SZLENK INDEX, UPPER ESTIMATES, AND EMBEDDING IN BANACH SPACES

A DISSERTATION

by

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ABSTRACT

We investigate the relationship between two notions, one which refers to a coordinate system and one which does not, of asymptotic domination by subsequences of a fixed basis. We use this relationship to prove the existence of a universal space with a coordinate system satisfying this asymptotic domination condition. Last, we relate this asymptotic domination notion to the Szlenk index and prove a result concerning the existence of a universal space for classes determined by Szlenk index. Each of these results also has a corresponding result for reflexive spaces.

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1. INTRODUCTION

1.1 History

In Banach space theory, coordinate systems play a very important role. In particular, Schauder bases and finite dimensional decompositions (abbreviated FDD) are of particular interest. This is because these coordinate systems, unlike Hamel bases, for example, are connected to the norm and the topology of the space. But Enflo's famous example of a Banach space failing the approximation property [8] is also a Banach space failing to have either a Schauder basis or a finite dimensional decomposition. For this reason, one often wishes to determine when a given Banach space can be realized as a subspace or a quotient of a space with a Schauder basis or FDD. For example, any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0,1]$, the continuous, scalar-valued functions on the unit interval $[0,1]$. Moreover, any separable Banach space is isometrically isomorphic to a quotient of ℓ_1, the space of absolutely summable scalar sequences. Both of these spaces have bases, so we know that any separable Banach space is isometrically a subspace and a quotient of a Banach space with a basis. These very general results, however, preserve very little information about the original Banach space. We come to one of the basic types of questions which we will address in this paper: Given a suitably nice Banach space which possesses some additional, coordinate free property P, does there exist a Banach space with a suitably nice coordinate system that possesses a property Q which is related to P, but which refers to the coordinate system? For us, the suitably nice Banach spaces will either be Banach spaces with separable dual, the class of which we denote **SD**, or the class of separable, reflexive Banach spaces, which we denote **REFL**. The suitably nice coordinate systems will be either a shrinking finite dimensional decomposition if the original space was only assumed to have a separable dual, and a shrinking and boundedly complete finite dimensional decomposition if the original space was separable and reflexive. Throughout, this process of witnessing our Banach space as a subspace or a quotient of a Banach space with an FDD will be referred to as "coordinatization."

One question of this type was answered by Zippin [29], without the assumption of the additional property P. Zippin proved that if X is a Banach space with separable dual, then there exists a Banach space Z with shrinking basis so that X is isometric to a subspace of Z. Another question of this type, answered by Davis, Figiel, Johnson, and Pełczyński [7], is that if X is a Banach space with separable dual, then there exists a Banach space Y with shrinking basis so that X is isometric to a quotient of Y. Together with another result of Davis, Figiel, Johnson, and Pełczyński, the result of Zippin implies that for any separable, reflexive Banach space X, there exist reflexive spaces Y, Z with bases so that X is isometrically a quotient of Y and a subspace of Z. Another set of examples, which we generalize in this paper, are the examples of subsequential upper tree and block estimates. We define these notions in Chapter II. It was shown by Odell, Schlumprecht, and Zsák [24] that any separable, reflexive Banach space X which satisfies subsequential $T^*_{\alpha,c}$ lower tree estimates and subsequential $T_{\alpha,c}$ upper tree estimates, where $T_{\alpha,c}$ is the Tsirelson space of order α and parameter c, then X is isomorphic to a quotient of a reflexive space Y and to a subspace of a reflexive space Z, both of which have finite dimensional decompositions satisfying $T^*_{\alpha,c}$ lower block estimates and $T_{\alpha,c}$ upper block estimates. By relating these estimates to the Szlenk index, these authors also showed that if the Szlenk index of X and the Szlenk index of X^* are both bounded above by $\omega^{\alpha\omega}$ for a countable ordinal α, then X embeds into a reflexive Banach space Z so that the Szlenk index of Z and the Szlenk index of Z^* are both bounded above by $\omega^{\alpha\omega}$. A similar result, due to Freeman, Odell, Schlumprecht, and Zsák [9], establishes that a separable Banach space has Szlenk index not exceeding $\omega^{\alpha\omega}$ if and only if there exists $c \in (0,1)$ so that X is isomorphic to a subspace of a Banach space Y with shrinking FDD satisfying $T_{\alpha,c}$ upper block estimates and a quotient of a Banach space Z with shrinking FDD satisfying $T_{\alpha,c}$ upper block estimates.

These results, as well as several others, have been completed with the aid of weakly null and w^* null trees. The notions of trees and branches will both be defined in Chapter II. For example, Johnson and Zheng [15] have shown that a separable, reflexive Banach space embeds into a reflexive Banach space with unconditional FDD if and only if every normalized, weakly null tree has an unconditional branch. They later showed [16] that a Banach space with separable dual embeds into a Banach space with unconditional, shrinking FDD if and only if every normalized, w^* null tree in X^* has an unconditional branch. The hypothesis that every normalized, weakly null tree has a branch with a certain property bears a resemblance to the hypothesis that every normalized, weakly null sequence has a subsequence with a certain property. In fact, the second hypothesis is implied by the first, since every sequence naturally yields a tree the branches of which are the subsequences of the given sequence. But the utility of trees is emphasized by the following two examples: It was shown by Odell and Zheng [25] that there exists a separable Banach space such that every normalized, weakly null sequence admits an unconditional subsequence, but this space does not embed into a Banach space with unconditional basis. Johnson [13] showed that if X is a subspace of L_p, $1 < p < \infty$, then X embeds into an ℓ_p sum of finite dimensional space if and only if every normalized, weakly null sequence has a subsequence equivalent to the unit vector basis of ℓ_p. Odell and Schlumprecht [22] showed that this sequence/subsequence hypothesis is not sufficient in general by constructing a separable, reflexive Banach space Y so that for any $\varepsilon > 0$, every normalized, weakly null sequence has a subsequence $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p, but so that Y does not embed in any Banach space which is an ℓ_p sum of finite dimensional spaces. This result was an example in the same paper where the authors showed that if Y is a separable, reflexive Banach space so that every normalized, weakly null tree in Y has a branch equivalent to the unit vector basis of ℓ_p, then Y embeds into an ℓ_p sum of finite dimensional spaces. It was in this paper that Odell and Schlumprecht began to frame questions of trees and branches in terms of two player games between players S, for subspace, and V, for vector. These games have several small variations, but roughly, player S chooses a space Y_1 of finite codimension in X, player V chooses a vector x_1 in the unit sphere of Y_1, player S chooses a second finite codimensional space Y_2, and so on. Player S wins if the resulting sequence (x_n) lies in a predetermined target set, and player V wins otherwise. As we will see, framing embedding questions and questions concerning the tree/branch hypothesis can be quite fruitful.

If $\mathcal{C}$ is a class of Banach spaces, we say that Z is universal for the class $\mathcal{C}$ if any member of $\mathcal{C}$ embeds isomorphically into Z. We have already mentioned that $\mathcal{C}[0,1]$ is universal for the class of separable Banach spaces, and, it is worth remarking, that $\mathcal{C}[0,1]$ is actually a member of this class. This gives one question of the form: If $\mathcal{C}$ is a class of Banach spaces, can we find a member of $\mathcal{C}$ which is universal for this class? If we cannot take it to be a member of $\mathcal{C}$, can we take the universal space to be a member of a class somehow related to $\mathcal{C}$? In the next paragraph, we will discuss the Szlenk index, answer some of these questions for the classes **REFL** and **SD**, and discuss related questions which will be a main result of this paper to answer. Other noteworthy examples of universal spaces are those due to Pełczyński and Schechtman. Pełczyński [26] proved the existence of a Banach space X_P with a basis so that if X is any Banach space with a basis, then X embeds into X_P so that the basis of X

is sent to a subsequence of the basis of X_P which spans a complemented subspace. Pełczyński also proved the existence of a Banach space X_u with an unconditional basis so that if X is any Banach space with unconditional basis, then X embeds into X_u so that the basis of X is sent to a subsequence of the basis of X_u which spans a complemented subspace. Last, and importantly for us to prove the existence of our universal spaces, is a space W constructed by Schechtman [27] which has a finite dimensional decomposition $F = (F_n)$ so that if X is any Banach space with finite dimensional decomposition, say $E = (E_n)$, then X embeds into W so that there exist natural numbers $k_1 < k_2 < \ldots$ so that the embedding takes E_n to F_{k_n} and so that the image of X under the embedding is complemented in W.

Ordinal indices have also been used fruitfully since the inception of Banach space theory. Our favorite index here will be the Szlenk index. Szlenk [28] originally constructed this index to prove the non-existence of a separable, reflexive Banach space universal for this class. Roughly speaking, for a separable Banach space, the Szlenk index measures the "degree" of separability of the dual space. To that end, the Szlenk index of a separable Banach space X is countable if and only if the dual X^* is separable. Szlenk also showed that there exist separable, reflexive spaces with arbitrarily high countable Szlenk index, and that if X, Y are Banach spaces so that Y embeds isomorphically into X, the Szlenk index of Y cannot exceed the Szlenk index of X. From these three properties of the Szlenk index, one can easily deduce that there does not exist a separable, reflexive Banach space universal for this class. Any such space would necessarily have countable Szlenk index. One can then find a separable, reflexive space with larger Szlenk index, which would necessarily embed into the universal space, which contradicts the third property of the Szlenk index mentioned above. The same argument proves that there does not exist a Banach space with separable universal for this class. Bourgain [4] introduced an index measuring the complexity of finite sequences in a given Banach space which are equivalent to the spline basis of $\mathcal{C}[0,1]$. Using a standard "overspill" argument, this index will be uncountable for a separable Banach space if and only if that space contains an infinite sequence equivalent to the spline basis. Equivalently, the index is uncountable if and only if the given space contains a copy of $\mathcal{C}[0,1]$. This argument, together with the fact that there exist separable, reflexive Banach spaces for which the previously mentioned index introduced by Bourgain takes arbitrarily high countable values, proves that any separable Banach space which is universal for the class of separable, reflexive Banach spaces must actually contain a copy of $\mathcal{C}[0,1]$, and therefore be universal for the class of separable Banach spaces. Another index, one which we will discuss in Chapter II, is the Bourgain ℓ_1 index. Again, this index measures the complexity of finite dimensional sequences in X which are equivalent to finite ℓ_1 bases. Preservation of these ℓ_1 and various other types of ℓ_1 structures will be a focal point of Chapter II. At the confluence of our discussion of universal spaces and ordinal indices will be the sets $\mathcal{C}_\alpha$ and $\mathcal{CR}_\alpha$. Here, $\mathcal{C}_\alpha$ consists of all Banach spaces from **SD** having Szlenk index not exceeding ω^α. The class $\mathcal{CR}_\alpha$ will consist of all **REFL** spaces X so that the Szlenk index of X and the Szlenk index of X^* are both bounded above by ω^α. A major result will be to prove the existence of $Y \in \mathcal{C}_{\alpha+1}$ universal for $\mathcal{C}_\alpha$ and the existence of $Z \in \mathcal{CR}_{\alpha+1}$ universal for $\mathcal{CR}_\alpha$.

1.2 Layout and results

In Chapter II, we discuss trees and branches. We introduce several important trees which will be used to measure complexity throughout the paper. We define prunings and prove results about duality of weakly null and w^* null trees using these prunings, as well as to characterize the Szlenk index for separable Banach spaces not containing ℓ_1. We also use these trees and others to define four different notions of ℓ_1 structure in Banach spaces. We then prove results about constant reduction and discuss a larger framework into which these results fit. We also prove several three space properties for each of these structures. We show how the constant reduction problem is related to certain distortion indices, which we also define.

In Chapter III, we define the necessary Banach space terminology required to relay our coordinatization and universality results. We also introduce the rules of our game and prove the main theorems. The main coordinatization theorems are as follows.

Theorem 1.1. *Let U be a Banach space with normalized, 1-unconditional, shrinking, right dominant basis (u_n) satisfying subsequential U upper block estimates in U. For $X \in \mathbf{SD}$, the following are equivalent.*

(i) X satisfies subsequential U upper tree estimates.

(ii) There exists a Banach space Y with shrinking FDD E which satisfies subsequential U upper block estimates in Y so that X is isomorphic to a closed subspace of Y.

(iii) There exists a Banach space Z with shrinking FDD F which satisfies subsequential U upper block estimates in Z so that X is isomorphic to a quotient of Z.

Theorem 1.2. *Suppose U, V are reflexive Banach spaces with normalized, 1-unconditional bases $(u_n), (v_n)$, respectively, so that (u_n) is right dominant and satisfies subsequential U upper block estimates in U, (v_n) is left dominant and satisfies subsequential V lower block estimates in V, and so that every normalized block of (v_n) is dominated by every normalized block of (u_n). Then for $X \in \mathbf{REFL}$, the following are equivalent.*

(i) *X satisfies subsequential V lower tree estimates and subsequential U upper tree estimates.*

(ii) *X is isomorphic to a subspace of a reflexive Banach space Y with FDD E satisfying subsequential V lower and subsequential U upper block estimates in Y.*

(iii) *X is isomorphic to a quotient of a reflexive Banach space Z with FDD F satisfying subsequential V lower and subsequential U upper block estimates in Z.*

Later in Chapter II, for each countable ordinal α, the Schreier space of order α, X_α, will be defined. We will also construct for each $1 \leq p \leq \infty$ the generalized Baernstein space of order α and parameter p, $X_{\alpha,p}$. These spaces will be the bridge between tree estimates and Szlenk index for us. The main results concerning this connection are as follows.

Theorem 1.3. *Let X be a separable Banach space and let α be a countable ordinal. If the Szlenk index of X does not exceed ω^α, then X satisfies subsequential X_α upper tree estimates. If, in addition to this, X is reflexive and the Szlenk index of X^* also does not exceed ω^α, then for any $1 < p \leq 2$, X satisfies subsequential $X^*_{\alpha,p}$ lower tree estimates and $X_{\alpha,p}$ upper tree estimates.*

Finally, the main universality results are as follows.

Theorem 1.4. *(i) If U is as in Theorem 1.1, then there exists a Banach space Y with shrinking FDD E satisfying subsequential U upper block estimates in Y such that if $X \in$ **SD** satisfies subsequential U upper tree estimates, then X embeds into Y.*

*(ii) If U, V are as in Theorem 1.2, then there exists a reflexive Banach space Z with FDD F satisfying subsequential V lower and subsequential U upper block estimates in Z such that if $X \in$ **REFL** satisfies subsequential V lower and subsequential U upper tree estimates, then X embeds into Z.*

Combining this theorem with facts from [14] and [24], we immediately deduce the follow.

Corollary 1.5. *(i) If α is a countable ordinal, then there exists $W \in \mathcal{C}_{\alpha+1}$ having a basis such that W is universal for $\mathcal{C}_\alpha$.*

(ii) If α is a countable ordinal, then there exists $W_0 \in \mathcal{CR}_{\alpha+1}$ having a basis such that W_0 is universal for $\mathcal{CR}_\alpha$.

2. TREES AND BRANCHES

In this chapter we discuss ordinal indices and the use of trees to compute these indices.

2.1 Trees, definitions, and notation

If S is any non-empty set, we let $[S]^{<\omega}$, $[S]$ denote the finite and infinite subsets of S, respectively. We let $S^{<\omega}$ and S^{ω} denote the finite and infinite sequences in S, respectively. We will identify elements of $[\mathbb{N}]^{<\omega}$ (resp. $[\mathbb{N}]$) with finite (resp. infinite) sequences of $\mathbb{N}$ listed in strictly increasing order. If $n, k \in \mathbb{N}$ with $k \leq n$ and $s = (s_1, \ldots, s_n) \in S^{<\omega}$, we let $s|_k = (s_1, \ldots, s_k)$, with a similar convention if $s \in S^{\omega}$. If $s = (s_1, \ldots, s_n)$, we let $|s| = n$ and refer to this as the *length* of s. By a *tree* on S, we will mean a subset T of $S^{<\omega}$ which is closed under taking initial segments. That is, if $s \in T$, $s|_k \in T$ for $1 \leq k \leq |s|$. We will call a tree *hereditary* if it contains all subsequences of its elements. We put a partial order, denoted $\preceq$, on $[S]^{<\omega}$, so that $s \preceq t$ if and only if s is an initial segment of t. That is, $s \preceq t$ if and only if $|s| \leq |t|$ and $s = t|_{|s|}$. If $s \preceq t$ or $t \preceq s$, we will say s and t are *comparable.* A *branch* of T will be a maximal linearly ordered subset of T. A *B-tree* on S will be a subset T of $S^{<\omega} \setminus \{\varnothing\}$ so that $\{\varnothing\} \cup T$ is a tree on S. If T is a tree on S, we will let $\widehat{T} = T \setminus \{\varnothing\}$ denote the B-tree associated to T. By convention, we will say that the empty set is both a tree and a B-tree (on any S). We also note that the intersection of trees on S is again a tree on S.

We next define the derived trees of T, denoted $(T^{\alpha})_{0 \leq \alpha < \omega_1}$. It makes sense to define the derived trees for uncountable ordinals, but in all applications below, we need only countably many derived trees. If T is a tree or a B-tree on S, we can define $T' = T \setminus MAX(T)$, where $MAX(T)$ is the set of maximal elements of T with respect to the order $\preceq$. Note that if T is a tree or a B-tree, T' is as well. We then define

$$T^0 = T,$$

$$T^{\alpha+1} = (T^\alpha)', \quad 0 \leq \alpha < \omega_1,$$

$$T^\alpha = \bigcap_{\beta<\alpha} T^\beta, \quad \alpha < \omega_1 \text{ is a limit ordinal.}$$

We then define the *order* of the tree T (resp. B-tree) to be $o(T) = \min\{\alpha : T^\alpha = \varnothing\}$ if this set of ordinals is non-empty, and $o(T) = \omega_1$ if there is no such α. The purpose of introducing trees is to compute the complexity of strucutures within our Banach spaces, where complexity is measured by the order of a tree.

If T, T_0 are trees, we say $\phi : T \to T_0$ is a *tree isomorphism* if ϕ is a bijection so that $s \preceq t$ if and only if $\phi(s) \preceq \phi(t)$. We say ϕ is an *isomorphic embedding* of T into T_0 if $\phi(T)$ is a tree and $\phi : T \to \phi(T)$ is a tree isomorphism. These notions have obvious analogies for B-trees.

In the case of the natural numbers, we let $\min \varnothing = \omega$, $\max \varnothing = 0$. For $E, F \in [\mathbb{N}]^{<\omega}$, we say $E < F$ if $\max E < \min F$. We write $n \leq F$ if $n \leq \min F$. If (E_i) is a (finite or infinite) sequence sets in $[\mathbb{N}]^{<\omega}$, we say this sequence is *successive* if $E_1 < E_2 < \ldots$. If $(m_i), (n_i) \in [\mathbb{N}]^{<\omega}$ or $[\mathbb{N}]$ have the same (finite or infinite) length so that $m_i \leq n_i$ for each i, we say (n_i) is a *spread* of (m_i). We say $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *spreading* if it contains all spreads of its members. If $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ contains all subsets of its members, we say $\mathcal{F}$ is *hereditary*. We let $\mathfrak{S}$ denote the set of all spreading, hereditary subsets of $[\mathbb{N}]^{<\omega}$. If $E < F$, we let $E^\frown F = E \cup F$. We emphasize the fact that this symbol is reserved only for the case that $E < F$. It will be convenient, although not necessary for any proofs, to assume that if $E \in \mathcal{F}'$ and $n = 1 + \max E$, $E^\frown n \in \mathcal{F}$. In all applications below, this will be true, so we adopt this assumption throughout. If $\mathcal{F}$ is a tree or B tree and $(x_E)_{E\in\mathcal{F}} \subset X$, we can treat this set $(x_E)_{E\in\mathcal{F}}$ as a tree with branches

$$\{(x_{E_1},\ldots,x_{E_n}) : (E_1,\ldots,E_n) \text{ is a branch of } \mathcal{F}\}.$$

If $\mathcal{F}$ is a B-tree, X is a Banach space, and $(x_E)_{E\in\mathcal{F}}$ is so that for each $E \in \mathcal{F}'$, $(x_{E\char`^n})_{E<n}$ is a weakly null sequence, we say $(x_E)_{E\in\mathcal{F}}$ is a *weakly null tree* (despite the fact that the structure may be only a a B-tree in X). We similarly define normalized trees, w^* null trees, etc.

If $E \in [\mathbb{N}]^{<\omega}$, $(m_n) = M \in [\mathbb{N}]$, we let $M(E) = (m_n : n \in E)$. If $\mathcal{F} \in [\mathbb{N}]^{<\omega}$, we let $\mathcal{F}(M) = \{M(E) : E \in \mathcal{F}\}$. If $\mathcal{F},\mathcal{G} \subset [\mathbb{N}]^{<\omega}$, we let

$$\mathcal{F} \oplus \mathcal{G} = \Big\{E\char`^F : E \in \mathcal{F}, F \in \mathcal{G}\Big\}$$

and

$$\mathcal{F}[\mathcal{G}] = \Big\{\bigcup_{i=1}^{n} E_i : n \in \mathbb{N}, E_1 < \ldots < E_n, E_i \in \mathcal{G}, (\min E_i)_{i=1}^n \in \mathcal{F}\Big\}.$$

We note that $(\mathcal{F},\mathcal{G}) \mapsto \mathcal{F} \oplus \mathcal{G}$, $(\mathcal{F},\mathcal{G}) \mapsto \mathcal{F}[\mathcal{G}]$ are associative operations from $\mathfrak{S}^2$ into $\mathfrak{S}$. These operations have the effect of adding and multiplying the orders of the associated B-trees of elements of $\mathfrak{S}$. That is, if $\mathcal{F},\mathcal{G} \in \mathfrak{S}$, $o(\widehat{\mathcal{F}\oplus\mathcal{G}}) = o(\widehat{\mathcal{G}}) + o(\widehat{\mathcal{F}})$ and $o(\widehat{\mathcal{F}[\mathcal{G}]}) = o(\widehat{\mathcal{G}})o(\widehat{\mathcal{F}})$.

Next, for each countable ordinal $\alpha \geq 0$, we define families $\mathcal{F}_\alpha$ and $\mathcal{S}_\alpha$, all of which lie in $\mathfrak{S}$. These families have easily computed order, and so will see much use as index sets for trees in our Banach spaces. The families $(\mathcal{F}_\alpha)_{0\leq\alpha<\omega_1}$ are called the *fine Schreier families*, and the families $(\mathcal{S}_\alpha)_{0\leq\alpha<\omega_1}$ are called the *Schreier families* [1]. We let

$$\mathcal{F}_0 = \{\varnothing\},$$

$$\mathcal{F}_1 = \{(n) : n \in \mathbb{N}\} \cup \{\varnothing\},$$

$$\mathcal{F}_{\alpha+1} = \{n^\frown E : n < E\} \cup \{\varnothing\} = \mathcal{F}_1 \oplus \mathcal{F}_\alpha, \quad \alpha < \omega_1.$$

If $\alpha < \omega_1$ is a limit ordinal and $\mathcal{F}_\beta$ has been defined for each $\beta < \alpha$, we choose a sequence of successors $\alpha_n \uparrow \alpha$ and define

$$\mathcal{F}_\alpha = \{E : \exists n \leq E \in \mathcal{F}_{\alpha_n}\} \cup \{\varnothing\}.$$

We next let

$$\mathcal{S}_0 = \{\varnothing\} \cup \big\{(n) : n \in \mathbb{N}\big\},$$

$$\mathcal{S}_1 = \big\{E : |E| \leq E\big\},$$

$$\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha], \quad \alpha < \omega_1.$$

If $\alpha < \omega_1$ is a limit ordinal and $\mathcal{S}_\beta$ has been defined for each $\beta < \alpha$, we choose a sequence of successors $\alpha_n \uparrow \alpha$ and let

$$\mathcal{S}_\alpha = \{E : \exists n \leq E \in \mathcal{S}_{\alpha_n}\}.$$

We note that these families depend on the choices of sequences we make at limit ordinals. Regardless of this choice, $\mathcal{F}_\alpha, \mathcal{S}_\alpha \in \mathfrak{S}$ for each $0 \leq \alpha < \omega_1$. It follows from easy induction arguments that we can make these choices to have the following properties.

Proposition 2.1. *For each countable limit ordinal α, we can choose the sequences $\beta_n + 1 = \alpha_n \uparrow \alpha$ in the construction of the Schreier and fine Schreier families so that for each $n \in \mathbb{N}$,*

(i) $\mathcal{F}_{\alpha_n} \subset \mathcal{F}_{\beta_{n+1}}$,

(ii) $\mathcal{S}_{\alpha_n} \subset \mathcal{S}_{\beta_{n+1}}$.

Neither of these properties is necessary for our applications, but they greatly simplify the proofs.

The fineness of the fine Schreier families is to our advantage during inductive constructions, since the family $\mathcal{F}_{\alpha+1}$ is only slightly different from $\mathcal{F}_\alpha$. This fineness is to our detriment when using these families to classify complexity, since typically the only significant changes in complexity occur at ordinals of the form ω^α. We note that $o(\mathcal{F}_\alpha) = \alpha + 1$ and $o(\widehat{\mathcal{F}_\alpha}) = \alpha$ for each $\alpha < \omega_1$. We also note that if we topologize $\mathcal{F}_\alpha$ or $\mathcal{S}_\alpha$ by identifying its members with their characteristic functions and considering these as elements of the Cantor set, then $\mathcal{F}_\alpha$ or $\mathcal{S}_\alpha$ is compact.

We will need the following facts about the fine Schreier families. We will use them repeatedly throughout the next sections. We note that [20] contains (i) and a result similar to (ii), (iii). We include proofs of (ii) and (iii), since the author is not aware of any proof in the literature.

Lemma 2.2. *(i) If $0 \le \alpha \le \beta < \omega_1$, there exists $n \in \mathbb{N}$ so that if $n \le E \in \mathcal{F}_\alpha$, then $E \in \mathcal{F}_\beta$.*

(ii) If $0 \le \alpha, \beta < \omega_1$, there exists $M \in [\mathbb{N}]$ so that $(\mathcal{F}_\alpha \oplus \mathcal{F}_\beta)(M) \subset \mathcal{F}_{\beta+\alpha}$.

(iii) If $1 \le \alpha, \beta < \omega_1$, there exists $M \in [\mathbb{N}]$ so that $\mathcal{F}_\alpha[\mathcal{F}_\beta](M) \subset \mathcal{F}_{\beta\cdot\alpha}$.

Proof. We prove both (ii) and (iii) by induction on α for a fixed β. We also note that if $\mathcal{A}, \mathcal{B} \subset [\mathbb{N}]^{<\omega}$ and $M \in [\mathbb{N}]$ are such that $\mathcal{B}$ is spreading and $\mathcal{A}(M) \subset \mathcal{B}$, then $\mathcal{A}(M') \subset \mathcal{B}$ for any spread M' of M. This implies that for any $N \in [\mathbb{N}]$, there exists $N' \in [N]$ so that $\mathcal{A}(N') \subset \mathcal{B}$, since $[N]$ contains a spread N' of M.

(ii) Note that $\mathcal{F}_0 \oplus \mathcal{F}_\beta = \mathcal{F}_\beta$, so we can take $M = \mathbb{N}$ in this case.

If $M \in [\mathbb{N}]$ is such that $(\mathcal{F}_\alpha \oplus \mathcal{F}_\beta)(M) \subset \mathcal{F}_{\beta+\alpha}$. Then

$$(\mathcal{F}_{\alpha+1} \oplus \mathcal{F}_\beta)(M) = (\mathcal{F}_1 \oplus \mathcal{F}_\alpha \oplus \mathcal{F}_\beta)(M) \subset \mathcal{F}_1 \oplus ((\mathcal{F}_\alpha \oplus \mathcal{F}_\beta)(M))$$
$$\subset \mathcal{F}_1 \oplus \mathcal{F}_{\beta+\alpha} = \mathcal{F}_{\beta+\alpha+1}.$$

The first inclusion above is easily checked. We see that M also works for $\alpha + 1$.

Suppose the result holds for each $\gamma < \alpha$, where α is a countable limit ordinal. Let $\alpha_n \uparrow \alpha$ be the ordinals used to define $\mathcal{F}_\alpha$. Note that $\beta + \alpha$ is also a limit. Take $\gamma_n \uparrow \beta + \alpha$ the ordinals used to define $\mathcal{F}_{\beta+\alpha}$. Choose a strictly increasing sequence of natural numbers n_k so that $\beta + \alpha_k < \gamma_{n_k}$. Choose ℓ_k strictly increasing natural numbers so that for all $k \in \mathbb{N}$, $n_k \leq \ell_k$ and $\ell_k \leq E \in \mathcal{F}_{\beta+\alpha_k}$ implies $E \in \mathcal{F}_{\gamma_{n_k}}$. We can do this by (i). Choose infinite sets $M_1 \supset M_2 \supset \ldots$ so that $\ell_k \leq M_k$ and so that $(\mathcal{F}_{\alpha_k} \oplus \mathcal{F}_\beta)(M_k) \subset \mathcal{F}_{\beta+\alpha_k}$. Note that by our choice of ℓ_k, this set is also contained in $\mathcal{F}_{\gamma_{n_k}} \cap [\ell_k, \infty)^{<\omega} \subset \mathcal{F}_{\gamma_{n_k}} \cap [n_k, \infty)^{<\omega} \subset \mathcal{F}_{\beta+\alpha}$. Choose $M = (m_k^k)$, where $M_k = (m_i^k)_i$. Then if $E \in \mathcal{F}_\alpha \oplus \mathcal{F}_\beta$, then $E = F^\frown G$ for some $F \in \mathcal{F}_\alpha$ and $G \in \mathcal{F}_\beta$. Then there exists $k \leq F$ so that $F \in \mathcal{F}_{\alpha_k}$. We deduce $k \leq E \in \mathcal{F}_{\alpha_k} \oplus \mathcal{F}_\beta$. Then $M(E)$ is a spread of $M_k(E)$, which lies in $\mathcal{F}_{\beta+\alpha}$ by choice of M_k.

(iii) Note that $\mathcal{F}_1[\mathcal{F}_\beta] = \mathcal{F}_\beta$, so we may take $M = \mathbb{N}$ in this case.

Suppose we have chosen M so that $\mathcal{F}_\alpha[\mathcal{F}_\beta](M) \subset \mathcal{F}_{\beta\alpha}$. Choose by (ii) some $N \in [\mathbb{N}]$ so that $(\mathcal{F}_\beta \oplus \mathcal{F}_{\beta\cdot\alpha})(N) \subset \mathcal{F}_{\beta\cdot\alpha+\beta}$. Let $N \circ M = (n_{m_k})_k$ so that $\mathcal{F}(N \circ M) = (\mathcal{F}(M))(N)$ for any $\mathcal{F} \in \mathfrak{S}$. Then

$$\begin{aligned}\mathcal{F}_{\alpha+1}[\mathcal{F}_\beta](N \circ M) &= (\mathcal{F}_\beta \oplus \mathcal{F}_\alpha[\mathcal{F}_\beta])(N \circ M) \\ &= \big((\mathcal{F}_\beta \oplus \mathcal{F}_\alpha[\mathcal{F}_\beta])(M)\big)(N) \subset \big(\mathcal{F}_\beta \oplus (\mathcal{F}_\alpha[\mathcal{F}_\beta](M))\big)(N) \\ &\subset (\mathcal{F}_\beta \oplus \mathcal{F}_{\beta\alpha})(N) \subset \mathcal{F}_{\beta\cdot\alpha+\beta} = \mathcal{F}_{\beta\cdot(\alpha+1)}.\end{aligned}$$

The limit ordinal case is very similar to that in (ii). Choose $\alpha_k \uparrow \alpha$, $\gamma_k \uparrow \beta \cdot \alpha$, and n_k strictly increasing natural numbers so that $\beta \cdot \alpha_k \leq \gamma_{n_k}$. Choose ℓ_k strictly increasing natural numbers so that for all $k \in \mathbb{N}$, $n_k \leq \ell_k$ and $\ell_k \leq E \in \mathcal{F}_{\beta \cdot \alpha_k}$ implies $E \in \mathcal{F}_{\gamma_{n_k}}$. Choose infinite sets $M_1 \supset M_2 \supset \ldots$ so that $\ell_k \leq M_k$ and $\mathcal{F}_{\alpha_k}[\mathcal{F}_\beta](M_k) \subset \mathcal{F}_{\beta \cdot \alpha_k}$, which again implies $\mathcal{F}_{\alpha_k}[\mathcal{F}_\beta](M_k) \subset \mathcal{F}_{\beta \cdot \alpha}$. Taking $M = (m_k^k)$, we deduce $\mathcal{F}_\alpha[\mathcal{F}_\beta](M) \subset \mathcal{F}_{\beta \cdot \alpha}$.

□

Before we begin the applications, we state a definition and a vital lemma which will be used repeatedly and give an application. This definition illustrates the usefulness of using spreading families as index sets.

Pruning If $\mathcal{F} \in \mathfrak{S}$, we say $\phi : \mathcal{F} \to \mathcal{F}$ is a *pruning* if

(i) $\phi(\varnothing) = \varnothing$,

(ii) for each $E \in \mathcal{F}'$, there exists a strictly increasing

$$\phi_E : (n : E < n) \to (n : \phi(E) < n)$$

so that $\phi(E^\frown n) = \phi(E)^\frown \phi_E(n)$.

Again, there is an analogous definition if $\mathcal{F}$ is a B-tree.

We describe the intuition behind this definition. Either E is maximal in $\mathcal{F}$ or it has a sequence of immediate successors of the form $(E^\frown n)_{n>E}$ for some n_0. Then if ϕ is a pruning, it has the effect of mapping the sequence of immediate successors of E to a subsequence of the immediate successors of $\phi(E)$. Thus if $(x_E)_{E \in \mathcal{F}} \subset X$ and $\phi : \mathcal{F} \to \mathcal{F}$ is a pruning, $(x_{\phi(E)})_{E \in \mathcal{F}}$ is obtained by passing to a subsequence $(n_k)_{k \in \mathbb{N}}$ of $(n)_{n \in \mathbb{N}}$, passing to a subsequence of each of the sequences of successors of (n_k), etc.

With this intuition, the following lemma is obvious. The idea is that if $(x_E)_{E\in\widehat{\mathcal{F}}} \subset X$ is such that for each E, the sequence $(x_{E^\wedge n})$ indexed by the sequence of immediate successors of E has a subsequence with some property (a property which is allowed to depend upon E and its immediate successors), then we can find a tree in X also indexed by $\widehat{\mathcal{F}}$ so that each sequence of immediate successors has the property which depends on their immediate predecessor (without having to pass to a subsequence).

Lemma 2.3. *Let S be a set, $\mathcal{F} \in \mathfrak{S}$, $(x_E)_{\widehat{\mathcal{F}}} \subset S$. For each $E \in \mathcal{F}'$, suppose $P_E \subset S^\omega$ is such that for each $E \in \mathcal{F}$ there exist $E < k_1 < k_2 < \ldots$ so that $(x_{E^\wedge k_n})_n \in P_E$. Then there exists a pruning $\phi : \mathcal{F} \to \mathcal{F}$ so that $(x_{\phi(E^\wedge n)}) \in P_{\phi(E)}$ for all $E \in \mathcal{F}$.*

Example Let (B, ρ) be a metric space, $x \in S$ a fixed element, and $(x_E)_{E\in\widehat{\mathcal{F}}} \subset B$ so that for each $E \in \mathcal{F}'$, $\lim_n x_{E^\wedge n} = x$. Let $\delta : \widehat{\mathcal{F}} \to (0,1)$ be any function. Then there exists a pruning $\phi : \mathcal{F} \to \mathcal{F}$ so that for each $E \in \mathcal{F}'$, $\rho(x, x_{\phi(E)}) < \delta(E)$. We will use this particular example when B is the unit ball of a Banach space which has separable dual, $x = 0$, and ρ is a metric which determines the weak topology on B.

Example Let X be a separable Banach space, $(x_E)_{E\in\widehat{\mathcal{F}}} \subset B_X$ be a weakly null tree so that $0 < \rho \leq \|x_E\|$ for all $E \in \widehat{\mathcal{F}}$. Then for any $\delta > 0$ and $\varepsilon_n > 0$, there exists $(y_E)_{E\in\widehat{\mathcal{F}}} \subset B_X$ and $(f_E)_{E\in\widehat{\mathcal{F}}} \subset B_{X^*}$ so that for $E, F \in \widehat{\mathcal{F}}$ comparable and not equal, $f_E(y_E) > \rho/2 - \delta$ and $|f_E(y_F)| < \min\{\varepsilon_{|E|}, \varepsilon_{|F|}\}$.

Sketch of proof. In the sketch of the proof, we repeatedly use the pruning lemma. When convenient, we will relabel between each application and assume that the previous trees had the property that the pruned tree possesses. Fix $T : X \to S_{X^*}$ so that $Tx(x) = \|x\|$. Let $g_E = Tx_E$. We first let P_E be the w^* Cauchy sequences in B_{X^*}. The hypotheses of Lemma 2.3 are satisfied by the w^* sequential compactness of B_{X^*}, so we can replace x_E with $x_{\phi(E)}$ and g_E with $g_{\phi(E)}$ and assume that $(g_E)_{E\in\widehat{\mathcal{F}}}$

is a w^* Cauchy tree. We can then let

$$P_E = \{(Tx_n) \in B_{X^*}^\omega : (x_n) \in X^\omega, |Tx_n(x_m)| < \delta \ \forall n < m, m, n \in \mathbb{N}\}.$$

Find an appropriate pruning and assume $(x_E), (g_E)$ already have the properties of the pruned tree. Then we replace x_E with $z_E = x_{2E}$ and $g_{E^\wedge n}$ with $h_{E^\wedge n} = (g_{2E^\wedge 2n} - g_{2E^\wedge 2n-1})/2$. Here, $2E = (2k : k \in E)$. By the previous pruning, $|g_{2E^\wedge 2n-1}(x_{2E^\wedge 2n})| < \delta$, so that

$$h_{E^\wedge n}(z_{E^\wedge n}) \geq g_{2E^\wedge 2n}(x_{2E^\wedge 2n})/2 - g_{2E^\wedge 2n-1}(x_{2E^\wedge 2n})/2 > \|x_{2E^\wedge 2n}\|/2 - \delta \geq \rho/2 - \delta.$$

We next let

$$P_E = \{(x_n) \in B_X^\omega : |h_F(x_n)| < \varepsilon_{|E|+1} \ \forall F \preceq E\},$$

pass to the appropriate prunings of both (z_E) and (h_E), and a similar pruning with

$$P_E = \{(x_n^*) \in B_{X^*}^\omega : |x_n^*(z_F)| < \varepsilon_{|E|+1} \ \forall F \preceq E\}.$$

Finally, one checks that the final tree resulting from the last pruning satisfies the desired properties. $\square$

Remark One can find trees (y_E) and (f_E) satisfying the same conclusions if one begins with $(g_E) \subset B_{X^*}$ w^* null so that $0 < \rho \leq \|g_E\|$ and assumes that X contains no copy of ℓ_1. The only difference is that instead of using w^* sequential compactness of B_{X^*} we use Rosenthal's ℓ_1 theorem to pass to weakly Cauchy subsequences. This example will be important in passing from trees with upper norm estimates to trees in the dual which have lower norm estimates by finding these "almost biorthogonal" trees.

2.2 Szlenk index

As mentioned in the introduction, the Szlenk index is an ordinal index introduced by Szlenk to deduce the non-existence of a Banach space $Z \in$ **REFL** which is universal for **REFL** and the non-existence of a Banach space $Z \in$ **SD** which is universal for **SD**. For a Banach space X, $\varepsilon > 0$, and $K \subset X^*$, we let

$$d_\varepsilon(K) = \{f \in K : \forall w^* \text{ open neighborhoods } V \text{ of } f, \operatorname{diam}_{\|\cdot\|}(V \cap K) > \varepsilon\}.$$

Note that if K is w^* closed, then $d_\varepsilon(K)$ is as well. We then let

$$d_\varepsilon^0(K) = K,$$

$$d_\varepsilon^{\alpha+1}(K) = d_\varepsilon(d_\varepsilon^\alpha(K)), \quad \alpha < \omega_1,$$

$$d_\varepsilon^\alpha(K) = \bigcap_{\beta<\alpha} d_\varepsilon^\beta(K), \quad \alpha < \omega_1 \text{ a limit ordinal.}$$

If there exists $\alpha < \omega_1$ so that $d_\varepsilon^\alpha(K) = \varnothing$, we let

$$\eta(K, \varepsilon) = \min\{\alpha : d_\varepsilon^\alpha(K) = \varnothing\},$$

and $\eta(K, \varepsilon) = \omega_1$ otherwise. We then let $Sz(X) = \sup_{\varepsilon>0} \eta(B_{X^*}, \varepsilon)$.

It is clear that if $f \in d_\varepsilon(K)$, then for any w^* neighborhood V of f, we can choose $g_V, h_V \in V \cap K$ with $\|g_V - h_V\| > \varepsilon$. Then for each V, either $\|g_V - f\| > \varepsilon/2$ or $\|h_V - f\| > \varepsilon/2$. This means we can choose $f_V \in \{g_V, h_V\}$ so that $\|f_V - f\| > \varepsilon/2$, and we have found a net $(f_V)_{V\in\mathcal{N}}$ converging w^* to f with $\liminf_{V\in\mathcal{N}} \|f_V - f\| \geq \varepsilon/2$. Here, $\mathcal{N}$ is a neighborhood basis for the w^* topology at f. Next, suppose that for

$K \subset X^*$ w^* compact, $f \in K$, and $\varepsilon > 0$ we have found a net $(f_\lambda)_\lambda \subset K$ with $f_\lambda \underset{w^*}{\to} f$ so that $\liminf_\lambda \|f_\lambda - f\| \geq \varepsilon$. Then for any $\delta \in (0, \varepsilon)$ and any w^* open neighborhood V of f, some element f_λ in a tail of the net must lie in V and satisfy $\|f_\lambda - f\| > \delta$. This means $\text{diam}_{\|\cdot\|}(V \cap K) > \delta$, and $f \in d_\delta^\alpha(K)$. This motivates the following definition.

For $K \subset X^*$, we let

$$D_\varepsilon(K) = \{f \in K : \exists \text{a net}(f_\lambda) \subset K, f_\lambda \underset{w^*}{\to} f, \liminf_\lambda \|f_\lambda - f\| \geq \varepsilon\}.$$

We then define $D_\varepsilon^\alpha(K)$ and $\eta_D(K, \varepsilon)$ as above. If X is separable, it is clear that we can use sequences instead of nets when K is a bounded set. Our remarks above show that for $\varepsilon > 0$, $K \subset X^*$, and $\delta \in (0, \varepsilon)$,

$$d_\varepsilon(K) \subset D_{\varepsilon/2}(K), D_\varepsilon(K) \subset d_\delta(K).$$

Thus $\sup_{\varepsilon>0} \eta(K, \varepsilon) = \sup_{\varepsilon>0} \eta_D(K, \varepsilon)$, and we can use either to determined the Szlenk index. Each definition affords its benefits, so we will use both.

We will make use of the following fact. The following observation can be easily shown by transfinite induction. A consequence is that the supremum $\sup_{\varepsilon>0} \eta(B_{X^*}, \varepsilon)$ is not attained.

Proposition 2.4. *[19] For any $\varepsilon > 0$ and $\alpha < \eta(B_{X^*}, \varepsilon)$,*

$$(1/2)B_{X^*} + (1/2)d_\varepsilon^\alpha(B_{X^*}) \subset d_{\varepsilon/2}^\alpha(B_{X^*}).$$

In particular, if $\alpha < \eta(B_{X^*}, \varepsilon)$, $\alpha \cdot 2 \leq \eta(B_{X^*}, \varepsilon/2) \geq$, and if $Sz(X) < \omega_1$, then there exists α countable so that $Sz(X) = \omega^\alpha$.

We wish to introduce alternate ways to compute $Sz(X)$ when X is a separable Banach space not containing ℓ_1. Suppose we have fixed some collection $S \subset X^\omega$. For $\mathcal{H} \subset B_X^{<\omega}$, we define

$$(\mathcal{H})_S^0 = \mathcal{H},$$

$$(\mathcal{H})_S^{\alpha+1} = \big\{\mathbf{x} \in \mathcal{H} : \exists (y_k) \in S | \mathbf{x}^\frown y_k \in (\mathcal{H})_S^\alpha \ \forall k\big\}, \quad \alpha < \omega_1,$$

$$(\mathcal{H})_S^\alpha = \bigcap_{\beta<\alpha} (\mathcal{H})_S^\beta, \quad \alpha < \omega_1 \text{ a limit ordinal.}$$

We let $I_S(\mathcal{H}) = \min\{\alpha : (\mathcal{H})_S^\alpha = \varnothing\}$ if this set is non-empty, and ω_1 otherwise. We will have several examples of this. In this section, we will apply this with S equal to all weakly null sequences in B_X. We will write $(\mathcal{H})_w^\alpha, I_w(\mathcal{H})$ in place of $(\mathcal{H})_S^\alpha$ and $I_S(\mathcal{H})$ in this case. In the next section, we will use the same notation to denote the index where S consists of all normalized, weakly null sequences. We will write $(\mathcal{H})_{\text{bl}}^\alpha, I_{\text{bl}}(\mathcal{H})$ if S consists of all normalized block sequences in a Banach space with fixed (understood) FDD. We will also consider $S \subset [\mathbb{N}]$ the collection of all strictly increasing sequences in $\mathbb{N}$. In this case, $I_{CB}(\mathcal{H})$ will denote the usual Cantor-Bendixson index of $\mathcal{H}$.

For $\varepsilon > 0$, we let

$$\mathcal{H}_\varepsilon^X = \Big\{(x_i)_{i=1}^n \in S_X^{<\omega} : n \in \mathbb{N}, \Big\|\sum_{i=1}^n a_i x_i\Big\| \geq \varepsilon \sum_{i=1}^n a_i \ \forall (a_i) \subset [0,\infty)\Big\}.$$

Note that $(x_i)_{i=1}^n \in \mathcal{H}_\varepsilon^X$ if and only if whenever x lies in the convex hull of $(x_i)_{i=1}^n$, $\|x\| \geq \varepsilon$. By the geometric version of the Hahn-Banach theorem, we see that this is equivalent to the existence of a functional $x^* \in B_{X^*}$ so that $x^*(x_i) \geq \varepsilon$ for each i. It is a result of Alspach, Judd, and Odell [2] that if X is a separable Banach space not containing ℓ_1, then $Sz(X) = \sup_{\varepsilon>0} I_w(\mathcal{H}_\varepsilon^X)$. Because it is instructive to later arguments, we include a proof.

Theorem 2.5. *If X is a separable Banach space not containing ℓ_1,*

$$Sz(X) = \sup_{\varepsilon > 0} I_w(\mathcal{H}_\varepsilon^X).$$

Proof. We will prove each quantity cannot exceed the other. We begin by proving $Sz(X) \geq \sup_{\varepsilon>0} I_w(\mathcal{H}_\varepsilon^X)$. For this direction, we only require the separability of X, and not the assumption that X does not contain a copy of ℓ_1. Fix $\varepsilon \in (0,1)$ and fix $\varepsilon_0 \in (0, \varepsilon)$. We prove by induction on $0 \leq \alpha < I_w(\mathcal{H}_\varepsilon^X)$ that for each $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathcal{H}_\varepsilon^X)_w^\alpha$ there exists $f_\mathbf{x} \in d_{\varepsilon_0}^\alpha(B_{X^*})$ so that $f_\mathbf{x}(x_i) \geq \varepsilon$ for $1 \leq i \leq n$. Note that this condition is trivial if $\mathbf{x}$ is the empty sequence.

The $\alpha = 0$ case is simply the geometric version of the Hahn-Banach theorem as we mentioned previously. Suppose $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathcal{H}_\varepsilon^X)_w^{\alpha+1}$. This means there exists a weakly null sequence $(y_k) \subset B_X$ so that $\mathbf{x}_k = (x_1, \ldots, x_n, y_k) \in (\mathcal{H}_\varepsilon)_w^\alpha$ for each k. Let $(f_{\mathbf{x}_k}) \subset d_{\varepsilon_0}^\alpha(B_{X^*})$ be as guaranteed by the inductive hypothesis. By passing to a subsequence and using the separability of X, we can assume $f_{\mathbf{x}_k}$ is w^* convergent to some functional, call it $f_\mathbf{x}$. By w^* compactess of $d_{\varepsilon_0}^\alpha(B_{X^*})$, we deduce $f_\mathbf{x} \in d_{\varepsilon_0}^\alpha(B_{X^*})$. Moreover, for $1 \leq i \leq n$,

$$f_\mathbf{x}(x_i) = \lim_k f_{\mathbf{x}_k}(x_i) \geq \varepsilon.$$

Moreover, since $y_k \in B_X$ and the sequence is weakly null,

$$\liminf_k \|f_{\mathbf{x}_k} - f_\mathbf{x}\| \geq \liminf_k (f_{\mathbf{x}_k} - f_\mathbf{x})(y_k) = \liminf_k f_{\mathbf{x}_k}(y_k) \geq \varepsilon > \varepsilon_0,$$

from which we deduce $f_{\mathbf{x}} \in d_{\varepsilon_0}^{\alpha+1}(B_{X^*})$.

If α is a limit ordinal, we can take any sequence $\alpha_k \uparrow \alpha$. If $\mathbf{x} = (x_1, \dots, x_n) \in (\mathcal{H}_\varepsilon^X)_w^\alpha$, then $\mathbf{x} \in (\mathcal{H}_\varepsilon^X)_w^{\alpha_k}$ for each $k \in \mathbb{N}$. We can choose $f_k \in d_{\varepsilon_0}^{\alpha_k}(B_{X^*})$ so that $f_k(x_i) \geq \varepsilon$ for $1 \leq i \leq n$ and each k. By passing to a w^* convergent subsequence (and using the fact that the sets $d_{\varepsilon_0}^{\alpha_k}(B_{X^*})$ are decreasing and α_k was chosen strictly increasing), we can assume f_k is w^* convergent to some functional, say $f_{\mathbf{x}}$. Clearly $f_{\mathbf{x}}(x_i) \geq \varepsilon$ for $1 \leq i \leq n$. Moreover, by w^* compactness, $f_{\mathbf{x}} = w^* \lim f_k \in d_{\varepsilon_0}^{\alpha_j}(B_{X^*})$ for each $j \in \mathbb{N}$, from which we deduce $f_{\mathbf{x}} \in \cap_j d_{\varepsilon_0}^{\alpha_j}(B_{X^*}) = d_{\varepsilon_0}^{\alpha}(B_{X^*})$.

Thus if $\alpha < I_w(\mathcal{H}_\varepsilon^X)$, $\eta(B_{X^*}, \varepsilon_0) > \alpha$, and $Sz(X) \geq \sup_{\varepsilon > 0} I_w(\mathcal{H}_\varepsilon^X)$.

Next, fix $\varepsilon > 0$.

Claim 2.6. *For any $K \subset B_{X^*}$ w^* compact, if $f \in D_\varepsilon^\alpha(K)$, then there exists a collection $(f_E)_{E \in \mathcal{F}_\alpha} \subset K$ such that $f = f_\varnothing$ and for each $E \in \mathcal{F}'_\alpha$, $f_{E^\frown n} \underset{w^*}{\to} f_E$ and $\liminf_n \|f_{E^\frown n} - f_E\| \geq \varepsilon$.*

The proof is, of course, by induction. If $f \in D_\varepsilon(K)$, this simply means there exists a sequence (f_n) in K with the two properties above in relation to f. We let $f_\varnothing = f$ and $f_{(n)} = f_n$.

In the successor case, we suppose $f \in D_\varepsilon^{\alpha+1}(K)$, we take $(f_n) \subset D_\varepsilon^\alpha(K)$ with the two properties above in relation to f. By the inductive hypothesis, there exists for each n some $(f_E^n)_{E \in \mathcal{F}_\alpha} \subset K$ with $f_\varnothing^n = f_n$ satisfying again the two properties on f_E^n and $f_{E^\frown k}^n$ for each $E \in \mathcal{F}'_\alpha$ and $k > E$. Let $f_\varnothing = f$, $f_{n^\frown E} = f_E^n \in K$, $n < E$. Here we note that if $F \in \mathcal{F}_{\alpha+1}$ is non-empty, F can be written uniquely as $n^\frown E$ for some $E \in \mathcal{F}_\alpha$, $n < E$, so the definition makes sense. Then if $F \in \mathcal{F}'_{\alpha+1}$ is non-empty, $F = n^\frown E$ for some $n \in \mathbb{N}$ and $n < E \in \mathcal{F}'_\alpha$. Then

$$f_{n^\frown F^\frown k} = f_{E^\frown k}^n \underset{w^*}{\to} f_E^n = f_{n^\frown E}$$

and

$$\liminf_k \|f_{n^\frown F^\frown k} - f_E\| = \liminf_k \|f^n_{E^\frown k} - f^n_E\| \geq \varepsilon.$$

These two conditions on $f_\varnothing$ and $(f_{(n)})$ follow from our choice of $f_n = f_{(n)}$.

In the limit case, let $\alpha_n \uparrow \alpha$ be the ordinals used to define $\mathcal{F}_\alpha$. Then $f \in D^{\alpha_n}_\varepsilon(K)$ for all $n \in \mathbb{N}$, which means we can find $(f^n_E)_{E\in\mathcal{F}_{\alpha_n}}$ to satisfy the desired conditions by the inductive hypothesis. Note that (B_{X^*}, w^*) is metrizable, and fix a metrx ρ which determines the w^* topology on B_{X^*}. We can choose for each n some $k_n \geq n$ so that $\rho(f, f^n_{(k_n)}) = \rho(f^n_\varnothing, f^n_{(k_n)}) < 1/n$. For $E \in \mathcal{F}_\alpha$ non-empty with $\min E = n$, let $f_E = f^n_{E+k_n-n}$. Here $E + m = (m + i : i \in E)$. Note that for such E, $E \in \mathcal{F}_{\alpha_n}$ by construction of $\mathcal{F}_\alpha$ and that since $k_n - n \geq 0$ and $\mathcal{F}_{\alpha_n}$ is spreading, this is well-defined. Moreover, since this map $E \mapsto E + m$ preserves immediate successors, the two properties

$$f_{E^\frown k} \underset{w^*}{\to} f_E$$

and

$$\liminf \|f_{E^\frown k} - f_E\| \geq \varepsilon$$

are verified similarly to the successor case whenever $E \neq \varnothing$. By our choice of k_n, the two desired properties for $f_\varnothing$ and $f_{(n)} = f^n_{k_n}$ are easily verified by choice of k_n. This completes the proof of the claim.

Next, fix $\alpha < \eta_D(B_{X^*}, \varepsilon)$. This means there must exist some $f \in D^\alpha_\varepsilon(B_{X^*})$ and some $(f_E)_{E\in\mathcal{F}_\alpha} \subset B_{X^*}$ with the properties stated in the claim. If $E \in \widehat{\mathcal{F}_\alpha}$, there exists a unique $F \in \mathcal{F}_\alpha$ so that $E = F^\frown n$. Let $g_E = f_E - f_F = f_{F^\frown n} - f_F$. Then $(g_E)_{E\in\widehat{\mathcal{F}_\alpha}} \subset 2B_{X^*}$ is a w^* null tree with the property that for each $E \in \mathcal{F}'_\alpha$,

$\liminf \|g_{E^\frown n}\| \geq \varepsilon$ and for each $E \in \widehat{\mathcal{F}_\alpha}$ and each $1 \leq i \leq |E|$,

$$\Big\|\sum_{j=1}^{i} g_{E|_j}\Big\| \leq 2.$$

This is because this sum telescopes to $f_{E|_i} - f_\varnothing$. By pruning, we can fix $\delta > 0$ and assume that $\|g_E\| \geq \varepsilon - \delta$ for all $E \in \widehat{\mathcal{F}_\alpha}$.

As discussed following the pruning lemma, we can assume (since the properties above are preserved by pruning) that we have some weakly null tree $(x_E)_{E\in\widehat{\mathcal{F}}_\alpha} \subset B_X$ weakly null so that $g_E(x_E) \geq \varepsilon - 2\delta$ and that $|g_E(x_F)| < \min\{\varepsilon_{|E|}, \varepsilon_{|F|}\}$ whenever $E \prec F$ or $F \prec E$. Here $\varepsilon_i \downarrow 0$ is chosen so that $\sum_{i=1}^\infty \sum_{j=i}^\infty \varepsilon_j < \delta$. Then for any $E \in \widehat{\mathcal{F}_\alpha}$ and $a_1, \ldots, a_{|E|} \geq 0$,

$$\begin{aligned}
\Big\|\sum_{i=1}^{|E|} a_i x_{E|_i}\Big\| &\geq 2^{-1}\Big(\sum_{i=1}^{|E|} g_{E|_i}\Big)\Big(\sum_{i=1}^{|E|} a_i x|_{E_i}\Big)\\
&\geq 2^{-1}\Big(\sum_{i=1}^{|E|} a_i g_{E|_i}(x_{E|_i}) - \sum_{i=1}^{|E|}\sum_{j\neq i} a_i |g_{E|_j}(x_{E|_i})|\Big)\\
&\geq 2^{-1}\Big((\varepsilon - 2\delta)\sum_{i=1}^{|E|} a_i - \sum_{i=1}^{|E|} a_i\delta\Big)\\
&\geq (\varepsilon - 3\delta)/2 \sum_{i=1}^{n} a_i.
\end{aligned}$$

From this, an easy induction proof shows that for each $0 \leq \beta \leq \alpha$ and any $\rho \in (0, \varepsilon/2)$, we can choose a δ so that this process results in $(x_E)_{E\in\widehat{\mathcal{F}_\alpha}}$ with

$$\{(x_{E|_1}, \ldots, x_{E|_{|E|}}) : E \in \mathcal{F}_\alpha^\beta\} \subset (\mathcal{H}_\rho^X)_w^\beta.$$

The $\beta = 0$ case is a direct consequence of the computation above. If we have the result for some $\beta < \alpha$ and if $E \in \mathcal{F}_\alpha^{\beta+1}$, $(x_{E^\frown n})_{n>E} \subset B_X$ is weakly null and

$$(x_{E|_1}, \ldots, x_{E|_{|E|}}, x_{E^\frown n}) = (x_{E^\frown n|_1}, \ldots, x_{E^\frown n|_{|E|}}, x_{E^\frown n}) \in (\mathcal{H}_\rho^X)_w^\beta$$

by the inductive hypothesis. Therefore $(x_{E|_1}, \ldots, x_{E|_{|E|}}) \in (\mathcal{H}_\rho^X)_w^{\beta+1}$.

If β is a limit, $E \in \mathcal{F}_\alpha^\beta$ if and only if $\mathcal{F}_\alpha^\gamma$ for each $\gamma < \beta$. This means for such E, $(x_{E|_1}, \ldots, x_{E|_{|E|}}) \in (\mathcal{H}_\rho^X)_w^\gamma$ for each $\gamma < \beta$, and we have the conclusion by definition of $(\mathcal{H}_\rho^X)_w^\beta$.

But $\varnothing \in \{(x_{E|_1}, \ldots, x_{E|_{|E|}}) : E \in \mathcal{F}_\alpha^\alpha\}$, so $(\mathcal{H}_\rho^X)_w^\alpha \neq \varnothing$, and we deduce $I_w(\mathcal{H}_\rho^X) > \alpha$.

□

2.3 The James technique, tight constants, three space problems

In this section, we discuss different ways of quantifying ℓ_1 and ℓ_1^+ structure in Banach spaces. This has applications in determining for which ordinals α we can find a separable Banach space X with $Sz(X) = \alpha$, as well as giving upper estimates for $Sz(X)$ in terms of $Sz(Y)$ and $Sz(X/Y)$, where Y is a closed subspace of X. Each of the arguments in this section has at its root the same idea as the original argument of James to prove that any Banach space which contains ℓ_1 isomorphically must contain ℓ_1 almost isometrically [12]. We discuss how this leads to a similar family of problems. The prototypical constant reduction argument is as follows.

Theorem 2.7. *Let X be a Banach space with separable dual. Then if $I_w(\mathcal{H}_\varepsilon^X) > \alpha^\omega$ for some $\varepsilon \in (0,1)$, then $I_w(\mathcal{H}_\delta^X) > \alpha$ for any $\delta \in (0,1)$.*

From this theorem, we can deduce the following corollary. It is similar to a result of Judd and Odell [18], which discussed the Bourgain ℓ_1 index, defined and discussed below, instead of the Szlenk index.

Corollary 2.8. *If $\alpha < \omega_1$ is a limit ordinal, there is no Banach space with $Sz(X) = \omega^{\omega^\alpha}$.*

Proof of Corollary 2.8. Recall that our definition of Szlenk index is not equivalent to the usual definition if X is not separable. But if $Sz(X)$ is countable, it is separably determined [19]. Thus if there exists a Banach space X with $Sz(X) = \omega^{\omega^\alpha}$, α a countable limit ordinal, we can assume X is separable with this Szlenk index. This means X must have separable dual, since a separable space has countable Szlenk index if and only if it has separable dual. Thus we can apply the first part of this problem. Take $\beta < \alpha$. Then $\beta + 1 < \alpha$, which means there exists $\varepsilon \in (0, 1)$ so that

$$I_w(\mathcal{H}_\varepsilon^X) > \omega^{\omega^{\beta+1}} = \omega^{\omega^\beta \cdot \omega} = (\omega^{\omega^\beta})^\omega.$$

This means that $I_w(\mathcal{H}_{1/2}^X) > \omega^{\omega^\beta}$. Since $\beta < \alpha$ was arbitrary, $I_w(\mathcal{H}_{1/2}^X) \geq \omega^{\omega^\alpha}$. But since the supremum $\sup_{\varepsilon > 0} I_w(\mathcal{H}_\varepsilon^X)$ is not attained, this means $Sz(X) > \omega^{\omega^\alpha}$. This contradiction completes the proof.

□

The prototypical three space argument is as follows.

Theorem 2.9. *Let X be a Banach space with separable dual, and let Y be a closed subspace. Then for $\varepsilon \in (0, 1)$, there exists $\delta = \delta(\varepsilon) \in (0, 1)$ so that $I_w(\mathcal{H}_\varepsilon^X) \leq I_w(\mathcal{H}_\delta^{X/Y}) I_w(\mathcal{H}_\delta^Y)$.*

We state these together because the general idea as well as the major step in the proof of both is the same. We think of the tree $\widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}$ as an $\widehat{\mathcal{F}_\alpha}$ with the vertices replaced by an $\widehat{\mathcal{F}_\beta}$ tree. Either one of these $\widehat{\mathcal{F}_\beta}$ trees has "good" branches (which means the convex combinations have some property with a good constant in one case, and the convex combinations have large quotient norms in the other case), or we can replace each $\widehat{\mathcal{F}_\beta}$ with a "bad" convex combination of one of its branches so that the remaining bad combinations will form an $\widehat{\mathcal{F}_\alpha}$ tree. We state this as

Lemma 2.10. *Let X be a Banach space. Let $A \subset X$. Then if $1 \le \beta, \alpha < \omega_1$ and $(x_E)_{E \in \widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}}$ is any tree, then either there exists a subtree $(y_E)_{E \in \widehat{\mathcal{F}_\beta}}$ of $(x_E)_{E \in \widehat{\mathcal{F}_\alpha[\mathcal{F}_\alpha]}}$ so that for each $E \in \widehat{\mathcal{F}_\beta}$, $A \cap co(y_{E|_1}, \ldots, x_{E|_{|E|}}) = \varnothing$, or there exist $(z_E)_{E \in \widehat{\mathcal{F}_\alpha}} \subset A$, $(F_E)_{\widehat{\mathcal{F}_\alpha}} \subset \widehat{\mathcal{F}_\beta}$ so that for each $E \in \widehat{\mathcal{F}_\alpha}$,*

(i) $F_{E|_1} < \ldots < F_E$,

(ii) $(\min F_{E|_n})_{n=1}^{|E|}$ *is a spread of E,*

(iii) $z_E \in co\Big(x_F : \bigcup_{n=1}^{|E|-1} F_{E|_n} \prec F \preceq \bigcup_{n=1}^{|E|} F_{E|_n}\Big)$.

Proof. Suppose that the first alternative does not hold. If there exists n so that for each $F \in \widehat{\mathcal{F}_\alpha} \cap [n, \infty)^{<\omega}$,

$$A \cap \mathrm{co}(x_G : \varnothing \prec G \preceq F) = \varnothing,$$

then we define $y_E = x_{E+n}$, where $E + n = (m + n : m \in E)$. Then $(y_E)_{E \in \widehat{\mathcal{F}_\beta}}$ fulfills the first alternative, and we have a contradiction. This means there exists no such n, and for each $k \in \mathbb{N}$, we can find natural numbers $n_1 < n_2 < \ldots$, $F_{(k)} \in \widehat{\mathcal{F}_\beta}$ with $\min F_{(k)} = n_k$, and $z_{(k)} \in A \cap \mathrm{co}(x_G : \varnothing \prec G \preceq F_{(k)})$.

Next, suppose that for some $1 < \ell \in \mathbb{N}$ and for each $E \in \widehat{\mathcal{F}_\alpha}$ with $|E| < \ell$, we have constructed z_E, F_E with the desired properties. If there exist no $E \in \widehat{\mathcal{F}_\alpha}$ with $|E| = \ell$, we are done. Otherwise, choose $E \in \widehat{\mathcal{F}_\alpha}'$ with $|E| = \ell - 1$. Let $G = \cup_{i=1}^{|E|} F_{E|_i}$. Let $m = \max G$, $m_0 = \max E$. If there exists $n \in \mathbb{N}$ so that for each $F \in \widehat{\mathcal{F}_\beta} \cap [m + n, \infty)^{<\omega}$,

$$A \cap \mathrm{co}(x_{G^\frown H} : \varnothing \prec H \preceq F) = \varnothing,$$

we let $y_F = x_{G^\frown(F+n)}$. Note that since E is non-maximal in $\widehat{\mathcal{F}_\alpha}$, then $(\min F_{E|_i})_{i=1}^{|E|}$ is also non-maximal in $\widehat{\mathcal{F}_\alpha}$. This means that for any $F \in \widehat{\mathcal{F}_\beta}$ with $m < F$, $G^\frown F \in \widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}$. Thus $(y_F)_{F\in\widehat{\mathcal{F}_\beta}}$ is well-defined and satisfies the first alternative. Thus no such $n \in \mathbb{N}$ can exist. This means we can find $n_{m_0+1} < n_{m_0+2} < \ldots$, $F_{E^\frown(m_0+1)}, F_{E^\frown(m_0+2)}, \dots \in \widehat{\mathcal{F}_\beta}$ with $\min F_{E^\frown(m_0+k)} = n_{m_0+k}$ and

$$z_{E^\frown(m_0+k)} \in \mathrm{co}\big(x_{G^\frown H} : \varnothing \prec H \preceq F_{E^\frown(m_0+k)}\big) = \mathrm{co}\big(x_H : G \prec H \preceq G^\frown F_{E^\frown(m_0+k)}\big).$$

This completes the recursive step. The trees $(z_E)_{E\in\widehat{\mathcal{F}_\alpha}} \subset A$, $(F_E)_{E\in\widehat{\mathcal{F}_\alpha}} \subset \widehat{\mathcal{F}_\beta}$ clearly fulfill the second alternative.

Remark Suppose that X is a Banach space with separable dual. We can choose a metric ρ on B_X which determines the weak topology so that the function $\phi(x) = \rho(0, x)$ is convex. Fix a function $f : [\mathbb{N}]^{<\omega} \to (0, 1)$ so that for each $\varepsilon > 0$, there exist only finitely many $E \in [\mathbb{N}]^{<\omega}$ with $f(E) > \varepsilon$.

Suppose $(u_E)_{E\in\widehat{\mathcal{F}_{\beta\cdot\alpha}}} \subset B_X$ is a weakly null tree. Choose according to Lemma 2.2 some $M \in [\mathbb{N}]$ so that $\widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}(M) \subset \widehat{\mathcal{F}_{\beta\cdot\alpha}}$. Then $w_E = u_{M(E)}$ is well-defined for each $E \in \widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}$, and $(w_E)_{E\in\widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}} \subset B_X$ is also weakly null. Let

$$P_E = \{(x_n) \in B_X^\omega : \phi(x_n) < f(E^\frown(n + \max E))\}.$$

If we apply Lemma 2.3 to $(w_E)_{E\in\widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}}$, we can find a pruning $(x_E)_{E\in\widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}} \subset B_X$ a weakly null tree so that for each $\varepsilon > 0$, there exist only finitely many $E \in \widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}$ such that $\phi(x_E) > \varepsilon$. Then if we apply Lemma 2.10, the tree which results from the dichotomy there must also be weakly null in the unit ball of X. This is because sequences of immediate successors of $(y_E)_{E\in\widehat{\mathcal{F}_\beta}} \subset B_X$ are also sequences of immediate successors in $(x_E)_{E\in\widehat{\mathcal{F}_\alpha[\mathcal{F}_\beta]}}$. In the second alternative, a sequence of immediate successors $(z_{E^\frown n})_{n>E}$ is such that $z_{E^\frown n} \in \text{co}(x_{G^\frown H} : \varnothing \prec H \preceq F_{E^\frown n})$. Since the sets $(G^\frown H : \varnothing \prec H \preceq F_{E^\frown n})$ are pairwise disjoint and ϕ is convex, the sequence $(z_{E^\frown n})_{E<n}$ is weakly null.

$\square$

For the proof of Theorem 2.7, we will use Proposition 5 of [24].

Proposition 2.11. *If X is a Banach space with separable dual and $\varepsilon \in (0,1)$, then $I_w(\mathcal{H}_\varepsilon^X) > \alpha$ if and only if there exists $(x_E)_{E\in\widehat{\mathcal{F}_\alpha}} \subset B_X$ weakly null so that $(x_{E|_1}, \ldots, x_{E|_{|E|}}) \in \mathcal{H}_\varepsilon^X$ for each $E \in \widehat{\mathcal{F}_\alpha}$.*

Proof of Theorem 2.7. First, suppose $I_w(\mathcal{H}_\varepsilon^X) > \alpha^2$. Then by Proposition 2.11, there exists a weakly null $(x_E)_{E\in\widehat{\mathcal{F}_{\alpha^2}}} \subset B_X$ with branches in $\mathcal{H}_\varepsilon^X$. We let $A = \text{int}(\varepsilon^{1/2}B_X)$. Applying Lemma 2.10 and the remark following it, we know we can find either a weakly null tree $(y_E)_{E\in\widehat{\mathcal{F}_\beta}} \subset B_X$ so that no convex hull of a branch of this tree intersects A, or we can find a weakly null tree $(z_E)_{E\in\widehat{\mathcal{F}_\alpha}} \subset A\cap B_X$ and $(F_E)_{E\in\widehat{\mathcal{F}_\alpha}}$ satisfying (i)-(iii) of 2.10. In the first case, the branches of the tree $(y_E)_{E\in\widehat{\mathcal{F}_\alpha}}$ lie in $\mathcal{H}_{\varepsilon^{1/2}}^X$. In the second case, each branch of the tree $(z_E)_{E\in\widehat{\mathcal{F}_\alpha}}$ is a convex blocking of a branch of $(x_E)_{E\in\widehat{\mathcal{F}_{\alpha^2}}}$, and therefore lies in $\mathcal{H}_\varepsilon^X$, and $\|z_E\| < \varepsilon^{1/2}$. Then $(\varepsilon^{-1/2}z_E)_{E\in\mathcal{F}_\alpha} \subset B_X$ is weakly null, and homogeneity implies the branches lie in $\mathcal{H}_{\varepsilon^{1/2}}^X$. Thus in either case of the dichotomy of Lemma 2.10, $I_w(\mathcal{H}_{\varepsilon^{1/2}}^X) > \alpha$.

Next, suppose $I_w(\mathcal{H}_\varepsilon^X) > \alpha^\omega$. Fix $\delta \in (0,1)$. We can take N so large that $\varepsilon^{1/2^N} > \delta$. Then $I_w(\mathcal{H}_\varepsilon^X) > \alpha^{2^N}$, and N applications of the first part gives that $I_w(\mathcal{H}_\delta^X) > \alpha$.

$\square$

For the proof of Theorem 2.9, we will need the following

Proposition 2.12. *Let X be a Banach space not containing ℓ_1, Y a closed subspace. Let $\delta \in (0, 1/2)$, and $(x_n) \subset B_X$ be a weakly null sequence so that $\|x_n\|_{X/Y} < \delta$ for all $n \in \mathbb{N}$. Then there exists $N \in [\mathbb{N}]$ and a weakly null sequence $(y_n)_{n\in N} \subset 2B_Y$ so that so that $\|x_n - y_n\| < 2\delta$ for all $n \in N$. There exists $(z_n)_{N\in N} \subset S_Y$ with $\|x_n - z_n\| < 4\delta$ for all $n \in N$.*

Proof. Choose for each $n \in \mathbb{N}$ some $u_n \in Y$ so that $\|x_n - u_n\| < \delta$. By Rosenthal's ℓ_1 theorem, we can find $N \in [\mathbb{N}]$ so that $(u_n)_{n\in N}$ is weakly Cauchy. Let $\varepsilon_n = \delta - \|x_n - u_n\|$ for each $n \in N$. For each $n \in N$, choose $I_n \in [N]^{<\omega}$ and a convex combination $v_n = \sum_{i\in I_n} a_i x_i$ so that $\|v_n\| < \varepsilon_n$ and so that $(I_n)_{n\in N}$ is successive. Let $w_n = \sum_{i\in I_n} a_i u_i$ and note that

$$\|w_n\| \leq \|v_n\| + \|v_n - w_n\| \leq \varepsilon_n + \sum_{i\in I_n} a_i \|x_i - u_i\| < \varepsilon_n + \delta.$$

Let $y_n = u_n - w_n$, so $(y_n)_{n\in N}$ is weakly null in Y and

$$\|x_n - y_n\| \leq \|x_n - u_n\| + \|w_n\| < \|x_n - u_n\| + \varepsilon_n + \delta = 2\delta.$$

To see that (y_n) is weakly null, fix $x^* \in X^*$. Then the convex blocking $\left(\sum_{i\in I_n} a_i x^*(u_i)\right)_{n\in N}$ of $(x^*(u_n))_{n\in N}$ must converge to the same limit as does the sequence $(x^*(u_i))_{n\in N}$, so that the differences $x^*(y_n) - \sum_{i\in I_n} a_i x^*(y_n)$ vanish as $N \ni n \to \infty$.

For the second statement, note that $y_n \neq 0$, so that if $z_n = y_n/\|y_n\|$,

$$\|x_n - z_n\| \leq \|z_n - y_n\| + \|y_n - x_n\| < 4\delta.$$

□

Proof of Theorem 2.9. Let X be a Banach space, Y a closed subspace, $\varepsilon \in (0,1)$. Let $\delta = \varepsilon/6$. Let $\beta = I_w(\mathcal{H}_\delta^{X/Y})$, $\alpha = I_w(\mathcal{H}_\delta^Y)$. If $I_w(\mathcal{H}_\varepsilon^X) > \beta \cdot \alpha$, Proposition 2.11 gives us a weakly null tree $(x_E)_{E \in \widehat{\mathcal{F}_{\beta\cdot\alpha}}}$ with branches in $\mathcal{H}_\varepsilon^X$. Let $Q : X \to X/Y$ be the quotient map and let $A = Q^{-1}\big(\delta \text{int} B_{X/Y}\big)$. In the first alternative of Lemma 2.10, we find a weakly null tree $(w_E)_{E\in\widehat{\mathcal{F}_\beta}}$ so that $A \cap \text{co}(w_{E|_i} : 1 \le i \le |E|) = \varnothing$ for each $E \in \widehat{\mathcal{F}_\beta}$. But this means that $(Qw_E)_{E\in\widehat{\mathcal{F}_\beta}} \subset B_{X/Y}$ is weakly null and the branches of this tree lie in $\mathcal{H}_\delta^{X/Y}$. But this would mean $I_w(\mathcal{H}_\delta^{X/Y}) > \beta$, a contradiction.

In the second alternative, we find a weakly null tree $(z_E)_{E\in\widehat{\mathcal{F}_\alpha}}$ so that each branch is a convex blocking of a branch of the tree $(x_E)_{E\in\widehat{\mathcal{F}_{\beta\cdot\alpha}}}$ so that $\|z_E\|_{X/Y} < \delta$ for each E, and therefore also lies in $\mathcal{H}_\varepsilon^X$. We apply a pruning, this time with

$$P_E = \{(x_n) \in B_X^\omega : \exists (y_n) \in (2B_Y)^\omega | (y_n) \text{ weakly null}, \|x_n - y_n\| < 4\delta \ \forall n\}.$$

We can apply Proposition 2.12 to find a pruning $(z'_E)_{E\in\widehat{\mathcal{F}_\alpha}}$ and a tree $(y_E)_{E\in\widehat{\mathcal{F}_\alpha}}$ so that $\|z'_E - y_E\| < 4\delta$ for each $E \in \widehat{\mathcal{F}_\alpha}$. Then $(y_E) \subset 2B_Y$ is a weakly null tree. Moreover, if $E \in \widehat{\mathcal{F}_\alpha}$ and $a_1, \ldots, a_{|E|} \ge 0$,

$$\begin{aligned}\Big\|\sum_{i=1}^{|E|} a_i y_{E|_i}\Big\| &\ge \Big\|\sum_{i=1}^{|E|} a_i z'_{E|_i}\Big\| - \sum_{i=1}^{|E|} a_i \|z'_{E|_i} - y_{E|_i}\| \\ &\ge (\varepsilon - 4\delta)\sum_{i=1}^{|E|} a_i > \varepsilon/3 \sum_{i=1}^{|E|} a_i.\end{aligned}$$

Then $(y_E/2)_{E\in\widehat{\mathcal{F}_\alpha}} \subset B_Y$ is weakly null with branches lying in $\mathcal{H}_\delta^Y$, a contradiction to the assumption that $I_w(\mathcal{H}_\delta^Y) = \alpha$.

□

Ordinals of the form ω^{ω^α} are characterized by the property that if $\beta, \gamma < \omega^{\omega^\alpha}$, then $\beta \cdot \gamma < \omega^{\omega^\alpha}$. Therefore Theorem 2.9 immediately gives the following

Corollary 2.13. *If X is a separable Banach space so that $Sz(X) > \omega^{\omega^\alpha}$, then either $Sz(Y) > \omega^{\omega^\alpha}$ or $Sz(X/Y) > \omega^{\omega^\alpha}$.*

This corollary means that being separable and having Szlenk index not exceeding ω^{ω^α} is a three space property. It is not known whether this holds for ordinals not of the particular form ω^{ω^α}.

The proof above is an adaptation of an argument due to James. His original argument was that if a Banach space contains vectors $(x_i)_{i=1}^{n^2}$ which are C-equivalent to the unit vector basis of $\ell_1^{n^2}$, then there exists a blocking $(u_i)_{i=1}^n$ of $(x_i)_{i=1}^{n^2}$ which is $C^{1/2}$-equivalent to the unit vector basis of ℓ_1^n. Consequently if X contains the ℓ_1^n spaces uniformly, it contains them almost isometrically. A similar proof shows that if X contains a copy of ℓ_1, then X contains a subspace which is $(1+\varepsilon)$-isomorphic to ℓ_1. This is also how one proves that ℓ_1 is not distortable. We now discuss different versions of the James argument with applications to three space problems, constant reduction, and distortion.

We next recall some results due to Judd and Odell [18]. For a Banach space X and $K \geq 1$, we let

$$T(X,K) = \Big\{(x_i)_{i=1}^n \in [B_X]^{<\omega} : K\Big\|\sum_{i=1}^n a_i x_i\Big\| \geq \sum_{i=1}^n |a_i| \ \forall (a_i)_{i=1}^n \subset \mathbb{F}\Big\}.$$

As usual, we define the derived trees $(T(X,K)^\alpha)_{\alpha<\omega_1}$ by transifinite induction. That is,

$$T(X,K)^0 = T(X,K), \quad T(X,K)^{\alpha+1} = (T(X,K)^\alpha)',$$

and

$$T(X,K)^\alpha = \cap_{\beta<\alpha} T(X,K)^\beta, \quad \alpha \text{ a limit ordinal.}$$

We let $I(X,K) = \min\{\alpha < \omega_1 : T(X,K)^\alpha = \varnothing\}$ if this set is non-empty, and

$I(X,K) = \omega_1$ otherwise. We let $I(X) = \sup\{I(X,K) : K \geq 1\}$. This is called the *Bourgain ℓ_1 index* of X, and it measures the complexity of the local ℓ_1 structure of X. The statement that $I(X,K) > \omega$ is simply the statement that X contains the ℓ_1^n spaces uniformly. This leads to the following theorems, analogous to Theorem 2.7 and Theorem 2.9. They both follow from a similar process to Lemma 2.10. If we consider trees so that each vertex has either no immediate successors or infinitely many, then among the trees of order $\alpha+1$, $\mathcal{F}_\alpha$ is minimal. By this, we mean that any such tree must contain a subtree isomorphic to $\mathcal{F}_\alpha$. If we do not restrict ourselves to trees such that each vertex has either zero or infinitely many immediate successors, we obtain a different family of minimal trees. These were denoted by (T_α) by Judd and Odell [18]. They also constructed trees, which they called replacement trees and denoted $T(\beta,\alpha)$, which were the analogues of $\mathcal{F}_\alpha[\mathcal{F}_\beta]$. They then prove that T_{α^2} and $T(\alpha,\alpha)$ are isomorphic to subtrees of each other. They then show that $o(T(X,K)) \geq \alpha^2$ if and only if one can find a tree $(x_t)_{t\in T_{\alpha^2}} \subset B_X$ so that any branch of this tree is K-equivalent to the unit vector basis of ℓ_1^n, where n is the length of the branch. They convert this to a tree $(u_t)_{t\in T(\alpha,\alpha)}$ and prove the existence of either a "good" or "bad" tree indexed by T_α and proceed as we did. This argument was somewhat simpler, since there is no weak nullity requirement. The next theorem was stated explicitly.

Theorem 2.14. *If $I(X,K) > \omega^{\alpha\cdot\omega}$, then for any $\varepsilon > 0$, $I(X, 1+\varepsilon) > \omega^\alpha$. Consequently, if $\alpha < \omega_1$ is a limit ordinal, there does not exist a Banach space X with $I(X) = \omega^{\omega^\alpha}$.*

The next theorem was not explicitly stated by Judd and Odell, but they proved that $T_{\beta\cdot\alpha}$ and $T(\beta,\alpha)$ are isomorphic to subtrees of each other. The next theorem is an easy consequence of their work, proved similarly to Theorem 2.9.

Theorem 2.15. *If X is a separable Banach space and Y is a closed subspace, then for $K \geq 1$, there exists $C = C(K)$ so that $I(X, K) \leq I(X/Y, C)I(Y, C)$.*

Remark Theorem 2.14 also has a c_0 analogue, also shown by Judd and Odell, modifying the corresponding result of James about the non-distortability of c_0.

Last, if $\mathcal{F} \in \mathfrak{S}$ is a set containing all singletons, we say a basic sequence $(x_n) \subset B_X$ in X is a K-$\ell_1^{\mathcal{F}}$ spreading model if for any $E \in \mathcal{F}$ and scalars $(a_n)_{n\in E}$,

$$K\Big\|\sum_{n\in E} a_n x_n\Big\| \geq \sum_{n\in E} |a_n|.$$

In the case that $\mathcal{F} = \mathcal{S}_\alpha$, we write ℓ_1^α in place of $\ell_1^{\mathcal{S}_\alpha}$.

From this we deduce two more theorems, again in line with the previous theme.

Theorem 2.16. *If X contains an $\ell_1^{\omega^\alpha}$ spreading model, then X contains a $(1+\varepsilon)$-$\ell_1^{\omega^\alpha}$ spreading model. If X contains an $\ell_1^{\omega^\alpha}$ spreading model, then X contains a $(1+\varepsilon)$-$\ell_1^{\omega^\alpha}$ spreading model.*

Theorem 2.17. *If X is a Banach space and Y is a closed subspace, then X contains an $\ell_1^{\omega^\alpha}$ spreading model if and only if either Y or X/Y does.*

To prove both of these theorems, we make a brief definition and state some easy facts.

If (x_i) is a sequence in a Banach space X, $E_1 < E_2 < \ldots$ are finite sets, and (a_i) are scalars such that $(a_i)_{i\in E_n} \in S_{\ell_1^{|E_n|}}$ for each n, we call the sequence $\left(\sum_{i\in E_n} a_i x_i\right)_n$ an *absolutely convex blocking* of (x_i). If E_n can be taken to lie in $\mathcal{F}$ for each n, we call this blocking an $\mathcal{F}$ *absolutely convex blocking.*

The following facts are routinely checked.

Proposition 2.18. *(i) If $\mathcal{F} \subset \mathcal{G}$ and (x_n) is a K-$\ell_1^{\mathcal{G}}$ spreading model, it is a K-$\ell_1^{\mathcal{F}}$ spreading model.*

(ii) If (x_n) is a K-$\ell_1^{\mathcal{F}[\mathcal{G}]}$ spreading model and (y_n) is a $\mathcal{G}$ absolutely bounded blocking of (x_n), then (y_n) is a K-$\ell_1^{\mathcal{F}}$ spreading model.

(iii) If $M = (2n)_n$, then $\mathcal{S}_\alpha[\mathcal{F}_2](M) \subset \mathcal{S}_\alpha$.

Remark If we search for ℓ_1^α spreading models in a Banach space X, the requirement that the sequence be basic is not a limitation. Suppose $(x_n) \subset B_X$ is a sequence in the Banach space X, $K > 1$ are such that

$$K\Big\|\sum_{n\in E} a_n x_n\Big\| \geq \sum_{n\in E} |a_n|$$

for all $E \in \mathcal{S}_\alpha$ and scalars $(a_n)_{n\in E}$. If this sequence has no weakly Cauchy subsequence, Rosenthal's ℓ_1 theorem implies that some subsequence must be equivalent to the unit vector basis of ℓ_1. In this case, we have a $(1+\varepsilon)$-$\ell_1^{[\mathbb{N}]^{<\omega}}$ spreading model for any $\varepsilon > 0$. Otherwise we can use Rosenthal's ℓ_1 theorem to pass to a weakly Cauchy subsequence of (x_n). We assume the sequence itself is weakly Cauchy. We then pass to the subsequence $(y_n) = (x_{2n})$ and the $\mathcal{F}_2$ absolutely convex blocking $(z_n) = ((y_{2n} - y_{2n-1})/2)$ to obtain a weakly null seminormalized sequence in B_X satisfying the appropriate lower norm estimates on $\mathcal{S}_\alpha$ sets. Any basic subsequence of this sequence is a K-ℓ_1^α spreading model.

We sketch the proofs of Theorems 2.16 and 2.17. If X, Y, or X/Y contains ℓ_1, the result is clear. Thus we assume none of these spaces contains ℓ_1. If $\alpha > 0$, we let $\alpha_n \uparrow \omega^\alpha$ be the ordinals used to define $\mathcal{S}_{\omega^\alpha}$. If $\alpha = 0$, we replace the families $\mathcal{S}_{\alpha_n}$ with $\mathcal{F}_n$ and the proof goes through the same. Suppose X contains a K-$\ell_1^{\omega^\alpha}$

spreading model (x_n). We say a subnormalized sequence (u_n) in X has property P_n (respectively, $P_n(C)$ for $C > K$) if for each $E \in \mathcal{S}_{\alpha_n}$ with $n \le E$ and all scalars $(a_i)_{i\in E}$,

$$K^{1/2}\Big\|\sum_{i\in E} a_i u_i\Big\| \ge \sum_{i\in E} |a_i|$$

respectively, if

$$C\Big\|\sum_{i\in E} a_i u_i\Big\|_{X/Y} \ge \sum_{i\in E} |a_i|.$$

We have the following dichotomy. Either for each $n \in \mathbb{N}$ and any $N \in [\mathbb{N}]$ there exists $M \in [N]$ so that $(x_i)_{i\in M}$ has property P_n (respectively, $P_n(C)$), or there exists $N \in [\mathbb{N}]$ and $n \in \mathbb{N}$ so that for each $M \in [N]$, the subsequence $(x_i)_{i\in M}$ fails to have property P_n ($P_n(C)$). In the first case, we find $N_1 \supset N_2 \supset N_3 \supset \ldots$ so that $(x_i)_{i\in N_j}$ has P_j ($P_j(C)$) for each j. One easily checks that if $N_j = (n^j_k)$, $n_j = n^j_j$, and $N = (n_j)$, then $(x_i)_{i\in N}$ is a $K^{1/2}$-$\ell_1^{\omega^\alpha}$ spreading model in X (or $(Qx_i)_{i\in N}$ has the appropriate C-$\ell_1^{\omega^\alpha}$ lower estimates in X/Y), and a blocking and subsequence arguments allows us to obtain a subnormalized basic sequence in X/Y which is a C-$\ell_1^{\omega^\alpha}$ spreading model. If not, then we assume that (x_i) has no subsequence with property P_n. We pass to the subsequence $(x_i)_{i\in M}$, where M is chosen so that $\mathcal{S}_{\omega^\alpha}[\mathcal{S}_{\alpha_n}](M) \subset \mathcal{S}_{\omega^\alpha}$, and argue that we can find an $\mathcal{S}_{\alpha_n}$ absolutely convex blocking (y_i) of (x_i) so that $\|y_i\| < K^{-1/2}$ for all $i \in \mathbb{N}$. Then an appeal to Proposition 2.18 yields that $(K^{1/2}y_i)$ is the desired $\mathcal{S}_{\omega^\alpha}$ spreading model. For the $P_n(C)$ argument, we pass to an $\mathcal{S}_{\alpha_n}$ absolutely convex blocking (z_i) of (x_i) so that $\|z_i\|_{X/Y} < C^{-1}$. Then (z_i) is also a K-$\ell_1^{\omega^\alpha}$ spreading model in X. We choose for each i some $y_i \in Y$ so that $\|z_i - y_i\| < C^{-1}$. Then for any $E \in \mathcal{S}_{\omega^\alpha}$ and any scalars $(a_i)_{i\in E}$,

$$\Big\|\sum_{i\in E} a_i y_i\Big\| \ge \Big\|\sum_{i\in E} a_i z_i\Big\| - \sum_{i\in E} |a_i|/C \ge (K^{-1} - C^{-1})\sum_{i\in E} |a_i|.$$

Thus $(y_i/2) \subset B_Y$ satisfies the appropriate lower estimates with constant $(K^{-1} - C^{-1})/2$, and passing to a blocking of a subsequence gives us a basic sequence in Y which is the $\ell_1^{\omega^\alpha}$ spreading model we sought.

We discuss a general framework into which each of these results fit. Suppose we have defined for each countable ordinal α and constant $K \geq 1$ some type of structure, say $P(\alpha, K)$, which may exist in a Banach space. For example, the structure $P(\alpha, K)$ may be weakly null tree $(x_E)_{E \in \widehat{\mathcal{F}_\alpha}} \subset B_X$ trees satisfying ℓ_1^+ estimates, or a K-$\ell_1^{\mathcal{F}_\alpha}$ spreading model. For this type of structure, we can try to verify the existence of a map $\phi : [1, \omega_1) \to [1, \omega_1)$ so that if any Banach space X contains a $P(\phi(\alpha), K)$ strucure, then X contains a $P(\alpha, 1 + \varepsilon)$ structure for any $\varepsilon > 0$. We could also ask for functions $\psi_1 : [1, \omega_1) \to [1, \omega_1)$ and $\psi_2 : [1, \infty) \to [1, \infty)$ so that if any Banach space X contains a $P(\psi_1(\alpha), K)$ structure, then either Y or X/Y contains a $P(\alpha, \psi_2(K))$ structure. We have seen three examples of such structures which admit positive answers for both types of questions under certain assumptions on the space. We have also seen that all three structures admit ordinals of the form ω^{ω^α} as fixed points of the functions ϕ and ψ_1. The author is currently investigating several questions within this framework, including determining if ordinals of this form are the only fixed points.

The theorems above corresponding to reduction of constants have analogues for ℓ_∞^n and c_0. The theme above, like the original proofs of James, have at their heart the fact that c_0 and ℓ_1 are extremes in some sense, and that one can use this extremity to force preservation of ℓ_1 or c_0 structure. This connects such structures with distortion, and this connection has been expanded upon.

If X is a Banach space, an equivalent norm $|\cdot|$ on X is said to be a *t-distortion* of X if for any infinite-dimensional subspace Y of X we can find $x, y \in S_Y = S_Y^{\|\cdot\|}$ with $|x|/|y| > t$. For this definition, it is necessary and sufficient to assume that the Y

above has a basis. Therefore we can equivalently reformulate this definition by saying that $|\cdot|$ is a t-distortion of X if for any basic sequence (x_n) there exists $E \in [\mathbb{N}]^{<\omega}$ and $x, y \in S_{[x_i]_{i\in E}}$ with $|x|/|y| > t$. This motivates the following definition, which is an attempt to measure the complexity required to witness distortion. We say an equivalent norm $|\cdot|$ on X is a t-$\mathcal{F}$-distortion of X if for any basic sequence (x_n) there exists $E \in \mathcal{F}$ and $x, y \in S_{[x_i]_{i\in E}}$ with $|x|/|y| > t$. In the spirit of James original proof that ℓ_1 and c_0 are not distortable, we have the following.

Theorem 2.19. *If X contains an $\ell_1^{\omega^\alpha}$ or $c_0^{\omega^\alpha}$ spreading model, then X is not $(1+\varepsilon)$-$\mathcal{S}_{\omega^\alpha}$-distortable for any $\varepsilon > 0$.*

3. COORDINATIZATION AND UNIVERSALITY*

3.1 Finite dimensional decompositions

A sequence of finite-dimensional normed spaces $E = (E_n)$ is called a *finite dimensional decomposition* (or FDD) for a Banach space Z if for each $z \in Z$ there exists a unique sequence (z_n) so that $z_n \in E_n$ and $z = \sum z_n$. As in the case of Schauder bases, for each $n \in \mathbb{N}$, the linear operator $P_n^E : Z \to E_n$, given by $P_n^E z = z_n$, where $z = \sum_m z_m$ is the unique representation of z with $z_m \in E_m$, is a bounded projection. The operator P_n^E is called the n^{th} *canonical projection*. We define the *support* of $z \in Z$ *with respect to* E by

$$\text{supp}_E(z) = \{n \in \mathbb{N} : P_n^E z \neq 0\}.$$

If no confusion is possible, we will write supp in place of supp_E. We let

$$c_{00}(E) = \{z \in Z : |\text{supp}(z)| < \infty\}.$$

We write $\text{ran}_E(z_n)$ to denote smallest interval of natural numbers which containing $\text{supp}_E(z_n)$. If $(z_n) \subset c_{00}(E)$ is a sequence of non-zero vectors such that $(\text{ran}_E(z_n))$ is successive, we call (z_n) a *block sequence with respect to* E.

For each $A \in [\mathbb{N}]^{<\omega}$, $P_A^E z = \sum_{n\in A} z_n$ is also a bounded linear operator. By the

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uniform boundedness principle, the *projection constant of* E in Z, given by

$$K = \sup\{\|P^E_{[m,n]}\| : 1 \leq m \leq n < \infty\} \geq 1,$$

is finite. If $K = 1$, we say E is a *bimonotone* FDD for Z. It is known that if E is an FDD for Z, we can endow Z with an equivalent norm making E a bimonotone FDD for Z with the new norm. We can think of E^*_n as being naturally embedded in Z^* via the map $z^* \mapsto z^* \circ (P^E_n)^*$, but this is not necessarily an isometric embedding if E is not bimonotone. We identify E^*_n with its image in Z^* and let $E^* = (E^*_n)$. We let $Z^{(*)} = \overline{c_{00}(E^*)}$, where the closure is taken in Z^*. Then E^* is an FDD for $Z^{(*)}$ with projection constant not exceeding the projection constant of E in Z.

An FDD E for Z is called *shrinking* if $Z^{(*)} = Z^*$, that is, if E^* is an FDD for Z^*. An FDD E for Z is called *boundedly complete* if whenever (z_n) is a sequence in Z so that $z_n \in E_n$ and $\sup_N \left\|\sum_{n=1}^N z_n\right\| < \infty$, then $\sum z_n$ converges in Z. If E is a boundedly complete FDD for Z, then Z is naturally a dual space. This is because in this case, E^* is a shrinking FDD for $Z^{(*)}$, so $(Z^{(*)})^* = Z$ via the natural map which takes $E_n \to E^{**}_n \subset (Z^{(*)})^*$. It is known that a Banach space Z with FDD E is reflexive if and only if E is both shrinking and boundedly complete. A proof of this fact, originally due to James, can be found in [10]. The proof there is given for the case of a Schauder basis, but the same proof works for FDDs.

If $(e_n), (f_n)$ are basic sequences in (possibly different) Banach spaces, we say (f_n) C-dominates (e_n) or that (e_n) is C-dominated by (f_n) if for all scalars $(a_n) \in c_{00}$,

$$\left\|\sum a_n e_n\right\| \leq C \left\|\sum a_n f_n\right\|.$$

We denote this by $(e_n) \lesssim_C (f_n)$. If we do not wish to specify the constant, we simply write $(e_n) \lesssim (f_n)$.

If (u_n) is a basis for a Banach space U, we say (u_n) is C-right dominant if for any $(m_n) \in [\mathbb{N}]$ and any spread (ℓ_n) of (m_n), $(u_{m_n}) \lesssim_C (u_{\ell_n})$. *Left dominant* is defined similarly. An easy duality argument gives that if (u_n) is normalized, 1-unconditional, (u_n) is C-right dominant if and only if (u_n^*) is C-left dominant.

Remark Fix a basis (u_n) for the Banach space U. Let

$$\mathcal{R} = \Big\{ \sum_{n=1}^{N} u_{p_n} \otimes u_{q_n}^* : 1 \leq p_1 < \ldots < p_N, 1 \leq q_1 < \ldots < q_N, p_n \leq q_n \Big\} \subset B(U),$$

$$\mathcal{L} = \Big\{ \sum_{n=1}^{N} u_{p_n} \otimes u_{q_n}^* : 1 \leq p_1 < \ldots < p_N, 1 \leq q_1 < \ldots < q_N, p_n \geq q_n \Big\} \subset B(U).$$

We note that (u_n) is right dominant if and only if $\sup_{T \in \mathcal{R}} \|T\| < \infty$, and this supremum is the smallest constant R so that (u_n) is R-right dominant. Similarly, (u_n) is left dominant if and only if $\sup_{T \in \mathcal{L}} \|T\| < \infty$, and this supremum is the smallest constant L so that (u_n) is L-left dominant. Note that both $\mathcal{R}$ and $\mathcal{L}$ are closed under composition, so that if (u_n) is R-right dominant, $|u| = \sup_{T \in \mathcal{R}} \|Tu\|$ defines an R-equivalent norm on U making the basis 1-right dominant. Moreover, if the basis was initially normalized and 1-unconditional, it will remain so under the new norm. A similar result holds for left dominance. Because of this, we are not limited by assuming that right (resp. left) dominant bases are 1-right (resp. 1-left) dominant.

If Z is a Banach space with FDD E and U is a Banach space with normalized, 1-unconditional basis (u_n), we say E satisfies *subsequential C-U upper block estimates* in Z if whenever (z_n) is a normalized block sequence with respect to E, $(z_n) \lesssim_C (u_{m_n})$, where $m_n = \min \operatorname{ran}_E(z_n)$. We define *subsequential C-U lower block estimates* similarly. An easy duality argument proves that E satisfies subsequential U upper (resp. lower) block estimates in Z if and only if E^* satisfies subsequential $U^{(*)}$ lower (resp. upper) block estimates in $Z^{(*)}$. If E is bimonotone in Z, the preceding statement remains true if we replace upper (lower) block estimates with C-upper (lower) block estimates.

The following relation between upper estimates and the Szlenk index is now quite clear.

Proposition 3.1. *Let Z be a Banach space with FDD F and U a Banach space with normalized, 1-unconditional, weakly null basis (u_n). If F satisfies subsequential C-U upper block estimates in Z, then for every $\varepsilon > 0$, $I_w(\mathcal{H}_\varepsilon^Z) \leq I_w(\mathcal{H}_{\varepsilon/C}^U)$. If F and (u_n) are shrinking, $Sz(Z) \leq Sz(U)$.*

Proof. For $\varepsilon > 0$ and $\alpha < I_w(\mathcal{H}_\varepsilon^Z)$, we can find $(z_E)_{E\in\widehat{\mathcal{F}_\alpha}} \subset B_Z$ a weakly null tree so that for each $E \in \widehat{\mathcal{F}_\alpha}$ and non-negative scalars $(a_i)_{i=1}^{|E|}$,

$$\Big\|\sum_{i=1}^{|E|} a_i z_{E|_i}\Big\| \geq \varepsilon \sum_{i=1}^{|E|} a_i.$$

By standard perturbation and pruning arguments, and by replacing ε with any strictly smaller constant, we can assume this tree is actually a block tree with respect to the FDD F. If we let $m(E) = \min \operatorname{ran}_F(z_E)$ for each $E \in \widehat{\mathcal{F}_\alpha}$, then the tree $(\|z_E\| u_{m(E)})_{E\in\widehat{\mathcal{F}_\alpha}}$, and by 1-unconditionality $(u_{m(E)})_{E\in\widehat{\mathcal{F}_\alpha}} \subset S_U$, witnesses the fact that $I_w(\mathcal{H}_{\varepsilon/C}^U) > \alpha$. This is because the tree is weakly null, since $m(E\char`^n) \to \infty$ as $n \to \infty$ and the basis (u_n) is weakly null. We also have for any $E \in \widehat{\mathcal{F}_\alpha}$ and non-negative scalars $(a_i)_{i=1}^{|E|}$,

$$\Big\|\sum_{i=1}^{n} a_i u_{m(E|_i)}\Big\| \geq C^{-1}\Big\|\sum_{i=1}^{n} a_i z_{E|_i}\Big\| \geq \varepsilon/C \sum_{i=1}^{|E|} a_i.$$

Since this holds for any $\alpha < I_w(\mathcal{H}_\varepsilon^Z)$, $I_w(\mathcal{H}_\varepsilon^Z) \leq I_w(\mathcal{H}_{\varepsilon/C}^U)$. If F and (u_n) are shrinking, then

$$Sz(Z) = \sup_{\varepsilon>0} I_w(\mathcal{H}_\varepsilon^Z) \leq \sup_{\varepsilon>0} I_w(\mathcal{H}_\varepsilon^U) = Sz(U).$$

□

The following proposition, which follows from a standard perturbation argument, will be used frequently throughout.

Proposition 3.2. *Let U be a Banach space with normalized, 1-unconditional basis (u_n) and let Z be a Banach space with FDD E which satisfies subsequential C-U upper (resp. lower) block estimates in Z. Then if (z_n) is a normalized block sequence in Z with respect to E and $(k_n) \in [\mathbb{N}]$ is so that*

$$k_n \leq \min \operatorname{ran}(z_{n+1}) < k_{n+1}$$

for all $n \in \mathbb{N}$, then (z_n) is C-dominated by (resp. C-dominates) (u_{k_n}).

The typical coordinatization method will involve making a given Banach space X a subspace or a quotient of a Banach space Z which has an FDD E, and then building from Z, E, and U a new space with FDD which has the appropriate block estimates so that X is still either a subspace or a quotient of this new space. We next introduce the method for building such new spaces from old. In the particular case that $V = \ell_p$, these spaces were considered in [22]. In the general case, these spaces were first considered in [23].

If Z is a Banach space with FDD E and V is a Banach space with normalized, 1-unconditional basis (v_n), we define a new norm on $c_{00}(E)$ by

$$\|z\|_{Z^V(E)} = \max\Big\{\Big\|\sum_{i=1}^{n} \|P^E_{[m_{i-1},m_i)} z\|_Z v_{m_{i-1}}\Big\|_V : 1 \le m_0 < \ldots < m_n, n, m_i \in \mathbb{N}\Big\}.$$

We let $Z^V(E)$ be the completion of $c_{00}(E)$ with this norm. We note that E is an FDD for $Z^V(E)$ which has projection constant in $Z^V(E)$ not exceeding the projection constant of E in Z. We can also connect some properties of the FDD E of Z and the basis (v_n) of U to the FDD E in $Z^V(E)$.

We would like to verify that the space $Z^V(E)$ does in fact possess the desired lower block estimates. In the case that we want simultaneous lower and upper block estimates, the scheme will be to first use a duality argument and the above method to achieve the upper estimates and then to use the above method again to achieve the lower estimates. Since the $Z^V(E)$ norm dominates the Z norm, it will be important in this situation to guarantee that when we use the above method to get the lower block estimates, we do not lose the upper estimates. Also, the embedding theorems we have will typically not yield a space $Z^V(E)$, but a space $Z^{V_M}(E)$, where $M = (m_n) \in [\mathbb{N}]$ and $V_M = [v_{m_n}]$. For this reason, we will need to "fill out" the FDD. We would also like to know that E is a shrinking or shrinking and boundedly complete FDD for $Z^V(E)$, depending on the case. The next five technical results will accomplish everything mentioned in this paragraph. The proofs below are slight generalizations of proofs appearing in [23].

Proposition 3.3. *Let V be a Banach space with normalized, 1-unconditional basis (v_n) and Z be a Banach space with FDD E. If (z_n) is any block sequence with respect to E, then there exists a block sequence (y_n) in V such that $2\|y_n\| \ge \|z_n\|_{Z^V(E)}$ and so that $(y_n) \lesssim_1 (z_n)$.*

Proof. For each n, choose $1 \leq k_0^n < k_1^n < \ldots < k_{\ell_n}^n$ so that

$$\|z_n\|_{Z^V(E)} = \Big\| \sum_{i=1}^{\ell_n} \|P^E_{[k_{i-1}^n, k_i^n)} z_n\|_Z v_{k_{i-1}^n} \Big\|_V .$$

We can assume

$$k_0^n \leq \min \operatorname{ran}(z_n) < k_1^n, k_{\ell_n}^n = \min \operatorname{ran}(z_{n+1}).$$

If $\|P^E_{[k_0^n, k_1^n)} z_n\|_Z \geq (1/2)\|z_n\|_{Z^V(E)}$, then we can replace $(k_0^n, \ldots, k_{\ell_n}^n)$ with $(\min \operatorname{ran}(z_n), k_1^n)$ and otherwise replace $(k_0^n, \ldots, k_{\ell_n}^n)$ with $(k_1^n, \ldots, k_{\ell_n}^n)$ and assume

$$\|z_n\|_{Z^V(E)} \leq 2 \Big\| \sum_{i=1}^{\ell_n} \|P^E_{[k_{i-1}^n, k_i^n)} z_n\|_Z v_{k_{i-1}^n} \Big\|_V$$

and that $k_0^n \geq \min \operatorname{ran}(z_n)$ for each $n \in \mathbb{N}$. Then if $(a_n) \in c_{00}$ and $z = \sum a_n z_n$, the concatenation of the sequences (k_i^n) and using 1-unconditionality of (v_n) implies

$$\|z\|_{Z^V(E)} \geq \Big\| \sum_n \sum_{i=1}^{\ell_n} a_n \|P^E_{[k_{i-1}^n, k_i^n)} z_n\|_Z v_{k_{i-1}^n} \Big\|_V .$$

Letting $y_n = \sum_{i=1}^{\ell_n} \|P^E_{[k_{i-1}^n, k_i^n)} z_n\|_Z v_{k_{i-1}^n}$ finishes the proof.

□

Remark The constant 2 above is sharp. This is because if $z \in c_{00}(E)$, if we wish to choose $1 \leq k_0 < k_1 < \ldots$ so that

$$\|z\|_{Z^V(E)} = \Big\| \sum \|P^E_{[k_{i-1}, k_i)} z\|_Z v_{k_{i-1}} \Big\|_V ,$$

we cannot necessarily assume $k_0 \geq \min \operatorname{ran}(z)$. Taking $Z = c_0$ with obvious FDD and $V = \mathbb{R} \oplus_1 c_0$, we observe that if $z = e_m + e_n$, $1 < m < n$, then $\|z\|_{Z^V(E)} = 2$, but

$$\left\| \sum \|P^E_{[k_{i-1},k_i)} z\|_Z v_{k_{i-1}} \right\|_V \leq 1$$

unless $k_0 = 1$.

One hypothesis that we will use frequently throughout is that a basis (v_n) for V satisfies subsequential V upper or lower block estimates in V. Formally, if (v_n) satisfies subsequential C-V upper block estimates in V, then any normalized block sequence (y_n) of (v_n) with $\min \operatorname{supp}(y_n) = m_n$, then $(y_n) \lesssim_C (v_{m_n})$. We consider an example of a Banach space failing to have this property. Fix $1 \leq p, q < \infty$. We let (e_n) denote the canonical basis of ℓ_p, (f_n) the canonical basis of ℓ_q. Then $(v_1, v_2, v_3, \ldots) = (e_1, f_1, e_2, \ldots)$ is a normalized, 1-unconditional basis for $\ell_p \oplus \ell_q$. Then $y_n = e_n + f_n$ is a normalized block sequence of (v_n), and $m_n = \min \operatorname{supp}(y_n) = 2n - 1$ is such that $v_{m_n} = e_m$. Then if $1 \leq q < p$, then (y_n) is isometrically equivalent to (f_n) and is not dominated by (e_n). Thus in this case, (v_n) fails to satisfy subsequential V upper block estimates. If $1 \leq p < q$, we can take real numbers $1 \geq t_n \downarrow 0$ so rapidly that the normalized block sequence $(y_n) = (t_n e_n + f_n)$ is equivalent to (f_n) and does not dominate (e_n) with any constant.

Lemma 3.4. *If V is a Banach space with normalized, 1-unconditional basis (v_n) which satisfies subsequential C-V lower block estimates in V, and Z is a Banach space with FDD E, then E satisfies subsequential $2C$-V lower block estimates in $Z^V(E)$.*

Proof. Choose a normalized block sequence (z_n) in $Z^V(E)$. Let $m_n = \min \operatorname{supp}\,(z_n)$. Choose a block sequence (y_n) according to Proposition 3.3 so that $(y_n) \lesssim_1 (z_n)$, $\|y_n\| \geq 1/2$ for all $n \in \mathbb{N}$, and recall from the proof that $m_n \leq \min \operatorname{ran}(y_n)$ for all

$n \in \mathbb{N}$. Then using Proposition 3.2 and the fact that (v_n) satisfies subsequential C-V lower block estimates in V,

$$\left\|\sum a_n v_{m_n}\right\| \leq 2C\left\|\sum a_n y_n\right\| \leq 2C\left\|\sum a_n z_n\right\|_{Z^V(E)}$$

for any $(a_n) \in c_{00}$.

$\square$

Lemma 3.5. *Let Z be a Banach space with FDD E. Let V, U be Banach spaces with normalized, 1-unconditional bases $(v_n), (u_n)$, respectively, so that every normalized block of (v_n) is dominated by every normalized block of (u_n). Then if E satisfies subsequential U upper block estimates in Z, E satisfies subsequential U upper block estimates in $Z^V(E)$.*

Proof. First, we observe that if every normalized block of (v_n) is dominated by every normalized block of (u_n), then there exists C such that every normalized block of (v_n) is C-dominated by every normalized block of (u_n). Let us assume also that E satisfies subsequential C-U upper block estimates in Z. We may also assume that E is bimonotone in Z.

Fix $(a_n) \in c_{00}$ and let $u = \sum_n a_n u_n$. Fix $1 \leq k_0 < k_1 < \ldots$. Let $N = \{n \in \mathbb{N} : P_{[k_{n-1},k_n)} u \neq 0\}$. For $n \in N$, let $x_n = P_{[k_{n-1},k_n)} u$, $y_n = x_n / \|x_n\|$, $c_n = \|x_n\|$. Then $u = \sum_{n \in N} c_n y_n$. Moreover,

$$\begin{aligned}\left\|\sum_n \|P_{[k_{n-1},k_n)} u\|_U v_{k_{n-1}}\right\|_V &= \left\|\sum_{n\in N} \|P_{[k_{n-1},k_n)} u\|_U v_{k_{n-1}}\right\|_V \\ &= \left\|\sum_{n\in N} c_n v_{k_{n-1}}\right\|_V \leq C\left\|\sum_{n\in N} c_n y_n\right\|_U = C\|u\|.\end{aligned}$$

This means the U and U^V norms are C-equivalent on c_{00}.

Fix a normalized block sequence (z_n) in $Z^V(E)$. Let $m_n = \min \operatorname{ran}_E(z_n)$. Fix $(a_n) \in c_{00}$ and let $z = \sum a_n z_n$. Choose $1 \le k_1 < \ldots < k_N$ so that

$$\|z\|_{Z^V(E)} = \Big\| \sum_{i=1}^{N} \|P^E_{[k_{i-1},k_i)} z\|_Z v_{k_{i-1}} \Big\|_V .$$

For $n \in \mathbb{N}$, let

$$I_n = \{ i \le N : [k_{i-1}, k_i) \subset [\min \operatorname{ran}_E(z_n), \min \operatorname{ran}_E(z_{n+1})) \}.$$

Let $I = \{1, \ldots, N\} \setminus \bigcup_n I_n$. For each $i \in I$, let

$$J_i = \{ n \in \mathbb{N} : [k_{i-1}, k_i) \cap \operatorname{ran}_E(z_n) \neq \varnothing \}.$$

Note that the $(I_n)_{n\in\mathbb{N}}$ are pairwise disjoint. The $(J_i)_{i\in I}$ need not be pairwise disjoint, but if $I = I' \cup I''$ is a partition of I so that neither I' nor I'' contains consecutive elements of I, $(J_i)_{i\in I'}$ are pairwise disjoint, and so are $(J_i)_{i\in I''}$. Then

$$\begin{aligned}
\|z\|_{Z^V(E)} &\le \Big\| \sum_{i\notin I} \|P^E_{[k_{i-1},k_i)} z\|_Z v_{k_{i-1}} \Big\|_V + \Big\| \sum_{i\in I} \|P^E_{[k_{i-1},k_i)} z\|_Z v_{k_{i-1}} \Big\|_V \\
&\le \Big\| \sum_{n} \sum_{i\in I_n} a_n \|P^E_{[k_{i-1},k_i)} z_n\|_Z v_{k_{i-1}} \Big\|_V + \Big\| \sum_{i\in I'} \Big\| P^E_{[k_{i-1},k_i)} \big(\sum_{n\in J_i} a_n z_n \big) \Big\|_Z v_{k_{i-1}} \Big\|_V \\
&+ \Big\| \sum_{i\in I''} \Big\| P^E_{[k_{i-1},k_i)} \big(\sum_{n\in J_i} a_n z_n \big) \Big\|_Z v_{k_{i-1}} \Big\|_V .
\end{aligned}$$

We will bound each term by a multiple of $\Big\| \sum a_n u_{m_n} \Big\|_U$. Let

$$y_n = \sum_{i\in I_n} \|P^E_{[k_{i-1},k_i)} z_n\|_Z v_{k_{i-1}}.$$

Then $\|y_n\|_V \le \|z_n\|_{Z^{V}(E)} \le 1$. Then

$$\Big\|\sum_n \sum_{i\in I_n} a_n \|P^E_{[k_{i-1},k_i)} z_n\|_Z v_{k_{i-1}}\Big\|_V = \Big\|\sum_n a_n y_n\Big\|_V \le C\Big\|\sum a_n u_{m_n}\Big\|_U.$$

Moreover, by bimonotonicity and the fact that E satisfies subsequential C-U upper block estimates in U, we can use Proposition 3.2 to deduce that for each $i \in I$,

$$\Big\|P^E_{[k_{i-1},k_i)}\big(\sum_{n\in J_i} a_n z_n\big)\Big\|_Z \le \Big\|\sum_{n\in J_i} a_n z_n\Big\|_Z \le \Big\|\sum_{n\in J_i} a_n u_{m_n}\Big\|_U.$$

Then

$$\begin{aligned}\Big\|\sum_{i\in I'}\Big\|P^E_{[k_{i-1},k_i)}\big(\sum_{n\in J_i} a_n z_n\big)\Big\|_Z v_{k_{i-1}}\Big\|_V &\le C\Big\|\sum_{i\in I'}\Big\|\sum_{n\in J_i} a_n u_{m_n}\Big\|_U v_{k_{i-1}}\Big\|_V\\ &\le C\Big\|\sum_{i\in I'}\sum_{n\in J_i} a_n u_{m_n}\Big\|_{U^V} \le C^2\Big\|\sum a_n u_{m_n}\Big\|_U.\end{aligned}$$

A similar estimate holds for the sum over I''.

□

Lemma 3.6. *Let $M \in [\mathbb{N}]$. Let Z be a Banach space with FDD E. Let V, U be Banach spaces with normalized, 1-unconditional bases $(v_n), (u_n)$, respectively, so that (u_n) satisfies subsequential C-U upper block estimates in U, (v_n) satisfies subsequential C-V lower block estimates in V, and so that every normalized block of (v_n) is C-dominated by any normalized block of (u_n). Suppose also that E satisfies subsequential C-(V_M, U_M) block estimates in Z. Then $W = Z \oplus_\infty V_{\mathbb{N}\setminus M}$ has an FDD which satisfies subsequential (V, U) block estimates.*

Proof. We will use Proposition 3.2 implicitly throughout the proof. Write $M = (m_k)$

and let

$$F_n = \begin{cases} E_k & : n = m_k \\ \operatorname{span} v_n & : n \notin M. \end{cases}$$

Fix a normalized block sequence (w_n) with respect to F. Let $w_n = z_n + y_n$ with $z_n \in Z$ and $y_n \in V_{\mathbb{N}\setminus M}$. Note that (z_n) is a subnormalized block sequence in Z, (y_n) is a subnormalized block sequence in V. Let $A = \{n : \|z_n\| = 1\}$, and $B = \mathbb{N} \setminus A$. Observe that if $n \in B$, $\|y_n\| = 1$.

Let $N' = \{n \in \mathbb{N} : z_n \neq 0\}$, $N'' = \{n \in \mathbb{N} : y_n \neq 0\}$. For $n \in N'$, let $p_n = \min \operatorname{ran}_E(z_n)$ for $n \in \mathbb{N}$ and note that $\min \operatorname{ran}_F(z_n) = m_{p_n}$. For $n \in N''$, let $\min \operatorname{ran}_F(y_n) = \min \operatorname{ran}_V(y_n) = q_n$. For each $n \in \mathbb{N}$, let $\min \operatorname{ran}_F(w_n) = r_n$. Choose $(a_n) \in c_{00}$, let $w = \sum a_n w_n$, $y = \sum a_n y_n$, $z = \sum a_n z_n$.

Since any normalized block of (v_n) is C-dominated by any normalized block of (u_n),

$$\|y\| \leq C \Big\| \sum a_n u_{r_n} \Big\|.$$

Since E satisfies subsequential C-U_M upper block estimates in Z and (u_n) satisfies subsequential C-U upper block estimates in U,

$$\|z\| \leq C \Big\| \sum_{n \in N'} a_n u_{m_{p_n}} \Big\| \leq C^2 \Big\| \sum a_n u_{r_n} \Big\|.$$

Thus F satisfies subsequential C^2-U upper block estimates in W.

Next, because (v_n) satisfies subsequential C-V lower block estimates in V and is 1-unconditional,

$$\|y\| \geq C^{-1} \Big\| \sum_{n \in N''} a_n \|y_n\| v_{q_n} \Big\| \geq C^{-1} \Big\| \sum_{n \in B} a_n v_{q_n} \Big\| \geq C^{-2} \Big\| \sum_{n \in B} a_n v_{r_n} \Big\|.$$

Because E satisfies subsequential C-V_M upper block estimates in Z, (v_n) satisfies subsequential C-V lower block estimates in V and is 1-unconditional,

$$\|z\| \geq C^{-1}\Big\|\sum_{n\in N'} a_n\|z_n\|v_{m_{p_n}}\Big\| \geq C^{-1}\Big\|\sum_{n\in A} a_n v_{m_{p_n}}\Big\| \geq C^{-2}\Big\|\sum_{n\in A} a_n v_{r_n}\Big\|.$$

Then

$$\|w\| = \max\{\|z\|,\|y\|\} \geq C^{-2}\max\Big\{\Big\|\sum_{n\in A} a_n v_{r_n}\Big\|,\Big\|\sum_{n\in B} a_n v_{r_n}\Big\|\Big\} \geq C^{-2}/2\Big\|\sum a_n v_{r_n}\Big\|.$$

Thus F satisfies subsequential $2C^2$-(V,U) block estimates in W.

$\square$

Proposition 3.7. *Let V be a Banach space with a normalized, 1-unconditional basis (v_n), and Z a Banach space with FDD E.*

(i) If (v_n) is boundedly complete, then E is a boundedly complete FDD for $Z^V(E)$.

(ii) If (v_n) is a shrinking basis for V and if E is a shrinking FDD for Z, then E is a shrinking FDD for $Z^V(E)$.

Proof. (i) follows easily from Proposition 3.3. If (x_n) is a block sequence in $Z^V(E)$ and $\varepsilon > 0$ is such that $\|x_n\|_{Z^V(E)} \geq \varepsilon$ for all $n \in \mathbb{N}$, we can find a block sequence (y_n) in V so that $\|y_n\| \geq \varepsilon/2$ for all $n \in \mathbb{N}$ and so that $(y_n) \lesssim_1 (x_n)$. Then

$$\sup_{N\in\mathbb{N}}\Big\|\sum_{n=1}^{N} x_n\Big\|_{Z^V(E)} \geq \sup_{N\in\mathbb{N}}\Big\|\sum_{n=1}^{N} y_n\Big\| = \infty.$$

This implies that E is boundedly complete. This is because if the series $\sum z_n$ fails to converge, there must exist $\varepsilon > 0$ and natural numbers $0 = k_0 < k_1 < \ldots$ so that

$\left\|\sum_{n=k_{i-1}+1}^{k_i} z_n\right\| \geq \varepsilon$ for all $i \in \mathbb{N}$. Then with $x_i = \sum_{n=k_{i-1}+1}^{k_i} z_n$,

$$\sup_{N\in\mathbb{N}}\left\|\sum_{n=1}^{N} z_n\right\|_{Z^V(E)} \geq \sup_{i\in\mathbb{N}}\left\|\sum_{n=1}^{k_i} z_n\right\|_{Z^V(E)} = \sup_{i\in\mathbb{N}}\left\|\sum_{n=1}^{i} x_n\right\|_{Z^V(E)} = \infty.$$

For (ii), we begin by assuming E is bimonotone in Z and hence also in $Z^V(E)$. Observe that if $1 \leq m_0 < m_1 < \ldots$, $(a_n) \subset \mathbb{R}$, and $(z_n^*) \subset B_{Z^*}$ are such that

$$\left\|\sum a_n v_{m_{n-1}}^*\right\| \leq 1$$

and

$$\operatorname{ran}(z_n^*) \subset [m_{n-1}, m_n),$$

then $\sum a_n z_n^*$ converges in $(Z^V(E))^*$ and has norm not exceeding 1. To see this, fix $M < N \in \mathbb{N}$. Fix $z \in c_{00}(E)$ with $\|z\|_{Z^V(E)} = 1$, $\operatorname{ran}(z) \subset [n_{M-1}, n_N)$ to norm $\sum_{n=M}^{N} a_n z_n^*$ in $(Z^V(E))^*$. Then

$$\begin{aligned}
\left\|\sum_{n=M}^{N} a_n z_n^*\right\| &= \sum_{n=M}^{N} a_n z_n^*(z) \\
&\leq \sum_{n=M}^{N} |a_n| \|P_{[m_{n-1},m_n)}^E z\|_Z \\
&= \Big(\sum_{n=M}^{N} |a_n| v_{m_{n-1}}^*\Big)\Big(\sum_{n=M}^{N} \|P_{[m_{n-1},m_n)}^E z\|_Z v_{m_{n-1}}\Big) \\
&\leq \left\|\sum_{n=M}^{N} a_n v_{m_{n-1}}^*\right\| \left\|\sum_{n=M}^{N} \|P_{[m_{n-1},m_n)}^E z\|_Z v_{m_{n-1}}\right\| \\
&\leq \left\|\sum_{n=M}^{N} a_n v_{m_{n-1}}^*\right\|.
\end{aligned}$$

This gives both convergence and the norm estimate.

Next, let

$$K = \left\{ \sum a_n z_n^* : \exists (m_n) \text{ finite or infinite}, \sum a_n v_{m_{n-1}}^* \in B_{V^*}, \right.$$
$$\left. z_n^* \in B_{Z^*}, \operatorname{ran}(z_n^*) \subset [m_{n-1}, m_n) \right\},$$

where if the sum is finite with largest index N, $m_N = \infty$ is also allowed. That is, the last element of a finite sum need not have finite support. Our above remark shows that $K \subset B_{Z^V(E)^*}$. It is clear that this is 1-norming for $Z^V(E)$. We claim that it is w^* compact. To see this, for each $k \in \mathbb{N}$, fix $(m_n^k)_{0 \le n} \in [\mathbb{N}]$, a block sequence $(z_{nk}^*)_{1 \le n} \subset B_{Z^*}$ and a sequence of scalars $(a_{nk})_{1 \le n}$ so that $\operatorname{ran}(z_{nk}^*) \subset [m_{n-1}^k, m_n^k)$ and $\left\| \sum a_{nk} v_{m_{n-1}^k}^* \right\| \le 1$. It is sufficient to consider only infinite sequences here. This is because for any element $\sum_{n=1}^N a_n z_n^*$ of K which is a finite sum, we can replace z_N^* with an arbitrarily small perturbation which has finite support. We can then let $a_n = z_n^* = 0$ for all $n > N$, and $\sum_{n=1}^\infty a_n z_n \in K$ is an arbitrarily small perturbation of $\sum_{n=1}^N a_n z_n$.

By fixing n, considering $(m_n^k)_k, (a_{nk})_k, (z_{nk}^*)_k$, and passing to a diagonal subsequence, we can pass to a subsequence and assume that for each appropriate n,

$$a_n = \lim_k a_{nk}, \quad m_n = \lim_k m_n^k$$

exists where $m_n = \omega$ is possible. We let $N = \max\{n : m_n < \infty\}$, noting that $N = \omega$ is possible. If this set is empty, then clearly $\sum_n a_{nk} z_{nk}^* \underset{w^*}{\to} 0$ as $k \to \infty$. Assume $N \in \mathbb{N} \cup \{\infty\}$. Then $1 \le m_0 < m_1 < \ldots$. Moreover,

$$\sum a_{nk} v_{m_{n-1}^k}^* \underset{w^*}{\to} \sum a_n v_{m_{n-1}}^*$$

as $k \to \infty$. Therefore this limit must also have norm not exceeding 1. Last, by passing to a further subsequence, we can assume that $z_{nk}^* \underset{w^*}{\overset{k\to\infty}{\to}} z_n$ for each appropriate n. If $1 \leq n < N < \infty$, then $\text{ran}(z_n^*) \subset [m_{n-1}, m_n)$. It is easy to see in this case that

$$\sum a_{nk} z_{nk}^* \underset{w^*}{\to} \sum a_n z_n^* \in K$$

as $k \to \infty$, which gives the claim.

This means we can embed $Z^V(E)$ isometrically into $\mathcal{C}(K)$. Since any bounded block sequence in $Z^V(E)$ must be pointwise null on K, it is weakly null in $Z^V(E)$, so E is a shrinking FDD for $Z^V(E)$.

□

Lemma 3.8. *[17] Let W, Z be Banach spaces with boundedly complete FDDs E, F, respectively, and let $T : W \to Z$ be a w^*-w^* continuous operator (since the FDDs are complete, both spaces are naturally dual spaces). Then for any sequence $(\varepsilon_n) \subset (0,1)$, there exist blockings G, H of E, F, respectively, so that if $w \in \oplus_{i=k+1}^{\ell-1} G_i$, $\|P_{[1,k)}^H Tw\| < \varepsilon_k \|w\|$ and $\|P_{[\ell,\infty)}^H Tw\| < \varepsilon_\ell \|w\|$.*

Proof. First, note that for $\varepsilon > 0$ and $p \in \mathbb{N}$, there exists $q \in \mathbb{N}$ so that if $w \in \oplus_{j=q+1}^{\infty} E_j$, $\|P_{[1,p]}^F Tw\| < \varepsilon \|w\|$. Moreover, for $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists $p \in \mathbb{N}$ so that if $w \in \oplus_{j=1}^{q} E_j$ and $r \geq p$, $\|P_{(r,\infty)}^F Tw\| < \varepsilon \|w\|$. To see the first, suppose not. This means there exists $\varepsilon > 0$, $p \in \mathbb{N}$, and a sequence $(w_q) \subset B_W$ such that $w_q \in \oplus_{j=q+1}^{\infty} E_j$ and $\|P_{[1,p]}^F Tw_q\| \geq \varepsilon$. Since $w_q \underset{w^*}{\to} 0$, w^*-w^* continuity of T implies $Tw_q \underset{w^*}{\to} 0$, and compactness of $P_{[1,p]}^F$ implies $P_{[1,p]}^F Tw_q \to 0$. This contradiction gives the first claim.

For the second, simply take an η-net $w_1, \ldots, w_s$ of $B_{\oplus_{j=1}^q E_j}$, where $(1 + K\|T\|)\eta < \varepsilon$, K the projection constant of F in Z. Choose p so large that for $r \geq p$ and

$1 \leq i \leq s$, $\|P^F_{(r,\infty)} T w_i\| < \eta$. Then for any $z \in B_{\oplus_{j=1}^q E_j}$, there exists $1 \leq i \leq s$ so that

$$\|P^F_{(r,\infty)} Tz\| \leq \|P^F_{(r,\infty)} Tz_i\| + \|P^F_{(r,\infty)}\|\|T\|\|z - z_i\| \leq (1 + K\|T\|)\eta < \varepsilon.$$

Next, let $0 = p_0 = q_0$ and choose recursively $q_1, p_1, q_2, p_2, \ldots$ so that if $w \in \oplus_{j=q_k+1}^{\infty} E_j$, $\|P^F_{[1,p_{k-1}]} Tw\| < \varepsilon_k \|w\|$ and if $w \in \oplus_{j=1}^{q_\ell} E_j$, $\|P^F_{(p_\ell,\infty)} Tw\| < \varepsilon_{\ell+1}\|w\|$. Let $G_i = \oplus_{j=q_{i-1}+1}^{q_i} E_j$ and $H_i = \oplus_{j=p_{i-1}+1}^{p_i} F_j$. Then if

$$w \in \oplus_{j=k+1}^{\ell-1} G_j = \oplus_{j=q_k+1}^{q_{\ell-1}} E_j,$$

$$\|P^H_{[1,k)} Tw\| = \|P^F_{[1,p_{k-1}]} Tw\| < \varepsilon_k \|w\| \quad \text{and} \quad \|P^H_{[\ell,\infty)} Tw\| = \|P^F_{(p_{\ell-1},\infty)} Tw\| < \varepsilon_\ell \|w\|.$$

□

Proposition 3.9. *Let X be a w^* closed subspace of a dual Banach space Z such that Z has boundedly complete FDD E having projection constant K. Let $(\delta_n) \subset (0,1)$ with $\delta_n \downarrow 0$. Then there exists $(s_n)_{n\geq 1} \in [\mathbb{N}]$ ($s_0 = 0$) such that the following holds. Given $(k_n)_{n\geq 0} \in [\mathbb{N}]$ and $x \in B_X$, for all $n \in \mathbb{N}$ there exists $x_n \in X$ and $t_n \in (s_{k_{n-1}-1}, s_{k_{n-1}})$ ($t_0 = 0$) such that*

(i) $x = \sum x_n$,

and for all $n \in \mathbb{N}$,

(ii) *either* $\|x_n\| < \delta_n$ *or* $\|x_n - P^E_{(t_{n-1},t_n)} x_n\| < \delta_n \|x_n\|$,

(iii) $\|x_n - P^E_{(t_{n-1},t_n)} x\| < \delta_n$,

(iv) $\|x_n\| < K + 1$,

(v) $\|P^E_{t_n} x\| < \delta_n$.

Proof. First, if $m \in \mathbb{N}$ and $\varepsilon > 0$, we can choose $r > m$ so that for any $z \in B_Z$, there exists $t \in (m, r)$ so that $\|P_t^E z\| < \varepsilon$. If it were not true, we could find a sequence $(z_r)_{r>m} \subset B_Z$ so that for all $t \in (m, r)$, $\|P_t^E z\| \geq \varepsilon$. If z is any w^* limit of a subsequence of this sequence and if $t > m$, $\|P_t^E z\| \geq \varepsilon$, an obvious contradiction. We then choose $0 = r_0 < r_1 < \dots$ recursively so that if $x \in B_X$ and $n \in \mathbb{N}$, there exists $t \in (r_{n-1}, r_n)$ with $\|P_t^E x\| < \varepsilon_n$. Here, $\varepsilon_n \downarrow 0$ is chosen so that $(1+K)\varepsilon_1 < \delta_1^2$ and $(1+K)(2\varepsilon_n + \varepsilon_{n-1}) < \delta_n^2$ for each $1 < n \in \mathbb{N}$.

Next, we recursively select $0 = j_0 < j_1 < \dots$ and set $s_n = r_{j_n}$, so that for each $n \in \mathbb{N}$ and each $x \in B_X$, there exists $t_n \in (j_{n-1}, j_n)$ so that $\|P_{t_n}^E x\| < \varepsilon_n$ and $d(P_{[1,t)}^E x, X) < \varepsilon_n$. If we cannot complete the recursive construction, assume we have chosen $0 = j_0 < \dots < j_{p-1}$ to satisfy this conclusion, but we cannot find an appropriate j_p. Let $j = j_{p-1}$ and $\varepsilon = \varepsilon_p$. If we cannot complete this step of the construction, this means that for any $i > j$ there must exist some $x_i \in B_X$ so that for each $t \in (r_j, r_i)$, either $\|P_t^E x_i\| \geq \varepsilon$ or $d(P_{[1,t)}^E x, X) \geq \varepsilon$. But we can choose for each $j < k \leq i$ some $t_{ik} \in (r_{k-1}, r_k)$ so that $\|P_{t_{ik}}^E x_i\| < \varepsilon_k \leq \varepsilon$. Therefore it must be that for each $j < k \leq i$, $d(P_{[1,t_{ik})}^E x_i, X) \geq \varepsilon$. We can pass to a subsequence so that for each $j < k$, $t_{ik} \underset{i\to\infty}{\to} t_k \in (r_{k-1}, r_k)$ and so that x_i is w^* convergent to some $x \in B_X$. Then $d(P_{[1,t_k)}^E x, X) = \lim_i d(P_{[1,t_{ik})}^E x_i, X) \geq \varepsilon$ for all $k > j$, which is absurd. Therefore we can complete the recursive construction.

Fix $x \in B_X$, $(k_n)_{n\geq 0} \in [\mathbb{N}]$. We can find for each $n \in \mathbb{N}$ some $t_n \in (s_{k_{n-1}-1}, s_{k_{n-1}})$ so that $d(P_{[1,t_n)}^E x, X) < \varepsilon_{k_{n-1}} \leq \varepsilon_n$ and so that $\|P_{t_n}^E x\| < \varepsilon_{k_{n-1}} \leq \varepsilon_n$. Note that (v) is satisfied by this choice. Choose $y_n \in X$ so that $\|y_n - P_{[1,t_n)}^E x\| < \varepsilon_n$. Let $x_1 = y_1$ and let $x_n = y_n - y_{n-1}$ for each $1 < n \in \mathbb{N}$. Then for each $N \in \mathbb{N}$,

$$x - \sum_{n=1}^{N} x_n = x - y_N = x - P_{[1,t_N)}^E x + P_{[1,t_N)}^E x - y_N \to 0.$$

Thus $x = \sum x_n$, which is (i). Let now $t_0 = 0$. Note that

$$\begin{aligned}\|x_n - P^E_{(t_{n-1},t_n)}x\| &= \|y_n - y_{n-1} - P^E_{[1,t_n)}x + P^E_{t_n}x + P^E_{[1,t_{n-1})}x\| \\ &\le 2\varepsilon_n + \varepsilon_{n-1} < \delta_n,\end{aligned}$$

which gives (iii). Since $\|P^E_{(t_{n-1},t_n)}x\| \le K$ and $\delta_n < 1$, (iii) implies (iv). For (ii), note that

$$\begin{aligned}\|x_n - P^E_{(t_{n-1},t_n)}x_n\| &\le \|x_n - P^E_{(t_{n-1},t_n)}x\| + \|P^E_{(t_{n-1},t_n)}\|\|P^E_{(t_{n-1},t_n)}x - x_n\| \\ &< (1+K)(2\varepsilon_n + \varepsilon_{n-1}) < \delta_n^2.\end{aligned}$$

Either $\|x_n\| < \delta_n$, or $\delta_n^2 \le \delta_n\|x_n\|$, as desired.

□

Lemma 3.10. *Let X and Z be Banach spaces, E an FDD for Z, and $Q : Z \to X$ a surjection. If $(x_k) \subset S_X$ is weakly null, $Q(CB_Z) \supset B_X$ for some $C > 0$, then for all $\varepsilon > 0$ and $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $z \in 2CB_Z$ with finite support such that $P^F_{[1,n]}z = 0$ and $\|Qz - x_m\| < \varepsilon$.*

Proof. It is sufficient to find a subsequence $(x_k)_{k\in N}$ of (x_k) and a sequence $(z_k)_{k\in N} \subset 2CB_Z$ so that $\|P^F_{[1,n]}z_k\| \underset{k\in N}{\to} 0$ and $\|x_k - Qz_k\| \underset{k\in N}{\to} 0$. This is because in this case, for all m in some tail $M \in [N]$ of N, z_m will have a small perturbation $z'_m \in 2CB_Z$ with $P^F_{[1,n]}z'_m = 0$ and $\|x_m - Qz'_m\| < \varepsilon$.

Choose a sequence $(w_k) \subset CB_Z$ so that for all $k \in \mathbb{N}$, $Qw_k = x_k$. By passing to a subsequence $N \in [\mathbb{N}]$, we can assume $(P^F_{[1,n]}w_k)_{k\in N}$ is norm convergent. Take a convex blocking $y_k = \sum_{i\in I_k} a_i x_i$ of $(x_k)_{k\in N}$ which is norm null. Let $z_k = w_k - \sum_{i\in I_k} a_i w_i$, so that $Qz_k = x_k - \sum_{i\in I_k} a_i x_i$. Note that $z_k \in 2CB_Z$. Last $P^F_{[1,n]} z_k \underset{k\in N}{\to} 0$ and

$$\|x_k - Qz_k\| = \Big\|\sum_{i\in I_k} a_i x_i\Big\| \underset{k\in N}{\to} 0.$$

As stated above, we can take z to be a small perturbation of z_m for large enough $m \in \mathbb{N}$.

□

The following lemma is essentially obvious, but is a matter of bookkeeping. In order to preserve clarity of later proofs, and because it will be used multiple times, we prove it separately. It is quite clear at this point that an FDD H for a Banach space Z satisfies subsequential V lower block estimates in Z if and only if $\|\cdot\|_Z$ and $\|\cdot\|_{Z^V(H)}$ are equivalent on $c_{00}(H)$. But if we have a Banach space B with FDD G and a subspace $B' = \overline{(\oplus_n G_{k_n})}$ for some $1 \le k_1 < k_2 < \ldots$, then the spaces $B^V(G)$ and $(B')^V(H)$ need not be the same. Here, $H_n = G_{k_n}$. This is because for an interval I, P^G_I and P^H_I are different, so that the coefficients $\|P^G_{[r_{n-1},r_n)} z\|_Z$ and $\|P^H_{[r_{n-1},r_n)} z\|_Z$ may be different. We would like to know that if H satisfies subsequential V lower block estimates in B', the norms $\|\cdot\|_{B^V(G)}$ and $\|\cdot\|_{B'}$ are equivalent on $c_{00}(H)$.

Lemma 3.11. *Let E be a Banach space with normalized, 1-unconditional basis (e_n) which is 1 left dominant and satisfies subsequential C-E lower block estimates in E. Let B be a Banach space with FDD G. Let $(k_n) \in [\mathbb{N}]$, $H_n = G_{k_n}$, and suppose that the FDD H satisfies subsequential K-E lower block estimates in $B' = \overline{(\oplus H_n)} = \overline{(\oplus G_{k_n})}$. Then the norms $\|\cdot\|_B$ and $\|\cdot\|_{B^E(G)}$ are CK-equivalent on $c_{00}(H)$.*

Proof. Let $b \in c_{00}(H)$, and assume $b \neq 0$. We can choose $N \in \mathbb{N}$ and intervals $I_1 < \ldots < I_N$ so that

$$\|b\|_{B^E(G)} = \Big\|\sum_{\ell=1}^{N} \|P^G_{I_\ell} b\|_B e_{i_\ell}\Big\|_E,$$

where $i_\ell = \min I_\ell$ for all $\ell \le N$. We can also assume that $P^G_{I_\ell} b \neq 0$. Let $J_\ell = \mathrm{ran}_H(P^G_{I_\ell} b)$ for $1 \le \ell \le N$. Letting $j_\ell = \min J_\ell$, we note that $P^H_{J_\ell} b = P^G_{I_\ell} b$ for $1 \le \ell \le N$ and

$$i_1 \le k_{j_1} < \ldots < i_N \le k_{j_N}.$$

Then using left dominance of (e_n) and lower block estimates of H and (e_n) in B' and E, respectively, we see that

$$\begin{aligned}\|b\|_{B^E(G)} &= \Big\|\sum_{\ell=1}^N \|P^G_{I_\ell} b\|_B e_{i_\ell}\Big\|_E \le C\Big\|\sum_{\ell=1}^N \|P^G_{I_\ell} b\|_B e_{k_{j_\ell}}\Big\|_E \\ &\le C\Big\|\sum_{\ell=1}^N \|P^G_{I_\ell} b\|_B e_{j_\ell}\Big\|_E = C\Big\|\sum_{\ell=1}^N \|P^H_{J_\ell} b\|_B e_{j_\ell}\Big\|_E \\ &\le CK\|b\|_B.\end{aligned}$$

Since $\|\cdot\|_B \le \|\cdot\|_{B^E(G)}$, we are done.

□

Proposition 3.12. *Suppose W, Z are Banach spaces with boundedly complete FDDs F, E, respectively. Suppose the projection constant of F in W is 1, and the projection constant of E in Z is at most K. Let $Q : W \to X$ be (isometrically) a w^*-w^* continuous quotient map onto a w^* closed subspace X of Z. Suppose $(\varepsilon_n) \subset (0,1)$, $\varepsilon_n \downarrow 0$ is fixed so that for $p < q$ and $w \in \oplus_{n\in(p,q)} F_n$, $\|P^E_{[1,p)} Qw\| < \varepsilon_p \|w\|/K$ and $\|P^E_{[q,\infty)} Qw\| < \varepsilon_q \|w\|/K$. Then there exist $0 = s_0 < s_1 < \ldots$ so that if for each $n \in \mathbb{N}$ we define*

$$C_n = \oplus_{i=s_{n-1}+1}^{s_n} F_i, \quad D_n = \oplus_{i=s_{n-1}+1}^{s_n} E_i,$$

$$L_n = \Big\{i \in \mathbb{N} : s_{n-1} < i \le (s_{n-1} + s_n)/2\Big\},$$

$$R_n = \Big\{i \in \mathbb{N} : (s_{n-1} + s_n)/2 < i \le s_n\Big\},$$

$$C_{n,L} = \oplus_{i \in L_n} F_i, \quad C_{n,R} = \oplus_{i \in R_n} F_i,$$

then the following holds. Let $x \in S_X$, $0 \leq m < n$ and $\varepsilon > 0$. Assume that $\|x - P^D_{(m,n)}x\| < \varepsilon$. Then there exists $w \in B_W$ with $w \in [C_{m,R} \cup (C_i)_{m<i<n} \cup C_{n,L}]$ (where $C_{0,R} = (0)$) and $\|Qw - x\| < 2K\varepsilon + 6K\varepsilon_m$. If $m = 0$, we can replace this last inequality with $\|Qw - x\| < K\varepsilon + 3K\varepsilon_1$.

By an isometric quotient map, we mean that X has the quotient norm induced by this map. That is, $Q : W \to X$ is a surjection so that for each $x \in X$, $\|x\| = \inf_{Qw=x} \|w\|$.

Proof. As in the proof of Proposition 3.9, we can deduce the existence of a sequence $0 = s_0 < s_1 < \ldots$ so that if for each $n \in \mathbb{N}$, L_n, R_n are defined as above, then for any $w \in W$ we can find $\ell_n \in L_n$, $r_n \in R_n$ so that $\|P^F_{\ell_n} w\|, \|P^F_{r_n} w\| < \varepsilon_n \|w\|/K$. Define C, D as above. Suppose $x \in S_X$ satisfies $\|P^D_{[1,m]\cup[n,\infty)} x\| < \varepsilon$. Let $w \in W$ have norm 1 with $Qw = x$. Choose $r_m \in R_m$ and $\ell_n \in L_n$ with $\|P^F_{r_m} w\| < \varepsilon_m/K$ and $\|P^F_{\ell_n} w\| < \varepsilon_n/K$. Let $w' = P^F_{(r_m,\ell_n)} w$. Note that $\|w'\| \leq 1$ and $w' \in [C_{m,R} \cup (C_i)_{m<i<n} \cup C_{n,L}]$. We also observe that

$$\|P^E_{[1,r_m)\cup[\ell_n,\infty)} Qw'\| < (\varepsilon_{r_m}/K + \varepsilon_{\ell_n}/K)\|w'\| \leq \varepsilon_{r_m} + \varepsilon_{\ell_n}.$$

Also,

$$\begin{aligned}\|P^E_{[r_m,\ell_n)} Q(w - w')\| &= \left\| P^E_{[r_m,\ell_n)} Q(P^F_{[1,r_m)} w + P^F_{r_m} w + P^F_{\ell_n} w + P^F_{(\ell_n,\infty)} w) \right\| \\ &< \varepsilon_{r_m} + \varepsilon_m + \varepsilon_n + \varepsilon_{\ell_n}.\end{aligned}$$

Since $\|P^E_{[1,r_m)\cup(\ell_n,\infty)}x\| < 2K\varepsilon$, we deduce

$$\begin{aligned}\|Qw' - x\| &\leq \|P^E_{[1,r_m)\cup[\ell_n,\infty)}(Qw'-x)\| + \|P^E_{[r_m,\ell_n)}(Qw'-x)\| \\ &< \varepsilon_{r_m} + \varepsilon_{\ell_n} + 2K\varepsilon + K(\varepsilon_{r_m} + \varepsilon_m + \varepsilon_n + \varepsilon_{\ell_n}) \\ &< 2K\varepsilon + 6K\varepsilon_m.\end{aligned}$$

For the last statement, we simply repeat the argument, except all indices on ε_k terms satisfy $k \geq 1$ and we now only have projections onto tails instead of both tails and initial segments.

□

3.2 Trees and games

Next, we define the uncoordinatized version of subsequential block estimates, which was first considered in [23]. Our notation differs slightly to remain consistent with notation from Chapter II.

We let $\mathcal{E} = \{E \in [\mathbb{N}]^{<\omega} : 0 < |E| \text{ is even}\}$. We call a tree indexed by $\mathcal{E}$ an *even tree*. Consider an even tree $(x_E)_{E\in\mathcal{E}}$ in a Banach space. For each $M = (m_n) \in [\mathbb{N}]$, the sequence $(m_{2n-1}, x_{M|_{2n}})_n$ is called a *branch* of the tree. The notions of *weakly null even tree*, *w^* null even tree*, and *block even tree* are defined similarly to in Chapter II.

We need the even tree analogue of our pruning lemma. If $\mathcal{F} \subset \mathcal{E}$ is closed under taking restrictions to non-empty initial segments so that for each $E \in \mathcal{F}$ and $m \in \mathbb{N}$ the set $\{n : E^\wedge m^\wedge n \in \mathcal{E}\}$ is either empty or infinite, and if the latter occurs for infinitely many values of m, then there exists a pruning $\phi : \mathcal{E} \to \mathcal{F}$ which is onto. It should be noted that the construction of an "almost biorthogonal" even tree works as well in this case. That is, if $(x_E)_{E\in\mathcal{E}} \subset S_X$ is a weakly null even tree in a Banach space with separable dual, for any $\delta > 0, \varepsilon_n > 0$, there exists a pruning $\phi : \mathcal{E} \to \mathcal{E}$ and an even tree $(f_E)_{E\in\mathcal{E}}$ so that for each $E \in \mathcal{E}$, $f_E(x_{\phi(E)}) > 1/2 - \delta$ and for each $E, F \in \mathcal{E}$ with $E \preceq F$ or $F \preceq E$, $|f_E(x_{\phi(E)})| < \min\{\varepsilon_{|E|}, \varepsilon_{|F|}\}$. The analogous result concerning a w^* null even tree $(f_E)_{E\in\mathcal{E}} \subset S_{X^*}$, where X is a Banach space containing no copy of ℓ_1, is also valid.

Let U be a Banach space with normalized, 1-unconditional basis (u_n) and let $C \geq 1$. Let X be an infinite dimensional Banach space. We say that X satisfies *subsequential C-U upper tree estimates* (resp. *C-U lower tree estimates*) if every normalized, weakly null even tree $(x_E)_{E\in\mathcal{E}}$ has a branch $(m_{2n-1}, x_{M|_{2n}})_n$ so that $(x_{M|_{2n}})_n \lesssim_C (u_{m_{2n-1}})_n$ (resp. $(u_{m_{2n-1}})_n \lesssim_C (x_{M|_{2n}})_n$). We say X satisfies *subsequential U upper tree estimates* (resp. *lower tree estimates*) if X satisfies subsequential C-U upper tree estimates (resp. lower tree estimates) for some $C \geq 1$. If X is a subspace of a dual space, we define w^* subsequential U and subsequential C-U upper or lower tree estimates similarly. If we have two spaces, V, U each of which has a normalized, 1-unconditional basis, we say X satisfies *subsequential C-(V, U) tree estimates* if X satisfies subsequential C-V lower tree estimates and subsequential C-U upper tree estimates. We define *subsequential (V, U) tree estimates* similarly.

For $C \geq 1$, let $\mathcal{A}_U(C)$ denote the class of Banach spaces in **SD** which satisfy subsequential C-U upper tree estimates and let $\mathcal{A}_U = \bigcup_C \mathcal{A}_U(C)$. If we have two spaces, U, V, each with a normalized, 1-unconditional basis, we let $\mathcal{A}_{V,U}(C)$ denote the class of all Banach spaces in **REFL** which satisfy subsequential C-(V, U) tree estimates. The class $\mathcal{A}_{V,U}$ is defined similarly. We will prove that under certain assumptions on the basis of U or the basis of U and V, the class $\mathcal{A}_U$ or $\mathcal{A}_{V,U}$ will contain universal elements.

The dualization of tree estimates is more complicated than the dualization of block estimates, but under certain assumptions, it can be done.

Lemma 3.13. *Let X be a Banach space with separable dual, U a Banach space with normalized, 1-unconditional, 1-right dominant basis. If X satisfies subsequential U upper tree estimates, then X^* satisfies w^* subsequential $U^{(*)}$ lower tree estimates.*

Proof. Fix a w^* null even tree $(f_E)_{E\in\mathcal{E}} \subset S_{X^*}$. By our pruning lemma from Chapter II, for any fixed $\delta \in (0, 1/2)$ and $\varepsilon_n > 0$ we can find a pruning $\phi : \mathcal{E} \to \mathcal{E}$ and $(x_E)_{E\in\mathcal{E}} \subset B_X$ weakly null such that for each $E \in \mathcal{E}$, $f_{\phi(E)}(x_E) > 1/2 - \delta$ and for each $E, F \in \mathcal{E}$ such that $E \preceq F$ or $F \preceq E$,

$$|f_{\phi(E)}(x_F)| \leq (1/2 - \delta) \min\{\varepsilon_{|E|}, \varepsilon_{|F|}\}.$$

By replacing x_E with $x_E/\|x_E\|$, we can assume $(x_E)_{E\in\widehat{[\mathbb{N}]^{<\omega}}} \subset S_X$, except now we know only that

$$|f_{\phi(E)}(x_F)| \leq \min\{\varepsilon_{|E|}, \varepsilon_{|F|}\}.$$

If X satisfies subsequential C-U upper tree estimates, we can find $M \in [\mathbb{N}]$ so that

$$(x_{M|_{2n}})_n \lesssim_C (u_{m_{2n-1}})_n.$$

Recall that by the definition of pruning, defining $\ell_n = \max \phi(M|_n)$ gives us $(\ell_n) = L \in [\mathbb{N}]$. Note that $m_n \leq \ell_n$ for all $n \in \mathbb{N}$. For any $\varepsilon > 0$, we could have chosen δ, ε_n so that the "almost biorthogonality" implies

$$(u^*_{m_{2n-1}})_n \lesssim_{2C+\varepsilon} (f_{L|_{2n}})_n,$$

whence

$$(u^*_{\ell_{2n-1}})_n \lesssim_{2C+\varepsilon} (f|_{L_{2n}})_n$$

by 1-left dominance of (u^*_n).

□

We finish this section with a proposition relating infinite asymptotic games to trees and branches. To present this proposition, we must delineate some notation and discuss the notion of infinite asymptotic games. We will frame our coordinatization result as a game between two players and use this interpretation to prove a key result. We let X be a Banach space, $\mathcal{A} \subset (\mathbb{N} \times S_X)^\omega$ and $\varepsilon > 0$. We let

$$\mathcal{A}_\varepsilon = \big\{(k_n, y_n) \in (\mathbb{N} \times S_X)^\omega : k_n < k_{n+1}, \exists (\ell_n, x_n) \in \mathcal{A}, \ell_n \leq k_n, \\ \|x_n - y_n\| < \varepsilon/2^n \ \forall n \in \mathbb{N}\big\}.$$

We will topologize $(\mathbb{N} \times S_X)^\omega$ with the product of the discrete topologies, and all closures $\overline{\mathcal{A}_\varepsilon}$ will be with respect to this topology. It is clear that if $(k_n, x_n) \in \overline{\mathcal{A}_\varepsilon}$ and $(p_n) \in [\mathbb{N}]$ satisfies $p_n \geq k_n$ for all $n \in \mathbb{N}$, $(p_n, x_n) \in \overline{\mathcal{A}_\varepsilon}$.

Let E be an FDD for a Banach space Z and let $\delta = (\delta_n) \subset (0,1)$ with $\delta_n \downarrow 0$. A sequence $(z_n) \subset S_Z$ is called a *δ-skipped block with respect to E* if there exist integers $1 \leq k_0 < k_1 < \ldots$ so that for all n,

$$\|y_n - P^E_{(k_{n-1}, k_n)} z_n\| < \delta_n.$$

Let Z be a Banach space with FDD E and assume Z contains our Banach space X isometrically. For each $m \in \mathbb{N}$, we let $Z_m = \overline{\big(\oplus_{n>m} E_n\big)}$. Let $\mathcal{A} \subset (\mathbb{N} \times S_X)^\omega$ be fixed. Given $\varepsilon \in (0,1)$, we consider the following game between players S (for subspace) and V (for vector). On the n^{th} move, player S picks $k_n, m_n \in \mathbb{N}$ and player V chooses $x_n \in S_X$ with $\|x + Z_{m_n}\|_{Z/Z_{m_n}} < \varepsilon 2^{-n}$. We say S wins the game if $(k_n, x_n) \in \overline{\mathcal{A}_{4\varepsilon}}$. If $(k_n, x_n) \notin \overline{\mathcal{A}_{4\varepsilon}}$, we say V wins. We refer to this as the $(\mathcal{A}, \varepsilon)$ game.

Proposition 3.14. *Let X be an infinite dimensional, w^* closed subspace of a Banach* *space Z with boundedly complete FDD E, where the w^* topology is that coming from the natural predual of Z. Let $\mathcal{A} \subset (\mathbb{N} \times S_X)^\omega$. Then following statements are equivalent.*

(i) For all $\varepsilon > 0$, S has a winning strategy for the $(\mathcal{A}, \varepsilon)$ game.

(ii) For all $\varepsilon > 0$ there exists $(k_n) \in [\mathbb{N}]$ and $(\delta_n) \subset (0,1)$ with $\delta_n \downarrow 0$ and a blocking F of E such that if $(x_n) \subset S_X$ is a $\overline{\delta}$-skipped block with respect to F in Z such that $\|x_n - P^F_{(r_{n-1},r_n)} x_n\| < \delta_n$ for all $n \in \mathbb{N}$, where $1 \le r_0 < r_1 < \ldots$, then $(k_{r_{n-1}}, x_n) \in \overline{\mathcal{A}_\varepsilon}$.

(iii) For all $\varepsilon > 0$, every normalized, w^* null even tree in X has a branch in $\overline{\mathcal{A}_\varepsilon}$.

Proof. (i)$\Rightarrow$ (ii) Fix $\varepsilon > 0$. Choose a winning strategy (f, g) for S in the $(\mathcal{A}, \varepsilon)$ game. That is, $f, g : S_X^{<\omega} \to \mathbb{N}$ are such that if $(\ell_n), (m_n) \in \mathbb{N}^\omega$ and $(x_n) \in S_X^\omega$ are such that $\left\| x_n + Z_{m_n} \right\| < \varepsilon 2^{-n}$, $m_n \ge g(x_1, \ldots, x_{n-1})$, and $\ell_n = f(x_1, \ldots, x_{n-1})$ for all $n \in \mathbb{N}$, then $(\ell_n, x_n) \in \overline{\mathcal{A}_{4\varepsilon}}$.

For each finite interval I of natural numbers and each $\delta > 0$, choose a finite 3δ-net $D(I, \delta)$ of $\{x \in S_X : \|x - P^E_I x\| < \delta\}$.

Choose $m_1 \ge g(\varnothing)$ and $m_2 > m_1$. Let $F_1 = \oplus_{j=1}^{m_1} E_j$ and $F_2 = \oplus_{j=m_1+1}^{m_2} E_j$. Next, suppose we have chosen $m_1 < \ldots < m_{i-1}$ and set $F_j = \oplus_{k=m_{j-1}+1}^{m_j} E_k$. Choose $m_i > m_{i-1}$ so large that if $\ell \in \mathbb{N}$, $1 \le r_0 < r_1 < \ldots < r_{\ell-1} \le i$, and $(x_j)_{j=1}^\ell \in S_X^{<\omega}$ such that

$$x_j \in D((m_{r_{j-1}}, m_{r_j - 1}], \varepsilon 2^{-j})$$

for all $1 \le j \le \ell$, then $m_i \ge g(x_1, \ldots, x_\ell)$. There are only finite many such ℓ, r_j, and x_j, so we can make this choice. Then let $F_i = \oplus_{j=m_{i-1}+1}^{m_i} E_j$.

Choose $f(\varnothing) \le k_1 < k_2 < \ldots$ so that if $\ell \in \mathbb{N}$, $1 \le r_0 < \ldots < r_\ell \le n$, and

$(x_j)_{j=1}^\ell \in S_X^\omega$ satisfies

$$x_j \in D((m_{r_{j-1}}, m_{r_j-1}], \varepsilon 2^{-j})$$

for all $1 \leq j \leq \ell$, then $\ell_n \geq f(x_1, \dots, x_\ell)$. Let $\delta_n = \varepsilon 2^{-n}$. We show that with $F_i = \oplus_{j=m_{i-1}+1}^{m_i} E_j$, $(m_0 = 0)$, this blocking fulfills the requirements of (ii) with ε replaced by 7ε. Since $\varepsilon > 0$ is arbitrary, this will finish the proof of the first implication.

Suppose (y_n) is a δ-skipped block with respect to F, and suppose $1 \leq r_0 < r_1 < \dots$ are such that $\|y_n - P^F_{(r_{n-1}, r_n)}\| < \delta_n$. Note that $P^F_{(r_{n-1},r_n)} = P^E_{(m_{r_{n-1}}, m_{r_n-1}]}$, so for each $n \in \mathbb{N}$, there exists $x_n \in D((m_{r_{n-1}}, m_{r_n-1}], \varepsilon 2^{-n})$ so that $\|x_n - y_n\| < 3\delta_n = 3\varepsilon 2^{-n}$. Let $\ell_n = f(x_1, \dots, x_{n-1})$. We will prove using the properties of f, g that $(\ell_n, x_n) \in \overline{\mathcal{A}_{4\varepsilon}}$. We will then prove from our choice of k_n, ℓ_n that $k_{r_{n-1}} \geq \ell_n$, so that $(k_{r_{n-1}}, y_n) \in \overline{\mathcal{A}_{7\varepsilon}}$.

Observe that $m_{r_{n-1}} \geq m_1 \geq g(\varnothing)$. For $j \geq 1$, $r_j \geq 3$, so that since $1 \leq r_0 < \dots < r_{n-1} = r_{n-1}$,

$$x_j \in D((m_{r_{n-1}}, m_{r_n-1}], \varepsilon 2^{-n})$$

for $1 \leq j \leq n-1$, $m_{r_{n-1}} \geq g(x_1, \dots, x_{n-1})$. This is simply a consequence of our choice of m_i with $i = r_{n-1}$ and $\ell = n-1$. We also observe that

$$\left\| x_n + Z_{m_{r_{n-1}}} \right\| \leq \left\| x_n - P^E_{(m_{r_{n-1}}, m_{r_n-1}]} x_n \right\| < \varepsilon 2^{-n}$$

and $\ell_n = f(x_1, \dots, x_{n-1})$ implies $(\ell_n, x_n) \in \overline{\mathcal{A}_{4\varepsilon}}$. For $n > 1$, a similar argument using the choice of k_n with n replaced by r_{n-1} and $\ell = n-1$, $\ell_{r_{n-1}} \geq f(x_1, \dots, x_{n-1}) = k_n$. Noting that $k_1 = f(\varnothing) \leq \ell_1 \leq \ell_{r_0}$ finishes the implication.

(ii)$\Rightarrow$ (iii) Fix $\varepsilon > 0$. Choose $(k_n) \in [\mathbb{N}]$, $(\delta_n) \subset (0,1)$ and F as in (ii). Let $(x_E)_{E \in \mathcal{E}}$ be a normalized, w^* null even tree in X. Let $r_0 = 1$. Next, assume $1 = r_0 < \dots < r_n$

and $1 \leq m_1 < m_2 < \ldots < m_{2n}$ have been chosen. Pick $m_{2n+1} > k_{r_n}, m_{2n}$ and $m_{2n+2} > m_{2n+1}$ so large that

$$\|P^F_{[1,r_n]} x_{(m_1,\ldots,m_{2n+2})}\| < \delta_{n+1}/2.$$

Choose $r_{n+1} > r_n$ so large that

$$\|P^F_{[r_{n+1},\infty)} x_{(m_1,\ldots,m_{2n+2})}\| < \delta_{n+1}/2.$$

Let $M = (m_n)$. By our construction, $(k_{r_{n-1}}, x_{M|_{2n}})\overline{\mathcal{A}_\varepsilon}$. Since $k_{r_{n-1}} \leq m_{2n-1}$, $(m_{2n-1}, x_{M|_{2n}}) \in \overline{\mathcal{A}_\varepsilon}$.

(iii)⇒ (i) We prove that if S fails to have a winning strategy for some ε, then there exists a w^* null even tree in X failing to have a branch in $\overline{\mathcal{A}_\varepsilon}$. Without loss of generality, we may assume $\varepsilon < 1$. Since $\overline{\mathcal{A}_\varepsilon}$ is closed, the $(\mathcal{A}, \varepsilon)$ game is determined. This means that if S fails to have a winning strategy, then V has a winning strategy. That is, there exists a function f defined on all non-empty sequences of natural numbers of even length taking values in S_X so that if $(k_n), (m_n) \in \mathbb{N}^\omega$ and if $(x_n) \in S_X^\omega$ are such that $x_n = f(k_1, m_1, \ldots, k_n, m_n)$, then $\left\|x_n + Z_{m_n}\right\| < \varepsilon 2^{-n}$ and $(x_n) \notin \overline{\mathcal{A}_{4\varepsilon}}$. Using this function, we will construct $(x_E)_{E\in\mathcal{E}}, (y_E)_{E\in\mathcal{E}} \subset S_X$ and $(m_E)_{E\in\mathcal{E}} \subset \mathbb{N}$ so that

(a) $(y_E)_{E\in\mathcal{E}}$ is w^* null,

(b) $\|x_E - y_E\| < 3\varepsilon 2^{|E|/2}$ for all $E \in \mathcal{E}$,

(c) if $E \in \mathcal{E}$, $E = (k_1, \ldots, k_{2n})$, then

$$x_E = f(k_1, m_{E|_2}, k_3, m_{E|_4}, \ldots, k_{|E|-1|}, m_E).$$

First, we see how this finishes the proof. Fix natural numbers $i_1 < i_2 < \dots$ and let $I = (i_n)$. Let $k_n = i_{2n-1}$ and $m_n = m_{I|2n}$. Then $x_{I|2n} = f(k_1, m_1, \dots, k_n, m_n)$. This means $(k_n, x_{I|2n}) \notin \overline{\mathcal{A}_{4\varepsilon}}$. But if $(k_n, y_{I|2n}) = (i_{2n-1}, y_{I|2n}) \in \overline{\mathcal{A}_\varepsilon}$, the fact that $\|x_{I|2n} - y_{I|2n}\| < 3\varepsilon 2^{-n}$ would imply that $(k_n, x_{I|2n}) \in \overline{\mathcal{A}_{4\varepsilon}}$. This means $(i_{2n-1}, y_{I_{2n}}) \notin \overline{\mathcal{A}_\varepsilon}$, and we have found a w^* null even tree in S_X with no branch in $\overline{\mathcal{A}_\varepsilon}$.

Fix $k \in \mathbb{N}$. For $i \in \mathbb{N}$, let $z_i = f(k, i)$. Choose $i_1 < i_2 < \dots$ and $x \in X$ so that $z_{i_j} \underset{w^*}{\to} z$. Note that since $\left\| z_{i_j} + Z_{i_j} \right\| < \varepsilon 2^{-1}$ for all $j \in \mathbb{N}$, $\left\| z + Z_{i_j} \right\| \leq \varepsilon 2^{-1}$ for all $j \in \mathbb{N}$. This means $\|z\| \leq \varepsilon 2^{-1}$. For each $j > k$, let

$$m_{(k,j)} = i_j, x_{(k,j)} = z_{i_j}, y_{(k,j)} = \frac{z_{i_j} - z}{\|z_{i_j} - z\|}.$$

Properties (a)-(c) are easily verified.

Next, assume that for some $1 < \ell \in \mathbb{N}$ and for each $E \in \mathcal{E}$ with $|E| < 2\ell$, x_E, y_E, m_E have been chosen. Fix E with $|E| = 2\ell - 2$. Fix $k > E$. For $i \in \mathbb{N}$, let

$$z_i = f(k_1, m_{E|2}, k_3, m_{E|4}, \dots, k, i),$$

where $E = (k_1, k_2, \dots, k_{2\ell-2})$. Again, choose $i_1 < i_2 < \dots$, $z \in X$ so that $z_{i_j} \underset{w^*}{\to} z$. Note that $\|z\| \leq \varepsilon 2^{-\ell}$. For all $j > k$, let

$$m_{E\frown k\frown j} = i_j, x_{E\frown k\frown j} = z_{i_j}, y_{E\frown k\frown j} = \frac{z_{i_j} - z}{\|z_{i_j} - z\|}.$$

Again (a)-(c) are easily verified.

□

Theorem 3.15. *Let V be a Banach space with normalized, 1-unconditional, 1-left dominant, boundedly complete basis (v_n). Suppose that X_0, X are separable Banach* *spaces with $X = X_0^*$. Suppose also that X satisfies subsequential V lower tree estimates.*

(i) *If Z is any Banach space with boundedly complete FDD E so that X embeds isomorphically into Z via a w^*-w^* continuous embedding, then there exists a blocking H of E, $M \in [\mathbb{N}]$, and a w^*-w^* continuous embedding of X into $Z^{V_M}(H)$, where X has the w^* topology induced by X_0.*

(ii) *There exist a Banach space $\tilde{W}$ with FDD $\tilde{H}$, $N \in [\mathbb{N}]$, and a w^*-w^* continuous surjection of $\tilde{W}^{V_N}(\tilde{G})$ onto X, where X has the w^* topology induced by X_0. If $X \in$ **REFL**, then $\tilde{G}$ can be taken to be shrinking for $\tilde{W}$.*

Proof. (i) By first equivalently renorming Z and then X, we can assume that E is bimonotone in Z and that X is isometrically isomorphic to a w^* closed subspace of Z. Let

$$\mathcal{A} = \Big\{(k_n, x_n) \in (\mathbb{N} \times S_X)^\omega : k_n < k_{n+1}, (v_{k_n}) \lesssim_C (x_n)\Big\}.$$

We can choose $\varepsilon > 0$ so small that

$$\overline{\mathcal{A}_\varepsilon} = \Big\{(k_n, x_n) \in (\mathbb{N} \times S_X)^\omega : k_n < k_{n+1}, (v_{k_n}) \lesssim_{2C} (x_n)\Big\}.$$

By Proposition 3.14, there exists a blocking F of E, a summable sequence $(\delta_n) \subset (0,1)$ which is strictly decreasing, and $(k_n) \in [\mathbb{N}]$ so that if $(x_n) \subset S_X$ and $1 \le r_0 < r_1 < \dots$ satisfy

$$\|x_n - P^F_{(r_{n-1}, r_n)} x_n\| < 2\delta_n$$

for all $n \in \mathbb{N}$, then $(v_{k_{r_{n-1}}}) \lesssim_{2C} (x_n)$.

Next, suppose D is a blocking of F with $D_n = \oplus_{i=j_{n-1}+1}^{j_n} F_i$. Suppose also that (I_n) are intervals, $1 \le r_0 < r_1 < \dots$ are such that $r_{n-1} + 1 = \min I_n < r_n$, and

$(x_n) \subset S_X$ is such that

$$\|x_n - P^D_{I_n} x_n\| < \delta_n$$

for all $n \in \mathbb{N}$. Then $(v_{k_{j_{r_{n-1}}}}) \lesssim_{2C} (x_n)$, because these conditions imply

$$\|x_n - P^F_{(j_{r_{n-1}}, j_{r_n})} x_n\| < 2\delta_n.$$

We will use without reference a similar fact in the proof of (ii) when the FDD F will not be bimonotone.

We can replace F with a blocking G so that for any subsequent blocking D of G, there exists $(e_n) \subset S_X$ so that $\|e_n - P^D_n e_n\| < \delta_n/2$. By the previous paragraph, there exists some subsequence (w_n) of (v_n) so that if $(x_n) \subset S_X$ and $1 \leq r_0 < r_1 < \dots$ are such that $\|x_n - P^G_{(r_{n-1}, r_n)} x_n\| < \delta_n$, then $(w_{r_{n-1}}) \lesssim_{2C} (x_n)$. To pass to the final blocking, choose $0 = s_0 < s_1 < \dots$ according to Proposition 3.9 applied to the FDD G and the sequence (δ_n). Let $H_n = \oplus_{i=s_{n-1}+1}^{s_n} G_i$. Let $(w'_n) = (w_{s_n})$ and pick $M \in [\mathbb{N}]$ so that $(w'_n) = (v_{m_n})$. We claim that the inclusion of X into Z also defines an isomorphic embedding of X into $Z^{V_M}(H)$, and that this is w^*-w^* continuous. To see this, fix $1 \leq n_0 < n_1 < \dots$ and $x \in S_X$. We will first find $A < \infty$ independent of the sequence (n_i) and the vector x so that

$$\Big\| \sum_i \|P^H_{[n_{i-1}, n_i)} x\|_Z w'_{n_{i-1}} \Big\|_V \leq A,$$

which will demonstrate that the inclusion is an isomorphic embedding.

Let $\Delta = \sum \delta_n$. Let $\ell_i = s_{n_i - 1}$. Observe that $P^G_{(\ell_{i-1}, \ell_i]} = P^H_{[n_{i-1}, n_i)}$. We seek A so that

$$\Big\| \sum_i \|P^G_{(\ell_{i-1}, \ell_i]} x w'_{n_{i-1}} \Big\|_V \leq A.$$

By our choice of (s_n), we can find $(x_i) \subset X$ and $(t_i) \in [\mathbb{N}]$ ($t_0 = 0$) to satisfy conclusions (i)-(v) of Proposition 3.9, with $t_i \in (\ell_{i-1}, s_{n_{i-1}}) = (s_{n_{i-1}-1}, s_{n_{i-1}})$.

If $\|x_{i+1}\| \geq \delta_{i+1}$, let $a_i = \|x_{i+1}\|$ and let $y_i = x_{i+1}$. If $\|x_{i+1}\| < \delta_{i+1}$, let $a_i = 0$ and let $y_i = e_{\ell_i}$ so that for each i,

$$\|y_i - P^G_{(t_i,t_{i+1})} y_i\| < \delta_i.$$

This means $(w_{t_i}) \lesssim_{2C} (y_i)$. Therefore

$$\begin{aligned} 1 = \|x\| = \left\|\sum x_i\right\| &\geq \left\|\sum a_i y_i\right\| - \|x_1\| - \Delta \\ &\geq \frac{1}{2C}\left\|\sum a_i w_{t_i}\right\| - 2 - \Delta \\ &\geq \frac{1}{2C}\left\|\sum \|x_{i+1}\| w_{t_i}\right\| - 2 - 2\Delta. \end{aligned}$$

From this we deduce that

$$\left\|\sum \|x_{i+1}\| w_{t_i}\right\| \leq 2C(3 + 2\Delta).$$

Moreover,

$$\|P^G_{(\ell_{i-1},\ell_i]} x\| \leq \|P^G_{(t_{i-1},t_{i+1})} x\| \leq \|x_i\| + \|x_{i+1}\| + 3\delta_i.$$

It follows that

$$\begin{aligned}\Big\|\sum \|P^G_{(\ell_{i-1},\ell_i]}x\|w'_{n_{i-1}}\Big\| &\le \Big\|\sum \|x_i\|w'_{n_{i-1}}\Big\| + \Big\|\sum \|x_{i+1}\|w'_{n_{i-1}}\Big\| + 3\Delta \\ &\le \Big\|\sum \|x_{i+1}\|w'_{n_i}\Big\| + \Big\|\sum \|x_{i+1}\|w'_{n_{i-1}}\Big\| + \|x_1\| + 3\Delta \\ &\le 2\Big\|\sum \|x_{i+1}\|w_{t_i}\Big\| + 2 + 3\Delta \\ &\le 4C(3+2\Delta) + 2 + 3\Delta.\end{aligned}$$

Here we used that $t_i < s_{n_{i-1}}$ and $w'_{n_{i-1}} = w_{s_{n_{i-1}}}$.

We last prove w^*-w^* continuity of this embedding. Note that the original w^*-w^* continuous isometric embedding $\iota : X \to Z$ must be the adjoint of a quotient map $Q : Z_0 \twoheadrightarrow X_0$, and Z_0 has H^* as a shrinking FDD. Moreover, H^* is also a shrinking FDD for the natural predual Y of $Z^{V_M}(H)$. Since $\iota : X \to Z^{V_M}(H)$ is coordinate-wise the same as the embedding $\iota : X \to Z$, the restriction of the adjoints of these maps to any H^*_n coincide, regardless of whether H^*_n is considered as a subspace of Z_0 or Y. Since $\iota^*|_{H^*_n} = Q|_{H^*_n}$ when H^*_n is considered as a subspace of Z_0, we deduce that ι^* maps $c_{00}(H^*) \subset Y$ into X_0. By density, ι^* maps Y into X_0, which implies ι is w^*-w^* continuous.

(ii) By Corollary 8 of [7], we can find Z_0 with shrinking FDD E_0 and a quotient map $Q : Z_0 \twoheadrightarrow X_0$. By a Lemma 3.1 of [21], we can find W_0 with shrinking FDD F_0 and an embedding $\iota : X_0 \to W_0$ so that $c_{00}(F_0) \cap X_0$ is dense in X_0 (identified with $\iota(X_0)$). By first renorming W_0, then X_0, then Z_0, we can assume F_0 is bimonotone in W_0, ι is an isometric embedding, and that $Q^* : X = X_0^* \to Z := Z_0^*$ is an isometric embedding. We will consider X as a subspace of Z and consider ι^* as mapping into either X or Z as is convenient. If X is reflexive, we can take the space W_0, Z_0 to be reflexive as well [21]. Fix C so large that X satisfies subsequential C-V lower tree estimates. Let K be the projection constant of E_0 in Z_0. Let $F = F_0^*, E = E_0^*$.

As in (i), we can choose $\varepsilon > 0$ so that if

$$\mathcal{A} = \Big\{(k_n, x_n) \in (\mathbb{N} \times S_X)^\omega : k_n < k_{n+1}, (v_{k_n}) \lesssim_C (x_n)\Big\},$$

then

$$\overline{\mathcal{A}_\varepsilon} = \Big\{(k_n, x_n) \in (\mathbb{N} \times S_X)^\omega : k_n < k_{n+1}, (v_{k_n}) \lesssim_{2C} (x_n)\Big\}.$$

By Proposition 3.14, there exists $(k_n) \in [\mathbb{N}]$, a blocking of E (which we also call E), and $(\delta_n) \subset (0,1)$ summable and decreasing so that if $(x_n) \subset S_X$ is a δ-skipped block with respect to E and $1 \leq r_0 < r_1 < \dots$ are such that

$$\|x_n - P^E_{(r_{n-1}, r_n)} x_n\| < 2K\delta_n$$

for all $n \in \mathbb{N}$, then $(v_{k_{r_{n-1}}}) \lesssim_{2C} (x_n)$. By making δ smaller if necessary, we can assume that if (x_n), (r_n) are as in the previous sentence and that if $(z_n) \subset Z$ is such that $\|z_n - x_n\| < \delta_n$ for all $n \in \mathbb{N}$, then (z_n) is basic, equivalent to (x_n), and has projection constant not exceeding $2K$. We also assume $\sum \delta_n < 1/7$. Choose $(\varepsilon_n) \subset (0,1)$ strictly decreasing and so small that for each $n \in \mathbb{N}$,

$$10K(K+1)\sum_{j=n}^{\infty} \varepsilon_j < \delta_n^2.$$

After blocking E if necessary, we may assume that for each further blocking D of E, there exists $(e_n) \subset S_X$ so that for each $n \in \mathbb{N}$, $\|e_n - P^D_n e_n\| < \varepsilon_{n+1}/2K$. After blocking F, we can assume that for each $n \in \mathbb{N}$, $\iota^*(F_n) \neq (0)$.

Using Lemma 3.8, we may block E, F and assume that for each $i < j$ and each

$w \in \oplus_{n\in(i,j)} F_n$,

$$\|P^E_{[1,i)}\iota^* w\| < \varepsilon_i \|w\|/K$$

and

$$\|P^E_{[j,\infty)}\iota^* w\| < \varepsilon_j \|w\|/K,$$

and that this property is preserved if we pass to a blocking of either E or F and to the corresponding blocking of the other.

Let C, D be the blockings of F, E, respectively, obtained from Proposition 3.12 with the sequence (ε_n). We apply Proposition 3.9 to the FDD D and the sequence (ε_n) to obtain $0 = s_0 < s_1 < \ldots$ to satisfy (i)-(v) of that proposition.

Let (v''_n) be any subsequence of (v_n) so that if $(x_n) \subset X$ is a δ-skipped block of D in Z with

$$\|x_n - P^D_{(r_{n-1},r_n)} x_n\| < \delta_n$$

for all $n \in \mathbb{N}$ and $1 \le r_0 < r_1 < \ldots$, then $(v''_{r_{n-1}}) \lesssim_{2C} (x_n)$. Such a sequence exists by an argument similar to that in (i). Let $v'_n = v''_{s_n}$, $G_n = \oplus_{i=s_{n-1}+1}^{s_n} C_i$, $H_n = \oplus_{i=s_{n-1}+1}^{s_n} D_i$. Let $N = (n_i) \in [\mathbb{N}]$ be such that $(v'_i) = (v_{n_i})$.

For $n \in \mathbb{N}$, let $\tilde{G}_n = G_n/\ker(\iota^*|_{G_n})$, endowed with the quotient norm $\|\tilde{w}_n\|_\sim = \|\iota^* w_n\|$. Note that $\tilde{G}_n \neq (0)$, since for each $k \in \mathbb{N}$, $\iota^*(F_k) \neq (0)$. Given $w = \sum w_n \in c_{00}(G)$, we set $\tilde{w} = \sum \tilde{w}_n \in c_{00}(\tilde{G})$. We set

$$\|\tilde{w}\|_\sim = \max_{i\le j} \Big\| \iota^* \big(\sum_{n=i}^{j} w_n \big) \Big\| = \max_{i\le j} \|\iota^* P^G_{[i,j]} w\|.$$

Clearly $\tilde{G}$ becomes a bimonotone FDD for the completion $\tilde{W}$ of $c_{00}(\tilde{G})$. Since G is bimonotone, $\|\tilde{w}\|_\sim \le \|w\|$ for all $w \in c_{00}(G)$, so that $w \mapsto \tilde{w}$ extends to a norm 1 operator from W into $\tilde{W}$. By the definition of $\|\cdot\|_\sim$,

$$\|\iota^* w\| \leq \|\tilde{w}\|_\sim$$

for each $w \in c_{00}(G)$. Thus $\tilde{w} \mapsto \iota^* w$ is a well-defined operator and extends to norm 1 operator $\tilde{\iota^*} : \tilde{W} \to X$. Moreover, $\iota^* w = \tilde{\iota^*}\tilde{w}$ for all $w \in W$. We need the following.

Claim 3.16. *(i) ι^* is a quotient map. More precisely, if $x \in S_X$ and $w \in S_W$ are such that $\iota^* w = x$, with $w = \sum w_n$, then $\tilde{w} = \sum \tilde{w}_n \in S_{\tilde{W}}$ and $\tilde{\iota^*}\tilde{w} = x$.*

(ii) If $(\tilde{w}_n)$ is a subnormalized block sequence in $\tilde{W}$ with respect to $\tilde{G}$ so that $(\iota^* \tilde{w}_n)$ is basic with projection constant at most $\tilde{K}$ and $a = \inf_n \|\tilde{\iota^*}\tilde{w}_n\| > 0$, then for all $(a_n) \in c_{00}$,

$$\Big\|\sum a_n \tilde{\iota^*}\tilde{w}_n\Big\| \leq \Big\|\sum a_n \tilde{w}_n\Big\|_\sim \leq \frac{3\tilde{K}}{a}\Big\|\sum a_n \tilde{\iota^*}\tilde{w}_n\Big\|.$$

(iii) In the reflexive case, $\tilde{G}$ is shrinking in $\tilde{W}$.

We postpone the proof of the claim until the end of Theorem 3.15. With $N = (n_i)$ as above, we will show that there exists $L < \infty$ so that for any $x \in S_X$, there exists $\tilde{w} \in \tilde{W}^{V_N}(\tilde{G})$ with $\|\tilde{w}\|_{\tilde{W}^{V_N}(\tilde{G})} \leq L$ so that $\|\tilde{\iota^*}\tilde{w} - x\| < 1/2$. This will prove that $\tilde{\iota^*} : \tilde{W}^{V_N}(\tilde{G}) \to X$ is onto. We note that since the $\tilde{W}^{V_N}(\tilde{G})$ norm dominates the $\tilde{W}$ norm, $\tilde{\iota^*}$ is a norm at most 1 operator on $c_{00}(\tilde{G})$ considered as a subspace of $\tilde{W}^{V_N}(\tilde{G})$, and so extends to a map on all of $\tilde{W}^{V_N}(\tilde{G})$ into X. We then prove that this map is w^*-w^* continuous, and then prove Claim 3.16.

Fix $x \in S_X$. Fix a sequence $(e_n) \subset S_X$ so that for each $n \in \mathbb{N}$, $\|e_n - P^D_n e_n\| < \varepsilon_{n+1}/2K$. Choose $(x_n) \subset X$ and $0 = t_0 < t_1 < \ldots$ according to Proposition 3.9 so that for each $n \in \mathbb{N}$, $t_n \in (s_{n-1}, s_n)$, $x = \sum x_n$, and either $\|x_n\| < \varepsilon_n$ or $\|x_n -$

$P^D_{(t_{n-1},t_n)}x_n\| < \varepsilon_n\|x_n\|$.

If $\|x_{n+1}\| \geq \varepsilon_{n+1}$, let $y_n = x_{n+1}/\|x_{n+1}\|$ and $a_n = \|x_{n+1}\|$. If $\|x_{n+1}\| < \varepsilon_{n+1}$, let $y_n = e_{s_n}$, $a_n = 0$. Note that $\|y_n - P^D_{(t_n,t_{n+1})}y_n\| < \varepsilon_{n+1}$ for all $n \in \mathbb{N}$. This means $(v''_{t_n}) \lesssim_{2C} (y_n)$. By Proposition 3.12, there exists a sequence $(w_n) \subset B_W$ with

$$w_n \in [C_{t_n,R} \cup (C_i)_{t_n<i<t_{n+1}} \cup C_{t_{n+1},L}]$$

such that

$$\|\iota^* w_n - y_n\| < 2K\varepsilon_{n+1} + 6K\varepsilon_n < 3K(K+1)\sum_{j=n}^{\infty}\varepsilon_j < \delta_n.$$

If $\|x_1\| < \varepsilon_1$, let $w_0 = 0$. Otherwise, use Proposition 3.12 again to find $w_0 \in W$ with $\|w_0\| < K + 1$ so that

$$w_0 \in [(C_i)_{0<i<t_1} \cup C_{t_1,L}]$$

such that

$$\|\iota^* w_0 - x_1\| \leq 4K\varepsilon_1\|x_1\| < 4K(K+1)\varepsilon_1.$$

Set $y = x_1 + \sum_{n=1}^{\infty} a_n y_n$. Note that this series converges and

$$\|x - y\| \leq \sum_{n=2}^{\infty}\varepsilon_n < 1/4.$$

By our choice of δ_n, and since $\|\tilde{\iota^*}\tilde{w}_n - y_n\| < \delta_n$, $(\tilde{\iota^*}\tilde{w}_n)$ is a basic sequence with projection constant not exceeding $2K$ and is equivalent to (y_n). Furthermore,

$$\inf_n \|\tilde{\iota^*}\tilde{w}_n\| \geq \inf_n \|y_n\| - \delta_n > 6/7,$$

and by Claim 3.16,

$$\left\|\sum c_n \tilde{\iota^*} \tilde{w}_n\right\| \leq \left\|\sum c_n \tilde{w}_n\right\| \leq 7K\left\|\sum c_n \tilde{\iota^*} \tilde{w}_n\right\| \tag{3.1}$$

for any $(c_n) \in c_{00}$. Thus $(\tilde{w}_n)$ is basic, equivalent to (y_n), and $\sum a_n \tilde{w}_n$ converges. Let $\tilde{w} = \tilde{w}_0 + \sum a_n \tilde{w}_n$. We have

$$\begin{aligned}\|\tilde{\iota^*}\tilde{w} - y\| &\leq \|\tilde{\iota^*}\tilde{w}_0 - x_1\| + \sum |a_n| \|\tilde{\iota^*}\tilde{w}_n - y_n\| \\ &\leq 10K(K+1)\sum \varepsilon_n < 1/4.\end{aligned}$$

Thus $\|\tilde{\iota^*}\tilde{w} - x\| < 1/2$. We next prove the norm estimate. Fix $1 \leq r_0 < r_1 < \ldots$. Note that $\tilde{w}_n \in \tilde{G}_n \oplus \tilde{G}_{n+1}$ and $\tilde{w}_0 \in \tilde{G}_1$. It follows that

$$\begin{aligned}\left\|\sum \|P^{\tilde{G}}_{[r_{n-1},r_n)}\tilde{w}\|_\sim v'_{r_{n-1}}\right\| &\leq \|\tilde{w}_0\|_\sim + \left\|\sum a_{r_{n-1}-1} v'_{r_{n-1}}\right\| \\ &\quad + \left\|\sum_n \left\|\sum_{i=r_{n-1}}^{r_n-1} a_i \tilde{w}_i\right\|_\sim v'_{r_{n-1}}\right\|,\end{aligned}$$

where we put $a_0 = 0$ if $r_0 = 1$. This is because $\tilde{w}_0$ may have non-zero image only under the first projection $P^{\tilde{G}}_{[r_0,r_1)}$, which accounts for the first term on the right. If $i \in [r_{n-1}, r_n - 1)$, $\tilde{w}_i$ may have non-zero image only under the projection $P^{\tilde{G}}_{[r_{n-1},r_n)}$. For $n \geq 1$, $\tilde{w}_{r_{n-1}-1}$ may have non-zero image under either $P^{\tilde{G}}_{[r_{n-1},r_n)}$ or $P^{\tilde{G}}_{[r_n,r_{n+1})}$. The images $P^{\tilde{G}}_{[r_n,r_{n+1})}\tilde{w}_{r_n-1}$ account for the second term, and the projections $P^{\tilde{G}}_{[r_{n-1},r_n)}\tilde{w}_i$ for $i \in [r_{n-1}, r_n)$ account for the second line. We will establish an upper bound for each term. The first term simply uses the fact that $\|x_1\| < K + 1$. The second term can be compared to $\left\|\sum a_n y_n\right\|$ using the skipped block condition. The third term will take more work, but it will be another application of the skipped block condition. For this case, we will consider the normalization of the blocking with terms $\sum_{i=r_{n-1}}^{r_n-1} a_i\tilde{w}_i$.

Since $(v''_{t_n}) \lesssim_{2C} (y_n)$ and since (v_n) is 1-left dominant,

$$\begin{aligned}\left\|\sum a_{r_{n-1}-1}v'_{r_{n-1}}\right\| \le \left\|\sum a_n v'_{n+1}\right\| &\le \left\|\sum a_n v''_{t_n}\right\| \\ &\le 2C\left\|\sum a_n y_n\right\| = 2C\|y - x_1\| \\ &\le 2C(\|x\| + \|x_1\| + \|x-y\|) < 2C(K+3).\end{aligned}$$

For each $n \in \mathbb{N}$, let

$$h_n = \sum_{i=r_{n-1}}^{r_n-1} a_i\tilde{w}_i, \quad g_n = \sum_{i=r_{n-1}}^{r_n-1} a_i y_i.$$

First note that (3.1) implies $\|h_n\|_\sim \le 7K\|\tilde{\iota^*}h_n\|$. Next, observe that

$$\begin{aligned}\|g_n - P^D_{(t_{r_{n-1}},t_{r_n})}g_n\| &\le \sum_{i=r_{n-1}}^{r_n-1} |a_i|2K\|y_i - P^D_{(t_i,t_{i+1})}\| \\ &< 2K(K+1)\sum_{i=r_{n-1}}^{\infty}\varepsilon_i < \delta_n^2.\end{aligned}$$

If $\|g_n\| \ge \delta_n$, let $f_n = g_n/\|g_n\|$, $b_n = \|g_n\|$. Otherwise let $f_n = y_{r_{n-1}}$ and $b_n = 0$. Then $(f_n) \subset S_X$ is such that

$$\|f_n - P^D_{(t_{r_{n-1}},t_{r_n})}f_n\| < \delta_n$$

for all $n \in \mathbb{N}$. This means (f_n) is a basic sequence with projection constant not exceeding $2K$ so that $(v''_{t_{r_{n-1}}}) \lesssim_{2C} (f_n)$. Then, with $\Delta = \sum \delta_n$ as before,

$$\begin{aligned}
\Big\|\sum \|h_n\|_{\sim} v'_{r_{n-1}}\Big\| &\le 7K\Big\|\sum \|\tilde{\iota^*} h_n\| v'_{r_{n-1}}\Big\| \\
&\le 7K\Big\|\sum \|g_n\| v'_{r_{n-1}}\Big\| + 7K\Delta \\
&\le 7K\Big\|\sum b_n v'_{r_{n-1}}\Big\| + 14K\Delta \\
&\le 7K\Big\|\sum b_n v''_{t_{r_{n-1}}}\Big\| + 14K\Delta \\
&\le 14CK\Big\|\sum b_n f_n\Big\| + 14K\Delta \\
&\le 14CK\Big\|\sum_{n=r_0}^{\infty} a_n y_n\Big\| + 14CK\Delta + 14K\Delta \\
&\le 28CK^2\Big\|\sum a_n y_n\Big\| + 14CK\Delta + 14K\Delta \\
&\le 28CK^2(K+3) + 14KC\Delta + 14K\Delta.
\end{aligned}$$

To prove w^*-w^* continuity, we first recall that $\tilde{G}$ is a boundedly complete FDD for $\tilde{W}^{V_N}(\tilde{G})$. This means this space has a natural predual, call it Y, for which $\tilde{G}^*$ is a shrinking FDD. We note that in this case, Y^{**} can be identified with all formal (not necessarily norm convergent) series $\sum y_n$, where $y_n \in \tilde{G}^*_n$ and $\sup_m \Big\|\sum_{n=1}^m y_n\Big\| < \infty$. If we choose $x \in X_0 = \iota(X_0) \subset W_0$ which has finite support with respect to G^*, and if we choose $n \notin \mathrm{supp}_{G^*}(x)$, then for any $\tilde{w} \in \tilde{G}_n$,

$$\langle (\tilde{\iota^*})^* x, \tilde{w}\rangle = \langle \tilde{\iota^*}\tilde{w}, x\rangle = \langle \iota^* w, x\rangle = \langle x, w\rangle = 0.$$

This means $\mathrm{supp}_{\tilde{G}^*}((\tilde{\iota^*})^* x) \subset \mathrm{supp}_{G^*}(x)$ is finite, and therefore $(\tilde{\iota^*})^* x \in Y$. Since $c_{00}(G^*) \cap X_0$ is dense in X_0, this gives w^*-w^* continuity.

□

Proof of Claim 3.16. (i) For any $x \in S_X$, w^* compactness of the unit ball of W and

w^*-w^* continuity of ι^* imply that there exists $w \in S_W$ with $\iota^* w = x$. Let $w = \sum w_n$ with $w_n \in G_n$. Then for any $i \leq j$ there exist $i \leq p \leq q \leq j$ so that

$$\Big\|\sum_{n=i}^{j} \tilde{w}_n\Big\|_\sim = \Big\|\tilde{\iota^*}\big(\sum_{n=p}^{q} \tilde{w}_n\big)\Big\| \leq \Big\|\sum_{n=p}^{q} w_n\Big\| \leq \Big\|\sum_{n=i}^{j} w_n\Big\|.$$

This gives convergence of $\tilde{w} = \sum \tilde{w}_n$ in $\tilde{W}$. Clearly $\tilde{\iota^*}\tilde{w} = x$.

(ii) The left inequality is clear, since $\tilde{\iota^*}$ has norm 1. Fix $(a_n) \in c_{00}$. Then there exist $i \leq j$ so that

$$\Big\|\sum a_n \tilde{w}_n\Big\|_\sim = \Big\|\tilde{\iota^*} P^{\tilde{G}}_{[i,j]} \sum a_n \tilde{w}_n\Big\|.$$

For all except perhaps two values of n, say $n_0 < n_1$, $P^{\tilde{G}}_{[i,j]} \tilde{w}_n$ is either 0 or $\tilde{w}_n$. Then

$$\begin{aligned}\Big\|\sum a_n \tilde{w}_n\Big\|_\sim &= \Big\|\tilde{\iota^*} P^{\tilde{G}}_{[i,j]} \sum a_n \tilde{w}_n\Big\| \\ &\leq |a_{n_0}|\|\tilde{w}_{n_0}\|_\sim + \Big\|\sum_{n\in(n_0,n_1)} a_n \tilde{\iota^*}\tilde{w}_n\Big\| + |a_{n_1}|\|\tilde{w}_{n_1}\|_\sim \\ &\leq \frac{3\tilde{K}}{a}\Big\|\sum a_n \tilde{\iota^*}\tilde{w}_n\Big\|.\end{aligned}$$

(iii) Take a normalized block sequence $(\tilde{w}_n)$ in $\tilde{W}$. Then $(\tilde{\iota^*}\tilde{w}_n)$ is bounded in X. It is also pointwise null on $c_{00}(G) \cap X_0$, by the same argument given in the proof of w^*-w^* continuity. But by density of this set in X_0, $(\tilde{\iota^*}\tilde{w}_n)$ is weakly null in X. By passing to a subsequence, we can assume that either $(\tilde{\iota^*}\tilde{w}_n)$ is either norm null or weakly null, bounded away from zero, and basic. In the second case, we have weak nullity of $(\tilde{w}_n)$ by (ii). In the first case, we can take sets $E_1 < E_2 < \ldots$ so that $|E_i| = i$ and $\|\tilde{\iota^*}\tilde{w}_n\| < 1/i$ for all $n \in E_i$. Then for fixed i and some $n_0 < n_1 \in E_i$, with $F_i = E_i \cap (n_0, n_1)$,

$$\begin{aligned}\Big\|\frac{1}{i}\sum_{n\in E_i}\tilde{w}_n\Big\|_\sim &\leq \|\tilde{w}_{n_0}\|_\sim/i + \Big\|i^{-1}\sum_{n\in F_i}\tilde{\iota^*}\tilde{w}_n\Big\| + \|\tilde{w}_{n_1}\|_\sim \\ &\leq 3/i.\end{aligned}$$

This proves $(\tilde{w}_n)$ is weakly null. Since this was an arbitrary normalized block sequence, $\tilde{G}$ must be shrinking.

□

Example We include an example which illustrates the necessity of the hypothesis that our embedding $\iota : X_0 \to W_0$ be such that $c_{00}(F) \cap X_0$ is dense in X_0. This hypothesis was used twice. Once to use w^*-w^* continuity of the map $\tilde{\iota^*} : \tilde{W}^{V_N}(\tilde{G}) \to X$, and once to prove that $\tilde{G}$ is shrinking in $\tilde{W}$. The idea behind both examples is that, while bounded block or pointwise null sequences in W have the desired properties (having w^* null images under ι^* in the w^*-w^* case or being weakly null in the shrinking case), we may have unbounded block or pointwise null sequences which fail that same property. When we pass from W to $\tilde{W}$, these unbounded block or pointwise null sequences may be sent to a bounded sequence in $\tilde{W}$ which also fails to have w^* null images or to be weakly null.

Let $X_0 = \ell_2$. Choose disjoint sets $M_n \in [\mathbb{N}]$ and let $M_n = (m_i^n)_i$. Let (c_i) be a sequence of positive scalars with $1 = \sum c_i^2$. Let $\iota : \ell_2 \to W_0 = \ell_2$ be the map satisfying $\iota e_n = \sum c_i e_{m_i^n}$. This is an isometric embedding. One easily checks that $\iota^* e^*_{m_i^n} = c_i e_n$. Thus for any fixed blocking G of the ℓ_2 basis, the sequence $(c_i^{-1} e^*_{m_i^n})_i$ will have a subsequence which is a block sequence. It is of course unbounded, but the images in $\tilde{\ell_2}$ are normalized in $\tilde{\ell_2}$. Since each element will have a singleton as its support, it will also be normalized in $\tilde{\ell_2}^{V_N}(\tilde{G})$, regardless of V, N. Thus the sequence cannot be weakly null, nor can it be sent to a w^* null sequence via $\tilde{\iota^*}$.

Going further into this example, let $V = \ell_2$. Fix $k \in \mathbb{N}$ and $\sum a_n e_n \in B_{\ell_2}$. Choose $i_1, i_2, \ldots$ so that $k < \min \operatorname{ran}_{\tilde{G}}(\tilde{e^*}_{m^n_{i_n}})$ for all n. Let $\tilde{w}_k = \sum a_n c_{i_n}^{-1} \tilde{e^*}_{m^n_{i_n}}$. One easily checks that for any $r \leq s$,

$$\Big\|\sum_{n=r}^{s} a_n c_{i_n}^{-1} \tilde{e^*}_{m^n_{i_n}}\Big\|_{\sim}, \Big\|\sum_{n=r}^{s} a_n c_{i_n}^{-1} \tilde{e^*}_{m^n_{i_n}}\Big\|_{\tilde{W}^V(\tilde{G})} \leq \big(\sum_{n=r}^{s} |a_n|^2\big)^{1/2}.$$

This implies that $\tilde{w}_k$ actually converges to a norm at most one element of $\tilde{W}^V(\tilde{G})$. Moreover, $\tilde{\iota^*}\tilde{w}_k = \sum a_n e_n$. Therefore we have shown that for any $x \in B_{\ell_2}$, we can find a subnormalized, pointwise null sequence $(\tilde{w}_k)$ in $\tilde{W}^V(\tilde{G})$ with $\tilde{\iota^*} w_k = x$ for all $k \in \mathbb{N}$. Let $\tilde{W}_n = \overline{\big(\oplus_{m>n} \tilde{G}_m\big)}$. We have shown that instead of deducing that $\cap_n \tilde{\iota^*}(B_{\tilde{W}_n}) = (0)$, as would be the case if $\tilde{\iota^*}$ were w^*-w^* continuous, this case gives the opposite extreme, $\cap_n \tilde{\iota^*}(B_{\tilde{W}_n}) = B_{\ell_2}$.

3.3 Schreier and Baernstein spaces

Recall the families $\mathcal{S}_\alpha$ introduced in Chapter II. For each $\alpha < \omega_1$, we will use the family $\mathcal{S}_\alpha$ to define the *Schreier space of order* α. For each $\alpha < \omega_1$ and $1 \leq p \leq \infty$, will also define the *Baernstein space of order* α *and parameter* p. These spaces and their duals will be the spaces U, V in the previous sections. Information concerning Schreier's original space and modified versions due to Baernstein and Seifert can be found in [6]. The transfinite versions of Schreier's space were first considered in [1].

For $E \in [\mathbb{N}]^{<\omega}$ and $x \in c_{00}$, let Ex be the projection of x onto E. For $x \in c_{00}$, let

$$\|x\|_\alpha = \max\big\{\|Ex\|_{\ell_1} : E \in \mathcal{S}_\alpha\big\}.$$

It is clear that the canonical c_{00} basis becomes a normalized, 1-unconditional basis for the completion of $X_\alpha = (c_{00}, \|\cdot\|_\alpha)$, which is the Schreier space of order α. For

$1 \le p < \infty$, we let $X_{\alpha,p} = X_\alpha^{\ell_p}$. For convenience, we will also let $X_{\alpha,\infty} = X_\alpha$. This is consistent with our previous notation, since $X_\alpha = X_\alpha^{c_0}$ isometrically. We could alternately define the norm on c_{00} as

$$\|x\|_{\alpha,p} = \left\{ \left\| \left(\|E_i x\|_{\ell_1} \right)_i \right\|_{\ell_p} : E_1 < E_2 < \ldots, E_i \in \mathcal{S}_\alpha \right\}$$

and then let $X_{\alpha,p}$ denote the completion. These are the Baernstein spaces of order α and parameter p. The following proposition collects some simple facts about these spaces.

Proposition 3.17. *Let $0 \le \alpha < \omega_1$, $1 \le p \le \infty$. Then the canonical basis of $X_{\alpha,p}$ is normalized, 1-unconditional, 1-right dominant, and satisfies subsequential $X_{\alpha,p}$ upper block estimates in $X_{\alpha,p}$. If $1 < p$, then the basis is shrinking. If $p < \infty$, then the basis is boundedly complete.*

Proof. That the basis is normalized and 1-unconditional is obvious. The 1-right dominance comes from the fact that the Schreier families are spreading. Indeed, suppose $M = (m_k), N = (n_k) \in [\mathbb{N}]$ are such that $m_k \le n_k$ for all k. Choose $(a_k) \in c_{00}$ and let $x = \sum a_k e_{m_k}$, $y = \sum a_k e_{n_k}$. Choose $E_1 < E_2 < \ldots$ so that $E_i \in \mathcal{S}_\alpha$ for each i. For convenience, we can assume that $E_i \subset M$, because $\|E_i\|_{\ell_1}$ will be unchanged by replacing E_i with $E_i \cap M$.

Observe that for each i there exists $A_i \in [\mathbb{N}]^{<\omega}$ so that

$$\|E_i x\|_{\ell_1} = \sum_{k \in A_i} |a_k|,$$

and $M(A_i) = E_i$. We note that $N(A_i)$ is a spread of $M(A_i)$, so $N(A_i) \in \mathcal{S}_\alpha$ for each i. Moreover, $A_1 < A_2 < \ldots$, so $N(A_1) < N(A_2) < \ldots$, and $\|E_i x\|_{\ell_1} = \|N(A_i) y\|_{\ell_1}$ for each i. Taking p norms and suprema over all successive sequences (E_i) in $\mathcal{S}_\alpha$

gives 1-right dominance.

We next show that if $M = (m_i), N = (n_i) \in [\mathbb{N}]$ are such that $m_i \leq n_i < m_{i+1}$, $(e_{n_i}) \lesssim_2 (e_{m_i})$. Let $(a_k) \in c_{00}$ and $x = \sum a_k e_{n_k}$. Choose $E_1 < E_2 < \ldots$, $E_i \in \mathcal{S}_\alpha$. As in the previous argument, we assume $\cup_i E_i \subset N$. Then for each i there exists $A_i \in [\mathbb{N}]^{<\omega}$ so that

$$\|E_i x\|_{\ell_1} = \sum_{k \in A_i} |a_k|$$

and $N(A_i) = E_i$. Let $A_i' = A_i \setminus (\min A_i)$, $A_i'' = (\min A_i)$. If

$$\sum_{k \in A_i'} |a_k| \geq (1/2) \sum_{k \in A_i} |a_k|,$$

let $B_i = A_i'$. Otherwise, let $B_i = A_i''$. Note that $\sum_{k \in B_i} |a_k| \geq (1/2)\|E_i x\|_{\ell_1}$ for each i and that $M(B_i) \in \mathcal{S}_\alpha$. This is because if $B_i = A_i'$, then $M(B_i)$ is a spread of $E_i \setminus (\max E_i)$, and otherwise $M(B_i)$ is a singleton. Note also that $M(B_1) < M(B_2) < \ldots$, since $B_1 < B_2 < \ldots$. Then

$$\|y\|_{\alpha,p}^p \geq \sum \|M(B_i)y\|_{\ell_1}^p \geq 2^{-p} \sum \|E_i x\|_{\ell_1}^p.$$

If $p = \infty$, we omit the exponents and replace the sums with maxima. Taking the supremum over appropriate (E_i) gives the claim.

Let (x_n) be a normalized block sequence in $X_{\alpha,p}$, fix $(a_n) \in c_{00}$, and $E_1 < E_2 < \ldots$, $E_i \in \mathcal{S}_\alpha$. Let $x = \sum a_n x_n$. We can assume without loss of generality that $\cup_i E_i \subset \cup_n \mathrm{supp}(x_n)$. For each n, let

$$A_n = \{i : E_i \subset \mathrm{ran}(x_n)\}, A = \cup_n A_n, B = \mathbb{N} \setminus A.$$

Observe that for each $n \in \mathbb{N}$,

$$\sum_{i\in A_n} \|E_i x\|_{\ell_1}^p = |a_n|^p \sum_{i\in A_n} \|E_i x_n\|_{\ell_1}^p \leq |a_n|^p \|x_n\|_{\alpha,p}^p \leq |a_n|^p.$$

Since $(e_n) \subset X_{\alpha,p}$ 1-dominates the ℓ_p unit vector basis for each α, we deduce that

$$\sum_{i\in A} \|E_i x\|_{\ell_1}^p \leq \sum_n |a_n|^p \leq \Big\|\sum a_n e_{m_n}\Big\|_{\alpha,p}^p,$$

for any $(m_n) \in [\mathbb{N}]$. Thus this also holds in the particular case that $\min \operatorname{ran}(x_n) = m_n$. If $p = \infty$, we again omit the exponents and replace the sums with maxima.

For each $i \in B$, let $B_i = \{n : E_i \cap \operatorname{ran}(x_n) \neq \varnothing\}$ and observe that each $n \in \mathbb{N}$ can be in B_i for at most two values of i. Indeed, if $n \in B_i \cap B_{i+1} \cap B_{i+2}$, $i+1 \in A$, a contradiction. Therefore we can partition $B = C \cup D$ so that $(B_i)_{i\in C}$ are pairwise disjoint, as are $(B_i)_{i\in D}$. Let $N = \cup_{i\in B} B_i$. Choose for each $i \in B$ and each $n \in B_i$ some $s_n \in E_i \cap \operatorname{ran}(x_n)$. Let

$$F_i = (s_n : n \in B_i) \subset E_i \in \mathcal{S}_\alpha.$$

Then $(F_i)_{i\in C}$, $(F_i)_{i\in D}$ are successive sequences of members of $\mathcal{S}_\alpha$. By pairwise disjointness of $(B_i)_{i\in C}$, we deduce that

$$\begin{aligned}
\sum_{i\in C} \|E_i x\|_{\ell_1}^p &= \sum_{i\in C} \Big\|E_i \sum_{n\in B_i} a_n x_n\Big\|_{\ell_1}^p \\
&= \sum_{i\in C} \Big(\sum_{n\in B_i} |a_n| \|E_i x_n\|_{\ell_1}\Big)^p \leq \sum_{i\in C} \Big(\sum_{n\in B_i} |a_n|\Big)^p \\
&= \sum_{i\in C} \Big\|F_i\big(\sum_{n\in B_i} a_n u_{s_n}\big)\Big\|_{\ell_1}^p \leq \Big\|\sum_{n\in N} a_n u_{s_n}\Big\|_{\alpha,p}^p \\
&\leq 2\Big\|\sum a_n u_{m_n}\Big\|_{\alpha,p}^p.
\end{aligned}$$

The same argument shows that

$$\sum_{i\in D}\|E_i x\|_{\ell_1}^p \leq 2\Big\|\sum a_n u_{m_n}\Big\|_{\alpha,p}^p.$$

Putting these estimates together and taking the supremum over all (E_i) gives that

$$\|x\| \leq 5\Big\|\sum a_n u_{m_n}\Big\|.$$

Again, if $p=\infty$, omitting exponents and replacing sums with maxima gives the same estimate with the same constant (and obviously a better constant is possible in this case).

We will prove later that the bases of these spaces are shrinking if $1<p$ when we compute the Szlenk indices of these spaces. The boundedly complete statement for $p<\infty$ is obvious since any normalized block of the $X_{\alpha,p}$ basis 1-dominates the ℓ_p unit vector basis.

□

We wish to use the weak ℓ_1^+ index to compute the Szlenk index of the space $X_{\alpha,p}$, $p>1$. For this, we will use the repeated averages hierarchy introduced in [3]. This will allow us to find within a sufficiently complex weakly null tree in $X_{\alpha,p}$, meaning a tree indexed by $\widehat{\mathcal{S}_{\alpha+1}}$, a branch which is dominated by ℓ_p^n for some n. For each $\alpha<\omega_1$, $M\in[\mathbb{N}]$, we will construct a convex blocking $(x_n^{\alpha,M})$ of the canonical c_{00} basis (e_n) so that for all α, M,

(i) $\bigcup_{n=1}^{\infty}\text{supp }(x_n^{\alpha,M})=M$,

(ii) $\text{supp }(x_n^{\alpha,M})\in MAX(\mathcal{S}_\alpha)$ for each $n\in\mathbb{N}$.

Let $M=(m_n)$. We let $x_n^{0,M}=e_{m_n}$. Suppose that for some $\alpha<\omega_1$ and $M\in[\mathbb{N}]$,

$(x_n^{\alpha,M})_n$ has been defined. Let $s_0 = 0$. Let $p_1 = \min \operatorname{supp}\,(x_1^{\alpha,M})$, $s_1 = p_1 + s_0$,

$$x_1^{\alpha+1,M} = p_1^{-1} \sum_{j=s_0+1}^{s_0+p_1} x_j^{\alpha,M} = p_1^{-1} \sum_{j=s_0+1}^{s_1} x_j^{\alpha,M}.$$

Suppose $p_1 < \ldots < p_n$, $s_0 < \ldots < s_n$, $x_1^{\alpha+1,M}, \ldots, x_n^{\alpha+1,M}$ have been chosen so that for $1 \leq i \leq n$, $\min \operatorname{supp}\,(x_i^{\alpha+1,M}) = p_i$, $s_i = s_{i-1} + p_i$,

$$x_i^{\alpha+1,M} = p_i^{-1} \sum_{j=s_{i-1}+1}^{s_i} x_j^{\alpha,M} = p_i^{-1} \sum_{j=s_{i-1}+1}^{s_{i-1}+p_i} x_j^{\alpha,M}.$$

Then let $p_{n+1} = \min \operatorname{supp}(x_{s_n+1}^{\alpha+1,M})$, $s_{n+1} = s_n + p_{n+1}$, and

$$x_{n+1}^{\alpha+1,M} = p_{n+1}^{-1} \sum_{j=s_n+1}^{s_{n+1}} x_j^{\alpha,M}.$$

Last, suppose $\alpha < \omega_1$ is a limit ordinal and for every $\beta < \alpha$ and every $N \in [\mathbb{N}]$, $(x_n^{\beta,N})_n$ has been defined. Let $\alpha_n \uparrow \alpha$ be the sequence used to define $\mathcal{S}_\alpha$. Fix $M \in [\mathbb{N}]$ and let $M_0 = M$. Let $p_1 = m_1$ and let $x_1^{\alpha,M} = x_1^{\alpha_{m_1},M}$. Next, assume $p_1 < \ldots < p_n$, $M_1, \ldots, M_n \in [\mathbb{N}]$, $x_1^{\alpha,M}, \ldots, x_n^{\alpha,M}$ have been chosen so that for $1 \leq i \leq n$, $\min \operatorname{supp}\,(x_i^{\alpha,M}) = p_i$,

$$M_i = M_{i-1} \setminus \bigcup_{j=1}^{i-1} \operatorname{supp}\,(x_j^{\alpha,M}), \quad \text{and} \quad x_i^{\alpha,M} = x_1^{\alpha_{p_i},M_i}.$$

Let

$$M_{n+1} = M_n \setminus \bigcup_{j=1}^{n} \operatorname{supp}\,(x_j^{\alpha,M}), \qquad p_{n+1} = \min M_{n+1}, \quad \text{and} \quad x_{n+1}^{\alpha,M} = x_1^{\alpha_{p_{n+1}},M_{n+1}}.$$

Lemma 3.18. *Let $M = (m_n) \in [\mathbb{N}]$ be such that $m_{n+1} \geq 3m_n$ for all $n \in \mathbb{N}$. For*

$0 \leq \alpha < \omega_1$ *and* $1 \leq p \leq \infty$, $(x_n^{\alpha,M})_n \subset X_{\alpha,p}$ *is equivalent to the unit vector basis of* ℓ_p *(or* c_0 *if* $p = \infty$*).*

Proof. Since $(x_n^{\alpha,M})$ is normalized in $X_{\alpha,p}$ for any M, even without the lacunary condition, the ℓ_p (resp. c_0 unit vector basis) is dominated by $(x_n^{\alpha,M})$. Thus we must simply prove domination by the ℓ_p (resp. c_0 basis).

We first prove the result with 2-equivalence in the $p = \infty$ case. Since $(x_n^{\alpha,M})$ is 1-unconditional, it is sufficient to prove that for any N, α,

$$\Big\|\sum_{n=1}^{N} x_n^{\alpha,M}\Big\|_{\alpha,\infty} \leq 2.$$

It is clear that $X_{0,\infty} = c_0$ isometrically, so the base case is trivial. Assume the result for some α. Recall that there exist $s_0 < s_1 < \ldots, < p_1 < \ldots$ so that

$$x_n^{\alpha+1,M} = p_n^{-1} \sum_{i=s_{n-1}+1}^{s_{n-1}+p_n} x_i^{\alpha,M}, p_n = \min \operatorname{ran}(x_n^{\alpha+1,M}).$$

Fix $N \in \mathbb{N}$ and $E \in \mathcal{S}_{\alpha+1}$, $E \subset \cup_{n=1}^{N} \operatorname{supp}(x_n^{\alpha+1,M})$. Write $E = \cup_{i=1}^{m} E_i$ with $m \leq E$ and $E_i \in \mathcal{S}_\alpha$. Let R be the minimum index n so that $E \cap \operatorname{ran}(x_n^{\alpha+1,M}) \neq \varnothing$. We observe that $p_{R+1} \geq 3m$ in this situation, and inductively, $p_{R+n} \geq 3^n m$. Then

$$\begin{aligned}
\Big\|E\Big(\sum_{n=1}^{N} x_n^{\alpha+1,M}\Big)\Big\|_{\ell_1} &= \Big\|E\Big(x_R^{\alpha+1,M} + \sum_{n=R+1}^{N} x_n^{\alpha+1,M}\Big)\Big\|_{\ell_1} \\
&\leq 1 + \sum_{i=1}^{m} \Big\|E_i \sum_{n=R+1}^{N} x_n^{\alpha+1,M}\Big\|_{\ell_1} \\
&\leq 1 + 2\sum_{i=1}^{m} \sum_{n=R+1}^{\infty} p_n^{-1} \\
&\leq 1 + \sum_{n=1}^{\infty} 2m/p_{R+n} \leq 1 + 2\sum_{n=1}^{\infty} 3^{-n} = 2.
\end{aligned}$$

Next, assume the result holds for all $\beta < \alpha$, α a countable limit ordinal. Fix $N \in \mathbb{N}$, $E \in \mathcal{S}_\alpha$, and let $m = \min E$. Then $E \in \mathcal{S}_{\alpha_m}$, where $\alpha_n \uparrow \alpha$ is the sequence used to define $\mathcal{S}_\alpha$. Recall from the construction that each α_n is a successor, say $\alpha_n = \beta_n + 1$, and that $\mathcal{S}_{\alpha_n} \subset \mathcal{S}_{\beta_{n+1}}$ for all n. This means that $E \in \mathcal{S}_{\alpha_m} \subset \mathcal{S}_{\beta_n}$ for each $n > m$. Recall also that for each n, if $p_n = \min \operatorname{ran}(x_n^{\alpha,M})$, there exists $M_n \in [M]$ (which also satisfies the lacunary condition) so that

$$x_n^{\alpha,M} = x_1^{\alpha_{p_n},M_n} = x_1^{\beta_{p_n}+1,M_n} = p_n^{-1} \sum_{i=1}^{p_n} x_i^{\beta_{p_n},M_n}.$$

If R is the smallest index n so that E intersects $\operatorname{ran}(x_n^{\alpha,M})$, we deduce that for all $n \in \mathbb{N}$, $p_{R+n}^{-1} \leq 3^{-n}$. Moreover, $m < p_{R+n}$ for all $n \in \mathbb{N}$, so the inductive hypothesis gives that

$$\|Ex_{R+n}^{\alpha,M}\|_{\ell_1} = p_{R+n}^{-1} \Big\| E\Big(\sum_{i=1}^{p_{R+n}} x_i^{\beta_{p_{R+n}},M_{R+n}} \Big) \Big\|_{\ell_1} \leq 2/p_{R+n} \leq 2/3^n,$$

since $E \in \mathcal{S}_{\alpha_m} \subset \mathcal{S}_{\beta_{p_{R+n}}}$. Then

$$\begin{aligned} \Big\| E\Big(\sum_{n=1}^{N} x_n^{\alpha,M} \Big) \Big\|_{\ell_1} &\leq 1 + \Big\| E\Big(\sum_{n=1}^{\infty} x_{R+n}^{\alpha,M} \Big) \Big\|_{\ell_1} \\ &\leq 1 + 2\sum_{n=1}^{\infty} 3^{-n} = 2. \end{aligned}$$

Next, observe that the $p = 1$ case is trivial, since all spaces are isometrically ℓ_1 in this case. We will last prove the $1 < p < \infty$ case from the $p = \infty$ case with constant equal to 6. Since it will not be by induction, we simply fix α and let $x_n = x_n^{\alpha,M}$.

Fix $(a_n) \in c_{00}$ and let $x = \sum a_n x_n$. Let $E_1 < E_2 < \ldots$, $E_i \in \mathcal{S}_\alpha$. Without loss of generality we can assume that for each n, there is at most one i so that $E_i \subset \operatorname{supp}(x_n)$. To see this, note that if $E_i, E_{i+1}, \ldots, E_j \subset \operatorname{supp}(x_n) \in \mathcal{S}_\alpha$, we can replace these sets with their union, call it E, which is also a member of $\mathcal{S}_\alpha$. Since

$$\sum_{k=1}^{i-1}\|E_kx\|_{\ell_1}^p+\sum_{k=i}^{j}\|E_kx\|_{\ell_1}^p+\sum_{k=j+1}^{\infty}\|E_kx\|_{\ell_1}^p\le\sum_{k=1}^{i-1}\|E_kx\|_{\ell_1}^p+\|Ex\|_{\ell_1}^p+\sum_{k=j+1}^{\infty}\|E_kx\|_{\ell_1}^p,$$

to compute the norm $\|x\|_{\alpha,p}$ it is sufficient to optimize over (E_i) of the indicated form.

Let $B_i=(n:E_i\cap\operatorname{ran}(x_n)\neq\varnothing)$. By our previous remark, we can find A_1,A_2,A_3 a partition of $\mathbb{N}$ so that $(B_i)_{i\in A_j}$ are pairwise disjoint for $j=1,2,3$. Choose for each i some $n_i\in B_i$ so that $|a_{n_i}|=\max_{n\in B_i}|a_n|$. Then for $j=1,2,3$,

$$\sum_{i\in A_j}\|E_ix\|_{\ell_1}^p=\sum_{i\in A_j}\Big\|E_i\Big(\sum_{n\in B_i}a_nx_n\Big)\Big\|_{\ell_1}^p\le 2^p\sum_{i\in A_j}|a_{n_i}|^p\le 2^p\sum_n|a_n|^p.$$

From here we deduce $\|x\|_{\alpha,p}\le 6\big(\sum|a_n|^p\big)^{1/p}$. □

The next theorem will imply that the canonical $X_{\alpha,p}$ basis is shrinking whenever $1<p$.

Theorem 3.19. *For $0\le\alpha<\omega_1$ and $1<p\le\infty$, $Sz(X_{\alpha,p})=\omega^{\alpha+1}$.*

Proof. First, assume $\varepsilon\in(0,1)$ and $(x_E)_{E\in\widehat{\mathcal{S}_{\alpha+1}}}$ is a normalized block tree in $X_{\alpha,p}$ such that for each $E\in\widehat{\mathcal{S}_{\alpha+1}}$,

$$\operatorname{co}(x_{E|_n}:1\le n\le|E|)\cap\operatorname{int}\big(\varepsilon B_{X_{\alpha,p}}\big)=\varnothing.$$

Observe that we can find such an ε and a tree if either $\ell_1\hookrightarrow X_{\alpha,p}$ or if $Sz(X_{\alpha,p})>\omega^{\alpha+1}$. Since (e_n) satisfies subsequential 5-$X_{\alpha,p}$ upper block estimates in $X_{\alpha,p}$, by letting $m(E)=\min\operatorname{ran}(x_E)$ and replacing ε with $\varepsilon/5$ and (x_E) with $(e_{m(E)})$, we can assume that each each x_E is actually a single basis element.

Choose $i_1 \in \mathbb{N}$ so that $12 i_1^{1/p} < \varepsilon i_1$ if $p < \infty$, and so that $12 < \varepsilon i_1$ if $p = \infty$. Next, suppose $i_1, \ldots, i_j$ have been chosen so that $(i_1, \ldots, i_j) \in \mathcal{S}_{\alpha+1}$, and for $1 \leq k < j$, $i_{k+1} > \max\{3i_k, m((i_1, \ldots, i_k))\}$. If $(i_1, \ldots, i_j) \in MAX(\mathcal{S}_{\alpha+1})$, let $E = (i_1, \ldots, i_j)$. Otherwise, choose $i_{j+1} > \max\{3i_j, m((i_1, \ldots, i_j))\}$. Because $\mathcal{S}_{\alpha+1}$ is compact, this process must terminate after finitely many steps. Once we have this $E \in MAX(\mathcal{S}_{\alpha+1})$, let $M = E^\frown(3^n i_{|E|} : n \in \mathbb{N})$.

Observe that $i_1 \leq m(E|_1) < i_2 \leq m(E|_2) < \ldots$, so that $(e_{i_n})_{n=1}^{|E|}$ 2-dominates $(e_{m(E|_n)})_{n=1}^{|E|}$, from the proof of Proposition 3.17. Let $(x_n) = (x_n^{\alpha,M})$ be the repeated averages hierarchy blocking corresponding to α, M. Observe that

$$\Big\|\sum_{n=1}^{i_1} x_n\Big\|_{\alpha,p} \leq 6 i_1^{1/p}, \tag{3.2}$$

if $p < \infty$, and if $p = \infty$,

$$\Big\|\sum_{n=1}^{i_1} x_n\Big\|_{\alpha,\infty} \leq 6. \tag{3.3}$$

We note that $\text{supp}(x_n) \in \mathcal{S}_\alpha$ for each n, so that $\cup_{n=1}^{i_1}\text{supp}(x_n) \in \mathcal{S}_{\alpha+1}$. This means $(x_n)_{n=1}^{i_1} \subset \text{co}(e_{i_n} : 1 \leq n \leq |E|)$. Suppose $E = \cup_{n=1}^{i_1} E_n$, $E_n = \text{supp}(x_n)$. We can write $x_n = \sum_{j \in A_n} a_j e_{i_j}$ for the appropriate A_n so that $E_n = M(A_n)$, $\sum_{j\in A_n} a_j = 1$, $a_j \geq 0$. Then because $(e_{i_n})_{n=1}^{|E|}$ 2-dominates $(e_{m(E|_n)})_{n=1}^{|E|}$,

$$2\Big\|\sum_{n=1}^{i_1} x_n\Big\|_{\alpha,p} = 2\Big\|\sum_{n=1}^{i_1}\sum_{j\in A_n} a_j e_{i_j}\Big\|_{\alpha,p} \geq \Big\|\sum_{n=1}^{i_1}\sum_{j\in A_n} a_j e_{m(E|_j)}\Big\|_{\alpha,p} \geq \varepsilon \sum_{n=1}^{i_1}\sum_{j\in A_n} a_j \geq \varepsilon i_1. \tag{3.4}$$

Equations (3.2) or (3.3) will contradict (3.4), which proves that the tree (x_E) cannot exist. Thus $X_{\alpha,p}$ does not contain a copy of ℓ_1, and the canonical basis is shrinking. Moreover, we have also shown that $Sz(X_{\alpha,p}) \leq \omega^{\alpha+1}$.

Next, observe that if we let $x_E = e_{\max E}$, the tree $(x_E)_{E\in\widehat{\mathcal{S}_\alpha}}$ witnesses the fact that

$Sz(X_{\alpha,p}) > \omega^\alpha$ by Proposition 2.14.

□

3.4 Coordinatization and universality

We begin with the **SD** case of our coordinatization theorem.

Theorem 3.20. *If U is a Banach space with normalized, 1-unconditional, shrinking, 1-right dominant basis (u_n) which satisfies U subsequential U upper estimates in U, and if $X \in$ **SD**, then the following are equivalent.*

(i) X satisfies subsequential U upper tree estimates.

(ii) There exists a Banach space Z with shrinking FDD satisfying subsequential U upper block estimates in Z such that X is isomorphic to a subspace of Z.

(iii) There exists a Banach space Z with shrinking FDD satisfying subsequential U upper block estimates in Z such that X is isomorphic to a quotient of Z.

We also have the **REFL** case of the coordinatization theorem.

Theorem 3.21. *Let U, V be reflexive Banach spaces with normalized, 1-unconditional bases $(u_n), (v_n)$, respectively, so that (u_n) is 1-right dominant and satisfies subsequential U upper block estimates in U, (v_n) is 1-left dominant and satisfies subsequential V lower block estimates in V, and every normalized block of (u_n) dominates every normalized block of (v_n). Then for $X \in$ **REFL**, the following are equivalent.*

(i) X satisfies subsequential (V, U) tree estimates.

*(ii) There exists $Z \in$ **REFL** with FDD E satisfying subsequential (V, U) block estimates in Z such that X is isomorphic to a subspace of Z.*

*(iii) There exists $Z \in$ **REFL** with FDD E satisfying subsequential (V, U) block estimates in Z such that X is isomorphic to a quotient of Z.*

Proof of Theorems 3.20, 3.21. (ii)$\Rightarrow$ (i) By equivalently renorming Z, we may assume that X is isometrically a subspace of a Banach space Z with FDD E satisfying subsequential C-U upper block estimates in Z. In the reflexive case, we can assume that E also satisfies subsequential C-V lower block estimates in Z. Let $(x_E)_{E\in\mathcal{E}}$ be a normalized, weakly null even tree in X. Choose $1 \leq m_1 < m_2 < \ldots$ and $1 = s_0 < s_1 < \ldots$ so that

$$\|x_{(m_1,\ldots,m_{2n})} - P^E_{[s_{n-1},s_n)} x_{(m_1,\ldots,m_{2n})}\| < \varepsilon_n$$

and so that $s_{n-1} \leq m_{2n-1} < s_n$, where $(\varepsilon_n) \subset (0,1)$ decreases to zero rapidly. For a sufficient choice of (ε_n), we can make $(x_{M|_{2n}})$ 2-equivalent to (z_n), where z_n is the normalization of $P^E_{[s_{n-1},s_n)} x_{M|_{2n}}$. Then by Proposition 3.2 and 1-right dominance,

$$(x_{M|_{2n}}) \lesssim_2 (z_n) \lesssim_C (u_{s_{n-1}}) \lesssim_1 (u_{m_{2n-1}}).$$

In the reflexive case, the same reasoning establishes that $(v_{m_{2n-1}}) \lesssim_C (z_n) \lesssim_2 (x_{M_{2n}})$.

(iii)$\Rightarrow$ (i) Suppose $Q : Z \to X$ is a norm 1 surjection, and Z has FDD E satisfying subsequential C-U upper block estimates. Assume also that $Q(CB_Z) \supset B_X$. Let $(x_F)_{F\in\mathcal{E}}$ be a normalized, weakly null even tree in X. By applying Lemma 3.10, we can find a pruning $\phi : \mathcal{E} \to \mathcal{E}$ and an even block tree $(z_F)_{F\in\mathcal{E}} \subset 2CB_Z$ so that $\|x_{\phi(F)} - Qz_F\| < \eta 2^{-|F|}$ and each branch of $(x_{\phi(M|_{2n})})$ is 2-basic, where $\eta \in (0,1)$ has been fixed. For a sufficiently small choice of $\eta \in (0,1)$, we can guarantee that if $M \in [\mathbb{N}]$, $(x_{\phi(M|_{2n})}) \lesssim_2 (Qz_{M|_{2n}})$. Recall that if we choose any $1 \leq m_1 < m_2 < \ldots$, $\ell_n = \max \phi(M|_n)$ determines an infinite subset $L = (\ell_n) \in [\mathbb{N}]$ so that $\ell_1 < \ell_2 < \ldots$,

$m_n \leq \ell_n$. Choose recursively $1 = m_1 < m_2 < \dots$ so that for all $n \in \mathbb{N}$,

$$m_{2n-1} \leq \min \operatorname{ran}_E(z_{M|2n}) \leq \max \operatorname{ran}_E(z_{M|2n}) < m_{2n+1}.$$

This condition with 1-right dominance guarantees that

$$(x_{L|2n}) = (x_{\phi(M|2n)}) \lesssim_2 (Qz_{M|2n}) \lesssim_1 (z_{M|2n}) \lesssim_(2C^2)(u_{m_{2n-1}}) \lesssim (u_{\ell_{2n-1}}).$$

In the reflexive case, X^* is isomorphic to a subspace of Z^*, which has FDD satisfying subsequential V^* upper block estimates. The implication (ii)$\Rightarrow$ (i) gives that X^* satisfies subsequential V^* upper tree estimates, since a duality argument allows us to check that the conditions required of (v_n^*) are satisfied in this case. Then Lemma 3.13 gives that $X = X^{**}$ satisfies $V = V^{**}$ lower tree estimates.

(i)$\Rightarrow$ (ii) By Lemma 3.13, X^* satisfies subsequential U^* lower tree estimates. By Theorem 3.15 (ii), there exists a Banach space $\tilde{W}$ with FDD $\tilde{H}$, shrinking in the case that X is reflexive, $M \in [\mathbb{N}]$, and a w^*-w^* continuous surjection of $\tilde{W}^{U_M^*}(\tilde{H})$ onto X^*. By Lemma 3.6, $Z_0 = \tilde{W}^{U_M^*}(\tilde{H}) \oplus U^*_{\mathbb{N}\setminus M}$ has an FDD, call it E, which satisfies subsequential U^* lower block estimates in Z_0. Moreover, E is a boundedly complete FDD for Z_0, and shrinking in the reflexive case, and X^* is also the image of Z_0 under a w^*-w^* continuous surjection. We deduce from this that there exists Z with shrinking FDD E^* which satisfies subsequential U upper block estimates in Z so that $Z^* = Z_0$, and Z can be taken to be reflexive in the reflexive case. Moreover, the w^*-w^* continuous surjection from Z_0 onto X^* is the adjoint of an embedding of X into Z. This finishes the non-reflexive case.

In the reflexive case, Theorem 3.15(i) implies there exists a blocking H of E^* and $P \in [\mathbb{N}]$ so that X naturally isomorphically embeds into $Z^{V_P}(H)$. Note that if

E^* satisfies subsequential U upper block estimates in Z and if $H_n = \oplus_{i=\ell_{n-1}+1}^{\ell_n} E_i^*$, $0 = \ell_0 < \ell_1 < \ldots$, then H satisfies subsequential U_L upper block estimates in Z. One easily checks that (u_{ℓ_n}) and (v_{p_n}) satisfy the hypotheses of Lemma 3.5, so H satisfies subsequential U_L upper block estimates in $Z^{V_P}(H)$. If we choose $r_n \geq \ell_n, p_n$, then right and left dominance, respectively, imply that H satisfies subsequential U_R upper block estimates and subsequential V_R lower block estimates in $Z^{V_P}(H)$. Once again, we fill out the FDD to deduce that X is isomorphic to a subspace of $Z^{V_P}(H) \oplus V_{\mathbb{N}\setminus P}$ which is reflexive with FDD satisfying subsequential (V, U) block estimates.

(i)$\Rightarrow$ (iii) For the reflexive case, the equivalence (i)$\Leftrightarrow$ (ii) implies that X embeds into a reflexive Banach space Y with FDD E satisfying subsequential (V, U) block estimates. This means X^* is a quotient of Y^*, which has FDD satisfying subsequential (U^*, V^*) block estimates. By (iii)$\Rightarrow$ (i), X^* satisfies subsequential (U^*, V^*) tree estimates (since the bases $(u_n^*), (v_n^*)$ satisfy the hypothesis as well), and we can use the reflexive case of (i)$\Rightarrow$ (ii) to deduce X^* is isomorphic to a subspace of a reflexive Banach space Z with FDD satisfying subsequential (U^*, V^*) block estimates. This means X is isomorhpic to a quotient of Z^*, which clearly has the required properties.

For the non-reflexive case, fix a Banach space Z_0 with shrinking FDD E and a quotient map $Q : Z_0 \to X$. This can be done by Corollary 8 of [7]. Then $Q^* : X^* \to Z = Z_0^*$ is an isometric embedding. Use Lemma 3.13 and Theorem 3.15(i) to deduce the existence of a blocking H of E^* and $M \in [\mathbb{N}]$ so that X^* embeds into $Z^{U_M^*}(H)$ via a w^*-w^* continuous embedding. Then let $W = Z^{U_M^*}(H) \oplus U_{\mathbb{N}\setminus M}^*$ and F the FDD for W guaranteed by Lemma 3.6. Note that there still exists a w^*-w^* continuous embedding of X^* into W, which is the adjoint of a surjection onto X from the natural predual W_0 of W. We last note that F^* is a shrinking FDD for W_0 and satisfies subsequential U upper block estimates in W_0.

$\square$

With these results, we can proceed to our universality results. Recall the following definitions.

$$\mathcal{A}_U = \{X \in \mathbf{SD} : X \text{ satisfies subsequential } U \text{ upper tree estimates}\},$$

$$\mathcal{A}_{V,U} = \{X \in \mathbf{REFL} : X \text{ satisfies subsequential } (V, U) \text{ tree estimates}\}.$$

Theorem 3.22. *(i) If U is a Banach space with normalized, 1-unconditional, shrinking, 1-right dominant basis (u_n) which satisfies subsequential U upper block estimates in U, then $\mathcal{A}_U$ contains a universal element.*

(ii) If U, V are reflexive Banach spaces with normalized, 1-unconditional bases (u_n), (v_n) so that (u_n) is 1-right dominant and satisfies subsequential U upper block estimates in U, (v_n) is 1-left dominant and satisfies subsequential V lower block estimates in V, and every normalized block of (u_n) dominates every normalized block of (v_n), then $\mathcal{A}_{V,U}$ contains a universal element.

Proof. We first prove (ii). By Theorem 1 of Schechtman [27], there exists a Banach space W with bimonotone FDD F so that if X is any Banach space with bimontone FDD E and $\varepsilon > 0$, there exists $(k_n) \in [\mathbb{N}]$ and a $(1+\varepsilon)$-embedding $T : X \to W$ so that $T(E_n) = F_{k_n}$ and so that $\sum P^F_{k_n}$ is a norm 1-projection of W onto $T(X)$. Since F^* is a bimonotone FDD for $W^{(*)}$ and the basis of (u^*_n) is boundedly complete, F^* is a bimonotone, boundedly complete FDD for $(W^{(*)})^{U^*}(F^*)$. This means there exists a Banach space Y for which F is a shrinking, bimonotone FDD for Y and so that $Y^* = (W^{(*)})^{U^*}(F^*)$. We last let $Z = Y^V(F)$. We claim that Z is the universal space we seek. We note that F is a bimonotone, shrinking, boundedly complete FDD for Z which satisfies subsequential (V, U) block estimates in Z. Thus it remains only to prove universality.

If $X \in \mathcal{A}_{(V,U)}$, we can first embed X by Theorem 3.21 into a reflexive Banach space with bimonotone FDD E satisfying subsequential (V,U) block estimates in that space, so we can assume X itself has such an FDD. Let $(k_n) \in [\mathbb{N}]$, $T : X \to W$ be a 2-isomorphic embedding such that $T(E_n) = F_{k_n}$ and so that $\sum P^F_{k_n} : W \to T(X)$ defines a norm 1 projection.

Observe that since X is isomorphic to $\overline{(\oplus F_{k_n})}^W$ via T, which takes the E onto (F_{k_n}), and since $T(X)$ is complemented in W, X^* is isomorphic to $\overline{(\oplus F^*_{k_n})}^{W^{(*)}}$ via an isomorphism which takes E^* onto $(F^*_{k_n})$. Since E satisfies subsequential U upper block estimates in X, E^* satisfies subsequential U^* lower block estimates in X^*, which means that $(F^*_{k_n})$ satisfies subsequential U^* lower block estimates in $\overline{(\oplus F^*_{k_n})}^{W^{(*)}}$. We can now apply Lemma 3.11 to deduce that $\|\cdot\|_{W^{(*)}}$ and $\|\cdot\|_{Y^*}$ are equivalent norms on $c_{00}\big(F^*_{k_n}\big)$. Since the Y^* norm dominates the $W^{(*)}$ norm, $\overline{(\oplus F^*_{k_n})}$ is also complemented in Y^*. This means $\|\cdot\|_W$ and $\|\cdot\|_Y$ are equivalent norms on $c_{00}\big(F_{k_n}\big)$. But since X is isomorphic to $\overline{(\oplus F_{k_n})}^W$ via T which takes E onto the FDD (F_{k_n}), (F_{k_n}) satisfies subsequential V lower block estimates in $\overline{(\oplus F_{k_n})}^W$, and by equivalence also in $\overline{(\oplus F_{k_n})}^Y$. We can again apply Lemma 3.11 to deduce that $\|\cdot\|_Y$ and $\|\cdot\|_Z$ are equivalent on $c_{00}\big(F_{k_n}\big)$. This means $\|\cdot\|_W, \|\cdot\|_Y, \|\cdot\|_Z$ are equivalent on $c_{00}\big(F_{k_n}\big)$, and the map

$$x = \sum x_n \mapsto \sum T x_n$$

is still an isomorphic embedding of X into Z.

The case (i) follows from appropriate modifications of this argument. We let Y be such that $Y^* = (W^{(*)})^{U^*}$ as in the previous case. By Theorem 3.20, it suffices to prove that any Banach space X with bimonotone, shrinking FDD E satisfying subsequential U upper block estimates in X, then X embeds into Y. Taking $(k_n) \in$

$[\mathbb{N}]$ and $T : X \to W$ as before, the first part of the previous case gives that $\|\cdot\|_W$ and $\|\cdot\|_Y$ are equivalent norms on $c_{00}\big(\oplus F_{k_n}\big)$, which means T is still an isomorphic embedding when considered as mapping X into Y.

□

3.5 Relation to Szlenk index

In this section we aim to connect the tree estimates of the previous section to quantitative Szlenk index estimates. For this, we begin by recalling a result of Gasparis from infinite Ramsey theory.

Theorem 3.23. *[11] If $\mathcal{F}, \mathcal{G} \subset [\mathbb{N}]^{<\omega}$ are hereditary and $N \in [\mathbb{N}]$, there exists $M \in [N]$ so that either*

$$\mathcal{F} \cap [M]^{<\omega} \subset \mathcal{G} \quad or \quad \mathcal{G} \cap [M]^{<\omega} \subset \mathcal{F}.$$

Next, if X is a separable Banach space and $\mathcal{A} \subset S_X^{<\omega}$, and $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1)$, we let

$$\mathcal{A}_{\overline{\varepsilon}}^X = \big\{(x_n)_{n=1}^N \in S_X^{<\omega} : N \in \mathbb{N}, \exists (y_n)_{n=1}^N \in \mathcal{A}, \|x_n - y_n\| \leq \varepsilon_n \ \forall 1 \leq n \leq N\big\}.$$

Let Z be a Banach space with FDD E and let $\mathcal{A}$ be a block tree of E in Z. We write $\Sigma(E, Z)$ for the set of all finite, normalized block sequences of E in Z. For $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1)$, we let

$$\mathcal{A}_{\overline{\varepsilon}}^{E,Z} = \mathcal{A}_{\overline{\varepsilon}}^Z \cap \Sigma(E, Z).$$

Finally, we define the *compression* $\tilde{\mathcal{A}}$ of $\mathcal{A}$ to be

$$\tilde{\mathcal{A}} = \big\{E \in [\mathbb{N}]^{<\omega} : \exists (z_n)_{n=1}^{|E|} \in \mathcal{A}, E = (\min \operatorname{ran}_E(z_n))_{n=1}^{|E|}\big\}.$$

We have already shown the following result in Chapter II, but we did not have the notation to relay it until now.

Proposition 3.24. *[24, Proposition 6] Let $X \subset Y$ be Banach spaces with separable duals and let $\mathcal{A} \subset S_X^{<\omega}$ be a tree on S_X. Then for all $\overline{\varepsilon} = (\varepsilon) \subset (0,1)$,*

$$I_w(\mathcal{A}_{\overline{\varepsilon}}^Y) \leq I_w(\mathcal{A}_{5\overline{\varepsilon}}^X).$$

Proposition 3.25. *[24, Proposition 8] Let Z be a Banach space with FDD E. Let $\mathcal{A}$ be a hereditary block tree of E in Z. Then for all $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1)$ and for all limit ordinals α, if $I_{\mathrm{bl}}(\mathcal{A}_{\overline{\varepsilon}}^{E,Z}) < \alpha$, then $I_{CB}(\tilde{\mathcal{A}}) < \alpha$.*

Theorem 3.26. *Let $\alpha < \omega_1$ and $C > 2$. Let Z be a Banach space with a shrinking, bimonotone FDD E and let X be an infinite dimensional closed subspace. If $Sz(X) \leq \omega^\alpha$, then there exists $M = (m_n)_{n\geq 0} \in [\mathbb{N}]$ with $1 = m_0 < m_1 < \ldots$ and $\delta = (\delta_n) \subset (0,1)$ so that if (x_n) is a normalized δ-block sequence with respect to the blocking H of E, defined by $H_n = \oplus_{i=m_{n-1}}^{m_n-1} E_i$ with $\|x_n - P^H_{(s_{n-1},s_n]} x_n\| < \delta_n$ for some $1 \leq s_0 < s_1 < \ldots$, then (x_n) is C-dominated by $(e_{m_{s_{n-1}}}) \subset X_\alpha$.*

Proof. Fix $2 < D < C$ and $\rho \in (0, 1/3)$ so small that $2(1-\rho)^2 < D$. Let

$$\mathcal{A}_n = \Big\{(x_j) \in S_X^{<\omega} : \Big\|\sum a_j x_j\Big\| \geq 2\rho^{n+1} \sum a_j \ \forall (a_j) \subset [0,\infty)\Big\}.$$

Observe that $\mathcal{A}_n$ is a hereditary tree on $S_X^{<\omega}$ for each $n \in \mathbb{N}$. For each n, fix $\overline{\varepsilon_n} = (\varepsilon_{i,n})_i \subset (0,1)$ so that $10 \sum_i \varepsilon_{i,n} < \rho^{n+1}$ and so that if we fix $n \in \mathbb{N}$, $i \mapsto \varepsilon_{i,n}$ is decreasing, and if we fix $i \in \mathbb{N}$, $n \mapsto \varepsilon_{i,n}$ is decreasing. We note that $10\sum_i \varepsilon_{i,n} \leq \rho^{n+1}$ implies

$$(\mathcal{A}_n)^Z_{10\overline{\varepsilon_n}} \subset \Big\{(z_j) \in S_Z^{<\omega} : \Big\|\sum a_j z_j\Big\| \geq \rho^{n+1} \sum a_j \ \forall (a_j) \subset [0,\infty)\Big\},$$

which means

$$I_w((\mathcal{A}_n)^X_{10\overline{\varepsilon_n}}) < Sz(X).$$

Let $\mathcal{B}_n = \Sigma(E,Z) \cap (\mathcal{A}_n)^Z_{\overline{\varepsilon_n}}$. This is a hereditary block tree of E in Z. Let $\tilde{\mathcal{B}}_n$ be its compression. By Proposition 3.24, for all $n \in \mathbb{N}$,

$$I_w((\mathcal{A}_n)^Z_{2\overline{\varepsilon_n}}) \leq I_w((\mathcal{A}_n)^X_{10\overline{\varepsilon_n}}) < Sz(X).$$

Since $(\mathcal{B}_n)^{E,Z}_{\overline{\varepsilon_n}} \subset (\mathcal{A}_n)^Z_{2\overline{\varepsilon_n}}$, and since the FDD E is shrinking, we deduce $I_{\mathrm{bl}}(\mathcal{B}_n)^{E,Z}_{\overline{\varepsilon_n}} \leq I_w((\mathcal{A}_n)^Z_{2\overline{\varepsilon_n}})$. Since $Sz(X)$ is a limit ordinal, Proposition 3.25 implies

$$I_{CB}(\tilde{\mathcal{B}}_n) < Sz(X) \leq \omega^\alpha.$$

Let $M_0 = \mathbb{N} \setminus (1)$. We note that $\mathcal{S}_\alpha$ and $\tilde{\mathcal{B}}_1$ are hereditary trees on $[\mathbb{N}]^{<\omega}$. By Theorem 3.23, there exists $M_1 \in [M_0 \setminus (\min M_0)]$ so that either

$$\mathcal{S}_\alpha \cap [M_1]^{<\omega} \subset \tilde{\mathcal{B}}_1 \ \text{ or } \ \tilde{\mathcal{B}}_1 \cap [M_1]^{<\omega} \subset \mathcal{S}_\alpha.$$

Since for any $M \in [\mathbb{N}]$,

$$I_{CB}(\mathcal{S}_\alpha \cap [M]^{<\omega}) = I_{CB}(\mathcal{S}_\alpha) = \omega^\alpha + 1 > I_{CB}(\tilde{\mathcal{B}}_1),$$

the first containment cannot hold.

Next, assume we have chosen $M_1 \supset M_2 \supset \ldots \supset M_k$ so that $M_n \in [M_{n-1} \setminus (\min M_{n-1})]$ and $\tilde{\mathcal{B}}_n \cap [M_n]^{<\omega} \subset \mathcal{S}_\alpha$ for $1 \leq n \leq k$. Applying Theorem 3.23, we can obtain $M_{k+1} \in [M_k \setminus (\min M_k)]$ so that either

$$\mathcal{S}_\alpha \cap [M_{k+1}]^{<\omega} \subset \tilde{\mathcal{B}}_{k+1} \text{ or } \tilde{\mathcal{B}}_{k+1} \cap [M_{k+1}]^{<\omega} \subset \mathcal{S}_\alpha.$$

By the same reasoning as in the base step, the first inclusion cannot hold.

For $n \geq 0$, let $m_n = \min M_n$. Note that $(m_i)_{i \geq n} \in [M_n]$ for each n.

Choose a strictly decreasing sequence $\overline{\delta} = (\delta_n) \subset (0,1)$ so that

$$3\delta_n < \min\{\varepsilon_{n,n}, \rho^{n+1}\}$$

for each $n \in \mathbb{N}$ and so that

$$3 \sum \delta_n < C - D.$$

Let $H_n = \oplus_{i=m_{n-1}}^{m_n - 1} E_i$ as in the statement of the theorem. Suppose $(x_n) \subset S_X$ is a $\overline{\delta}$-block sequence with respect to H and $1 \leq s_0 < s_2 < \ldots$ are such that $\|x_n - P^H_{(s_{n-1}, s_n]} x_n\| < \delta_n$. Define

$$z_n = \frac{P^H_{(s_{n-1}, s_n]} x_n}{\|P^H_{(s_{n-1}, s_n]} x_n\|}.$$

Then $\|z_n - x_n\| < 2\delta_n$ for all $n \in \mathbb{N}$. Let us now choose a normalized block sequence (w_n) so that $\operatorname{ran}_H(w_n) \subset (s_{n-1}, s_n]$, $\|z_n - w_n\| < \delta_n$, and $\min \operatorname{ran}_E(w_n) = m_{s_{n-1}}$. Then $\|x_n - w_n\| < 3\delta_n$ for each $n \in \mathbb{N}$. By our choice of $\overline{\delta}$, it is sufficient to show that (w_n) is D-dominated by $(e_{m_{s_{n-1}}})$ in order to show that (x_n) is C-dominated by $(e_{m_{s_{n-1}}})$.

Fix $(a_n) \in c_{00}$ and choose $w^* \in S_{Z^*}$ so that $w^*\left(\sum a_n w_n\right) = \left\|\sum a_n w_n\right\|$. Let

$MS = (m_{s_0}, m_{s_1}, \ldots)$. For each $j \in \mathbb{N}$, let

$$I_{j,+} = \{n \in \mathbb{N} : n < j, \rho^j < w^*(w_n) \leq \rho^{j-1}\},$$

$$I_{j,-} = \{n \in \mathbb{N} : n < j, \rho^j < -w^*(w_n) \leq \rho^{j-1}\},$$

$$J_{j,+} = \{n \in \mathbb{N} : n \geq j, \rho^j < w^*(w_n) \leq \rho^{j-1}\},$$

$$J_{j,-} = \{n \in \mathbb{N} : n \geq j, \rho^j < -w^*(w_n) \leq \rho^{j-1}\}.$$

Ssince E is shrinking, these sets are finite. We will show that $MS(J_{j,\pm}) \in \mathcal{S}_\alpha$ for each $j \in \mathbb{N}$. Note that $s_{n-1} \geq n$ for all $n \in \mathbb{N}$, which means

$$MS(J_{j,\pm}) \subset (m_n)_{n \geq j} \subset M_j.$$

It is clear that $(w_n)_{n \in MS(J_{j,+})} \in \Sigma(E, Z)$. For each $n \in MS(J_{j,+})$,

$$w^*(x_n) \geq w^*(w_n) - w^*(w_n - x_n) > \rho^j - 3\delta_j \geq \rho^j - \rho^{j+1} > 2\rho^{j+1}.$$

By the geometric version of the Hahn-Banach theorem, this means $(x_n)_{n \in J_{j,+}} \in \mathcal{A}_j$. If $n \in J_{j,+}$, then $n \geq j$, which means

$$\|x_n - w_n\| < 3\delta_n \leq \varepsilon_{n,n} \leq \varepsilon_{j,n}.$$

Thus $(w_n)_{n \in J_{j,+}}$ is an $\overline{\varepsilon_j}$-perturbation of $(x_n)_{n \in J_{j,+}}$, and $(w_n)_{n \in J_{j,+}} \in \mathcal{B}_j$. This means $MS(J_{j,+}) \in \tilde{\mathcal{B}}_j$, and

$$MS(J_{j,+}) \in \tilde{\mathcal{B}}_j \cap [M_j]^{<\omega} \subset \mathcal{S}_\alpha.$$

A similar argument replacing w^* with $-w^*$ yields $MS(J_{j,-}) \in \mathcal{S}_\alpha$.

Note that

$$\sum_{n\in J_{j,+}} |a_n w^*(w_n)| \le \rho^{j-1} \sum_{n\in J_{j,+}} |a_n| \le \rho^{j-1} \Big\| \sum a_n e_{m_{s_{n-1}}} \Big\|_{X_\alpha},$$

and the same holds if we replace $J_{j,+}$ by $J_{j,-}$. By 1-unconditionality,

$$|a_k| \le \|\sum a_n e_{m_{s_{n-1}}}\|_{X_\alpha}$$

for all $k \in \mathbb{N}$. Because $|I_{j,\pm}| < j$, it follows that

$$\sum_{n\in I_{j,\pm}} |a_n w^*(w_n)| \le \rho^{j-1}(j-1) \Big\| \sum a_n e_{m_{s_{n-1}}} \Big\|_{X_\alpha}.$$

Consequently,

$$\begin{aligned}
\Big\| \sum a_n w_n \Big\| &= \sum_j \Big(\sum_{n\in J_{j,\pm}} a_n w^*(w_n) + \sum_{n\in I_{j,\pm}} a_n w^*(w_n) \Big) \\
&\le \Big\| \sum a_n e_{m_{s_{n-1}}} \Big\|_{X_\alpha} \sum_j \big(2\rho^{j-1} + 2(j-1)\rho^{j-1} \big) \\
&= 2 \Big\| \sum a_n e_{m_{s_{n-1}}} \Big\|_{X_\alpha} \sum_j j\rho^{j-1} = \frac{2}{(1-\rho)^2} \Big\| \sum a_n e_{m_{s_{n-1}}} \Big\|_{X_\alpha} \\
&< D \Big\| \sum a_n e_{m_{s_{n-1}}} \Big\|_{X_\alpha}.
\end{aligned}$$

□

We now have

Corollary 3.27. *Let $\alpha < \omega_1$. If $X \in$ **SD** and $Sz(X) \le \omega^\alpha$, X satisfies subsequential X_α upper tree estimates. If $X \in$ **REFL** and $Sz(X), Sz(X^*) \le \omega^\alpha$, then X satifies subsequential $(X^*_{\alpha,2}, X_{\alpha,2})$ tree estimates.*

Proof. By the main theorem of [29], X embeds into a Banach space Z with shrinking bimonotone FDD E. By equivalently renorming X, we can assume that X embeds isometrically into Z. Fix $C > 2$. Let $(m_n) \in [\mathbb{N}]$, (δ_n), and H be as in Theorem 3.26. Let $(x_E)_{E\in\mathcal{E}}$ be a weakly null even tree in S_X. Let $s_0 = 1$.

Next, suppose we have chosen $1 = s_0 < s_1 < \ldots < s_k$ and $1 \leq p_1 < \ldots < p_{2k}$ so that $m_{s_{n-1}} \leq p_{2n-1}$ and

$$\|x_{(p_1,\ldots,p_{2n})} - P^H_{(s_{n-1},s_n]}x_{(p_1,\ldots,p_{2n})}\| < \delta_n$$

for each $n \in \mathbb{N}$. Choose $p_{2k+1} > m_{s_k}, p_{2k}$ and $p_{2k+2} > p_{2k+1}$ so large that

$$\|P^H_{[1,s_k]}x_{(p_1,\ldots,p_{2k+2})}\| < \delta_{k+1}/2.$$

Choose $s_{k+1} > s_k$ so that

$$\|P^H_{(s_{k+1},\infty)}x_{(p_1,\ldots,p_{2k+2})}\| < \delta_{k+1}/2.$$

Then

$$\|x_{(p_1,\ldots,p_{2k+2})} - P^H_{(s_k,s_{k+1}]}x_{(p_1,\ldots,p_{2k+2})}\| < \delta_{k+1}.$$

This completes the recursive construction. Note that by the properties of m_n, δ_n, and H, $(x_{P|2n})_n \lesssim_C (e_{m_{s_{n-1}}})$. By 1-right dominance, and since $p_{2n-1} \geq m_{s_{n-1}}$, $(x_{P|2n})_n \lesssim_C (e_{p_{2n-1}})$. This gives the first claim.

For the second claim, note that since the $X_{\alpha,2}$ basis 1-dominates the X_α basis, the above argument yields that any separable, reflexive Banach space X with $Sz(X), Sz(X^*) \leq \omega^\alpha$ must be such that both X, X^* satisfy subsequential X_α, hence also $X_{\alpha,2}$, upper tree estimates. By Lemma 3.13, $X = X^{**}$ satisfies subsequential

$X^*_{\alpha,2}$ lower tree estimates.

□

For $\alpha < \omega_1$, recall that

$$\mathcal{C}_\alpha = \{X \in \mathbf{SD} : Sz(X) \leq \omega^\alpha\},$$

$$\mathcal{CR}_\alpha = \{X \in \mathbf{REFL} : Sz(X), Sz(X^*) \leq \omega^\alpha\}.$$

Our work can be combined with a result of Johnson, Rosenthal, and Zippin [14] to obtain

Corollary 3.28. *For $\alpha < \omega_1$, there exists $W \in \mathcal{C}_{\alpha+1}$ with a basis so that if $X \in \mathcal{C}_\alpha$, X embeds into W. There exists $W_0 \in \mathcal{CR}_{\alpha+1}$ with a basis such that if $X \in \mathcal{CR}_\alpha$, X embeds into W_0.*

Proof. Let Z be the universal space for the class $\mathcal{A}_{X_\alpha}$ with shrinking FDD E. By Corollary 4.12 of Johnson, Rosenthal, and Zippin [14] we can find for each n a finite dimensional normed space H_n so that if $H = \left(\oplus_n H_n\right)_2$, $W = Z \oplus H$ has a basis. Since H satisfies ℓ_2 upper block estimates, $Sz(H) \leq Sz(\ell_2) = \omega$. By a result of Schlumprecht, Odell, and Zsák [24], we know that $Sz(W) = \max\{Sz(Z), Sz(H)\} = \omega^{\alpha+1}$. This means W^* must be separable, which gives the first statement.

For the second statement, the argument is similar. We simply replace Z with Z_0, a reflexive space with FDD which is universal for the class $\mathcal{A}_{(X^*_{\alpha,2}, X_{\alpha,2})}$. Choose finite dimensional spaces G_n so that if $G = \left(\oplus_n G_n\right)_2$, $W_0 = Z_0 \oplus G$ has a basis. Note that W_0 is reflexive. Since G^* is also an ℓ_2 sum of finite dimensional spaces, $Sz(W_0), Sz(W_0^*) = \omega^{\alpha+1}$ follows as in the proof of the first statement.

□

Remark We can see now that for each $\alpha < \omega_1$,

$$\mathcal{C}_\alpha \subsetneq \mathcal{A}_{X_\alpha} \subsetneq \mathcal{C}_{\alpha+1}.$$

The strict containments follow from the observations that $X_\alpha \in \mathcal{A}_{X_\alpha} \setminus \mathcal{C}_\alpha$ and $X_{\alpha,2} \in \mathcal{C}_{\alpha+1} \setminus \mathcal{A}_{X_\alpha}$.

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