Stable Controller Interpolation and Controller Switching for LPV Systems

This paper examines the gain-scheduling problem with a particular focus on controller interpolation with guaranteed stability of the nonlinear closed-loop system. For linear parameter varying model representations, a method of interpolating between controllers utilizing the Youla parametrization is proposed. Quadratic stability despite fast scheduling is guaranteed by construction, while the characteristics of individual controllers designed a priori are recovered at critical design points. Methods for reducing the state dimension of the interpolated controller are also given. The capability of the proposed approach to guarantee stability despite arbitrarily fast transitions leads naturally to application to switched linear systems. The efficacy of the method is demonstrated in simulation using a multi-input, multi-output, nonminimum-phase system, while interpolating between two controllers of different sizes and structures.

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1 Introduction

Many physical systems exhibit dynamics sufficiently nonlinear that a single linear controller may fail to achieve acceptable performance throughout the envelope of conditions. Gain-scheduling is one of the most popular approaches in the industry for controlling a nonlinear system, often by interpolating a family of local controllers, thus dividing the nonlinear control design problem into several smaller problems where linear design tools are employed [1].

The principal challenge facing gain-scheduling research is guaranteeing stability of the nonlinear closed-loop system. The simplicity in design, where linear controllers and ad hoc interpolation methods are used, is contrasted with difficulties in analysis, where guaranteeing the stability of the resulting nonlinear closed-loop system can be extremely challenging. Moreover, the presence of “hidden coupling” terms or “scheduling dynamics” due to the interpolation functions [2] can create unanticipated stability problems. This paper proposes a method of controller interpolation for linear parameter varying (LPV) systems that guarantee quadratic stability of the closed-loop system, while recovering the local controllers states and facilitating practical implementation. Section 5 then presents a special case, where a particular choice of nominal controller leads to further reduction in the controller state dimension. Finally, Sec. 6 provides an illustrative example where dissimilar controllers are blended using the proposed framework and simulated on the quadruple tank system.

2 Background

Gain-scheduling is a general term encompassing a wide variety of approaches. A rough categorization of techniques would include (1) gain-scheduling “the LPV way” [6] and (2) linear controller interpolation. As the proposed approach draws upon ideas from both of these areas, a brief overview of existing techniques is given to clarify the novelty of this work.

2.1 LPV Gain-Scheduling. For LPV systems there are many reported efforts to synthesize a single LPV controller based on specified performance criteria. These “self-scheduled” [7] approaches guarantee closed-loop system stability via a common Lyapunov function, or more commonly using a parameter dependent Lyapunov function to reduce conservatism, and, as a result, require bounds on the rate of change in the scheduling parameters [7,6,8]. This requirement can be restrictive, and in some cases, inappropriate. For systems whose nonlinearities are more appropriately captured by internal signals, rate limit assumptions would be difficult to guarantee. Attempts to develop gain-scheduled controllers for fast scheduling changes construct controllers dependent not only on the scheduling variable \( \theta \), but also on \( \dot{\theta} \) [9,10].

Moreover, computational approaches to the LPV synthesis problems may not be guaranteed to find a solution even if one exists [7]. These problems are typically convex, but infinite di-
2.2 Interpolation Gain-Scheduling. The second general gain-scheduling approach, linear controller interpolation, operates on a “divide-and-conquer” [1] paradigm. Local linear models, either linearized first principles models or data-driven identified models, are used in conjunction with standard design tools to create a family of linear controllers. Thus the controllers at critical design points can be tuned to achieve high performance, and the gain-scheduling problem becomes a problem of determining an interpolation strategy that ensures stable transitions between critical design points. Thus the interpolation paradigm offers a distinct advantage over the self-scheduled approaches for systems with distinctly different operation points and control objectives.

Published interpolation methods are varied, including interpolating between transfer function poles/zeroes [14], solutions to Riccati equations [15], state feedback and observer gains [16], etc. There also exist methods for interpolating between controllers of different sizes and structures; controller blending uses a weighted average of the individual controller outputs (Fig. 1(a)). Sometimes termed a local controller network (LCN) [17] this ad hoc method uses weighting functions that are a function of the scheduling variable(s) \( \theta \), as \( \alpha = f(\theta) \). While selection of the scheduling variables is based on a physical understanding of which variables most accurately capture the system nonlinearities, and is thus situation dependent, selection of the weighting functions is by design. Industrial applications of this approach include vehicle dynamics [18], power plants [19], and hydraulic systems [20].

Despite the simplicity in implementing interpolated controllers, guaranteeing stability of the resulting closed-loop system can pose a challenging problem. Consider the following example: Let a fixed plant and two stabilizing controllers be defined as in Eq. (1). An interpolated controller could be defined as in Eq. (2), where \( \alpha \in [0,1] \). Although both \( K_1 \) and \( K_2 \) stabilize the plant, the blended controller \( K_0 \) destabilizes the plant for the majority of the intermediate values \( \alpha \in [0.25,1] \).

Interpolation methods that guarantee stability for any fixed value of the scheduling parameter, known as frozen parameter stability, have been termed stability-preserving interpolation methods [21,8]. However, with these techniques, no scheduling dynamics are considered and global stability can only be inferred by assuming slowly varying scheduling variables.

2.3 Youla Parameter Gain-Scheduling. One proposed method of guaranteeing frozen parameter stability is to use a Youla parameter-based gain-scheduling approach [21–24]. The Youla parametrization is a well-known method for characterizing the set of all stabilizing controllers, and the interested reader is referred to Refs. [25,26] for a detailed discussion. In summary, assuming controller \( K_0 \) stabilizes a plant \( P_0 \), then the set of all stabilizing controllers can be parametrized as \( K(Q) = (U_0 + M_0Q) \times (V_0 + N_0Q)^{-1}, \) where the controller and plant are decomposed into coprime factors as \( K_0 = U_0V_0^{-1} \) and \( P_0 = N_0M_0^{-1}. \) and \( Q \in RH_\infty \). This decomposition can be represented by the lower linear fractional transformation of the interconnection system \( J_k(s) \) (Eq. (3)) and the system \( Q(s) \), as \( K(Q) = F(J_k, Q). \)

\[
J_k(s) = \left[ \begin{array}{c} U_0V_0^{-1} \\ V_0^{-1} - V_0^{-1}N_0 \end{array} \right]
\]  

Several authors have suggested different methods for using the Youla parametrization as a basic framework for gain-scheduling, using the terms \( J-Q \) interpolation [27], blending of the Youla parameters [22], or local \( Q \)-network (LQN) [23]. The simplest method is depicted in Fig. 1(b), where systems \( Q_0 \) are blended, instead of directly blending the controllers (Fig. 1(a)). This structure is a general representation for controller blending, and recovers the classical blending approach (Fig. 1(a)) as a special case [23].

This framework offers several advantages over the classical controller blending. First, it permits the scheduling of open-loop unstable controllers [28], and \( Q_0 \) can be designed to recover the characteristics of particular controllers \( K_i \) at specified design points. Second, this framework has the intuitive appeal of isolating common controller elements in the function \( J_k(s) \) and blending only the differences between the individual controllers. Finally, this framework guarantees frozen parameter stability by construction. Because \( K(Q) \) stabilizes \( P_0 \) for any \( Q \in RH_\infty \), then \( K(\Sigma \alpha Q_i) \) also stabilizes \( P_0 \) for every frozen value of \( \alpha \), since if \( Q_i \in RH_\infty \) then \( \Sigma \alpha Q_i \in RH_\infty \) also.

Although controller interpolation within a Youla parameter-based framework offers several distinct advantages compared with common ad hoc controller blending methods, two fundamental issues remain. First is the issue of closed-loop stability despite...
scheduling dynamics. Although many gain-scheduled approaches assume rate limits on scheduling inputs to guarantee stability, this is only appropriate for exogenous scheduling signals. For systems whose nonlinearities are more appropriately captured by endogenous signals, such as a system state/output, rate limit assumptions would be inappropriate. Efforts by Niemann and Stoustrup [21,22] to implement Youla-based interpolation schemes focus on frozen parameter stability, but do not address scheduling dynamics. The notable work by Stillwell and co-worker [24,29,8] utilizes a Youla-based approach to implement LPV gain-scheduled controllers, but requires an explicit rate limit on the scheduling variable. The extreme case of fast scheduling can be viewed as a switching control problem, and several researchers have utilized Youla-based schemes for stable controller switching, but assumed a fixed linear time invariant plant [30,31]. Recent work has utilized a Youla-based scheme with guaranteed stability for fast scheduling, but focused on synthesis of a single LPV controller, not interpolation among fixed controllers [32].

The second fundamental issue with controller interpolation schemes is limiting the number of dynamic states. While LPV or self-scheduled control schemes “share” controller states, many controller interpolation approaches (e.g., LCN) rely on implementing many controllers in parallel, resulting in significant computational requirements in practice. Additionally, the Youla-based scheduling framework results in additional state variables depending on the choice of coprime factorization. A partial solution is found by implementing only a subset of the controllers at any given time, and using bumpless transfer techniques to ensure smooth transitions [33]. For Youla-based scheduling, all of the previous efforts listed above utilize a state feedback/observer structure, which reduces the number of states in the coprime factorizations, and the resulting controllers.

2.4 Summary. There is a clear gap in literature for gain-scheduling interpolation schemes that minimize the number of states and are applicable to endogenously scheduled systems, where stability must be guaranteed for arbitrarily fast variations in the scheduling variables. In summary, the objective of this research is to construct an interpolation scheme that possesses the following:

1. guarantees stability of the nonlinear closed-loop system despite arbitrarily fast transitions of the scheduling variable
2. is applicable to a broad class of systems, including those with multiple inputs/outputs, open-loop unstable dynamics, and controllers of different sizes and structures
3. recovers the characteristics of the given local controllers at specified operating conditions with a minimal number of controller states

3 Preliminaries

In this section, a few mathematical and notational preliminaries are presented. The transfer function for a LPV system is denoted as

\[ G(\theta) = \begin{bmatrix} A_G(\theta) & B_G(\theta) \\ C_G(\theta) & D_G(\theta) \end{bmatrix} \]

The search for such a common quadratic Lyapunov function typically would require gridding the variable over its predefined range [11] and solving the finite number of associated LMIs. The number of LMIs is generally reduced considerably for a polytopic LPV system [35], where the system matrices are defined in terms of vertices, at which the LPV system is evaluated at a particular operation point, e.g.,

\[ G_{polytopic}(\theta) = C^T \begin{bmatrix} A_G(\theta) & B_G(\theta) \\ C_G(\theta) & D_G(\theta) \end{bmatrix} C \quad \forall \ j = 1 \cdots m \]

At these points the dynamics are denoted simply as

\[ \dot{x}_G = A_G x_G + B_G u \\
\]

\[ y = C_G x_G + D_G u \]

This set of all real, rational, proper, and stable transfer functions (real rational subspace of \( \mathcal{H}_\infty \)) is denoted as \( \mathcal{RH}_\infty \) [36]. A square matrix \( A \) is called a Hurwitz matrix if every eigenvalue of \( A \) has a strictly negative real part, i.e., \( \text{Re}(\lambda(A)) < 0 \).

4 General LPV Control With Local Controller Recovery

When nonlinear system models are constructed using first principles, the state variables generally remain tied to system physics. In many cases, this naturally leads to LPV models, \( P(s, \theta) \), where linear models at different operating points share the same state variables, and the state-space system matrices are parametrized in terms of the scheduling variable \( \theta \) as follows:

\[ x_p = A_p(\theta) x_p + B_p(\theta) u \]

\[ y = c_p(\theta) x_p \]

This is in contrast with a set of controllers, defined a priori, where there is no physical relationship between state variables. We assume that these local controllers have been designed for a set of critical operating conditions, with plant dynamics defined by Eq. (6) with \( \theta = \theta_i \). The associated controllers \( K_i(s) \) are represented in state-space form as

\[ \dot{x}_{i1} = A_{i1} x_{i1} + B_{i1} z_2 \\
\]

\[ z_1 = C_{i1} x_{i1} + D_{i1} z_2 \]

The input/output notation is defined consistent with the general feedback control diagram shown in Fig. 2.

4.1 Youla Parameter-Based Gain-Scheduling. To create a controller interpolation scheme that satisfies the aforementioned
objectives we employ a Youla parameter-based framework. First it is necessary to select a nominal controller $K_0(s, \theta)$ as follows:

$$\dot{x}_0 = A_{00}(s) x_0 + B_{00}(s) z_2$$
$$u = C_{00}(s) x_0 + D_{00}(s) z_2$$

(8)

Note that this controller need not be a LPV controller. It simply needs to be any controller (linear time-invariant or LPV) that stabilizes the LPV plant over the range defined by the scheduling parameter, such that a quadratic CQLF exists. In most cases, this nominal controller would be designed for robustness, not performance, as the local controllers can be designed to achieve high performance at critical operating points. This aspect of the design process, including particular choices for $K_0(s, \theta)$, will be discussed later in Sec. 5.

Next we decompose the plant and nominal controller into the left and right coprime factors as

$$\begin{bmatrix} M(\theta) & U_0(\theta) \\ N(\theta) & V_0(\theta) \end{bmatrix} = \begin{bmatrix} A_p(\theta) + B_p(\theta) F_p(\theta) & 0 \\ 0 & A_{00}(\theta) + B_{00}(\theta) F_{00}(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} F_p(\theta) & C_{00}(\theta) + D_{00}(\theta) F_{00}(\theta) \\ C_p(\theta) & F_{00}(\theta) \end{bmatrix}$$

(12)

where $F_p(\theta)$ and $F_{00}(\theta)$ are feedback gains chosen such that the matrices $A_p(\theta)+B_p(\theta) F_p(\theta)$ and $A_{00}(\theta)+B_{00}(\theta) F_{00}(\theta)$ are Hurwitz. A particular technique for selecting $F_p(\theta)$ and $F_{00}(\theta)$, which guarantee stability for time-varying $\theta$, will be given later in Eqs. (34) and (35).

The set of all stabilizing controllers for the LPV plant can then be formulated in terms of the interconnection system

$$J_k(\theta) = \begin{bmatrix} U_0(\theta) V_0(\theta)^{-1} & \tilde{V}_0(\theta)^{-1} \\ V_0(\theta)^{-1} N(\theta) \end{bmatrix}$$

(14)

and a Youla parameter $Q$, as shown in Fig. 3. For implementation, state-space representations of these elements are given by

$$U_0(\theta) V_0(\theta)^{-1} = K_0(\theta) = \begin{bmatrix} A_{00}(\theta) & B_{00}(\theta) \\ C_{00}(\theta) & D_{00}(\theta) \end{bmatrix}$$

$$\tilde{V}_0(\theta)^{-1} = \begin{bmatrix} A_p(\theta) + B_p(\theta) F_p(\theta) & 0 \\ B_p(\theta) & A_{00}(\theta) \end{bmatrix} - \begin{bmatrix} F_p(\theta) & C_{00}(\theta) + D_{00}(\theta) F_{00}(\theta) \\ C_p(\theta) & F_{00}(\theta) \end{bmatrix}$$

$$V_0(\theta)^{-1} = \begin{bmatrix} A_{00}(\theta) & B_{00}(\theta) \\ C_{00}(\theta) & D_{00}(\theta) \end{bmatrix}$$

(15)

(16)

(17)

$$V_0(\theta)^{-1} N(\theta) = \begin{bmatrix} A_p(\theta) + B_p(\theta) F_p(\theta) & 0 \\ B_p(\theta) C_p(\theta) & A_{00}(\theta) \end{bmatrix} - \begin{bmatrix} F_p(\theta) C_p(\theta) & A_{00}(\theta) \\ C_p(\theta) & D_{00}(\theta) \end{bmatrix}$$

(18)

$$K_0(s, \theta) = U_0(s, \theta) V_0(s, \theta)^{-1} = \tilde{V}_0(s, \theta)^{-1} \tilde{U}_0(s, \theta)$$

(9)

and

$$P(s, \theta) = N(s, \theta) M(s, \theta)^{-1} = \tilde{M}(s, \theta)^{-1} \tilde{N}(s, \theta)$$

(10)

For the remainder of the paper, we will drop the $K_0(s, \theta)$ notation for the more compact $K_0(\theta)$. These coprime factors are constructed such that $U_0(\theta), \tilde{U}_0(\theta), V_0(\theta), \tilde{V}_0(\theta) \in RH_{\infty}$. $N(\theta), \tilde{N}(\theta), M(\theta), \tilde{M}(\theta) \in RH_{\infty}$, and such that they satisfy the double Bezout identity

$$\begin{bmatrix} M(\theta) U_0(\theta) \\ N(\theta) V_0(\theta) \end{bmatrix} = \begin{bmatrix} V_0(\theta) - \tilde{U}_0(\theta) \\ -\tilde{N}(\theta) \tilde{M}(\theta) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

(11)

One such factorization can be constructed from the state-space representations as

4.2 Local Controller Recovery. The set of all stabilizing controllers is parametrized in terms of the system $Q$. Assuming that the local controller $K_l$ stabilizes the LPV plant with $\theta=\theta_l$, then the local controller can be recovered as $K_l = F_l / J_l Q_l$, where $F_l$ is the lower fractional transformation. $Q_l$ is defined as $Q_l = \tilde{U}_l V_0(\theta) - \tilde{V}_l U_0(\theta) = \tilde{V}_l (K_l - K_0) V_0(\theta)$, and the necessary coprime factors are defined as given below to satisfy a similar Bezout identity as in Eq. (11). Note that by construction $Q_l \in RH_{\infty}$ and each $Q_l$ is stable.

$$Q_l = -\begin{bmatrix} \tilde{V}_l - \tilde{U}_l \end{bmatrix} \begin{bmatrix} U_0(\theta) \\ V_0(\theta) \end{bmatrix}$$

(19)
The closed-loop system is stable under arbitrarily fast variations in $\theta$ if there exist CQLFs $V(x) = x^T P_G x > 0$, $\forall x \neq 0$ and $P_G A_G(\theta) + A_G^T(\theta) P_G < 0$, $\forall x \neq 0$. Noting the block diagonal structure of the closed-loop system state matrix (Eq. (28)), the stability of the system can be guaranteed by ensuring the stability of each sub-block.
First, the state matrix $A_{T21}(\theta)$ is simply the closed loop describing the interaction between the nominal controller and the LPV plant. By an earlier assumption, the nominal controller, fixed or LPV, stabilizes the LPV plant, such that there exists a corresponding CQLF

$$P_{T21}A_{T21}(\theta) + A_{T21}^T(\theta)P_{T21} < 0 \quad (33)$$

Second, assuming that the state feedback gains for the coprime factors $F_p(\theta)$ and $F_k(\theta)$ are chosen such that $F_p(\theta) = X(\theta)P^{-1}_p$ and $F_k(\theta) = Y(\theta)P^{-1}_k$, such that the following LMs are satisfied, then $A_{T21}(\theta)$ is also guaranteed to be quadratically stable:

$$A_{\beta}(\theta)P_{\beta} + P_{\beta}A_{\beta}(\theta)^T + B_{\beta}(\theta)X(\theta) + X(\theta)^TB_{\beta}(\theta)^T < 0 \quad (34)$$

(A Note that for polytopic LPV systems, these conditions can be covered at the corresponding operating point $\theta$.) Finally, we assume that the interpolation scheme is designed such that there exists a quadratic Lyapunov function $P_G$ such that

$$P_GA_G + A_G^T P_G < 0 \quad (36)$$

Methods for creating such an interpolation scheme will be given in Sec. 4.4. As the state matrix of the closed-loop system is block diagonal, then the block diagonal CQLF $P_G = \text{diag}(P_1, P_2, P_3, P_4, P_5)$ is sufficient for guaranteeing stability of the system under arbitrarily fast transitions.

### 4.4 Construction of LPV-Q system

The previous sections discuss how to form $Q_i$ such that each local controller $K_i$ is recovered at the corresponding operating point (Sec. 4.2), and how the stability of the resulting closed-loop system can be guaranteed, assuming there exists a CQLF for the interpolated $Q$ (Sec. 4.3). In this section, we present a method of interpolating between these $Q_i$ while guaranteeing the existence of a CQLF for the interpolated $Q$, and limiting the state dimension of the eventual controller.

The standard LCN approach to interpolation would be to simply create a weighted average of the output signals from each $Q_i$ based on the current operating point (Eq. (37)). The stability of the resulting polytopic system can be established with a simple CQLF (Eq. (38)).

$$\begin{bmatrix} x_{Q1} \\ \vdots \\ x_{Qn} \end{bmatrix} = \begin{bmatrix} A_{Q1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{Qn} \end{bmatrix} \begin{bmatrix} x_{Q1} \\ \vdots \\ x_{Qn} \end{bmatrix} + \begin{bmatrix} B_{Q1} \\ \vdots \\ B_{Qn} \end{bmatrix} z$$

$$u = \begin{bmatrix} \alpha_1 C_{Q1} \\ \vdots \\ \alpha_n C_{Qn} \end{bmatrix} \begin{bmatrix} x_{Q1} \\ \vdots \\ x_{Qn} \end{bmatrix} + \sum_{j=1}^{n} \alpha_j D_{Qj} z \quad (37)$$

$$P_{Q} = \text{diag}(P_{Q1}, P_{Q2}, \ldots, P_{Qn}) \quad (38)$$

However, utilizing a large number of local controllers within a LCN/LQN can lead to significant computational problems, as the state dimension of the resulting nonlinear controller can become very large. In contrast to plant models where the physical nature of the dynamic states naturally leads to LPV representations, the states of a set of local controllers do not share any physical significance. Thus LPV controllers generally only arise from direct synthesis, and do not allow separate control design at specified operating conditions. However, a notable advantage of the LPV controller implementation is the limited number of dynamic state variables required. Thus we propose an alternative formulation of the gain-scheduled interpolated controller where states are shared, leading to a LPV controller formulation, which enjoys significantly lower state dimension than the local controller network approach, but retains the benefits of local controller recovery.

First, note that if the local controllers $K_i$ have different state dimensions, the corresponding $Q_i$ would also have different state dimensions. In this case, stable, unobservable/uncontrollable states are augmented to the state-space representations of $Q_i$ such that the augmented $\hat{Q}_i$ has equal state dimensions to the highest dimensional $Q_i$.

$$\dot{\hat{Q}} = \begin{bmatrix} A_{Q1} & 0 & B_{Q1} \\ 0 & -\mu & 0 \\ C_{Q1} & 0 & D_{Q1} \end{bmatrix}$$

Each $\dot{\hat{Q}}$ is guaranteed to be stable by construction, and therefore there exists an associated QLF $P_{\hat{Q}_i}$. For the purposes of this paper, the following finite set of LMs will be solved to obtain the matrices $P_{\hat{Q}_i}$ that guarantee stability and also ensure that $\left\| \dot{\hat{Q}}_i \right\| < \gamma_i$:

$$P_{\hat{Q}_i}A_{\hat{Q}_i} + A_{\hat{Q}_i}^T P_{\hat{Q}_i} + B_{\hat{Q}_i}^T C_{\hat{Q}_i} P_{\hat{Q}_i} \left[ B_{\hat{Q}_i} + C_{\hat{Q}_i} + D_{\hat{Q}_i} \right] < 0 \quad (40)$$

Next, note that a state transformation can be applied to each augmented system $\hat{Q}_i$ without affecting its input-output nature. Applying a similarity transformation defined by $(P_{\hat{Q}_i})^{-1/2}$ to each $\hat{Q}_i$ system yields

$$\tilde{\hat{Q}}_i = \begin{bmatrix} P_{\hat{Q}_i}^{1/2} A_{\hat{Q}_i} P_{\hat{Q}_i}^{-1/2} & P_{\hat{Q}_i}^{1/2} B_{\hat{Q}_i} \end{bmatrix}$$

It is straightforward to verify that under this similarity transformation, the LMI norm bound becomes

$$\begin{bmatrix} A_{\hat{Q}_i} + A_{\hat{Q}_i}^T C_{\hat{Q}_i} B_{\hat{Q}_i} + C_{\hat{Q}_i} D_{\hat{Q}_i} \\ B_{\hat{Q}_i} + D_{\hat{Q}_i} C_{\hat{Q}_i} + D_{\hat{Q}_i} D_{\hat{Q}_i} \end{bmatrix} < 0 \quad (42)$$

Thus the polytopic system formed by the transformed systems (Eq. (43)) is guaranteed stable with CQLF $F_{\hat{Q}_i}(\theta)=I$, and with guaranteed norm bound $\|Q(\theta)\|_\infty = \max \|\dot{\hat{Q}}_i\| < \gamma_{\max}$. (notations such as $\tilde{\hat{Q}}_i$, $\gamma_{\max}$ are interpo-

$$Q(\theta) = C_{\hat{Q}_i} \begin{bmatrix} A_{\hat{Q}_i} & B_{\hat{Q}_i} \\ C_{\hat{Q}_i} & D_{\hat{Q}_i} \end{bmatrix} \quad (43)$$

Please note that the stability of the nonlinear closed loop does not depend on how the state matrices of the particular $\hat{Q}_i$ are interpolated to form $Q(\theta)$; this offers an additional element of design freedom. In general the weighting functions are designed such that $\alpha_i \in [0, 1]$ and $\sum_i \alpha_i = 1$, with the magnitude based on the relative distance to the respective design point in the scheduling space (e.g., Fig. 4).

A summary of the design procedure is given as follows:

**Step 1.** Design fixed linear controllers at key operating conditions $K_i(s)$.

**Step 2.** Select a nominal controller $K_0(\theta)$ that stabilizes the system for the entire operating envelope (see Sec. 5 for a detailed discussion).

**Step 3.** Utilizing the LPV representation of the system dynamics, solve Eqs. (34) and (35) for feedback gains as $F_p(\theta) = X(\theta)P^{-1}_p$ and $F_k(\theta) = Y(\theta)P^{-1}_k$. These are used to construct the coprime factors given in Eqs. (12) and (13).

**Step 4.** Formulate the interconnection system $J_k(\theta)$ given in Eq. (14) and the individual Youla parameters $Q(\theta)$ as given in Eq. (22).

**Step 5.** As necessary, augment the states of the individual $Q_i$ systems as given in Eq. (39) to ensure equal state dimension among controllers.

**Step 6.** Use the LMIs given in Eq. (40) to determine the state transformation specified in Eq. (41), and formulate the LPV representation of $\hat{Q}(\theta)$ as in Eq. (43), and select the weighting func-
4.5 Application to Control of Switched Linear Systems.

The capability of the proposed interpolation approach to guarantee closed-loop stability for arbitrarily fast changes in the scheduling variable leads naturally to application to switched linear systems. If the transitions between critical operating conditions occur infinitely fast (instantly), the LPV plant model can be represented by a switched linear system, using standard notation [37]

\[
\dot{x}_p = A_{p,\sigma} x_p + B_{p,\sigma} u
\]

\[
y = C_{p,\sigma} x_p
\]

where \( \sigma \) denotes the switching signal. Application of the techniques presented above results in stable switching between controllers of arbitrary size/structure, if there exists a nominal control strategy \( K_{\sigma}(\sigma) \) that stabilizes the switched system, such that there exists a common quadratic Lyapunov function

\[
P A_{p K,\sigma} + A_{p K,\sigma}^T P < 0 \quad (45)
\]

where

\[
A_{p K,\sigma} = \begin{bmatrix} A_{p,\sigma} + B_{p,\sigma} D_{p,\sigma} C_{p,\sigma} & B_{p,\sigma} C_{k,\sigma} \\ B_{k,\sigma} C_{p,\sigma} & A_{k,\sigma} \end{bmatrix} \quad (46)
\]

If any such nominal stabilizing controller exists, then the above framework allows local control strategies to be parametrized in terms of this nominal controller, such that the characteristics of the local controllers are recovered exactly, but the stability of the closed-loop system is guaranteed. This technique provides a promising alternative to standard switched systems' control methodologies, such as dwell-time or switching sequences approaches [37].

5 Special Cases: Choice of Nominal Controller

The approach outlined in Sec. 4 provides a general method of constructing a gain-scheduled controller for LPV systems with local controller recovery. Additionally, an interpolation approach is given that results in a LPV controller, significantly reducing the large state dimensions resulting from simple controller blending. However, a prerequisite to this design methodology is the existence of a nominal controller \( K_{\sigma}(\sigma) \) that stabilizes the plant for the entire range of operating conditions. The synthesis of a single fixed controller that meets these conditions is a well-known, and provably difficult, simultaneous stabilization problem, while the synthesis of a stabilizing LPV controller relies on the existing design methods available in literature. However, in practice, a far simpler method involves selecting one of the fixed controllers and attempting to verify stability by determining an appropriate common Lyapunov function. In this section, two specific choices of nominal controller are discussed, where the approach for verifying stability is more formulaic, and the state dimensions of the final controller can be reduced further. These special cases result when the nominal controller is selected as a state feedback/estimator controller or as a simple static output feedback controller.

5.1 State Estimate/State Feedback Controller. A common choice when implementing Youla-based controller interpolation schemes is to use state estimate/state feedback controllers. If the nominal controller is selected as such, the result is similar to that presented in Ref. [32], where the authors use a Youla-based LPV controller, and self-schedule the \( Q \)-parameter to optimize \( L_2 \)-gain performance. However, instead of focusing on LPV controller synthesis, we will examine this choice of nominal control from the perspective of ensuring local controller recovery.

Assuming a state estimate/feedback controller \( K_{\sigma}(\sigma) \) of the form

\[
\dot{x}_{\sigma} = [A_{p}(\sigma) + B_{p}(\sigma) F_{p}(\sigma) + H_{p}(\sigma) C_{p}(\sigma)] x_{\sigma} - H_{p}(\sigma) z_{\sigma}
\]

\[
u = F_{p}(\sigma) x_{\sigma}
\]

where the state feedback and observer gains are calculated as

\[
A_{p}(\sigma) P_{\sigma} + P_{\sigma} A_{p}(\sigma)^T + B_{p}(\sigma) X(\sigma) + X(\sigma)^T B_{p}(\sigma)^T < 0 \quad (48)
\]

\[
P_{\sigma} A_{p}(\sigma) + A_{p}(\sigma)^T P_{\sigma} + W(\sigma) C_{\sigma}(\sigma) + C_{\sigma}(\sigma)^T W(\sigma)^T < 0 \quad (49)
\]

As before, for polytopic LPV systems, these conditions can be written as a finite set of LMIs, and a feasible solution to the LMIs is necessary to guarantee stability.

For controllers of this form, a doubly coprime factorization satisfying the Bezout identities for the LPV plant and nominal LPV controller, \( K_{\sigma}(\sigma) \), can be constructed as

\[
\begin{bmatrix} M(\sigma) & U_0(\sigma) \\ N(\sigma) & V_0(\sigma) \end{bmatrix} = \begin{bmatrix} A_{p}(\sigma) + B_{p}(\sigma) F_{p}(\sigma) & B_{p}(\sigma) & - H_{p}(\sigma) \\ F_{p}(\sigma) & I & 0 \\ C_{p}(\sigma) & 0 & I \end{bmatrix} \quad (50)
\]

and

\[
\begin{bmatrix} \bar{V}_0(\sigma) - \bar{U}_0(\sigma) \\ \bar{N}(\sigma) - \bar{M}(\sigma) \end{bmatrix} = \begin{bmatrix} A_{p}(\sigma) + H_{p}(\sigma) C_{p}(\sigma) & - B_{p}(\sigma) & H_{p}(\sigma) \\ F_{p}(\sigma) & I & 0 \\ C_{p}(\sigma) & 0 & I \end{bmatrix} \quad (51)
\]

With only the state estimate/feedback controller, the resulting closed-loop LPV system is guaranteed to be quadratically stable by construction. But to recover the local controller behavior at each operating condition, we define the coprime factors at the \( i \)th operating condition as given previously in Eqs. (20) and (21), and the individual Youla parameters as
The system \( J_k(\theta) \) is constructed simply as
\[
J_k(\theta) = \begin{bmatrix}
U_0(\theta) V_0(\theta)^{-1} & -\bar{V}_0(\theta)^{-1} \\
V_0(\theta)^{-1} & V_0(\theta)^{-1} N(\theta)^{-1}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
A_p(\theta) + B_p(\theta) D_{\text{col}}(\theta) C_p(\theta) & -H_p(\theta) B_p(\theta) \\
F_p(\theta) & -C_p(\theta)
\end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
\]
(53)

5.2 Static Output Feedback. Another technique for reducing the dimension of the interpolated controller is to select the nominal controller to be as simple as possible, namely, static output feedback control: \( K_0(s, \theta) = D_{\text{col}}(\theta) \). Although many nonlinear systems can be stabilized by static output feedback control (constant or scheduled with \( \theta \), the synthesis problem of solving Eq. (54) is nonconvex in general. Thus for the purposes of this approach proposed in this paper, the authors advocate selecting a static output feedback gain and verifying stability in place of attempting to synthesize a stabilizing gain.

\[
[A_p(\theta) + B_p(\theta) D_{\text{col}}(\theta) C_p(\theta)]^T P + P[A_p(\theta) + B_p(\theta) D_{\text{col}}(\theta) C_p(\theta)] < 0
\]
(54)

This choice of controller does, in fact, simplify the resulting interpolated controller. Assuming this choice for the nominal stabilizing controller, the associated coprime factorizations would be
\[
\begin{bmatrix} M(\theta) & U_0(\theta) \\ N(\theta) & V_0(\theta) \end{bmatrix} = \begin{bmatrix} A_p(\theta) + B_p(\theta) F_p(\theta) & B_p(\theta) \\ F_p(\theta) & C_p(\theta) \end{bmatrix} \begin{bmatrix} 0 & I \\ I & D_{\text{col}}(\theta) \end{bmatrix}
\]
(55)

The interconnection system is then given as
\[
J_k(\theta) = \begin{bmatrix}
U_0(\theta) V_0(\theta)^{-1} & -\bar{V}_0(\theta)^{-1} \\
V_0(\theta)^{-1} & V_0(\theta)^{-1} N(\theta)^{-1}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
A_p(\theta) + B_p(\theta) D_{\text{col}}(\theta) C_p(\theta) & -B_p(\theta) B_p(\theta) D_{\text{col}}(\theta) \\
F_p(\theta) - D_{\text{col}}(\theta) C_p(\theta) & C_p(\theta)
\end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
\]
(56)

Again, we define the coprime factors at the \( i \)th operating condition as given previously in Eqs. (20) and (21), and the individual Youla parameters as
\[
Q_i = -\bar{V}_i - \bar{U}_i
\]
(57)

6 Gain-Scheduled Control of a Quadruple Tank System
To demonstrate efficacy of the gain-scheduling framework, a simulated quadruple tank system is selected. This system is a well-known multivariable control example and has been discussed.
in detail in Ref. [5]. A schematic of the system is shown in Fig. 5(a). The two inputs to the system are the pump inputs, and the two outputs of interest are the fluid levels in tanks 1 and 2. Two valves divide the flow from each of the pumps to the upper and lower tanks. The cross flow from the pumps to the upper/lower tanks as dictated by the valves creates a rich dynamic behavior. Although the dynamics will change with tank height, the valve positions are selected as the scheduling variables, as these will be used to dictate whether the system dynamics are minimum phase or nonminimum phase. Thus changes in valve position will change the underlying system dynamics and act as a disturbance to the closed-loop system attempting to regulate the fluid height of the lower tanks. Two operating points in a quasi-LPV model [5] are selected: One is in the minimum-phase region and the other in the nonminimum-phase region (Fig. 5(b)).

This system can be modeled simply using mass balances and Bernoulli’s law for orifice flow. The resulting nonlinear model is given in Eq. (59) where \( A \) is the tank cross-sectional area, \( h \) is the fluid level, and \( u \) is the pump input with a scalar gain \( k_u \). The valve parameters \( \gamma \in [0,1] \) determine the flow to each tank. The selected outputs are the fluid levels of tanks 1 and 2 and are measured with a scalar gain \( k_y \).

\[
A_1 \dot{h}_1 = -a_1 \sqrt{2gh_1} + a_3 \sqrt{2gh_3} + \gamma_1 k_u u_1 \\
A_2 \dot{h}_2 = -a_2 \sqrt{2gh_2} + a_4 \sqrt{2gh_4} + \gamma_2 k_u u_2 \\
A_3 \dot{h}_3 = -a_3 \sqrt{2gh_3} + (1 - \gamma_3) k_u u_2 \\
A_4 \dot{h}_4 = -a_4 \sqrt{2gh_4} + (1 - \gamma_4) k_u u_1 
\]

(59)

For control design purposes, a quasi-LPV representation of the dynamics can be constructed from the nonlinear model as shown in Eq. (60), with equilibrium defined by Eq. (61).

\[
\begin{bmatrix}
\dot{h}_1 \\
\dot{h}_2 \\
\dot{h}_3 \\
\dot{h}_4 \\
\end{bmatrix} = 
\begin{bmatrix}
-\left(\frac{a_1}{A_1}\right) \sqrt{\frac{2g}{h_1}} & 0 & \left(\frac{a_3}{A_3}\right) \sqrt{\frac{2g}{h_3}} & 0 \\
0 & -\left(\frac{a_2}{A_2}\right) \sqrt{\frac{2g}{h_2}} & 0 & \left(\frac{a_4}{A_4}\right) \sqrt{\frac{2g}{h_4}} \\
0 & 0 & -\left(\frac{a_3}{A_3}\right) \sqrt{\frac{2g}{h_3}} & 0 \\
0 & 0 & 0 & -\left(\frac{a_4}{A_4}\right) \sqrt{\frac{2g}{h_4}} \\
\end{bmatrix}
\begin{bmatrix}
h_1 \\
h_2 \\
h_3 \\
h_4 \\
\end{bmatrix} + 
\begin{bmatrix}
\gamma_1 k_y u_1 \\
\gamma_2 k_y u_2 \\
0 \\
(1 - \gamma_3) k_y u_2 \\
(1 - \gamma_4) k_y u_1 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix} 
\]

(60)

\[
\begin{bmatrix}
u_1^0 \\
u_2^0 \\
\end{bmatrix} = \frac{1}{k_y (1 - \gamma_1 - \gamma_2)} \begin{bmatrix}
1 - \gamma_2 \\
1 - \gamma_1 \\
\end{bmatrix}
\]

(61)

A slight variation of the parameter values published in Ref. [5] is used for the simulations presented here. The values of tank and orifice areas, and input/output scaling are given in Table 1. The gravity is given as 9.81 m/s², and the steady state values at the chosen operating conditions are given in Table 2.

For the minimum-phase operating point, a set of proportional-integral (PI) controllers are used (Eq. (59)). As advocated in Ref. [5], for the nonminimum-phase operating point an \( \mathcal{H}_\infty \) controller is designed using standard procedures (Eq. (60)). These controllers are not necessarily selected for optimal performance, but to demonstrate the full capabilities of the proposed interpolation approach. Not only do the two controllers have different state dimensions, but they are also designed for fundamentally different plant dynamics, and the PI controller has pure integrators that prevent the controller from being strictly stable. Both controllers perform adequately around their respective design points, and easily track reference changes in the desired fluid height (Fig. 6). Although the closed-loop system around the second operating condition displays significant undershoot, this is to be expected given the strong nonminimum-phase nature of the plant.

<table>
<thead>
<tr>
<th>Table 1 Model parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
</tr>
<tr>
<td>( A_2 )</td>
</tr>
<tr>
<td>( A_3 )</td>
</tr>
<tr>
<td>( A_4 )</td>
</tr>
<tr>
<td>( a_1 )</td>
</tr>
<tr>
<td>( a_2 )</td>
</tr>
<tr>
<td>( a_3 )</td>
</tr>
<tr>
<td>( a_4 )</td>
</tr>
<tr>
<td>( k_y )</td>
</tr>
<tr>
<td>( k_u )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2 Chosen operating conditions (minimum and nonminimum phases)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_1^0 )</td>
</tr>
<tr>
<td>( h_2^0 )</td>
</tr>
<tr>
<td>( h_3^0 )</td>
</tr>
<tr>
<td>( h_4^0 )</td>
</tr>
<tr>
<td>( u_1^0 )</td>
</tr>
<tr>
<td>( u_2^0 )</td>
</tr>
<tr>
<td>( \gamma_1 )</td>
</tr>
<tr>
<td>( \gamma_2 )</td>
</tr>
</tbody>
</table>
These controllers, however, are not effective at controlling the system at off-design conditions, and, in fact, are destabilizing. The closed-loop system poles for the four possible combinations of plant/controller are given in Table 3.

A principal advantage to the LQN framework for controller interpolation is to interpolate controllers of different sizes and structures, or unstable controllers. As described in Secs. 4 and 5, a nominal LPV observer-based controller (Eqs. (47)–(49)) is designed for the system, and Q parameters are calculated such that the original MIMO-PI and $\mathcal{H}_\infty$ controller are recovered at the associated design points, blended by an exponential weighting function. As these controllers were designed using a linearized model, the output of the interpolated controller is added to the nominal control effort $u_{1,2}$, which defines the equilibrium point assumed in the design procedure. The resulting controller is applied to the nonlinear model of the system dynamics, and evaluated in simulation. The interpolated controller retains the abilities of the local controller designs, and is capable of rejecting disturbances and regulating the lower tank fluid heights to the desired levels at both the nonminimum-phase and minimum-phase design conditions. For example, Fig. 7 shows the closed-loop system response to instantaneous disturbances applied to individual tank fluid heights. More importantly, the interpolated controller can transition smoothly and stably from one design point to another.

Table 3  Closed-loop system poles

<table>
<thead>
<tr>
<th>Type of Controller</th>
<th>Closed-loop poles</th>
</tr>
</thead>
<tbody>
<tr>
<td>PI controller with minimum-phase plant:</td>
<td>$-0.095, -0.033, -0.040 \pm 0.032j, -0.022 \pm 0.012j$</td>
</tr>
<tr>
<td>PI controller with nonminimum-phase plant:</td>
<td>$0.016, -0.023, -0.012 \pm 0.035j, -0.034 \pm 0.009j$</td>
</tr>
<tr>
<td>$\mathcal{H}_\infty$ controller with nonminimum-phase plant:</td>
<td>$-0.087, -0.009, -0.037 \pm 0.067j, -0.018 \pm 0.025j, -0.017 \pm 0.001j$</td>
</tr>
<tr>
<td>$\mathcal{H}_\infty$ controller with minimum-phase plant:</td>
<td>$0.074, -0.220, -0.009, -0.014, -0.056 \pm 0.026j, -0.029 \pm 0.019j$</td>
</tr>
</tbody>
</table>

Fig. 6  Step response of PI and $\mathcal{H}_\infty$ controlled systems at associated design points

$$K_{PID}(s) = \begin{bmatrix} 1.26 + \frac{0.042}{s} & 0 \\ 0 & 1.29 + \frac{0.029}{s} \end{bmatrix}$$ (62)

$$K_{H_\infty}(s) = \begin{bmatrix} -0.00125 & -0.0061 & 0.01279 & -0.00201 & -0.1116 & 0.0143 \\ -0.00037 & -0.0181 & 0.01026 & -0.00557 & -0.0218 & 0.1110 \\ 0.00878 & 0.01859 & -0.16460 & 0.15750 & 0.6230 & -0.7261 \\ 0.00207 & 0.00549 & -0.15760 & -0.00086 & 0.0450 & -0.0456 \\ -0.02317 & -0.11190 & 0.68900 & -0.05296 & -0.1998 & 0.6442 \\ 0.11010 & 0.01625 & -0.66390 & 0.03595 & -0.7687 & -0.2333 \end{bmatrix}$$ (63)
Fig. 7 Disturbance rejection at design conditions

Fig. 8 Tracking during transition between design conditions

Fig. 9 Tracking during transition between design and off-design conditions
Figure 8 shows the system response to rapid changes in $\gamma_1$ and $\gamma_2$, which both induce disturbances on the system and change the underlying system dynamics from minimum phase to nonminimum phase. As the scheduling variables change, the exponential weighting factors allow smooth transitioning between the two $Q$ functions. The control input voltages remain within reasonable bounds, and fluid heights in the two lower tanks are effectively regulated. Finally, Fig. 9 demonstrates that the interpolated controller performs adequately at off-design conditions as well, with some degradation in the tracking error. This simulation also illustrates that while the interpolation scheme is guaranteed stable for infinitely fast transitions, slower changes in the scheduling variable require less control effort.

7 Conclusions

This paper presents a method for stable controller interpolation for LPV systems using the Youla parametrization. The existence of a quadratic common $H_2$ Lyapunov function given by construction, guaranteeing stability of the closed-loop system despite arbitrarily fast transitions in the scheduling variable. A particular state transformation is used to allow the interpolated $Q$-system to share state variables, significantly reducing the number of states required for controller implementation. The approach has the advantage that controllers of different sizes and structures can be interpolated smoothly and stably, with the performance of the local controllers recovered exactly at critical operating points. The efficacy of the method is demonstrated using a multi-input, multi-output, nonminimum-phase system.

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