conditions. To understand these conditions, recall that Brownian motion is the unique stochastic process having stationary, independent increments and continuous sample paths; unbounded variation follows as a consequence of these primitive properties. Also note that the total variation of a netput process over any given interval equals the sum of potential input and potential output over that interval. If such a netput process is to be well approximated by Brownian motion, both potential input and potential output must be large for intervals of moderate length, but their difference (netput itself) must be moderate in value. We may express this state of affairs by saying that we have a system of balanced high-volume flows. Pulling together several times, we conclude that Brownian motion may reasonably approximate the netput process for a system of stationary, continuous, balanced high-volume flow, where netput increments during non-overlapping intervals are approximately independent"* (Harrison, 1985). This argument has been confirmed in practice by heavy traffic conditions in queueing theory that lead to diffusion approximations.

The first models applying diffusion processes for inventory systems and dams are done in Bather (1966) and Bather (1968). For work using Brownian motion, see Harrison and Taylor (1978), Harrison and Taksar (1983), Harrison (1985), Vickson (1986), Lam and Lou (1987), and Dixit (1991). Harrison and Taksar (1983) consider a storage system whose content follows a drifted Brownian motion. The content level can be increased or decreased instantaneously with a proportional cost without a fixed cost. The objective is to minimize the expected discounted costs of holding costs and controls costs. The optimal policy turns out to be the one which keeps the controlled content level within certain boundaries, exerting the minimal effort required to do so. In fact, the cumulative input and output controls are continuous but not absolutely continuous, increasing on a set of Lebesgue measure zero. This instantaneous control belongs to singular control problems.

Impulsive control theory is developed to deal with optimal control problems where there is a fixed cost associated with each control (Richard, 1977; Bensoussan and Tapiero, 1982; Bensoussan and Lions, 1984). By using dynamic programming, the impulsive control problem would boil down to solving quasi-variational inequalities. Bather (1966) is the first paper that applies impulsive control to prove the optimality of an (s, S) policy in inventory models. For applications of impulsive control technique in the optimality of an (s, S) policy in inventory models, see Sulem (1986), Beyer and Sethi (1998), Bensoussan et al. (2005), Presman and Sethi (2006), Benkherouf (2008), Benkherouf and Bensoussan (2009).

Beyer and Sethi (1998) provide a rigorous proof for EOQ formula using quasi-variational inequalities. Presman and Sethi (2006) prove for the first time that an (s, S) policy is optimal in the case that demand is a compound Poisson process plus a constant rate component, with both the average and discounted cost criteria. Benkherouf and Bensoussan (2009) and Bensoussan et al. (2005) assume the demand is a mixture of a drifted Brownian motion and a compound Poisson process.

Harrison et al. (1983) consider a storage system whose content follows a drifted Brownian motion without control, and the storage level can be adjusted by any desirable level at any time as long as the content is kept nonnegative. Implementing positive or negative jumps incurs a fixed plus variable costs. It is shown that a control band policy minimizes expected discounted costs by using impulsive control. Ormeci et al. (2008) prove the control band policy is optimal even with constrains on the maximum inventory level and on the sizes of the adjustments to the inventory by using Lagrangian relaxation techniques.

Cadenillas et al. (2010) assume that a company's inventory level is a mean-reverting process, and aims at keep the inventory as close as possible to a given level. The manager can purchase or sell an amount of the goods to adjust the inventory level with both fixed and proportional costs. The total cost is minimized by determining the optimal stopping times and adjustment magnitudes.

Impulsive control theory is extensively applied in cash management (Constantinides, 1976; Constantinides and Richard, 1978; Baccarin, 2002, 2009). For an excellent review about impulsive control applications in mathematical finance, see Korn (1999).

In all previous work in shipment consolidation, only practical consolidation policies have been investigated. We apply stochastic impulsive control theory to prove the optimality of a certain policy among a general class of admissible policies in shipment consolidation area by assuming the demand process is a drifted Brownian motion. See Chapter 2. As far as we know, this is the first work in proving optimality in the shipment consolidation setting.

Another motivation of our research comes from control policies in queueing theory. Yadin and Naor (1963) introduce the concept of a controllable queueing system. Yadin and Naor (1963) and Heyman (1968) study the N-policy, where the server restarts providing service until there are N waiting customers in the system after the end of last busy period. Heyman (1977) introduces the T-policy, where the server reactivates T units time after his removal when there are no customers in the system, and shows that the optimal N-policy performs better than the optimal T-policy in terms of average cost. Balachandran (1973) and Balachandran and Tijms (1975) introduce the D-policy, which is to turn the server on when the total workload for all customers in the waiting line reaches D. Boxma (1976) shows that the optimal D-policy performs better than the optimal N-policy if the holding cost is the waiting cost per unit workload per unit time. Feinberg and Kella (2002) consider a class of regenerative policies, and shows that the D-policy is the best one within the class, by applying optimal stopping arguments. Artalejo (2002) shows that the D-policy is not necessarily superior than the N-policy, if the holding cost is based on the expected number of customers in the system. Gakis et al. (1995) consider the distributions and first moments of the busy and idle periods in controllable M/G/1 queueing systems operating under simple and dyadic policies. Artalejo (2001) and Chae and Park (2001) focus on the queue length analysis of the M/G/1 queue under the D-policy. Lillo and Martin (2000) consider a $(\mathscr{P}+\tau)$-policy, which is to turn on the server at a random time τ later than $\mathscr{P}$. It investigates necessary and sufficient conditions such that the $(\mathscr{P}+\tau)$-policy performs better than the $\mathscr{P}$-policy, in terms of average cost criteria, where the holding cost is the waiting cost per unit time per customer. Lee and Seo (2008) study the performance of the M/G/1 queue under the dyadic Min(N,D)-policy and its cost optimization. In particular, the optimal Min(N,D)-policy is compared with the optimal N-policy and the optimal D-policy under two linear models, one based on the accumulated workload and the other one based on the customers number. Inspired by Lillo and Martin (2000), we propose a $(Q+\tau)$-policy that dispatches the consolidated load at an independent random time τ after the time it takes to accumulate Q, in the multi-item shipment consolidation model with drifted Brownian motion demand. We show that τ should be a constant in the $(Q+\tau)$-policy model when the average cost is minimized. Further, we provide the sufficient and necessary conditions such that the $(Q+\tau)$-policy achieves lower average cost than the Q-policy. Furthermore, we show the jointly optimal $(Q+\tau)$-policy can only be either quantity-based policy or time-based policy, depending on the parameter values. For the multi-item model with Poisson process input as a special case, we show that the jointly optimal $(Q+\tau)$-policy is a quantity-based policy, see Chapter 3. Further, we propose an instantaneous rate policy, which is shown to be the optimal one among a large class of renewal type clearing policies, in terms of average cost, by applying a martingale argument. See Chapter 5. As far as we know, this is the first work that considers the multi-item shipment consolidation model.

Our results for the joint optimality of $(Q + \tau)$-policy and the optimality of the instantaneous rate policy, demonstrate the value of information in optimization and control for dynamic system, especially for stochastic dynamic systems. Specifically, in a stochastic dynamic system where there is a fixed cost associated with each control, the optimal policy can only be triggered by a threshold limit. That is, the optimal policy can only be a closed-loop policy, and cannot be an open-loop policy. That is why a time-based policy cannot be optimal. Further, in a multi-item system, the optimal policy should be one which requires tracking each state associated with each item. It can not be the one that just tracks the sum of all states. That is why a quantity-based policy can not be optimal, while the instantaneous rate policy can be optimal.

Another line of related research lies in vehicle dispatching, which is also one application of stochastic clearing models. Readers are referred to Ross (1969), Tapiero and Zuckerman (1979), Zuckerman and Tapiero (1980) and Robin and Tapiero (1982). Ross (1969) considers an optimal dispatching problem for a Poisson process $N(t)$ with rate λ, where all items are dispatched at time T. An intermediate dispatch time needs to be selected to minimize the total waiting time of all items. It is shown that the optimal intermediate dispatch time should be the smallest t, such that $N(t) \geq \lambda(T-t)$. Robin and Tapiero (1982) study the vehicle dispatching policy with non-stationary Poisson arrival and provides the quasi-variational inequalities for optimality of dispatching policies by applying impulsive control technique. Tapiero and Zuckerman (1979) propose three policies for vehicle dispatching, (i) a C-capacity policy; (ii) a dispatching frequency policy T; (iii) a (T, C) policy. The average cost models are derived under the three policies, and they consider the competition issue between two firms adopting certain vehicle dispatching policies. Zuckerman and Tapiero (1980) consider a random vehicle dispatching problem with options to send rented vehicles, and determines the firm's optimal fleet size to minimize the average cost.

It is worth noting that Higginson and Bookbinder (1994) and Çetinkaya et al. (2006) consider the service performance of the practical shipment consolidation policies introduced above. According to the simulation result in Higginson and Bookbinder (1994), QP achieves lower average cost than TP2 and HP2. However, in terms of average waiting time, HP2 outperforms QP and TP2 when parameter values are fixed. Using simulation, in the integrated inventory/shipment consolidation setting, Çetinkaya et al. (2006) reveal that, although HP is not superior to QP in terms of the cost criteria, it is superior in terms of a service measure: average waiting time. As we have emphasized, however, the observations in Higginson and Bookbinder (1994) and Çetinkaya et al. (2006) are based on detailed simulation studies.

Very recently, Çetinkaya et al. (2014) attempt to provide an analytical comparison for the maximum waiting time (MWT) and the average waiting time per order (AOD). Specifically, they show that under fixed policy parameters, q and/or T, HP outperforms QP and TP, in terms not only of $P(MWT > t)$, but also of AOD. Under the fixed expected consolidation cycle length, QP achieves the least AOD, compared with all other practical policies.

In terms of service measure, the existing work focuses on computing the average order delay for each practical policy under Poisson process input. We propose a unified method based on a martingale point of view to calculate the average order delay for a general class of policies, both for Poisson process input and drifted Brownian motion input. See Chapter 4 and 5. We show that QP achieves the lowest AOD, compared with all other renewal type clearing policies, with a fixed dispatching frequency. In particular, we demonstrate that the AOD under HPs can be expressed in terms of truncated random variables. Therefore, the essential difficulty underlying comparisons among QP, TPs and HPs is uncovering more refined properties about truncated random variables. A noteworthy result is that, with a fixed dispatching frequency, the hybrid policy achieves less AOD and less average cost than the time-based policy, which justifies the advantage of the hybrid policy.

For the integrated inventory/shipment consolidation model, some service measures are proposed. The first measure is the average inventory holding rate (AIR), the second measure is the average order delay (AOD), and the last measure is the average squared order delay (ASOD), which is proposed if the customers are not patient and place more penalty on longer time delay. We have shown that under the same expected replenishment and consolidation cycle length, QP performs the best, TP performs the worst in terms of AIR, and HP lies between QP and TP. Moreover, after identifying certain properties of Poisson random variable, we provide comparison results in terms of AOD and AOSD. Finally, based on the comparison results in terms of service criteria, we obtain insight into comparisons of average cost among the three integrated models. See Chapter 6.

2. ON THE OPTIMALITY OF QUANTITY-BASED POLICY IN THE SINGLE-ITEM SHIPMENT CONSOLIDATION MODEL: A QVI METHOD

In the chapter, we consider the single-item shipment consolidation problem when the demand is a drifted Brownian motion. We provide a rigorous proof to show the optimal quantity-based policy achieves the minimum of the long-run average cost among a large class of admissible policies by using quasi-variational inequalities method. In particular, we derive the quasi-variational inequalities corresponding with the problem and construct the solution, which provides an average optimal dispatching policy.

2.1 Problem Formulation

Assume that the demand process of the item $N(t)$ is a Brownian motion with drift given by $N(t) = Dt + \sigma B(t)$, where $D > 0$, $\sigma > 0$ are two constants, denoting drift coefficient and diffusion coefficient, respectively. $B(t)$ is a standard Brownian motion.

Parameters:

A_D: Fixed cost of dispatching

c: Unit transportation cost

ω: customer waiting cost for the item per unit per unit time, which represents the loss-of-goodwill penalty.

We define the class of admissible dispatching policies. Let $\mathscr{F}_t, t > 0$ be the σ filtration generated by the $\{N(s) : 0 < s \leq t\}$ and $\mathscr{F}_0$=$\{\emptyset, \Omega\}$. Let $\theta_i \geq 0, i = 1, 2, \ldots$ be a sequence of $\{\mathscr{F}_t\}$-stopping time, $\theta_n \nearrow +\infty$ as $n \nearrow \infty$, and let $q_i > 0, i = 1, 2, \ldots$ be a sequence of impulse values such that for each $i = 1, 2, \ldots$, q_i takes positive value but not greater than the current consolidated load and q_i is measurable with respect to $\mathscr{F}_{\theta_i}$. Clearly, θ_i and q_i denote the $i-th$ dispatching time point and dispatching quantity, respectively. $U = \{\theta_1, q_1; \theta_2, q_2; \ldots\}$ is an admissible policy. We denote the set of admissible policies by $\mathscr{U}$. Clearly, the common used shipment consolidation policies (time-based policy, quantity-based policy, and hybrid policy) are included in the admissible policies.

The consolidated load process $z(t)$ is continuous almost everywhere except at the dispatching time points $t = \theta_1, \theta_2, \ldots$. Denote $z(\theta_i-)$ as the left limit of the consolidated load process and obviously, $z(\theta_i) = z(\theta_i-) - q_i$.

Definition 2.1. *Let $U = (\theta_1, q_1, \theta_2, q_2, \ldots)$ be an admissible policy and $z(.)$ the associated consolidated load process. U is called stable with respect to the function $u(.)$, if*

$$\lim_{n\to\infty} \frac{E[u(z(\theta_n-))]}{E[\theta_n]} = 0.$$

The consolidated load process corresponding with an admissible policy $U = (\theta_1, q_1, \theta_2, q_2, \ldots)$ is described as follows:

$$\begin{cases} z(t) = x + \lambda t + \sigma B(t) - \sum_{\{i:\theta_i \le t\}} q_i, \\ z(0-) = x. \end{cases}$$

The average cost functional is as follows:

$$F_0(x, U) = \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T w(z(s))ds + \sum_{\{i:\theta_i<T\}} c(q_i)\right],$$

where $w(z) = \omega z$ and $c(q) = A_D 1_{q>0} + cq$ denote the waiting cost rate and dispatching cost, respectively.

2.2 Q-policy Model

In this section, we consider the quantity-based consolidation policy, which dispatches a consolidated load when an economical dispatch quantity q is available. Since the demand $N(t)$ is continuous, the dispatching quantity is exactly q.

Define $T_q = \inf\{t > 0 : N(t) \geq q\}$,which is a stopping time w.r.t the filtration generated by $B(t)$. Clearly, the successive outbound shipping time intervals $S_1, S_2 \ldots$ are i.i.d, each has the same distribution as the random variable T_q. We have the following result which characterizes T_q.

Lemma 2.2. *For $s > 0$,*

$$E[\exp(-sT_q)] = \exp\left(-\frac{\sqrt{D^2 + 2s\sigma^2} - D}{\sigma^2} q\right),$$

$$E[T_q] = \frac{q}{D}, \quad E[T_q^2] = \frac{q^2}{D^2} + \frac{\sigma^2 q}{D^3}.$$

In fact, T_q has the inverse Gaussian distribution.

We compute the cumulative amount waiting per consolidation cycle.

$$\begin{aligned}
& E[\text{Cumulative Waiting per Consolidation Cycle}] \\
= \quad & E\left[\int_0^{T_q} N(t)dt\right] \\
= \quad & E\left[tN(t) \mid_{t=0}^{T_q}\right] - E\left[\int_0^{T_q} tdN(t)\right] \\
= \quad & qE[T_q] - \frac{1}{2} DE[T_q^2] - E\left[\int_0^{T_q} \sigma t dB(t)\right].
\end{aligned}$$

We compute the third term in the following. Let

$$g(s) = \int_0^s tdB(t) \quad \text{and} \quad \eta(s) = \left(\int_0^s tdB(t)\right)^2 - \frac{1}{3}s^3.$$

Clearly, $g(s)$ and $\eta(s)$ are two martingales. So,

$$E[\eta(T_q \wedge s)] = E[\eta(0)] = 0, \quad \text{for any } s \geq 0,$$

that is,

$$E\left[\left(\int_0^{T_q \wedge s} t dB(t)\right)^2\right] = \frac{1}{3}E[(T_q \wedge s)^3] \leq \frac{1}{3}E[T_q^3] < \infty,$$

which implies that

$$g(T_q \wedge s) = \int_0^{T_q \wedge s} t dB(t)$$

is a square integrable martingale, thus a uniformly integrable martingale. Therefore,

$$E[g(T_q)] = E[\int_0^{T_q} t dB(t)] = 0$$

by optional stopping theorem, i.e., the third term is 0. Therefore,

$$\begin{aligned} & E[\text{Cumulative Waiting per Consolidation Cycle}] \\ = \quad & E\left[\int_0^{T_q} N(t)dt\right] = qE[T_q] - \frac{1}{2}DE[T_q^2] = \frac{q}{2D}\left(q - \frac{\sigma^2}{D}\right). \end{aligned}$$

The expected total long-run average cost per unit-time is

$$C(q) = \frac{A_D + cq + \omega(\frac{q^2}{2D} - \frac{\sigma^2 q}{2D^2})}{q/D} = \frac{A_D D}{q} + \frac{1}{2}\omega q + cD - \frac{\omega\sigma^2}{2D}.$$

Minimizing $C(q)$ over q, we obtain the optimal dispatching quantity

$$q^{opt} = \sqrt{\frac{2A_D D}{\omega}},$$

and the minimized average cost

$$C(q^{opt}) = \sqrt{2\omega A_D D} + cD - \frac{\omega\sigma^2}{2D}.$$

2.3 Optimal Dispatching Policy and Quasi-Variational Inequalities

The discounted cost functional is as follows:

$$F_r(x, U) = E\left[\int_0^\infty e^{-rs} w(z(s))ds + \sum_i c(q_i)e^{-r\theta_i} 1_{\{\theta_i<\infty\}}\right].$$

Denote $V_r(x) = \inf_{U\in\mathscr{U}} F_r(x, U)$.

From the definition of $V_r(x)$ and the dynamic programming principle, we have the following inequalities

$$\begin{cases} V_r(x) \le E\left[\int_0^\theta e^{-rs}w(z(s))ds + V_r(z(\theta-))e^{-r\theta}\right], \forall \text{ stopping time } \quad \theta \ge 0, \\ V_r(x) \le c(q) + V_r(x-q), \forall 0 \le q \le x. \end{cases} \tag{2.1}$$

For our analysis for average cost problem, we define the potential function

$$u_r(x) = V_r(x) - V_r(0).$$

The first inequality of formula (2.1) can be written as

$$u_r(x) \le E\left[\int_0^\theta e^{-rs} w(z(s))ds + u_r(z(\theta-))e^{-r\theta}\right] - V_r(0)E[1 - e^{-r\theta}].$$

Assume $u_r(x) \to u(x)$ and $rV_r(0) \to h$ as $r \to 0$, then we have

$$u(x) \le E\left[\int_0^\theta w(z(s))ds + u(z(\theta-)) - h\theta\right]. \tag{2.2}$$

By applying $It\hat{o}'s$ formula to $u(z(\theta-))$, we obtain

$$E[u(z(\theta-))] = u(x) + E\left[\int_0^\theta \mathcal{L}u(z(s))ds\right], \tag{2.3}$$

where $\mathcal{L}u(z) = Du'(z) + \frac{1}{2}\sigma^2 u''(z)$ is the infinitesimal generator of the consolidated load process, which is a drifted Brownian motion.

By replacing (2.3) into (2.2), we have

$$E\left[\int_0^\theta w(z(s))ds + \int_0^\theta \mathcal{L}u(z(s))ds - h\theta\right] \geq 0.$$

Dividing by θ and taking limits as $\theta \to 0$ yields

$$w(x) + \mathcal{L}u(x) - h \geq 0.$$

In addition, the second inequality of formula (2.1) under the variable change is

$$u(x) \leq c(q) + u(x-q), \forall 0 \leq q \leq x.$$

In sum, formula (2.1) can be written as

$$\begin{cases} w(x) + \mathcal{L}u(x) - h \geq 0, \forall x, \\ u(x) \leq c(q) + u(x-q), \forall 0 \leq q \leq x. \end{cases}$$

Theorem 2.3. *Suppose* $(u(z), h)$ *satisfies*

$$\begin{cases} w(z) + \mathcal{L}u(z) - h \geq 0, \forall z, \\ u(z) \leq c(q) + u(z-q), \forall 0 \leq q \leq z, \end{cases} \tag{2.4}$$

then for any admissible policy U *which is stable with respect to* $u(.)$,

$$F_0(x, U) \geq h.$$

Proof. Let any $U = (\theta_1, q_1, \theta_2, q_2, \ldots)$, by using $It\hat{o}'s$ formula and the first inequality of (2.4),

$$\begin{aligned}
& E[u(z(\theta_{k+1}-))] - E[u(z(\theta_k))] \\
= \quad & E\left[\int_{\theta_k}^{\theta_{k+1}} \mathcal{L}u(z(s))ds\right] \\
\geq \quad & hE[\theta_{k+1} - \theta_k] - E\left[\int_{\theta_k}^{\theta_{k+1}} w(z(s))ds\right]. \qquad (2.5)
\end{aligned}$$

Further, from the second inequality of (2.4),

$$E[u(z(\theta_k-))] \leq E[c(q_k) + u(z(\theta_k))]. \qquad (2.6)$$

Combining (2.5) and (2.6), we have for $k = 1, 2, \ldots$

$$hE[\theta_{k+1} - \theta_k] \leq E\left[\int_{\theta_k}^{\theta_{k+1}} w(z(s))ds + u(z(\theta_{k+1}-)) - u(z(\theta_k-)) + c(q_k)\right]. \qquad (2.7)$$

During the time before the first dispatch, by using $It\hat{o}'s$ formula and the first inequality of (2.4),

$$E[u(z(\theta_1-))] - u(x) = E\left[\int_0^{\theta_1} \mathcal{L}u(z(s))ds\right] \geq E\left[\int_0^{\theta_1} (h - w(z(s)))ds\right],$$

which is

$$hE[\theta_1] \leq E\left[\int_0^{\theta_1} w(z(s))ds + u(z(\theta_1-))\right] - u(x), \qquad (2.8)$$

adding over $k = 1, 2, \ldots, n$ in (2.7) with (2.8), and dividing by $E[\theta_{n+1}]$,we have

$$h \leq \frac{E\left[\int_0^{\theta_{n+1}} w(z(s))ds + \sum_{k=1}^{n} c(q_k)\right]}{E[\theta_{n+1}]} - \frac{u(x)}{E[\theta_{n+1}]} + \frac{E[u(z(\theta_{n+1}-))]}{E[\theta_{n+1}]}.$$

As $n \to \infty$, the last two terms converge to 0 because of the stability of U with respect to $u(.)$ and $\theta_n \to +\infty$. Therefore,

$$h \leq \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T w(z(s))ds + \sum_{\{i:\theta_i<T\}} c(q_i)\right] = F_0(x, U), \forall U \in \mathscr{U}.$$

$\square$

Theorem 2.4. *Let $(u(z), h)$ be a solution of (2.4), and there exists a q^* such that*

$$w(z) + \mathcal{L}u(z) - h = 0, \forall z \in D \triangleq \{z|z < q^*\}, \tag{2.9}$$

$$u(z) = \inf_{0<q\leq z}\{c(q) + u(z-q)\}, \forall z \in D^c = \{z|z \geq q^*\}, \tag{2.10}$$

and denote $q^*(z) = \arg\inf\{c(q) + u(z-q)\}, \forall z \geq q^*$. Let U^* be the policy that dispatching $q^*(z)$ if the current consolidated load $z \geq q^*$ and no dispatch if $z < q^*$. Assume that U^* is stable with respect to $u(.)$. Then, $F_0(x, U^*) = h$.

Proof. Let $z^*(s)$ be the consolidated load process under the policy U^*. Applying $It\hat{o}'s$ formula to $u(z)$ over the time $[\theta_k, \theta_{k+1})$

$$E[u(z^*(\theta_{k+1}-))] - E[u(z^*(\theta_k))] = E\left[\int_{\theta_k}^{\theta_{k+1}} \mathcal{L}u(z^*(s))ds\right]. \tag{2.11}$$

Further, we have

$$E[u(z^*(\theta_k-))] = E[c(q_k^*) + u(z^*(\theta_k))]. \tag{2.12}$$

Combining (2.11) and (2.12), we have for $k = 1, 2, \ldots$

$$E[u(z^*(\theta_{k+1}-)) - u(z^*(\theta_k-)) + c(q_k^*)] = E\left[\int_{\theta_k}^{\theta_{k+1}} \mathcal{L}u(z^*(s))ds\right]. \tag{2.13}$$

During the time before the first dispatch, by using $It\hat{o}'s$ formula,

$$E[u(z^*(\theta_1-))] - u(x) = E\left[\int_0^{\theta_1} \mathcal{L}u(z^*(s))ds\right]. \tag{2.14}$$

Adding over $k = 1, 2, \ldots, n$ in (2.13) with (2.14), and adding

$$E\left[\int_0^{\theta_{n+1}} w(z^*(s))ds\right] - hE[\theta_{n+1}]$$

on both sides, we have

$$\begin{aligned} & E\left[\int_0^{\theta_{n+1}} w(z^*(s))ds + \sum_{k=1}^{n} c(q_k^*)\right] + E[u(z^*(\theta_{n+1}-))] - u(x) - hE[\theta_{n+1}] \\ = \quad & E\left[\int_0^{\theta_{n+1}} (w(z^*(s)) + \mathcal{L}u(z^*(s)) - h)ds\right]. \end{aligned}$$

The state process z^* is always moved instantaneously back to D whenever it exits the region D, which implies that z^* spends 0 time outside of D in the sense that the Lebesgue measure of that z^* is outside of D is 0 and it follows

$$E\left[\int_0^{\theta_{n+1}} w(z^*(s))ds + \sum_{k=1}^{n} c(q_k^*)\right] + E[u(z^*(\theta_{n+1}-))] - u(x) - hE[\theta_{n+1}] = 0.$$

Dividing by $E[\theta_{n+1}]$ on both sides we have

$$h = \frac{E\left[\int_0^{\theta_{n+1}} w(z^*(s))ds + \sum_{k=1}^{n} c(q_k^*)\right]}{E[\theta_{n+1}]} + \frac{E[u(z^*(\theta_{n+1}-))]}{E[\theta_{n+1}]} - \frac{u(x)}{E[\theta_{n+1}]}.$$

As $n \to \infty$, the last two terms converge to 0 because of the stability of U^* with respect to $u(.)$ and $\theta_n \nearrow +\infty$. Therefore,

$$h = \limsup_{T \to \infty} \frac{1}{T} E \left[\int_0^T w(z^*(s))ds + \sum_{\{i:\theta_i < T\}} c(q_i^*) \right] = F_0(x, U^*).$$

$\square$

Remark 2.5. *Theorem 2.3 and Theorem 2.4 together imply that the policy U^*, if it exists, is optimal in terms of average cost criterion. In the next theorem, we construct a policy that satisfies Theorem 2.3 and Theorem 2.4, and therefore is optimal.*

Remark 2.6. *Formula (2.4), (2.9) and (2.10) are called quasi-variational inequalities(QVI) in the control theory literature. In the following, we show that indeed the QVI has a solution, and the solution provides an average optimal dispatching policy.*

Theorem 2.7. *Let*

$$\begin{aligned} h^* &= \min_{q>0} \frac{\omega E[\int_0^{T_q} N(s)ds] + c(q)}{E[T_q]}, \\ q^* &= \arg\min_{q>0} \frac{\omega E[\int_0^{T_q} N(s)ds] + c(q)}{E[T_q]}, \\ u^*(z) &= \begin{cases} \frac{h^* z}{D} - \frac{\omega z^2}{2D} + \frac{\omega \sigma^2 z}{2D^2}, z < q* \\ A_D + cz, z \geq q^*, \end{cases} \end{aligned}$$

where $N(s)$ is the drifted Brownian motion starting from 0. Then the policy U^* that dispatches all consolidated load if the consolidated load is equal or greater than q^* and no dispatch if the consolidated load is less than q^* is optimal among all the admissible policies that are stable with respect to $u^*(.)$, in terms of the long-run average cost criterion and further, $F_0(x, U^*) = h^*$.

Proof. From the quantity based policy section, we obtain

$$\begin{aligned} q^* &= \sqrt{2A_D D/\omega}, \\ h^* &= \sqrt{2\omega A_D D} + c\lambda - \frac{\omega\sigma^2}{2D}. \end{aligned}$$

It is straightforward to verify that

$$w(z) + \mathcal{L}u^*(z) - h^* = 0, \quad \text{if } z < q*,$$

$$w(z) + \mathcal{L}u(z) - h > 0, \quad \text{if } z \geq q^*,$$

and $u^*(z)$ is smooth (continuously differentiable) at $z = q^*$.

Further, if $z < q*$,

$$\begin{aligned} & u^*(z-q) - u^*(z) \\ = \quad & -\frac{h^* q}{\lambda} - \frac{\omega\sigma^2 q}{2\lambda^2} + \frac{\omega}{2\lambda}(z^2 - (z-q)^2). \end{aligned}$$

From the definition of h^* and q^*, for $0 < q \leq z$, we obtain

$$\frac{h^* q}{\lambda} < \frac{\omega q^2}{2\lambda} - \frac{\omega\sigma^2 q}{2\lambda^2} + c(q).$$

Using the above inequality and

$$z^2 - (z-q)^2 \geq q^2, \quad \text{if } 0 < q \leq z,$$

we have

$$u^*(z-q) - u^*(z) > -c(q), \quad \text{if } 0 < q \leq z.$$

If $z \geq q*$, it is easy to verify that

$$u(z) = \inf_{0<q\leq z}\{c(q) + u(z-q)\},$$

and

$$\arg\inf_{0<q\leq z}\{c(q) + u(z-q)\} = z,$$

which implies dispatching all the consolidated load is the optimal choice.

Finally, by Theorem 2.3 and Theorem 2.4, we end the proof.

□

Remark 2.8. *The general solution for the second order ODE*

$$\frac{1}{2}\sigma^2 u''(z) + Du'(z) + \omega z - h^* = 0$$

is

$$u(z) = \frac{h^* z}{D} - \frac{\omega z^2}{2D} + \frac{\omega\sigma^2 z}{2D^2} - C_1\frac{\sigma^2}{2D}e^{-\frac{2Dz}{\sigma^2}} + C_2,$$

where C_1, C_2 are two constants. By using $u(0) = 0$ and the smoothing pasting condition, we have $C_1 = C_2 = 0$.

The condition that the function $u^*(z)$ is smooth(continuously differentiable) at $z = q^*$ is referred to as the smoothing pasting condition.

Notice that

$$h^*E[T_z] - \omega E[\int_0^{T_z} N(s)ds] = \frac{h^* z}{D} - \frac{\omega z^2}{2D} + \frac{\omega\sigma^2 z}{2D^2},$$

where $N(s)$ is the drifted Brownian motion starting from 0 and $T_z = \inf\{t > 0 : N(t) \geq z\}$.

Remark 2.9.

$$u^*(z) = \begin{cases} \frac{h^*z}{D} - \frac{\omega z^2}{2D} + \frac{\omega\sigma^2 z}{2D^2}, z < q*, \\ A_D + cz, z \geq q^*. \end{cases}$$

It is straightforward to verify the quantity based policy with any parameter q, the time based policy with any parameter T, and the hybrid policy with any parameters q and T are stable with respect to $u^*(.)$.

Remark 2.10. *On one hand, if the initial consolidated load $x \geq q^*$, we dispatch all the load immediately; on the other hand, if the initial consolidated load $x < q^*$, we do not dispatch until the load is accumulated to q^*, and afterward we dispatch q^* whenever the consolidated load reaches q^*.*

Usually, the initial value $x = 0$, the optimal dispatching policy is the quantity based policy with q^*.

The long run average cost corresponding with U^* is the average cost of one shipment consolidation cycle when we adopt the quantity based consolidation policy with q^*. Further, the long run average cost is independent of the initial value x.

3. $(Q+\tau)$-POLICY IN MULTI-ITEM SHIPMENT CONSOLIDATION MODEL

In this chapter, we consider a multi-item shipment consolidation problem with drifted Brownian motion demand, where a shipper implementing a quantity-based consolidation policy and an alternative of dispatching the consolidated load at an independent random time τ later than it takes to accumulate Q. We call the former Q-policy and the latter (alternative) $(Q+\tau)$-policy. We provide the necessary and sufficient conditions such that the $(Q+\tau)$-policy achieves lower average cost than the Q-policy. Furthermore, we show the jointly optimal $(Q+\tau)$-policy can only be either a quantity-based policy or a time-based policy.

3.1 Mathematical Preliminaries

Assume $N_i(t) = D_i t + \sigma_i B_i(t)$, where $i = 1, 2, \ldots, n$, $D_i > 0$, $\sigma_i > 0$ are the drift coefficient and diffusion coefficient, respectively. $B_1(t), B_2(t), \ldots, B_n(t)$ are independent standard Brownian motions.

Define

$$N(t) = \sum_{i=1}^{n} N_i(t) = (\sum_{i=1}^{n} D_i)t + \sum_{i=1}^{n} \sigma_i B_i(t),$$

and

$$T_Q = \inf\{t > 0 : N(t) \geq Q\},$$

which is a stopping time w.r.t. the filtration generated by $B_1(t)$,$B_2(t)$,..., $B_n(t)$. We have the following result that characterizes the statistical property of T_Q.

Lemma 3.1. *For $s > 0$,*

$$E[\exp(-sT_Q)] = \exp(-\frac{\sqrt{D^2 + 2s\sigma^2} - D}{\sigma^2} Q),$$

$$E[T_Q] = \frac{Q}{D}, \quad E[T_Q^2] = \frac{Q^2}{D^2} + \frac{\sigma^2 Q}{D^3},$$

where

$$D = \sum_{i=1}^{n} D_i, \quad and \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

In fact, T_Q has the inverse Gaussian distribution.

The next result gives the joint moment generation function for $(B_i(T_Q), T_Q)$, which would be used in next section.

Lemma 3.2. *For $s_1^2 + 2s_2 < 0$,*

$$E[\exp(s_1 B_i(T_Q) + s_2 T_Q)] = \exp(\frac{s_1\sigma_i + D - \sqrt{(s_1\sigma_i + D)^2 - (s_1^2 + 2s_2)\sigma^2}}{\sigma^2} Q),$$

$$E[B_i(T_Q)T_Q] = -\frac{\sigma_i Q}{D^2},$$

where

$$D = \sum_{i=1}^{n} D_i, \quad and \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

.

Proof. From $\sum_{i=1}^{n} D_i T_Q + \sum_{i=1}^{n} \sigma_i B_i(T_Q) = Q$, we can arrive at

$$B_n(T_Q) = \frac{Q - \sum_{i=1}^{n} D_i T_Q - \sum_{i=1}^{n-1} \sigma_i B_i(T_Q)}{\sigma_n},$$

then we have

$$\begin{aligned} & \sum_{i=1}^{n} a_i B_i(T_Q) - \frac{1}{2}\sum_{i=1}^{n} a_i^2 T_Q \\ = \quad & \sum_{i=1}^{n-1} a_i B_i(T_Q) + a_n \frac{Q - \sum_{i=1}^{n} D_i T_Q - \sum_{i=1}^{n-1} \sigma_i B_i(T_Q)}{\sigma_n} - \frac{1}{2}\sum_{i=1}^{n} a_i^2 T_Q \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}(a_i - \frac{a_n}{\sigma_n}\sigma_i)B_i(T_Q) - \sum_{i=1}^{n}(\frac{a_n}{\sigma_n}D_i + \frac{1}{2}a_i^2)T_Q + \frac{a_n}{\sigma_n}Q \\
&\triangleq s_1B_1(T_Q) + s_2T_Q + \frac{a_n}{\sigma_n}Q. \qquad (3.1)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{cases}
a_1 - \frac{a_n}{\sigma_n}\sigma_1 = s_1 \\
a_2 - \frac{a_n}{\sigma_n}\sigma_2 = 0 \\
\vdots \\
a_{n-1} - \frac{a_n}{\sigma_n}\sigma_{n-1} = 0 \\
-\sum_{i=1}^{n}(\frac{a_n}{\sigma_n}D_i + \frac{1}{2}a_i^2) = s_2
\end{cases}
\Longrightarrow
\begin{cases}
a_1 = \frac{a_n}{\sigma_n}\sigma_1 + s_1 \\
a_2 = \frac{a_n}{\sigma_n}\sigma_2 \\
\vdots \\
a_{n-1} = \frac{a_n}{\sigma_n}\sigma_{n-1} \\
\sum_{i=1}^{n}\frac{1}{2}a_i^2 + \frac{a_n}{\sigma_n}\sum_{i=1}^{n}D_i + s_2 = 0,
\end{cases}
\qquad (3.2)
$$

and arrive at

$$
\begin{aligned}
&\frac{1}{2}(\frac{a_n}{\sigma_n}\sigma_1 + s_1)^2 + \frac{1}{2}(\frac{a_n}{\sigma_n})^2\sum_{i=2}^{n}\sigma_i^2 + \frac{a_n}{\sigma_n}\sum_{i=1}^{n}D_i + s_2 \\
= \; &\frac{1}{2}\sigma^2(\frac{a_n}{\sigma_n})^2 + (s_1\sigma_1 + D)\frac{a_n}{\sigma_n} + (\frac{1}{2}s_1^2 + s_2) = 0.
\end{aligned}
$$

We take the positive root $\frac{a_n}{\sigma_n} = \frac{-(s_1\sigma_1+D)+\sqrt{(s_1\sigma_1+D)^2-(s_1^2+2s_2)\sigma^2}}{\sigma^2} > 0.$

Since $s_1^2 + 2s_2 < 0$, there exist $\epsilon > 0, \delta > 0$ such that

$$
(1+\epsilon)(1+\delta)s_1^2 + 2s_2 = 0.
$$

For any fixed $t \geq 0$,

$$
\begin{aligned}
& E[(\exp\{\sum_{i=1}^{n} a_i B_i(T_Q \wedge t) - \sum_{i=1}^{n} \frac{1}{2} a_i^2 (T_Q \wedge t)\})^{1+\delta}] \\
= \quad & E[\exp((1+\delta)\sum_{i=1}^{n} a_i B_i(T_Q \wedge t) - (1+\delta)\sum_{i=1}^{n} \frac{1}{2} a_i^2 (T_Q \wedge t))] \\
= \quad & E[\exp\{(1+\delta)\frac{a_n}{\sigma_n}\sigma_1 B_1(T_Q \wedge t) + (1+\delta)s_1 B_1(T_Q \wedge t) \\
& +(1+\delta)\frac{a_n}{\sigma_n}\sum_{i=2}^{n-1}\sigma_i B_i(T_Q \wedge t) + (1+\delta)a_n B_n(T_Q \wedge t) \\
& +(1+\delta)(\frac{a_n}{\sigma_n}\sum_{i=1}^{n} D_i + s_2)(T_Q \wedge t)\}] \\
= \quad & E[\exp\{(1+\delta)\frac{a_n}{\sigma_n}[\sum_{i=1}^{n} D_i(T_Q \wedge t) + \sum_{i=1}^{n}\sigma_i B_i(T_Q \wedge t)] + (1+\delta)s_1 B_1(T_Q \wedge t) \\
& +(1+\delta)s_2(T_Q \wedge t)\}] \\
\leq \quad & \{E[\exp\{\frac{(1+\epsilon)(1+\delta)}{\epsilon}\frac{a_n}{\sigma_n}(\sum_{i=1}^{n} D_i(T_Q \wedge t) + \sum_{i=1}^{n}\sigma_i B_i(T_Q \wedge t))\}]\}^{\frac{\epsilon}{1+\epsilon}} \\
& .\{E[\exp\{(1+\epsilon)(1+\delta)s_1 B_1(T_Q \wedge t) + (1+\epsilon)(1+\delta)s_2(T_Q \wedge t)\}]\}^{\frac{1}{1+\epsilon}} \\
\leq \quad & (\exp\{\frac{(1+\epsilon)(1+\delta)}{\epsilon}\frac{a_n}{\sigma_n}Q\})^{\frac{\epsilon}{1+\epsilon}}.\{E[\exp\{(1+\epsilon)(1+\delta)s_1 B_1(T_Q \wedge t) \\
& +(1+\epsilon)(1+\delta)s_2(T_Q \wedge t)\}]\}^{\frac{1}{1+\epsilon}} \\
= \quad & \exp\{(1+\delta)\frac{a_n}{\sigma_n}Q\}.\{E[\exp\{(1+\epsilon)(1+\delta)s_1 B_1(T_Q \wedge t) \\
& -\frac{1}{2}(1+\epsilon)^2(1+\delta)^2 s_1^2(T_Q \wedge t)\}]\}^{\frac{1}{1+\epsilon}} \\
= \quad & \exp\{(1+\delta)\frac{a_n}{\sigma_n}Q\} \\
\leq \quad & \infty.
\end{aligned}
$$

The second equality comes from replacing (3.2); the first inequality derives from Hölder's inequality (Theorem 3.1.11, Athreya and Lahiri (2006), p.87), where $p = \frac{1+\epsilon}{\epsilon}, q = 1+\epsilon$; the second inequality derives from $\frac{a_n}{\sigma_n} > 0$ and $\sum_{i=1}^{n} D_i(T_Q \wedge t) + \sum_{i=1}^{n}\sigma_i B_i(T_Q \wedge t) \leq Q$; the penultimate equality comes from $(1+\epsilon)(1+\delta)s_1^2 + 2s_2 = 0$

and the reason for the last equality: let $a = (1+\epsilon)(1+\delta)s_1$, $\{e^{aB_1(t)-\frac{1}{2}a^2t}\}_{t\geq 0}$ is a martingale with respect to the filtration generated by $B_1(t), B_2(t), \ldots, B_n(t)$, and $T_Q \wedge t$ is a bounded stopping time for any fixed $t \geq 0$, we have

$$E[\exp\{(1+\epsilon)(1+\delta)s_1 B_1(T_Q \wedge t) - \frac{1}{2}(1+\epsilon)^2(1+\delta)^2 s_1^2 (T_Q \wedge t)\}] = 1,$$

by optional stopping theorem.

Therefore, according to Proposition 2.5.7(ii) (Athreya and Lahiri (2006), p.65),

$$\{\exp(\sum_{i=1}^{n} a_i B_i(T_Q \wedge t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2 (T_Q \wedge t))\}_{t\geq 0}$$

is a uniformly integrable martingale.

Further, by optional stopping theorem,

$$E[\exp(\sum_{i=1}^{n} a_i B_i(T_Q) - \frac{1}{2}\sum_{i=1}^{n} a_i^2 T_Q)] = 1,$$

therefore, by reminding (3.1), we have

$$\begin{aligned} & E[\exp(s_1 B_1(T_Q) + s_2 T_Q)] \\ = \quad & \exp(-\frac{a_n}{\sigma_n} Q) \\ = \quad & \exp\{\frac{(s_1\sigma_1 + D) - \sqrt{(s_1\sigma_1 + D)^2 - (s_1^2 + 2s_2)\sigma^2}}{\sigma^2} Q\}, \end{aligned}$$

which is the joint moment generation function for $(B_1(T_Q), T_Q)$. Then

$$E[B_1(T_Q)T_Q] = \frac{\partial^2 E[\exp(s_1 B_1(T_Q) + s_2 T_Q)]}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} = -\frac{\sigma_1 Q}{D^2}.$$

By the same reasoning, we can obtain

$$E[\exp(s_1 B_i(T_Q) + s_2 T_Q)] = \exp(\frac{s_1\sigma_i + D - \sqrt{(s_1\sigma_i + D)^2 - (s_1^2 + 2s_2)\sigma^2}}{\sigma^2} Q),$$

$$E[B_i(T_Q)T_Q] = -\frac{\sigma_i Q}{D^2}.$$

□

3.2 Q-Policy Model

Assume there are n different kinds of items, and the cumulative demand of the ith item $N_i(t)$ is a Brownian motion with drift given by $N_i(t) = D_i t + \sigma_i B_i(t)$, where $i = 1, 2, \ldots, n$ is the index associated with each item. $D_i > 0$, $\sigma_i > 0$ are the drift coefficient and diffusion coefficient, respectively. $B_1(t), B_2(t), \ldots, B_n(t)$ are independent standard Brownian motions. The total demand process can be expressed as $N(t) = \sum_{i=1}^n N_i(t) = (\sum_{i=1}^n D_i)t + \sum_{i=1}^n \sigma_i B_i(t)$.

The different items share the same freight when they are dispatched. We assume that different items have different unit transportation cost, and different waiting costs per unit per unit time since customers have a distinctly different waiting sensitivity for different items. The different items would be packaged at the collection depot and await the delivery.

We take into account for the following parameters:

A_D: Fixed cost of dispatching

c_i: transportation cost for one unit i-th item

ω_i: customer waiting cost for the i-th item per unit per unit time, which represents the loss-of-goodwill penalty.

We adopt a quantity-based consolidation policy, which dispatches a consolidated load when an economical dispatch quantity Q is available. Since the demand $N(t)$

is continuous, the dispatch quantity is exactly Q.

Clearly, the successive outbound shipping time intervals $S_1, S_2 \ldots$ are independent identically distributed, and each one has the same distribution as the random variable T_Q.

Figure 3.1 illustrates how to calculate the cumulative waiting time for the i-th item within one consolidation cycle, which is area of the shaded portion.

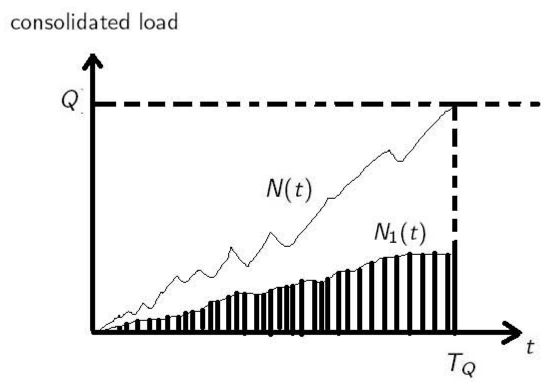

Figure 3.1: Cumulative Waiting Time for the i-th Item within One Cycle.

The next result gives the expectation of cumulative waiting time for i-th item within one consolidation cycle.

Theorem 3.3. *Under the quantity-based policy, the cumulative waiting time for i-th item within one consolidation cycle is* $\frac{D_i Q^2}{2D^2} + \frac{D_i \sigma^2 Q}{2D^3} - \frac{\sigma_i^2 Q}{D^2}$.

Proof.

$$
\begin{aligned}
& E[\text{Cumulative Waiting time for } 1st \text{ item per Consolidation Cycle}] \\
= \quad & E\left[\int_0^{T_Q} N_1(t)dt\right] \\
= \quad & E[tN_1(t)|_{t=0}^{T_Q}] - E\left[\int_0^{T_Q} tdN_1(t)\right] \\
= \quad & E[D_1T_Q^2] + E[\sigma_1 T_Q B_1(T_Q)] - E\left[\int_0^{T_Q} tD_1dt\right] - E\left[\int_0^{T_Q} t\sigma_1 dB_1(t)\right] \\
= \quad & \frac{1}{2}D_1E[T_Q^2] + \sigma_1 E[T_Q B_1(T_Q)] - \sigma_1 E\left[\int_0^{T_Q} tdB_1(t)\right].
\end{aligned}
$$

The first term and the second term can be obtained from Lemma 3.1 and Lemma 3.2, respectively.

To computer the third term, we denote

$$
g(s) = \int_0^s tdB_1(t), \eta(s) = (\int_0^s tdB_1(t))^2 - \frac{1}{3}s^3,
$$

which are two martingales with respect to the filtration generated by $B_1(t)$, $B_2(t)$, $\ldots$, $B_n(t)$. So, $E[\eta(T_Q \wedge s)] = E[\eta(0)] = 0$ for any $s \geq 0$, i.e.,

$$
E\left[(\int_0^{T_Q \wedge s} tdB_1(t))^2\right] = \frac{1}{3}E[(T_Q \wedge s)^3] \leq \frac{1}{3}E[T_Q^3] < \infty,
$$

which implies that $g(T_Q \wedge s) = \int_0^{T_Q \wedge s} tdB_1(t)$ is a square integrable martingale, thus a uniformly integrable martingale. Therefore,

$$
E[g(T_Q)] = E[\int_0^{T_Q} tdB_1(t)] = 0,
$$

by optional stopping theorem, i.e., the third term is 0.

$$
\begin{aligned}
& E[\text{Cumulative Waiting Time for } 1st \text{ item per Consolidation Cycle}] \\
= \quad & \frac{1}{2} D_1 E[T_Q^2] + \sigma_1 E[T_Q B_1(T_Q)] = \frac{D_1 Q^2}{2D^2} + \frac{D_1 \sigma^2 Q}{2D^3} - \frac{\sigma_1^2 Q}{D^2}.
\end{aligned}
$$

By the same reasoning, the cumulative waiting time for i-th item within one consolidation cycle is $\frac{D_i Q^2}{2D^2} + \frac{D_i \sigma^2 Q}{2D^3} - \frac{\sigma_i^2 Q}{D^2}$.

□

By the Renewal Reward Theorem, the expected total long-run average cost per unit-time is

$$C(Q) = \frac{E[\text{Consolidation Cycle Cost}]}{E[\text{Consolidation Cycle Length}]},$$

where the consolidation cycle cost has two components: shipment costs and waiting costs, denoted by C_s and C_w, respectively.

$$E[C_s] = A_D + \frac{Q}{D} \sum_{i=1}^{n} c_i D_i,$$

$$E[C_w] = \sum_{i=1}^{n} \omega_i \left(\frac{D_i Q^2}{2D^2} + \frac{D_i \sigma^2 Q}{2D^3} - \frac{\sigma_i^2 Q}{D^2} \right),$$

$$E[\text{Consolidation Cycle Length}] = E[T_Q] = \frac{Q}{D}.$$

Thus

$$
\begin{aligned}
C(Q) &= \frac{A_D + \frac{Q}{D} \sum_{i=1}^{n} c_i D_i + \sum_{i=1}^{n} \omega_i \left(\frac{D_i Q^2}{2D^2} + \frac{D_i \sigma^2 Q}{2D^3} - \frac{\sigma_i^2 Q}{D^2} \right)}{Q/D} \\
&= \frac{A_D D}{Q} + \frac{Q}{2D} \sum_{i=1}^{n} \omega_i D_i + \sum_{i=1}^{n} c_i D_i - \sum_{i=1}^{n} \omega_i \left(\frac{\sigma_i^2}{D} - \frac{D_i \sigma^2}{2D^2} \right).
\end{aligned}
$$

We obtain the optimal dispatch quantity value

$$Q^{opt} = \sqrt{\frac{2A_D D^2}{\sum_{i=1}^n \omega_i D_i}},$$

and the associated average cost

$$C(Q^{opt}) = \sqrt{2A_D \sum_{i=1}^n \omega_i D_i} + \sum_{i=1}^n c_i D_i - \sum_{i=1}^n \omega_i \left(\frac{\sigma_i^2}{D} - \frac{D_i \sigma^2}{2D^2} \right).$$

In particular, we consider the single-item case, i.e., $n = 1$, $D = D_1$, $\sigma^2 = \sigma_1^2$, we have the expectation of cumulative waiting time within one consolidation cycle is

$$E[\int_0^{T_Q} N(t)dt] = \frac{D_1 Q^2}{2D^2} + \frac{D_1 \sigma^2 Q}{2D^3} - \frac{\sigma_1^2 Q}{D^2} = \frac{Q}{2D}(Q - \frac{\sigma^2}{D}),$$

and the expected total long-run average cost per unit-time is

$$\begin{aligned} C(Q) \quad &= \frac{A_D + C_D Q + \omega(\frac{Q^2}{2D} - \frac{\sigma^2 Q}{2D^2})}{Q/D} \\ &= \frac{A_D D}{Q} + \frac{1}{2}\omega Q + cD - \frac{\omega\sigma^2}{2D}. \end{aligned}$$

We obtain the optimal dispatch quantity value

$$Q^{opt} = \sqrt{\frac{2A_D D}{\omega}},$$

and the associated average cost

$$C(Q^{opt}) = \sqrt{2\omega A_D D} + cD - \frac{\omega\sigma^2}{2D}.$$

Remark 3.4. *Çetinkaya et al. (2006) show that, for Poisson process with arrive* *rate λ, the total cumulative waiting time within a consolidation cycle by adopting the quantity-based policy is $\frac{(Q-1)Q}{2\lambda}$. We can approximate a Poisson process with rate λ by a drifted Brownian motion with $D = \sigma^2 = \lambda$. Therefore, the expected cumulative waiting time within one consolidation cycle is*

$$\frac{Q}{2D}(Q - \frac{\sigma^2}{D}) = \frac{(Q-1)Q}{2D} = \frac{(Q-1)Q}{2\lambda}.$$

From this point, the analysis in the case of Brownian motion with drift is more generalized, from which the case of pure poisson process demand is a special one.

Suppose the demand of the i-th item $N_i(t)$ is a Poisson process with rate λ_i, $i = 1, 2, \ldots, n$, we deduce the cumulative waiting time for the i-th item within a consolidation cycle:

$$\begin{aligned} E\left[\int_0^{T_Q} N_i(t)dt\right] \quad &= E[tN_i(t)|_{t=0}^{T_Q}] - E\left[\int_0^{T_Q} tdN_i(t)\right] \\ &= E[T_Q N_i(T_Q)] - E\left[\int_0^{T_Q} tdN_i(t)\right]. \end{aligned}$$

Clearly, the total demand $N(t)$ is a Poisson process with rate $\lambda = \sum_{i=1}^n \lambda_i$, and T_Q is a random variable having gamma(Q,λ) distribution, which has mean $\frac{Q}{\lambda}$ and variance $\frac{Q}{\lambda^2}$. Notice that

$$E[T_Q N_i(T_Q)] = E[T_Q E[N_i(T_Q)|T_Q]] = \frac{\lambda_i}{\lambda} Q E[T_Q] = \frac{\lambda_i Q^2}{\lambda^2}.$$

Further, $\int_0^t s dN_i(s) - \int_0^t \lambda_i s ds$ is a square integrable martingale if $N_i(t)$ is a Poisson process, then by optional stopping theorem, we have that

$$E\left[\int_0^{T_Q} tdN_i(t)\right] = \frac{1}{2}\lambda_i E[T_Q^2] = \frac{1}{2}\lambda_i(\frac{Q}{\lambda^2} + \frac{Q^2}{\lambda^2}).$$

Therefore, the cumulative waiting time for the i-th item within a consolidation cycle is

$$E\left[\int_0^{T_Q} N_i(t)dt\right] = \frac{\lambda_i(Q-1)Q}{2\lambda^2}.$$

By approximating $N_i(t)$ by drifted Brownian motion with $D_i = \sigma_i^2 = \lambda_i$, the cumulative waiting time for i-th item within one consolidation cycle is

$$\frac{D_i Q^2}{2D^2} + \frac{D_i \sigma^2 Q}{2D^3} - \frac{\sigma_i^2 Q}{D^2} = \frac{\lambda_i(Q-1)Q}{2\lambda^2}.$$

3.3 $(Q + \tau)$-Policy Model

In this section, we discuss the $(Q + \tau)$-policy.

Given a quantity-based consolidation policy with parameter Q, we consider a modified policy, denoted as $(Q + \tau)$-policy, which dispatches the consolidated load at a nonnegative random time τ later than it takes to accumulate Q, where τ is independent of the demand processes. Let $\tau_1 = E[\tau]$, $\tau_2 = E[\tau^2]$. Figure 3.2 provides the illustration for $(Q + \tau)$-policy.

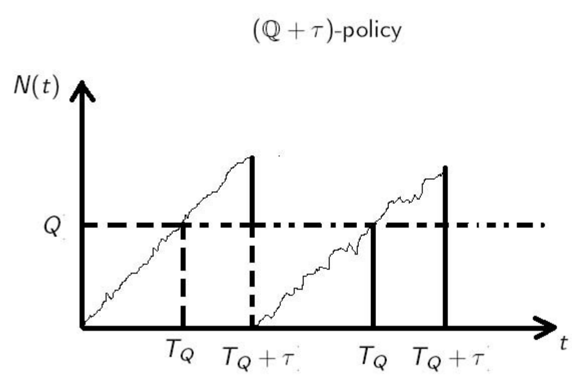

Figure 3.2: $(Q + \tau)$-Policy.

Theorem 3.5. *Under the $(Q+\tau)$-policy, the cumulative waiting time for i-th item within one consolidation cycle is* $\frac{D_iQ^2}{2D^2}+\frac{D_i\sigma^2 Q}{2D^3}-\frac{\sigma_i^2 Q}{D^2}+\frac{D_iQ}{D}\tau_1+\frac{1}{2}D_i\tau_2$.

Proof. The expectation of the cumulative waiting time for a particular item within one consolidation cycle can be computed as follows,

$$
\begin{aligned}
& E\left[\int_0^{T_Q+\tau} N_1(t)dt\right] \\
= \quad & E[tN_1(t)|_{t=0}^{T_Q+\tau}] - E\left[\int_0^{T_Q+\tau} tdN_1(t)\right] \\
= \quad & E[(T_Q+\tau)N_1(T_Q+\tau)] - E\left[\int_0^{T_Q+\tau} tD_1dt\right] - E\left[\int_0^{T_Q+\tau} t\sigma_1 dB_1(t)\right] \\
= \quad & \frac{1}{2}D_1E[(T_Q+\tau)^2] + \sigma_1 E[(T_Q+\tau)B_1(T_Q+\tau)] - \sigma_1 E\left[\int_0^{T_Q+\tau} tdB_1(t)\right].
\end{aligned}
$$

For the first term, we have

$$\frac{1}{2}D_1E[(T_Q+\tau)^2] = \frac{1}{2}D_1\left(E[T_Q^2]+2E[T_Q]\tau_1+\tau_2\right),$$

since τ and T_Q are independent.

For the second term, by using the strong Markov property of Brownian motion and τ is independent of the demand process, we have

$$
\begin{aligned}
E[T_QB_1(T_Q+\tau)] = \quad & E[T_Q(B_1(T_Q+\tau)-B_1(T_Q))] + E[T_QB_1(T_Q)] \\
= \quad & E[T_Q]E[B_1(T_Q+\tau)-B_1(T_Q)] + E[T_QB_1(T_Q)].
\end{aligned}
$$

Further,

$$E[B_1(T_Q+\tau)] = E[E[B_1(T_Q+\tau)|\tau]] = 0,$$

and

$$E[B_1(T_Q)] = 0,$$

by noticing that τ is independent of the demand process and applying optional stopping theorem. Further,

$$E[\tau B_1(T_Q+\tau)] = E[E[\tau B_1(T_Q+\tau)|\tau] = E[\tau E[B_1(T_Q+\tau)|\tau]] = 0,$$

by applying optional stopping theorem. Therefore, we obtain the second term

$$E[(T_Q+\tau)B_1(T_Q+\tau)] = E[T_Q B_1(T_Q)].$$

For the third term,

$$E\left[\int_0^{T_Q+\tau} t dB_1(t)\right] = E\left[E[\int_0^{T_Q+\tau} t dB_1(t)|\tau]\right],$$

we can obtain

$$E\left[\int_0^{T_Q+\tau} t dB_1(t)|\tau = z\right] = E\left[\int_0^{T_Q+z} t dB_1(t)|\tau = z\right] = E\left[\int_0^{T_Q+z} t dB_1(t)\right] = 0.$$

The penultimate equality holds since τ is independent of the demand process $B_1(t)$ and the last equality is derived by applying optional stopping theorem. Thus, $E\left[\int_0^{T_Q+\tau} t dB_1(t)\right] = 0$.

By Lemma 3.1 and Lemma 3.2, we obtain

$$\begin{aligned} & E[\text{Cumulative Waiting for 1st item per Consolidation Cycle}] \\ = \quad & \frac{D_1 Q^2}{2D^2} + \frac{D_1\sigma^2 Q}{2D^3} - \frac{\sigma_1^2 Q}{D^2} + \frac{D_1 Q}{D}\tau_1 + \frac{1}{2}D_1\tau_2. \end{aligned}$$

□

From the above result, the expected total cost during one consolidation cycle is

$$
\begin{aligned}
E[C] = \quad & A_D + \left(\frac{Q}{D} + \tau_1\right) \sum_{i=1}^{n} c_i D_i \\
& + \sum_{i=1}^{n} \omega_i \left(\frac{D_i Q^2}{2D^2} + \frac{D_i \sigma^2 Q}{2D^3} - \frac{\sigma_i^2 Q}{D^2} + \frac{D_i Q}{D}\tau_1 + \frac{1}{2} D_i \tau_2 \right),
\end{aligned}
$$

and the expected length of a consolidation cycle is

$$E[L] = E[T_Q + \tau] = \frac{Q}{D} + \tau_1.$$

By the Renewal Reward Theorem, the expected total long-run average cost per unit-time is

$$C(Q, \tau) = \frac{E[C]}{E[L]}.$$

We notice that $\tau_2 \geq \tau_1^2$, to minimize $C(Q, \tau)$, the optimal choice of τ is to take $\tau_2 = \tau_1^2$, which implies that τ is a constant. so we can denote the average cost per time-unit by $C(Q, \tau_1)$,

$$C(Q, \tau_1) = \frac{Q + D\tau_1}{2D} \sum_{i=1}^{n} \omega_i D_i + \sum_{i=1}^{n} c_i D_i + \frac{A_D D - \sum_{i=1}^{n} \omega_i (\frac{\sigma_i^2}{D} - \frac{D_i \sigma^2}{2D^2}) Q}{Q + D\tau_1}.$$

We consider the single item case, i.e., $n = 1$, $D = D_1$, $\sigma^2 = \sigma_1^2$, we can obtain the average cost per time-unit

$$C(Q, \tau_1) = \frac{1}{2}\omega(Q + D\tau_1) + cD + \frac{A_D D - \frac{\omega\sigma^2 Q}{2D}}{Q + D\tau_1}.$$

3.4 Main Results

The following result states that for a fixed value Q, $(Q+\tau)$-policy may achieve less average cost than Q-policy.

Theorem 3.6. *For multi-item case, the Q-policy can be improved by a $(Q+\tau)$-policy if and only if Q satisfies the following:*

$$\sum_{i=1}^{n} \omega_i D_i Q^2 + \sum_{i=1}^{n} \omega_i (2\sigma_i^2 - \frac{D_i \sigma^2}{D}) Q - 2A_D D^2 < 0, \tag{3.3}$$

which is

$$Q < \frac{-\sum_{i=1}^{n} \omega_i (2\sigma_i^2 - \frac{D_i\sigma^2}{D}) + \sqrt{[\sum_{i=1}^{n} \omega_i (2\sigma_i^2 - \frac{D_i\sigma^2}{D}]^2) + 8A_D D^2 \sum_{i=1}^{n} \omega_i D_i}}{2\sum_{i=1}^{n} \omega_i D_i} \triangleq Q^{**}.$$

In particular, for single item case, the Q-policy can be improved by a $(Q+\tau)$-policy if and only if Q satisfies the following:

$$Q^2 + \frac{\sigma^2}{D} Q - \frac{2A_D D}{\omega} < 0, \tag{3.4}$$

which is

$$0 \leq Q < -\frac{\sigma^2}{2D} + \sqrt{\frac{\sigma^4}{4D^2} + \frac{2A_D D}{\omega}}.$$

Proof. $C(Q) > C(Q, \tau_1)$, which is

$$\begin{aligned}
&\frac{A_D D}{Q} + \frac{Q}{2D} \sum_{i=1}^{n} \omega_i D_i + \sum_{i=1}^{n} c_i D_i - \sum_{i=1}^{n} \omega_i (\frac{\sigma_i^2}{D} - \frac{D_i \sigma^2}{2D^2}) \\
&> \frac{Q + D\tau_1}{2D} \sum_{i=1}^{n} \omega_i D_i + \sum_{i=1}^{n} c_i D_i + \frac{A_D D - \sum_{i=1}^{n} \omega_i (\frac{\sigma_i^2}{D} - \frac{D_i\sigma^2}{2D^2}) Q}{Q + D\tau_1}.
\end{aligned}$$

After some algebraic manipulation, we arrive at

$$DQ\sum_{i=1}^{n}\omega_i D_i\tau_1 < 2A_D D^2 - \sum_{i=1}^{n}\omega_i(2\sigma_i^2 - \frac{D_i\sigma^2}{D})Q - \sum_{i=1}^{n}\omega_i D_i Q^2.$$

$(Q+\tau)$-policy improves Q-policy if and only if we can choose a positive value of τ_1 such that the above inequality is satisfied. This is always possible if

$$\sum_{i=1}^{n}\omega_i D_i Q^2 + \sum_{i=1}^{n}\omega_i(2\sigma_i^2 - \frac{D_i\sigma^2}{D})Q - 2A_D D^2 < 0.$$

Since $Q \geq 0$, we only consider the positive root of the quadratic equation. Thus, the Q-policy can be improved if and only if

$$0 \leq Q < \frac{-\sum_{i=1}^{n}\omega_i(2\sigma_i^2 - \frac{D_i\sigma^2}{D}) + \sqrt{[\sum_{i=1}^{n}\omega_i(2\sigma_i^2 - \frac{D_i\sigma^2}{D}]^2) + 8A_D D^2\sum_{i=1}^{n}\omega_i D_i}}{2\sum_{i=1}^{n}\omega_i D_i}.$$

and the proof is completed. □

In practice, the conditions in the above theorem are easy to verify.

Corollary 3.7. *For multi-item case, if Q satisfies (3.3) , the optimal $(Q+\tau)$-policy is $(Q+\tau_1^{opt})$, where $\tau_1^{opt} = \sqrt{\frac{2A_D - \sum_{i=1}^{n}\omega_i(2\sigma_i^2 Q/D^2 - D_i\sigma^2 Q/D^3)}{\sum_{i=1}^{n}\omega_i D_i}} - \frac{Q}{D}$. In particular, for single-item case, if Q satisfies (3.4), the optimal $(Q+\tau)$-policy is $(Q+\tau_1^{opt})$, where $\tau_1^{opt} = \sqrt{\frac{2A_D}{\omega D} - \frac{\sigma^2 Q}{D^3}} - \frac{Q}{D}$.*

Proof. To optimize $C(Q,\tau_1)$ as a function of τ_1, we have to solve the equation $dC(Q,\tau_1)/d\tau_1 = 0$, which is to solve

$$\frac{1}{2}\sum_{i=1}^{n}\omega_i D_i - \frac{A_D D - \sum_{i=1}^{n}\omega_i(\sigma_i^2 Q/D - \frac{1}{2}D_i\sigma^2 Q/D^2)}{(Q+D\tau_1)^2}D = 0,$$

that is,

$$(Q+D\tau_1)^2 = \frac{2A_D D^2 - \sum_{i=1}^n \omega_i(2\sigma_i^2 Q - D_i\sigma^2 Q/D)}{\sum_{i=1}^n \omega_i D_i}.$$

Hence

$$\tau_1^{opt} = \sqrt{\frac{2A_D - \sum_{i=1}^n \omega_i(2\sigma_i^2 Q/D^2 - D_i\sigma^2 Q/D^3)}{\sum_{i=1}^n \omega_i D_i}} - \frac{Q}{D}.$$

Since Q satisfies (3.3), we can see $\tau_1^{opt} > 0$. Differentiating twice, we prove that the minimum is achieved and therefore a value $\tau_1^{opt} > 0$ exists. □

For optimal Q-policy with parameter Q^{opt}, are there any $(Q^{opt}+\tau)$-policies achieving less average cost than it?

Theorem 3.8. *For multi-item case: If* $\sum_{i=1}^n \omega_i(2D\sigma_i^2 - D_i\sigma^2) > 0$*, the optimal* Q*-policy,* Q^{opt} *can not be improved by any* $(Q^{opt}+\tau)$*-policy; If* $\sum_{i=1}^n \omega_i(2D\sigma_i^2 - D_i\sigma^2) < 0$*, the optimal* Q*-policy,* Q^{opt} *can be improved by some* $(Q^{opt}+\tau)$*-policy. In particular, for single-item case, the optimal* Q*-policy,* Q^{opt} *can not be improved by any* $(Q^{opt}+\tau)$*-policy.*

Proof. From $C(Q)$ we can obtain that $Q^{opt} = \sqrt{\frac{2A_D D^2}{\sum_{i=1}^n \omega_i D_i}}$.

Case 1: $\sum_{i=1}^n \omega_i(2D\sigma_i^2 - D_i\sigma^2) > 0$,

$$\sum_{i=1}^n \omega_i D_i (Q^{opt})^2 + \sum_{i=1}^n \omega_i(2\sigma_i^2 - \frac{D_i\sigma^2}{D})Q^{opt} - 2A_D D^2 > 0,$$

which implies that (3.3) does not hold for $Q = Q^{opt}$, i.e., Q^{opt} can not be improved by any $(Q^{opt}+\tau)$-policy.

Case 2: $\sum_{i=1}^n \omega_i(2D\sigma_i^2 - D_i\sigma^2) < 0$,

$$\sum_{i=1}^n \omega_i D_i (Q^{opt})^2 + \sum_{i=1}^n \omega_i(2\sigma_i^2 - \frac{D_i\sigma^2}{D})Q^{opt} - 2A_D D^2 < 0,$$

which implies that (3.3) holds for $Q = Q^{opt}$, i.e., Q^{opt} can be improved by some

$(Q^{opt}+\tau)$-policy.

□

All the previous work show optimal quantity policy achieve the lowest average cost (Higginson and Bookbinder, 1994; Mutlu et al., 2010), but the above result points out it may not be true in multi-item case with drifted Brownian motion demand.

The following result characterizes the optimality of $(Q+\tau)$-policy, optimizing jointly on Q and τ_1. It shows that either a quantity policy or a time policy is optimal, depending on whether $\sum_{i=1}^{n}\omega_i(2D\sigma_i^2 - D_i\sigma^2)$ is positive or negative.

Theorem 3.9. *For multi-item case: If $\sum_{i=1}^{n}\omega_i(2D\sigma_i^2 - D_i\sigma^2) > 0$, the optimal $(Q+\tau)$-policy is the Q^{opt}-policy; If $\sum_{i=1}^{n}\omega_i(2D\sigma_i^2 - D_i\sigma^2) < 0$, the optimal $(Q+\tau)$-policy is the τ_1^{opt}-policy, which is a pure time-based policy. In particular, for single-item case, the jointly optimal $(Q+\tau)$-policy is the Q^{opt}-policy.*

Proof. Differentiating in $C(Q,\tau_1)$ w.r.t. τ_1 and equating to 0, we obtain an equation for the optimal τ_1 as function of Q,

$$\tau_1^{opt}(Q) = \sqrt{\frac{2A_D - \sum_{i=1}^{n}\omega_i(2\sigma_i^2 Q/D^2 - D_i\sigma^2 Q/D^3)}{\sum_{i=1}^{n}\omega_i D_i}} - \frac{Q}{D}.$$

$\tau_1 > 0$ if and only if (3.3) holds, that is, if the Q-policy can be improved.

If (3.3) does not hold, i.e $Q \geq Q^{**}$, the optimal $(Q+\tau)$-policy is the Q-policy.

We focus on the values of Q that satisfy (3). For such Q, the optimal cost is determined by the pair $(Q, \tau_1^{opt}(Q))$ as follows:

$$C(Q,\tau_1^{opt}(Q)) = \sqrt{[2A_D - \sum_{i=1}^{n}\omega_i(2\sigma_i^2/D^2 - D_i\sigma^2/D^3)Q](\sum_{i=1}^{n}\omega_i D_i)} + C_D D. \quad (3.5)$$

Case 1: $\sum_{i=1}^{n}\omega_i(2D\sigma_i^2 - D_i\sigma^2) > 0$.

In this case, (3.5) is a decreasing function of Q. Since $0 \leq Q < Q^{**}$, take Q as close from the left side to Q^{**} as possible to minimize (3.5), which yields $\tau_1^{opt} = 0$. Therefore, the optimal $(Q+\tau)$-policy is the Q^{opt}-policy, $Q^{opt} = \sqrt{\frac{2A_D D^2}{\sum_{i=1}^n \omega_i D_i}}$ and the optimal average cost per time-unit is

$$C(Q^{opt}, 0) = \sqrt{2A_D \sum_{i=1}^n \omega_i D_i} + \sum_{i=1}^n c_i D_i - \sum_{i=1}^n \omega_i (\frac{\sigma_i^2}{D} - \frac{D_i \sigma^2}{2D^2}).$$

Case 2: $\sum_{i=1}^n \omega_i (2D\sigma_i^2 - D_i \sigma^2) < 0$.

In this case, (3.5) is an increasing function of Q. Since $0 \leq Q < Q^{**}$, take $Q = 0$ to minimize (3.5), which implies that the optimal $(Q+\tau)$-policy is the τ_1^{opt}-policy, i.e., pure time-based policy, $\tau_1^{opt} = \sqrt{\frac{2A_D}{\sum_{i=1}^n \omega_i D_i}}$ and the optimal average cost per time-unit is

$$C(0, \tau_1^{opt}) = \sqrt{2A_D \sum_{i=1}^n \omega_i D_i} + \sum_{i=1}^n c_i D_i.$$

□

The result is somewhat surprising, which claims the jointly optimal $(Q+\tau)$-policy can only be either quantity policy or time policy, which are the two extreme policies.

If the waiting cost for all the items are the same, $\omega_1 = \omega_2 = ... = \omega_n$, $\sum_{i=1}^n \omega_i (2D\sigma_i^2 - D_i \sigma^2)$ is positive, which implies the quantity policy is optimal.

Remark 3.10. *Mutlu et al. (2010) shows that in the single item Poisson demand case, quantity policy achieve the lowest average cost. Actually, we can generalize the result to multi-item case as follows.*

For Poisson demand processes case, we can obtain the

$$\begin{aligned} C(Q) = & \frac{A_D + \frac{Q}{\lambda}\sum_{i=1}^n c_i\lambda_i + \sum_{i=1}^n \omega_i \frac{\lambda_i Q(Q-1)}{2\lambda^2}}{Q/\lambda} \\ = & \frac{A_D\lambda}{Q} + \frac{Q}{2\lambda}\sum_{i=1}^n \omega_i\lambda_i + \sum_{i=1}^n c_i\lambda_i - \sum_{i=1}^n \omega_i \frac{\lambda_i}{2\lambda}. \end{aligned}$$

In fact, we can obtain it by letting $D_i = \lambda_i$, $\sigma_i^2 = \lambda_i$ *in the expression of* $C(Q)$ *in the Brownian motion case. Thus we obtain*

$$Q^{opt} = \sqrt{\frac{2A_D\lambda^2}{\sum_{i=1}^n \omega_i\lambda_i}},$$

and

$$C(Q^{opt}) = \sqrt{2A_D\sum_{i=1}^n \omega_i\lambda_i} + \sum_{i=1}^n c_i\lambda_i - \sum_{i=1}^n \omega_i\frac{\lambda_i}{2\lambda}.$$

By applying the same methodology, we have the following results for the multi-item Poisson demand case: (1) The optimal Q-policy, Q^{opt} *can not be improved by any* $(Q^{opt}+\tau)$*-policy; (2) The optimal* $(Q+\tau)$*-policy is the* Q^{opt}*-policy. In fact, we can approximate Poisson process by drifted Brownian motion with* $D_i = \lambda_i$, $\sigma_i^2 = \lambda_i$*, and then we have*

$$\sum_{i=1}^n \omega_i(2D\sigma_i^2 - D_i\sigma^2) = D\sum_{i=1}^n \omega_i D_i > 0.$$

From Theorem 3.9, we arrive at the conclusion.

4. ON THE SERVICE PERFORMANCE IN SHIPMENT CONSOLIDATION SYSTEM

This chapter revists the problem in Çetinkaya et al. (2014), where they consider the service performance of alternative shipment consolidation policies. Firstly, we provide a unified method to calculate Average Order Delay (AOD) under any consolidation policy by applying the martingale theory. Next, we develop some more refined properties of truncated random variables. Based on these properties, we complete some comparative results among different consolidation policies in terms of AOD, which are not proved in Çetinkaya et al. (2014).

4.1 Average Order Delay

Customer waiting occurs when consolidation policies are implemented, since a prolonged order holding is needed to accumulate a large load. One important service measure indicator is average order delay, which is the average delay of orders before delivery (Çetinkaya et al., 2014). Under any renewal-type consolidation policy, the consolidated load forms a regenerative process. So, the average order delay can be obtained by applying the Renewal Reward Theorem, i.e.,

$$AOD = \frac{E[\text{Cumulative waiting per consolidation cycle}]}{E[\text{Number of orders arriving in a consolidation cycle}]} = \frac{E[W]}{\lambda E[C]},$$

where W denotes the sum of the waiting times of the orders within a consolidation cycle, and C denotes the consolidation cycle length. We index AOD, W, and C by policy type as needed.

We assume the arrival process follows a Poisson process $N(t)$ with rate λ. In this section, based on a martingale associated with Poisson process, we provide a unified method to calculate the AOD for any shipment consolidation policy. The following lemma reveals a martingale associated with Poisson process, which is the foundation for the unified method to calculate the expectation of the cumulative waiting per consolidation cycle under any renewal-type consolidation policy.

Lemma 4.1. *Let $N(t)$ is a Poisson process with rate λ, then*

$$W(t) - \frac{1}{2\lambda}N^2(t) + \frac{1}{2\lambda}N(t)$$

is a martingale with respect to $N(t)$, where $W(t) = \int_0^t N(u)du$.

Proof. Let $\{\mathcal{G}_t\}$ be the natural filtration for $N(t)$. Then, for $s < t$,

$$\begin{aligned} E\Big[\int_0^t N(u)du \mid \mathcal{G}_s\Big] &= \int_0^s N(u)du + E\Big[\int_s^t N(u)du \mid \mathcal{G}_s\Big] \\ &= \int_0^s N(u)du + (t-s)N(s) + E\Big[\int_0^{t-s} N(u)du\Big] \\ &= \int_0^s N(u)du + (t-s)N(s) + \frac{1}{2}\lambda(t-s)^2, \end{aligned}$$

$$\frac{1}{2\lambda}E[N^2(t) \mid \mathcal{G}_s] = \frac{1}{2\lambda}\Big(N^2(s) + 2\lambda(t-s)N(s) + \lambda(t-s) + \lambda^2(t-s)^2\Big),$$

and

$$\frac{1}{2\lambda}E[N(t) \mid \mathcal{G}_s] = \frac{1}{2\lambda}(N(s) + \lambda(t-s)).$$

We obtain

$$\begin{aligned} &E\Big[\int_0^t N(u)du - \frac{1}{2\lambda}N^2(t) + \frac{1}{2\lambda}N(t) \mid \mathcal{G}_s\Big] \\ &= \int_0^s N(u)du - \frac{1}{2\lambda}N^2(s) + \frac{1}{2\lambda}N(s), \end{aligned}$$

which shows that $W(t) - \frac{1}{2\lambda}N^2(t) + \frac{1}{2\lambda}N(t)$ is a martingale. $\square$

By applying the optional stopping theorem, for any stopping time τ and $\forall t > 0$,

$$E[W(\tau \wedge t)] = \frac{1}{2\lambda} E[N^2(\tau \wedge t) - N(\tau \wedge t)], \tag{4.1}$$

$$E[N(\tau \wedge t)] = \lambda E[\tau \wedge t].$$

The cumulative waiting time within one consolidation cycle of any clearing shipment consolidation policy with dispatching stopping time τ is

$$W(\tau) = \int_0^\tau N(u)du.$$

Figure 4.1 illustrates how to calculate the cumulative waiting time within one consolidation cycle for a general clearing shipment consolidation policy, which is area of the shaded portion.

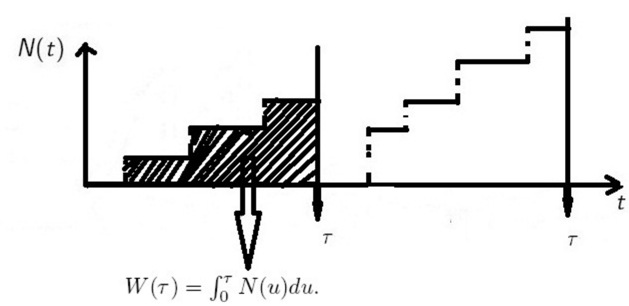

Figure 4.1: The Cumulative Waiting Time within One Consolidation Cycle.

Assume τ is with finite mean, from monotone convergence theorem,

$$\lim_{t\to\infty} E[W(\tau \wedge t)] = E[W(\tau)],$$

$$\lim_{t\to\infty} E[N(\tau \wedge t)(N(\tau \wedge t) - 1)] = E[N(\tau)(N(\tau) - 1)].$$

Noticing (4.1), we have

$$E[W(\tau)] = \frac{1}{2\lambda} E[N^2(\tau) - N(\tau)], \tag{4.2}$$

Similarly, we have

$$E[C_\tau] = E[\tau] = \frac{1}{\lambda} E[N(\tau)], \tag{4.3}$$

where C_τ denotes the length of a consolidation cycle of the shipment consolidation policy with dispatching time τ.

From the above discussion, we can deduce AOD for any renewal-type shipment consolidation policy. In fact, we can notice the view of martingale is a useful idea in stochastic calculation.

Now we calculate the AOD under the practical shipment consolidation policies:

1. QP with parameter q: $\tau = \tau_q$, the time until the $q - th$ order, q is a positive integer; $N(\tau_q) = q$. So,

$$E[W_{QP}] = E[W(\tau_q)] = \frac{1}{2\lambda} q(q-1), \quad E[C_{QP}] = E[\tau_q] = \frac{q}{\lambda}.$$

2. TP1 with parameter T: $\tau = T$, a constant; $N(T) \sim Poisson(\lambda T)$. So

$$E[W_{TP1}] = E[W(T)] = \frac{1}{2\lambda}E[N^2(T) - N(T)] = \frac{1}{2}\lambda T^2, \quad E[C_{TP1}] = T.$$

3. TP2 with parameter T: $\tau = \tau_1 + T$; $N(\tau_1 + T) \stackrel{d}{=} 1 + N(T)$. So

$$\begin{aligned} E[W_{TP2}] &= E[W(\tau_1 + T)] = \frac{1}{2\lambda}E[N^2(T) + N(T)] = \frac{1}{2}\lambda T^2 + T, \\ E[C_{TP2}] &= \frac{1}{\lambda} + T. \end{aligned}$$

4. HP1 with parameters q and T: $\tau = \tau_q \wedge T$; $N(\tau_q \wedge T) = N(T) \wedge q$. Define $Y_q = Y \wedge q = N(T) \wedge q$, where $Y \sim Poisson(\lambda T)$ and thus Y_q is a truncated Poisson random variable. So

$$E[W_{HP1}] = E[W(\tau_q \wedge T)] = \frac{1}{2\lambda}E[Y_q(Y_q - 1)], \quad E[C_{HP1}] = \frac{1}{\lambda}E[Y_q].$$

5. HP2 with parameters q and T: $\tau = \tau_q \wedge (\tau_1 + T)$.

$$N(\tau_q \wedge (\tau_1 + T)) \stackrel{d}{=} (1 + N(T)) \wedge q = Y_{q-1} + 1.$$

We have

$$\begin{aligned} E[W_{HP2}] &= E[W(\tau_q \wedge (\tau_1 + T))] = \frac{1}{2\lambda}E[Y_{q-1}(Y_{q-1} + 1)], \\ E[C_{HP2}] &= \frac{1}{\lambda}E[Y_{q-1} + 1]. \end{aligned}$$

In Table 4.1, we summarize the AOD for different consolidation policies. We would notice that the expressions of AOD_{HPs} involve the truncated Poisson random variables, which are much simplified than the expressions in Çetinkaya et al. (2014).

Note that under TP1 and HP1, the consolidation cycle clock starts over, even if no order arrives within the previous cycle. We consider the correspondingly revised policies in Appendix A, which do not allow empty dispatches.

$$
\begin{array}{lcl}
AOD_{\tau} & = & \frac{E[W(\tau)]}{\lambda E[C_{\tau}]} = \frac{E[N^2(\tau) - N(\tau)]/(2\lambda)}{E[N(\tau)]} \\
AOD_{QP} & = & \frac{E\left[W_{QP}\right]}{\lambda E\left[C_{QP}\right]} = \frac{(q-1)q/2\lambda}{q} = \frac{q-1}{2\lambda} \\
AOD_{TP1} & = & \frac{E[W_{TP1}]}{\lambda E[C_{TP1}]} = \frac{\lambda T^2/2}{\lambda T} = \frac{1}{2}T \\
AOD_{TP2} & = & \frac{E[W_{TP2}]}{\lambda E[C_{TP2}]} = \frac{T + \lambda T^2/2}{1 + \lambda T} \\
AOD_{HP1} & = & \frac{E[W_{HP1}]}{\lambda E[C_{HP1}]} = \frac{E[Y_q(Y_q - 1)]/(2\lambda)}{E[Y_q]} \\
AOD_{HP2} & = & \frac{E[W_{HP2}]}{\lambda E[C_{HP2}]} = \frac{E[Y_{q-1}(Y_{q-1} + 1)]/(2\lambda)}{E[Y_{q-1} + 1]}
\end{array}
$$

Table 4.1: Summary of the Expressions of AOD.

4.2 Some Properties on Truncated Random Variables

In this section, we investigate the properties of truncated random variables, which are connected to the comparison of different consolidation policies in terms of AOD.

In the following, given a random variable X and a real number N, we denote $X_N = \min(X, N)$, which is a truncated random variable.

Lemma 4.2. *Given an integer valued random variable Y, and a positive integer M, we have*

$$VAR[Y] - VAR[Y_M] = VAR[Y - Y_M] + 2(M - E[Y_M])(E[Y] - E[Y_M]) > 0.$$

Proof. First, notice

$$Y - Y_M = (Y - M)1_{Y \geq M+1},$$

and

$$E[Y_M(Y - Y_M)] = ME[(Y - M)1_{Y \geq M+1}] = M(E[Y] - E[Y_M]).$$

Therefore,

$$\begin{aligned} COV(Y_M, Y - Y_M) = \quad & E[Y_M(Y - Y_M)] - E[Y_M]E[Y - Y_M] \\ = \quad & (M - E[Y_M])(E[Y] - E[Y_M]). \end{aligned}$$

We have

$$\begin{aligned} VAR[Y] = \quad & VAR[Y_M + (Y - Y_M)] \\ = \quad & VAR[Y_M] + VAR[Y - Y_M] + 2COV(Y_M, Y - Y_M) \\ = \quad & VAR[Y_M] + VAR[Y - Y_M] + 2(M - E[Y_M])(E[Y] - E[Y_M]). \end{aligned}$$

□

The following result is useful in the comparison between the general class of HPs and the general class of counterpart TPs in terms of AOD.

Lemma 4.3. *Given an integer-valued random variable Y with $VAR[Y] \leq E[Y] < \infty$, for any positive integer N, we have $VAR[Y_N] < E[Y_N]$. In particular, $VAR[Y_N] < E[Y_N]$, if Y is a Poisson random variable.*

Proof. Noticing

$$Y_N = \min(Y_{N+1}, N), \quad \text{and} \quad Y_{N+1} - Y_N = 1_{Y \geq N+1},$$

and applying Lemma 4.2, we have

$$\begin{aligned}
& (VAR[Y_{N+1}] - E[Y_{N+1}]) - (VAR[Y_N] - E[Y_N]) \\
= \quad & (VAR[Y_{N+1}] - VAR[Y_N]) - (E[Y_{N+1}] - E[Y_N]) \\
= \quad & VAR[Y_{N+1} - Y_N] + 2(N - E[Y_N])(E[Y_{N+1}] - E[Y_N]) - (E[Y_{N+1}] - E[Y_N]) \\
= \quad & VAR[Y_{N+1} - Y_N] + (2N - 2E[Y_N] - 1)(E[Y_{N+1}] - E[Y_N]) \\
= \quad & P(Y \geq N+1)P(Y \leq N) + (2N - 2E[Y_N] - 1)P(Y \geq N+1) \\
= \quad & (2E[\max(N - Y, 0)] - P(Y \geq N+1))P(Y \geq N+1).
\end{aligned}$$

Obviously,

$$2E[\max(N - Y, 0)] - P(Y \geq N+1)$$

is increasing with respect to N, which implies that $f(N+1) - f(N)$ changes sign at most once with respect to N: either from negative to positive or always positive, where $f(N) = VAR[Y_N] - E[Y_N]$

Further,

$$\lim_{N\to\infty} f(N) = \lim_{N\to\infty} (VAR[Y_N] - E[Y_N]) = VAR[Y] - E[Y] \leq 0.$$

In particular, when $N = 1$, we have $Y_N = Y_1 = 1_{Y\geq 1}$, then $VAR[Y_1] = P(Y \geq 1)(1 - P(Y \geq 1))$ and $E[Y_1] = P(Y \geq 1)$, so $f(1) < 0$.

Therefore, $f(N) < 0$ for all N, i.e. $VAR[Y_N] < E[Y_N]$. □

Next, we provide a result which would be essential in comparing the same type HP with different parameters in terms of AOD, with a given expected consolidation cycle length $E[C]$.

Lemma 4.4. *X, Y are two integer valued random variables, and X is stochas-*

tically larger than Y. *If* $E[X_q] \leq E[Y_{q+1}]$, *where* q *is a positive integer, then* $E[X_q^2] \leq E[Y_{q+1}^2]$.

Proof. From

$$Y_{q+1}^2 - Y_q^2 = (2q+1)(Y_{q+1} - Y_q),$$

we have

$$E[Y_{q+1}^2] - E[Y_q^2] = (2q+1)(E[Y_{q+1}] - E[Y_q]) \geq (2q+1)(E[X_q] - E[Y_q]).$$

Therefore,

$$\begin{aligned} E[Y_{q+1}^2] - E[X_q^2] \geq & \quad E[Y_q^2] - E[X_q^2] + (2q+1)(E[X_q] - E[Y_q]) && (4.4) \\ = & \quad E[(X_q - Y_q)(2q+1 - X_q - Y_q)]. && (4.5) \end{aligned}$$

From the observation of (4.4), the value of $E[Y_{q+1}^2] - E[X_q^2]$ depends on the probability distributions of X_q and Y_q while does not depend on joint distribution of X_q and Y_q.

Since X is stochastically larger than Y, X_q is also stochastically larger than Y_q. From Proposition 9.2.2 in Ross (1996) (p. 410), we always can find two random variables X' and Y', such that X' has the same probability distribution as X_q, Y' has the same probability distribution as Y_q, and $X' \geq Y'$ almost surely.

From (4.5), and notice $X' \leq q$, $Y' \leq q$ almost surely, we have

$$E[Y_{q+1}^2] - E[X_q^2] = E[(X' - Y')(2q+1 - X' - Y')] \geq 0.$$

□

The following lemma characterizes how the ratio between the second moment and the first moment of a truncated Poisson random variable change with respect to the Poisson rate parameter, which would be used when we compare HP1 and HP2 under the fixed policy parameters, in terms of AOD.

Lemma 4.5. *Suppose $X \sim Poisson(\lambda)$ and N is a positive integer, then $\frac{E[X_N^2]}{E[X_N]}$ is increasing with respect to λ.*

Proof. Let $Y \sim Poisson(\lambda_1)$, $Z \sim Poisson(\lambda_2)$, where $\lambda_1 < \lambda_2$. Denote $Y_N = Y \wedge N$, and $Z_N = Z \wedge N$.

When $k < m < N$,

$$\begin{aligned} & P(Z_N = m)P(Y_N = k) - P(Y_N = m)P(Z_N = k) \\ = \quad & \frac{e^{-\lambda_1 - \lambda_2}}{m!k!}(\lambda_2^m \lambda_1^k - \lambda_1^m \lambda_2^k) > 0, \end{aligned} \tag{4.6}$$

and when $k < N$,

$$\begin{aligned} & P(Z_N = N)P(Y_N = k) - P(Y_N = N)P(Z_N = k) \\ = \quad & \sum_{j \geq N} (P(Z = j)P(Y = k) - P(Y = j)P(Z = k)) \\ = \quad & \sum_{j \geq N} \frac{e^{-\lambda_1 - \lambda_2}}{j!k!}(\lambda_2^j \lambda_1^k - \lambda_1^j \lambda_2^k) > 0. \end{aligned} \tag{4.7}$$

Note that for any non-negative integer valued random variable W, we have

$$E[W^2] = \sum_{m=1}^{\infty} m^2 P(W = m) = \sum_{m=1}^{\infty} \sum_{j=1}^{m} mP(W = m) = \sum_{j=1}^{\infty} \sum_{m=j}^{\infty} mP(W = m).$$

Therefore, we obtain

$$\begin{aligned}
& E[Z_N^2]E[Y_N] - E[Y_N^2]E[Z_N] \\
= \quad & \sum_{j=1}^{N}\sum_{m=j}^{N} mP(Z_N = m)\sum_{k=1}^{N} kP(Y_N = k) - \sum_{j=1}^{N}\sum_{m=j}^{N} mP(Y_N = m)\sum_{k=1}^{N} kP(Z_N = k) \\
= \quad & \sum_{j=1}^{N}\sum_{m=j}^{N}\sum_{k=1}^{j-1} mk[P(Z_N = m)P(Y_N = k) - P(Y_N = m)P(Z_N = k)] \\
> \quad & 0,
\end{aligned}$$

where the second equality comes from

$$\sum_{m=j}^{N}\sum_{k=j}^{N} mk[P(Z_N = m)P(Y_N = k) - P(Y_N = m)P(Z_N = k)] = 0,$$

and the last inequality holds since (4.6) and (4.7).

Therefore,

$$\frac{E[Z_N^2]}{E[Z_N]} - \frac{E[Y_N^2]}{E[Y_N]} = \frac{E[Z_N^2]E[Y_N] - E[Y_N^2]E[Z_N]}{E[Y_N]E[Z_N]} > 0,$$

which implies that $\frac{E[X_N^2]}{E[X_N]}$ is increasing with respect to λ. □

4.3 Comparison of AOD under Fixed Expected Cycle Length

In **(O10)** of Çetinkaya et al. (2014), it is observed numerically that for a given $E[C]$, the QP performs the best and TPs perform the worst in terms of AOD.

In this section, we analytically show that for a given $E[C]$, QP provides superior service compared with any other shipment consolidation policy in terms of AOD, not limited to HPs and TPs. Further, we provide the rigorous justification about the comparison between HPs and TPs in terms of AOD, for a given $E[C]$. In addition, for a given $E[C]$, we provide the comparison of the same type HP with different parameters, in terms of AOD.

Theorem 4.6. *For a given expected consolidation cycle length, QP dominates all the other consolidation policies in terms of AOD.*

Proof. From Table 4.1, we know AOD of a shipment consolidation policy with dispatching time τ is $AOD_\tau = \frac{E[N^2(\tau)-N(\tau)]/(2\lambda)}{E[N(\tau)]}$.

From (4.3), the fixed $E[\tau]$ implies $E[N(\tau)]$ is fixed. Then

$$AOD_\tau = \frac{1}{2\lambda}(\frac{E[N^2(\tau)]}{E[N(\tau)]} - 1) \geq \frac{1}{2\lambda}(E[N(\tau)] - 1),$$

the equality holds if and only if $N(\tau)$ is a constant, which implies QP achieves the least AOD with a fixed consolidation cycle length. $\square$

Remark 4.7. *If there is a consolidation policy with dispatching time τ, which has the same expected cycle length as a quantity-based policy with parameter q, that is $E[\tau] = \frac{q}{\lambda}$, the average cost associated with this policy is*

$$\frac{A_D + c[N(\tau)] + \omega E[W(\tau)]}{E[\tau]},$$

where A_D is the fixed cost for each dispatch, c is the unit transportation cost, and ω is the waiting cost per unit per unit time.

With fixed $E[\tau]$, $E[N(\tau)]$ is also fixed. From Theorem 4.6, we can conclude that the corresponding quantity-based policy achieves less average cost than this policy.

Theorem 4.8. *For a given expected consolidation cycle $E[C]$, HP1 performs better than TP1, and HP2 performs better than TP2 in terms of AOD.*

Proof. We consider a fixed $E[C]$ and use the following notation for the corresponding policy parameters under this $E[C]$ value: TP1 with parameter T_1, TP2 with parameter T_2, HP1 with parameters q_{H1} and T_{H1}, and HP2 with parameters q_{H2} and T_{H2}. Recalling the $E[C]$ expressions in Table 4.1, we note that, by assumption,

$$\frac{1}{\lambda}E[X_{q_{H1}}] = T_1, \tag{4.8}$$

$$\frac{1}{\lambda}E[1 + Z_{q_{H2}-1}] = \frac{1}{\lambda} + T_2, \tag{4.9}$$

where $X \sim Poisson(\lambda T_{H1})$, $Z \sim Poisson(\lambda T_{H2})$.

Next, recalling the results in Table 4.1 and the assumption of fixed $E[C]$ values for all the policies of interest, we need to show that

$$E[X_{q_{H1}}(X_{q_{H1}} - 1)] < \lambda^2 T_1^2, \tag{4.10}$$

$$E[Z_{q_{H2}-1}(Z_{q_{H2}-1} + 1)] < 2\lambda T_2 + \lambda^2 T_2^2, \tag{4.11}$$

In fact, by recalling (4.8) and (4.9), we have

$$\begin{aligned} E[X_{q_{H1}}(X_{q_{H1}} - 1)] = &\quad VAR[X_{q_{H1}}] + E^2[X_{q_{H1}}] - E[X_{q_{H1}}] \\ < &\quad E^2[X_{q_{H1}}] = \lambda^2 T_1^2 \end{aligned}$$

$$\begin{aligned} E[Z_{q_{H2}-1}(Z_{q_{H2}-1} + 1)] \quad &= VAR[Z_{q_{H2}-1}] + E^2[Z_{q_{H2}-1}] + E[Z_{q_{H2}-1}] \\ &< 2E[Z_{q_{H2}-1}] + E^2[Z_{q_{H2}-1}] \\ &= 2\lambda T_2 + \lambda^2 T_2^2 \end{aligned}$$

where the inequalities are derived from Lemma 4.3. □

From Lemma 4.4, we can perceive a stronger result:

Theorem 4.9. *For a fixed expected consolidation cycle length $E[C]$, the HP1* *with larger quantity parameter would achieve larger AOD than the HP1 with smaller quantity parameter, and the same result for HP2.*

Proof. We consider a fixed $E[C]$ and use the following notation for the corresponding policy parameters under this $E[C]$ value: the first HP1 with parameters q_H and T_H, the second HP1 with parameters $q_H + 1$ and T'_H. Recalling the $E[C]$ expressions in Table 4.1, we note that, by assumption,

$$E[X_{q_H}] = E[Y_{q_H+1}], \tag{4.12}$$

where $X \sim Poisson(\lambda T_H)$, $Y \sim Poisson(\lambda T'_H)$. Clearly, $T_H > T'_H$.

Next, recalling the results in Table 4.1 and the assumption of fixed $E[C]$ values for all the policies of interest, we need to show that

$$E[X_{q_H}(X_{q_H} - 1)] < E[Y_{q_H+1}(Y_{q_H+1} - 1)]. \tag{4.13}$$

From Lemma 4.4 and recalling (4.12), we have

$$E[X_{q_H}^2] \leq E[Y_{q_H+1}^2],$$

so that (4.13) is verified.

The same procedure can be applied to prove the similar results between two HP2 policies. □

4.4 Comparison of AOD under Fixed Parameters

In Çetinkaya et al. (2014), it is analytically shown that under fixed parameters, the general class of HPs outperform the general classes of counterpart QP and TPs in terms of AOD. In this section, we provides another simplified proof of the above statement based on the rewritten expressions in Table 4.1. Further, we show under fixed parameters, HP1 outperforms HP2 in terms of AOD.

Theorem 4.10. *With fixed parameters q, T, HP1 performs better than QP and TP1 in terms of AOD.*

Proof. On one aspect, we need to show HP1 performs better than QP in terms of AOD with the same parameters q, T, from Table 4.1, that is,

$$\frac{E[Y_q(Y_q - 1)]}{E[Y_q]} < q - 1.$$

In fact,

$$(q-1)E[Y_q] - E[Y_q(Y_q - 1)] = qE[Y_q] - E[Y_q^2] = E[(q - Y_q)Y_q] > 0.$$

On the other aspect, we need to show HP1 performs better than TP1 in terms of AOD with the same parameters q, T, from Table 4.1, that is,

$$\frac{E[Y_q(Y_q - 1)]}{E[Y_q]} < \lambda T.$$

In fact, from Lemma 4.3, we have $VAR[Y_q] < E[Y_q]$, which can written as

$$E[Y_q(Y_q - 1)] < E^2[Y_q].$$

It is sufficient to show

$$E[Y_q] < \lambda T,$$

which holds since $Y \sim Poisson(\lambda T)$. □

Theorem 4.11. *With fixed parameters q, T, HP2 performs better than QP and TP2 in terms of AOD.*

Proof. On one aspect, we need to show HP2 performs better than QP in terms of AOD with the same parameters q, T, from Table 4.1, that is,

$$\frac{E[Y_{q-1}(Y_{q-1}+1)]}{E[1+Y_{q-1}]} < q-1.$$

In fact,

$$(q-1)E[1+Y_{q-1}] - E[Y_{q-1}(Y_{q-1}+1)] = E[(q-1-Y_{q-1})(Y_{q-1}+1)] > 0.$$

On the other aspect, we need to show HP2 performs better than TP2 in terms of AOD with the same parameters q, T, from Table 4.1, that is,

$$\frac{E[Y_{q-1}(Y_{q-1}+1)]}{E[1+Y_{q-1}]} < \frac{2\lambda T + \lambda^2 T^2}{1+\lambda T}.$$

In fact, from Lemma 4.3, we have $VAR[Y_{q-1}] < E[Y_{q-1}]$, which can written as

$$E[Y_{q-1}(Y_{q-1}+1)] < E^2[Y_{q-1}+1] - 1.$$

It is sufficient to show

$$E[Y_{q-1}+1] - \frac{1}{E[Y_{q-1}+1]} < (\lambda T + 1) - \frac{1}{\lambda T + 1},$$

which holds since $E[Y_{q-1}] < \lambda T$. $\square$

Theorem 4.12. *With fixed parameters q and T, HP1 performs better than HP2 in terms of AOD.*

Proof. From Table 4.1, we need to show

$$\frac{E[Y_q(Y_q-1)]}{E[Y_q]} < \frac{E[Y_{q-1}(Y_{q-1}+1)]}{E[1+Y_{q-1}]}.$$

After simplification, it suffices to show

$$\frac{E[Y_q^2]}{E[Y_q]} < \frac{E[(Y_{q-1}+1)^2]}{E[Y_{q-1}+1]}. \tag{4.14}$$

Note for $X \sim Poisson(\mu)$, we have

$$\frac{d}{d\mu}E[g(X)] = E[g(X+1)] - E[g(X)],$$

for any appropriate function $g(x)$.

Let $\mu = \lambda T$, $g_1(x) = (x \wedge q)^2$, and $g_2(x) = x \wedge q$, we have

$$\begin{aligned} \frac{d}{d\mu}E[Y_q^2] = \frac{d}{d\mu}E[g_1(Y)] = & \quad E[g_1(Y+1)] - E[g_1(Y)] \\ = & \quad E[((Y+1)\wedge q)^2] - E[(Y\wedge q)^2] \\ = & \quad E[(Y_{q-1}+1)^2] - E[Y_q^2], \end{aligned}$$

$$\begin{aligned} \frac{d}{d\mu}E[Y_q] = \frac{d}{d\mu}E[g_2(Y)] = & \quad E[g_2(Y+1)] - E[g_2(Y)] \\ = & \quad E[(Y+1)\wedge q] - E[Y\wedge q] \\ = & \quad E[Y_{q-1}+1] - E[Y_q]. \end{aligned}$$

Hence,

$$\begin{aligned}\frac{d}{d\mu}\frac{E[Y_q^2]}{E[Y_q]} &= \frac{(E[(Y_{q-1}+1)^2] - E[Y_q^2])E[Y_q] - E[Y_q^2](E[Y_{q-1}+1] - E[Y_q])}{E^2[Y_q]} \\ &= \frac{E[(Y_{q-1}+1)^2]E[Y_q] - E[Y_q^2]E[Y_{q-1}+1]}{E^2[Y_q]}.\end{aligned}$$

From Lemma 4.5, we know $\frac{d}{d\mu}\frac{E[Y_q^2]}{E[Y_q]} > 0$, thus

$$E[(Y_{q-1}+1)^2]E[Y_q] - E[Y_q^2]E[Y_{q-1}+1] > 0,$$

which implies (4.14) is satisfied.

□

5. ON A NEW POLICY IN SHIPMENT CONSOLIDATION MODEL

In this chapter, we reconsider the problem in Chapter 3, the multi-item shipment consolidation model with drifted Brownian motion demands. We show that among $(Q+\tau)$-type policy, either quantity-based or time-based policy is the best one in terms of average cost in the long run. The natural question is, can we find some other type policy, which achieves lower average cost in the long run than the optimal $(Q+\tau)$-type policy?

We need to decide a sequence of increasing stopping times $\theta_1, \theta_2, \dots,$ at which the consolidated load of the n items are dispatched. Denote the policy as $U = (\theta_1, \theta_2, \dots,)$.

The average cost functional is as follows:

$$\begin{aligned} & F_0(x_1, x_2, \dots, x_n, U) \\ = \quad & \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T w(z_1(s), \dots, z_n(s))ds + \sum_{\{k:\theta_k<T\}} c(z_1(\theta_k), \dots, z_n(\theta_k))\right], \end{aligned}$$

where $z_i(t)$ is the consolidated load of the $i-$th item, $z_i(0) = x_i$, $i = 1, 2, \dots, n$. Clearly, between two consecutive dispatching, $z_i(t)$ is a drifted Brownian motion. $w(z_1, z_2, \dots, z_n) = \sum_{i=1}^n \omega_i z_i$ and $c(z_1, z_2, \dots, z_n) = A_D + \sum_{i=1}^n c_i z_i$ denote the waiting cost and dispatching cost, respectively.

Further, we propose a service measure about average waiting penalty rate before delivery. Under any renewal-type consolidation policy, the consolidated load forms a regenerative process. So, under the clearing policy with cycle τ, the average waiting penalty rate (AWPR) can be obtained by applying the Renewal Reward Theorem, i.e.,

$$\begin{aligned} AWPR &= \frac{E[\text{Cumulative weighted waiting delay per consolidation cycle}]}{E[\text{Consolidation cycle length}]} \\ &= \frac{E\left[\sum_{i=1}^{n} \int_0^{\tau} \omega_i N_i(u)du\right]}{E[\tau]}. \end{aligned}$$

We provide comparison results among different policies in terms of AWPR.

5.1 Average Cost Model

5.1.1 Mathematical Preliminaries

Assume $N_i(t) = D_i t + \sigma_i B_i(t)$, where $i = 1, 2, \ldots, n$, $D_i > 0$, $\sigma_i > 0$ are the drift coefficient and diffusion coefficient, respectively. $B_1(t), B_2(t), \ldots, B_n(t)$ are independent standard Brownian motions.

Define $\tau_M = \inf\{t > 0 : \sum_{i=1}^{n} \omega_i N_i(t) \geq M\}$, which is a stopping time w.r.t the filtration generated by $B_1(t), B_2(t), \ldots, B_n(t)$. We have the following results that characterize the statistical property of τ_M.

Lemma 5.1. *For $s > 0$,*

$$E[\exp(-s\tau_M)] = \exp\left(\frac{\sum_{i=1}^{n} \omega_i D_i - \sqrt{(\sum_{i=1}^{n} \omega_i D_i)^2 + 2s \sum_{i=1}^{n} \omega_i^2 \sigma_i^2}}{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2} M\right),$$

$$E[\tau_M] = \frac{M}{\sum_{i=1}^{n} \omega_i D_i}, \quad E[\tau_M^2] = \frac{M^2}{(\sum_{i=1}^{n} \omega_i D_i)^2} + \frac{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2 M}{(\sum_{i=1}^{n} \omega_i D_i)^3}.$$

The next result gives joint moment generation function for $(B_i(\tau_M), \tau_M)$.

Lemma 5.2. *For $s_1^2 + 2s_2 < 0$,*

$$\begin{aligned} & E[\exp(s_1 B_i(\tau_M) + s_2 \tau_M)] \\ = \quad & \exp\left(\frac{s_1 \omega_i \sigma_i + \sum_{i=1}^{n} \omega_i D_i - \sqrt{(s_1 \omega_i \sigma_i + \sum_{i=1}^{n} \omega_i D_i)^2 - (s_1^2 + 2s_2) \sum_{i=1}^{n} \omega_i^2 \sigma_i^2}}{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2} M\right), \end{aligned}$$

$$E[B_i(\tau_M)\tau_M] = -\frac{\omega_i \sigma_i M}{(\sum_{i=1}^n \omega_i D_i)^2}.$$

Proof. The proof is similar as Lemma 3.2. □

5.1.2 Instantaneous Rate Policy

We propose a new policy, where a clearing is trigger whenever the instantaneous waiting penalty rate hits a threshold value, i.e., a clearing is made as long as $\sum_{i=1}^n \omega_i N_i(t) = M$, M is a threshold value we need to optimize. We call this new policy as an instantaneous rate policy. Recalling that under a quantity-based policy with parameter Q, we clear the system as long as the total consolidated load reaches Q, and under a time-based policy with parameter T, the system is cleared every T units time. Clearly, under a quantity-based policy, we just need to track the total demand process as a whole. Under a time-based policy, we do not need to track any process at all. On contrast, we need to track each demand process associated with each item, when we implement an instantaneous rate policy.

The motivation of the new policy is as follows: suppose the demands are discrete and arrives one by one, if the first arriving item is with large waiting sensitivity, we should not hold the consolidated load for a long time; while if the first arriving item is with small waiting sensitivity, we can prolong the holding time of the consolidated load. Upon this observation, we should realize that the optimal policy requires tracking each demand process associated with each item.

By using Lemma 5.1 and Lemma 5.2, we can obtain the following result which provides the expected waiting time for the i-th item and the total waiting cost for all the items within one dispatch cycle.

Theorem 5.3. *Under the instantaneous rate policy with parameter M, the cu-*

mulative waiting time for i-th item within one consolidation cycle is

$$
\begin{aligned}
& E\left[\int_0^{\tau_M} N_i(t)dt\right] \\
= \quad & \frac{1}{2}\frac{D_i}{(\sum_{i=1}^n \omega_i D_i)^2}M^2 + \frac{1}{2}\frac{D_i \sum_{i=1}^n \omega_i^2\sigma_i^2}{(\sum_{i=1}^n \omega_i D_i)^3}M - \frac{\omega_i\sigma_i^2}{(\sum_{i=1}^n \omega_i D_i)^2}M,
\end{aligned}
$$

and the expected total waiting cost for all items within one dispatch cycle is

$$
\sum_{i=1}^n \omega_i E\left[\int_0^{\tau_M} N_i(t)dt\right] = \frac{1}{2\sum_{i=1}^n \omega_i D_i}M^2 - \frac{\sum_{i=1}^n \omega_i^2\sigma_i^2}{2(\sum_{i=1}^n \omega_i D_i)^2}M.
$$

Proof. The proof is similar as Theorem 3.3. □

Further, the expected transportation cost each shipping is

$$
A_D + E[\sum_{i=1}^n c_i N_i(\tau_Q)] = A_D + \frac{M}{\sum_{i=1}^n \omega_i D_i}\sum_{i=1}^n c_i D_i.
$$

Therefore, we can obtain the average cost under the instantaneous rate policy with parameter M is

$$
\begin{aligned}
& AC^{IRP}(M) \\
= \quad & \frac{A_D + \frac{M}{\sum_{i=1}^n \omega_i D_i}\sum_{i=1}^n c_i D_i + \frac{1}{2\sum_{i=1}^n \omega_i D_i}M^2 - \frac{\sum_{i=1}^n \omega_i^2\sigma_i^2}{2(\sum_{i=1}^n \omega_i D_i)^2}M}{\frac{M}{\sum_{i=1}^n \omega_i D_i}} \\
= \quad & \frac{A_D \sum_{i=1}^n \omega_i D_i}{M} + \frac{1}{2}M + \sum_{i=1}^n c_i D_i - \frac{\sum_{i=1}^n \omega_i^2\sigma_i^2}{2\sum_{i=1}^n \omega_i D_i}
\end{aligned}
$$

Minimizing $AC^{IRP}(M)$, we get

$$
M^{OPT} = \sqrt{2A_D \sum_{i=1}^n \omega_i D_i},
$$

and

$$AC^{IRP}(M^{OPT}) = \sqrt{2A_D \sum_{i=1}^{n} \omega_i D_i} + \sum_{i=1}^{n} c_i D_i - \frac{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2}{2\sum_{i=1}^{n} \omega_i D_i}.$$

From Chapter 3, we obtain that the average cost of the optimal quantity-based policy is

$$\sqrt{2A_D \sum_{i=1}^{n} \omega_i D_i} + \sum_{i=1}^{n} c_i D_i - \sum_{i=1}^{n} \omega_i (\frac{\sigma_i^2}{D} - \frac{D_i \sigma^2}{2D^2}).$$

In the following, we adopt a time based policy, which dispatches the consolidated load every T units time.

$$\begin{aligned} & E[\text{Cumulative Waiting time for } i\text{-th item per Consolidation Cycle}] \\ = \quad & E\left[\int_0^T N_i(t)dt\right] = \int_0^T D_i t dt = \frac{1}{2} D_i T^2. \end{aligned}$$

By the Renewal Reward Theorem, the expected total long-run average cost per unit-time is

$$\begin{aligned} AC^{TP}(T) = \quad & \frac{A_D + \sum_{i=1}^{n} c_i D_i T + \frac{1}{2}\sum_{i=1}^{n} \omega_i D_i T^2}{T} \\ = \quad & \frac{A_D}{T} + \frac{1}{2}\sum_{i=1}^{n} \omega_i D_i T + \sum_{i=1}^{n} c_i D_i. \end{aligned}$$

We obtain the optimal time parameter

$$T^* = \sqrt{\frac{2A_D}{\sum_{i=1}^{n} \omega_i D_i}},$$

and the associated average cost

$$AC^{TP}(T^*) = \sqrt{2A_D \sum_{i=1}^{n} \omega_i D_i} + \sum_{i=1}^{n} c_i D_i.$$

From the equality

$$
\begin{aligned}
& (\sum_{i=1}^n \omega_i^2\sigma_i^2)(\sum_{i=1}^n D_i)^2 - [2\sum_{i=1}^n \omega_i\sigma_i^2 \sum_{i=1}^n D_i - \sum_{i=1}^n \omega_i D_i \sum_{i=1}^n \sigma_i^2] \sum_{i=1}^n \omega_i D_i \\
= \quad & \sum_{k=1}^n \sigma_k^2 [\omega_k \sum_{i=1}^n D_i - \sum_{i=1}^n \omega_i D_i]^2 \geq 0,
\end{aligned}
$$

we can see that the optimal instantaneous rate policy achieves lower average cost than both of the optimal quantity-based policy and the optimal time-based policy.

Remark 5.4. *In the multi-item model, under TP, we do not need to track any process realization; under QP, we need to track the realization of the total cumulative process of all items; under instantaneous rate policy, we need to track the realization of each input process. In a stochastic dynamic system, the optimal policy must be the one taking advantage of full information. That is the value of information.*

5.1.3 Martingale Argument for the Optimality of Instantaneous Rate Policy

Lemma 5.5. *Let $N(t) = Dt + \sigma B(t)$, then*

$$\int_0^t N(u)du - \frac{1}{2D}N^2(t) + \frac{\sigma^2}{2D^2}N(t)$$

is a martingale with respect to the filtration generated by $N(t)$.

Proof. Let $\{\mathcal{G}_t\}$ be the natural filtration for $N(t)$. Then, for $s < t$,

$$
\begin{aligned}
E\Big[\int_0^t N(u)du \mid \mathcal{G}_s\Big] &= \int_0^s N(u)du + E\Big[\int_s^t N(u)du \mid \mathcal{G}_s\Big] \\
&= \int_0^s N(u)du + (t-s)N(s) + E\Big[\int_0^{t-s} N(u)du\Big] \\
&= \int_0^s N(u)du + (t-s)N(s) + \frac{1}{2}D(t-s)^2,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2D}E[N^2(t) \mid \mathcal{G}_s] \\
= \quad & \frac{1}{2D}\Big(E[(N(t)-N(s))^2 \mid \mathcal{G}_s] + 2N(s)E[N(t)-N(s) \mid \mathcal{G}_s] + N^2(s)\Big) \\
= \quad & \frac{1}{2D}\Big(\sigma^2(t-s) + D^2(t-s)^2 + 2D(t-s)N(s) + N^2(s)\Big),
\end{aligned}
$$

and

$$\frac{\sigma^2}{2D^2}E[N(t) \mid \mathcal{G}_s] = \frac{\sigma^2}{2D^2}\Big(N(s) + D(t-s)\Big).$$

We obtain

$$
\begin{aligned}
& E\Big[\int_0^t N(u)du - \frac{1}{2D}N^2(t) + \frac{\sigma^2}{2D^2}N(t) \mid \mathcal{G}_s\Big] \\
& = \int_0^s N(u)du - \frac{1}{2D}N^2(s) + \frac{\sigma^2}{2D^2}N(s),
\end{aligned}
$$

which shows that $\int_0^t N(u)du - \frac{1}{2D}N^2(t) + \frac{\sigma^2}{2D^2}N(t)$ is a martingale. $\square$

From $\sum_{i=1}^n \omega_i N_i(t) = \sum_{i=1}^n \omega_i D_i t + \sum_{i=1}^n \omega_i \sigma_i B_i(t)$ is also a drifted BM, we have the following result.

Lemma 5.6.

$$\sum_{i=1}^n \int_0^t \omega_i N_i(u)du - \frac{1}{2\sum_{i=1}^n \omega_i D_i}(\sum_{i=1}^n \omega_i N_i(t))^2 + \frac{\sum_{i=1}^n \omega_i^2 \sigma_i^2}{2(\sum_{i=1}^n \omega_i D_i)^2}\sum_{i=1}^n \omega_i N_i(t)$$

is a martingale with respect to the filtration generated by $N_1(t), N_2(t), \ldots, N_n(t)$.

Applying optional stopping theorem and martingale convergence theorem in L_1, for stopping times τ taking forms of τ_M, T or T_Q (corresponding to IRP, TP, and

QP, respectively), we have

$$E\left[\sum_{i=1}^{n}\int_0^{\tau}\omega_i N_i(u)du\right] = \frac{1}{2\sum_{i=1}^{n}\omega_i D_i}E[(\sum_{i=1}^{n}\omega_i N_i(\tau))^2] - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{2(\sum_{i=1}^{n}\omega_i D_i)^2}E\left[\sum_{i=1}^{n}\omega_i N_i(\tau)\right]. \tag{5.1}$$

Suppose we dispatch the consolidated load every τ units of time, the average cost in the long run should be:

$$\begin{aligned}
& \frac{A_D + \sum_{i=1}^{n} c_i E[N_i(\tau)] + E[\sum_{i=1}^{n}\int_0^{\tau}\omega_i N_i(u)du]}{E[\tau]} \\
= \quad & \frac{A_D + \sum_{i=1}^{n} c_i E[N_i(\tau)] + \frac{1}{2\sum_{i=1}^{n}\omega_i D_i}E[(\sum_{i=1}^{n}\omega_i N_i(\tau))^2]}{E[\tau]} \\
& - \frac{\frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{2(\sum_{i=1}^{n}\omega_i D_i)^2}E[\sum_{i=1}^{n}\omega_i N_i(\tau)]}{E[\tau]} \\
\geq \quad & \frac{A_D + \sum_{i=1}^{n} c_i E[N_i(\tau)] + \frac{1}{2\sum_{i=1}^{n}\omega_i D_i}E^2[\sum_{i=1}^{n}\omega_i N_i(\tau)]}{E[\tau]} \\
& - \frac{\frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{2(\sum_{i=1}^{n}\omega_i D_i)^2}E[\sum_{i=1}^{n}\omega_i N_i(\tau)]}{E[\tau]},
\end{aligned}$$

where the last inequality comes from $E[X^2] \geq E^2[X]$, and equality holds if and only if $\sum_{i=1}^{n}\omega_i N_i(\tau)$ is a constant a.s.

Also, we notice that, if we fixed $E[\tau]$, the numerator of last term in the formula is also fixed.

Therefore, we show that among a general class of renewal type dispatch policies, if the expected cycle length $E[\tau]$ is fixed, the best policy should be the instantaneous rate policy.

In sum, if we limit to consider the renewal type clearing policies and notice that each policy corresponds to a stopping rule, we can apply martingale arguments (together with the well celebrated optional stopping theorem) to conquer the optimal control problem, avoiding the dynamic programming, which is the fundamental tool in various optimal control theoretic frameworks.

Remark 5.7. *From Lemma 5.5, we notice that*

$$\sum_{i=1}^{n}\omega_i\int_0^t N_i(u)du-\frac{1}{2}\sum_{i=1}^{n}\frac{\omega_i}{D_i}N_i^2(t)+\frac{1}{2}\sum_{i=1}^{n}\frac{\sigma_i^2}{D_i^2}\omega_iN_i(t)$$

is a martingale with respect to the filtration generated by $N_1(t), N_2(t), \dots, N_n(t)$.

Together with Lemma 5.6, we deduce

$$\sum_{i=1}^{n}\frac{\omega_i}{D_i}N_i^2(t)-\sum_{i=1}^{n}\frac{\sigma_i^2}{D_i^2}\omega_iN_i(t)+\frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{(\sum_{i=1}^{n}\omega_iD_i)^2}\sum_{i=1}^{n}\omega_iN_i(t)$$
$$-\frac{1}{\sum_{i=1}^{n}\omega_iD_i}(\sum_{i=1}^{n}\omega_iN_i(t))^2$$

is also a martingale.

So, applying optional stopping theorem, for some stopping time τ, *we have*

$$\begin{aligned}
& E[\sum_{i=1}^{n}\frac{\omega_i}{D_i}N_i^2(\tau)] \\
= \quad & \sum_{i=1}^{n}\frac{\sigma_i^2}{D_i^2}E[\omega_iN_i(\tau)]-\frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{(\sum_{i=1}^{n}\omega_iD_i)^2}E[\sum_{i=1}^{n}\omega_iN_i(\tau)] \\
& +\frac{1}{\sum_{i=1}^{n}\omega_iD_i}E[(\sum_{i=1}^{n}\omega_iN_i(\tau))^2] \\
\geq \quad & \sum_{i=1}^{n}\frac{\sigma_i^2}{D_i^2}E[\omega_iN_i(\tau)]-\frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{(\sum_{i=1}^{n}\omega_iD_i)^2}E[\sum_{i=1}^{n}\omega_iN_i(\tau)] \\
& +\frac{1}{\sum_{i=1}^{n}\omega_iD_i}E^2[\sum_{i=1}^{n}\omega_iN_i(\tau)].
\end{aligned}$$

With $E[\tau]$ fixed, $\sum_{i=1}^{n} \frac{\sigma_i^2}{D_i^2} E[\omega_i N_i(\tau)]$ and $E[\sum_{i=1}^{n} \omega_i N_i(\tau)]$ is also fixed.

The equality holds if and only if $\sum_{i=1}^{n} \omega_i N_i(\tau)$ is a constant. In sum, we solve an interesting optimization problem as follows: To minimize $E[\sum_{i=1}^{n} \frac{\omega_i}{D_i} N_i^2(\tau)]$ subjected to a constant $E[\tau]$, τ should be τ_M-type stopping time.

5.2 Average Weighted Delay Rate

Customer waiting occurs when consolidation policies are implemented, since a prolonged order holding is needed to accumulate a large load. One important service measure indicator is average weighted delay per unit time before delivery. Under any renewal-type consolidation policy, the consolidated load forms a regenerative process. So, under the clearing policy with cycle τ, the average weighted delay rate can be obtained by applying the Renewal Reward Theorem, i.e.,

$$\begin{aligned} AWDR &= \frac{E[\text{Cumulative weighted waiting delay per consolidation cycle}]}{E[\text{Consolidation cycle length}]} \\ &= \frac{E[W]}{E[L]} = \frac{E\left[\sum_{i=1}^{n} \int_0^{\tau} \omega_i N_i(u) du\right]}{E[\tau]}, \end{aligned}$$

where W denotes the cumulative weighted waiting delay within one consolidation cycle, and L denotes the consolidation cycle length. We index $AWDR$, W, and L by policy type as needed.

Recalling (5.1), we have

$$AWDR = \frac{\frac{1}{2\sum_{i=1}^{n} \omega_i D_i} E[(\sum_{i=1}^{n} \omega_i N_i(\tau))^2] - \frac{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2}{2(\sum_{i=1}^{n} \omega_i D_i)^2} E[\sum_{i=1}^{n} \omega_i N_i(\tau)]}{E[\tau]},$$

which provides an unified method to calculate the average weighted delay per unit load under any renewal-type consolidation policy.

From the above discussion, we can deduce AWDR for any renewal-type clearing policy. We focus on instantaneous rate policy (IRP), time based policy (TP), and instantaneous rate hybrid policy (IRHP). Instantaneous rate hybrid policy is a combination of IRP and TP. Stated formally, under IRHP with parameter M and T, the goal is to implement an instantaneous rate policy with parameter M. However, if until time T since the last shipment epoch, $\sum_{i=1}^{n} \omega_i N_i(t)$ hasn't reached M, then a shipment decision is made.

1. IRP with parameter M: $\tau = \tau_M$, $\sum_{i=1}^{n} \omega_i N_i(\tau_M) = M$. So,

$$\begin{aligned} E[W_{IRP}] &= \frac{1}{2\sum_{i=1}^{n} \omega_i D_i} M^2 - \frac{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2}{2(\sum_{i=1}^{n} \omega_i D_i)^2} M, \\ E[L_{WQP}] &= E[\tau_M] = \frac{M}{\sum_{i=1}^{n} \omega_i D_i}. \end{aligned}$$

2. TP with parameter T: $\tau = T$, and

$$\sum_{i=1}^{n} \omega_i N_i(T) \sim Normal(\sum_{i=1}^{n} \omega_i D_i T, \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 T).$$

So,

$$E[W_{TP}] = \frac{1}{2} \sum_{i=1}^{n} \omega_i D_i T^2, E[L_{TP}] = T.$$

3. IRHP with parameters M and T: $\tau = \tau_M \wedge T$.

$$\begin{aligned} E[W_{WHP}] = &\quad \frac{1}{2\sum_{i=1}^{n} \omega_i D_i} E\left[(\sum_{i=1}^{n} \omega_i N_i(\tau_M \wedge T))^2 \right] \\ &- \frac{\sum_{i=1}^{n} \omega_i^2 \sigma_i^2}{2(\sum_{i=1}^{n} \omega_i D_i)^2} E\left[\sum_{i=1}^{n} \omega_i N_i(\tau_M \wedge T) \right], \\ E[L_{HP}] = &\quad E[\tau_M \wedge T]. \end{aligned}$$

In Table 5.1, we summarize the AWDR for different consolidation policies.

$$
\begin{aligned}
AWDR_{\tau} &= \frac{\frac{1}{2\sum_{i=1}^{n}\omega_i D_i}E[(\sum_{i=1}^{n}\omega_i N_i(\tau))^2] - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{2(\sum_{i=1}^{n}\omega_i D_i)^2}E[\sum_{i=1}^{n}\omega_i N_i(\tau)]}{E[\tau]} \\
AWDR_{IRP} &= \frac{\frac{1}{2\sum_{i=1}^{n}\omega_i D_i}M^2 - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{2(\sum_{i=1}^{n}\omega_i D_i)^2}M}{\frac{M}{\sum_{i=1}^{n}\omega_i D_i}} = \frac{M - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{\sum_{i=1}^{n}\omega_i D_i}}{2} \\
AWDR_{TP} &= \frac{\frac{1}{2}\sum_{i=1}^{n}\omega_i D_i T^2}{T} = \frac{\sum_{i=1}^{n}\omega_i D_i T}{2} \\
AWDR_{IRHP} &= \frac{\frac{1}{2\sum_{i=1}^{n}\omega_i D_i}E[(\sum_{i=1}^{n}\omega_i N_i(\tau_M \wedge T))^2] - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{2(\sum_{i=1}^{n}\omega_i D_i)^2}E[\sum_{i=1}^{n}\omega_i N_i(\tau_M \wedge T)]}{E[\tau_M \wedge T]}
\end{aligned}
$$

Table 5.1: Summary of the Expressions of $AWDR$.

5.2.1 A Key Inequality

Lemma 5.8. *Let $N(t) = \lambda t + \sigma B(t)$ be a Brownian motion with drift and denote its hitting times $\tau_q = \min\{t : N(t) = q\}$ for $q > 0$. Fix $T > 0$, then*

$$\begin{aligned} & \frac{\sigma^2}{\lambda} E[N(\tau_q \wedge T)] - var[N(\tau_q \wedge T)] \\ = \quad & \lambda^2 (var[\tau_q \wedge T] + 2E[(\tau_q - T)_+]E[(T - \tau_q)_+]) > 0. \end{aligned}$$

Proof. Using Wald's first two equations and then simplifying,

$$\begin{aligned} & \frac{\sigma^2}{\lambda} E[N(\tau_q \wedge T)] - var[N(\tau_q \wedge T)] \\ = \quad & \sigma^2 E[\tau_q \wedge T] - var[N(\tau_q \wedge T)] \\ = \quad & E[(N(\tau_q \wedge T) - \lambda(\tau_q \wedge T))^2] - E[(N(\tau_q \wedge T))^2] + \lambda^2 (E[\tau_q \wedge T])^2 \\ = \quad & \lambda^2 E[(\tau_q \wedge T)^2] + \lambda^2 (E[\tau_q \wedge T])^2 - 2\lambda E[(\tau_q \wedge T) N(\tau_q \wedge T)]. \end{aligned} \tag{5.2}$$

Next,

$$E[\tau_q \wedge T] = T - E[(T - \tau_q) 1_{\tau_q \leq T}] = T - E[(T - \tau_q)_+]. \tag{5.3}$$

Likewise,

$$\begin{aligned} E[(\tau_q \wedge T)^2] &= T^2 - E[(T^2 - \tau_q^2) 1_{\tau_q \leq T}] = T^2 - E[(T + \tau_q)(T - \tau_q)_+] \\ &= T^2 - 2TE[(T - \tau_q)_+] + E[(T - \tau_q)_+^2], \end{aligned} \tag{5.4}$$

having noted that $T - \tau_q = (T - \tau_q)_+ - (\tau_q - T)_+$ and $(T - \tau_q)_+ (\tau_q - T)_+ = 0$. Applying the strong Markov property and using Wald's first equation again,

$$E[(\tau_q \wedge T) N(\tau_q \wedge T)] = E[TN(T) + (q\tau_q - TN(T)) 1_{\tau_q \leq T}]$$

$$
\begin{aligned}
&= \lambda T^2 + E[(q(\tau_q - T) - T(N(T) - N(\tau_q)))1_{\tau_q \le T}] \\
&= \lambda T^2 + E[(q(\tau_q - T) - \lambda T(T - \tau_q))1_{\tau_q \le T}] \\
&= \lambda T^2 - \lambda E[(T + E[\tau_q])(T - \tau_q)_+] \\
&= \lambda T^2 - 2\lambda T E[(T - \tau_q)_+] + \lambda E[T - \tau_q]E[(T - \tau_q)_+]. \qquad (5.5)
\end{aligned}
$$

Putting (5.3)–(5.5) into (5.2),

$$
\begin{aligned}
& \frac{\sigma^2}{\lambda} E[N(\tau_q \wedge T)] - var[N(\tau_q \wedge T)] \\
= \quad & \lambda^2 (E[(T - \tau_q)_+^2] + (E[(T - \tau_q)_+])^2 - 2E[T - \tau_q]E[(T - \tau_q)_+]) \\
= \quad & \lambda^2 (E[(T - \tau_q)_+^2] - (E[(T - \tau_q)_+])^2 + 2E[(\tau_q - T)_+]E[(T - \tau_q)_+]) \\
= \quad & \lambda^2 (var[(T - \tau_q)_+] + 2E[(\tau_q - T)_+]E[(T - \tau_q)_+]) \\
= \quad & \lambda^2 (var[\tau_q \wedge T] + 2E[(\tau_q - T)_+]E[(T - \tau_q)_+]) > 0.
\end{aligned}
$$

□

Lemma 5.9. *Let $N_i(t) = D_i t + \sigma_i B_i(t)$ be n independent Brownian motions with drift, where $i = 1, 2, \ldots, n$, and denote $\tau_Q = \min\{t : \sum_{i=1}^n \omega_i N_i(t) = Q\}$ for $Q > 0$. Fix $T > 0$, then*

$$
\frac{\sum_{i=1}^n \omega_i^2 \sigma_i^2}{\sum_{i=1}^n \omega_i D_i} E[\sum_{i=1}^n \omega_i N_i(\tau_Q \wedge T)] - var[\sum_{i=1}^n \omega_i N_i(\tau_Q \wedge T)] > 0.
$$

Proof. Treating $\sum_{i=1}^n \omega_i N_i(t)$ as a one dimensional drifted Brownian motion with drift $\sum_{i=1}^n \omega_i D_i$ and diffusion coefficient $\sqrt{\sum_{i=1}^n \omega_i^2 \sigma_i^2}$, and applying Lemma 5.8, we arrive at the conclusion. □

5.2.2 Comparison of AWDR under Fixed Expected Cycle Length

Theorem 5.10. *For a given expected consolidation cycle length, IRP dominates all the other consolidation policies in terms of AWDR.*

Proof. From Table 5.1, we know AWDR of a consolidation policy with dispatching time τ is

$$AWDR_\tau = \frac{\frac{1}{2\sum_{i=1}^n \omega_i D_i} E[(\sum_{i=1}^n \omega_i N_i(\tau))^2] - \frac{\sum_{i=1}^n \omega_i^2 \sigma_i^2}{2(\sum_{i=1}^n \omega_i D_i)^2} E[\sum_{i=1}^n \omega_i N_i(\tau)]}{E[\tau]}.$$

Noticing the fixed $E[\tau]$ implies $E[\sum_{i=1}^n \omega_i N_i(\tau)]$ is fixed, we have

$$AWDR_\tau \geq \frac{\frac{1}{2\sum_{i=1}^n \omega_i D_i} E^2[\sum_{i=1}^n \omega_i N_i(\tau)] - \frac{\sum_{i=1}^n \omega_i^2 \sigma_i^2}{2(\sum_{i=1}^n \omega_i D_i)^2} E[\sum_{i=1}^n \omega_i N_i(\tau)]}{E[\tau]},$$

the equality holds if and only if $E[\sum_{i=1}^n \omega_i N_i(\tau)]$ is a constant, which implies IRP achieves the least AWDR with a fixed consolidation cycle length. □

Theorem 5.11. *For a given expected consolidation cycle, IRHP performs better than TP, in terms of AWDR.*

Proof. We consider a fixed $E[\tau]$ and use the following notation for the corresponding policy parameters under this $E[\tau]$ value: TP with parameter T, and IRHP with parameters M_H and T_H. Recalling the $E[\tau]$ expressions for different policies in Table 5.1, we note that, by assumption,

$$E[\tau_{M_H} \wedge T_H] = T,$$

which implies

$$E[\sum_{i=1}^n \omega_i N_i(\tau_{M_H} \wedge T_H)] = \sum_{i=1}^n \omega_i D_i T. \tag{5.6}$$

Next, recalling the results in Table 5.1 and the assumption of fixed $E[\tau]$ values for all the policies of interest, we need to show that

$$
\begin{aligned}
& E\left[(\sum_{i=1}^{n}\omega_i N_i(\tau_{M_H}\wedge T_H))^2\right] - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{\sum_{i=1}^{n}\omega_i D_i} E\left[\sum_{i=1}^{n}\omega_i N_i(\tau_{M_H}\wedge T_H)\right] \\
< \quad & (\sum_{i=1}^{n}\omega_i D_i)^2 T^2. \qquad (5.7)
\end{aligned}
$$

In fact, by recalling (5.6) and Lemma 5.9, (5.7) is verified. □

Remark 5.12. *From Lemma 5.11, we can conclude that, given any TP and IRHP, as long as they have the same expected consolidation cycle, the IRHP achieves less average cost than the TP.*

5.2.3 Comparison of AWDR under Fixed Parameters

Theorem 5.13. *With fixed parameters M, T, IRHP performs better than TP, in terms of AWDR.*

Proof. From Table 5.1, and noticing that

$$
E\left[\sum_{i=1}^{n}\omega_i N_i(\tau_M\wedge T)\right] = \sum_{i=1}^{n}\omega_i D_i E[\tau_M\wedge T],
$$

we need to show

$$
\frac{E[(\sum_{i=1}^{n}\omega_i N_i(\tau_M\wedge T))^2]}{E[\sum_{i=1}^{n}\omega_i N_i(\tau_M\wedge T)]} - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{\sum_{i=1}^{n}\omega_i D_i} < \sum_{i=1}^{n}\omega_i D_i T.
$$

Furthermore, noticing that

$$
\sum_{i=1}^{n}\omega_i D_i T > \sum_{i=1}^{n}\omega_i D_i E[\tau_M\wedge T] = E\left[\sum_{i=1}^{n}\omega_i N_i(\tau_M\wedge T)\right],
$$

it is enough to show that

$$\frac{E\left[\left(\sum_{i=1}^{n}\omega_i N_i(\tau_M \wedge T)\right)^2\right]}{E\left[\sum_{i=1}^{n}\omega_i N_i(\tau_M \wedge T)\right]} - \frac{\sum_{i=1}^{n}\omega_i^2\sigma_i^2}{\sum_{i=1}^{n}\omega_i D_i} < E\left[\sum_{i=1}^{n}\omega_i N_i(\tau_M \wedge T)\right],$$

which is verified by Lemma 5.9, immediately. $\square$

6. A COMPARISON ANALYSIS OF AN INTEGRATED INVENTORY/SC MODEL

Recently, the development in supply chain management focus on the coordination of different functional specialties (D. Simchi-Levi, 2003). In this work, we revisit a joint inventory replenishment and outbound dispatch scheduling problem which arises in the context of Vendor-Managed Inventory (Çetinkaya and Lee, 2000; Çetinkaya et al., 2006, 2008), which is a supply-chain initiative the supplier is authorized to manage inventories of agreed-upon stock-holding units at retail locations.

In this two-echelon setting, the upper echelon is a vendor serving a group of downstream members, and the vendor has to optimally schedule the upstream replenishment as well as the outbound shipments to the downstream. Usually, customer demands should be dispatched immediately, but the vendor has the right to consolidate small orders from the retailers until an agreeable dispatching time. This practice is known as temporal shipment consolidation (Higginson and Bookbinder, 1994, 1995). In this way, this model is a push-pull system, where some stages of the supply chain, typically the initial stages, are operated in a push-based manner while the remaining stages employ a pull-based strategy (D. Simchi-Levi, 2003).

In this chapter, we provide the analytical model for the integrated inventory/quantity-time-based shipment consolidation problem, and propose two service measures in the general integrated inventory/transportation model. Based on the service criteria, the impact of alternative shipment consolidation policies in this problem setting is investigated. Specifically, under the same replenishment and consolidation cycle length, we compare the performance in terms of the service criteria among the three integrated models with quantity-based, time-based, and quantity-time-based consolidation policy, respectively. The results are useful for designing the inventory/transportation systems.

6.1 The Integrated Inventory/Quantity-Time-based Dispatch Model

Till now, there is no exact model about the integrated inventory/quantity-time-based dispatch model. In this section, by applying renewal theory, we obtain the exact mathematical expression for this problem. Assume the demand is a Poisson process with rate λ and let q_H and T_H denote the parameters associated with the a quantity-time-based consolidation policy, under which a dispatch decision is taken every $\tau_{q_H} \wedge T_H$ time units, where τ_{q_H} is first hitting time of q_H with respect to the Poisson demand. The vendor employs a special kind of (s, S) policy, with $s = -1$ and $S = Q_H$. Thus, there is no need to make an order if inventory is nonnegative immediately after a shipment is dispatched; a replenishment order is placed only if the on-hand inventory is not enough to clear the outstanding orders. Figure 6.1 provides the illustration of inventory dynamics under the integrated inventory/quantity-time-based dispatch model.

Let $I(t)$ denote the inventory level at time t and $L(t)$ is the realization of the consolidation process, which represents the size of the accumulative load, i.e. the amount of the outstanding demands, at time t. $Z(t)$ is the replenishment order quantity,

$$Z(t) = \begin{cases} Q_H + L(t) - I(t), & \text{if} \quad I(t) < L(t), \\ 0, & \text{if} \quad I(t) \geq L(t). \end{cases}$$

$Y(t)$ is the inventory amount when a new shipment-consolidation cycle begins,

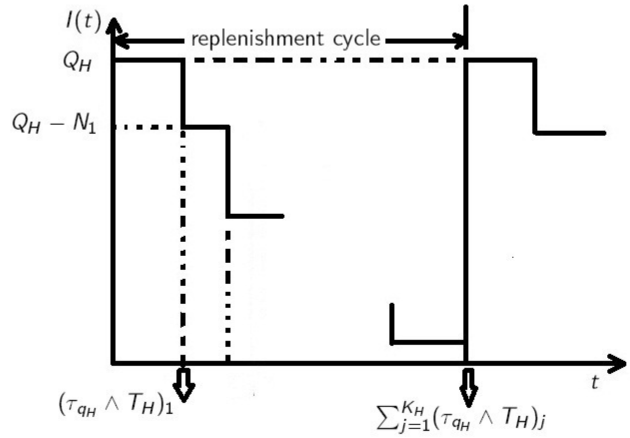

Figure 6.1: Inventory under an Integrated Inventory/Quantity-Time-based Dispatch Model.

$$Y(t) = \begin{cases} Q_H, & \text{if} \quad I(t) < L(t), \\ I(t) - L(t), & \text{if} \quad I(t) \geq L(t), \end{cases}$$

where $t = \tau_{q_H} \wedge T_H, 2(\tau_{q_H} \wedge T_H), 3(\tau_{q_H} \wedge T_H), \ldots$.

The consolidation system is cleared and a new shipment-consolidation cycle begins every $\tau_{q_H} \wedge T_H$ time units. Therefore, $L(j(\tau_{q_H} \wedge T_H)), j = 1, 2, 3, \ldots$ is a sequence of random variables representing the dispatching quantities.

Let $N_j(\tau_{q_H} \wedge T_H) = L(j(\tau_{q_H} \wedge T_H)), j = 1, 2, 3, \ldots$. Clearly, $N_j(\tau_{q_H} \wedge T_H)$ denotes the demand process realized by the inventory system under the quantity-time-based dispatching policy. Since the demand process $N(t)$ is a Poisson process with rate λ, $\{N_j(\tau_{q_H} \wedge T_H)\}_{j=1,2,\ldots}$ are identically independent distributed, each has the same distribution as the random variable $N(\tau_{q_H} \wedge T_H)$, which has the same distribution as Y_{q_H}, where $Y \sim Poisson(\lambda T_H)$.

Notice $N(t) - \lambda t$ is a martingale with respect to $N(t)$, and $\tau_{q_H} \wedge T_H$ is a bounded stopping time, using optional stopping theorem, we have

$$E[N(\tau_{q_H} \wedge T_H)] = \lambda E[\tau_{q_H} \wedge T_H],$$

thus, the expected consolidation cycle is

$$E[L^C_{HP}] = E[\tau_{q_H} \wedge T_H] = \frac{1}{\lambda} E[N(\tau_{q_H} \wedge T_H)] = \frac{1}{\lambda} E[Y_{q_H}]. \tag{6.1}$$

Define

$$K_H = \min\{k \textit{ is a positive integer} : \sum_{j=1}^{k} N_j(\tau_{q_H} \wedge T_H) \geq Q_H + 1\},$$

where K_H is a random variable representing number of dispatch decisions within an inventory replenishment cycle under the hybrid consolidation policy. Thus the length of an inventory replenishment cycle under the hybrid policy with parameters q_H and T_H is

$$L^R_{HP} = \sum_{j=1}^{K_H} (\tau_{q_H} \wedge T_H)_j,$$

where $(\tau_{q_H} \wedge T_H)_j$ denotes the $j-th$ consolidation cycle within one replenishment cycle.

Notice $\sum_{j=1}^{K_H} (\tau_{q_H} \wedge T_H)_j$ is a finite stopping time with respect to $N(t)$, and for $\forall t > 0$,

$$|N(\sum_{j=1}^{K_H} (\tau_{q_H} \wedge T_H)_j \wedge t) - \lambda(\sum_{j=1}^{K_H} (\tau_{q_H} \wedge T_H)_j \wedge t)| \leq q_H K_H + \lambda T_H K_H \in L_1,$$

which implies (by Proposition 2.5.7(iii) in Athreya and Lahiri (2006), p. 65)

$$\{N(\sum_{j=1}^{K_H}(\tau_{q_H} \wedge T_H)_j \wedge t) - \lambda(\sum_{j=1}^{K_H}(\tau_{q_H} \wedge T_H)_j \wedge t)\}_{t\geq 0}$$

is a uniformly integrable martingale, thus,

$$E[N(\sum_{j=1}^{K_H}(\tau_{q_H} \wedge T_H)_j)] = \lambda E[\sum_{j=1}^{K_H}(\tau_{q_H} \wedge T_H)_j].$$

So, we have

$$\begin{aligned}
E[L_{HP}^R] = \quad & E[\sum_{j=1}^{K_H}(\tau_{q_H} \wedge T_H)_j] \\
= \quad & \frac{1}{\lambda}E[N(\sum_{j=1}^{K_H}(\tau_{q_H} \wedge T_H)_j)] \\
= \quad & \frac{1}{\lambda}E[\sum_{j=1}^{K_H} N_j(\tau_{q_H} \wedge T_H)] \\
= \quad & \frac{1}{\lambda}E[N(\tau_{q_H} \wedge T_H)]E[K_H] \\
= \quad & \frac{1}{\lambda}E[K_H]E[Y_{q_H}], \qquad (6.2)
\end{aligned}$$

where the penultimate equation comes from Wald equation since K_H is a stopping time for the sequence $N_j(\tau_{q_H} \wedge T_H), j = 1, 2, \ldots$.

From the definition of K_H, we have

$$\{K_H \geq k\} \Leftrightarrow \{\sum_{j=1}^{k-1} N_j(\tau_{q_H} \wedge T_H) \leq Q_H\}.$$

Let $G(.)$ as the distribution function of Y_{q_H} and $G^{(k)}(.)$ as the k-fold convolution of $G(.)$. Then we have

$$P(K_H \geq k) = G^{(k-1)}(Q_H),$$

and

$$\begin{aligned} E[K_H] \quad &= \sum_{k=1}^{\infty} P(K_H \geq k) = \sum_{k=1}^{\infty} G^{(k-1)}(Q_H) \\ &= 1 + \sum_{k=1}^{\infty} G^{(k)}(Q_H) = 1 + M_G(Q_H), \end{aligned} \tag{6.3}$$

where $M_G(i) = \sum_{k=1}^{\infty} G^{(k)}(i)$ is the renewal function associated with $G(.)$.

6.1.1 Expected Inventory Carrying per Replenishment Cycle

Under the quantity-time-based dispatch policy, the inventory dynamics within a replenishment cycle is as follows,

$$I(t) = \begin{cases} Q_H, & 0 \leq t \leq (\tau_{q_H} \wedge T_H)_1, \\ Q_H - N_1(\tau_{q_H} \wedge T_H), & (\tau_{q_H} \wedge T_H)_1 < t \leq \sum_{j=1}^{2} (\tau_{q_H} \wedge T_H)_j, \\ \vdots & \\ Q_H - \sum_{j=1}^{K_H - 1} N_j(\tau_{q_H} \wedge T_H), & \sum_{j=1}^{K_H - 1} (\tau_{q_H} \wedge T_H)_j < t \leq \sum_{j=1}^{K_H} (\tau_{q_H} \wedge T_H)_j. \end{cases}$$

Let

$$E[H_{HP}] = H(Q_H, q_H, T_H) = E[\int_0^{\sum_{j=1}^{K_H} (\tau_{q_H} \wedge T_H)_j} I(t) dt],$$

which denotes the expected inventory holding within one replenishment cycle.

Using the renewal argument, we have

$$
\begin{aligned}
& H(Q_H, q_H, T_H | N_1(\tau_{q_H} \wedge T_H) = i) \\
= \quad & \begin{cases} E[\tau_{q_H} \wedge T_H] Q_H, & \text{if} \quad i \geq Q_H + 1, \\ E[\tau_{q_H} \wedge T_H] Q_H + H(Q_H - i, q_H, T_H), & \text{if} \quad i \leq Q_H, \end{cases}
\end{aligned}
$$

thus

$$
H(Q_H, q_H, T_H) = E[\tau_{q_H} \wedge T_H] Q_H + \sum_{i=0}^{Q_H} H(Q_H - i, q_H, T_H) g(i),
$$

where $g(.)$ denotes the probability mass function of Y_{q_H}.

The above expression for $H(Q_H, q_H, T_H)$ is a renewal type equation, its solution is given as

$$
\begin{aligned}
E[H_{HP}] \quad & = H(Q_H, q_H, T_H) \\
& = E[\tau_{q_H} \wedge T_H] Q_H + E[\tau_{q_H} \wedge T_H] \sum_{i=0}^{Q_H} (Q_H - i) m_g(i) \\
& = \frac{1}{\lambda} E[Y_{q_H}] Q_H + \frac{1}{\lambda} E[Y_{q_H}] \sum_{i=0}^{Q_H} (Q_H - i) m_g(i), \qquad (6.4)
\end{aligned}
$$

where $m_g(i) = \sum_{k=1}^{\infty} g^{(k)}(i)$ is the renewal density associated with $g(.)$, $g^{(k)}(.)$ denotes the k-fold convolution of $g(.)$.

Denote $E[HCost_{HP}]$ as the expected inventory holding cost within one replenishment cycle under the hybrid policy with parameters q_H and T_H.

It follows that

$$\begin{aligned} E[HCost_{HP}] = \quad & hH(Q_H, q_H, T_H) \\ = \quad & \frac{h}{\lambda} E[Y_{q_H}] Q_H + \frac{h}{\lambda} E[Y_{q_H}] \sum_{i=0}^{Q_H} (Q_H - i) m_g(i), \end{aligned} \tag{6.5}$$

where h represents the inventory carrying cost per unit per unit time.

6.1.2 Expected Linear Delay per Replenishment Cycle

As in the previous work, we assume that the customer waiting penalty is linear to the customer waiting time.

Notice the shipment consolidation length of hybrid policy with parameters q_H, T_H is $\tau_{q_H} \wedge T_H$, and the cumulative linear delay within one shipment consolidation cycle is $W_{HP} = \int_0^{\tau_{q_H} \wedge T_H} N(t) dt$.

The expected cumulative customer linear delay within one shipment consolidation cycle can be calculated as

$$E[W_{HP}] = E[\int_0^{\tau_{q_H} \wedge T_H} N(t) dt].$$

From Chapter 4, the expected cumulative waiting time within one consolidation cycle under hybrid policy is

$$\begin{aligned} E[W_{HP}] = E[W(\tau_{q_H} \wedge T_H)] \quad & = \frac{1}{2\lambda} E[N^2(\tau_{q_H} \wedge T_H) - N(\tau_{q_H} \wedge T_H)] \\ & = \frac{1}{2\lambda} E[Y_{q_H}(Y_{q_H} - 1)]. \end{aligned} \tag{6.6}$$

Denote $E[WCost_{HP}]$ as the linear delay cost per replenishment cycle.

Since K_H is the number of shipment consolidation cycles within one replenishment cycle, it follows

$$\begin{aligned} E[WCost_{HP}] &= \omega E[K_H]E[W_{HP}] \\ &= \frac{\omega}{2\lambda}E[K_H]E[Y_{q_H}(Y_{q_H}-1)], \end{aligned} \tag{6.7}$$

where ω denotes the waiting cost per unit per unit time.

6.1.3 Expected Squared Delay per Replenishment Cycle

In the customer linear delay case, the waiting penalty is linear to the time delay. However, in practice, due to customer impatience, the waiting penalty is unlikely to be linear in time or units. In this subsection, we consider the case with squared delay penalty, where the waiting penalty is proportional to the square of the waiting time encountered by the customer.

The expected cumulative customer squared delay within one shipment consolidation cycle can be calculated as

$$E[W'_{HP}] = E[\int_0^{\tau_{q_H} \wedge T_H} (\tau_{q_H} \wedge T_H - t)^2 dN(t)],$$

where q_H and T_H are the parameters of the adopted hybrid policy.

In Appendix B, we provide the computation for the expression of $E[W'_{HP}]$. We use the expression directly as follows.

$$E[W'_{HP}] = \frac{1}{3\lambda^2}E[Y_{q_H+1}(Y_{q_H+1}-1)(Y_{q_H+1}-2)],$$

where $Y \sim Poisson(\lambda T_H)$.

In particular, quantity-time-based policy with parameters q and T degenerates to QP with parameter q when $T \to \infty$, while degenerates to TP with parameter T

when $q \to \infty$.

$$E[W'_{QP}] = \lim_{T\to\infty} E[W'_{HP}] = \frac{1}{3\lambda^2}(q^3 - q), \tag{6.8}$$

$$E[W'_{TP}] = \lim_{q\to\infty} E[W'_{HP}] = \frac{1}{3\lambda^2}E[Y(Y-1)(Y-2)] = \frac{1}{3}\lambda T^3. \tag{6.9}$$

6.1.4 *Expected Inventory Replenishment Costs per Replenishment Cycle*

Denote $E[RCost_{HP}]$ as the replenishment cost per replenishment cycle, A_R the fixed cost of replenishing the inventory, C_R the unit procurement cost.

Since K_H is a stopping time for the sequence $N_j(\tau_{q_H} \wedge T_H), j = 1, 2, \ldots$, by Wald equation, we have

$$E[Order\ \ Quantity] = E[\sum_{j=1}^{K_H} N_j(\tau_{q_H} \wedge T_H)] = E[N(\tau_{q_H} \wedge T_H)]E[K_H],$$

thus,

$$\begin{aligned} E[RCost_{HP}] = & \quad A_R + C_R E[Order\ \ Quantity] \\ = & \quad A_R + C_R E[N(\tau_{q_H} \wedge T_H)]E[K_H] \\ = & \quad A_R + C_R E[K_H]E[Y_{q_H}]. \end{aligned} \tag{6.10}$$

6.1.5 *Expected dispatch Costs per Replenishment Cycle*

Denote $E[DCost_{HP}]$ as the dispatch cost per replenishment cycle, A_D the fixed cost of dispatching, C_D the unit shipment cost.

All outstanding demands are dispatched every $\tau_{q_H} \wedge T_H$ units of time, and K_H is the number of shipment consolidation cycles within one replenishment cycle, thus,

$$
\begin{aligned}
E[DCost_{HP}] = &\quad A_D E[K_H] + C_D E[\sum_{j=1}^{K_H} N_j(\tau_{q_H} \wedge T_H)] \\
= &\quad A_D E[K_H] + C_D E[K_H] E[Y_{q_H}]. \qquad (6.11)
\end{aligned}
$$

6.1.6 Average Cost per Unit Time

From the derivation of the previous subsections, we have the total cost within one replenishment cycle is

$$
E[TCost_{HP}] = \quad E[HCost_{HP}] + E[WCost_{HP}] + E[RCost_{HP}] + E[DCost_{HP}].
$$

Let $AC_{HP}(Q_H, q_H, T_H)$ denote the expected long-run average cost per unit time. By the Renewal Reward Theorem, we have

$$
AC_{HP}(Q_H, q_H, T_H) = \quad \frac{E[TCost_{HP}]}{E[L_{HP}^R]}. \qquad (6.12)
$$

From the definition of K_H, we have

$$
E[\sum_{j=1}^{K_H} N_j(\tau_{q_H} \wedge T_H)] = E[K_H]E[Y_{q_H}] \geq Q_H + 1,
$$

$$
E[\sum_{j=1}^{K_H-1} N_j(\tau_{q_H} \wedge T_H)] = E[K_H]E[Y_{q_H}] - E[Y_{q_H}] \leq Q_H,
$$

thus, $\frac{Q_H}{E[Y_{q_H}]} + 1 \geq E[K_H] \geq \frac{Q_H}{E[Y_{q_H}]} + \frac{1}{E[Y_{q_H}]}$.

If we treat K_H as a continuous random variable, we would have

$$E[\sum_{j=1}^{K_H} N_j(\tau_{q_H} \wedge T_H)] = E[K_H]E[Y_{q_H}] = Q_H + 1,$$

then,

$$E[K_H] = \frac{Q_H + 1}{E[Y_{q_H}]}. \tag{6.13}$$

From (6.2),

$$E[L_{HP}^R] = \frac{Q_H + 1}{\lambda}.$$

From (6.3), we have

$$M_G(Q_H) = \frac{Q_H + 1}{E[Y_{q_H}]} - 1,$$

and

$$m_g(i) = M_G(i) - M_G(i-1) = \frac{1}{E[Y_{q_H}]}. \tag{6.14}$$

Thus, from (6.4),

$$E[H_{HP}] = \frac{1}{\lambda}E[Y_{q_H}]Q_H + \frac{1}{2\lambda}(Q_H + 1)Q_H. \tag{6.15}$$

Therefore,

$$\begin{aligned} & AC_{HP}(Q_H, q_H, T_H) \\ = \quad & \frac{A_R\lambda}{Q_H+1} + C_R\lambda + \frac{A_D\lambda}{E[Y_{q_H}]} + C_D\lambda + \frac{hQ_H E[Y_{q_H}]}{Q_H+1} + \frac{1}{2}hQ_H \\ & +\frac{1}{2}\omega(\frac{E^2[Y_{q_H}]}{E[Y_{q_H}]} - 1). \end{aligned} \tag{6.16}$$

6.2 Comparison of Service Performance and Average Cost under Various Consolidation Policies for VMI Systems

Çetinkaya and Lee (2000) consider the integrated model with time-based shipment consolidation policy for VMI system. This work is the first one providing a framework to synchronize inventory and transportation decision.

Later, Çetinkaya et al. (2006) study the integrated model with quantity-based shipment consolidation policy in VMI setting and present numerical results showing that the quantity-based policies can achieve cost savings, compared with time-based and hybrid-based policies. However, hybrid policy is superior to quantity-based policy in terms of average waiting time, although it is not superior to quantity-based policy in terms of cost criterion.

In this section, we propose two service measures and analytically compare the three integrated models with different shipment consolidation policies. Based on the comparative results about service criteria, we can obtain some perception about the average cost comparison.

In the following, we cite the results directly relating to our work from the above two papers.

Let T and Q_T denote the consolidation cycle and the vendor's order-up-to level in the integrated model with time-based shipment consolidation policy for VMI system (see Çetinkaya and Lee (2000)), the expected cumulative inventory holding within one replenishment cycle is $E[H_{TP}] = TQ_T + \frac{Q_T(Q_T+1)}{2\lambda}$ and the expected cumulative linear delay within one consolidation cycle is $E[W_{TP}] = \frac{1}{2}\lambda T^2$.

Let n denote the number of consolidation cycles within an inventory replenishment cycle and q denote the consolidation quantity threshold value for the quantity-based dispatch model in Çetinkaya et al. (2006). The expected cumulative inventory holding within one replenishment cycle is $E[H_{QP}] = \frac{1}{2\lambda}n(n-1)q^2$ and the expected cumulative linear delay within one consolidation cycle is $E[W_{QP}] = \frac{1}{2\lambda}(q-1)q$.

The expected replenishment cycle lengths $E[L^R]$ and the expected consolidation cycle lengths $E[L^C]$ are summarized in Table 6.1.

$E[L^R_{QP}] = \frac{nq}{\lambda}$	$E[L^C_{QP}] = \frac{q}{\lambda}$
$E[L^R_{TP}] = \frac{Q_T+1}{\lambda}$	$E[L^C_{TP}] = T$
$E[L^R_{HP}] = \frac{Q_H+1}{\lambda}$	$E[L^C_{HP}] = \frac{E[Y_{q_H}]}{\lambda}$

Table 6.1: Summary of Expected Consolidation Cycle Length and Replenishment Cycle Length.

The expected replenishment cost $E[RCost]$ and dispatch cost $E[DCost]$ in one replenishment cycle are summarized in Table 6.2, where n, K_T and K_H denote the number of consolidation cycles within one replenishment cycle under quantity-based, time-based, and hybrid-based shipment consolidation policies, respectively.

$E[RCost_{QP}] = A_R + C_R nq$
$E[DCost_{QP}] = nA_D + C_D nq$
$E[RCost_{TP}] = A_R + C_R E[K_T]\lambda T$
$E[DCost_{TP}] = A_D E[K_T] + C_D E[K_T]\lambda T$
$E[RCost_{HP}] = A_R + C_R E[K_H]E[Y_{q_H}]$
$E[DCost_{HP}] = A_D E[K_H] + C_D E[K_H]E[Y_{q_H}]$

Table 6.2: Summary of Expected Replenishment Cost and Dispatch Cost in One Replenishment Cycle.

6.2.1 Average Inventory Rate, AIR

The first service measure, AIR, takes into account the average inventory holding per time unit. It can be obtained by applying the Renewal Reward Theorem, i.e.,

$$AIR = \frac{E[\text{Cumulative inventory holding per replenishment cycle}]}{E[\text{Replenishment cycle length}]} = \frac{E[H]}{E[L^R]}.$$

We index AIR, H, and L^R by policy type as needed.

The expressions of average inventory rate under different policies are summarized in Table 6.3.

Theorem 6.1. *Under the same expected consolidation length $E[L^C]$ and the same replenishment cycle length $E[L^R]$, $AIR_{TP} = AIR_{HP} > AIR_{QP}$.*

Proof. We consider fixed $E[L^C]$ & $E[L^R]$, and all possible policies under the $E[L^C]$ & $E[L^R]$ values.

Recalling the $E[L^C]$ & $E[L^R]$ expressions in Table 6.1, we note that, by assump-

$$
\begin{aligned}
AIR_{QP} &= \frac{E\big[H_{QP}\big]}{E\big[L^R_{QP}\big]} = \frac{n(n-1)q^2/(2\lambda)}{nq/\lambda} = \frac{(n-1)q}{2} \\
AIR_{TP} &= \frac{E[H_{TP}]}{E\big[L^R_{TP}\big]} = \frac{TQ_T + Q_T(Q_T+1)/(2\lambda)}{(Q_T+1)/\lambda} = \frac{Q_T(2\lambda T + Q_T + 1)}{2(Q_T+1)} \\
AIR_{HP} &= \frac{E[H_{HP}]}{E\big[L^R_{HP}\big]} = \frac{E[Y_{q_H}]Q_H/\lambda + (Q_H+1)Q_H/(2\lambda)}{(Q_H+1)/(\lambda)} = \frac{Q_H(2E[Y_{q_H}] + Q_H + 1)}{2(Q_H+1)}
\end{aligned}
$$

Table 6.3: Summary of the Expressions of AIR.

tion,

$$\frac{E[Y_{q_H}]}{\lambda} = T = \frac{q}{\lambda}, \tag{6.17}$$

$$Q_H + 1 = Q_T + 1 = nq. \tag{6.18}$$

Next, recalling the results in Table 6.3 and reiterating the assumption of fixed $E[L^C]$ & $E[L^R]$ values for all the policies of interest, we can see

$$AIR_{TP} = AIR_{HP},$$

and

$$
\begin{aligned}
AIR_{TP} - AIR_{QP} = & \quad \frac{Q_T(2\lambda T + Q_T + 1)}{2(Q_T+1)} - \frac{(n-1)q}{2} \\
= & \quad \frac{(nq-1)(n+2)}{2n} - \frac{(n-1)q}{2} \\
= & \quad \frac{2(nq-1) + n(q-1)}{2n} > 0.
\end{aligned}
$$

□

6.2.2 Average Order Delay, AOD

In this subsection, we consider the second service criterion, which pertains to the average waiting time of an order. It can be obtained by applying the Renewal Reward Theorem, i.e.,

$$AOD = \frac{E[\text{Cumulative delay per consolidation cycle}]}{E[\text{Number of orders arriving in a consolidation cycle}]} = \frac{E[W]}{\lambda E[L^C]}.$$

Again, we index AOD, W, and L^C by policy type as needed.

The expressions of average order delay under different policies are summarized in Table 6.4.

$$\begin{aligned}
AOD_{QP} &= \frac{E\left[W_{QP}\right]}{\lambda E\left[L_{QP}^C\right]} = \frac{(q-1)q/2\lambda}{q} = \frac{q-1}{2\lambda} \\
AOD_{TP} &= \frac{E[W_{TP}]}{\lambda E\left[L_{TP}^C\right]} = \frac{\lambda T^2/2}{\lambda T} = \frac{T}{2} \\
AOD_{HP} &= \frac{E[W_{HP}]}{\lambda E\left[L_{HP}^C\right]} = \frac{E[Y_{q_H}(Y_{q_H}-1)]/(2\lambda)}{E[Y_{q_H}]}
\end{aligned}$$

Table 6.4: Another Summary of the Expressions of AOD.

In the following, we compare AOD under the same expected consolidation cycle length among the three shipment consolidation policies. Before we provide the comparison result, we need to dig out some more refined properties about Poisson random variable.

Lemma 6.2. *Suppose $X \sim Poisson(\mu)$, $V_n \sim gamma(n, 1)$ for integer $n \geq 1$,* *we have*

$$E[X_q^{(k)}] \triangleq E[X_q(X_q-1)\cdots(X_q-k+1)] = E[V_{q-k+1}^k \wedge \mu^k],$$

$$\frac{d}{d\mu}E[X_q^{(k)}] = k\mu^{k-1}P(X \leq q-k),$$

where $X_q = \min(X, q)$ *for integer positive valued* q*,* k *is positive integer valued and* $k \leq q$*.*

Proof. Using the relationship between Poisson and gamma distribution,

$$\begin{aligned}
E[X_q^{(k)}] = & \sum_{x=0}^{q} x^{(k)} \frac{e^{-\mu}\mu^x}{x!} + \sum_{x=q+1}^{\infty} q^{(k)} \frac{e^{-\mu}\mu^x}{x!} \\
= & \mu^k P(X \leq q-k) + q^{(k)} P(X \geq q+1) \\
= & \mu^k P(V_{q-k+1} > \mu) + q^{(k)} P(V_{q+1} \leq \mu) \\
= & \int_{\mu}^{\infty} \mu^k \frac{v^{q-k}e^{-v}}{(q-k)!} dv + \int_0^{\mu} v^k \frac{v^{q-k}e^{-v}}{(q-k)!} dv \\
= & E[V_{q-k+1}^k \wedge \mu^k].
\end{aligned}$$

Using the two properties of Poisson random variable,

$$\mu^k P(X = q-k) = q^{(k)} P(X = q),$$

and

$$\frac{d}{d\mu} P(X \leq q) = -P(X = q),$$

it is straightforward to show that

$$\begin{aligned}
\frac{d}{d\mu}E[X_q^{(k)}] = & k\mu^{k-1}P(X \leq q-k) - \mu^k P(X = q-k) + q^{(k)} P(X = q) \\
= & k\mu^{k-1}P(X \leq q-k).
\end{aligned}$$

□

Lemma 6.3. *Suppose $X \sim Poisson(\mu)$, then $\frac{E^2[X_q]}{E[X_q^{(2)}]}$ is increasing in μ and $E^2[X_q] > E[X_q^{(2)}]$, where $E[X_q^{(2)}] \triangleq E[X_q(X_q-1)]$.*

Proof. Using Lemma 6.2 and

$$\mu P(X = n-1) = nP(X = n),$$

it is straightforward to obtain

$$\begin{aligned}
& \frac{d}{d\mu}\frac{E^2[X_q]}{E[X_q^{(2)}]} \\
= \quad & \frac{2E[X_q]}{E^2[X_q^{(2)}]}[P(X \leq q-1)E[X_q^{(2)}] - \mu E[X_q]P(X \leq q-2)] \\
= \quad & \frac{2qE[X_q]P(X \geq q+1)}{E^2[X_q^{(2)}]}[(q-1)P(X \leq q-1) - \mu P(X \leq q-2)] \\
= \quad & \frac{2qE[X_q]P(X \geq q+1)}{E^2[X_q^{(2)}]}\sum_{n=0}^{q-1}(q-1-n)P(X = n) > 0.
\end{aligned}$$

Using Lemma 6.2,

$$\lim_{\mu \downarrow 0}\frac{E^2[X_q]}{E[X_q^{(2)}]} = \lim_{\mu \downarrow 0}\frac{E^2[(V_q/\mu) \wedge 1]}{E[(V_{q-1}/\mu)^2 \wedge 1]} = 1,$$

thus, $E^2[X_q] > E[X_q^{(2)}]$.

□

Theorem 6.4. *Under the same expected consolidation cycle length $E[L^C]$,*

$$AOD_{QP} < AOD_{HP} < AOD_{TP}.$$

Proof. We consider a fixed $E[L^C]$ and use the following notation for the corresponding policy parameters under this $E[L^C]$ value: QP with parameter q, TP with parameter T, HP with parameters q_H and T_H. Recalling the $E[L^C]$ expressions in Table 6.1, we note that, by assumption,

$$\begin{aligned} \frac{1}{\lambda}E[Y_{q_H}] &= \frac{q}{\lambda}, &(6.19)\\ \frac{1}{\lambda}E[Y_{q_H}] &= T. &(6.20)\end{aligned}$$

Next, recalling the results in Table 6.4 and reiterating the assumption of fixed $E[L^C]$ values for all the policies of interest, we proceed with showing that

$$\begin{aligned} (q-1)q &< E[Y_{q_H}(Y_{q_H}-1)], &(6.21)\\ E[Y_{q_H}(Y_{q_H}-1)] &< \lambda^2T^2. &(6.22)\end{aligned}$$

In fact, recalling the assumption in (6.19),

$$E[Y_{q_H}(Y_{q_H}-1)] = E[Y_{q_H}^2] - q = VAR[Y_{q_H}] + E^2[Y_{q_H}] - q > q^2 - q.$$

From Lemma 6.3, and recalling the assumption in (6.20), we have

$$E[Y_{q_H}(Y_{q_H}-1)] < E^2[Y_{q_H}] = \lambda^2T^2.$$

□

6.2.3 Average Order Squared Delay, AOSD

From the definition of AOD, the waiting penalty is assume to be linear to the time delay. However, in some situation, due to the impatience of the customer, we should put more penalty on longer time delay. In this subsection, we assume the waiting penalty is proportional to the square of the waiting time encountered by the customer. The corresponding service criteria is defined as follows.

$$AOSD = \frac{E[\text{Cumulative squared delay per consolidation cycle}]}{E[\text{Number of orders arriving in a consolidation cycle}]} = \frac{E[W']}{\lambda E[L^C]},$$

Again, we index $AOSD$, W', and L^C by policy type as needed.

The computation for average order squared delay is in Appendix B. We summarize the expressions under different policies in Table 6.5. Notice that $AOSD_{HP} \to AOSD_{QP}$ as $T \to \infty$, and $AOSD_{HP} \to AOSD_{TP}$ as $q \to \infty$. With the expressions of AOSD under different policies, we try to provide comparative result in terms of AOSD, after revealing some refined properties of truncated Poisson random variable.

$$\begin{array}{lcl}
AOSD_{QP} & = & \dfrac{E\left[W'_{QP}\right]}{\lambda E\left[L^C_{QP}\right]} = \dfrac{(q^3-q)/(3\lambda^2)}{q} = \dfrac{q^2-1}{3\lambda^2} \\
AOSD_{TP} & = & \dfrac{E\left[W'_{TP}\right]}{\lambda E\left[L^C_{TP}\right]} = \dfrac{\lambda T^3/3}{\lambda T} = \dfrac{T^2}{3} \\
AOSD_{HP} & = & \dfrac{E\left[W'_{HP}\right]}{\lambda E\left[L^C_{HP}\right]} = \dfrac{E[Y_{q_H+1}(Y_{q_H+1}-1)(Y_{q_H+1}-2)]/(3\lambda^2)}{E[Y_{q_H}]}
\end{array}$$

Table 6.5: Summary of the Expressions of $AOSD$.

We need the following properties about Poisson random variable to prove the main result in this subsection.

Lemma 6.5. *Suppose $X \sim Poisson(\mu)$, and $A \subset \mathbb{Z}_+$, then*

$$\frac{d}{d\mu}E[X|X \in A] = \frac{1}{\mu}Var[X|X \in A] \geq 0.$$

with equality to 0 only if A is a singleton set.

Proof. It is straightforward to obtain

$$\begin{aligned}
\frac{d}{d\mu}E[X|X \in A] = & \quad \frac{d}{d\mu}\frac{\sum_{x\in A} x\frac{\mu^x}{x!}}{\sum_{x\in A}\frac{\mu^x}{x!}} \\
= & \quad \frac{\sum_{x\in A} x^2\frac{\mu^{x-1}}{x!}\sum_{x\in A}\frac{\mu^x}{x!} - \sum_{x\in A} x\frac{\mu^x}{x!}\sum_{x\in A} x\frac{\mu^{x-1}}{x!}}{(\sum_{x\in A}\frac{\mu^x}{x!})^2} \\
= & \quad \frac{\sum_{x\in A} x^2\frac{\mu^x}{x!}}{\mu\sum_{x\in A}\frac{\mu^x}{x!}} - \frac{(\sum_{x\in A} x\frac{\mu^x}{x!})^2}{\mu(\sum_{x\in A}\frac{\mu^x}{x!})^2} \\
= & \quad \frac{1}{\mu}Var[X|X \in A] \geq 0.
\end{aligned}$$

□

Lemma 6.6. *Suppose $X \sim Poisson(\mu)$, then there exists some $\hat{\mu}$, such that $\frac{E^3[X_q]}{E[X^{(3)}_{q+1}]}$ is increasing on $(0, \hat{\mu})$ and decreasing on $(\hat{\mu}, \infty)$, and $E^3[X_q] > E[X^{(3)}_{q+1}]$, for all $\mu > 0$, where $E[X^{(3)}_{q+1}] \triangleq E[X_{q+1}(X_{q+1} - 1)(X_{q+1} - 2)]$.*

Proof. Using Lemma 6.2, it is straightforward to obtain

$$\begin{aligned}
& \frac{d}{d\mu}\frac{E^3[X_q]}{E[X^{(3)}_{q+1}]} \\
= \quad & \frac{3E^2[X_q]}{E^2[X^{(3)}_{q+1}]}(P(X \leq q-1)E[X^{(3)}_{q+1}] - \mu^2 E[X_q]P(X \leq q-2)) \\
= \quad & \frac{3\mu q E^2[X_q]P(X \geq q+2)P(X \leq q-2)}{E^2[X^{(3)}_{q+1}]}\Big(\frac{(q^2-1)P(X \leq q-1)}{\mu P(X \leq q-2)} \\
& - \frac{\mu P(X \geq q+1)}{P(X \geq q+2)}\Big).
\end{aligned}$$

From Lemma 6.5, we note that

$$\frac{\mu P(X \leq q-2)}{P(X \leq q-1)} = \frac{\mu \sum_{x=0}^{q-2} \frac{\mu^x}{x!}}{\sum_{x=0}^{q-1} \frac{\mu^x}{x!}} = \frac{\mu \sum_{x=0}^{q-1} x \frac{\mu^x}{x!}}{\sum_{x=0}^{q-1} \frac{\mu^x}{x!}} = E[X|X \leq q-1]$$

is increasing in μ (from 0 to $q-1$). Likewise,

$$\frac{\mu P(X \geq q+1)}{P(X \geq q+2)} = E[X|X \geq q+2]$$

is increasing in μ (from $q+2$ to ∞).

Therefore, the last expression in parenthesis regarding $\frac{d}{d\mu}\frac{E^3[X_q]}{E[X_{q+1}^{(3)}]}$ is decreasing in μ, positive (and unbounded) for small μ and negative (and unbounded) for large μ, which implies that there exists some $\hat{\mu}$, $\frac{E^3[X_q]}{E[X_{q+1}^{(3)}]}$ is increasing on $(0, \hat{\mu})$ and decreasing on $(\hat{\mu}, \infty)$.

Using Lemma 6.2,

$$\lim_{\mu \downarrow 0} \frac{E^3[X_q]}{E[X_{q+1}^{(3)}]} = \lim_{\mu \downarrow 0} \frac{E^3[(V_q/\mu) \wedge 1]}{E[(V_{q-1}/\mu)^3 \wedge 1]} = 1,$$

and

$$\lim_{\mu \to \infty} \frac{E^3[X_q]}{E[X_{q+1}^{(3)}]} = \frac{q^3}{(q+1)^{(3)}} = \frac{q^2}{q^2-1} > 1,$$

thus, $E^3[X_q] > E[X_{q+1}^{(3)}]$, for all $\mu > 0$.

□

Theorem 6.7. *Under the same expected consolidation cycle length $E[L^C]$,*

$$AOSD_{QP} < AOSD_{TP}, \textit{ and } \ AOSD_{HP} < AOSD_{TP}.$$

Proof. We consider a fixed $E[L^C]$ and use the following notation for the corresponding policy parameters under this $E[L^C]$ value: QP with parameter q, TP with parameter T, HP with parameters q_H and T_H. Recalling the $E[L^C]$ expressions in Table 6.1, we note that, by assumption,

$$\frac{q}{\lambda} = T, \tag{6.23}$$

$$\frac{E[Y_{q_H}]}{\lambda} = T. \tag{6.24}$$

Next, recalling the results in Table 6.5 and reiterating the assumption of fixed $E[L^C]$ value for all the policies of interest, we proceed with showing that

$$\frac{q^3 - q}{3\lambda^2} < \lambda T^3/3, \tag{6.25}$$

$$\frac{E[Y^{(3)}_{q_H+1}]}{3\lambda^2} < \lambda T^3/3. \tag{6.26}$$

Recalling the assumption in (6.23), we can easily see that (6.25) holds.

From Lemma 6.6, and recalling the assumption in (6.24), we have

$$E[Y^{(3)}_{q_H+1}] < E^3[Y_{q_H}] = (\lambda T)^3,$$

which verifies (6.26). □

6.2.4 Average Cost

Based on the previous service criteria comparison among different models, we provide the comparison results in terms of average cost criteria under the same expected consolidation length $E[L^C]$ and the same replenishment cycle length $E[L^R]$

Denote AC as the average cost per unit time and we index it by policy type as needed.

Theorem 6.8. *In the linear delay penalty case, under the same expected consolidation length $E[L^C]$ and the same replenishment cycle length $E[L^R]$,*

$$AC_{QP} < AC_{HP} < AC_{TP}.$$

Proof. By the Renewal Reward Theorem,

$$AC = \frac{E[TCost]}{E[L^R]},$$

where $E[TCost] = E[HCost] + E[WCost] + E[RCost] + E[DCost]$.

Since $E[K] = \frac{E[L^R]}{E[L^C]}$ for all the three models, and from the assumption that $E[L^R_{QP}] = E[L^R_{TP}] = E[L^R_{HP}]$ and $E[L^C_{QP}] = E[L^C_{TP}] = E[L^C_{HP}]$, we have

$$E[K_T] = E[K_H] = n.$$

So that

$$E[RCost_{QP}] = E[RCost_{HP}] = E[RCost_{TP}],$$

$$E[DCost_{QP}] = E[DCost_{HP}] = E[DCost_{TP}].$$

Further, from Theorem 6.1 and Theorem 6.4,

$$E[HCost_{QP}] < E[HCost_{HP}] = E[HCost_{TP}],$$

$$E[WCost_{QP}] < E[WCost_{HP}] < E[WCost_{TP}].$$

Therefore, $AC_{QP} < AC_{HP} < AC_{TP}$. □

Remark 6.9. *By the same idea, in the squared waiting penalty case, under the* *same expected consolidation length* $E[L^C]$ *and the same replenishment cycle length* $E[L^R]$*, from Theorem 6.1 and Theorem 6.7, we have* $AC_{QP} < AC_{TP}$ *and* $AC_{HP} < AC_{TP}$*.*

Remark 6.10. *We need to notice the approximation of (6.13) comes from treating* K_H *as continuous. Actually, this approximation technique is used in Çetinkaya and Lee (2000) and Wald (1944). Axsäter (2001) points out the approximation is reasonable in Çetinkaya and Lee (2000) except in the cases when there is only a single consolidation cycle in a replenishment cycle, so is in the integrated inventory/hybrid consolidation model. If there is only one consolidation cycle within a replenishment cycle, the integrated inventory/shipment consolidation model degenerates to a pure consolidation model, where no inventory is held at the vendor's warehouse, i.e. the vendor's warehouse acts as a transshipment point for consolidating orders. In this case, the comparison results in terms of the service measure AOD/AOSD are still true, and so that the comparison results in terms of the average cost criteria are also true.*

7. SUMMARY AND CONCLUSIONS

This work generalizes the existing work in several aspects. Specifically, we consider generalized control policies, generalized demand patterns, multi-item systems, and alternative performance criteria.

In Chapter 2, we consider the single-item shipment consolidation problem with drifted Brownian motion demand. We provide a rigorous proof to show the optimal quantity-based policy achieves the minimum of the long-run average cost among a large class of admissible policies by using a quasi-variational inequalities method. In particular, we derive the quasi-variational inequalities corresponding with the problem and construct the solution, which provides an average optimal dispatching policy.

In Chapter 3, we generalize the shipment consolidation problem by considering multi items with drifted Brownian motion demands. We derive the expectation of customer waiting cost for the items within one consolidation cycle by applying the optional stopping theorem for some suitable uniformly martingale. In the $(Q+\tau)$-model, we show that τ should be a constant, which reduces the model into a simpler one where we only need to characterize two parameters Q and $E[\tau]$. The result indicates in the single-item case, the optimal $(Q+\tau)$-policy is a quantity-based policy. While in the multi-item case, the optimal $(Q+\tau)$-policy is either a quantity-based policy or a time-based policy, depending on whether $\sum_{i=1}^{n}\omega_i(2D\sigma_i^2-D_i\sigma^2)$ is positive or negative. In particular, if the different item demands are Poisson processes, the optimal $(Q+\tau)$-policy is a quantity-based policy.

In Chapter 4, we first provide a unified method to calculate AOD (average order delay) for any consolidation policy based on a martingale associated with a Poisson process and the celebrated optional stopping theorem. Next, we point out that under the same expected consolidation cycle length, QB dominates any other renewal type consolidation policy in terms of AOD, not limited to HPs and TPs. Further, we complete the proof for the comparison between HPs and TPs under the same expected consolidation cycle length and provide a simplified proof for the comparison among HPs, TPs and QP in terms of AOD under fixed parameters, which are related to a property of truncated Poisson random variables: for a truncated Poisson random variable Y_N, $VAR[Y_N] < E[Y_N]$. Moreover, we provide the stronger comparative results between two HPs of the same type under the same expected consolidation cycle length, which deeply rely on a property of truncated random variables: given two integer valued random variables X and Y, X is stochastically larger than Y, if $E[X_q] = E[Y_{q+1}]$, where q is a positive integer, then $E[X_q^2] \leq E[Y_{q+1}^2]$. Finally, we analytically show HP1 performs better than HP2 in terms of AOD under fixed parameters, which is equivalent to another property of truncated Poisson random variables: $X \sim Poisson(\mu)$, then $\frac{E[X_N^2]}{E[X_N]}$ is increasing with respect to μ.

In Chapter 5, we first propose an instantaneous rate policy(IRP) and provide the average cost model associated with it. Next, we show that the optimal instantaneous rate policy achieves less average cost than the optimal quantity-based policy and time-based policy. Further, by applying a martingale argument, we show among a large class of renewal type clearing policies, the optimal instantaneous rate policy achieves the least average cost. Moreover, for a given expected consolidation cycle length, the instantaneous rate policy dominates a large class of consolidation policies, and the instantaneous rate hybrid policy performs better than TP, in terms of the average weighted delay rate.

In Chapter 6, two aspects are contributed: (1) an analytical model of integrated inventory/hybird consolidation problem is provided; (2) two service measures in the integrated problem are proposed and some interesting and insightful comparison results in terms of the service criterion are obtained. By using renewal theory, we derive the expected inventory holding within one replenishment cycle. Further, in the integrated inventory/shipment consolidation problem setting, we propose AIR, AOD/AOSD as two service measures. In particular, AOSD is useful if the waiting penalty is proportional to the square of the waiting time encountered by the customer due to the impatience of the customer. We have shown that under the same expected replenishment and consolidation cycle length, QP performs the best, TP performs the worst in terms of AIR and HP lies between QP and TP. Moreover, after revealing some more refined properties of Poisson random variables, we provide the comparison results in terms of AOD and AOSD. Finally, from comparison results in terms of the service criteria, we obtain insights into the comparison of average cost among the three integrated models.

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APPENDIX A

In Chapter 4, we consider average order delay under different policies using a unified method and provide comparative results in terms of average order delay. Under TP1 and HP1, there may be empty shipments, which happens when $N(T) = 0$. In this Appendix, we consider revised TP1 and revised HP1, which do not allow empty shipments.

Specifically speaking, under the revised TP1 with parameter T, a clearing is made every T units of time as long as the consolidated load is not 0. However, if there is no order arriving within T units of time since the last shipment, we do not dispatch, but consolidate another multiple of T units of time and dispatch until the consolidated load is positive.

Under the revised HP1 with parameter q, T, the goal is to consolidate a load of size q. However, if the time since the last shipment epoch exceeds T and the consolidated load is positive, then the load is dispatched; on the other hand, if the time since the last shipment exceeds T and the consolidated load is zero, we do not dispatch and the system restarts.

Under the revised HP1 with parameters q, T, the following recursion equation about the expected consolidation cycle length $E[C_{RHP1}]$ is satisfied:

$$\begin{aligned} E[C_{RHP1}] = \quad & P(N(T) \geq 1)E[(\tau_q \wedge T)|N(T) \geq 1] \\ & + P(N(T) = 0)(T + E[C_{RHP1}]). \end{aligned} \tag{1}$$

The equation means if no order arrives within T units time, which happens with probability $P(N(T) = 0)$, the consolidation cycle restarts; if there are orders arriving within T units time, which happens with probability $P(N(T) \geq 1)$, the load is dispatched at stopping time $\tau_q \wedge T$.

By noticing

$$\begin{aligned} & E[(\tau_q \wedge T)] \\ = \quad & P(N(T) \geq 1)E[(\tau_q \wedge T)|N(T) \geq 1] \\ & +P(N(T) = 0)E[(\tau_q \wedge T)|N(T) = 0] \\ = \quad & P(N(T) \geq 1)E[(\tau_q \wedge T)|N(T) \geq 1] + P(N(T) = 0)T, \end{aligned}$$

we have

$$P(N(T) \geq 1)E[(\tau_q \wedge T)|N(T) \geq 1] = E[(\tau_q \wedge T)] - P(N(T) = 0)T. \tag{2}$$

Replacing (2) into (1), and recalling $E[C_{HP1}]$ in Table 4.1, we have

$$E[C_{RHP1}] = \frac{E[(\tau_q \wedge T)]}{1 - P(N(T) = 0)} = \frac{1}{\lambda} \frac{E[Y_q]}{1 - P(Y = 0)} = \frac{E[C_{HP1}]}{1 - P(Y = 0)}, \tag{3}$$

where $Y \sim Poisson(\lambda T)$.

Next, we calculate the expected cumulative delay within one consolidation cycle under the revised HP1, which is denoted as $E[W_{RHP1}]$.

The following recursion equation is satisfied:

$$\begin{aligned} E[W_{RHP1}] = \quad & P(N(T) \geq 1)E[\int_0^{\tau_q \wedge T} N(t)dt|N(T) \geq 1] \\ & +P(N(T) = 0)E[W_{RHP1}]. \end{aligned} \tag{4}$$

The equation means if no order arrives within T units time, which happens with probability $P(N(T) = 0)$, the consolidation system restarts; if there are orders arriving within T units time, which happens with probability $P(N(T) \geq 1)$, the cumulative delay of within one consolidation cycle is $\int_0^{\tau_q \wedge T} N(t)dt$.

By noticing

$$
\begin{aligned}
& E[\int_0^{\tau_q \wedge T} N(t)dt] \\
= \quad & P(N(T) \geq 1)E[\int_0^{\tau_q \wedge T} N(t)dt | N(T) \geq 1] \\
& + P(N(T) = 0)E[\int_0^{\tau_q \wedge T} N(t)dt | N(T) = 0] \\
= \quad & P(N(T) \geq 1)E[\int_0^{\tau_q \wedge T} N(t)dt | N(T) \geq 1], \qquad (5)
\end{aligned}
$$

and replacing (5) into (4), together with recalling $E[W_{HP1}]$ in Table 4.1, we have

$$
E[W_{RHP1}] = \frac{E[\int_0^{\tau_q \wedge T} N(t)dt]}{1 - P(N(T) = 0)} = \frac{E[W_{HP1}]}{1 - P(Y = 0)} = \frac{1}{2\lambda} \frac{E[Y_q(Y_q - 1)]}{1 - P(Y = 0)}, \qquad (6)
$$

where $Y \sim Poisson(\lambda T)$.

Define a new random variable $\tilde{Y}$, which has the same distribution of $Y \mid Y > 0$. In this way, we can rewrite

$$
E[C_{RHP1}] = \frac{1}{\lambda} E[\tilde{Y}_q], \qquad (7)
$$

$$
E[W_{RHP1}] = \frac{1}{2\lambda} E[\tilde{Y}_q(\tilde{Y}_q - 1)], \qquad (8)
$$

where $\tilde{Y}_q = \tilde{Y} \wedge q$.

Similarly, we can obtain the expected cycle length under the revised TP1 with parameters T is

$$E[C_{RTP1}] = \frac{E[C_{TP1}]}{1 - P(N(T) = 0)} = \frac{T}{1 - e^{-\lambda T}}, \tag{9}$$

and the cumulative delay with one consolidation cycle under the revised TP1 with parameters T is

$$E[W_{RTP1}] = \frac{E[W_{TP1}]}{1 - P(N(T) = 0)} = \frac{\lambda T^2}{2(1 - e^{-\lambda T})}. \tag{10}$$

From (3), (6), (9) and (10) and the definition of AOD, we know AOD of the revised HP1 is the same as HP1, AOD of the revised TP1 is the same as TP1 if the parameters q, T are fixed. From Theorem 4.10, with fixed parameters q, T, the revised HP1 also performs better than QP and revised TP1 in terms of AOD.

From Theorem 4.6, we can conclude that for a given expected consolidation cycle length, QP performs better than the revised HP1 and the revised TP1 in terms of AOD. In the following, we provide the comparison between the revised HP1 and the revised TP1 with a given expected consolidation cycle length.

Suppose $Y_i \sim Poisson(\lambda_i)$, $i = 1, 2$ and $\lambda_1 > \lambda_2$, we know Y_1 is stochastically larger than Y_2. Define $\tilde{Y_i} \stackrel{d}{=} Y_i \mid Y_i > 0$, we show $\tilde{Y_1}$ is also stochastically larger than $\tilde{Y_2}$ in the following result.

Lemma .1. *Let $Y \sim Poisson(\lambda)$, $\tilde{Y}$ is distributed as $Y \mid Y > 0$, then $P(\tilde{Y} > n)$ is increasing in λ, for any integer $n \geq 1$.*

Proof. Notice $\frac{d}{d\lambda}P(Y > n) = P(Y = n)$. Then for $n \geq 1$,

$$\begin{aligned}
\frac{d}{d\lambda}P(\tilde{Y} > n) = & \quad \frac{d}{d\lambda}\frac{P(Y > n)}{P(Y > 0)} \\
= & \quad \frac{P(Y = n)P(Y > 0) - P(Y > n)P(Y = 0)}{(P(Y > 0))^2} \\
= & \quad \frac{P(Y = n) - e^{-\lambda}P(Y \geq n)}{(P(Y > 0))^2}.
\end{aligned}$$

In addition, by using $P(Y = k) = \frac{\lambda}{k}P(Y = k-1)$, we have

$$\begin{aligned}
P(Y = n) - e^{-\lambda}P(Y \geq n) = & \quad \frac{\lambda}{n}P(Y = n-1) - e^{-\lambda}\sum_{k=n}^{\infty}\frac{\lambda}{k}P(Y = k-1) \\
> & \quad \frac{\lambda}{n}(P(Y = n-1) - e^{-\lambda}P(Y \geq n-1)).
\end{aligned}$$

Since $P(Y = 0) - e^{-\lambda}P(Y \geq 0) = 0$, it follows by induction that

$$P(Y = n) - e^{-\lambda}P(Y \geq n) > 0.$$

Therefore, $\frac{d}{d\lambda}P(\tilde{Y} > n) > 0$.

□

Theorem .2. *For a given expected consolidation cycle length $E[C]$, the revised HP1 with larger quantity parameter would achieve larger AOD than the revised HP1 with smaller quantity parameter, in terms of AOD. In particular, the revised HP1 performs better than the revised TP1 in terms of AOD, under a given expected consolidation cycle length $E[C]$.*

Proof. We consider a fixed $E[C]$ and use the following notation for the corresponding policy parameters under this $E[C]$ value: a revised HP1 with parameters q_H and T_H, the other revised HP1 with parameters $q_H + 1$ and T'_H. Recalling (7) and by assumption that the two revised HP1 have the same expected cycle length, we have,

$$E[\tilde{U}_{q_H}] = E[\tilde{V}_{q_H+1}], \tag{11}$$

where $\tilde{U}$ is distributed as $U \mid U > 0$, $U \sim Poisson(\lambda T_H)$, and $\tilde{V}$ is distributed as $V \mid V > 0$, $V \sim Poisson(\lambda T'_H)$. Clearly, $T_H > T'_H$. From lemma .1, $\tilde{U}$ is stochastically larger than $\tilde{V}$.

Next, recalling (8) and reiterating the assumption of fixed $E[C]$, we proceed to show that

$$E[\tilde{U}_{q_H}(\tilde{U}_{q_H} - 1)] \leq E[\tilde{V}_{q_H+1}(\tilde{V}_{q_H+1} - 1)]. \tag{12}$$

From Lemma 4.4, and recalling (11), we have

$$E[\tilde{U}^2_{q_H}] \leq E[\tilde{V}^2_{q_H+1}],$$

so that (12) is verified.

The revised TP1 can be seen as the revised HP1 with quantity parameter ∞, therefore, under the same expected consolidation cycle $E[C]$, the revised HP1 performs better than the revised TP1 in terms of AOD.

$\square$

APPENDIX B

In Section 6.2.3, we define average order squared delay and provide comparative results under different policies. In this Appendix, we provide the computation for the expression of average order squared delay.

$N(t)$ is a Poisson process with rate λ. τ_n is the first hitting time for n with respect to the demand process $N(t)$, where n is an positive integer. Clearly τ_n is distributed as $gamma(n, \lambda)$. Let q and $T > 0$ be the two parameters of HP.

The expected cumulative squared delay penalty within one shipment consolidation cycle of HP with parameters q and T can be calculated as

$$
\begin{aligned}
E[W'_{HP}] = \quad & E[\int_0^{\tau_q \wedge T} (\tau_q \wedge T - t)^2 dN(t)] \\
= \quad & E[\int_0^{\tau_q \wedge T} (\tau_q \wedge T)^2 dN(t)] - 2E[\int_0^{\tau_q \wedge T} t(\tau_q \wedge T) dN(t)] \\
& + E[\int_0^{\tau_q \wedge T} t^2 dN(t)].
\end{aligned} \tag{13}
$$

The three terms are calculated as follows one by one.

$$
\begin{aligned}
& E[\int_0^{\tau_q \wedge T} (\tau_q \wedge T)^2 dN(t)] \\
= \quad & E[(\tau_q \wedge T)^2 N(\tau_q \wedge T)] \\
= \quad & qE[\tau_q^2 1_{\tau_q \le T}] + T^2 E[N(T) 1_{N(T) \le q-1}] \\
= \quad & \frac{q^2(q+1)}{\lambda^2} P(N(T) \ge q+2) + T^2 \sum_{n=0}^{q-1} nP(N(T) = n),
\end{aligned} \tag{14}
$$

where the last equality comes from $E[\tau_q^2 1_{\tau_q \le T}] = \frac{q(q+1)}{\lambda^2} P(N(T) \ge q+2)$.

$$
\begin{aligned}
& 2E[\int_0^{\tau_q \wedge T} t(\tau_q \wedge T) dN(t)] \\
= \quad & 2\int_0^T E[\int_0^{\tau_q} t\tau_q dN(t)|\tau_q = s] f_{\tau_q}(s) ds + 2TE[\int_0^T t dN(t) 1_{N(T)\leq q-1}] \\
= \quad & 2\int_0^T s(E[\sum_{i=1}^{q-1} \tau_i | \tau_q = s] + s) f_{\tau_q}(s) ds + 2T\sum_{n=0}^{q-1} E[\sum_{i=1}^{n} \tau_i | N(T) = n] P(N(T) = n) \\
= \quad & 2\int_0^T s((q-1)\frac{s}{2} + s) f_{\tau_q}(s) ds + 2T\sum_{n=0}^{q-1} n\frac{T}{2} P(N(T) = n) \\
= \quad & (q+1)E[\tau_q^2 1_{\tau_q \leq T}] + T^2 \sum_{n=0}^{q-1} nP(N(T) = n) \\
= \quad & \frac{q(q+1)^2}{\lambda^2} P(N(T) \geq q+2) + T^2 \sum_{n=0}^{q-1} nP(N(T) = n), \qquad (15)
\end{aligned}
$$

where the third equality is derived from Lemma 4.5.1 and Theorem 4.5.2 in Resnick (2002)(p. 322, 325).

Since $g(t) = \int_0^t s^2 dN(s) - \frac{1}{3}\lambda t^3$ is a martingale with respect to $N(t)$ and $\tau_q \wedge T$ is a bounded stopping time, then applying optional stopping theorem, we have

$$
\begin{aligned}
& E[\int_0^{\tau_q \wedge T} t^2 dN(t)] = \frac{1}{3}\lambda E[(\tau_q \wedge T)^3] \\
= \quad & \frac{1}{3}\lambda E[\tau_q^3 1_{\tau_q \leq T}] + \frac{1}{3}\lambda T^3 P(N(T) \leq q-1) \\
= \quad & \frac{(q+2)(q+1)q}{3\lambda^2} P(N(T) \geq q+3) + \frac{1}{3\lambda^2}\sum_{m=0}^{q+2} m(m-1)(m-2)P(N(T) = m) \\
= \quad & \frac{1}{3\lambda^2} E[Y_{q+2}(Y_{q+2}-1)(Y_{q+2}-2)], \qquad (16)
\end{aligned}
$$

where the second equality comes from

$$E[\tau_q^3 1_{\tau_q \leq T}] = \frac{(q+2)(q+1)q}{\lambda^3} P(N(T) \geq q+3),$$

$$\lambda T^3 P(N(T) = n) = \frac{(n+3)(n+2)(n+1)}{\lambda^2} P(N(T) = n+3).$$

Substituting (14), (15) and (16) in (13), we obtain

$$\begin{aligned} E[W'_{HP}] = & \quad \frac{(q+1)q(q-1)}{3\lambda^2} P(N(T) \geq q+2) + \frac{1}{3\lambda^2} \sum_{m=0}^{q+1} m(m-1)(m-2) P(N(T) = m) \\ = & \quad \frac{1}{3\lambda^2} E[Y_{q+1}(Y_{q+1} - 1)(Y_{q+1} - 2)], \end{aligned} \tag{17}$$

where $Y \sim Poisson(\lambda T)$ and $Y_{q+1} = \min(Y, q+1)$.