COMPACTNESS OF THE $\bar{\partial}$-NEUMANN OPERATOR ON THE INTERSECTION DOMAINS IN $\mathbb{C}^N$

A Dissertation

by

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ABSTRACT

In this dissertation we are concerned with a problem which asks whether the compactness of the $\bar{\partial}$-Neumann operator is preserved under the intersection of two bounded pseudoconvex domains in $\mathbb{C}^n$ with the mild assumption that their intersection is connected. Our solutions to this problem in this dissertation can be grouped into two affirmative main results.

The first of these two main results provides a solution under the assumption that the intersection of the boundaries of the (intersecting) domains satisfies McNeal's property $(\tilde{P})$. More precisely, let Ω_1 and Ω_2 be bounded (not necessarily smooth) pseudoconvex domains in $\mathbb{C}^n$ which intersect each other in a domain Ω. If the $\bar{\partial}$-Neumann operators $N_q^{\Omega_1}$ and $N_q^{\Omega_2}$ are compact and the compact set $b\Omega_1 \cap b\Omega_2$ satisfies property $(\tilde{P}_q)$ for some $1 \leq q \leq n$, then the $\bar{\partial}$-Neumann operator N_q^{Ω} is also compact. We discuss some examples of pseudoconvex domains Ω_1 and Ω_2 for which the assumption "$b\Omega_1 \cap b\Omega_2$ satisfies property $(\tilde{P}_q)$" actually holds.

The second main result provides a partial solution to the problem when the intersecting domains have smooth boundaries which intersect each other real transversally. More precisely, let Ω_1 and Ω_2 be bounded smooth pseudoconvex domains in $\mathbb{C}^n$ whose boundaries intersect real transversally and let Ω be the intersection domain. If the $\bar{\partial}$-Neumann operators $N_q^{\Omega_1}$ and $N_q^{\Omega_2}$ are compact for some $1 \leq q \leq n-1$, then N_{n-1}^{Ω} is also compact. In particular, when $n = 2$, compactness of the $\bar{\partial}$-Neumann operator is preserved under the real transversal intersection of two smooth bounded pseudoconvex domains in $\mathbb{C}^2$. We also discuss some by-products of the problem when the domains are smooth and intersect real transversally.

To my parents and to my sister and to my niece

(Anneme, babama, ablama ve yeğenime)

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1. INTRODUCTION

Compactness of the $\bar{\partial}$-Neumann operator is an important property which can be verified on a large class of domains in $\mathbb{C}^n$. Apart from its applications, its importance is due to the fact that it provides a path towards the global regularity of the $\bar{\partial}$-Neumann operator. However, the natural question "Which domains in $\mathbb{C}^n$ can support a compact $\bar{\partial}$-Neumann operator?" remains to be solved. As there are some sufficient conditions which guarantee the compactness, there are also some obstructions which prevent $\bar{\partial}$-Neumann operator from being compact. Nevertheless, the compactness has been fully understood in some special class of domains in terms of some sufficient conditions such as property (P) or its formally weaker version property $(\tilde{P})$.

This dissertation is concerned with the following simple question:

> If two bounded pseudoconvex domains in $\mathbb{C}^n$ intersect each other in a domain and corresponding $\bar{\partial}$-Neumann operators are compact, does it follow that the $\bar{\partial}$-Neumann operator corresponding to the intersection domain is also compact?

A positive result is mostly encouraged by the localization of the compactness of the $\bar{\partial}$-Neumann operator and it forms an important solution of the problem when one of the domains is strictly pseudoconvex:

Localization theorem. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. If for any point in $b\Omega$ there exists a strictly pseudoconvex neighborhood so that this neighborhood intersects Ω in a connected set and this intersection has compact $\bar{\partial}$-Neumann operator, then the $\bar{\partial}$-Neumann operator on Ω is compact. Conversely, if the $\bar{\partial}$-Neumann* *operator on Ω is compact, then for any strictly pseudoconvex domain intersecting Ω in a connected set, the intersection has compact $\bar{\partial}$-Neumann operator.*

The Localization theorem is essentially folklore but see [25] and the monograph [56] for a proof. Problems similar to the one presented here but with stronger assumptions imposed on the intersecting domains were answered by Henkin and Iordan ([29]) and Henkin, Iordan and Kohn ([30]) by using the Bochner-Martinelli kernels, by Michel and Shaw ([41]) by using strictly plurisubharmonic exhaustion functions. Straube considered the similar problem in [53] on piecewise smooth weakly pseudoconvex domains of finite type and obtained an affirmative result. In his dissertation [12], Çelik considered an example of a non-transversal intersection and gave an affirmative answer to the problem. Moreover, he observed that the proof of Localization theorem gives a positive answer to the problem if, on top of the assumptions of the problem, one assumes that the intersecting domains are smooth and one of them satisfies Catlin's property (P). There are also several other articles working on problems of this kind with stronger assumptions made on the convexity of the domains (see [60], [28]).

In order to have more insight on the facts that lie behind the compactness of the $\bar{\partial}$-Neumann operator, the question stated above is of fundamental importance. More precisely, a positive or negative answer will shed some light to characterize the compactness of the $\bar{\partial}$-Neumann operator.

In the analysis of $\bar{\partial}$-Neumann operator, techniques from the theory of partial differential equations and its tools are always of great help. In particular, in proving several properties of $\bar{\partial}$-Neumann operator (such as its compactness), the works are reduced to a neighborhood of the boundary. Therefore, if there is an obstruction to the compactness of the $\bar{\partial}$-Neumann operator, thanks to these tools and techniques, the obstruction is related to the boundary. Thus, in order to understand what an obstruction could be, one needs to investigate the boundary and properties therein. Furthermore, the obstruction (if one exists) is a local property of the boundary, because the compactness of the $\bar{\partial}$-Neumann operator localizes as given by Localization theorem.

A way of testing whether there is some reasonable notion of obstruction is looking at the intersection of pseudoconvex domains as provided in the problem above. Indeed, if two pseudoconvex domains intersect in $\mathbb{C}^n$ and their boundaries lack some notion of obstruction, then the same notion is expected to be absent in the boundary of the intersection domain. So, in order to understand the compactness of the $\bar{\partial}$-Neumann operator, a satisfactory answer must be given to the question.

In this dissertation, there are two main results. Both results give an affirmative answer to the question. In the first result (see Theorem 4.1.2), we assume that the intersection of boundaries $b\Omega_1$ and $b\Omega_2$ satisfies property $(\tilde{P})$. Examples include intersection of domains where at least one of the domains satisfies property $(\tilde{P})$; and property $(\tilde{P})$ is satisfied for instance on strictly pseudoconvex domains, on smooth pseudoconvex domains of finite type or more generally on those domains which satisfy property (P). More examples can be given under weaker assumptions (see Section 4). In the second result (see Theorem 4.2.3), we assume that boundaries are smooth and they intersect real transversally. Under this assumption, N_{n-1} is always compact. In particular, when $n = 2$, the problem is solved when the intersecting domains are smooth and their boundaries intersect real transversally.

The organization of this dissertation is as follows: we will start Section 2 with an introductory language and notation which are necessary to us in this dissertation. The relevant background for the $\mathcal{L}^2$-theory of $\bar{\partial}$ needed for further sections is also provided in Section 2. Section 3 discusses the compactness of $\bar{\partial}$-Neumann operator in general, then lists some of the results needed in proving the main results of this dissertation. In Section 3, we also provide proofs of some useful facts that are only implicit in the literature. In Section 4, we prove our main results and also discuss some interesting by-products of the problem. The content of the dissertation is finalized with a summarizing section and the references are listed at the very end.

2. $\mathcal{L}^2$-THEORY OF THE CAUCHY-RIEMANN OPERATOR

A researcher working in the theory of $\bar{\partial}$-Neumann problem or more generally in Several Complex Variables needs languages of analysis, geometry and partial differential equations. Introducing a sufficient background in each of these fields will necessitate a detailed writing and doing so, we would end up with a lengthy introduction. To keep things shorter, in the first part of this section, we will introduce notation only from some parts of Several Complex Variables and proceed with a review of the weighted $\mathcal{L}^2$-theory of the Cauchy-Riemann operator $\bar{\partial}$. For more information on the geometry, analysis or partial differential equations, we refer to the books [47], [37], [15], [49], [59].

2.1 Notation and basic tools

For a positive integer n, the Euclidean space of complex dimension n is denoted by $\mathbb{C}^n$; that is, $\mathbb{C}^n$ consists of n-tuples $(z_1, \cdots, z_n)$, where $z_j \in \mathbb{C}$ for each $j = 1, \cdots, n$. Each z_j is written as $x_j + iy_j$, where x_j and y_j are the real and the imaginary parts of z_j respectively. Via the mapping $(z_1, \cdots, z_n) \mapsto (x_1, y_1, \cdots, x_n, y_n)$, $\mathbb{C}^n$ becomes isomorphic to the Euclidean space $\mathbb{R}^{2n}$ of real dimension $2n$. When considered as the product of n-copies of $\mathbb{C}$, the topologies on $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ are equal, which in turn gives the advantage of seeing a given open set in one of them also open in the other. The norm on $\mathbb{C}^n$ is inherited via the Hermitian inner product $\langle \cdot, \cdot \rangle$ defined for the vector space $\mathbb{C}$. That is, for $z \in \mathbb{C}^n$, the norm $|z|$ of z is given by

$$|z| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} \langle z_j, z_j \rangle \right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} x_j^2 + y_j^2 \right)^{\frac{1}{2}}.$$

Let Ω be a bounded domain in $\mathbb{R}^m$, $m \geq 2$; that is, Ω is a bounded, connected open set in $\mathbb{R}^m$. The boundary of Ω is denoted by $b\Omega$. For $1 \le k \le \infty$, Ω is called a C^k domain or said to have a C^k boundary if there exists a C^k function $\rho : \mathbb{R}^m \to \mathbb{R}$ such that

(i) $\Omega = \{x \in \mathbb{R}^m : \rho(x) < 0\}$,

(ii) $b\Omega = \{x \in \mathbb{R}^m : \rho(x) = 0\}$, and

(iii) the gradient of ρ does not vanish on $b\Omega$, i.e., $\nabla\rho(p) \neq 0$ for $p \in b\Omega$.

Such a function ρ for a given domain Ω is called a defining function for Ω. Ω is called a smooth domain or a bounded domain with smooth boundary if the conditions $(i), (ii), (iii)$ are satisfied by a C^∞ function, i.e., a smooth function. Similarly, a domain is said to have a Lipschitz boundary if its boundary can locally be written as the graph of a Lipschitz function. That is, given $p \in b\Omega$, there exists a neighborhood $U = U_p$ of p such that, after a rotation, the intersection $\Omega \cap U$ is given by the set

$$\{(x_1, \cdots, x_{m-1}, x_m) \in U | x_m > \lambda(x_1, \cdots, x_{m-1})\}$$

where $\lambda : \mathbb{R}^{m-1} \to \mathbb{R}$ is a Lipschitz function, i.e., there exists an $M > 0$ such that $|\lambda(x) - \lambda(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}^{m-1}$.

The partial derivatives with respect to complex variables z_j or $\bar{z}_j$ are similar to the ones we have in one complex variable case:

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

A bounded C^2 domain $\Omega \subset \mathbb{C}^n$ is called a *pseudoconvex domain* if the complex Hessian of its defining function ρ, when restricted on its boundary, is nonnegative on those vectors that are orthogonal (in the Hermitian inner product in $\mathbb{C}^n$) to the complex normal $(\frac{\partial\rho}{\partial\bar{z}_1}, \cdots, \frac{\partial\rho}{\partial\bar{z}_n})$ to the boundary; i.e., Ω is pseudoconvex if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho(z)}{\partial z_j \partial\bar{z}_k} w_j \bar{w}_k \geq 0 \quad \text{for } z \in b\Omega, w \in \mathbb{C}^n \text{ with } \quad \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(z) w_j = 0. \tag{2.1}$$

If the inequality in (2.1) is strict for all nonzero vectors w that satisfy the equality $\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(z) w_j = 0$, then Ω is called a *strictly pseudoconvex domain*. In that case, there exists a positive constant $C > 0$ such that $\sum_{j,k=1}^{n} \frac{\partial^2\rho(z)}{\partial z_j \partial\bar{z}_k} w_j \bar{w}_k \geq C|w|^2$ for all $z \in b\Omega, w \in \mathbb{C}^n$ with $\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(z) w_j = 0$. If the domain is strictly pseudoconvex at a boundary point p, then p is called a *strictly pseudoconvex point*; otherwise, it is called a *weakly pseudoconvex point*. When at least one weakly pseudoconvex point exists in its boundary, a pseudoconvex domain is sometimes called a *weakly pseudoconvex domain* in order to emphasize that it has a weakly pseudoconvex point. If a domain Ω does not have a sufficient boundary regularity; that is, if it has a C^k boundary with $k < 2$ or its boundary is not the graph of a differentiable function, it is still called a *pseudoconvex domain* if there exists an exhaustion of Ω by strictly pseudoconvex domains that are compactly contained in Ω. In other words, such an Ω is called a pseudoconvex domain if there exists a nested sequence of strictly pseudoconvex domains $\{\Omega_\nu\}_{\nu=1}^{\infty}$ with $\Omega_\nu \Subset \Omega$ for each $\nu = 1, 2, \cdots$ such that $\sup_{\nu \geq 1} \Omega_\nu = \Omega$.

Let Ω be an open subset of $\mathbb{R}^m$ and $x_0 \in \Omega$. A function $f : \Omega \to [-\infty, \infty)$ is said to be *upper semi-continuous at* x_0 if for every $M > f(x_0)$ there exists a neighborhood U of x_0 such that $M > f(x)$ for all $x \in U \cap \Omega$. f is called *upper semi-continuous* if it is upper semi-continuous at each $x \in \Omega$. Equivalently, f is upper semi-continuous if for every $x \in \Omega$, $\limsup_{y \to x} f(y) \leq f(x)$. An upper semi-continuous function $f : \Omega \to [-\infty, \infty)$ is called a *subharmonic function* if at any $z \in \Omega$ it satisfies the sub-mean value property:

$$f(z) \leq \frac{1}{A_m r^{m-1}} \int_{S(z,r)} f(\xi) d\sigma(\xi) \text{ for all } r > 0 \text{ with } S(z,r) \subset \Omega.$$

Here, A_m denotes the surface area of the unit sphere in $\mathbb{R}^m$, $d\sigma(\xi)$ denotes the surface area measure and the integration is taken over any sphere $S(z,r)$ (with center z and radius r) that is contained in Ω. When $\Omega \subset \mathbb{C}^n (n \geq 2)$ is open, an upper semi-continuous function $f : \Omega \to [-\infty, \infty)$ is called *plurisubharmonic* if for any $z \in \Omega$ and $w \in \mathbb{C}^n$, $f(z + \tau w)$ is subharmonic in $\tau \in \mathbb{C}$ whenever $\{z + \tau w : \tau \in \mathbb{C}\}$ is contained in Ω; that is, f is plurisubharmonic on Ω if it is subharmonic on the intersection of every complex line with Ω. A C^2 real-valued function φ on Ω is plurisubharmonic if and only if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0 \text{ for all } w = (w_1, \cdots, w_n) \in \mathbb{C}^n \text{ and } z \in \Omega. \tag{2.2}$$

Another way of saying a C^2 function is plurisubharmonic is that its complex Hessian at each point of its domain is positive semi-definite on $\mathbb{C}^n$. If the inequality in (2.2) is strict for nonzero vectors w, then φ is called a *strictly plurisubharmonic function* on Ω. So, in particular, if a bounded domain in $\mathbb{C}^n$ has a plurisubharmonic C^2 defining function, then it is a pseudoconvex domain.

When a domain in $\mathbb{C}^n$ does not have any boundary regularity, one can still decide whether it is pseudoconvex or not by checking the existence of a particular function defined on it as follows: a domain $\Omega \subset \mathbb{C}^n$ is said to be *pseudoconvex* if there exists a continuous plurisubharmonic function ρ on Ω such that $\{z \in \Omega : \rho(z) < c\}$ is a relatively compact subset of Ω for any $c > 0$. Note that this last definition of the pseudoconvexity is equivalent to the one that we introduced before. For a complete treatment of pseudoconvexity or more generally the topics in Several Complex Variables, we refer to the books [47] and [37].

2.2 The weighted $\mathcal{L}^2$-theory of the Cauchy-Riemann operator

We now introduce briefly some parts of the $\mathcal{L}^2$ machinery behind the Cauchy-Riemann operator. For a complete treatment of the theory, we refer to the book [15] and the monograph [56] from which we benefited to a great extent (see also [22]).

Let Ω be a bounded domain in $\mathbb{C}^n$ (unless stated otherwise, we take $n \geq 2$). For $1 \leq q \leq n$, we represent a $(0,q)$-form u on Ω by $\sum'_{|J|=q} u_J d\bar{z}_J$. The sum is taken over strictly increasing q-tuples and u_J are functions defined on Ω. In case $q = 0$, u is simply a function defined on Ω. Let ϕ be a continuous function on $\overline{\Omega}$. The form u is said to be in $\mathcal{L}^2_{(0,q)}(\Omega, \phi)$ if

$$\|u\|^2_{\mathcal{L}^2_{(0,q)}(\Omega,\phi)} := \sum_{|J|=q}{}' \int_\Omega |u_J(z)|^2 e^{-\phi(z)} dV(z) < \infty.$$

Defined this way, the weighted space $\mathcal{L}^2_{(0,q)}(\Omega, \phi)$ is a Hilbert space with associated inner product $(u, v)_\phi = \sum'_{|J|=q} \int_\Omega u_J(z)\overline{v_J(z)} e^{-\phi(z)} dV(z)$. Notice that, since Ω is bounded, the unweighted Lebesgue space $\mathcal{L}^2_{(0,q)}(\Omega)$ of $(0,q)$-forms (this corresponds to $\phi \equiv 0$) is equal to $\mathcal{L}^2_{(0,q)}(\Omega, \phi)$. When $q = 0$, the corresponding space is the space of square integrable functions defined on Ω and it is denoted by $\mathcal{L}^2(\Omega)$. If Γ is a function space defined on a set $E \subset \mathbb{C}^n$, we simply write $u \in \Gamma_{(0,q)}(E)$ to mean that the functions u_J belong to $\Gamma(E)$ for each J.

2.2.1 The Cauchy-Riemann operator and its adjoint

For $0 \leq q \leq n$, we define the weighted Cauchy-Riemann operator, or simply the weighted D-bar operator, $\bar{\partial}_{q,\phi} : \mathcal{L}^2_{(0,q)}(\Omega, \phi) \to \mathcal{L}^2_{(0,q+1)}(\Omega, \phi)$ by

$$\bar{\partial}_{q,\phi}\left(\sum_{|J|=q}{}' u_J d\bar{z}_J\right) := \sum_{j=1}^{n} \sum_{|J|=q}{}' \frac{\partial u_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_J$$

with $\text{dom}(\bar{\partial}_{q,\phi}) = \{u \in \mathcal{L}^2_{(0,q)}(\Omega, \phi) | \ \bar{\partial}_{q,\phi} u \in \mathcal{L}^2_{(0,q+1)}(\Omega, \phi)\}$. Here, the derivatives are taken in the distributional sense. $\bar{\partial}_{q,\phi}$ is a linear, closed and densely defined operator. Note that $\text{Ran}(\bar{\partial}_{n,\phi}) = \{0\}$ and a simple calculation shows that $\bar{\partial}_{q+1,\phi}\bar{\partial}_{q,\phi} = 0$. That is, the operator $\bar{\partial}_{q,\phi}$ forms a complex, which we call the (weighted) $\bar{\partial}$-complex. The domain of $\bar{\partial}_{q,\phi}$ is same with the domain of $\bar{\partial}_q$ (the latter is the corresponding operator when $\phi \equiv 0$); therefore, most of the time, we will suppress the weight notation in the subscripts and just write $\bar{\partial}_q$ instead of $\bar{\partial}_{q,\phi}$. The formal adjoint of $\bar{\partial}_{q,\phi}$ is denoted by $\vartheta_{q,\phi}$. Hilbert space theory of unbounded operators gives that the adjoint of $\bar{\partial}_{q,\phi}$, which we denote by $\bar{\partial}^*_{q,\phi}$, is also linear, closed and densely defined. We denote the null spaces of $\bar{\partial}_q$ and $\bar{\partial}^*_{q,\phi}$ by $\ker(\bar{\partial}_q)$ and $\ker(\bar{\partial}^*_{q,\phi})$ respectively; and for $0 \leq q \leq n$, the orthogonal projection $P_{q,\phi} : \mathcal{L}^2_{(0,q)}(\Omega, \phi) \to \ker(\bar{\partial}_q)$ is called the (weighted) Bergman projection.

We recall that an abstract definition for a form $u \in \mathcal{L}^2_{(0,q+1)}(\Omega)$ to be in $\text{dom}(\bar{\partial}^*_{q,\phi})$ is as follows: there exists a constant $C > 0$ such that $|(u, \bar{\partial}_q \alpha)_\phi| \leq C||\alpha||_{\mathcal{L}^2_{(0,q)}(\Omega,\phi)}$ whenever $\alpha \in \text{dom}(\bar{\partial}_q)$. When $\phi \equiv 0$ on Ω, an integration by parts argument shows that the action of the formal adjoint ϑ_q on a form u (when derivatives are taken in the distributional sense) is given by

$$\vartheta_q u = -\sum_{|K|=q}{}' \left(\sum_{j=1}^{n} \frac{\partial u_{jK}}{\partial z_j} \right) d\bar{z}_K. \tag{2.3}$$

Here, we use the notation u_{jK} as follows: let $j \in \{1, \cdots, n\}$ and $K := (k_1, \cdots, k_q)$ with $1 \leq k_1 < \cdots < k_q \leq n$ be fixed. Then

$$u_{jK} := \begin{cases} 0 & \text{if } j = k_s \text{ for some } s = 1, \cdots, q; \\ u_{(j,k_1,\cdots,k_q)} & \text{if } j < k_1; \\ (-1)^r u_{(k_1,\cdots,k_r,j,k_{r+1},\cdots,k_q)} & \text{if } k_r < j < k_{r+1} \text{ for some } r \in \{1, \cdots, q-1\}; \\ (-1)^q u_{(k_1,\cdots,k_q,j)} & \text{if } j > k_q. \end{cases}$$

We go back to (2.3) and note also that if u is in $\mathrm{dom}(\bar{\partial}_q^*)$, then $\vartheta_q u = \bar{\partial}_q^* u$. However, a remark is also in order: for a given $(0, q+1)$-form u, $\vartheta_q u \in \mathcal{L}^2_{(0,q)}(\Omega)$ does not necessarily imply that $u \in \mathrm{dom}(\bar{\partial}_q^*)$. Indeed, if the same integration by parts argument used in showing (2.3) is considered on a C^1 domain, then a form $u \in C^1_{(0,q+1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}_q^*)$ has to satisfy

$$\sum_{j=1}^{n} u_{jK}(z) \frac{\partial \rho}{\partial z_j}(z) = 0 \quad \text{ for all } K \text{ and } z \in b\Omega. \tag{2.4}$$

When $\phi \in C^1(\overline{\Omega})$, integration by parts methods give that

$$\vartheta_{q,\phi} u = \vartheta_q u + \sum_{j=1}^{n} \sum_{|K|=q}{}' \frac{\partial \phi}{\partial z_j} u_{jK} d\bar{z}_K.$$

Furthermore, we have $\mathrm{dom}(\bar{\partial}_{q,\phi}^*) = \mathrm{dom}(\bar{\partial}_q^*)$. Thus, the operators $\bar{\partial}_q^*$ and $\bar{\partial}_{q,\phi}^*$ have the same domain and they differ by an operator of order zero.

When a domain is smooth enough, there is another way to see whether a form is in $\mathrm{dom}(\bar{\partial}^*_{q,\phi})$ other than checking (2.4). For this, one needs to construct the so called special boundary frame and in doing that we follow Section 2.2 in [56]. Let Ω be a C^2 domain, ρ be its defining function and p be in $b\Omega$. The vectors $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$ which satisfy $\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(p) w_j = 0$ are called complex tangent vectors and the space of these vectors is denoted by $H_p(b\Omega)$. $H_p(b\Omega)$ is the maximal subspace of the tangent space to $b\Omega$ at p which stays invariant under multiplication by i. By Gram-Schmidt process, one can then choose (near p) an orthonormal set of vector fields $L_1, \cdots, L_{n-1}$ so that near p, $L_1, \cdots, L_{n-1}$ form a basis for the complex tangent space $H_p(b\Omega_\delta)$, where $b\Omega_\delta$ for $\delta > 0$ denotes the set of $z \in \Omega$ that satisfy $\rho(z) = -\delta$. If we add the normalized complex normal L_n to this set and consider the set of orthonormal dual forms $\omega_1, \cdots, \omega_n$, then we obtain a basis for the $(1,0)$-forms near p. When $q > 1$, we can take wedge products of the ω_j's and obtain a local basis for the $(q,0)$-forms near p. $\{\omega_1, \cdots, \omega_n\}$ is called a special boundary frame. The upshot is if $u = \sum'_J u_J \bar{\omega}_J$, where $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_q}$ and $u_J \in C^1(\overline{\Omega})$, then

$$u \in \mathrm{dom}(\bar{\partial}^*) \quad \text{if and only if} \quad u_J = 0 \text{ on } b\Omega \text{ whenever } n \in J. \tag{2.5}$$

The following density result is essentially due to Hörmander ([31]), but see also Lemma 4.3.2 in [15] and Proposition 2.3 in [56].

Lemma 2.2.1 (Density lemma)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$, $\phi \in C^1(\overline{\Omega})$ and $1 \leq q \leq n$.*

i) $C^\infty_{0,(0,q)}(\Omega)$ is dense in dom$(\bar{\partial}^*_{q-1,\phi})$ in the graph norm

$$u \mapsto (||u||^2_{\mathcal{L}^2_{(0,q)}(\Omega,\phi)} + ||\bar{\partial}^*_{q-1,\phi} u||^2_{\mathcal{L}^2_{(0,q-1)}(\Omega,\phi)})^{\frac{1}{2}}.$$

ii) If Ω is a Lipschitz domain, $C^\infty_{(0,q)}(\overline{\Omega})$ is dense in dom$(\bar{\partial}_q)$ in the graph norm

$$u \mapsto \left(||u||^2_{\mathcal{L}^2_{(0,q)}(\Omega,\phi)} + ||\bar{\partial}_q u||^2_{\mathcal{L}^2_{(0,q+1)}(\Omega,\phi)} \right)^{\frac{1}{2}}.$$

iii) If Ω is a C^{k+1} domain for some $k \in [1,\infty]$, then $C^k_{(0,q)}(\overline{\Omega}) \cap$ dom$(\bar{\partial}^*_{q-1,\phi})$ is dense in dom$(\bar{\partial}_q) \cap$ dom$(\bar{\partial}^*_{q-1,\phi})$ in the graph norm

$$u \mapsto \left(||u||^2_{\mathcal{L}^2_{(0,q)}(\Omega,\phi)} + ||\bar{\partial}_q u||^2_{\mathcal{L}^2_{(0,q+1)}(\Omega,\phi)} + ||\bar{\partial}^*_{q-1,\phi} u||^2_{\mathcal{L}^2_{(0,q-1)}(\Omega,\phi)} \right)^{\frac{1}{2}}.$$

2.2.2 $\bar{\partial}$-Neumann problem and the weighted basic estimate

Definition 2.2.2. *The weighted complex Laplacian $\Box_{q,\phi}$ is defined by $\bar{\partial}_{q-1}\bar{\partial}^*_{q-1,\phi} + \bar{\partial}^*_{q,\phi}\bar{\partial}_q$ with*

$$dom(\Box_{q,\phi}) := \{u \in dom(\bar{\partial}_q) \cap dom(\bar{\partial}^*_{q-1,\phi}) : \bar{\partial}_q u \in dom(\bar{\partial}^*_q), \bar{\partial}^*_{q-1,\phi} u \in dom(\bar{\partial}_{q-1})\}.$$

The $\bar{\partial}$-Neumann problem is then to invert $\Box_q$. If the inverse exists, it is called the $\bar{\partial}$-Neumann operator and denoted by $N_{q,\phi}$.

One can show by chasing the definitions of $\bar{\partial}$ and ϑ, and using the multi-linear algebra that the complex Laplacian acts on $(0,q)$-forms as a constant multiple of the usual Laplacian:

$$\bar{\partial}\vartheta + \vartheta\bar{\partial} = -\frac{1}{4} {\sum_J}' (\Delta u_J) d\bar{z}_J. \tag{2.6}$$

However, the boundary conditions $u \in \text{dom}(\bar{\partial}^*)$ and $\bar{\partial}u \in \text{dom}(\bar{\partial}^*)$ in the $\bar{\partial}$-Neumann problem make the problem *non-elliptic*; and these boundary conditions, for this reason, distinguish the $\bar{\partial}$-Neumann problem from the usual Dirichlet or Neumann problems for Laplacians.

The $\bar{\partial}$-Neumann problem was solved on pseudoconvex domains by Hörmander using the weighted $\mathcal{L}^2$ theory ([31]):

Theorem 2.2.3 (Hörmander)**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, $\phi \in C^2(\overline{\Omega})$ and suppose that $1 \leq q \leq n$. The weighted complex Laplacian $\Box_{q,\phi}$ is an unbounded, self-adjoint, surjective operator on $\mathcal{L}^2_{(0,q)}(\Omega, \phi)$ with a bounded, self-adjoint inverse $N_{q,\phi}$ defined by $j_{q,\phi} \circ j^*_{q,\phi}$, where $j_{q,\phi}$ denotes the imbedding of $dom(\bar{\partial}_q) \cap dom(\bar{\partial}^*_{q-1,\phi})$ into $\mathcal{L}^2_{(0,q)}(\Omega, \phi)$. Moreover, when ϕ is also plurisubharmonic, the operator norm of $N_{q,\phi}$ is at most $\frac{D^2 e}{q}$, where D denotes the diameter of Ω and e is the base of logarithm.*

An immediate important application of the existence of $\bar{\partial}$-Neumann operator is that it provides solutions to $\bar{\partial}$ and $\bar{\partial}^*$ problems. More precisely, for $1 \leq q \leq n$, if $\bar{\partial}_q u = 0$, then $\bar{\partial}^*_{(q-1,\phi)} N_{(q,\phi)} u$ gives the solution f with minimal $\mathcal{L}^2_{(0,q-1)}(\Omega, \phi)$-norm to the equation $\bar{\partial}_{(q-1,\phi)} f = u$; and if $\bar{\partial}^*_{(q-1,\phi)} u = 0$, then $\bar{\partial}_q N_{(q,\phi)} u$ gives the solution f with minimal $\mathcal{L}^2_{(0,q+1)}(\Omega, \phi)$-norm to the equation $\bar{\partial}^*_{(q,\phi)} f = u$. The operators $\bar{\partial}^*_{(q-1,\phi)} N_{(q,\phi)}$ and $\bar{\partial}_q N_{(q,\phi)}$ are called (weighted) canonical (solution) operators. Moreover, for $1 \leq q \leq n$, one has (see [46]) the following relation (also called Range's formula):

$$N_{q,\phi} = (\bar{\partial}^*_\phi N_{q,\phi})^* (\bar{\partial}^*_\phi N_{q,\phi}) + \bar{\partial}^*_\phi N_{(q+1,\phi)} (\bar{\partial}^*_\phi N_{(q+1,\phi)})^*. \tag{2.7}$$

There are two main approaches to show the existence of $N_{q,\phi}$. One approach passes through showing $\Box_{q,\phi}$ has closed range. The other approach makes use of the Hilbert space theory of unbounded operators via symmetric quadratic forms. Both approaches have a common ground, the so called (weighted) basic estimates. Construction of basic estimates for the smooth forms goes at least back to Morrey's work

[42] (see also [33] for the history of the theory from the point of view of a contributor and witness). We will state the basic estimates below as we will frequently make use of them. However, before moving further, let us note that from now on, when the form level q or the space on which a norm is taken are understood, we might drop q or the space notation from the subscripts of the operators and norms for the economy of notation. Sometimes, there will be only a set notation or function notation in the norms' subscripts such as $||.||_\Omega$ or $||.||_\phi$. What is meant by these either will be clear from the context or the adopted notation will be briefly explained.

2.2.3 The twisted Kohn-Morrey-Hörmander formula and its applications

The importance of the weighted theory comes into prominence especially after one has the following theorem (see [56], [40]):

Theorem 2.2.4. *(The twisted Kohn-Morrey-Hörmander formula)[56]*
Let Ω be a bounded C^2 domain in $\mathbb{C}^n$ and ρ be its defining function; f and ϕ be two real-valued functions such that $f, \phi \in C^2(\overline{\Omega})$ and $f \geq 0$. If u is a $(0,q)$-form $(1 \leq q \leq n)$ with $u \in dom(\bar{\partial}^*_{q-1,\phi}) \cap C^1_{(0,q)}(\overline{\Omega})$, then the following formula holds:

$$\begin{aligned}
||\sqrt{f}\bar{\partial}u||^2_\phi + ||\sqrt{f}\bar{\partial}^*_\phi u||^2_\phi = & \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_{b\Omega} f \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}} e^{-\phi} \frac{d\sigma}{|\nabla \rho|} \\
& + \sum_{|J|=q}{}' \sum_{j=1}^{n} \int_\Omega f \left| \frac{\partial u_J}{\partial \bar{z}_j} \right|^2 e^{-\phi} dV \\
& + 2\,\mathrm{Re} \left(\sum_{|K|=q-1}{}' \sum_{j=1}^{n} u_{jK} \frac{\partial f}{\partial z_j} d\bar{z}_K, \bar{\partial}^*_\phi u \right)_\phi \\
& + \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \left(f \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} - \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \right) u_{jK} \overline{u_{kK}} e^{-\phi} dV.
\end{aligned} \tag{2.8}$$

The twisted Kohn-Morrey-Hörmander formula can be proved by an application of integration by parts. In [7], the authors achieved an elegant way to deduce basic estimates on bounded pseudoconvex domains from the twisted Kohn-Morrey-Hörmander formula when $\phi \equiv 0$. We will use the methods of [7], but this time we will carry along the weight function ϕ.

Let Ω be a pseudoconvex domain with C^2 boundary and $b \in C^2(\overline{\Omega})$ with $b \leq 0$ on $\overline{\Omega}$. Set $f := 1 - e^b$. Then $0 \leq f \leq 1$ and therefore (since the domain Ω is pseudoconvex) the boundary integral on the right hand side of (2.8) is nonnegative. So, from Theorem 2.2.4, we obtain

$$||\sqrt{f}\bar{\partial}u||^2_\phi + ||\sqrt{f}\bar{\partial}^*_\phi u||^2_\phi \geq 2\,\mathrm{Re}\left(\sum_{|K|=q-1}{}' \sum_{j=1}^{n} u_{jK}\frac{\partial f}{\partial z_j}d\bar{z}_K, \bar{\partial}^*_\phi u\right)_\phi + \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \left(f\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} - \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}\right) u_{jK}\overline{u_{kK}}e^{-\phi}dV. \tag{2.9}$$

Substituting the definition of f on the right hand side of (2.9), we obtain

$$\begin{aligned}||\sqrt{f}\bar{\partial}u||^2_\phi + ||\sqrt{f}\bar{\partial}^*_\phi u||^2_\phi \geq{}& -2\,\mathrm{Re}\left(e^b\sum_{|K|=q-1}{}' \sum_{j=1}^{n} u_{jK}\frac{\partial b}{\partial z_j}d\bar{z}_K, \bar{\partial}^*_\phi u\right)_\phi \\ &+ \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega f\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}}e^{-\phi}dV \qquad (2.10) \\ &+ \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega e^b\left(\frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} + \frac{\partial b}{\partial z_j}\frac{\partial b}{\partial \bar{z}_k}\right) u_{jK}\overline{u_{kK}}e^{-\phi}dV.\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the inner product on the right hand side of (2.10) and then using the basic inequality $2|ab| \leq |a|^2 + |b|^2$, we obtain

$$||\sqrt{f}\bar{\partial}u||_\phi^2 + ||\sqrt{f}\bar{\partial}_\phi^* u||_\phi^2 \geq -||e^{\frac{b}{2}}\bar{\partial}_\phi^* u||_\phi^2 + \sum_{|K|=q-1}' \sum_{j,k=1}^n \int_\Omega f \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} e^{-\phi} dV + \sum_{|K|=q-1}' \sum_{j,k=1}^n \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} e^{-\phi} dV. \quad (2.11)$$

Now, taking the weighted norm on the right hand side of (2.11) to the left hand side and observing that $f + e^b = 1$ and recalling that $0 \leq f \leq 1$, we get

$$||\bar{\partial}u||_\phi^2 + ||\bar{\partial}_\phi^* u||_\phi^2 \geq \sum_{|K|=q-1}' \sum_{j,k=1}^n \int_\Omega f \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} e^{-\phi} dV + \sum_{|K|=q-1}' \sum_{j,k=1}^n \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} e^{-\phi} dV. \quad (2.12)$$

Furthermore, if ϕ is a plurisubharmonic function on Ω, then the first integral on the right hand side of (2.12) is nonnegative and hence we obtain

$$||\bar{\partial}u||_\phi^2 + ||\bar{\partial}_\phi^* u||_\phi^2 \geq \sum_{|K|=q-1}' \sum_{j,k=1}^n \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} e^{-\phi} dV. \quad (2.13)$$

Now, we set $b(z) := -1 + \frac{|z-P|^2}{D^2}$, where $P \in \Omega$ and D is the diameter of Ω. Then $e^b \geq \frac{1}{e}$ and $\frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} = \frac{\delta_{jk}}{D^2}$. So, we obtain from (2.13) that

$$||u||_\phi^2 \leq \frac{D^2 e}{q}\left(||\bar{\partial}u||_\phi^2 + ||\bar{\partial}_\phi^* u||_\phi^2\right). \quad (2.14)$$

Note that we obtained the estimate in (2.14) for forms $u \in C^1_{(0,q)}(\overline{\Omega}) \cap \text{dom}(\bar{\partial}^*_{q-1,\phi})$. However, since Ω is a C^2 domain, (iii) in Lemma 2.2.1 applies and we get (2.14) for any $u \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*_\phi) \subset \mathcal{L}^2_{(0,q)}(\Omega, \phi)$. We call these estimates weighted basic estimates.

Actually, more is true for the weighted basic estimate. It holds on *any* bounded pseudoconvex domain no matter how regular its boundary is. In order to show this, one can bring the Hilbert space tools as in [56] (pp. $26-27$) and imitate the work there for our case to obtain an equivalence of two conditions: weighted basic estimates hold for a form $u \in \text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{(q-1),\phi})$ if and only if every square integrable $(0,q)$-form u can be written as $\bar{\partial}_{q-1}v + \bar{\partial}^*_{q,\phi}w$ for some $v \in \ker(\bar{\partial}_{q-1})^\perp$ and $w \in \ker(\bar{\partial}^*_{q,\phi})^\perp$ whose weighted L^2-norms are dominated by that of u. Once such an equivalence is at hand, one can obtain inequality (2.14) on bounded pseudoconvex domains in $\mathbb{C}^n$ by proving that the decomposition and estimates in the second condition of the equivalence are preserved under increasing union of subdomains of Ω. Note that since Ω is pseudoconvex, we have an exhaustion of Ω by strictly pseudoconvex C^2 domains from the inside, therefore weighted basic estimates are already available on each exhausting subdomain. We state the existence of the weighted basic estimates in Proposition 2.2.5 below and skip its proof. A proof is technically same as in the unweighted case. For a proof of the latter, we refer to the proof of Proposition 2.7 in [56].

Proposition 2.2.5. *Suppose Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\phi \in C^2(\overline{\Omega})$ is a plurisubharmonic function on Ω. Then for all $u \in dom(\bar{\partial}_q) \cap dom(\bar{\partial}^*_{q-1,\phi}) \subset \mathcal{L}^2_{(0,q)}(\Omega, \phi)$, we have*

$$||u||^2_\phi \leq \frac{D^2 e}{q}\left(||\bar{\partial}u||^2_\phi + ||\bar{\partial}^*_\phi u||^2_\phi\right). \tag{2.15}$$

Remark 2.2.6. *Note that estimates* (2.15) *give a bound for the norm of weighted $\bar{\partial}$-Neumann operator as claimed in the last statement of Hörmander's theorem (Theorem 2.2.3).*

On the other hand, observe that in (2.12) if we choose $f \equiv 1$ (i.e., $b \equiv 0$), then for any $u \in \text{dom}(\bar{\partial}^*_{q-1,\phi}) \cap C^1_{(0,q)}(\overline{\Omega})$, we have

$$||\bar{\partial}u||^2_\phi + ||\bar{\partial}^*_\phi u||^2_\phi \geq \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}} e^{-\phi} dV \tag{2.16}$$

Note that Ω has to be at least C^2 domain by Theorem 2.2.4. Thus, in view of Lemma 2.2.1, the inequality (2.16) is valid for any $\text{dom}(\bar{\partial}_q) \cap \text{dom}(\bar{\partial}^*_{q-1,\phi})$. Observe that the inequality (2.16) does not involve any boundary integrals. At first sight, it seems that one can prove this inequality on any bounded pseudoconvex domain by restricting forms to exhausting subdomains. However, a form that is in $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*_\phi)$ when restricted to an exhausting subdomain does not have to be in $\text{dom}(\bar{\partial}^*_\phi)$. Straube overcame this problem by developing a regularization procedure in [53] (see also Corollary 2.13 in [56] for a more detailed proof). The proof given in [56] for the unweighted case works in the weighted case as well; hence we skip the proof here. This inequality for those forms in $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*_\phi)$ on any bounded pseudoconvex domain Ω in $\mathbb{C}^n$ will be essential in proving Theorem 4.1.2. Therefore, we give its formal statement here:

Proposition 2.2.7. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\phi \in C^2(\overline{\Omega})$. If $u = \sum'_{|J|=q} u_J d\bar{z}_J$ is in dom$(\bar{\partial}) \cap$ dom$(\bar{\partial}^*_\phi) \subset \mathcal{L}^2_{(0,q)}(\Omega, \phi)$, then*

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} u_{jK} \overline{u_{kK}} e^{-\phi} dV \leq ||\bar{\partial}u||^2_\phi + ||\bar{\partial}^*_\phi u||^2_\phi. \tag{2.17}$$

3. COMPACTNESS IN THE $\bar{\partial}$-NEUMANN PROBLEM

In this section, we will provide the tools that are important in understanding the compactness of the $\bar{\partial}$-Neumann operator, review some of the results related to the compactness and provide proofs to some of the well-known facts which do not seem to have proofs in the literature. We first recall the definition of a compact operator.

Definition 3.0.8. *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be Banach spaces and $L : \mathcal{B}_1 \to \mathcal{B}_2$ be a bounded operator. L is called compact, if for every bounded sequence in $\mathcal{B}_1$, the image under L of the sequence has a convergent subsequence in $\mathcal{B}_2$.*

There are several equivalent ways of verifying whether an operator is compact or not. Among many others, the following lemma, especially in the context of $\bar{\partial}$-Neumann problem, has proved to be very practical.

Lemma 3.0.9. *Let H_1, H_2 and H_3 be Hilbert spaces over the field of complex numbers. Suppose that $K : H_1 \to H_2$ is a linear, compact operator and $L : H_1 \to H_3$ is a linear, injective, bounded operator. Then, for every $\varepsilon > 0$, there exists a constant C_ε such that*

$$||Kx||_{H_2} \leq \varepsilon ||x||_{H_1} + C_\varepsilon ||Lx||_{H_3} \text{ for all } x \in H_1. \tag{3.1}$$

Conversely, let H_1, H_2 be Hilbert spaces over the field of complex numbers and $K : H_1 \to H_2$ be a linear operator. Suppose that for every $\varepsilon > 0$ there are a Hilbert space $\mathcal{H}_\varepsilon$, a linear, compact operator $\mathcal{K}_\varepsilon : H_1 \to \mathcal{H}_\varepsilon$, and a constant C_ε such that

$$||Kx||_{H_2} \leq \varepsilon ||x||_{H_1} + C_\varepsilon ||\mathcal{K}_\varepsilon x||_{\mathcal{H}_\varepsilon} \text{ for all } x \in H_1. \tag{3.2}$$

Then, K is compact.

Remark 3.0.10. *The inequalities* (3.1) *and* (3.2) *can be also stated where all corresponding norms have squares. The equivalence can be shown in one direction by choosing an appropriate scaled* ε *and then applying the basic inequality* $2ab \leq (a^2+b^2)$*; and by completing the right hand side into a square and then taking square roots of both sides in the other direction.*

Following a historical trail of Lemma 3.0.9 or its variants takes one to the works of late 50's or early 60's in which at some instances proving the interpolation inequalities was the main issue (see Theorem 1.4.3.3 in [27] and the references before and after this theorem). A variant of Lemma 3.0.9 in the context of Banach spaces can be found in [40]. A proof of the lemma we stated above or its variants in the literature can be found in [56], [20] and [35]. Second statement in Lemma 3.0.9 can be proved via a diagonal subsequence argument. Surprisingly, in all of the references that we provided here for the proofs, estimate (3.1) is proved via the contradiction argument. It would be interesting to see a direct proof of the first statement which could shed some light on the quantitative dependence of C_ε on ε and the operator norms involved.

Compactness of the $\bar{\partial}$-Neumann operator is useful in several ways. Historically, its first use is due to the fact that it implies the regularity in Sobolev spaces. The other applications include (see the introduction of Chapter 4 in [56]) "...the Fredholm theory of Toeplitz operators, existence and non-existence of Henkin-Ramirez type kernels for solving $\bar{\partial}$ and certain C^*-algebras of operators naturally associated to a domain".

Recall that the complex Laplacian $\Box_q$ is defined by $\bar{\partial}_{q-1}\bar{\partial}^*_{q-1} + \bar{\partial}^*_q\bar{\partial}_q$ with

$$\operatorname{dom}(\Box_q) = \{u \in \mathcal{L}^2_{(0,q)}(\Omega) | u \in \operatorname{dom}(\bar{\partial}) \cap \operatorname{dom}(\bar{\partial}^*), \bar{\partial}u \in \operatorname{dom}(\bar{\partial}^*), \bar{\partial}^* u \in \operatorname{dom}(\bar{\partial})\}.$$

Recall also from Theorem 2.2.3 that, if exists, N_q was given by $j_q \circ j_q^*$. We will now bring Lemma 3.0.9 into our context. With the notation of Lemma 3.0.9, set $H_1 = \mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_q)$ (with the graph norm $u \mapsto ||u||^2 + ||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2$), $H_2 = \mathcal{L}^2_{(0,q)}(\Omega)$, $H_3 = W^{-1}_{(0,q)}(\Omega)$, $\mathcal{H}_\varepsilon = H_3$ for all $\varepsilon > 0$. Let K be the inclusion $j_q : \mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_q) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega)$, L be the composition of inclusions $\mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_q) \hookrightarrow \mathcal{L}^2_{(0,q)}(\Omega) \hookrightarrow W^{-1}_{(0,q)}(\Omega)$ and $\mathcal{K}_\varepsilon = K$ for all $\varepsilon > 0$. After absorbing the $\varepsilon||u||$ terms on the right hand side to the left hand side, we obtain the equivalence of (iii) and (iv) in the lemma below (see [56]):

Lemma 3.0.11. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \le q \le n$. Then the following are equivalent:*

(i) *N_q is compact as an operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to itself;*

(ii) *N_q is compact as an operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to dom$(\bar{\partial}_q) \cap$ dom$(\bar{\partial}^*_q)$;*

(iii) *the embedding of dom$(\bar{\partial}_q) \cap$ dom$(\bar{\partial}^*_q)$ into $\mathcal{L}^2_{(0,q)}(\Omega)$ is compact;*

(iv) *for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that the following* compactness estimates *hold:*

$$||u||^2 \le \varepsilon(||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2) + C_\varepsilon ||u||^2_{-1} \text{ for } u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*); \tag{3.3}$$

(v) *the canonical solution operators $\bar{\partial}^* N_q : \mathcal{L}^2_{(0,q)}(\Omega) \cap \ker(\bar{\partial}_q) \to \mathcal{L}^2_{(0,q-1)}(\Omega)$ and $\bar{\partial}^* N_{q+1} : \mathcal{L}^2_{(0,q+1)}(\Omega) \cap \ker(\bar{\partial}_{q+1}) \to \mathcal{L}^2_{(0,q)}(\Omega)$ are compact.*

(vi) *there exists a compact solution operator for $\bar{\partial}$ on $(0,q)$-forms; i.e., there exists a linear compact operator $T_q : \mathcal{L}^2_{(0,q)}(\Omega) \cap \ker(\bar{\partial}_q) \to \mathcal{L}^2_{(0,q-1)}(\Omega)$ such that $\bar{\partial}_{q-1} T_q u = u$ for all $u \in \ker(\bar{\partial}_q)$.*

The equivalence of $(i), (ii)$ and (iii) are from definition and construction of the $\bar{\partial}$-Neumann operator (i.e., $N_q = j_q \circ j^*_q$) and the fact that a linear operator A in the form TT^* is compact if and only if T and T^* are compact. A similar discussion can be made for the equivalence of (i) and (v) by observing Range's formula (2.7) and noting that the operators on the right hand side of Range's formula are nonnegative. That (v) implies (vi) is by definition of canonical solution operators and that (vi) implies (v) is because compactness is preserved by projecting onto $\ker(\bar{\partial}_q)^{\perp}$.

Compactness of N_q enjoys several important properties. Among these are the facts that compactness of N_q and those of the canonical solution operators percolate up the complex ([13]). That is, if N_q is compact, so is N_{q+1} and similarly for the canonical solution operators. Having already a handful of several equivalent properties for the compactness of the unweighted $\bar{\partial}$-Neumann operator, one might wonder about compactness of the weighted $\bar{\partial}$-Neumann operator. However, compactness of the $\bar{\partial}$-Neumann operator is independent of the metric (see [12], [14]).

3.1 Sufficient conditions for the compactness of N_q

Instead of direct verification of compactness of the $\bar{\partial}$-Neumann operator as in Lemma 3.0.11, one can use several sufficient conditions which guarantee the compactness of the $\bar{\partial}$-Neumann operator or make reasonable reductions on the space worked.

3.1.1 Reduction of compactness estimates to harmonic forms

To prove the compactness of N_q, it suffices to verify the compactness estimates (3.3) for those forms in $\text{dom}(\bar{\partial})\cap\text{dom}(\bar{\partial}^*)$ with harmonic components. This is explicit in [53] where the same reduction to forms with harmonic components was used in the context of subelliptic estimates and the idea there can be traced back to [43]. A full proof in terms of the compactness estimates does not seem to have appeared elsewhere; therefore, we present a proof of this observation here:

Proposition 3.1.1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then, the* *compactness estimates for forms in dom*$(\bar\partial)\cap$*dom*$(\bar\partial^*)$ *(as in* (3.3)*) hold if and only if the same estimates hold for the forms in the same space with harmonic components.*

Proof. One direction is trivial. For the reverse direction, we will follow the strategy of [53].

Let ϑ be the formal adjoint of $\bar\partial$. The operator $\bar\partial\vartheta+\vartheta\bar\partial$ acts on the appropriate forms componentwise as a constant multiple of the usual Laplacian (cf. (2.6)). Therefore, if $u=\sum'_{|J|=q}u_J d\bar z_J\in\text{dom}(\bar\partial)\cap\text{dom}(\bar\partial^*)$, then for each strictly increasing q-tuple J, we have

$$||\triangle u_J||_{-1}\le C(||\bar\partial u||+||\bar\partial^* u||) \tag{3.4}$$

for some positive constant $C>0$ that depends only on n and Ω. On a bounded domain D of $\mathbb{R}^m$, the Laplace operator defines an isomorphism from $W^1_0(D)$ onto $W^{-1}(D)$ (see Theorem 23.1 in [59] or Proposition 1.1 in Chapter 5 of [58]). So, for each strictly increasing q-tuple J, let v_J be the (unique) function from $W^1_0(\Omega)$ such that $\triangle v_J=\triangle u_J$ and set $v:=\sum'_{|J|=q}v_J d\bar z_J$. Since $v_J\in W^1_0(\Omega)$, we have $v\in\text{dom}(\bar\partial)\cap\text{dom}(\bar\partial^*)$. Therefore, given a $u\in\text{dom}(\bar\partial)\cap\text{dom}(\bar\partial^*)$, we can always find a $v\in W^1_{0,(0,q)}(\Omega)$ such that $\triangle u=\triangle v$.

The Sobolev 1-norm of v is controlled by the norm of the Laplacian of u. Using this and (3.4), we obtain

$$||v||_1\le C_1||\triangle u||_{-1}\le C_2(||\bar\partial u||+||\bar\partial^* u||), \tag{3.5}$$

with C_1 depending only on Ω and C_2 depending on n and Ω. We will invoke first part of Lemma 3.0.9. To this end, set $H_1:=\text{dom}(\bar\partial)\cap\text{dom}(\bar\partial^*)$ with the graph norm and set $H_2:=\mathcal{L}^2_{(0,q)}(\Omega)$. Define $T_1:H_1\to W^1_{0,(0,q)}(\Omega)$ to be the operator whose action is given by $T_1(u) = v$. Observe that T_1 is well defined and linear. Moreover, by (3.5), T_1 is continuous. Denote by T_2 the embedding of $W^1_{0,(0,q)}(\Omega)$ into H_2. Then, by Rellich's lemma, T_2 is compact. Since T_1 is continuous and T_2 is compact, the composition map $K := T_2 \circ T_1$ is a linear, compact operator from H_1 to H_2. Let L be the embedding of H_1 into H_2 composed with the embedding of H_2 into $H_3 := W^{-1}_{(0,q)}(\Omega)$. Now, by the first part of Lemma 3.0.9, for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$||v||_{\mathcal{L}^2_{(0,q)}(\Omega)} \leq \varepsilon \left(||\bar{\partial}u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar{\partial}^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)} \right) + C_\varepsilon ||u||_{-1,\Omega}. \tag{3.6}$$

In fact, the last Sobolev -1 norm can be taken as $\mathcal{L}^2$-norm since the first part of the lemma requires the operator L we used to be linear, injective and continuous rather than the stronger compactness property.

By the same token, if we keep T_1 same and but extend T_2 to be an embedding of $W^1_{0,(0,q)}(\Omega)$ into H_3, then we obtain for any $\varepsilon' > 0$ a positive number $C_{\varepsilon'}$ so that

$$||v||_{-1} \leq \varepsilon' \left(||\bar{\partial}u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar{\partial}^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)} \right) + C_{\varepsilon'} ||u||_{-1,\Omega}. \tag{3.7}$$

On the other hand, observe that $u - v \in \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ and since $\triangle u = \triangle v$, the components of $u - v$ are harmonic. So, if there exist compactness estimates for forms in $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with harmonic components (this is our hypothesis in the reverse direction), then by applying these estimates to $u - v$, we get

$$||u - v||_{\mathcal{L}^2_{(0,q)}(\Omega)} \leq \varepsilon \left(||\bar{\partial}(u - v)||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar{\partial}^*(u - v)||_{\mathcal{L}^2_{(0,q-1)}(\Omega)} \right) + C_\varepsilon ||u - v||_{-1,\Omega}. \tag{3.8}$$

The operators $\bar{\partial}$ and $\bar{\partial}^*$ are linear. So, the norms of $u - v$ under $\bar{\partial}$ and $\bar{\partial}^*$ can be estimated above by those of u and v. Moreover, the $\mathcal{L}^2$-norms of the forms under $\bar\partial$ and $\bar\partial^*$ are controlled by the Sobolev norm $||.||_1$. This can be applied for v and resulting Sobolev 1-norm of v can be estimated via inequality (3.5). We use these observations on the right hand side of (3.8) as shown below and get

$$\begin{aligned} ||u-v||_{\mathcal{L}^2_{(0,q)}(\Omega)} \leq \varepsilon &\left(||\bar\partial u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar\partial^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)}\right) \\ &+ \varepsilon \left(||\bar\partial v||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar\partial^* v||_{\mathcal{L}^2_{(0,q-1)}(\Omega)}\right) + C_\varepsilon ||u-v||_{-1,\Omega} \\ \lesssim \varepsilon &\left(||\bar\partial u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar\partial^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)}\right) + \varepsilon ||v||_{1,\Omega} + C_\varepsilon ||u-v||_{-1,\Omega} \\ \lesssim \varepsilon &\left(||\bar\partial u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar\partial^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)}\right) + C_\varepsilon ||u-v||_{-1,\Omega}. \\ \lesssim \varepsilon &\left(||\bar\partial u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar\partial^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)}\right) + C_\varepsilon ||u||_{-1,\Omega} + C_\varepsilon ||v||_{-1,\Omega}. \end{aligned} \tag{3.9}$$

Here, we used the standard notation $a \lesssim b$ to mean that there exists a constant $c > 0$ independent of a and b such that $a \leq cb$. The term $C_\varepsilon ||v||_{-1,\Omega}$ on the right side of (3.9) can be estimated using (3.7). Indeed, if we let ε' in (3.7) to be $\frac{\varepsilon}{C_\varepsilon}$, then we get

$$C_\varepsilon ||v||_{-1,\Omega} \leq \varepsilon \left(||\bar\partial u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar\partial^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)}\right) + K_{\varepsilon,\varepsilon'} ||u||_{-1,\Omega}. \tag{3.10}$$

Note that $K_{\varepsilon,\varepsilon'}$ is a constant given by the multiplication of C_ε and $C_{\varepsilon'}$; and ε' depends on ε. Therefore, $K_{\varepsilon,\varepsilon'}$ depends only on ε and may be denoted by K_ε. By an abuse of notation, we denote the sum of C_ε on the right side of (3.9) and K_ε by C_ε again. Then, using (3.10) on the right side of (3.9), we get

$$||u-v||_{\mathcal{L}^2_{(0,q)}(\Omega)} \lesssim \varepsilon \left(||\bar\partial u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar\partial^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)}\right) + C_\varepsilon ||u||_{-1,\Omega}. \tag{3.11}$$

Writing $u = (u-v) + v$ and then using inequalities (3.11) and (3.6) after a triangle inequality, we obtain

$$||u||_{\mathcal{L}^2_{(0,q)}(\Omega)} \lesssim \varepsilon \left(||\bar{\partial} u||_{\mathcal{L}^2_{(0,q+1)}(\Omega)} + ||\bar{\partial}^* u||_{\mathcal{L}^2_{(0,q-1)}(\Omega)} \right) + C_\varepsilon ||u||_{-1,\Omega} \tag{3.12}$$

which is the compactness estimates desired for u. This finishes the proof of Proposition 3.1.1. □

3.1.2 Property (P) and property $(\tilde{P})$

In [11], Catlin introduced a (by now classical) condition under the name property (P), which guarantees the compactness of N. Its relaxed version property $(\tilde{P})$ was introduced by McNeal ([40]).

Definition 3.1.2. *For a bounded pseudoconvex domain Ω in $\mathbb{C}^n$, we say that $b\Omega$ satisfies property (P_q) if for every $M > 0$, there exist a neighborhood $U = U_M$ of $b\Omega$ and a C^2 smooth function $\lambda = \lambda_M$ on U such that*

(i) $0 \le \lambda(z) \le 1$, for $z \in U$; and

(ii) for any $z \in U$, the sum of any q (equivalently: the smallest q) eigenvalues of the Hermitian form $\left(\frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k}(z)\right)_{j,k}$ is at least M; that is, for any $(0,q)$-form u at $z \in U$,

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k}(z) u_{jK}(z) \overline{u_{kK}(z)} \ge M |u(z)|^2. \tag{3.13}$$

We say that $b\Omega$ satisfies property $(\tilde{P}_q)$ if there is a positive constant C such that for all $M > 0$, there exist a neighborhood $U = U_M$ of $b\Omega$ and a C^2 smooth function $\lambda = \lambda_M$ on U such that the following hold for any $(0,q)$-form u at $z \in U$:

(i)

$$\sum_{|K|=q-1}{}' \Big|\sum_{j=1}^{n} \frac{\partial\lambda}{\partial z_j}(z)u_{jK}(z)\Big|^2 \le C \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\lambda}{\partial z_j \partial \bar{z}_k}(z)u_{jK}(z)\overline{u_{kK}(z)}, \quad (3.14)$$

(ii)

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\lambda}{\partial z_j \partial \bar{z}_k}(z)u_{jK}(z)\overline{u_{kK}(z)} \ge M|u(z)|^2. \quad (3.15)$$

That is, in property ($\tilde{P}_q$), the uniform boundedness condition of λ on U is replaced by the self-bounded gradient property *of the function λ.*

Remark 3.1.3. *One can define property (P) and property ($\tilde{P}$) more generally on compact subsets of $\mathbb{C}^n$. This can be done simply by replacing the boundary notion in Definition 3.1.2 by the compact set on which definitions are desired.*

Both property (P) and property ($\tilde{P}$) percolate up the complex. That is, if $b\Omega$ satisfies property (P_q) or property ($\tilde{P}_q$), then it also satisfies property (P_{q+1}) or property ($\tilde{P}_{q+1}$), respectively. The following lemma is an equivalent formulation of the second condition in definition of property (P) (see Lemma 4.7 in [56]) and it will be useful in proving Proposition 3.1.7:

Lemma 3.1.4. *Let λ be as in Definition 3.1.2 and fix z. Let $1 \le q \le n$. Then the following are equivalent:*

(i) For any $u \in \Lambda_z^{(0,q)}$; that is, for any skew symmetric q-linear functional u on $\mathbb{C}^n$,

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2\lambda(z)}{\partial z_j \partial \bar{z}_k} u_{jK}\overline{u_{kK}} \ge M|u|^2. \quad (3.16)$$

(ii) The sum of any q (equivalently; the smallest q) eigenvalues of $\left(\frac{\partial^2\lambda}{\partial z_j \partial \bar{z}_k}(z)\right)_{j,k}$ is at least M.

(iii) $\sum_{s=1}^{q}\sum_{j,k=1}^{n}\frac{\partial^2\lambda(z)}{\partial z_j\partial\bar{z}_k}(t^s)_j\overline{(t^s)_k}\geq M$ *whenever* $t^1,\cdots,t^q$ *are orthonormal in* $\mathbb{C}^n$.

We will also find the following result useful in the applications of Theorem 4.1.2 (see Corollary 4.14 in [56]):

Lemma 3.1.5. *A compact set in* $\mathbb{C}^n$ *satisfies property* (P_q) *if it can be written as a countable union of compact sets each of which satisfies property* (P_q)*.*

Compactness of N and property (P) are equivalent on bounded locally convexifiable domains of $\mathbb{C}^n$ (see [24], [25]). In [52], Sibony took a systematic study of property (P) on compact subsets of $\mathbb{C}^n$ under the name of B-regularity. The sufficient condition property $(\tilde{P})$ is a relaxed version of property (P). It was introduced by McNeal in [40]. It is known that property (P) implies property $(\tilde{P})$ ([40]). The equivalence of property (P) and property $(\tilde{P})$ on Hartogs domains in $\mathbb{C}^2$ was shown in [26] and the equivalence of compactness of N and property (P) on some Hartogs domains in $\mathbb{C}^2$ was shown in [17]. There is another sufficient condition for compactness introduced by Takegoshi in [57] which implies property $(\tilde{P}_1)$ (see Remark 2.2 in [55] for a discussion).

Remark 3.1.6. *In the original definition of property* $(\tilde{P}_q)$*, one seeks a function* $\lambda_M\in C^2(\overline{\Omega})$ *which is plurisubharmonic on* Ω*, satisfying the condition* (i) *on all of* Ω *with constant* C *replaced by* 1 *and satisfying condition* (ii) *only on the boundary. However, the smoothness conditions on* λ *and* $b\Omega$ *may be eliminated to present it as we already did in Definition 3.1.2. This was essentially observed in [53] for property* (P)*. A similar discussion also exists in [40].*

Another condition, which guarantees the compactness of N was introduced by Straube in [54] for domains in $\mathbb{C}^2$. This geometric condition was generalized for domains in $\mathbb{C}^n$ by Munasinghe and Straube in [44]. In what generality all of these sufficient conditions for compactness of N stated above are related to the compactness of N or to each other is an open problem.

3.1.3 Property (P) and null space of the Levi form

Recall that verifying compactness estimates for the forms in $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with harmonic components rather than for all forms in $\text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ is sufficient to show that the $\bar{\partial}$-Neumann operator is compact (Proposition 3.1.1). An analogous relation exists between the null space of the Levi form and property (P).

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with smooth boundary. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a defining function for Ω. Denote by $H^{(1,0)}(b\Omega)$ the *holomorphic tangent bundle* on $b\Omega$. For $p \in b\Omega$, set

$$\mathcal{N}_p = \{\xi \in H^{(1,0)}(b\Omega) \Big| \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\xi_j \bar{\xi_k} = 0\},$$

the null space of the Levi form. A proof of the fact that property $(\tilde{P})$ for $b\Omega$ restricted to the null space of the Levi form is equivalent to property $(\tilde{P})$ for $b\Omega$ was given by Çelik in his dissertation [12]. We show below that an analogous equivalence also holds for property (P) for $b\Omega$. In the proof, we basically follow the techniques given in [12].

Proposition 3.1.7. *Property (P) for $b\Omega$ restricted to $\mathcal{N}_z$ is equivalent to property (P) for $b\Omega$.*

Proof. One direction is trivial: if we have the property (P) for $b\Omega$, then we trivially have it on the null space of the Levi form. For the other direction, suppose property (P) for $b\Omega$ restricted to $\mathcal{N}_z$ holds. We want to show that this is equivalent to property (P) holding for $b\Omega$ in general. By our hypothesis, we have the following: for every $M > 0$, there exist a neighborhood $U = U_M$ of $b\Omega$ and a function $\lambda = \lambda_M : U \to \mathbb{R}$

such that $\lambda \in C^2(\overline{\Omega \cap U})$, $0 \le \lambda \le 1$ on $\overline{\Omega \cap U}$ and

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k}(z)\xi_j \bar{\xi_k} \ge M|\xi|^2 \text{ for } z \in \overline{\Omega \cap U} \text{ and } \xi \in \mathcal{N}_z. \tag{3.17}$$

Let $\mathbb{S}H^{(1,0)}(b\Omega) \subset H^{(1,0)}(b\Omega)$ be the unit sphere bundle. The fiber over a point $p \in b\Omega$ is the set of all unit $(1,0)$-vectors in $H^{(1,0)}(b\Omega)$. Define

$$K := \{(p,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega) | \xi \in \mathcal{N}_p\}.$$

Note that (3.17) is also valid on K. In particular, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k}(z)\xi_j \bar{\xi_k} > \frac{4M}{5}|\xi|^2 \text{ for } (z,\xi) \in K. \tag{3.18}$$

Note that (3.18) is a strict inequality, i.e., it is an open condition and K is a compact set. Thus, (3.18) holds in a neighborhood $\tilde{U}$ of K in $\mathbb{S}H^{(1,0)}(b\Omega)$. Let $\tilde{U_1}$ be open such that $K \subset\subset \tilde{U_1} \subset\subset \tilde{U}$ and set

$$\alpha := \min\left\{ \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k}(z)\xi_j \bar{\xi_k} | (z,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega) \backslash \tilde{U_1} \right\},$$

and

$$\beta := \min\left\{ \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z)\xi_j \bar{\xi_k} | (z,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega) \backslash \tilde{U_1} \right\}.$$

Note that $\beta > 0$. Now, given $M > 0$ already above to determine α and β, define $V_M := U \cap V_{M,\alpha,\beta}$ where

$$V_{M,\alpha,\beta} = \Omega \backslash \left\{ z \in \Omega | \rho(z) \le \frac{-\beta}{8(M + |\alpha|)} \right\}.$$

Let $\tilde{\lambda}_M : \mathbb{C}^n \to \mathbb{R}$ be a function defined explicitly by $\dfrac{2(M+|\alpha|)}{\beta}\rho(z) + \dfrac{5}{4}\lambda(z) + \dfrac{5}{4}$ when $z \in \overline{V_M}$. Observe that $\tilde{\lambda}_M \in C^2(\overline{V_M})$. Note that for $z \in \overline{V_M}$, we also have $0 \le \frac{5}{4}\lambda(z) \le \frac{5}{4}$ and $-\frac{1}{4} \le \frac{2(M+|\alpha|)}{\beta}\rho(z) \le 0$. Therefore, $1 \le \tilde{\lambda}_M \le \frac{5}{2}$ on $\overline{V_M}$.

Observe that on $\tilde{U}$ we have

$$\begin{aligned}
\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\lambda}_M(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k &= \frac{2(M+|\alpha|)}{\beta}\sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k + \frac{5}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k \\
&\ge \frac{5}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k, \qquad \text{(since } \Omega \text{ is pseudoconvex)} \\
&> \frac{5}{4}\frac{4M}{5}|\xi|^2 = M|\xi|^2.
\end{aligned}$$

Similarly, on $\mathbb{S}H^{(1,0)}(b\Omega)\backslash\tilde{U}_1$ we have

$$\begin{aligned}
\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\lambda}_M(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k &= \frac{2(M+|\alpha|)}{\beta}\sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k + \frac{5}{4}\sum_{j,k=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k \\
&\ge \frac{2(M+|\alpha|)}{\beta}\beta|\xi|^2 + \frac{5}{4}\alpha|\xi|^2 \\
&= \left(2M + 2|\alpha| + \frac{5\alpha}{4}\right)|\xi|^2 \\
&> M|\xi|^2.
\end{aligned}$$

Consider $Y := \{b\Omega\} \times \{\xi \in \mathbb{C}^n \big| |\xi| = 1\}$. Then, $\mathbb{S}H^{(1,0)}(b\Omega)$ embeds into Y and it is a compact subset of Y. Thus by continuity, we have again

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\lambda}_M(z)}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k > M|\xi|^2$$

in an open neighborhood $\tilde{W}$ of $\mathbb{S}H^{(1,0)}(b\Omega)$ in Y.

Let $\tilde{W}_1$ be open in Y such that $\mathbb{S}H^{(1,0)}(b\Omega) \subset\subset \tilde{W}_1 \subset\subset \tilde{W} \subset\subset Y$. Set

$$\gamma := \min\left\{\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\lambda}_M}{\partial z_j \partial \bar{z_k}}(z)\xi_j\bar{\xi_k} | (z,\xi) \in Y \backslash \tilde{W}_1\right\},$$

and

$$\delta := \min\left\{\Big|\sum_{j=1}^{n} \frac{\partial \rho(z)}{\partial z_j}\xi_j\Big|^2 \ \big|(z,\xi) \in Y\backslash\tilde{W}_1\right\}.$$

Note that $\delta > 0$. Define $W_M := V_M \cap \Omega \cap W_{M,\gamma,\delta}$, where

$$W_{M,\gamma,\delta} = \Omega\backslash\left\{z \in \Omega | \rho(z) \leq -\sqrt{\frac{\delta}{48(M+|\gamma|)}}\right\}.$$

Let $\phi_M : \mathbb{C}^n \to \mathbb{R}$ be a function defined explicitly by $\frac{1}{3}\left(\frac{2(M+|\gamma|)}{\delta}\rho^2(z) + \frac{7}{6}\tilde{\lambda}(z)\right)$ when $z \in \overline{W_M}$. Observe that $\phi_M \in C^2(\overline{W_M})$. Note that for $z \in \overline{W_M}$, we also have $\frac{7}{6} \leq \frac{7}{6}\tilde{\lambda}(z) \leq \frac{35}{12}$ and $0 \leq \frac{2(M+|\gamma|)}{\delta}\rho^2(z) \leq \frac{1}{24}$. Therefore, $\frac{1}{3} < \frac{7}{18} \leq \phi_M \leq \frac{71}{72} < 1$ on $\overline{W_M}$.

Observe that on $\tilde{W}$ we have

$$\begin{aligned}\sum_{j,k=1}^{n} \frac{\partial^2 \phi_M(z)}{\partial z_j \partial \bar{z_k}}\xi_j\bar{\xi_k} &= \frac{1}{3}\left(\frac{2(M+|\gamma|)}{\delta}\sum_{j,k=1}^{n} \frac{\partial^2 \rho^2(z)}{\partial z_j \partial \bar{z_k}}\xi_j\bar{\xi_k} + \frac{7}{6}\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\lambda}(z)}{\partial z_j \partial \bar{z_k}}\xi_j\bar{\xi_k}\right) \\ &= \frac{4(M+|\gamma|)}{3\delta}\rho(z)\sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z_k}}\xi_j\bar{\xi_k} \\ &\qquad + \frac{4(M+|\gamma|)}{3\delta}\Big|\sum_{j,k=1}^{n} \frac{\partial \rho(z)}{\partial z_j}\xi_j\Big|^2 + \frac{7}{18}\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\lambda}(z)}{\partial z_j \partial \bar{z_k}}\xi_j\bar{\xi_k}, \\ &\geq \frac{7}{18}\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\lambda}(z)}{\partial z_j \partial \bar{z_k}}\xi_j\bar{\xi_k} > \frac{7}{18}M|\xi|^2 > \frac{M}{3}|\xi|^2.\end{aligned}$$

Similarly, on $Y\backslash\tilde{W}_1$ we have

$$\sum_{j,k=1}^{n}\frac{\partial^2\phi_M(z)}{\partial z_j\partial\bar{z}_k}\xi_j\bar{\xi_k} \geq \frac{4(M+|\gamma|)}{3\delta}\delta|\xi|^2+\frac{7}{18}\gamma|\xi|^2$$
$$\geq \frac{1}{3}\left(4(M+|\gamma|)+\frac{7}{6}\gamma\right)|\xi|^2$$
$$\geq \frac{4M}{3}|\xi|^2 > M|\xi|^2.$$

So, given $M>0$, we have a function $\phi_M:\mathbb{C}^n\to\mathbb{R}$ such that $\phi_M\in C^2(\overline{\Omega\cap W_M})$ for some neighborhood W_M of $b\Omega$, $\frac{1}{3}<\phi_M<1$ on $\overline{\Omega\cap W_M}$ and

$$\sum_{j,k=1}^{n}\frac{\partial^2\phi_M}{\partial z_j\partial\bar{z}_k}(z)\xi_j\bar{\xi_k} > \frac{M}{3}|\xi|^2 \text{ for } (z,\xi)\in b\Omega\times\{\xi\in\mathbb{C}^n\big||\xi|=1\}.$$

Therefore, given $M>0$ we can take ϕ_{3M} so that property (P) holds on the set Y and this suffices for property (P) to hold for $b\Omega$ in view of Lemma 3.1.4. □

3.1.4 Subsets of finite type points and property (P)

A remarkable example for the existence of property (P) defined by Catlin in [11] is that smooth bounded pseudoconvex domains of finite type satisfy property (P). However, Catlin's work reveals more: compact subsets of the set of finite type points in the boundary of a smooth pseudoconvex bounded domain in $\mathbb{C}^n$ satisfy property (P). This fact, although well-known by many experts in the field, does not seem to be proved elsewhere. Since we will make use of this observation later in giving examples, we will prove this observation here; and in proving it, we imitate Catlin's fundamental work [11] and modify it whenever necessary. The main steps in Catlin's work for our purposes are as follows: a definition of being weakly regular is presented for the boundary of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ which is *of finite type* is shown to have a weakly regular boundary and finally, weakly regular boundary of a smooth bounded pseudoconvex domain is shown to satisfy property (P). We recall first the definition of a domain of finite type (see [21], [19]) and to do this let us fix a notation: if λ is a smooth vector-valued function defined near the origin of the complex plane, we denote by $\nu(\lambda)$ the order of vanishing of λ at the origin.

Definition 3.1.8. *Let Ω be a smooth bounded domain in $\mathbb{C}^n$. Let $z_0 \in b\Omega$ and r be a local defining for $b\Omega$ at z_0. If there exists a constant τ such that $\frac{\nu(r(\gamma))}{\nu(\gamma - z_0)} \leq \tau$ whenever γ is a nonconstant, $\mathbb{C}^n$-valued germ of a holomorphic function around $0 \in \mathbb{C}$ satisfying $\gamma(0) = z_0$, then z_0 is called a finite type point. The infimum of such τ's for the point z_0 is denoted by $\tau(z_0)$ and called the type of z_0. The domain Ω is called a domain of finite type if every point in $b\Omega$ is a finite type point.*

Before stating the result, let us give the definition of property (P) for a compact subset K of $b\Omega$ in the same way Catlin defined ([11]):

Definition 3.1.9. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let K be a compact subset of the boundary $b\Omega$. We say that K satisfies property (P) if for every $M > 0$ there exists a plurisubharmonic function $\lambda_M \in C^\infty(\overline{\Omega})$ such that $0 \leq \lambda_M \leq 1$ and such that for all $z \in K$ the following holds:*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k \geq M|t|^2.$$

Remark 3.1.10. *The definition of property (P) when taken in the sense of Catlin implies property (P_1) we defined in Definition 3.1.2.*

We want to prove a result which states that a compact subset of the set of finite type points of a bounded smooth pseudoconvex domain in $\mathbb{C}^n$ satisfies property (P). We will achieve this result for the closure of a relatively compact open subset of the set of finite type points in the boundary. Assume that we can achieve the result in this form. We first recall that the set of finite type points is an open subset of the boundary ([19]). The observation is then a given compact subset of the set of finite type points is contained in the closure of a relatively compact open subset of the set of finite type points. Since the latter satisfies property (P) by our assumption, then any of its compact subsets also satisfies property (P). So, the result will be proved once it is proved with the compact subset in its assumption is particularly taken to be the closure of a relatively compact open subset of the set of finite type points.

We first modify Catlin's definition of being "weakly regular".

Definition 3.1.11. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let D be a domain in $\mathbb{C}^n$ such that $K := \overline{D \cap b\Omega}$ (the closure being in the topology of $\mathbb{C}^n$) is a proper subset of the set of finite type points in $b\Omega$. We shall say that K is weakly regular if there exists a finite number of compact subsets S_i of K, $i = 0, 1, \cdots, N$ such that*

(i) $\emptyset = S_N \subset S_{N-1} \subset \cdots \subset S_1 \subset S_0 = K$;

(ii) if $z \in S_i$, but $z \notin S_{i+1}$, then there are a neighborhood U of z and a submanifold M of $U \cap K$ with $z \in M$ such that the holomorphic dimension of M is equal to zero and such that $S_i \cap U \subset M$.

Recall that a submanifold of $b\Omega$ with constant CR dimension has holomorphic dimension zero if the Levi form of $b\Omega$ applied to nonzero complex tangential vector fields of type $(0, 1)$ is positive definite.

Let Γ_n denote the set of all n-tuples of extended numbers $\Lambda = (\lambda_n, \cdots, \lambda_1)$ such that $1 \leq \lambda_i \leq +\infty$ and $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$. An element of Γ_n is called a *weight*. A lexicographic order can be put on Γ_n: if $\mathcal{L} = (l_n, \cdots, l_1)$ and $\mathcal{L}' = (l'_n, \cdots, l'_1)$ are two weights, then $\mathcal{L} <_{lex} \mathcal{L}'$ if

i) for some j with $1 \leq j \leq n$, we have $l_i = l'_i$ for all $i > j$; but

ii) $l_j < l'_j$.

For instance, when $n = 3$ we have

$$(1,1,1) < \cdots < (1,1,2) < \cdots < (1,1,3) < \cdots < (1,1,4) < \cdots < (1,1,+\infty)$$
$$\cdots < (1,2,3) < \cdots < (1,2,4) < \cdots < (1,2,5) < \cdots < (1,2,+\infty)$$
$$\cdots < (1,3,4) < \cdots < (1,3,5) < \cdots$$

A given weight $\mathcal{L} = (l_n, \cdots, l_1)$ is called *distinguished* if there exists holomorphic coordinates $(z_1, z_2, \cdots, z_n)$ about z_0 with z_0 mapped to the origin such that

$$D^\alpha \bar{D}^\beta r(0) := \frac{\partial^{\alpha_1+\cdots+\alpha_n+\beta_1+\cdots+\beta_n}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}} r(0) = 0 \quad \text{whenever} \quad \sum_{j=1}^{n} \frac{\alpha_j + \beta_j}{l_j} < 1.$$

Definition 3.1.12. *The multi-type $\mathcal{M}(z_0)$ is defined to be the least weight $(m_n \cdots, m_1)$ in lexicographical order such that $\mathcal{L} \leq \mathcal{M}$ for every distinguished weight $\mathcal{L}$.*

Example 3.1.13. *Here are some examples from [10] and [11]:*

1. *If $\mathcal{M}(z_0) = (m_n, \cdots, m_1)$, since $dr(z_0) \neq 0$, we should have $m_n = 1$.*

2. *If z_0 is strictly pseudoconvex, then $\mathcal{M}(z_0) = (1, 2, \cdots, 2)$.*

3. *More generally, if the Levi form of $b\Omega$ has rank p at z_0, then $m_i = 2$ for $n-1 \geq i \geq n-p$ and $m_i > 2$ for $i < n-p$.*

4. *In general, the multi-type $\mathcal{M}(z_0)$ gives a measure of the order of vanishing of the boundary-defining function by assigning a weight m_i to the coordinate direction z_i.*

The important properties of multi-type of a point is summarized as follows:

Theorem 3.1.14 (Catlin; [10], [11])**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary near a given boundary point z_0. Let the multi-type invariant $\mathcal{M}(z)$ be defined for all z in $b\Omega$ near z_0. Then the multi-type invariant has the following properties:*

(1) If $\mathcal{M}(z) = (m_n, \cdots, m_1)$, then $m_n \le m_{n-1} \le \cdots \le m_1$.

(2) $\mathcal{M}(z)$ is upper semicontinuous with respect to the lexicographic ordering: there is a neighborhood U about z_0 such that for all $z \in U \cap b\Omega$, $\mathcal{M}(z) \le \mathcal{M}(z_0)$.

(3) There are a neighborhood U of z_0 and a submanifold M of $U \cap b\Omega$ of holomorphic dimension zero, with $z_0 \in M$, such that $\{z \in U \cap b\Omega : \mathcal{M}(z) = \mathcal{M}(z_0)\} \subset M$.

(4) If $\mathcal{M}(z_0) = (m_n, \cdots, m_1)$, then there exist coordinates $(z_1, \cdots, z_n)$ about z_0 such that $D^\alpha \bar{D}^\beta r(0) = 0$ if $\displaystyle\sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{m_i} < 1$. Furthermore for each q, $q = 1, \cdots, n$, there exist multi-indices $\alpha = (0, \cdots, \alpha_q, \cdots, \alpha_n)$ and $\beta = (0, \cdots, \beta_q, \cdots, \beta_n)$ with $\alpha_q + \beta_q > 0$ and $\displaystyle\sum_{i=q}^{n} \frac{\alpha_i + \beta_i}{m_i} = 1$ such that $D^\alpha \bar{D}^\beta r(0) \neq 0$.

(5) If $\mathcal{M}(z_0) = (m_n, \cdots, m_1)$, then $m_1 \le \tau(z_0)$, the type of z_0 in the sense of D'Angelo.

Remark 3.1.15. *If a point is of finite type; that is, if m_1 is bounded by some number $T < \infty$, then the numbers m_i, $i = 1, \cdots, n$ can take on only a finite number of rational values. This is a result of the fourth item in Catlin's theorem, see [11] for the discussion.*

The work by Catlin above and a theorem of D'Angelo can be combined to derive that the closure of a relatively compact, open subset of the set of finite type points is weakly regular. We will first list D'Angelo's result and then discuss this claim.

Theorem 3.1.16 (D'Angelo; [18]). *Suppose that Ω is a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary near z_0. Assume that z_0 is a point of finite type. Then there is a neighborhood U of z_0 such that $\tau(z)$, the type of the point $z \in U \cap b\Omega$, is bounded above by $\frac{(\tau(z_0))^{n-1}}{2^{n-2}}$.*

We now restate and prove our claim:

Lemma 3.1.17. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and let K be the closure of a relatively compact, open subset of the set of finite type points in $b\Omega$. Then K is weakly regular.*

Proof. By the result of D'Angelo above, the set K can be covered by a finite set of neighborhoods such that the type is uniformly bounded in each one, and hence, in all of them by some constant $T > 0$. This and the last item in Catlin's theorem gives that the first coordinate m_n of the multi-type at any point $z \in K$ is at most T. Hence, by Remark 3.1.15, the number of possible different multi-types $\mathcal{M}$ at any point $z \in K$ is finite. Let $\mathcal{M}_0 < \mathcal{M}_1 < \cdots < \mathcal{M}_{N-1}$ be the lexicographic ordering of these finitely many possible different multi-types with $\mathcal{M}_0 = (1, 2, \cdots, 2)$. For each $j = 0, 1, \cdots, N-1$, define

$$S_j = \{z \in K : \mathcal{M}(z) \geq \mathcal{M}_j\}$$

and set $S_N = \emptyset$. By the second item in Catlin's theorem, each S_j is compact. Moreover, since $\mathcal{M}_0 < \mathcal{M}_1 < \cdots < \mathcal{M}_{N-1}$, we have

$$\emptyset = S_N \subset S_{N-1} \subset \cdots \subset S_1 \subset S_0 = K.$$

Therefore, it remains to show that the second property in Definition 3.1.11 is satisfied. Observe that if $z_0 \in S_j$ but $z_0 \notin S_{j+1}$, then $z_0 \in \{z \in K : \mathcal{M}(z) = \mathcal{M}_j\}$. That is,

$\mathcal{M}(z_0) = \mathcal{M}_j$.

The third item in Catlin's theorem gives a neighborhood U around the point z_0 and a submanifold M of $U \cap b\Omega$ with holomorphic dimension zero such that $z_0 \in M$ and $\{z \in U \cap b\Omega : \mathcal{M}(z) = \mathcal{M}(z_0)\} \subset M$. By the second item, one can assume that $\mathcal{M}(z) \leq \mathcal{M}(z_0)$ for any $z \in U$. Otherwise, one can replace U by a smaller neighborhood V of z_0 if necessary and in this case M is replaced by $M \cap V$ (with holomorphic dimension still being zero). We need to show that if $z_0 \in S_i \backslash S_{i+1}$, there exist a neighborhood U of z_0 and a submanifold M' of $U \cap K$ with holomorphic dimension zero such that $z_0 \in M'$ and $S_i \cap U \subset M'$. We consider the same neighborhood U that is provided by the third item in Catlin's theorem. Note that $U \cap b\Omega \cap K = U \cap K$ and $M' := M \cap K$ is a submanifold of $U \cap K$. Furthermore, the holomorphic dimension of M' is still zero and $\{z \in U \cap K : \mathcal{M}(z) = \mathcal{M}(z_0)\} \subset M'$. What remains to show is that $S_i \cap U \subset M'$.

Recall from the discussion above that $z_0 \in S_i \backslash S_{i+1}$ implies $\mathcal{M}(z_0) = \mathcal{M}_i$. Now, if $z \in S_i$, then $\mathcal{M}(z) \geq \mathcal{M}_i = \mathcal{M}(z_0)$; and if $z \in U$, then $\mathcal{M}(z) \leq \mathcal{M}(z_0)$ (by the discussion above). Therefore, if $z \in S_i \cap U$, then z must be in $\{z \in U \cap K : \mathcal{M}(z) = \mathcal{M}(z_0)\}$. But this last is contained in M' by the modifications we made on the third item of Catlin's theorem. Thus, we showed that $z \in S_i \cap U$ implies $z \in M'$. □

We now restate and prove our result:

Theorem 3.1.18. *A compact subset of the set of finite type points of a bounded smooth pseudoconvex domain in $\mathbb{C}^n$ satisfies property (P_1).*

Proof. By the remark after Definition 3.1.9, it suffices to show that any compact subset of the set of finite type points in the boundary satisfies property (P) as in Definition 3.1.9. Recall also from the discussion made in the paragraph after Definition 3.1.9 that the theorem will be proved once it is proved with the compact subset in its assumption is particularly taken to be the closure of a relatively compact open subset of the set of finite type points. So, let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^n$ and K be the closure of a relatively compact, open subset of the set of finite type points in $b\Omega$. In Lemma 3.1.17, we showed that K is weakly regular. Therefore, it suffices to prove the statement "if K is weakly regular, then it satisfies property (P)". In order to do this, we shall prove by induction the following statement:

$$\text{"Let } S \text{ be any compact subset of } K \text{ with } S \cap S_i = \emptyset. \text{ Then } S \text{ has property } (P)\text{."} \tag{3.19}$$

This will prove the theorem because $S_N = \emptyset$ and hence any compact subset S has empty intersection with S_N. The basis of the induction trivially holds: for $i = 0$, we have $S_0 = K$ and if $S \cap S_0 = \emptyset$, then $S = \emptyset$. So, we assume now that the statement (3.19) is true for i, and we will prove it for $i + 1$.

Let S' be a compact subset of K with $S' \cap S_{i+1} = \emptyset$. Let z_0 be a given point of $S' \cap S_i$ (if $S' \cap S_i$ is also empty, then S' satisfies property (P) by the induction assumption; so we work with the non-empty case). Since K is weakly regular, we have a neighborhood U of z_0 and a submanifold M of $U \cap K$ such that M has holomorphic dimension zero and $S_i \cap U \subset M$. If l is the CR-dimension of M, then after shrinking U if necessary, we can find functions $\rho_{l+1}, \cdots, \rho_n$ with ρ_n being the defining function for Ω such that $M \subset \{z \in U : \rho_k(z) = 0, k = l + 1, \cdots, n\}$. Moreover, the set of vectors $\left\{\sum_{j=1}^n \frac{\partial \rho_k}{\partial z_j}(z)\frac{\partial}{\partial z_j} : k = l + 1, \cdots, n\right\}$ is linearly independent at each point $z \in U$. Since the manifold M has holomorphic dimension zero, we have

$$\sum_{j,k=1}^n \frac{\partial^2 \rho_n}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k > 0 \quad \text{for } t \in \mathbb{C}^n \text{ such that } \quad \sum_{j=1}^n \frac{\partial \rho_k}{\partial z_j}(z) t_j = 0. \tag{3.20}$$

Here, $k = l+1, \cdots, n$ and $z \in U$. The Levi form is nonnegative; therefore we have for $\tau > 0$

$$\frac{\tau}{2}\sum_{k=l+1}^{n-1}\left|\sum_{j=1}^{n}\frac{\partial\rho_k}{\partial z_j}(z)t_j\right|^2 + \tau e^{\tau\rho_n(z)}\sum_{j,k=1}^{n}\frac{\partial^2\rho_n}{\partial z_j\partial\bar{z}_k}(z)t_j\bar{t}_k \geq C\tau|t|^2 \tag{3.21}$$

for all t satisfying $\sum_{j=1}^{n}\frac{\partial\rho_n}{\partial z_j}t_j = 0$ and all $z \in U \cap K$, where C is a constant independent of τ, z and t. Replacing C by $\frac{C}{2}$, this last becomes an open condition. Therefore, if we take t from a conical neighborhood

$$\left\{t : \left|\sum_{j=1}^{n}\frac{\partial\rho_n}{\partial z_j}t_j\right| < a|t|\right\},$$

with a small enough, we can obtain the following: there exists a constant C (we adopt the usual convention that the constant C may change in each occurrence) such that for all $t \in \mathbb{C}^n$ and all $z \in U \cap K$ and sufficiently large τ, we have

$$\begin{aligned}\frac{\tau}{2}\sum_{k=l+1}^{n-1}\left|\sum_{j=1}^{n}\frac{\partial\rho_k}{\partial z_j}(z)t_j\right|^2 &+ \tau e^{\tau\rho_n(z)}\sum_{j,k=1}^{n}\frac{\partial^2\rho_n}{\partial z_j\partial\bar{z}_k}(z)t_j\bar{t}_k \\ &+ \tau^2 e^{\tau\rho_n(z)}\left|\sum_{j=1}^{n}\frac{\partial\rho_n}{\partial z_j}(z)t_j\right|^2 \geq C\tau|t|^2.\end{aligned} \tag{3.22}$$

Now let V be a relatively compact subset of U and choose a smooth cutoff function ϕ that is compactly supported in U such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $\overline{V}$. Define a new function

$$f_\tau(z) := \phi(z) + \tau\phi(z)\left(\sum_{k=l+1}^{n-1}\rho_k^2(z)\right) + e^{\tau\rho_n(z)}.$$

Computing the Hessian of f_τ applied on the vectors $t \in \mathbb{C}^n$, we obtain

$$\begin{aligned}\sum_{j,k=1}^{n} \frac{\partial^2 f_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k = &\left(1 + \tau \sum_{m=l+1}^{n-1} \rho_m^2(z)\right) \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k \\ &+ 2\tau \sum_{j,k=1}^{n} \sum_{m=l+1}^{n-1} \rho_m(z) \left(\frac{\partial \phi(z)}{\partial z_j} \frac{\partial \rho_m(z)}{\partial \bar{z}_k} + \frac{\partial \rho_m(z)}{\partial z_j} \frac{\partial \phi(z)}{\partial \bar{z}_k} + \phi(z) \frac{\partial^2 \rho_m(z)}{\partial z_j \partial \bar{z}_k} \right) t_j \bar{t}_k \\ &+ 2\tau\phi(z) \sum_{m=l+1}^{n-1} \left| \sum_{j=1}^{n} \frac{\partial \rho_m}{\partial z_j}(z) t_j \right|^2 + \tau e^{\tau \rho_n(z)} \sum_{j,k=1}^{n} \frac{\partial^2 \rho_n}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k \\ &+ \tau^2 e^{\tau \rho_n(z)} \left| \sum_{j=1}^{n} \frac{\partial \rho_n}{\partial z_j}(z) t_j \right|^2 .\end{aligned}$$

Let $A_\tau = \{z \in U \cap K : \sum_{k=l+1}^{n-1} \rho_k^2(z) \leq \frac{1}{\tau}\}$. Observe that the factor ρ_m appears squared in the first sum and as it is in the second sum. Therefore, there exists a constant C such that whenever $z \in A_\tau$, the absolute value of the first two sums is bounded by $C(\sqrt{\tau} + 1)$. Now, by (3.22), if $z \in A_\tau \cap \{z \in U : \phi(z) \geq \frac{1}{4}\}$ and τ is large, the Hessian of f_τ at z is bounded below by $C\tau|t|^2$.

Choose a smooth function $\chi(s)$ with $\chi(s) = 0$ for $s < \frac{5}{4}$, with $\chi''(s) > 0$ for $\frac{5}{4} < s \leq 3$, $\chi(s) \equiv 0$ for $s \geq 4$, and $\chi \leq 1$. By definition of f_τ and A_τ, $f_\tau(z) \geq \frac{5}{4}$ when $z \in A_\tau \cap \{z : \phi(z) \geq \frac{1}{4}\}$. Therefore, it follows that for large τ, the composition function $\chi(f_\tau)$ is plurisubharmonic in a neighborhood of A_τ. Also, $\chi(f_\tau)$ is supported in U. Thus, there is a compact subset S of U, disjoint from M (because M is contained in A_τ) such that the set of points where $\chi(f_\tau)$ is non-plurisubharmonic is contained in S. Define

$$N_\tau = \sup \left\{ - \sum_{j,k=1}^{n} \frac{\partial^2 \chi(f_\tau(z))}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k : z \in S, |t| = 1 \right\}.$$

Note that $S \cap K$ is compact. Also, since $S \subset U$ and K is weakly regular, we have $S_i \cap S \subset S_i \cap U \subset M$. But S is disjoint from M. Thus, $S_i \cap (S \cap K) = \emptyset$. Now, by the induction hypothesis there exists a plurisubharmonic function $\lambda_\tau \in C^\infty(\overline{\Omega})$ such that $0 \leq \lambda_t \leq 1$ and for $z \in S \cap K$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq (N_\tau + \tau)|t|^2. \tag{3.23}$$

We set $g_\tau = \lambda_\tau + \chi(f_\tau)$. Then g_τ is smooth in $\mathbb{C}^n$, plurisubharmonic in a neighborhood of $b\Omega$; and for $z \in S \cap K$ it satisfies

$$\sum_{j,k=1}^{n} \frac{\partial^2 g_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq \tau|t|^2. \tag{3.24}$$

That it is smooth is clear by definition of g_τ. To see that (3.24) holds, recall that $\chi(f_\tau)$ is plurisubharmonic in a neighborhood of A_τ which is a subset of $U \cap b\Omega$. The set of points in U where $\chi(f_\tau)$ fails to be plurisubharmonic is a compact subset of U and it was denoted by S. On $S \cap K$, the est value that its complex Hessian when applied on the vectors $t \in \mathbb{C}^n$ can get is $-N_\tau|t|^2$. However, the estimate (3.23) compensate this and in turn gives that the complex Hessian of g_τ applied to the vectors $t \in \mathbb{C}^n$ is at least $\tau|t|^2$. This gives (3.24). To see that g_τ is plurisubharmonic in a neighborhood of $b\Omega$, recall that λ_τ is plurisubharmonic on the closure of Ω and that $\chi(f_\tau)$ is supported in U. Therefore, g_τ is plurisubharmonic outside of the support of $\chi(f_\tau)$. But $\chi(f_\tau)$ is plurisubharmonic in a neighborhood of A_τ. So, what remains to be verified is that g_τ is plurisubharmonic in a neighborhood of $S \cap K$. However, we have inequality (3.24) on $S \cap K$. Replacing τ by $\frac{\tau}{2}$, (3.24) becomes an open condition. Since $S \cap K$ is compact, then (3.24) with τ on the right hand side changed (say with $\frac{\tau}{4}$) continues to hold in a neighborhood of $S \cap K$; that is, g_τ is plurisubharmonic in a neighborhood of $S \cap K$ and hence in a neighborhood of $b\Omega$. So, we have showed that g_τ is smooth in $\mathbb{C}^n$, plurisubharmonic in a neighborhood of

$b\Omega$; and for $z \in S \cap K$ it satisfies (3.24).

On the other hand, recall that $\phi \equiv 1$ on $\overline{V}$. Therefore, for any $z \in \overline{V} \cap A_\tau$, we have $2 \leq f_\tau \leq 3$. Thus, for some constant C, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 \chi(f_\tau(z))}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq C\tau |t|^2$$

for all $z \in \overline{V} \cap A_\tau$, and large τ. So, since λ_τ is plurisubharmonic, we obtain

$$\sum_{j,k=1}^{n} \frac{\partial^2 g_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq C\tau |t|^2 \tag{3.25}$$

for all $z \in \overline{V} \cap A_\tau$, and large τ. Note that $\overline{V} \cap K \subset \big((\overline{V} \cap A_\tau \cap K) \cup (\overline{V} \cap S \cap K) \big)$. So, combining (3.24) and (3.25), we obtain that there exists a constant C such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 g_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq C\tau |t|^2 \tag{3.26}$$

whenever $z \in \overline{V} \cap K$, and τ is large enough.

We summarize what we have obtained so far: for any $z_0 \in S' \cap S_i$, there are a neighborhood V with $z_0 \in V$ and a family of functions g_τ, $0 \leq g_\tau \leq 2$, such that g_τ is plurisubharmonic in a neighborhood of $b\Omega$ and such that for all $z \in \overline{V} \cap K$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 g_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq C\tau |t|^2.$$

Choose finitely many points $z_1, \cdots, z_p$ of $S_i \cap S'$ such that the associated neighborhoods $V_1, \cdots, V_p$ cover $S_i \cap S'$. Set $h_\tau(z) = \frac{1}{4p} \sum_{\nu=1}^{p} g_\tau^\nu(z)$, where $g_\tau^\nu(z)$ is the family of functions constructed as above for the point z_ν. The function h_τ satisfies

$$\sum_{j,k=1}^{n} \frac{\partial^2 h_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq C\tau |t|^2 \text{ for all } z \in \cup_{\nu=1}^{p} V_\nu. \tag{3.27}$$

By construction, h_τ is plurisubharmonic near $b\Omega$ and $0 \leq h_\tau \leq \frac{1}{2}$. Set

$$S'' = (S' \setminus (\cup_{\nu=1}^{n} V_\nu)) \cap K. \tag{3.28}$$

By construction, S'' is compact and $S'' \cap S_i = \emptyset$. Therefore, by the induction hypothesis again, there exists a plurisubharmonic function $\mu_\tau \in C^\infty(\overline{\Omega})$ such that $0 \leq \mu_\tau \leq \frac{1}{2}$ and for all $z \in S''$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \mu_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq \tau |t|^2. \tag{3.29}$$

Now, we set $p_\tau(z) := \frac{1}{2}\{h_\tau(z) + \mu_\tau(z) + \frac{|z|^2}{D^2}\}$ where D is the supremum of $|z|$'s as z runs over $\overline{\Omega}$. Then, we have $0 \leq p_\tau \leq 1$ and p_τ is strictly plurisubharmonic in a neighborhood of $b\Omega$. Furthermore, for some C and large τ, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 p_\tau(z)}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq C\tau |t|^2 \quad z \in S'.$$

What remains to do is to extend plurisubharmonicity of p_τ from a neighborhood of $b\Omega$ to $\overline{\Omega}$. However, by Proposition 3.1.6 in [9] and its proof (this proposition and its proof is based on Theorem 3.7 of [34] and its proof) we have a plurisubharmonic function $\tilde{p}_\tau$ such that $p_\tau \leq \tilde{p}_\tau$ and $\tilde{p}_\tau = p_\tau$ on $b\Omega$. Moreover, the complex Hessian of $\tilde{p}_\tau$ at a boundary point z applied to the vectors $t \in \mathbb{C}^n$ dominates that of p_τ. Finally, in order to obtain a new function with uniform bounds, choose a smooth function

$\psi(s)$ with $\psi(s) \equiv 0$ for $s \leq -1$, $\psi''(s) > 0$ for $s \geq -1$, and $\psi(s) \leq 1$ for $s \leq 1$. Now, set $\Lambda_\tau = \psi(\tilde{p}_\tau)$. Then Λ_τ is plurisubharmonic and smooth on $\overline{\Omega}$ and it has a big Hessian on S' as required by the definition of property (P). We have proved S' has property (P); this finishes the induction and hence the proof is complete. □

3.2 Obstructions to compactness of the $\bar{\partial}$-Neumann operator

The discussions made about the compactness of $\bar{\partial}$-Neumann operator so far were in the positive direction. However, there are also some domains for which the compactness of the $\bar{\partial}$-Neumann operator fails. For instance, a polydisc or a worm domain is an example of domain which has a noncompact $\bar{\partial}$-Neumann operator (see [38], [36] for the polydisc). To give a short explanation why compactness on worm domains fails, we note that compactness of N on a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ implies that N is exactly and globally regular. However, by the work of Christ ([16]) (see also [2]), we know that N corresponding to worm domains is not globally regular. Therefore, $\bar{\partial}$-Neumann operators corresponding to worm domains cannot be compact.

The most basic tool to produce examples of domains on which the $\bar{\partial}$-Neumann operator is not compact is the analytic discs. We recall that an analytic disc is a nontrivial holomorphic map from an open set around the origin of the complex plane into a complex Euclidean space. A folklore result (the smooth case is generally attributed to Catlin) states that if a bounded pseudoconvex domain in $\mathbb{C}^2$ with a Lipschitz boundary contains an analytic disc in its boundary, then it cannot have a compact $\bar{\partial}$-Neumann operator. A proof of this can be found in [25]. One can see from the proof of this result that the analytic discs can be replaced by complex manifolds of complex dimension $n-1$ in the general case. We record this for further use:

Proposition 3.2.1. *A bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$ with Lipschitz* *boundary and compact N_1 repels $(n-1)$-complex dimensional manifolds from its boundary.*

Although Proposition 3.2.1 states an obstruction in any complex dimension $n \geq 2$, whether an analytic disc in the boundary of a bounded pseudoconvex domain is an obstruction to the compactness of the $\bar{\partial}$-Neumann operator is not fully known. It is known, however, to be an obstruction in the case of locally convexifiable domains in $\mathbb{C}^n$. A partial result to the most general case is due to Şahutoğlu and Straube who showed in [50] that a complex manifold M in the boundary of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ is indeed an obstruction to the compactness of the $\bar{\partial}$-Neumann operator, provided that at some point of the manifold, the Levi form has the maximal possible rank $n-1-\dim(M)$ (i.e. the domain is strictly pseudoconvex in the directions transverse to M). When $\dim(M) = 1$, this gives that an analytic disc in the boundary is an obstruction to compactness of $\bar{\partial}$-Neumann operator when it has a point at which the boundary is strictly pseudoconvex in the $(n-2)$ transverse directions (to the disc).

In the reverse direction, one can also ask whether nonexistence of analytic discs implies compactness of the $\bar{\partial}$-Neumann operator. Matheos proved in his dissertation [39] that nonexistence of analytic discs in the boundary does not necessarily imply the compactness of the $\bar{\partial}$-Neumann operator (see also [25] for a simplified proof). For more information, we refer to the survey paper [25] and the monograph [56].

4. COMPACTNESS OF $\bar{\partial}$-NEUMANN OPERATOR ON THE INTERSECTION DOMAINS

We first recall the problem that was stated in the Introduction:

Problem. *Let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbb{C}^n$ which intersect each other and assume that the intersection set, say Ω, is a domain, i.e., connected. Suppose that the $\bar{\partial}$-Neumann operators on Ω_1 and Ω_2 at some form level are compact. Is the $\bar{\partial}$-Neumann operator of the intersection domain Ω compact at the same form level?*

As discussed before, a positive result is mostly encouraged by the localization of the compactness of the $\bar{\partial}$-Neumann operator. It reads as follows:

Theorem 4.0.2 (Localization)**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. If for any point in $b\Omega$ there exists a strictly pseudoconvex neighborhood so that this neighborhood intersects Ω as a connected set and this intersection has compact $\bar{\partial}$-Neumann operator, then the $\bar{\partial}$-Neumann operator on Ω is compact. Conversely, if the $\bar{\partial}$-Neumann operator on Ω is compact, then for any strictly pseudoconvex domain intersecting Ω in a connected set, the intersection has compact $\bar{\partial}$-Neumann operator.*

The theorem is essentially folklore but see [25] and the monograph [56] for a proof. Observe that since the intersecting domains Ω_1 and Ω_2 in the problem have compact $\bar{\partial}$-Neumann operators, then thanks to the localization theorem, the connected intersections of (small, open) balls centered at the boundary points of the domains Ω_1 and Ω_2 satisfy compactness estimates. This observation is useful in order to reduce the amount of work if one wants to prove the compactness of the $\bar{\partial}$-Neumann operator of the intersection domain Ω in the problem. One can start with considering the points in $b\Omega$ which are away from the intersection of the boundaries $b\Omega_1$ and $b\Omega_2$ so that the points that are considered belong to either only $b\Omega_1$ or only $b\Omega_2$. Then, one can just take a small ball around such a point so that the intersection of this ball with Ω is actually the intersection of this ball with either Ω_1 or Ω_2. The observation made for Ω_1 and Ω_2 now can be used to deduce that if one wants to use the localization theorem for the problem then the points which are away from $b\Omega_1 \cap b\Omega_2$ are benign for the problem: the intersection of small open balls around these points with the domain Ω always satisfies compactness estimates. Therefore, one has to focus on an analysis of the points in $b\Omega$ which are common to $b\Omega_1$ and $b\Omega_2$.

Let us denote the intersection of $b\Omega_1$ and $b\Omega_2$ by S. That is, S is given by $b\Omega_1 \cap b\Omega_2$. If the boundaries $b\Omega_1$ and $b\Omega_2$ overlap on S; that is, if the closure of interior of S in $b\Omega_1$ and $b\Omega_2$ topology is itself, then the approach taken via the localization theorem can be used to deduce that $\bar{\partial}$-Neumann operator is compact. For this particular case, one can also accommodate some cutoff functions around those boundary portions that are disjoint from S and can achieve the same result. The specific result proved in [12] is in this direction.

If some proper subset of S is an overlap of the boundaries, then one can similarly eliminate the work required to deal with this subset. As a consequence, the problematic parts of S are those where the boundaries are non-overlapping. From this point of view, the problem is most difficult when S has an empty interior with respect to one of the boundaries. An example of this is the case when boundaries intersect transversally. A transversal intersection of the boundaries would result in a closed manifold which has real codimension 1 in any of the boundaries and it would have empty interior in any of the boundaries. Since the problematic part is S, positive results for the problem may be expected when some assumptions are made on S. A positive result with an assumption on S is provided in Theorem 4.1.2. Before moving further on the results, it should be noted that affirmative results to analogous problems in different settings were considered before. We list these predecessor results:

1) If Ω is a piecewise smooth strictly pseudoconvex domain in $\mathbb{C}^n$ (if defined to be piecewise smooth strictly pseudoconvex in the sense of [48]), then $N_q : \mathcal{L}^2_{(0,q)}(\Omega) \to \mathcal{L}^2_{(0,q)}(\Omega)$ gains $\frac{1}{2}$ derivative and is compact ([41]);

2) if the domain is a piecewise smooth pseudoconvex domain of finite type in the sense of D'Angelo, then it also satisfies subelliptic estimates and hence the $\bar{\partial}$-Neumann operator is compact ([53]);

3) if both Ω_1 and Ω_2 have property (P_q), then $b\Omega$ satisfies property (P_q); and hence N_q^{Ω}, $1 \leq q \leq n$ is compact (see [52], [25], [56]);

4) if one of $b\Omega_1$ and $b\Omega_2$ has property (P_q), then N_q^{Ω}, $1 \leq q \leq n$ is compact (see proof of Localization theorem in [25] or [56] for a proof of this).

5) An example of a non-transversal intersection of two bounded pseudoconvex domains in $\mathbb{C}^n$ with compact $\bar{\partial}$-Neumann operators was investigated in [12] and the appropriate forms defined on the intersection domain was shown to satisfy compactness estimates.

4.1 Results on the general intersection case

The following lemma will be useful in proving Theorem 4.1.2:

Lemma 4.1.1. *Let ϕ be a smooth cutoff function which is identically equal to 1 in a small neighborhood of $S := b\Omega_1 \cap b\Omega_2$. If $N_q^{\Omega_1}$ and $N_q^{\Omega_2}$ are compact, then for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,\phi} > 0$ such that*

$$||(1-\phi)u||^2_{\Omega} \leq \varepsilon \left(||\bar{\partial}u||^2_{\Omega} + ||\bar{\partial}^* u||^2_{\Omega} \right) + C_{\varepsilon,\phi}||u||^2_{-1,\Omega}, \quad (4.1)$$

whenever $u \in dom(\bar{\partial}) \cap dom(\bar{\partial}^*) \subset \mathcal{L}^2_{(0,q)}(\Omega)$.

Before proving the lemma, we will introduce some notation. Let $b\Omega_+ := b\Omega_1 \cap \Omega_2$ and $b\Omega_- := b\Omega_2 \cap \Omega_1$. That is, $b\Omega_-$ and $b\Omega_+$ are open subsets of $b\Omega$ that lie in Ω_1 and Ω_2 respectively. Observe that $b\Omega_- \cup b\Omega_+ \cup S = b\Omega$.

Proof. Let U_1, U_2 be small neighborhoods of $b\Omega_+ \cap$ supp$(1-\phi)$ and $b\Omega_- \cap$ supp$(1-\phi)$ respectively that are also disjoint from S. We choose a relatively compact open subset U_0 of Ω so that supp$(1-\phi) \cap \Omega$ is compactly contained in $U_0 \cup U_1 \cup U_2$. That is, the sets U_0, U_1 and U_2 form an open cover of supp$(1-\phi) \cap \Omega$. Let ψ_0, ψ_1, ψ_2 be a partition of unity on supp$(1-\phi) \cap \Omega$ subordinate to the covering U_0, U_1, U_2; i.e., ψ_0, ψ_1, ψ_2 are smooth cutoff functions in $\mathbb{C}^n$ such that supp $\psi_0 \Subset U_0$, supp $\psi_1 \Subset U_1$, supp $\psi_2 \Subset U_2$ and their sum at a point of supp$(1-\phi) \cap \Omega$ is 1. Set $\varphi := 1 - \phi$. Then, for $u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$, we have

$$
\begin{aligned}
||\varphi u||^2_\Omega = ||\varphi u||^2_{\mathrm{supp}(\varphi)} &\leq 4\left(||\psi_0 \varphi u||^2_{\mathrm{supp}(\varphi)} + ||\psi_1 \varphi u||^2_{\mathrm{supp}(\varphi)} + ||\psi_2 \varphi u||^2_{\mathrm{supp}(\varphi)}\right) \\
&\leq 4\left(||\psi_0 \varphi u||^2_\Omega + ||\psi_1 \varphi u||^2_\Omega + ||\psi_2 \varphi u||^2_\Omega\right). \qquad (4.2)
\end{aligned}
$$

Observe that the forms $\psi_2 \varphi u$ and $\psi_1 \varphi u$ are still in $\mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$. By definition of ψ_2, the form $\psi_2 \varphi u$ is zero outside of the set $\Omega \cap U_2$ in Ω. Observe that $\Omega \cap U_2$ is away from $b\Omega_+$ and S. Although the form u is defined only on Ω, multiplying by the smooth cutoff function ψ_2 gives a well-defined form in Ω_2: the form $\psi_2 \varphi u$ nicely vanishes in a neighborhood of the set $\overline{b\Omega_+} \cup S$ and hence we can view the form $\psi_2 \varphi u$ as a form in $\mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1}) \subset \mathcal{L}^2_{(0,q)}(\Omega_2)$. That $\psi_2 \varphi u \in \mathrm{dom}(\bar{\partial})$ of Ω_2 is immediate by extending it to be zero outside of Ω. That it is also in $\mathrm{dom}(\bar{\partial}^*)$ of Ω_2 when extended to be zero outside of Ω can be seen by pairing $\psi_2 \varphi u$ with $\bar{\partial} v$ for any

$v \in \mathrm{dom}(\bar{\partial}_{\Omega_2})$:

$$|(\psi_2\varphi u, \bar{\partial}v)_{\Omega_2}| = |(\psi_2\varphi u, \bar{\partial}v)_{\Omega}| \leq C||v||_{\Omega} \leq C||v||_{\Omega_2}.$$

Here, the equality is due to extending the form to be zero outside of Ω. The first inequality is because a form on Ω_2 which is in $\mathrm{dom}(\bar{\partial})$ when restricted to Ω is also in $\mathrm{dom}(\bar{\partial})$ corresponding to Ω. The second inequality is just by the increasing property of norms. By what was discussed, we have $\psi_2\varphi u \in \mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1}) \subset \mathcal{L}^2_{(0,q)}(\Omega_2)$. By similar arguments, $\psi_1\varphi u$ can be seen $\mathrm{dom}(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1}) \subset \mathcal{L}^2_{(0,q)}(\Omega_1)$.

Now, since $N^{\Omega_1}_q$ and $N^{\Omega_2}_q$ are compact, we can apply the compactness estimates to the forms $\psi_2\varphi u$ and $\psi_1\varphi u$: for any $\varepsilon' > 0$ (to be specified below), there exists a $C_{\varepsilon'} > 0$ such that

$$\begin{aligned}
&||\psi_0\varphi u||^2_{\Omega} + ||\psi_2\varphi u||^2_{\Omega} + ||\psi_1\varphi u||^2_{\Omega} \\
&\qquad = ||\psi_0\varphi u||^2_{\Omega} + ||\psi_2\varphi u||^2_{\Omega_2} + ||\psi_1\varphi u||^2_{\Omega_1} \\
&\qquad \leq ||\psi_0\varphi u||^2_{\Omega} + \varepsilon' \left(||\bar{\partial}(\psi_2\varphi u)||^2_{\Omega_2} + ||\bar{\partial}^*(\psi_2\varphi u)||^2_{\Omega_2}\right) \\
&\qquad\qquad + \varepsilon' \left(||\bar{\partial}(\psi_1\varphi u)||^2_{\Omega_1} + ||\bar{\partial}^*(\psi_1\varphi u)||^2_{\Omega_1}\right) \\
&\qquad\qquad + C_{\varepsilon'} \left(||\psi_2\varphi u||^2_{-1,\Omega_2} + ||\psi_1\varphi u||^2_{-1,\Omega_1}\right). \qquad (4.3)
\end{aligned}$$

The term $||\psi_0\varphi u||^2_{\Omega}$ can be estimated via interior elliptic regularity by

$$\varepsilon' \left(||\bar{\partial}(\psi_0\varphi u)||^2_{\Omega} + ||\bar{\partial}^*(\psi_0\varphi u)||^2_{\Omega}\right), \qquad (4.4)$$

so that (by also bringing the inequality (4.2)) we get

$$
\begin{aligned}
||\varphi u||_{\Omega}^2 \leq 4\varepsilon' &\left(||\bar{\partial}(\psi_0\varphi u)||_{\Omega}^2 + ||\bar{\partial}^*(\psi_0\varphi u)||_{\Omega}^2\right) \\
&+ 4\varepsilon' \left(||\bar{\partial}(\psi_2\varphi u)||_{\Omega_2}^2 + ||\bar{\partial}^*(\psi_2\varphi u)||_{\Omega_2}^2\right) \\
&\quad + 4\varepsilon' \left(||\bar{\partial}(\psi_1\varphi u)||_{\Omega_1}^2 + ||\bar{\partial}^*(\psi_1\varphi u)||_{\Omega_1}^2\right) \\
&\quad + C_{\varepsilon'} \left(||\psi_2\varphi u||_{-1,\Omega_2}^2 + ||\psi_1\varphi u||_{-1,\Omega_1}^2\right). \qquad (4.5)
\end{aligned}
$$

We can estimate $||\bar{\partial}(\psi_2\varphi u)||_{\Omega_2}^2$ as follows:

$$
\begin{aligned}
||\bar{\partial}(\psi_2\varphi u)||_{\Omega_2}^2 &= ||(\bar{\partial}\psi_2) \wedge (\varphi u) + \psi_2\bar{\partial}(\varphi u)||_{\Omega_2}^2 \\
&\leq 2||(\bar{\partial}\psi_2) \wedge (\varphi u)||_{\Omega_2}^2 + 2||\psi_2\bar{\partial}(\varphi u)||_{\Omega_2}^2 \\
&= 2||(\bar{\partial}\psi_2) \wedge (\varphi u)||_{\Omega}^2 + 2||\psi_2\bar{\partial}(\varphi u)||_{\Omega}^2 \\
&\leq 2^{q+1}(n-q)\left(\sup_{\Omega} |\nabla\psi_2|^2\right) ||\varphi u||_{\Omega}^2 + 2||\bar{\partial}(\varphi u)||_{\Omega}^2 \\
&\leq 2^{2n+1}\left(\sup_{\Omega} |\nabla\psi_2|^2\right) ||\varphi u||_{\Omega}^2 + 2||\bar{\partial}(\varphi u)||_{\Omega}^2 \\
&\leq \left(2 + 2^{2n+1}\frac{D^2 e}{q}\left(\sup_{\Omega} |\nabla\psi_2|^2\right)\right) \left(||\bar{\partial}(\varphi u)||_{\Omega}^2 + ||\bar{\partial}^*(\varphi u)||_{\Omega}^2\right). \qquad (4.6)
\end{aligned}
$$

Similarly, we obtain

$$
||\bar{\partial}^*(\psi_2\varphi u)||_{\Omega_2}^2 \leq \left(2 + 2^{2n}\left(\sup_{\Omega} |\nabla\psi_2|^2\right)\right) \left(||\bar{\partial}(\varphi u)||_{\Omega}^2 + ||\bar{\partial}^*(\varphi u)||_{\Omega}^2\right). \qquad (4.7)
$$

Inequalities (4.6) and (4.7) together give

$$
||\bar{\partial}(\psi_2\varphi u)||_{\Omega_2}^2 + ||\bar{\partial}^*(\psi_2\varphi u)||_{\Omega_2}^2 \leq C_{\psi_2}\left(||\bar{\partial}(\varphi u)||_{\Omega}^2 + ||\bar{\partial}^*(\varphi u)||_{\Omega}^2\right), \qquad (4.8)
$$

where $C_{\psi_2} = 4 + 2^{2n+2}\frac{D^2 e}{q}\left(\sup_{\Omega} |\nabla\psi_2|^2\right)$.

Repeating the above for $\psi_1\varphi u$, we obtain

$$||\bar{\partial}(\psi_1\varphi u)||^2_{\Omega_1} + ||\bar{\partial}^*(\psi_1\varphi u)||^2_{\Omega_1} \leq C_{\psi_1}\left(||\bar{\partial}(\varphi u)||^2_{\Omega} + ||\bar{\partial}^*(\varphi u)||^2_{\Omega}\right), \tag{4.9}$$

where $C_{\psi_1} = 4 + 2^{2n+2}\frac{D^2 e}{q}\left(\sup_\Omega |\nabla\psi_1|^2\right)$.

Similar calculations can be made for the norms on right hand side of (4.4) to get

$$||\bar{\partial}(\psi_0\varphi u)||^2_{\Omega} + ||\bar{\partial}^*(\psi_0\varphi u)||^2_{\Omega} \leq C_{\psi_0}\left(||\bar{\partial}(\varphi u)||^2_{\Omega} + ||\bar{\partial}^*(\varphi u)||^2_{\Omega}\right), \tag{4.10}$$

where $C_{\psi_0} = 4 + 2^{2n+2}\frac{D^2 e}{q}\left(\sup_\Omega |\nabla\psi_0|^2\right)$.

Having the $\mathcal{L}^2$-norms of φu on the right hand side of each, we substitute (4.8), (4.9), (4.10) into (4.5) and obtain

$$\begin{aligned} ||\varphi u||^2_{\Omega} \leq\ & 4\varepsilon' M_\psi\left(||\bar{\partial}(\varphi u)||^2_{\Omega} + ||\bar{\partial}^*(\varphi u)||^2_{\Omega}\right) \\ & + C_{\varepsilon'}\left(||\psi_2\varphi u||^2_{-1,\Omega_2} + ||\psi_1\varphi u||^2_{-1,\Omega_1}\right). \end{aligned} \tag{4.11}$$

where $M_\psi := \max\{C_{\psi_0}, C_{\psi_2}, C_{\psi_1}\}$. Computing $\bar{\partial}(\varphi u)$ and $\bar{\partial}^*(\varphi u)$ and estimating similarly, we get

$$\begin{aligned} ||\varphi u||^2_{\Omega} \leq\ & 4\varepsilon' M_\psi K_\varphi\left(||\bar{\partial} u||^2_{\Omega} + ||\bar{\partial}^* u||^2_{\Omega}\right) \\ & + C_{\varepsilon'}\left(||\psi_2\varphi u||^2_{-1,\Omega_2} + ||\psi_1\varphi u||^2_{-1,\Omega_1}\right). \end{aligned} \tag{4.12}$$

with K_φ a constant depending on the supremum of the gradient of φ on Ω. The (-1)-norms of $\psi_1\varphi u$ on Ω_1 and $\psi_2\varphi u$ on Ω_2 can be estimated by their (-1)-norms on Ω. The arguments for estimating both of these norms will be similar. Thus, in what follows, we will discuss estimating only (-1)-norm of $\psi_2\varphi u$ on Ω_2. Let γ_2 be a smooth cutoff function that is identically equal to 1 on the support of ψ_2 and has compact support in Ω_1. Then, γ_2 is a (continuous) multiplier from $W^1_0(\Omega_2)$ to $W^1_0(\Omega)$, hence from $W^{-1}_0(\Omega)$ to $W^{-1}_0(\Omega_2)$ (recall that this is possible because for a linear continuous map T between Banach spaces X and Y, there is a linear and continuous transpose map T^* from Y^* to X^* defined by $T^*f = f \circ T$ and with $||T^*|| = ||T||$). But observe that $\gamma_2\psi_2\varphi u = \psi_2\varphi u$. Thus, $||\psi_2\varphi u||^2_{-1,\Omega_2}$ is dominated by $||\psi_2\varphi u||^2_{-1,\Omega}$, the constant depending on just the supremum of the gradient of γ_2 (and hence on ψ_2) and hence on φ. Moreover, ψ_2 is a continuous multiplier on $W^1_{0,(0,q)}(\Omega)$ with the operator norm depending on the 1-norm of ψ_2 on Ω. So, to summarize, $||\psi_2\varphi u||^2_{-1,\Omega_2} \lesssim ||\varphi u||^2_{-1,\Omega}$ with constant depending only on φ and Ω. Similar arguments apply to estimate $||\psi_1\varphi u||^2_{-1,\Omega_1}$ and we obtain $||\psi_1\varphi u||^2_{-1,\Omega_1} \lesssim ||\varphi u||^2_{-1,\Omega}$ with constant depending on Ω and φ.

On the other hand, the function φ is a continuous multiplier on $W^1_0(\Omega)$. Therefore, $||\varphi u||^2_{-1,\Omega} \lesssim ||u||^2_{-1,\Omega}$ with a constant depending on φ. Summarizing now, we get

$$\begin{aligned} ||\varphi u||^2_{\Omega} \leq 4\varepsilon' M_\psi K_\varphi \left(||\bar{\partial}u||^2_{\Omega} + ||\bar{\partial}^*u||^2_{\Omega}\right) \\ + \tilde{C}_{\varepsilon',\varphi}||u||^2_{-1,\Omega}. \end{aligned} \tag{4.13}$$

Now, we choose ε' such that $4\varepsilon' M_\psi K_\varphi \leq \varepsilon$. Note that partition of unity functions ψ_j's were depending on φ by our construction. Thus, we obtain

$$||(1-\phi)u||^2_{\Omega} \leq \varepsilon \left(||\bar{\partial}u||^2_{\Omega} + ||\bar{\partial}^*u)||^2_{\Omega}\right) + C_{\varepsilon,\phi}||u||^2_{-1,\Omega}, \tag{4.14}$$

whenever $u \in \mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*) \subset \mathcal{L}^2_{(0,q)}(\Omega)$. This completes the proof of the lemma. □

Theorem 4.1.2. *Suppose that Ω_1 and Ω_2 are two bounded pseudoconvex domains in $\mathbb{C}^n, n \geq 2$ which nontrivially intersect each other. Denote by Ω be the intersection* *domain (connected). If $N_{q_1}^{\Omega_1}$ and $N_{q_2}^{\Omega_2}$ are compact and S satisfies property $(\tilde{P}_{q_3})$, then the $\bar{\partial}$-Neumann operator N_j^{Ω} is compact for $j \geq \max\{q_1, q_2, q_3\}$.*

Remark 4.1.3. *Note that no boundary regularity was assumed for any of the domains.*

Proof. The proof of Theorem 4.1.2 is essentially same with the proof of Theorem 4.29 in [56] except at one point we need to invoke Lemma 4.1.1 rather than bringing the interior elliptic regularity argument. For convenience, we are discussing the complete details below.

Set $q := \max\{q_1, q_2, q_3\}$. Since compactness of N and property $(\tilde{P})$ percolate up the complex, it suffices to assume that $N_q^{\Omega_1}$ and $N_q^{\Omega_1}$ are compact and property $(\tilde{P}_q)$ holds in a neighborhood of S. We know by Lemma 3.0.11 that N_q is compact if and only if $\bar{\partial}^* N_q$ and $\bar{\partial}^* N_{q+1}$ are compact. Also, Çelik and Şahutoğlu ([13]) showed that for $1 \leq q \leq n-1$, if $\bar{\partial}^* N_q$ is compact, then $\bar{\partial}^* N_{q+1}$ is compact. Thus, it suffices to show that $\bar{\partial}^* N_q$ is compact. But $\bar{\partial}^* N_q$ is compact if and only if its adjoint $(\bar{\partial}^* N_q)^*$ is compact. Therefore, the theorem will be proved once we can show that $(\bar{\partial}^* N_q)^*$ is compact. To do this, we will show the compactness estimates for $(\bar{\partial}^* N_q)^*|_{\ker(\bar{\partial}_{q-1})^\perp}$ in view of Lemma 3.0.9. A few observations are in order to explain why we are restricting the operator $(\bar{\partial}^* N_q)^*$ onto $\ker(\bar{\partial}_{q-1})^\perp$ and how the compactness estimates will look like.

We first observe that $(\bar{\partial}_{q-1}^* N_q)^* = 0$ on $\ker(\bar{\partial}_{q-1})$ explaining the restriction to $\ker(\bar{\partial}_{q-1})^\perp$. Indeed, if $v \in \ker(\bar{\partial}_{q-1})$ and $u \in \mathcal{L}^2_{(0,q)}(\Omega)$, then

$$((\bar{\partial}_{q-1}^* N_q)^* v, u)_\Omega = (v, (\bar{\partial}_{q-1}^* N_q)u)_\Omega = (\bar{\partial}_{q-1} v, N_q u)_\Omega = 0.$$

We now want to understand how the compactness estimates will look like for the restricted operator; i.e. $(\bar{\partial}^* N_q)^*|_{\ker(\bar{\partial}_{q-1})^\perp}$. Our first observation is $(\bar{\partial}_{q-1}^* N_q)^* =$

$\bar{\partial}_{q-1}N_{q-1}$ on $\ker(\bar{\partial}_{q-1})^{\perp}$. To see this, observe first that if $w \in \ker(\bar{\partial}_{q-1})^{\perp}$, then $w = \bar{\partial}^*_{q-1}(\bar{\partial}_{q-1}N_{q-1})w$. Also, we have $(\bar{\partial}^*_{q-1}N_q)^*w \in \ker(\bar{\partial}_q)$. Therefore, to show $(\bar{\partial}^*_{q-1}N_q)^* = \bar{\partial}_{q-1}N_{q-1}$ on $\ker(\bar{\partial}_{q-1})^{\perp}$, it suffices to pair $(\bar{\partial}^*_{q-1}N_q)^*w$ with $u \in \ker(\bar{\partial}_q)$. These two observations then give us

$$\begin{aligned}((\bar{\partial}^*_{q-1}N_q)^*w, u) &= (w, \bar{\partial}^*_{q-1}N_q u)\\ &= (\bar{\partial}^*_{q-1}(\bar{\partial}_{q-1}N_{q-1})w, \bar{\partial}^*_{q-1}N_q u)\\ &= (\bar{\partial}_{q-1}N_{q-1}w, \bar{\partial}_{q-1}\bar{\partial}^*_{q-1}N_q u)\\ &= (\bar{\partial}_{q-1}N_{q-1}w, u).\end{aligned}$$

We have used in passing to the last equality that if $u \in \ker(\bar{\partial}_q)$, then it can be written as $\bar{\partial}_{q-1}\bar{\partial}^*_{q-1}N_q u$. If $w \in \ker(\bar{\partial}_{q-1})^{\perp}$, then $w = \bar{\partial}^*_{q-1}v$ for some $v \in \ker(\bar{\partial}_q) \cap \mathrm{dom}(\bar{\partial}^*_{q-1})$ and moreover, $\bar{\partial}_{q-1}N_{q-1}w = v$. Therefore, in the light of the observations above, it suffices to prove that $(\bar{\partial}^*N_q)^*$, restricted to $\ker(\bar{\partial})^{\perp}$, is compact and by Lemma 3.0.9, theorem will be proved if for every $\epsilon > 0$, we can find a $C_\epsilon > 0$ and a linear, compact $S_\epsilon : \ker(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*) \to W^{-1}_{(0,q)}(\Omega)$ such that

$$||v||^2 \leq \epsilon||\bar{\partial}^*v||^2 + C_\epsilon||S_\epsilon(\bar{\partial}N_{q-1})w||^2_{-1,\Omega}$$

for $v \in \ker(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ with $(\bar{\partial}N_{q-1})w = v$. Note that for any given $\varepsilon > 0$, we can let $M := \frac{C}{\varepsilon}$ for some constant $C > 0$. Therefore, given small ε, we can let M as we defined and work with M's. So, for any $M > 0$, denote by λ_M the function from the definition of property $(\tilde{P}_q)$. We may assume that λ_M is a C^2 function on a neighborhood of $\overline{\Omega}$ (replacing U_M by a smaller set if necessary; the conditions (3.14) and (3.15) are still assumed only near S); this function is still denoted by λ_M. The starting point is Proposition 2.2.7: if $u \in \ker(\bar\partial) \cap \operatorname{dom}(\bar\partial^*) \subset \mathcal{L}^2_{(0,q)}(\Omega)$, we obtain

$$\int_\Omega \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar z_k}(z) u_{jK}\overline{u_{kK}} e^{-\lambda_M} \leq ||\bar\partial^*_{\lambda_M} u||^2_{\lambda_M}. \tag{4.15}$$

We recall that

$$\bar\partial^*_{\lambda_M} u = \bar\partial^* u + \sum_{|K|=q-1}{}' \left(\sum_{j=1}^{n} \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar z_K, \tag{4.16}$$

and observe that

$$\begin{aligned} e^{-\frac{\lambda_M}{2}} \bar\partial^*_{\lambda_M} u &= e^{-\frac{\lambda_M}{2}} \bar\partial^* u + e^{-\frac{\lambda_M}{2}} \sum_{|K|=q-1}{}' \left(\sum_{j=1}^{n} \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar z_K \\ &= -e^{-\frac{\lambda_M}{2}} \sum_{|K|=q-1}{}' \left(\sum_{j=1}^{n} \frac{\partial u_{jK}}{\partial z_j} \right) d\bar z_K + 2 \left(\frac{1}{2} e^{-\frac{\lambda_M}{2}} \sum_{|K|=q-1}{}' \left(\sum_{j=1}^{n} \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar z_K \right) \\ &= \bar\partial^* \left(e^{-\frac{\lambda_M}{2}} u \right) + \frac{1}{2} e^{-\frac{\lambda_M}{2}} \sum_{|K|=q-1}{}' \left(\sum_{j=1}^{n} \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar z_K. \end{aligned}$$

Taking squares of both sides, integrating on Ω and combining with (4.15), we obtain

$$\begin{aligned} &\int_\Omega \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar z_k}(z) u_{jK}\overline{u_{kK}} e^{-\lambda_M} \\ &\qquad \leq C_1 ||\bar\partial^* \left(e^{-\frac{\lambda_M}{2}} u \right)||^2_\Omega + C_1 \int_\Omega \sum_{|K|=q-1}{}' \left| \sum_{j=1}^{n} \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right|^2 e^{-\lambda_M}, \end{aligned} \tag{4.17}$$

where C_1 is a constant independent of M. Using (3.14) in the integral on the right hand side of (4.17) for $z \in \Omega \cap U_M$, the resulting terms can be absorbed on the left hand side. Observe that we use here the fact that the constant C in (3.14) can be taken as small as we want (replacing the function λ_M by $\frac{\lambda_M}{A}$ if A is wanted). In particular, we may assume that $C_1 C \leq 1/2$ and the resulting inequality is

$$\int_\Omega \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k}(z) u_{jK} \overline{u_{kK}} e^{-\lambda_M} \leq 2C_1 ||\bar{\partial}^* \left(e^{-\frac{\lambda_M}{2}} u \right) ||_\Omega^2 + C_M ||u||^2_{\mathcal{L}^2_{(0,q)}(\Omega \setminus \overline{U_M})}. \tag{4.18}$$

Now we apply (3.15) on the left hand side of (4.18) for $z \in U_M \cap \Omega$ and observe that the integrals involving u (but not the derivatives of u) over $(\Omega \setminus \overline{U_M})$ can be moved to the right hand side and estimated by $C_M ||u||_{\mathcal{L}^2_{(0,q)}(\Omega \setminus \overline{U_M})}$. This gives the estimate

$$||e^{-\frac{\lambda_M}{2}} u||_\Omega^2 \leq \frac{2C_1}{M} ||\bar{\partial}^* \left(e^{-\frac{\lambda_M}{2}} u \right) ||_\Omega^2 + C_M ||u||^2_{\mathcal{L}^2_{(0,q)}(\Omega \setminus \overline{U_M})}. \tag{4.19}$$

The arguments discussed so far were same with the arguments discussed in the proof of Theorem 4.29 in [56]. In [56], $||u||^2_{\mathcal{L}^2_{(0,q)}(\Omega \setminus \overline{U_M})}$ can be estimated via interior elliptic regularity as the set $\Omega \setminus \overline{U_M}$ there is compactly contained in Ω. However, in our case, $\Omega \setminus \overline{U_M}$ is a subset of Ω that only stays away from S and does not have to be compactly contained in Ω. To estimate $||u||^2_{\mathcal{L}^2_{(0,q)}(\Omega \setminus \overline{U_M})}$, we will invoke Lemma 4.1.1 (this is the only argument in our proof that differs from the arguments in the proof of Theorem 4.29 in [56]). To this end, let $\chi = \chi_M$ be a smooth cutoff function that is identically equal to 1 in a neighborhood of $\Omega \setminus \overline{U_M}$ and compactly supported off a neighborhood of S. Taking the function ϕ in Lemma 4.1.1 to be $1 - \chi_M$, we obtain (note that $u \in \ker(\bar{\partial})$)

$$||e^{-\frac{\lambda_M}{2}} u||_\Omega^2 \leq \frac{2C_1}{M} ||\bar{\partial}^* \left(e^{-\frac{\lambda_M}{2}} u \right) ||_\Omega^2 + \tilde{C}_M \varepsilon (||\bar{\partial}^* u||_\Omega^2) + C_2 ||u||^2_{-1,\Omega}, \tag{4.20}$$

where $\tilde{C}_M$ depends only on M and C_2 depends on M and ε (hence on M as soon as

ε is chosen). Using the definition of $\bar{\partial}^*$ we obtain

$$\bar{\partial}^* u = \frac{1}{2} \sideset{}{'}\sum_{|K|=q-1} \left(\sum_{j=1}^{n} \frac{\partial \lambda_M}{\partial z_j} u_{jK} \right) d\bar{z}_K - e^{\frac{\lambda_M}{2}} \bar{\partial}^* (e^{-\frac{\lambda_M}{2}} u). \tag{4.21}$$

In (4.21), taking first the squared norms of both sides on Ω, and then invoking the inequality $||a+b||^2 \leq 2||a||^2 + 2||b||^2$ and finally using the fact that $\lambda_M \in C^2(\overline{\Omega})$ give that

$$||\bar{\partial}^* u||_\Omega^2 \leq K_M (||u||_\Omega^2 + ||\bar{\partial}^* (e^{-\frac{\lambda_M}{2}} u)||_\Omega^2), \tag{4.22}$$

where K_M is a constant coming from the maximum of the norms of the gradient of λ_M and $e^{-\frac{\lambda_M}{2}}$ on Ω and hence depending only on M. Substituting (4.22) into (4.20) gives

$$||e^{-\frac{\lambda_M}{2}} u||_\Omega^2 \leq \frac{2C_1}{M} ||\bar{\partial}^* \left(e^{-\frac{\lambda_M}{2}} u \right) ||_\Omega^2 + \tilde{K}_M \varepsilon (||\bar{\partial}^* (e^{-\frac{\lambda_M}{2}} u)||_\Omega^2 + ||u||_\Omega^2) + C_2 ||u||_{-1,\Omega}^2. \tag{4.23}$$

Note that the left hand side of (4.23) is for $e^{-\frac{\lambda_M}{2}} \ker(\bar{\partial})$ rather than for $\ker(\bar{\partial})$. In order to avoid this, we use the Bergman projection $P_q : \mathcal{L}^2_{(0,q)}(\Omega) \to \ker(\bar{\partial})$, and its weighted variant $P_{q,\frac{\lambda_M}{2}}$ (the orthogonal projection with respect to $(\cdot,\cdot)_{\frac{\lambda_M}{2}}$). We recall that for any element v in $\ker(\bar{\partial})$, we can write

$$v = P_q \left(e^{-\frac{\lambda_M}{2}} (P_{q,\frac{\lambda_M}{2}})(e^{\frac{\lambda_M}{2}} v) \right).$$

This can be verified by pairing $e^{-\frac{\lambda_M}{2}} (P_{q,\frac{\lambda_M}{2}})(e^{\frac{\lambda_M}{2}} v)$ with a $\bar{\partial}$-closed form (see p. 117 in [56]). Note that $u = (P_{q,\frac{\lambda_M}{2}})(e^{\frac{\lambda_M}{2}} v) \in \ker(\bar{\partial})$. Moreover, $u \in \text{dom}(\bar{\partial}^*)$ provided $v \in \text{dom}(\bar{\partial}^*)$ because the domains of $\bar{\partial}^*$ and $\bar{\partial}^*_{\frac{\lambda_M}{2}}$ agree and they are preserved under the corresponding Bergman projections. Since the Bergman projection is norm-nonincreasing and $\bar{\partial}^* g = \bar{\partial}^* P_q g$ for any $g \in \text{dom}(\bar{\partial}^*)$, for $v \in \ker(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$, we get from (4.23) that

$$\begin{aligned}
||v||^2_\Omega &= ||P_q(e^{-\frac{\lambda_M}{2}} u)||^2_\Omega \\
&\leq ||e^{-\frac{\lambda_M}{2}} u||^2_\Omega \\
&\leq \frac{2C_1}{M} ||\bar{\partial}^* \left(e^{-\frac{\lambda_M}{2}} u\right)||^2_\Omega + \tilde{K}_M \varepsilon (||\bar{\partial}^*(e^{-\frac{\lambda_M}{2}} u)||^2_\Omega + ||u||^2_\Omega) + C_2 ||u||^2_{-1,\Omega} \\
&\leq \frac{2C_1}{M} ||\bar{\partial}^* \left(P_q(e^{-\frac{\lambda_M}{2}} u)\right)||^2_\Omega + \tilde{K}_M \varepsilon (||\bar{\partial}^*(P_q(e^{-\frac{\lambda_M}{2}} u))||^2_\Omega + ||u||^2_\Omega) + C_2 ||u||^2_{-1,\Omega} \\
&\leq \frac{2C_1}{M} ||\bar{\partial}^* v||^2_\Omega + \tilde{K}_M \varepsilon (||\bar{\partial}^* v||^2_\Omega + ||u||^2_\Omega) + C_2 ||(P_{q,\frac{\lambda_M}{2}})(e^{\frac{\lambda_M}{2}} v)||^2_{-1,\Omega}. \qquad (4.24)
\end{aligned}$$

Furthermore, since $P_{q,\frac{\lambda_M}{2}}$ is the orthogonal projection with respect to $(\cdot,\cdot)_{\frac{\lambda_M}{2}}$, we have

$$\begin{aligned}
||u||^2_\Omega &= ||e^{\frac{\lambda_M}{2}} (P_{q,\frac{\lambda_M}{2}})(e^{\frac{\lambda_M}{2}} v)||^2_{\lambda_M} \\
&\leq \left(\sup_{\overline{\Omega}} e^{\lambda_M}\right) ||(P_{q,\frac{\lambda_M}{2}})(e^{\frac{\lambda_M}{2}} v)||^2_{\lambda_M} \\
&\leq \left(\sup_{\overline{\Omega}} e^{\lambda_M}\right) ||e^{\frac{\lambda_M}{2}} v||^2_{\lambda_M} \\
&= \left(\sup_{\overline{\Omega}} e^{\lambda_M}\right) ||v||^2_\Omega. \qquad (4.25)
\end{aligned}$$

Now using (4.25) in the last line of (4.24), and choosing ε small enough and finally absorbing the term $||v||^2_\Omega$, we obtain

$$||v||^2 \lesssim \frac{1}{M} ||\bar{\partial}^* v||^2_\Omega + C_2 ||(P_{q,\frac{\lambda_M}{2}})(e^{\frac{\lambda_M}{2}} v)||^2_{-1,\Omega}. \qquad (4.26)$$

The canonical solution operator to $\bar{\partial}^*$ is continuous in $\mathcal{L}^2$-norms. Therefore, the norm in the last term of (4.26) is compact not only with respect to $||v||$, but also with respect to $||\bar{\partial}^* v||$. Since M was arbitrary, this implies in view of compactness estimates that $(\bar{\partial}^* N_q)^*$, restricted to $\ker(\bar{\partial})^{\perp}$, is compact. But $(\bar{\partial}^* N_q)^*$ vanishes on $\ker(\bar{\partial})$, so it is compact on $\mathcal{L}^2_{(0,q-1)}(\Omega)$. This completes the proof for $j = q$. Now, using the fact that the compactness of N_q implies the compactness of N_{q+1}, we get the compactness of N_j for all $j \geq q$. □

Because the proof we presented here is different than the one presented for the Localization theorem in [25] and [56], an immediate result of Theorem 4.1.2 is the second part of Localization theorem:

Corollary 4.1.4. *If the $\bar{\partial}$-Neumann operator corresponding to a bounded pseudoconvex domain Ω is compact and U is a strictly pseudoconvex domain which intersects Ω in a connected set, then the $\bar{\partial}$-Neumann operator corresponding to $U \cap \Omega$ is compact.*

4.1.1 When does intersection of boundaries satisfy property $(\tilde{P})$?

It is of interest in view of Theorem 4.1.2 to ask when the intersection of the boundaries satisfies property $(\tilde{P})$. However, in the literature, the examples of sets or more generally domains which are known to satisfy property $(\tilde{P})$ also satisfy property (P). Nevertheless, the latter is formally weaker: property (P) implies property $(\tilde{P})$. So, we can still obtain that the intersection of the boundaries satisfies property $(\tilde{P})$ by verifying that it satisfies property (P). Motivated by this point of view, it is our aim now to present that there are examples where the set S satisfies property (P) and hence property $(\tilde{P})$

4.1.1.1 Examples with respect to type of points in S

We will now consider the type of points in S (recall that S is $b\Omega_1 \cap b\Omega_2$) and list some cases in which the assumptions of Theorem 4.1.2 are satisfied. Note that we mention finite or infinite type points; the smoothness of the boundaries must naturally be assumed; however, we still do not assume any special kind of intersection.

a) If Ω_1 and Ω_2 have smooth boundaries and any point of S is of finite type with respect to $b\Omega_1$, then S satisfies property (P_1). Indeed, since S is a compact set and by the assumption it consists of finite type points, Theorem 3.1.18 applies. As a result, S satisfies property (P_1). By symmetry, the same assumption can be made on $b\Omega_2$ as well.

b) More generally, if $S = F_1 \cup F_2$, where F_j denotes the set of points in S which are of finite type with respect to $b\Omega_j$, then S still has property (P_1). This can be seen as follows: one writes the set F_1 as countable union of compact sets each of which satisfies property (P_1). This is possible because we know that F_1 consists of finite type points and its compact subsets satisfy property (P_1) by Theorem 3.1.18. The remaining set in S, that is, $S \backslash F_1$ must be set of infinite type points in S with respect to $b\Omega_1$. However, this set is covered by F_2 and can also be written as the union of compact sets, again each of which satisfies property (P_1). The union of these two countable unions is again countable and it gives S, which is compact. Consequently, S satisfies property (P_1) by Lemma 3.1.5.

By what was listed above, there remains the case where S has a nonempty subset which consists of infinite type points with respect to both boundaries. Let $\mathcal{K}$ denote the set of points in S, which consists of boundary points of infinite type with respect to boundaries $b\Omega_1$ *and* $b\Omega_2$. We observe that $\mathcal{K}$ is closed (and hence compact) because S is closed and the sets of infinite type points in the boundaries are closed by D'Angelo's result (see [19]). Similar to the discussions above, we can write $S = \mathcal{K} \cup (S \backslash \mathcal{K})$. We can exhaust the set $S \backslash \mathcal{K}$ by compact subsets which satisfy property (P) (again by Theorem 3.1.18). Therefore, if $\mathcal{K}$ satisfies property (P), then invoking Lemma 3.1.5, we get that S satisfies property (P).

Sibony proved in [52] (Remarque on p. 310), when the set of infinite type points in the boundary of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ has two-dimensional Hausdorff measure 0, then the $\bar{\partial}$-Neumann operator is compact. After Sibony's remark, Boas built up an explicit construction in [5] and showed that if a subset of the set of infinite type points has two-dimensional Hausdorff measure 0 and if this set has a neighborhood which consists of finite type points only, then this set satisfies property (P). From Boas' proof, we see that he actually proves that a compact set which has 2-dimensional Hausdorff measure zero satisfies property (P_1). So, we can give some examples which take their assumptions in view of Sibony's and Boas' works:

i) If $\mathcal{K}$ is finite, then it has property (P_1), and as a consequence S has property (P_1).

ii) If $\mathcal{K}$ is a 1-dimensional (continuous) curve or is formed as a countable union of such curves, then it has property (P_1) (in view Lemma 3.1.5) and hence S has property (P_1).

iii) If $\mathcal{K}$ has 2-dimensional Hausdorff measure zero, then it has property (P_1). As a result, S has property (P_1).

We state the most general form of these examples (whose proof we already discussed) as a corollary to Theorem 4.1.2:

Corollary 4.1.5. *If $\mathcal{K}$ satisfies property (P_q), then N_q is compact. This happens, for example, for all q, when $\mathcal{K}$ has two-dimensional Hausdorff measure* 0*.*

Remark 4.1.6. *The assumption of Corollary 4.1.5 stimulates the following question: if $\mathcal{K}$ has property $(\tilde{P})$, then does it follow that S satisfies property $(\tilde{P})$? Although we don't know how to answer this question yet, the techniques in the proof of Theorem*

4.1.2 combined with the results obtained by Straube in [53] can be used to prove that N is compact.

The set of infinite type boundary points is necessarily contained in the set of weakly pseudoconvex boundary points. Moreover, a result by Şahutoğlu and Straube ([50]) says that the set of weakly pseudoconvex boundary points must have empty interior in the boundary topology if N_1 is compact. Therefore, when N_1 is compact, the set of infinite type points must have also empty interior. Thus, when S has nonempty interior in the subspace topology of (one of the) boundaries; then $\mathcal{K}$, because it has an empty interior, must be a proper subset of S. An example of when S has empty interior in any boundary topology is given by the transversal intersection of the boundaries.

4.1.1.2 An analysis of transversal intersections

Suppose that Ω_1 and Ω_2 have some boundary regularities and they intersect in the general position. More precisely, let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbb{C}^n$ with twice continuously differentiable boundaries which intersect each other real transversally. Suppose also that the $\bar{\partial}$-Neumann operators at the initial form levels are compact, i.e., $N_1^{\Omega_1}$ and $N_1^{\Omega_2}$ are compact. The real transversal intersection means that if ρ_1 and ρ_2 are defining functions for Ω_1 and Ω_2 respectively, then

$$d\rho_1(z) \wedge d\rho_2(z) \neq 0 \text{ when } z \in S, \tag{4.27}$$

where S is the common zero set of ρ_1 and ρ_2, i.e., $S := \{z \in \mathbb{C}^n : \rho_1(z) = 0 = \rho_2(z)\}$. With this assumption, the set S becomes a C^1-manifold with $\dim_{\mathbb{R}} S = 2n-2$. Since S has codimension 2 in $\mathbb{C}^n$, $H_p(S)$ (the complex tangent space to S at a point p) satisfies

$$2n - 4 \leq \dim_{\mathbb{R}} H_p(S) \leq 2n - 2. \tag{4.28}$$

Since the complex tangent spaces are even dimensional and also the manifold S is even dimensional, the case where $H_p(S)$ is equal to the whole tangent space $T_p(S)$ is of interest.

Definition 4.1.7. *A point $p \in S$ is called a complex tangent point if $H_p(S) = T_p(S)$. The set of complex tangent points in S is denoted by K. When $n = 2$, a point which is not complex tangent is called a totally real point.*

An immediate result is as follows (see Example 5 in [12] for $n = 2$ case):

Lemma 4.1.8. *The set K (as defined in Definition 4.1.7) is nowhere dense in S.*

Proof. If the set K were not nowhere dense in S, there would be an open subset of S which could be contained in K. Consisting of the points at which the tangent space is equal to the complex tangent space, this subset would be a complex manifold of complex dimension $n-1$. However, in $\mathbb{C}^n$, $(n-1)$-dimensional complex manifolds in the boundary are obstructions to the compactness of the $\bar{\partial}$-Neumann operator at the initial form level (see Proposition 3.2.1). Hence, K is a nowhere dense subset of S. □

Remark 4.1.9. *Note that the proof of the lemma did not use the compactness of $\bar{\partial}$-Neumann operator on both domains. To get such a result it suffices to assume that one of the domains has compact $\bar{\partial}$-Neumann operator and the other just be pseudoconvex with the same boundary regularities as in the lemma. More generally, as the proof reveals already, this is a specific case of a more general fact: if there are two domains in $\mathbb{C}^n$ with sufficiently regular boundaries and these boundaries intersect real transversally, then the set K is nowhere dense as long as $(n-1)$-dimensional complex manifolds in the boundary are obstructions to* some *property that one of the domains possesses.*

Suppose now that $n = 2$ and also that Ω_1 and Ω_2 have smooth boundaries. S is now a two real dimensional smooth submanifold in $\mathbb{C}^2$. Putting it another way, S is a real surface in the complex surface. Because the tangent space to S at a point is two-dimensional, the complex tangent space is either the whole tangent space or it is trivial.

Investigation of complex tangent points' behavior in a real surface or what properties exist for the real surface around the complex tangent points has been an area of intensive research since Bishop's foundational work [4]. Talking about such surfaces requires an introduction of terminology and in what follows, Chapter 9 of [23] is intensively used. From Bishop's work, we know that around any point p of S, there exist local holomorphic coordinates such that S can locally be parameterized by the graph $\{z_2 = f(z_1)\} \subset \mathbb{C}^2$. Here, f is a complex-valued smooth function in a domain of $\mathbb{C}$. Then $p = (a, f(a))$ is a complex tangent point of S if and only if $\frac{\partial f}{\partial \bar{z}_1}(a) = 0$. One can further assume that the point $p = (0, 0)$ and $T_{(0,0)}S = \{w = 0\}$ (equivalently $df_0 = 0$). If the second order Taylor polynomial of f does not identically vanish at the origin, then the complex tangent point is called non-degenerate. If furthermore $\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1}(0) \neq 0$, there exist local holomorphic coordinates at $(0, 0)$ in which S is given by

$$z_2 = |z_1|^2 + \lambda(z_1^2 + \bar{z}_1^2) + o(|z_1|^2) \tag{4.29}$$

for some $\lambda \geq 0$. The number λ is called the Bishop invariant. If $\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1}(0) = 0$ but the second order Taylor polynomial still does not vanish, then S is given near the origin by

$$z_2 = z_1^2 + \bar{z}_1^2 + o(|z_1|^2); \tag{4.30}$$

and this case corresponds to $\lambda = +\infty$ in (4.29). A non-degenerate complex tangent point is called elliptic, parabolic or hyperbolic if $\lambda \in [0, \frac{1}{2})$, $\lambda = \frac{1}{2}$ and $\lambda \in (\frac{1}{2}, \infty]$

respectively. A compact smooth real surface in $\mathbb{C}^2$ is called a Bishop surface if the complex tangent points are either elliptic or hyperbolic. We call a real surface non-degenerate if all of its complex points are non-degenerate; i.e., any complex point can be classified as of elliptic, parabolic or hyperbolic type.

A real surface which does not have any complex tangent points is called totally real and totally real surfaces are good sets for the compactness of N. Indeed, the square of the distance function to the totally real surface is twice continuously differentiable and strictly plurisubharmonic in a neighborhood of the totally real surface (see Lemma 17.2 in [1]). Therefore, totally real surfaces satisfy property (P_1). However, in our case, it is instructive to keep in mind that even in the transversal intersection of the boundaries of two balls, there are exactly two complex tangent points and these are of elliptic type ([4], see also [3]). Nevertheless, elliptic and hyperbolic points of a real two dimensional surface of class C^2 embedded in a complex surface are always isolated. Therefore, if S is a Bishop surface, then because S is compact, elliptic and hyperbolic points are finitely many. Since the set K is closed (and hence compact), we can write the totally real part of S as the countable union of compact sets each of which satisfies property (P_1). Therefore, in view of Lemma 3.1.5, S satisfies property (P_1) and hence property $(\tilde{P}_1)$. Thus, in case S is a Bishop surface, N_1^Ω is compact by Theorem 4.1.2.

Remark 4.1.10. *The observation that "if K satisfies property (P), then S satisfies property (P)" was made earlier in Chapter V of Çelik's dissertation [12] (see Example* 3*). He also listed some conditions in which K satisfies property (P) (Example* 4 *on p.* 58*). However, what is new here is the deduction that N is compact in these examples and there exist examples of manifolds S, such as Bishop surfaces, which satisfy the conditions listed in Example* 4 *of Çelik's dissertation.*

A smooth compact real surface in $\mathbb{C}^2$ is homotopic (by a generic continuous perturbation) to a compact real surface with isolated complex tangent points. However, if the surface is compact and the complex tangent points are isolated, then there are finitely many complex tangent points. Therefore, in the generic case, there are only finitely many complex tangent points. Because a finite set satisfies property (P), we have the following corollary :

Corollary 4.1.11. *If Ω_1 and Ω_2 are bounded smooth pseudoconvex domains in $\mathbb{C}^2$ which intersect real transversally and if $N_1^{\Omega_j}$, $j = 1, 2$ are compact, then "generically" N_1^{Ω} is compact.*

More generally, we have the following result:

Corollary 4.1.12. *Suppose that Ω_1 and Ω_2 are bounded smooth pseudoconvex domains in $\mathbb{C}^2$ which intersect each other real transversally. Suppose also that $N_1^{\Omega_1}$ and $N_1^{\Omega_2}$ are compact. If S is a non-degenerate surface, then N_1^{Ω} is compact.*

Proof. We have observed already that in case there are finitely many complex tangent points we have the compactness of N_1. In case there are countably many complex tangent points, then Lemma 3.1.5 applies (note that K is compact). Therefore, we should consider the case where there are uncountably many complex tangent points. We first observe that because the surface is non-degenerate, the set of parabolic points is closed and hence compact. Indeed, any limit of a sequence consisting of the parabolic points must be again parabolic because elliptic and hyperbolic points are isolated.

Recall that S can be locally represented after a holomorphic change of coordinates by $\{z_2 = f(z_1)\} \subset \mathbb{C}^2$, where f is a smooth function defined on a domain D near the origin. Consider the local representations around each parabolic point so that the set of parabolic points is contained in finitely many of them. We claim that we can show inside each such local representation, the complex tangent points are contained in a C^1-smooth curve and therefore has property (P_1). Since property (P_1) is invariant under holomorphic change of coordinates, we obtain that the set of parabolic points satisfies property (P_1). But the remaining complex points will be isolated and because they are isolated, they will be finitely many. Therefore, the set of complex tangent points satisfies property (P_1).

Now, we are ready to prove our claim and to do this we are using an idea that is contained in [51]. Since S is non-degenerate by our assumption, by (4.29), we have

$$f(z_1) = |z_1|^2 + \lambda(z_1^2 + \bar{z}_1^2) + o(|z_1|^2), \quad z_1 \in D. \tag{4.31}$$

Elliptic and hyperbolic points are isolated; so we can look at the case of a parabolic point; i.e., $\lambda = \frac{1}{2}$. In this case, letting $z_1 = x + iy$, we obtain from (4.31) that $\frac{\partial f}{\partial \bar{z}_1} = 2x + o(|z_1|)$. By implicit function theorem, we get that $\sigma = \{z_1 \in D : \mathrm{Re}\frac{\partial f}{\partial \bar{z}_1}(z_1) = 0\}$ is a C^1 smooth curve and locally the set of complex tangent points is given by $K_{loc} = \{(z_1, f(z_1)) : z \in \sigma, \mathrm{Im}\frac{\partial f}{\partial \bar{z}_1}(z_1) = 0\}$ and hence a closed subset of a C^1 smooth curve. A C^1 smooth curve has 2-dimensional Hausdorff measure zero (see also Example 4(b.) in [12] where the totally realness of such a curve is discussed); therefore K_{loc} has property (P). So, K has property (P). In view of Remark 4.1.10 or the paragraph preceding, S satisfies property (P) and so Theorem 4.1.2 applies. □

4.2 A result on the transversal intersection case

In this part, we assume that Ω_1 and Ω_2 are bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries which intersect real transversally. We carry the notation from previous parts: Ω is the intersection of Ω_1 and Ω_2, $b\Omega_- := b\Omega_2 \cap \Omega_1$, $b\Omega_+ :=$

$b\Omega_1 \cap \Omega_2$ and $S := b\Omega_1 \cap b\Omega_2$.

In our main result (Theorem 4.2.3), we will assume the existence of a function χ defined on the union of Ω_1 and Ω_2. χ will be a nonnegative smooth function in $\Omega_1 \cup \Omega_2$ such that $\chi \equiv 1$ on an open neighborhood of $\Omega_1 \backslash \Omega_2$ (in Ω_1-topology), $\chi \equiv 0$ on an open neighborhood of $\Omega_2 \backslash \Omega_1$ (in Ω_2-topology) and it will be bounded above by 1 in the remaining region which lies in Ω. Observe that by what was already said, χ is not smooth up the boundary. It has a sharp singularity on S and the support of its gradient is a proper subdomain of Ω whose boundary contains also S. With our set up, χ is an $\mathcal{L}^2$-function, but its gradient is not square integrable. Therefore, $\bar{\partial}\chi$ is not in $\mathcal{L}^2_{(0,1)}(\Omega_1 \cup \Omega_2)$; and hence is not in $\text{dom}(\bar{\partial})$. However, despite the fact that it lacks certain nice properties already mentioned, such a function χ will play a crucial role in the proof of Theorem 4.2.3. We will show first that such a function χ exists when Ω_1 and Ω_2 have smooth boundaries and intersect real transversally.

Lemma 4.2.1. *Let Ω_1 and Ω_2 be bounded smooth pseudoconvex domains in $\mathbb{C}^n$ whose boundaries intersect real transversally and whose intersection is a domain Ω. Then, there exists a nonnegative smooth function χ defined in $\Omega_1 \cup \Omega_2$ such that χ is bounded above by 1, $\chi \equiv 1$ on an open neighborhood of $\Omega_1 \backslash \Omega_2$ (in Ω_1-topology) and $\chi \equiv 0$ on an open neighborhood of $\Omega_2 \backslash \Omega_1$ (in Ω_2-topology). Moreover, if S denotes the set $b\Omega_1 \cap b\Omega_2$ and $\delta_S(z)$ denotes the distance of a point $z \in \mathbb{C}^n$ to S, then there exists a conic region in Ω near S on which $\delta_S \nabla \chi$ is bounded.*

Proof. Let ρ_1 and ρ_2 be defining functions of $b\Omega_1$ and $b\Omega_2$ respectively. Without loss of generality, we may assume that the gradients of the defining functions are normalized on the corresponding boundaries. The real transversal intersection assumption means that $d\rho_1(z) \wedge d\rho_2(z) \neq 0$ when $z \in S$ and this is equivalent to say that the gradients of the defining functions must be linearly independent when evaluated at the same points of S. On the other hand, because $b\Omega_1$ and $b\Omega_2$ intersect real transversally, S is a smooth manifold of real dimension $2n-2$. At a point p of S, the normal space to S at p, which is defined as the orthogonal complement of the tangent space T_pS in $\mathbb{C}^n$, is a linear space of real dimension 2 and spanned by $\{\nabla\rho_1(p), \nabla\rho_2(p)\}$ as these vectors are linearly independent. Therefore, if $\mathbb{D}_r$ is a plane disc centered at the origin with some sufficiently small radius $r > 0$, then the map sending $p \in S$ and $(x, y) \in \mathbb{D}_r$ to $p + x\nabla\rho_1(p) + y\nabla\rho_2(p)$ is a diffeomorphism of $S \times \mathbb{D}_r$ onto a tubular neighborhood U of S. We denote this diffeomorphism by H.

Having prepared a geometric setup around S, we now start constructing the function χ. Our first observation is that it suffices to construct the desired function χ on $U \cap \Omega$. Note that we take U small enough so that $U \cap \Omega$ is a proper subset of Ω. If such a function χ exists on $U \cap \Omega$, then there exists a conic region $\mathcal{C}$ in Ω whose boundary contains S and which separates $U \cap \Omega$ into three disjoint regions. These regions will be $\mathcal{C}$ itself on which $\delta_S \nabla\chi$ is bounded, an open set (say $\tilde{V}_1$) on which χ is identically equal to 1 and another open set (say $\tilde{V}_2$) on which χ is identically equal to 0. We can first take a proper subdomain of Ω, say $\tilde{\Omega}$, so that $\mathcal{C} \cap \tilde{\Omega} = \mathcal{C}$ and $b\tilde{\Omega} \cap b\Omega = S$. That is, we extend the conic region $\mathcal{C}$ in Ω to be a proper subdomain $\tilde{\Omega}$ of Ω so that $\tilde{\Omega}$ is same as $\mathcal{C}$ inside $U \cap \Omega$ and the boundary points of $\tilde{\Omega}$ which are not contained in $\overline{U}$ stay away from the boundary portions $b\Omega_-$ and $b\Omega_+$. The boundary of $\tilde{\Omega}$, similar to what the conic neighborhood $\mathcal{C}$ does to $U \cap \Omega$, will separate Ω into three disjoint regions. These regions will be $\tilde{\Omega}$ itself, an open set (say V_1) whose boundary has a portion common with $b\Omega_-$ and another open set (say V_2) whose boundary has a portion common with $b\Omega_+$. That is, as $\tilde{\Omega}$ was an extension of $\mathcal{C}$ in Ω, V_1 and V_2 are extensions of $\tilde{V}_1$ and $\tilde{V}_2$ in Ω respectively. Let $U_1 = V_1 \cup (\overline{\Omega_1 \backslash \Omega})$, where we take the closure in the Ω_1-topology and $U_2 = V_2 \cup (\overline{\Omega_2 \backslash \Omega})$, where we take the closure in the Ω_2-topology. We can extend the function χ to be identically equal to 1 on U_1 and to be identically equal to 0 on U_2. Thus, we obtain a smooth function on an open subset $A = U_1 \cup U_2 \cup \mathcal{C}$ of $\Omega_1 \cup \Omega_2$, which satisfies all the properties we desire apart from the fact that it is not defined on the whole union $\Omega_1 \cup \Omega_2$. However, smooth version of Urysohn's lemma applies: if B is a relatively compact subset of $(\Omega_1 \cup \Omega_2)\backslash\overline{A}$, the closure being taken in $\Omega_1 \cup \Omega_2$ topology, then there exists a smooth (Urysohn) function on $\Omega_1 \cup \Omega_2$ which is identically equal to 1 on A and 0 on B. This smooth (Urysohn) function when multiplied by the extended χ gives the desired smooth function on $\Omega_1 \cup \Omega_2$ which satisfies the properties of the function we want to construct.

By what was discussed above, we will construct the desired function on $U \cap \Omega$. Recall that the gradients of the defining functions are normalized and they are linearly independent by the transversal intersection. Therefore, on S, we have $|\langle \nabla\rho_1, \nabla\rho_2 \rangle| < 1$. Thus, we have

$$\langle \nabla\rho_1 + \nabla\rho_2, \nabla\rho_1 \rangle = 1 + \langle \nabla\rho_2, \nabla\rho_1 \rangle > 0 \tag{4.32}$$

and

$$\langle \nabla\rho_1 + \nabla\rho_2, \nabla\rho_2 \rangle = \langle \nabla\rho_1, \nabla\rho_2 \rangle + 1 > 0. \tag{4.33}$$

Inequalities (4.32) and (4.33) give that for each point p of S and $0 < t < \frac{r}{\sqrt{2}}$, $p + t\nabla\rho_1(p) + t\nabla\rho_2(p)$ is a point outside of $\Omega_1 \cup \Omega_2$ and $p - t\nabla\rho_1(p) - t\nabla\rho_2(p)$ is a point inside of Ω. So, for a fixed point p of S, we can find a sector in the first quadrant whose main axis bisecting its subtended angle is the line $y = x$ and whose image under H (when p is fixed) is contained outside of $\Omega_1 \cup \Omega_2$. Similarly, there is a sector in the third quadrant whose main axis bisecting its subtended angle is the line $y = x$ and whose image under H (when p is fixed) is contained inside Ω. By shrinking one of the subtended angles if necessary, we may assume without loss of generality that these sectors are symmetric with respect to the origin having same subtended angles. Moreover, because S is compact and boundaries intersect transversally, the subtended angles of the sectors can be taken same for all points of S. We take these angles to be 2α for some $\alpha > 0$ and let S_α and $\tilde{S}_\alpha$ denote the sectors that lie within the third quadrant and the first quadrant respectively.

To construct the function χ on $U \cap \Omega$, we will first construct a smooth function on $\mathbb{D}_r$ which is identically equal to 1 on one of the two regions that lie between the sectors S_α and $\tilde{S}_\alpha$ and which is identically equal to 0 on the other remaining region. Moreover, this smooth function will decrease from 1 to 0 on S_α . To find such a function, we will need a modified version of the argument function. Recall that the usual argument function arg takes values in $[-\pi, \pi)$ and for $(x+iy) \in \mathbb{C}\backslash\{0\}$, it is defined by $\arg(x+iy) = \arctan(\frac{y}{x})$. Note that arg is smooth on the slit plane where the slit is taken to be nonpositive real axis. We now define a new argument function A on $\mathbb{C}\backslash\{0\}$ by taking $A(x+iy) = \arg(e^{\frac{3\pi}{4}}(x+iy))$ and note that A is smooth everywhere on $\mathbb{C}\backslash\{t+it : t \geq 0\}$ with $|\nabla A(x+iy)| = \frac{1}{\sqrt{x^2+y^2}}$. Let χ_α be a nonnegative smooth function on $\mathbb{R}$ which is bounded above by 1, identically equal to 1 on $(-\infty, -\alpha]$, identically equal to 0 on $[\alpha, \infty)$ and strictly decreasing on $(-\alpha, \alpha)$. We have $|\chi_\alpha'| \leq \frac{C}{\alpha}$, where C is a constant independent of α. Let $\tilde{\chi}_\alpha$ be the function defined on $\mathbb{D}_r\backslash\{0\}$ by $\tilde{\chi}_\alpha(x+iy) = \chi(A(x+iy))$. By its definition, $\tilde{\chi}_\alpha$ is smooth on $\mathbb{D}_r\backslash\{t+it : t \in [0, \frac{r}{\sqrt{2}}]\}$. Also, by the way we constructed the functions χ_α and A, $\tilde{\chi}_\alpha$ is identically equal to 1 on the region between S_α and $\tilde{S}_\alpha$ which nontrivially intersects the second quadrant and identically equal to 0 on the remaining region between S_α and $\tilde{S}_\alpha$ which nontrivially intersects the fourth quadrant. Furthermore, by the chain rule, the gradient of $\tilde{\chi}_\alpha$ at a point $x+iy$ is bounded by $\frac{C_\alpha}{\sqrt{x^2+y^2}}$, where C_α is a constant that depends only on α.

We are now ready to define χ. For $w \in U \cap \Omega$, let $p_w \in S$ and $u_w + iv_w \in \mathbb{D}_r$ so that $H(p_w, u_w + iv_w) = w$. We define χ at the point w by setting $\chi(w) = \tilde{\chi}_\alpha(u_w + iv_w)$. By what was discussed, $\chi(w)$ is a smooth function on $U \cap \Omega$; it is zero 0 or 1 depending on which side of $\mathcal{C}$ it belongs to. Moreover, for $w \in \mathcal{C}$, we have

$$|\delta_S(w)\nabla\chi(w)| \leq \sqrt{u_w^2 + v_w^2}\frac{C_{H,\alpha}}{\sqrt{u_w^2 + v_w^2}}.$$

Here, $C_{H,\alpha}$ is a constant that depends on a bound on the determinant of Jacobian of H and the angle α. Therefore, $C_{H,\alpha}$ is independent of w and this finishes the proof of Lemma 4.2.1. □

As stated before Lemma 4.2.1, the function χ will play an important role in proving Theorem 4.2.3.

Lemma 4.2.2. *Let Ω_1 and Ω_2 be bounded smooth pseudoconvex domains in $\mathbb{C}^n$ whose boundaries intersect real transversally and which form a domain Ω. Let χ be the smooth function in $\Omega_1 \cup \Omega_2$ as in Lemma 4.2.1 and let $1 \leq q \leq n-1$. Then, for any $\alpha \in \ker(\bar{\partial}_q)$, we have $\bar{\partial}\chi \wedge \alpha \in W^{-1}_{(0,q+1)}(\Omega_1 \cup \Omega_2)$.*

Proof. By definition of χ, the $(0,1)$-form $\bar{\partial}\chi$ has a support contained in Ω and the boundary of its support contains S. However, $\bar{\partial}\chi = 0$ on $b\Omega_-$ and $b\Omega_+$. Therefore, we can extend $\bar{\partial}\chi \wedge \alpha$ to $\Omega_1 \cup \Omega_2$ by setting it to be zero componentwise on $(\Omega_1 \cup \Omega_2)\backslash\Omega$. We need to show that $\bar{\partial}\chi \wedge \alpha$ is a linear functional on $W^1_{0,(0,q+1)}(\Omega_1 \cup \Omega_2)$. Linearity being obvious, it suffices to check that the $\mathcal{L}^2(\Omega_1 \cup \Omega_2)$-pairing between $\bar{\partial}\chi \wedge \alpha$ and a compactly supported smooth $(0, q+1)$-form ϕ on $\Omega_1 \cup \Omega_2$ is bounded by some constant depending on χ times $\mathcal{L}^2$-norm of α on Ω times Sobolev 1-norm of ϕ.

Indeed, if $\alpha = \sum'_{|J|=q} \alpha_J d\bar{z}_J \in \ker(\bar{\partial}_q)$, we have

$$\bar{\partial}\chi \wedge \alpha = \left(\sum_{j=1}^{n} \frac{\partial \chi}{\partial \bar{z}_j} d\bar{z}_j \right) \wedge \left(\sum_{|J|=q}' \alpha_J d\bar{z}_J \right) = \sum_{|K|=q+1}' \beta_K d\bar{z}_K,$$

where

$$\beta_K = \sum_{\substack{|J|=q \\ \{j\}\cup J = K}}' \varepsilon_K^{jJ} \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J.$$

Here, $\varepsilon_K^{jJ} = \pm 1$ depending on the permutation that makes $\{j\} \cup J$ equal to K and $\beta_K = 0$ at a point where α is not defined. So, if $\phi = \sum'_{|K|=q+1} \phi_K d\bar{z}_K \in C^{\infty}_{0,(0,q+1)}(\Omega_1 \cup \Omega_2)$, we then have

$$\begin{aligned} \left| (\bar{\partial}\chi \wedge \alpha, \phi)_{\mathcal{L}^2_{(0,q+1)}(\Omega_1 \cup \Omega_2)} \right| &= \left| \sum_{|K|=q+1}' \int_{\Omega_1 \cup \Omega_2} \beta_K \bar{\phi}_K dV \right| \leq \sum_{|K|=q+1}' \int_{\Omega_1 \cup \Omega_2} |\beta_K \bar{\phi}_K| dV \\ &\leq \sum_{|K|=q+1}' \sum_{\substack{|J|=q \\ \{j\}\cup J = K}}' \int_{\Omega} \left| \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J \bar{\phi}_K \right| dV. \end{aligned}$$

Therefore, it suffices to estimate the integrals of the form $\int_{\Omega} \left| \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J \bar{\phi}_K \right| dV$ by some constant (depending on χ and α) times the Sobolev 1-norm of ϕ on $\Omega_1 \cup \Omega_2$. We fix j, J and K for the moment.

Let $\delta(z) = \delta_{b(\Omega_1 \cup \Omega_2)}(z)$ denote the distance of a point $z \in \mathbb{C}^n$ to $b(\Omega_1 \cup \Omega_2)$. Since $\delta(z) > 0$ on Ω, we can write

$$\int_{\Omega} \left| \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J \bar{\phi}_K \right| dV = \int_{\Omega} \delta \left| \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J \right| \frac{|\phi_K|}{\delta} dV. \tag{4.34}$$

Recall that χ is a smooth function whose gradient when multiplied by δ_S (distance to the manifold S) is bounded in a conic neighborhood of S. S is a subset of $b(\Omega_1 \cup \Omega_2)$;

so, we have $\delta \leq \delta_S$. Therefore, the gradient of χ when multiplied by δ is also bounded in the conic region. Furthermore, away from the conic neighborhood of S, $\delta \frac{\partial \chi}{\partial \bar{z}_j}$ is bounded by a constant depending on χ and the diameter of $\Omega_1 \cup \Omega_2$. So, as a result $\delta \frac{\partial \chi}{\partial \bar{z}_j}$ is bounded on Ω; and hence

$$\int_\Omega \delta^2 \left| \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J \right|^2 dV \leq C(\chi, \Omega) ||\alpha_J||^2_{\mathcal{L}^2(\Omega)}. \tag{4.35}$$

On the other hand, the function $\frac{|\phi_K|}{\delta}$ is in $\mathcal{L}^2(\Omega_1 \cup \Omega_2)$ and its norm is bounded by some constant times the Sobolev 1-norm of ϕ_K on $\Omega_1 \cup \Omega_2$. Indeed, a result by Boas and Straube (see Proposition on p. 174 of [6] with $\alpha = 1$ in their notation) states that if D is a bounded domain in $\mathbb{R}^m$ whose boundary is locally the graph of a Lipschitz function, then for $1 < p < \infty$ and $u \in C_0^\infty(D)$, we have

$$||\delta_{bD}^{-\varepsilon - \frac{1}{p}} u||_{\mathcal{L}^p(D)} \leq C ||\delta_{bD}^{1-\varepsilon-\frac{1}{p}} \nabla u||_{\mathcal{L}^p(D)} \text{ whenever } 0 < \varepsilon \leq 1 - \frac{1}{p}. \tag{4.36}$$

Note that the domains Ω_1 and Ω_1 are bounded and have smooth boundaries which intersect real transversally. Therefore, the assumption on the boundary in Boas-Straube result is satisfied when $D = \Omega_1 \cup \Omega_2$. Letting $u = \phi_K$, $p = 2$ and $\varepsilon = \frac{1}{2}$ in (4.36), we obtain

$$||\delta^{-1}\phi_K||_{\mathcal{L}^2(\Omega)} \leq ||\delta^{-1}\phi_K||_{\mathcal{L}^2(\Omega_1 \cup \Omega_2)} \leq C||\nabla \phi_K||_{\mathcal{L}^2(\Omega_1 \cup \Omega_2)} \leq C||\phi_K||_{W^1(\Omega_1 \cup \Omega_2)}. \tag{4.37}$$

Now, applying Cauchy-Schwarz inequality in (4.34) to the functions $\delta \left| \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J \right|$ and $\frac{|\phi_K|}{\delta}$ and using inequalities (4.35), (4.37), we obtain the desired estimate for the integrals

$$\int_\Omega \left| \frac{\partial \chi}{\partial \bar{z}_j} \alpha_J \bar{\phi}_K \right| dV.$$

Since each of these estimates is independent of j, J and K, summing up over all possible j and strictly increasing tuples J, K, we obtain

$$\left|(\bar{\partial}\chi \wedge \alpha, \phi)_{\mathcal{L}^2_{(0,2)}(\Omega_1 \cup \Omega_2)}\right| \leq C(\chi)||\alpha||_{\mathcal{L}^2_{0,q}(\Omega)}||\phi||_{W^1(\Omega_1 \cup \Omega_2)} \tag{4.38}$$

proving that $\bar{\partial}\chi \wedge \alpha$ is indeed in $W^{-1}_{(0,q+1)}(\Omega_1 \cup \Omega_2)$ with $||\bar{\partial}\chi \wedge \alpha||_{-1,\Omega_1 \cup \Omega_2}$ bounded by some constant (depending on χ) times the $\mathcal{L}^2(\Omega)$-norm of α. This completes the proof of Lemma 4.2.2. □

Theorem 4.2.3. *Let Ω_1 and Ω_2 be smooth bounded pseudoconvex domains in $\mathbb{C}^n$ which intersect each other real transversally and form a domain Ω. If the $\bar{\partial}$-Neumann operators $N^{\Omega_1}_{q_1}$ and $N^{\Omega_2}_{q_2}$ are compact for some $1 \leq q_1, q_2 \leq n-1$, then the $\bar{\partial}$-Neumann operator N^{Ω}_{n-1} is compact.*

Remark 4.2.4. *Theorem 4.2.3 gives the solution of the problem at the form level $n-1$ when domains are smooth and intersect real transversally. In particular, when $n = 2$, the problem is solved under smooth boundary and transversal intersection assumptions.*

Proof. In view of Lemma 3.0.11, it suffices to find a compact solution operator for $\bar{\partial}$ on $(0, n-1)$-forms. That is, we need to find a linear compact operator T : $\mathcal{L}^2_{(0,n-1)}(\Omega) \cap \ker(\bar{\partial}_{n-1}) \to \mathcal{L}^2_{(0,n-2)}(\Omega)$ such that $\bar{\partial}_{n-2}Tu = u$ for all $u \in \ker(\bar{\partial}_{n-1})$.

We recall that on a bounded domain D of $\mathbb{R}^m$, the Laplace operator $\triangle$ defines an isomorphism from $W^1_0(D)$ onto $W^{-1}(D)$ (see Theorem 23.1 in [59] or Proposition 1.1 in Chapter 5 of [58]). Set $D := \Omega_1 \cup \Omega_2$ and denote by $\triangle^{-1}$ (uniquely defined) inverse of the Laplacian on D. Then, $\triangle^{-1}$ maps $W^{-1}(D)$ onto $W^1_0(D)$. Let χ be as in Lemma 4.2.2 and $\alpha \in \ker(\bar{\partial}_{n-1}) \subset \mathcal{L}^2_{(0,n-1)}(\Omega)$ be arbitrary. We define a $(0, n-1)$-form γ on

D by setting

$$\gamma = -4(\bar{\partial}^*_D \triangle^{-1}(-\bar{\partial}\chi \wedge \alpha)), \tag{4.39}$$

where $\triangle^{-1}$ acts to the unique component of the $(0, n)$-form $-\bar{\partial}\chi \wedge \alpha$. Observe that γ is well-defined. By Lemma 4.2.2, $\bar{\partial}\chi \wedge \alpha \in W^{-1}_{(0,q+1)}(D)$ and by what was said above, $\triangle^{-1}(-\bar{\partial}\chi \wedge \alpha) \in W^1_0(D) \subset \mathrm{dom}(\bar{\partial}^*_D)$. Moreover, because $\bar{\partial}^*$ is a differential operator of order 1, we have $\gamma \in \mathcal{L}^2_{(0,n-1)}(D)$.

Recall from (2.6) that, for a $(0, n)$-form $u = u_{(12\cdots n)} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$, we have

$$\bar{\partial}_{n-1}\bar{\partial}^*_{n-1} u = \left[-\frac{1}{4} \triangle u_{(12\cdots n)} \right] d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

Therefore, by our construction of γ, we obtain

$$\bar{\partial}_{n-1}\gamma = \triangle\triangle^{-1}(-\bar{\partial}\chi \wedge \alpha) = -\bar{\partial}\chi \wedge \alpha \quad \text{on } D. \tag{4.40}$$

On the other hand, extending by 0 componentwise off their supports, $(1-\chi)\alpha$ and $\chi\alpha$ become well-defined $(0, n-1)$-forms on Ω_1 and Ω_2. Now, we let $\beta_1 := (1-\chi)\alpha - \gamma$ on Ω_1 and $\beta_2 := \chi\alpha + \gamma$ on Ω_2 so that

$$\alpha = \beta_1|_\Omega + \beta_2|_\Omega. \tag{4.41}$$

Moreover, for $j = 1, 2$, we have $\beta_j \in \ker(\bar{\partial}^{\Omega_j}_{n-1})$. Indeed, by (4.40) and the fact that $\alpha \in \ker(\bar{\partial})$, we have

$$\bar{\partial}\beta_1 = \bar{\partial}((1-\chi)\alpha) - \bar{\partial}\gamma = -\bar{\partial}\chi \wedge \alpha + (1-\chi)\bar{\partial}\alpha - (-\bar{\partial}\chi \wedge \alpha) = 0;$$

and similarly,

$$\bar{\partial}\beta_2 = \bar{\partial}(\chi\alpha) + \bar{\partial}\gamma = \bar{\partial}\chi \wedge \alpha + \chi\bar{\partial}\alpha - \bar{\partial}\chi \wedge \alpha = 0.$$

Since $\beta_1 \in \ker(\bar{\partial}_{n-1}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega_1)$ and $N^{\Omega_1}_{n-1}$ is compact by our hypothesis, in view of Lemma 3.0.11, there exists a linear compact operator $T_1 : \ker(\bar{\partial}_{n-1}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega_1) \to \mathcal{L}^2_{(0,n-2)}(\Omega_1)$ such that

$$\bar{\partial}_{\Omega_1} T_1 \beta_1 = \beta_1.$$

Similarly, there exists a linear compact operator $T_2 : \ker(\bar{\partial}_{n-1}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega_2) \to \mathcal{L}^2_{(0,n-2)}(\Omega_2)$ such that

$$\bar{\partial}_{\Omega_2} T_2 \beta_2 = \beta_2.$$

For $j = 1, 2$, restriction operators $R_j : \mathcal{L}^2_{(0,n-2)}(\Omega_j) \to \mathcal{L}^2_{(0,n-2)}(\Omega)$ defined by $R_j u = u|_{\Omega_j}$ are linear and bounded (as $\Omega \subset \Omega_j$). Therefore, the composition $R_j T_j$ is linear and compact. Moreover, a form which is in $\operatorname{dom}(\bar{\partial}) \subset \mathcal{L}^2_{(0,n-2)}(\Omega_j)$, when restricted to Ω, remains in $\operatorname{dom}(\bar{\partial}) \subset \mathcal{L}^2_{(0,n-2)}(\Omega)$. Thus, for $j = 1, 2$,

$$\beta_j|_\Omega = \left(\bar{\partial}_{\Omega_j} T_j \beta_j\right)|_\Omega = \bar{\partial}_\Omega (R_j T_j \beta_j).$$

So, from (4.41) we obtain

$$\alpha = \beta_1|_\Omega + \beta_2|_\Omega = \bar{\partial}_\Omega (R_1 T_1 \beta_1 + R_2 T_2 \beta_2).$$

Therefore, if we can show that the linear operators $S_j : \ker(\bar{\partial}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega) \to \mathcal{L}^2_{(0,n-1)}(\Omega_j)$ defined by $S_j \alpha = \beta_j$ are bounded, then $R_1 T_1 S_1 + R_2 T_2 S_2$ will be our compact solution operator. Without loss of generality, we will show S_2 is bounded.

Observe that

$$\begin{aligned}||\beta_2||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)} &= ||\chi\alpha - 4(\bar{\partial}^*_D\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha))||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)} \\ &\leq 2||\chi\alpha||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)} + 8||\bar{\partial}^*_D\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)} \\ &\leq 2||\alpha||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega)} + 8||\bar{\partial}^*_D\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)}.\end{aligned}$$

So, it suffices to estimate the norm of $||\bar{\partial}^*_D\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)}$. This norm is less than or equal to the norm over the union. So, we get

$$\begin{aligned}||\bar{\partial}^*_D\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega_2)} &\leq ||\bar{\partial}^*_D\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)||^2_{\mathcal{L}^2_{(0,n-1)}(D)} \\ &= \frac{1}{4}\left(\bar{\partial}\chi\wedge\alpha, \triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)\right)_D.\end{aligned}$$

However, note that $\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)\in W^1_{0,(0,n)}(D)$ and $\bar{\partial}\chi\wedge\alpha\in W^{-1}_{(0,n)}(D)$. Therefore, the pairing we have is estimated by

$$\left(\bar{\partial}\chi\wedge\alpha, \triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)\right)_D \leq ||\bar{\partial}\chi\wedge\alpha||_{-1,D}||\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)||_{1,D}.$$

But Sobolev 1-norm of a form whose components belong to $W^1_0(D)$ is controlled by the Sobolev -1 norm of its Laplacian. Therefore, what we get is

$$||\triangle^{-1}(-\bar{\partial}\chi\wedge\alpha)||^2_{W^1_{0,(0,n-1)}(D)} \leq C||\bar{\partial}\chi\wedge\alpha||^2_{-1,D}.$$

The proof of Lemma 4.2.2 gives that there exists a constant C_χ such that $||\bar{\partial}\chi\wedge\alpha||^2_{-1,D}\leq C_\chi||\alpha||^2_{\mathcal{L}^2_{(0,n-1)}(\Omega)}$. Thus, we have shown that S_2 defined by sending α to β_2 is a bounded linear operator. Similarly, S_1 is a bounded operator. Hence, there exists a linear compact operator

$$T := R_1T_1S_1 + R_2T_2S_2 : \mathcal{L}^2_{(0,n-1)}(\Omega) \cap \ker(\bar{\partial}_{n-1}) \to \mathcal{L}^2_{(0,n-2)}(\Omega) \tag{4.42}$$

such that $\bar{\partial}T\alpha = \alpha$ whenever $\alpha \in \ker(\bar{\partial}_{n-1}) \cap \mathcal{L}^2_{(0,n-1)}(\Omega)$ as desired. This finishes the proof of Theorem 4.2.3. $\square$

4.3 Vanishing of sufficiently smooth forms

When the boundaries of the intersecting domains are assumed to be sufficiently smooth and also assumed to intersect real transversally, we can obtain some by-product results about the forms in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$ with sufficiently smooth components. In this last part of the section, it is our purpose to exhibit these interesting results. To this end, let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbb{C}^n$ with C^2 boundaries which intersect (real) transversally. As before, we denote by S the intersection of the boundaries. Recall from Lemma 4.1.8 that the set of complex tangent points is a nowhere dense subset of S. An analogous result is as follows:

Lemma 4.3.1. *Let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbb{C}^n$ with C^2 boundaries which intersect (real) transversally. If one of the $\bar{\partial}$-Neumann operators $N_1^{\Omega_1}$ and $N_1^{\Omega_2}$ is compact, then the set of points in S at which the vectors $\partial\rho_1$ and $\partial\rho_2$ are linearly dependent is a nowhere dense subset of S. That is, the set*

$$\tilde{K} := \{p \in S | \exists a_p \in \mathbb{C}\backslash\{0\} \text{ such that } \frac{\partial\rho_1}{\partial z_j}(p) = a_p\frac{\partial\rho_2}{\partial z_j}(p) \quad \forall j = 1, \cdots, n\} \tag{4.43}$$

is a nowhere dense subset of S.

Proof. The set $\tilde{K}$ consists of those points $p \in S$ at which the matrix $(\partial\rho_1(p), \partial\rho_2(p))$ has rank 1. If there is a point $p \in \tilde{K}$ such that there is an open neighborhood U_p in

S of p on which the matrix $(\partial\rho_1(p), \partial\rho_2(p))$ has constant rank 1, then the tangent spaces at each point of U_p are invariant under multiplying by complex numbers. This means that each point of U_p also belongs to the set K of complex tangent points in S (see Definition 4.1.7). But U_p is an open set in S and we know from Lemma 4.1.8 that K cannot accept any open subsets. This is a contradiction. Hence, $\tilde{K}$ is a nowhere dense subset of S. □

Lemma 4.3.2. *When $n \geq 2$, the forms in $C^2_{(0,n-1)}(\overline{\Omega}) \cap dom(\bar{\partial}^*_{(n-2)}) \subset \mathcal{L}^2_{(0,n-1)}(\Omega)$ vanish on S.*

Proof. Let $u \in C^2_{(0,n-1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*_{(n-2)})$. We can represent u on the boundary locally by a special boundary chart. For $j = 1, 2$, let $\omega_{1,j}, \cdots, \omega_{n-1,j}, \omega_{n,j} = \partial\rho_j$ be a special boundary chart on $b\Omega_j$. Since $u \in \mathrm{dom}(\bar{\partial}^*_{(n-2)})$, using the special boundary charts we can write $u = u_2\bar{\omega}_{1,2} \wedge \cdots \wedge \bar{\omega}_{n-1,2}$ on $b\Omega_2 \cap \Omega_1$ (see (2.5)). Similarly, we can write $u = u_1\bar{\omega}_{1,1} \wedge \cdots \wedge \bar{\omega}_{n-1,1}$ on $b\Omega_1 \cap \Omega_2$. By continuity these representations continue to hold on S. That is, we have

$$u_1(z)(\bar{\omega}_{1,1} \wedge \cdots \wedge \bar{\omega}_{n-1,1})(z) = u_2(z)(\bar{\omega}_{1,2} \wedge \cdots \wedge \bar{\omega}_{n-1,2})(z) \quad z \in S. \tag{4.44}$$

On the other hand, for $j = 1, 2$, there exist nonzero constants $a_j \in \mathbb{C}$ such that $\bar{\omega}_{1,j} \wedge \cdots \wedge \bar{\omega}_{n-1,j} \wedge dz_1 \wedge \cdots \wedge dz_n = a_j \star (\omega_{n,j})$, where $\star$ is the $\mathbb{C}$-linear Hodge-star operator in $\mathbb{C}^n$ (see Lemma 3.3 and Corollary 3.5 in Chapter III of [47] for the exact statements and the Appendix in [45] for a similar application). Therefore, if $z \in S$, then the equality in (4.44) becomes

$$\star(a_1 u_1(z)\omega_{n,1}(z)) = \star(a_2 u_2(z)\omega_{n,2}(z)). \tag{4.45}$$

However, if $z \in S\backslash\tilde{K}$, where $\tilde{K}$ is defined as in Lemma 4.3.1, then this equality holds if and only if $u_1(z) = 0 = u_2(z)$ (recall that a_j's are nonzero constants). Thus, u_1 and u_2 vanish on $S \backslash \tilde{K}$. Since $\tilde{K}$ is nowhere dense in S by Lemma 4.3.1, then by continuity u_1 and u_2 vanish on S. Therefore, u vanishes on S. □

When $n = 2$, we can avoid using Lemma 3.5 and Corollary 3.5 of [47] and give a more direct proof of Lemma 4.3.2. The argument is as follows: observe that when $n = 2$, we have a nontrivial complex tangent space at a point $p \in S$ if and only if the complex normals of the boundaries at p are linearly dependent over $\mathbb{C}$. That is, $p \in S$ is a complex tangent point if and only if

$$\partial\rho_1(p) \wedge \partial\rho_2(p) = 0. \tag{4.46}$$

Using the special boundary charts, we can write a form $u \in C^2_{(0,1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*)$ near a point $p \in (b\Omega_j \cap b\Omega) \backslash S$ as $u = u_{j1}\bar{\omega}_{j1} + u_{j2}\bar{\omega}_{j2}$, where $\bar{\omega}_{j1} = -\frac{\partial\rho_j}{\partial\bar{z}_2} d\bar{z}_1 + \frac{\partial\rho_j}{\partial\bar{z}_1} d\bar{z}_2$ and $\bar{\omega}_{j2} = \frac{\partial\rho_j}{\partial z_1} d\bar{z}_1 + \frac{\partial\rho_j}{\partial z_2} d\bar{z}_2$ in a small neighborhood of p, and u_{jk}'s are continuous in this neighborhood. The condition for u to be in $\mathrm{dom}(\bar{\partial}^*)$ (cf. (2.5)) gives us that $u_{j2} = 0$ on $(b\Omega_j \cap b\Omega) \backslash S$. Moreover, by continuity these vanishing coordinates continue vanishing on S. Writing u in a special coordinate chart of each domain at a point $p \in S$ will give that $u = u_{11}\bar{\omega}_{11} = u_{21}\bar{\omega}_{21}$. But this equality gives that u should vanish on $S \backslash K$ by (4.46). Since the set K is nowhere dense by Lemma 4.1.8, then by continuity these coordinates also vanish on S.

Remark 4.3.3. *Note that the approach we use does not work if we take the domains in $\mathbb{C}^n$ for $n \geq 3$ and want to show that corresponding $(0,1)$-forms vanish on S. This is simply due to the fact that in our approach there are n equations resulting from equating the components of $d\bar{z}_1, \cdots, d\bar{z}_n$ on S and $2(n-1)$ unknowns $u_{11}, u_{21}, \cdots, u_{1,n-1}, u_{2,n-1}$ which are the components in the special boundary charts.*

More generally, considering $(0,q)$*-forms there will be* $\binom{n}{q}$ *equations. The number of unknowns will be* $2\left(\binom{n}{q}-\binom{n-1}{q-1}\right)$ *because there will be* $\binom{n}{q}$ *unknowns in the special boundary chart with respect to one boundary but those components of the wedge products which contain* $\bar{\omega}_n$ *will vanish (the number of such components is* $\binom{n-1}{q-1}$*). Therefore, if we start with the assumption "*$N_1^{\Omega_j}$ for at least one of $j=1,2$ *is compact" and consider the fact that* $(n-1)$*-dimensional complex manifolds are obstructions to the compactness of* $N_1^{\Omega_j}$*, then we should expect to see that our approach works as long as* $\binom{n}{q}-2\left(\binom{n}{q}-\binom{n-1}{q-1}\right)\geq 0$ *which is equivalent to saying that* $2q\geq n$*.*

Observe that in the proofs of Theorem 4.1.2 and Theorem 4.2.3, we verified some sort of compactness estimates and the existence of a compact solution operator for $\bar{\partial}$ respectively. In doing so, we considered the appropriate spaces of forms without any smoothness assumptions on the components. This was because of the fact that in our setting, a density result in the graph norm as in *iii*) of Lemma 2.2.1 was not accessible to us as of the date this dissertation was written. Nevertheless, Lemma 4.3.2 yields an interesting result (Lemma 4.3.5) on the density (in the graph norm) of $(0,n-1)$-forms when the intersecting domains have smooth boundaries which intersect real transversally. In order to prove Lemma 4.3.5, we will need smooth cutoff functions supported in a neighborhood of S. Recall that the set S is the intersection of the boundaries $b\Omega_1$ and $b\Omega_2$; and as such, it is a compact set. So, for a given $\varepsilon>0$, we can find a smooth cutoff function which is identically 1 on S and which vanishes outside of an ε-neighborhood of S. Moreover, such a function will have its gradient bounded by some constant (independent of the compact set S) times $\frac{1}{\varepsilon}$. We skip the details of constructing such a smooth function here as the construction can be done via classical techniques for any compact set in $\mathbb{C}^n$ (see, for instance, the introductory chapters of [32], [8]). However, for convenience, we state the existence smooth cutoff functions in the following lemma:

Lemma 4.3.4. *Given a compact set K and $\varepsilon > 0$, there exists a smooth cutoff function which is identically equal to 1 on K and vanishes outside of an ε-neighborhood of K. Furthermore, such a function can be constructed in a way that its gradient bounded by some constant (independent of the compact set K) times $\frac{1}{\varepsilon}$.*

We now state and prove our density result:

Lemma 4.3.5. *Let Ω_1 and Ω_2 be smooth bounded pseudoconvex domains in $\mathbb{C}^n$ whose boundaries intersect real transversally. If one of the $\bar{\partial}$-Neumann operators $N_1^{\Omega_1}$ and $N_1^{\Omega_2}$ is compact, then the forms in $C^2_{(0,n-1)}(\overline{\Omega}) \cap dom(\bar{\partial}^*)$ that are supported away from S are dense in $C^2_{(0,n-1)}(\overline{\Omega}) \cap dom(\bar{\partial}^*)$ in the graph norm.*

Proof. For $\varepsilon > 0$ sufficiently small, let U_ε be a tubular neighborhood of S which consists of those points in $\mathbb{C}^n$ that have distance to S less than ε. We may take $U_\varepsilon := \bigcup_{z\in S} B_\varepsilon(z)$, where $B_\varepsilon(z)$ denotes a ball of radius ε centered at $z \in \mathbb{C}^n$. By Lemma 4.3.4, we can find a smooth cutoff function φ_ε which is identically 1 on $U_{\varepsilon/2}$ and which vanishes outside of $\overline{U_{3\varepsilon/4}}$. Moreover, by Lemma 4.3.4 again, the gradient of φ_ε will be bounded by some constant independent of S times $\frac{1}{\varepsilon}$.

Let $u \in C^2_{(0,n-1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*)$. Observe that multiplying by a smooth function is an invariant operation for being in $\mathrm{dom}(\bar{\partial}) \cap \mathrm{dom}(\bar{\partial}^*)$. So, $\varphi_\varepsilon u$ is still in $C^2_{(0,n-1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*)$. Thus, for each sufficiently small $\varepsilon > 0$, $(1-\varphi_\varepsilon)u$ is in $C^2_{(0,n-1)}(\overline{\Omega}) \cap \mathrm{dom}(\bar{\partial}^*)$ and supported away from S. Therefore, if we set $\alpha_\varepsilon := (1-\varphi_\varepsilon)u - u = -\varphi_\varepsilon u$, it suffices to show that

$$||\alpha_\varepsilon||_\Omega := \left(||\alpha_\varepsilon||^2_\Omega + ||\bar{\partial}\alpha_\varepsilon||^2_\Omega + ||\bar{\partial}^*\alpha_\varepsilon||^2_\Omega\right)^{\frac{1}{2}}$$

tends to zero as $\varepsilon \to 0$. In order to do this, it is enough to show that each of $||\varphi_\varepsilon u||_\Omega$,

$||\bar{\partial}(\varphi_\varepsilon u)||_\Omega$ and $||\bar{\partial}^*(\varphi_\varepsilon u)||_\Omega$ go in $\mathcal{L}^2$-norms to 0 as $\varepsilon \to 0$. The convergence of the first norm does not require anything special about compactness. Indeed, observe that

$$||\varphi_\varepsilon u||^2_\Omega = \sum_{|J|=n-1}{}' \int_\Omega |\varphi_\varepsilon u_J|^2 dV \ \leq n \max_{\substack{|J|=n-1 \\ z\in\overline{\Omega}}} \{|u_J(z)|^2\} Vol(U_\varepsilon \cap \Omega). \tag{4.47}$$

The number n in the right side of the inequality (4.47) comes from the fact that there are $\binom{n}{n-1} = n$ strictly increasing $(n-1)$-tuples. The right hand side of the inequality (4.47) goes to 0 as ε goes to 0 since u has components continuous up to the boundary and the volume of the sets $U_\varepsilon \cap \Omega$ tends to 0. Now, we focus on the second norm and its convergence.

$$\begin{aligned} ||\bar{\partial}(\varphi_\varepsilon u)||^2_\Omega &\leq 2||\varphi_\varepsilon \bar{\partial} u||^2_\Omega + 2||\bar{\partial}\varphi_\varepsilon \wedge u||^2_\Omega \\ &\leq \left(n2^n \max_{\substack{|J|=n-1, \\ k=1,\cdots,n \\ z\in\overline{\Omega}}} \left\{ \left| \frac{\partial u_J}{\partial \bar{z}_k}(z) \right|^2 \right\} Vol(U_\varepsilon \cap \Omega) \right) \\ &\qquad + 2\left(\sum_{|J|=n-1}{}' \sum_{j=1}^n \int_\Omega \left| \frac{\partial \varphi_\varepsilon}{\partial \bar{z}_j} u_J \right|^2 dV \right) \\ &\lesssim Vol(U_\varepsilon \cap \Omega) + \frac{2n}{\varepsilon^2} \left(\sum_{|J|=n-1}{}' \int_{U_\varepsilon \cap \Omega} |u_J|^2 dV \right) \\ &\lesssim Vol(U_\varepsilon \cap \Omega) + \frac{4n^2}{\varepsilon^2} Vol(U_\varepsilon \cap \Omega) \max_{\substack{|J|=n-1 \\ z\in\overline{U_\varepsilon \cap \Omega}}} \{|u_J(z)|^2\}. \end{aligned} \tag{4.48}$$

Passing to the first terms on right hand side of the the second and third inequalities in (4.48), similar reasons as in (4.47) were used. The second term on the right hand side of the second inequality is by definition and passing to the third inequality, we used Lemma 4.3.4. The first term on the right hand side of (4.48) obviously goes to zero in the limit. For the second term, recall that the smooth manifold S has codimension 2 in $\mathbb{C}^n$. Thus, the volume of the tubular neighborhood has volume comparable to surface area of S times ε^2. Therefore, the volume of $U_\varepsilon \cap \Omega$ divided by ε^2 is bounded (but may not tend to zero in the limit). However, maximum of a continuous function is continuous. Thus the second term on the right hand side of (4.48) goes in the limit to maximum of the point-evaluations of the coefficients of u on S. But we know by Lemma 4.3.2 that u_J's are 0 on S. Thus, the second term on the right hand side of (4.48) goes to zero in the limit as well.

The convergence of the third norm $||\bar{\partial}^* u||$ to 0 in the limit uses the similar reason as in the last step of (4.48). Indeed,

$$\begin{aligned} ||\bar{\partial}^*(\varphi_\varepsilon u)||^2_\Omega &= \sum_{|K|=n-2}{}' \int_\Omega \left| - \sum_{j=1}^n \frac{\partial(\varphi_\varepsilon u_{jK})}{\partial z_j} \right|^2 dV \\ &= \sum_{|K|=n-2}{}' \int_\Omega \left| \sum_{j=1}^n \varphi_\varepsilon \frac{\partial u_{jK}}{\partial z_j} + \sum_{j=1}^n u_{jK} \frac{\partial \varphi_\varepsilon}{\partial z_j} \right|^2 dV \\ &\leq 2^n \sum_{|K|=n-2}{}' \sum_{j=1}^n \left(\int_\Omega \left| u_{jK} \frac{\partial \varphi_\varepsilon}{\partial z_j} \right|^2 dV + \int_\Omega \left| \varphi_\varepsilon \frac{\partial u_{jK}}{\partial z_j} \right|^2 dV \right) \\ &\lesssim \frac{1}{\varepsilon^2} Vol(U_\varepsilon \cap \Omega) \max_{\substack{jK \\ z \in \overline{U_\varepsilon \cap \Omega}}} \{|u_{jK}(z)|\} + Vol(U_\varepsilon \cap \Omega) \max_{\substack{jK \\ z \in \overline{U_\varepsilon \cap \Omega}}} \left\{ \left| \frac{\partial u_{jK}}{\partial \bar{z}_j}(z) \right|^2 \right\}. \end{aligned} \tag{4.49}$$

The first term on the right hand side of (4.49) goes to zero by the similar reasons to that of the second term on the right hand side of (4.48). That the second term on the right hand side of (4.49) goes to 0 in the limit is clear by boundedness of the partial derivatives of coefficients of u. This finishes the convergence of the third norm and therefore finishes the proof of the lemma. □

Remark 4.3.6. *For sufficiently small $\varepsilon' > 0$, one can take a family of functions*

$\{\varphi_{\epsilon'}\}$ *as in the proof of Lemma 4.3.5 and deduce by setting* $\phi = \varphi_{\varepsilon'}$ *in Lemma 4.1.1 that estimates* (4.1) *hold for each fixed* ε'. *However, if one further applies Lemma 4.3.5 (as* ε' *goes to zero) to obtain compactness estimates for* $C^2_{(0,n-1)}(\overline{\Omega}) \cap dom(\bar{\partial}^*)$, *there is no guarantee that the numbers* $C_{\varepsilon,\varepsilon'}$ *in estimates* (4.1) *will stay bounded. From this point of view, it would be interesting to know about how* $C_{\varepsilon,\varepsilon'}$ *depends on* ε, ε' *and the norms involved (see also the discussion at the end of paragraph after Remark 3.0.10).*

5. SUMMARY

In the first section, I introduced and motivated the problem of seeking compactness on the intersection of pseudoconvex domains in $\mathbb{C}^n$.

In the second section, I first introduced the notation and language that was used throughout this dissertation and then gave the necessary background for the set-up of the $\bar{\partial}$-Neumann problem. The section concluded with the twisted Kohn-Morrey Hörmander formula and its applications which were used in the subsequent sections.

In the third section, I gave a general glimpse of the compactness in the $\bar{\partial}$-Neumann problem. Definitions, results, properties and tools related the compactness of the $\bar{\partial}$-Neumann problem, which were used in proving the main results in this dissertation, were provided in this section. I also gave explicit proofs to some well-known facts in the field. These facts either have only implicit proofs in the literature or proofs for them were not provided elsewhere.

In the fourth section, I stated two main results and proved them. The first main result gives the solution of the problem under the assumption that the intersection of the boundaries of the intersecting domains satisfies property $(\tilde{P})$. Examples where this assumption is actually realized include smooth pseudoconvex domains in $\mathbb{C}^n$ whose $\bar{\partial}$-Neumann operators are compact and whose subset of infinite type points contained in the boundary intersection has Hausdorff measure zero. In particular, if all points in the boundary intersection are finite type points with respect to at least one of the boundaries, then the $\bar{\partial}$-Neumann operator corresponding to the intersection domain is compact. We also discussed some examples in $\mathbb{C}^2$ under the assumption that the domains have smooth boundaries and they intersect transversally. However, these examples are also covered by the second main result which gives a partial solution to the general problem: if the intersecting domains in $\mathbb{C}^n$ have smooth boundaries which intersect real transversally, then the $\bar{\partial}$-Neumann operator at the $(0, n-1)$-form level is compact. This means, when $n = 2$, the problem is solved if the domains are smooth and their boundaries intersect real transversally. We concluded the fourth section with some further discussion about some by-product results related to the vanishing of forms in the domain of $\bar{\partial}^*$ which are sufficiently smooth up to the boundary.

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