MAHLER MEASURES OF HYPERGEOMETRIC FAMILIES OF CALABI-YAU VARIETIES

A Dissertation

by

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ABSTRACT

The logarithmic Mahler measure of a nonzero n-variable Laurent polynomial $P \in \mathbb{C}\left[X_1^{\pm 1}, \ldots, X_n^{\pm 1}\right]$, denoted by $m(P)$, is defined to be the arithmetic mean of $\log|P|$ over the n-dimensional torus. It has been proved or conjectured that the logarithmic Mahler measures of some classes of polynomials have connections with special values of L-functions. However, the precise interpretation of $m(P)$ in terms of L-values is not clearly known, so it has become a new trend of research in arithmetic geometry and number theory to understand this phenomenon. In this dissertation, we study Mahler measures of certain families of Laurent polynomials of two, three, and four variables, whose zero loci define elliptic curves, $K3$ surfaces, and Calabi-Yau threefolds, respectively. On the one hand, it is known that these Mahler measures can be expressed in terms of hypergeometric series and logarithms. On the other hand, we derive explicitly that some of them can be written as linear combinations of special values of Dirichlet and modular L-functions, which potentially carry some arithmetic information of the corresponding algebraic varieties. Our results extend those of Boyd, Bertin, Lalín, Rodriguez Villegas, Rogers, and many others. We also prove that Mahler measures of those associated to families of $K3$ surfaces are related to the elliptic trilogarithm defined by Zagier. This can be seen as a higher dimensional analogue of relationship between Mahler measures of bivariate polynomials and the elliptic dilogarithm known previously by work of Guillera, Lalín, and Rogers.

DEDICATION

To my beloved grandparents, Boonchan Samart, Chamras Samart, Inn Yokteng, and Sai Yokteng.

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1. INTRODUCTION

1.1 Historical background

For any nonzero n-variable Laurent polynomial $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, the logarithmic Mahler measure of P is defined by

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| \, d\theta_1 \cdots d\theta_n. \tag{1.1}$$

Conventionally, one sometimes extends the definition of the logarithmic Mahler measure to include $m(0) = \infty$, but this convention is not necessary here. It can be shown that the integral in (1.1) is always a real number and is non-negative if the coefficients of P are integers [28, Lem. 3.7]. For any $P \in \mathbb{C}[x] \setminus \{0\}$, we can find an explicit formula for $m(P)$ using a standard result in complex analysis.

Theorem 1.1 (Jensen's Formula)**.** *Let $\alpha \in \mathbb{C}$. Then*

$$\int_0^1 \log |e^{2\pi i \theta} - \alpha| d\theta = \log \max\{1, |\alpha|\}.$$

As an immediate consequence of Jensen's formula, if $P(x) = (x - \alpha_1) \cdots (x - \alpha_m)$, where $\alpha_i \in \mathbb{C}$, then

$$m(P) = \sum_{j=1}^{m} \max\{0, \log |\alpha_j|\}. \tag{1.2}$$

If $P(x)$ is assumed further to be in $\mathbb{Q}[x]$ and is irreducible, then the quantity $m(P)/m$ is known as the *Weil height* of the algebraic numbers α_j. In fact, the exponential of $m(P)$, namely

$$M(P) := \exp(m(P)) = \prod_{j=1}^{m} \max\{1, |\alpha_j|\},$$

was introduced first by Lehmer [40] in the 1930s. One of the motivations of Lehmer's work is to construct large prime numbers by generalizing the notion of Mersenne prime. About three decades later, Mahler [46] then gave the extended definition (1.1) for the multi-variable case, which seemingly has the right arithmetic properties. Although the term *Mahler measure* refers to $M(P)$ in some parts of the literature, from now on *Mahler measure* will always mean $m(P)$.

It is known by Kronecker's theorem that if $P(x) \in \mathbb{Z}[x]\backslash\{0\}$, then $m(P) = 0$ if and only if every root of P is either zero or a root of unity. However, it is unknown whether the Mahler measures of other one-variable polynomials with integral coefficients is bounded below by a positive absolute constant. This problem is called Lehmer's conjecture, which is a famous open problem in number theory.

Conjecture 1.2 (Lehmer's Conjecture)**.** *The set*

$$\mathcal{M} := \{m(P) \mid P(x) \in \mathbb{Z}[x] \textit{ and } m(P) > 0\}$$

is bounded below by some constant $\alpha > 0$.

The smallest known element of $\mathcal{M}$ is

$$m\left(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1\right) \approx 0.162357612007738139432\cdots,$$

discovered by Lehmer himself. Actually, it is widely believed that the constant above is the greatest lower bound of $\mathcal{M}$.

In contrast to the univariate case, no easily calculable universal formula for $m(P)$ is known when P has two or more variables. (Observe that the zero set of $P(x_1, \ldots, x_n)$ is no longer discrete, if $n > 1$.) Despite lack of general formulas, Mahler measures of certain polynomials are surprisingly related to special values of L-functions, which are quantities of interest in number theory. The first known such examples, proved by Smyth [74], include

$$\begin{aligned} m(x + y + 1) &= \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1), \\ m(x + y + z + 1) &= \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function and $\chi_D(n) = \left(\frac{D}{n}\right)$ is the Dirichlet character associated to $\mathbb{Q}(\sqrt{D})$. By a result of Chinburg [20] and Dirichlet's class number formula [55], the Mahler measures of some families of univariate polynomials are also known to be related to special values of Artin L-functions and Dedekind zeta functions of real quadratic fields.

Another prototypical example is the family of two-variable polynomials

$$P_k(x, y) := x + x^{-1} + y + y^{-1} + k, \tag{1.3}$$

where $k \in \mathbb{C}$. This is one of the first examples of *tempered* polynomials, whose Mahler measures are predicted to have connections with L-values. To defined a tempered polynomial, one considers the Newton polygon corresponding to the polynomial and associates a univariate polynomial to each side of the polygon. If the zeroes of these univariate polynomials are roots of unity, we say that the original polynomial is tempered. For a precise definition of temperedness, see [55, §2]. For any $k \notin \{0, \pm 4\}$ the projective closure of $P_k = 0$ becomes an elliptic curve, whose geometry is well studied. Moreover, for a number of values of $k \in \mathbb{Z}$, $m(P_k)$ is expressible in terms of a special value of the L-function of the corresponding elliptic curve. For instance, using the Bloch-Beilinson conjectures, Deninger [24] hypothesized that

$$m(P_1) = c\frac{15}{(2\pi)^2}L(E_1, 2) = cL'(E_1, 0), \tag{1.4}$$

where E_k is the elliptic curve determined by $P_k = 0$, and $c \in \mathbb{Q}$. Boyd [12] subsequently verified numerically that $c = 1$ and also discovered a number of similar conjectural formulas for $k \in \mathbb{Z}\backslash\{\pm 4\}$, namely

$$m(P_k) \stackrel{?}{=} c_k L'(E_k, 0), \tag{1.5}$$

where $\stackrel{?}{=}$ means that the equality holds to at least 50 decimal places. More detailed explanations about these results will be given in Section 3.

Recall that elliptic curves are one-dimensional Calabi-Yau varieties. Therefore, one of the possible higher-dimensional analogues of elliptic curves are $K3$ surfaces, the compact simply connected Calabi-Yau twofolds. Examples of polynomials whose zero loci define $K3$ hypersurfaces include the following families:

$$Q_k(x, y, z) := x + x^{-1} + y + y^{-1} + z + z^{-1} - k,$$

$$R_k(x, y, z) := Q_k(x, y, z) + xy + (xy)^{-1} + zy + (zy)^{-1} + xyz + (xyz)^{-1}.$$

The geometries of $K3$ surfaces defined by these polynomials were investigated in [78, §7], and their Mahler measures were first studied by Bertin [6]. Subsequently, Bertin and others [5, 8, 9] showed that, for some values of $k \in \mathbb{Z}$, the Mahler measures of Q_k have L-value expressions analogous to the elliptic curve case. More precisely, they proved some formulas of the form

$$m(Q_k) = c_1 L'(\mathrm{T}(X_k), 0) + c_2 L'(\chi, -1), \tag{1.6}$$

where $c_1, c_2 \in \mathbb{Q}$, X_k is the $K3$ surface defined by the zero locus of Q_k with the transcendental lattice $\mathrm{T}(X_k)$, and χ is a quadratic character. (For more information about $K3$ surfaces, see Section 2.) There are much fewer examples of Mahler measures in higher-dimensional cases related to special L-functions known so far. More importantly, it is still unclear what are the precise ways that Mahler measures are related to the polynomials and the associated varieties.

1.2 An outline of this thesis

We mainly study Mahler measures of certain families of Laurent polynomials, many of which were introduced previously in the literature, with an emphasis on the three-variable case. As we will see in the next sections, in most cases, these polynomials define families of Calabi-Yau varieties, and their Mahler measures can be expressed in terms of hypergeometric series. In Section 2, we will give necessary definitions and preliminary results, which will be used in the subsequent sections. Sections 3, 4, 5, and 6 are constituted of four papers, three of which were written solely by the author [63, 64, 65]. A result on the Mahler measure of a four-variable polynomial in Section 6 is joint work of Matthew Papanikolas, Mathew Rogers, and the author [52].

In Section 3, we begin with some known results about Mahler measures of families of bivariate polynomials which were first investigated by Rodriguez Villegas. We will describe his crucial ideas which led to proofs of some formulas conjectured by Boyd. While each of all previously known Mahler measure formulas of these polynomials involves at most one modular L-value, we are able to give some examples of two-variable Mahler measures each of which is a linear combination of L-values of multiple modular forms. Main results of this section include the following formulas (see Theorem 3.3 and Theorem 3.6):

$$m(P_{\sqrt{8\pm 6\sqrt{2}}}) = \frac{1}{2}(L'(f_{64},0) \pm L'(f_{32},0)),$$

$$m(x^3+y^3+1-k_0xy) = \frac{1}{2}(L'(f_{108},0) + L'(f_{36},0) - 3L'(f_{27},0)),$$

where $k_0 = \sqrt[3]{6-6\sqrt[3]{2}+18\sqrt[3]{4}}$, and f_N denotes a normalized CM newform of weight 2 and level N with rational Fourier coefficients. The fact that these examples are corresponding to elliptic curves defined over some number fields rather than $\mathbb{Q}$ partly explains why our results are slightly different from those of Rodriguez Villegas.

We then investigate Mahler measures of certain families of three-variable polynomials introduced by Rogers in Section 4. It will be proved in Section 4.1 that, parameterized by some elliptic functions, these Mahler measures can be written in terms of Eisenstein-Kronecker series. Then we deduce from this result some formulas involving L-values of CM weight 3 newforms and those of Dirichlet characters, as stated in Theorem 4.3 and

Theorem 4.10. Similar to the bivariate case, if we allow the value of each parameter to be in some number fields, then the number of modular L-values appearing in the formula seems to depend on the shape of this algebraic value. In Section 4.3, we discuss the arithmetic of $K3$ surfaces defined by these polynomials. More precisely, these families of $K3$ surfaces are of generic Picard number 19. Then we explicitly construct families of elliptic curves which give rise to their Shioda-Inose structures. All formulas which we discovered using numerical computation will be tabulated at the end of the section. Furthermore, it turns out that all of these formulas correspond to singular $K3$ surfaces; i.e., those with Picard number 20.

In order to make progress towards generalizing three-variable Mahler measure formulas, we establish in Section 5 that when the parameters are real, with some exceptions, the Mahler measures of families of polynomials considered in Section 4 have expressions in terms of the elliptic trilogarithm, introduced by Zagier and Gangl [89, §10]. This result can be seen as a higher dimensional analogue of relationship between two-variable Mahler measures and the elliptic dilogarithm, which were studied by Bertin, Guillera, Lalín, and Rogers [7, 32, 39]. It also reveals some interesting connections between the families of $K3$ surfaces corresponding to our polynomials and families of elliptic curves in their Shioda-Inose structures given in Section 4.3. For the full statement of the result, see Theorem 5.3. In addition, we point out some connections between the elliptic trilogarithm and special values of L-functions obtained from our proved results and computational experiments in Section 5.4.

In the final section, we suggest some possible generalizations of our results in higher dimensional cases. We will also address in Theorem 6.1 a proved formula of the Mahler measure of a four-variable polynomial, which involves an L-value of a non-CM weight 4

newform h and a special value of the Riemann zeta function, namely,

$$m((x+x^{-2})(y+y^{-1})(z+z^{-1})(w+w^{-1})-16)=8L'(h,0)-28\zeta'(-2).$$

However, no other four-variable Mahler measure formulas expressible as modular L-values have been found, even numerically.

2. PRELIMINARIES

2.1 Hypergeometric families of Calabi-Yau varieties

Definition 2.1. *A smooth projective variety X of dimension k is called a Calabi-Yau variety if it satisfies the following conditions:*

1. *For every $0 < i < k$, $H^i(X, \mathcal{O}_X) = 0$, where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X.*

2. *The canonical bundle $K_X := \bigwedge^k \Omega_X$ of X is trivial.*

Calabi-Yau varieties of dimension one are elliptic curves; i.e., smooth curves of genus one with fixed base points. A two-dimensional Calabi-Yau variety is called a $K3$ surface. Calabi-Yau varieties have been studied extensively in physics, especially string theory, as they are mathematical objects used to explain a phenomenon called *mirror symmetry*. They are also known to be enriched with nice arithmetic properties when defined over number fields or finite fields. Their arithmetic information is usually encoded by the attached L-series, which will be discussed later. As mentioned in Section 1, the polynomials we will deal with define certain families of Calabi-Yau hypersurfaces in the corresponding projective spaces. Furthermore, the families of polynomials in $2, 3,$ and 4 variables which we consider are, in a certain sense, associated to hypergeometric series. We will gradually clarify this in the present section.

Definition 2.2. *For non-negative integers p and q, and $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q \in \mathbb{C}$, where b_j are not nonpositive integers, the hypergeometric series ${}_pF_q\left(\begin{smallmatrix} a_1,\ a_2,\ \ldots,\ a_p \\ b_1,\ b_2,\ \ldots,\ b_q \end{smallmatrix}; x\right)$ is defined by*

$${}_pF_q\left(\begin{matrix} a_1, & a_2, & \ldots, & a_p \\ b_1, & b_2, & \ldots, & b_q \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!},$$

where for any $c \in \mathbb{C}$*,*

$$(c)_n = \begin{cases} 1, & \textit{if } n = 0, \\ c(c+1)\cdots(c+n-1), & \textit{if } n \geq 1. \end{cases}$$

In [54], Rodriguez Villegas introduced the notion of a *hypergeometric weight system*, which is a formal linear combination

$$\gamma = \sum_{\nu \geq 1} \gamma_\nu [\nu],$$

where $\gamma_\nu \in \mathbb{Z}$ are zero for all but finitely many ν, satisfying the following conditions:

(i) $\displaystyle\sum_{\nu \geq 1} \nu \gamma_\nu = 0,$

(ii) $\displaystyle d = d(\gamma) := -\sum_{\nu \geq 1} \gamma_\nu > 0.$

We denote by Γ the set of all hypergeometic weight systems. The number d is called the *dimension* of γ. To each $\gamma \in \Gamma$, we associate the function

$$u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n, \quad \text{where } u_n = \prod_{\nu \geq 1} (\nu n)!^{\gamma_\nu}.$$

It can be checked that for some minimal $r \in \mathbb{N}$ we can write $u(\lambda)$ as

$$u(\lambda) = {}_rF_{r-1}\left(\begin{matrix} a_1, \ a_2, \ \ldots, \ a_r \\ b_1, \ b_2, \ \ldots, \ b_{r-1} \end{matrix} ; \frac{\lambda}{\lambda_0} \right), \quad \text{where } \lambda_0^{-1} = \prod_{\nu \geq 1} \nu^{\nu \gamma_\nu},$$

and $0 \leq a_i < 1$ and $0 \leq b_i < 1$ are rational numbers for all i. In other words, $u(\lambda)$ is a hypergeometric series with rational parameters. The number $r = r(\gamma)$ is called the *rank* of γ. For any $\gamma \in \Gamma$, we have $d(\gamma) \leq r(\gamma)$. In the special case when $d = r$, Rodriguez Villegas showed that there are only finitely many $\gamma \in \Gamma$ of each fixed dimension. He also listed all such γ for $d \leq 4$, which can be described as follows. Following his notation, we let $\Gamma_{\text{uni}} = \{\gamma \in \Gamma \mid d(\gamma) = r(\gamma)\}$.

There is only one $\gamma \in \Gamma_{\text{uni}}$ such that $d(\gamma) = 1$, namely $\gamma = [2] - 2[1]$. When $d = 2$, we have γ as tabulated in Table 2.1. Also, for each of them, we have the corresponding

γ	a_1	a_2	λ_0^{-1}
$2[2]-4[1]$	$\frac{1}{2}$	$\frac{1}{2}$	16
$[3]-3[1]$	$\frac{1}{3}$	$\frac{2}{3}$	27
$[4]-[2]-2[1]$	$\frac{1}{4}$	$\frac{3}{4}$	64
$[6]-[3]-[2]-[1]$	$\frac{1}{6}$	$\frac{5}{6}$	432

Table 2.1: $\gamma \in \Gamma_{\text{uni}}$ with $d(\gamma) = 2$

hypergeometric function

$$u(\lambda) = {}_2F_1\left(\begin{matrix} a_1,\ a_2 \\ 1 \end{matrix}; \frac{\lambda}{\lambda_0}\right),$$

where a_1, a_2, and λ_0^{-1} are given next to each γ.

When $d = 3$, we again have four $\gamma \in \Gamma_{\text{uni}}$, each of which arises from the previous case (see Table 2.2).

γ	a_1	a_3	λ_0^{-1}
$3[2]-6[1]$	$\frac{1}{2}$	$\frac{1}{2}$	64
$[3]+[2]-5[1]$	$\frac{1}{3}$	$\frac{2}{3}$	108
$[4]-4[1]$	$\frac{1}{4}$	$\frac{3}{4}$	256
$[6]-[3]-3[1]$	$\frac{1}{6}$	$\frac{5}{6}$	1728

Table 2.2: $\gamma \in \Gamma_{\text{uni}}$ with $d(\gamma) = 3$

In this case, the corresponding hypergeometric series are

$$u(\lambda) = {}_3F_2\left(\begin{matrix} a_1,\ \frac{1}{2},\ a_3 \\ 1,\ 1 \end{matrix}; \frac{\lambda}{\lambda_0}\right).$$

Finally, when $d = 4$, we have 14 different γ, which are listed in Table 2.3. Now the function $u(\lambda)$ is of the form

$$u(\lambda) = {}_4F_3\left(\begin{matrix} a_1,\ a_2,\ a_3,\ a_4 \\ 1,\ 1,\ 1 \end{matrix}; \frac{\lambda}{\lambda_0}\right).$$

Recall that the second condition in Definition 2.1 implies that every k dimensional Calabi-Yau variety X admits a nowhere-vanishing holomorphic k-form ω_X, unique up to

γ	a_1	a_2	a_3	a_4	λ_0^{-1}
$4[2]-8[1]$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	2^8
$[4]+[3]-[2]-5[1]$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$2^6\cdot 3^3$
$[4]+[2]-6[1]$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	2^{10}
$[5]-5[1]$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	5^5
$2[3]-6[1]$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	3^6
$2[4]-2[2]-4[1]$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	2^{12}
$[3]+2[2]-7[1]$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$2^4\cdot 3^3$
$[6]+[2]-[3]-5[1]$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{6}$	$2^8\cdot 3^3$
$[6]-[2]-4[1]$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$2^4\cdot 3^6$
$[8]-[4]-4[1]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	2^{16}
$[6]+[4]-[3]-2[2]-3[1]$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{6}$	$2^{10}\cdot 3^3$
$[10]-[5]-[2]-3[1]$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{7}{10}$	$\frac{9}{10}$	$2^8\cdot 5^5$
$2[6]-2[3]-2[2]-2[1]$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$2^8\cdot 3^6$
$[12]+[2]-[6]-[4]-4[1]$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{11}{12}$	$2^{12}\cdot 3^6$

Table 2.3: $\gamma \in \Gamma_{\text{uni}}$ with $d(\gamma)=4$

a scalar multiple. For any k-cycle γ on X, we call the quantity

$$\int_\gamma \omega_X$$

a period of X. To each hypergeometric weight system in the three tables above, one can associate a one-parameter family X_λ of Calabi-Yau varieties in such a way that $u(\lambda)$ becomes a period of X_λ. For $d = 2, 3,$ and 4, these Calabi-Yau varieties are elliptic curves, $K3$ surfaces, and Calabi-Yau threefolds, respectively. Examples of these varieties can be found in [41] (for $d = 2, 3$) and [3] (for $d = 4$). Note that, in these papers, they are constructed from hypersurfaces or complete intersections of hypersurfaces in (weighted) projective spaces. In fact, each family can be identified uniquely with the associated *Picard-Fuchs equation*, which is the linear differential equation satisfied by the function $u(\lambda)$. However, to study Mahler measures, we will consider certain Laurent polynomials which are models of Calabi-Yau hypersurfaces whose periods are the hypergeometric series given above.

One of the main reasons why we choose these polynomials is that their Mahler measures are also expressible in terms of hypergeometric series, which can be computed easily using standard computer algebra systems. In addition, some of them have been introduced previously in the literature and evidently yield special L-values potentially related to the corresponding varieties. We will briefly review some necessary definitions and results about Calabi-Yau varieties of dimensions 1 and 2 below. The polynomials which are of consideration will be listed in the forthcoming sections.

2.2 Elliptic curves

For a standard reference to the basic facts about the arithmetic of elliptic curves, the reader may consult [73]. As mentioned in Section 2.1, one may think of elliptic curves as one-dimensional Calabi-Yau varieties. Nevertheless, it is more common to define an elliptic curve as a smooth projective algebraic curve of genus one with a fixed rational point $\mathcal{O}$. Since, in most cases, we will be dealing with elliptic curves defined over number fields or the field of complex numbers, throughout this section an elliptic curve is assumed to be defined over a field K of characteristic zero. Let E be an elliptic curve. Then E has a Weierstrass equation of the form

$$E : y^2 = x^3 + Ax + B, \qquad A, B \in K,$$

and its fixed rational point is the point at infinity. The discriminant Δ and the j-invariant of E are then defined by

$$\Delta := -16\left(4A^3 + 27B^2\right), \qquad j(E) := -1728\frac{(4A)^3}{\Delta}.$$

For any field L containing A and B, we set

$$E(L) := \{[x, y, 1] \in \mathbb{P}^2(L) \mid y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}.$$

Every elliptic curve is endowed with a group structure making it become an abelian variety. A morphism $\phi : E \to E'$ between two elliptic curves such that $\phi(\mathcal{O}) = \mathcal{O}$ is called an *isogeny*. It can be shown that an isogeny is also a group homomorphism, and we say that two elliptic curves are *isogenous* if there is a non-constant isogeny between them. The set $\mathrm{End}(E)$ of the isogenies from E to itself forms a ring under usual addition and composition. More precisely, we have that $\mathrm{End}(E)$ can be either $\mathbb{Z}$ or an order in an imaginary quadratic field. If $\mathrm{End}(E) \ncong \mathbb{Z}$, then E is said to have *complex multiplication* (or CM for short).

If E is an elliptic curve defined over $\mathbb{C}$, then one can associate to E a lattice in $\mathbb{C}$; i.e., a subgroup of $\mathbb{C}$ of the form

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\},$$

where ω_1 and ω_2 form an $\mathbb{R}$-basis for $\mathbb{C}$. Any two lattices Λ_1 and Λ_2 are said to be *homothetic* if $\Lambda_1 = \alpha\Lambda_2$ for some $\alpha \in \mathbb{C}^*$.

Definition 2.3. *For any lattice $\Lambda \subset \mathbb{C}$, the Weierstrass $\wp$-function relative to Λ is defined by*

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

The following result explains how elliptic curves defined over $\mathbb{C}$ are related to lattices.

Theorem 2.4. *Let $\Lambda \subset \mathbb{C}$ be a lattice. Then there exist $g_2, g_3 \in \mathbb{C}$, depending only on Λ, such that $g_2^3 - 27g_3^2 \neq 0$, and the following relation holds for all $z \in \mathbb{C} \setminus \Lambda$:*

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3.$$

Furthermore, if E is the elliptic curve

$$E : y^2 = 4x^2 - g_2x - g_3,$$

then the map $\phi : \mathbb{C}/\Lambda \longrightarrow E(\mathbb{C})$ given by $\phi(z) = [\wp_\Lambda(z), \wp'_\Lambda(z), 1]$ is a complex analytic isomorphism of complex Lie groups. Conversely, let $E/\mathbb{C}$ be an elliptic curve. Then there exists a lattice $\Lambda \subset \mathbb{C}$, unique up to homothety, such that $E(\mathbb{C})$ is isomorphic to $\mathbb{C}/\Lambda$ via the map ϕ defined above.

What follows from the above theorem is that any two elliptic curves defined over $\mathbb{C}$ are isomorphic if and only if their period lattices are homothetic. If $E/\mathbb{C}$ is an elliptic curve, then a lattice $\Lambda \in \mathbb{C}$ such that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ is called a *period lattice* of E. In fact, for any two basis elements γ_1 and γ_2 of $H_1(E, \mathbb{Z})$ and any holomorphic form ω on E, the periods

$$\omega_1 = \int_{\gamma_1} \omega, \qquad \omega_2 = \int_{\gamma_2} \omega,$$

generate a period lattice of E. Multiplying a lattice period by a proper constant, we have immediately that there exists τ in the upper half plane $\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ such that $\Lambda := \mathbb{Z} + \mathbb{Z}\tau$ is a period lattice of E. In other words, we can always identify an elliptic curve defined over $\mathbb{C}$ with an element in $\mathcal{H}$. This fact will be constantly used in Section 5.

We conclude this section by giving a brief overview of the L-function of an elliptic curve. Let $E/\mathbb{Q}$ be an elliptic curve. Then it has a minimal Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad a_1, a_2, a_3, a_4, a_6 \in \mathbb{Z}.$$

Then for any prime p the curve E can be reduced to that defined over the finite field $\mathbb{F}_p$ of p elements. For each prime p, we define

$$a_p(E) := p + 1 - \#E(\mathbb{F}_p).$$

If $p \nmid \Delta$, then $E/\mathbb{F}_p$ is also an elliptic curve; i.e., it is smooth. In this case, we say that p is a prime of good reduction, and the quantity $a_p(E)$ is called the *trace of Frobenius* of E over $\mathbb{F}_p$. If $p|\Delta$, then $E/\mathbb{F}_p$ is a singular curve, and p is said to be a prime of bad reduction.

Now we define the L-series of E by the following Euler product:

$$L(E,s) := \prod_{p|\Delta}(1-a_p(E)p^{-s})^{-1}\prod_{p\nmid\Delta}(1-a_p(E)p^{-s}+p^{1-2s})^{-1} = \sum_{n=1}^{\infty}\frac{a_n}{n^s}.$$

If $E/\mathbb{F}_p$ is singular, then it has only one singular point, say P, which can be either a node or a cusp. Another important quantity associated to E, which measures how bad the reduction modulo p is, is the *conductor* of E. It can be defined by

$$N_E := \prod_{p|\Delta} p^{f_p(E)},$$

where $f_p(E) = 1$ if $E/\mathbb{F}_p$ has a node and $f_p(E) = 2 + \delta_p$ for some integer $0 \leq \delta_p \leq 3$ if $E/\mathbb{F}_p$ has a cusp. In the latter case, $f_p(E)$ is a little complicated to define. A precise definition and a formula of this number can be found in [72, Chap. IV]. Moreover, if p is of bad reduction, then we have that

$$a_p(E) = \begin{cases} 0, & \text{if } P \text{ is a cusp,} \\ 1, & \text{if } P \text{ is a node and the slopes of the tangent lines to } E \text{ at } P \text{ are in } \mathbb{F}_p, \\ -1, & \text{otherwise.} \end{cases}$$

The series $L(E,s)$ converges absolutely if $\mathrm{Re}(s) > 3/2$, and we shall see later by the celebrated results of Wiles [81] and Breuil, Conrad, Diamond, and Taylor [13] that $L(E,s)$ has an analytic continuation to an entire function, called the L-function of E. Given an elliptic curve E over $\mathbb{Q}$, one can also define another class of L-functions called the *symmetric power* L- functions, denoted by $L(\mathrm{Sym}^n(E),s), n \in \mathbb{N}$. They arise naturally from the symmetric n^{th}-powers of the dual representation of the l-adic representation attached to E, and are generally defined by the Euler products

$$L(\mathrm{Sym}^n(E),s) := \prod_{p|N_E} L_p(s) \times \prod_{p\nmid N_E}\prod_{j=0}^{n}(1-\alpha_p^j\beta_p^{n-j}p^{-s})^{-1},$$

where, for each prime p of good reduction, $\alpha_p + \beta_p = a_p(E)$ and $\alpha_p\beta_p = p$, and $L_p(s)$

is the Euler factor depending on each prime p of bad reduction. If $n = 1$, then we obtain the traditional L-series of E defined previously. Many important results about symmetric square elliptic curve L-functions can be found in a seminal paper of Coates and Schmidt [21]. In the higher power cases, the L-functions become more complicated and less well understood.

2.3 $K3$ surfaces

In this section, we discuss some basic facts about the geometry and arithmetic of $K3$ surfaces. The reader is referred to [68] and [69] for further details. By the definition above, a $K3$ surface is a smooth projective surface X which has a nowhere-vanishing holomorphic 2-form, unique up to scalar multiplication, and $H^1(X, \mathcal{O}_X) = 0$. Again, unless otherwise stated, we assume that the field of definition of a $K3$ surface is always of characteristic zero. Examples of $K3$ surfaces include:

- a smooth quartic surface in $\mathbb{P}^3$,
- a double cover of $\mathbb{P}^2$ branched along a sextic curve,
- a Kummer surface,
- an elliptic surface defined over a field K of characteristic $\neq 2, 3$ with a minimal Weierstrass form

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

  where $a_i(t) \in K[t]$, $\deg a_i \leq 2i$ for all i, and $\deg a_i > i$ for some i.

Let us recall the definition of divisors, which are important objects used in studying the geometry of varieties.

Definition 2.5. *Let X be an smooth compact variety. A divisor on X is a formal sum*

$$D = \sum_i n_i C_i,$$

where $n_i \in \mathbb{Z}$ are zero for all but finitely many i, and C_i are irreducible subvarieties of X of codimension 1.

The divisors on X form a free abelian group, denoted by $\mathrm{Div}(X)$. In particular, if C is a smooth curve, a divisor on C is a formal sum

$$D = \sum_{P \in C} n_P(P),$$

where $n_P \in \mathbb{Z}$ are zero for all but finitely many $P \in C$, and we define the *degree* of D to be

$$\deg D = \sum_{P \in C} n_P.$$

The *Picard group* of X, denoted $\mathrm{Pic}(X)$, is defined by $\mathrm{Div}(X)$ modulo linear equivalence. The classes in $\mathrm{Pic}(X)$ which are algebraic equivalent to 0 form a subgroup of $\mathrm{Pic}(X)$, denoted $\mathrm{Pic}^0(X)$. The *Néron-Severi group* is then defined as the quotient

$$\mathrm{NS}(X) := \mathrm{Pic}(X)/\operatorname{Pic}^0(X).$$

From now until the end of this section, let X be a $K3$ surface. Then we have $\mathrm{Pic}^0(X) = 0$ and $\mathrm{NS}(X) \cong \mathrm{Pic}(X)$. It can be shown that $H^2(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22, and the symmetric bilinear form given by the cup product defines a lattice structure on $H^2(X, \mathbb{Z})$. Furthermore, the group $\mathrm{NS}(X)$ can be embedded into $H^2(X, \mathbb{Z})$, so it is also a free $\mathbb{Z}$-module. The rank of $\mathrm{NS}(X)$ is called the *Picard number* of X, denoted $\rho(X)$. It is known that $0 \leq \rho(X) \leq 20$, and if $\rho(X)$ attains its maximum; i.e., $\rho(X) = 20$, then X is said to be a *singular* $K3$ surface. The orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$ is called the *transcendental lattice* of X, which we will denote $\mathrm{T}(X)$. It follows immediately that if X is a singular $K3$ surface, then $\mathrm{rank}(\mathrm{T}(X)) = 2$, and $\mathrm{T}(X)$ can be realized as a binary quadratic form. In this case, we define the *discriminant* of X to be the discriminant of the quadratic form corresponding to $\mathrm{T}(X)$. As we shall see in the next paragraph, in some sense, singular $K3$ surfaces can be seen as higher dimensional analogues of CM elliptic curves.

Let E and E' be elliptic curves defined by Weierstrass equations

$$E : y^2 = f(x), \qquad E' : y'^2 = g(x').$$

Then the Kummer surface $\mathrm{Km}(E \times E')$ can be defined as the desingularization of the surface

$$w^2 = f(x)g(x').$$

As mentioned above, $\mathrm{Km}(E \times E')$ is a $K3$ surface. Moreover, its Picard number is bounded below by 18 and depends directly on the curves E and E'. More precisely, one has that

$$\rho(\mathrm{Km}(E \times E')) = \begin{cases} 18, & \text{if } E \text{ is not isogenous to } E', \\ 19, & \text{if } E \text{ is isogenous to } E' \text{ and both are non-CM}, \\ 20, & \text{if } E \text{ is isogenous to } E' \text{ and both are CM.} \end{cases}$$

Shioda and Inose [71] proved that, for every singular $K3$ surface X, there exist isogenous CM elliptic curves E and E' such that $\mathrm{T}(X) \cong \mathrm{T}(\mathrm{Km}(E \times E'))$ together with the following diagram:

$$\begin{array}{ccc} X & & E \times E' \\ & \searrow \quad \swarrow & \\ & \mathrm{Km}(E \times E') & \end{array}$$

where the two arrows denote rational maps of degree 2. Morrison [49] then extended this result by proving that every $K3$ surface with Picard number 19 also has this property, but the two elliptic curves in the diagram do not have complex multiplication. Regardless of whether or not E and E' are CM, the diagram above is generally known as a *Shioda-Inose structure*.

As lattices, we have that $H^2(X,\mathbb{Z})$ is isometric to $E_8(-1)^2 \oplus U^3$, where U is the hyperbolic lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $E_8(-1)$ denotes the even unimodular negative definite lattice of rank 8. Let $M = \mathbb{Z}v$ where $\langle v, v\rangle = 2n$ for some $n \in \mathbb{N}$. Then M can be thought of as a primitive sublattice of U; i.e., U/M is free, and we denote the lattice M by $\langle 2n\rangle$. The orthogonal complement of M in $E_8(-1)^2 \oplus U^2$ is then isometric to

$$M_n := E_8(-1)^2 \oplus U \oplus \langle -2n\rangle,$$

which is of rank 19. We define an *M_n-polarized $K3$ surface* by a pair (X, φ), where X is a $K3$ surface and $\varphi : M_n \hookrightarrow \mathrm{NS}(X)$ is a primitive lattice embedding.

Let ω_X be a holomorphic 2-form on X which vanishes nowhere. Note that there is a natural isomorphism $H^2(X,\mathbb{Z}) \cong H_2(X,\mathbb{Z})$; i.e., $H_2(X,\mathbb{Z})$ is also a free $\mathbb{Z}$-module of rank 22. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_{22}\}$ be a basis for $H_2(X,\mathbb{Z})$. Then the period

$$\int_{\gamma_i} \omega_X$$

vanishes if and only if $\gamma_i \in \mathrm{Pic}(X)$; i.e., when γ_i is an algebraic cycle. Recall that if Y_t is a one-parameter family of varieties, then its periods satisfy the Picard-Fuchs differential equation of Y_t. In the case of elliptic curves, a Picard-Fuchs equation is a homogeneous ordinary linear differential equation of order two. For any second order differential operator

$$L := \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t), \qquad p(t), q(t) \in \mathbb{C}(t),$$

the *symmetric square* of L is defined by

$$\mathrm{Sym}^2(L) := \frac{d^3}{dt^3} + 3p(t)\frac{d^2}{dt^2} + (2p(t)^2 + 4q(t) + p'(t))\frac{d}{dt} + (4p(t)q(t) + 2q'(t)).$$

If Y_t is a family of $K3$ surfaces with generic Picard number l, then the order of its Picard-Fuchs equations is $22 - l$. If we assume further that Y_t is a family of M_n-polarized $K3$ surfaces, then it follows immediately that it has generic Picard number 19, whence the associated Picard-Fuchs equation is of order three, and, for each t, Y_t admits a Shioda-Inose structure. Moreover, we have the following result:

Theorem 2.6 (Doran, Long [26, 45])**.** *The Picard-Fuchs equation of a family Y_t of M_n polarized $K3$ surface is the symmetric square of an order two ordinary linear Fuchsian differential equation. Moreover, up to a change of variables, the order two differential equation is the Picard-Fuchs equation of some family E_t of elliptic curves which gives rise to a Shioda-Inose structure on Y_t.*

Similar to elliptic curves, one can attach L-series to $K3$ surfaces defined over $\mathbb{Q}$ by piecing the local arithmetic information together. We first recall the notion of zeta functions of varieties. Let V be a smooth projective variety of dimension k over a finite field $\mathbb{F}_q$ of q elements. Suppose further that V is geometrically irreducible; i.e., it is irreducible over $\bar{\mathbb{F}}_q$. For each $n \in \mathbb{N}$, let $N_n = \#V(\mathbb{F}_{q^n})$, where $\mathbb{F}_{q^n}$ is a degree n field extension of $\mathbb{F}_q$. Then the *zeta function* attached to V is defined by

$$Z_V(T) := \exp\left(\sum_{n\geq 1} N_n \frac{T^n}{n}\right).$$

Weil conjectures assert that $Z_V(T) \in \mathbb{Q}(T)$ and it satisfies the Riemann hypothesis; i.e.,

$$Z_V(T) = \frac{P_1(T)\cdots P_{2k-1}(T)}{P_0(T)P_2(T)\cdots P_{2k}(T)},$$

where, for $0 \leq i \leq 2k$, $P_i(T) \in \mathbb{Z}[T]$. We also have that $P_0(T) = 1-T$, $P_{2k}(T) = 1-q^kT$, and

$$P_i(T) = \prod_{j=1}^{b_i}(1 - \alpha_{ij}T),$$

with $|\alpha_{ij}| = q^{\frac{i}{2}}$ and $b_i = \dim_{\mathbb{C}} H^i(V, \mathbb{C})$. The rationality of $Z_V(T)$ and the Riemann hypothesis were proved by Dwork (1960) and Deligne (1973), respectively. If $E/\mathbb{F}_q$ is an elliptic curve, then we have that

$$Z_E(T) = \frac{1 - a_q(E)T + qT^2}{(1-T)(1-qT)},$$

where $a_q(E)$ is the trace of Frobenius of E over $\mathbb{F}_q$. If X is an algebraic $K3$ surface over $\mathbb{Q}$, then for all but finitely many primes p, the reduction of X modulo p, after desingularization, is also a $K3$ surface, which we denote by X_p. In this case, the zeta function of X_p is of the form

$$Z_{X_p}(T) = \frac{1}{(1-T)P_2(T)(1-p^2T)},$$

where $P_2(T)$ is a polynomial of degree 22 and $P_2(0) = 1$. Furthermore, $P_2(T)$ can be written as a product

$$P_2(T) = Q_p(T)R_p(T),$$

where $Q_p(T)$ and $R_p(T)$ come from the transcendental and algebraic cycles, respectively. In particular, if X is a singular $K3$ over $\mathbb{Q}$, then $\deg Q_p(T) = 2$ and $\deg R_p(T) = 20$. Then we define the L-series of $\mathrm{T}(X)$ by

$$L(\mathrm{T}(X), s) := (*) \prod_{p \text{ good}} \frac{1}{Q_p(p^{-s})} = \sum_{n \geq 1} \frac{a_n}{n^s},$$

where $(*)$ is the product of the Euler factors corresponding to the primes of bad reduction.

2.4 Modular forms and the modularity theorem

We end this section with the basic theory of classical modular forms, which will be used throughout this thesis. The main references used in writing this section include [36] and [50].

For a positive integer N, we define the *level N congruence subgroups* $\Gamma_0(N), \Gamma_1(N)$, and $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \text{ and } c \equiv 0 \mod N \right\},$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \text{ and } b \equiv c \equiv 0 \mod N \right\}.$$

These subgroups are known to have finite indices in $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathrm{GL}_2^+(\mathbb{R}) = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) \mid \det\gamma > 0\}$. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, an integer k, and a meromorphic function f on $\mathcal{H}$, we define the *slash operator* $|_k$ by

$$(f|_k\gamma)(z) := (\det\gamma)^{k/2}(cz+d)^{-k}f(\gamma z),$$

where

$$\gamma z := \frac{az+b}{cz+d}, \tag{2.1}$$

called the *fractional linear transformation*. An orbit in $\mathbb{Q} \cup \{\infty\}$ under the action (2.1) of a congruence subgroup Γ is called a *cusp* of Γ.

Definition 2.7. *Let Γ be a level N congruence subgroup and let $k \in \mathbb{Z}$. A holomorphic function f on $\mathcal{H}$ is called a modular form for Γ of weight k if f satisfies the following:*

(i) $f|_k\gamma = f$ for all $\gamma \in \Gamma$,

(ii) f is holomorphic at all cusps of Γ.

If a modular form f vanishes at all cusps, then f is said to be a cusp form.

Let Γ be congruence subgroup of level N. If f is a meromorphic function whose poles are supported on the cusps of Γ and $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$, then f is said to be a *modular function*. We also have that $\Gamma\backslash\mathcal{H}$ becomes an algebraic curve, which can be compactified by adding the cusps of Γ to the quotient. The compactified curve is called a *modular curve*. The modular curves corresponding to $\Gamma(N), \Gamma_1(N)$, and $\Gamma_0(N)$ are denoted by $X(N), X_1(N)$, and $X_0(N)$, respectively. Furthermore, if a modular curve is of genus zero, then its function field can be generated by a single modular function, called a *Hauptmodul* for Γ. For example, a Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$ is the modular j-function

$$j(z) := \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots, \qquad q = e^{2\pi i z}.$$

The modular forms (resp. cusp forms) for Γ of weight k form a $\mathbb{C}$-vector space $M_k(\Gamma)$ (resp. $S_k(\Gamma)$). By the definition of a modular form, we have that if Γ is a congruence subgroup containing $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then every modular form f for Γ has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n,$$

where $q = e^{2\pi i z}$, and $a_0 = 0$ if f is a cusp form.

Definition 2.8. *Let χ be a Dirichlet character modulo N. Then a modular form $f(z) \in M_k(\Gamma_1(N))$ is said to have Nebentypus character χ if*

$$f(\gamma z) = \chi(d)(cz + d)^k f(z),$$

for all $z \in \mathcal{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

The modular forms (resp. cusp forms) having Nebentypus character χ form a $\mathbb{C}$-vector space, denoted by $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$). Moreover, $S_k(\Gamma_0(N), \chi)$ equipped with the *Petersson inner product*, given by

$$\langle f, g \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]} \int_{\Gamma_1 \backslash \mathcal{H}} f(x + iy) \overline{g(x + iy)} y^k \frac{dx dy}{y^2},$$

is a finite dimensional Hilbert space.

Let $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)$. Then, for each $m \in \mathbb{N}$, we have the *Hecke* *operator* T_m acting on $f(z)$ by

$$(T_m f)(z) := \sum_{n=0}^{\infty} \left(\sum_{d|(m,n)} \chi(d) d^{k-1} a(mn/d^2) \right) q^n,$$

where (m, n) denotes the greatest common divisor of m and n. It can be shown that, for every $m \in \mathbb{N}$, T_m preserves $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$. We say that $f(z) \in M_k(\Gamma_0(N), \chi)$ is a *Hecke eigenform* if it is an eigenvector of T_m for all m.

If M and N are positive integers such that M divides N properly and $f \in S_k(\Gamma_0(M), \chi)$, then we can obtain $g \in S_k(\Gamma_0(N), \chi)$ from f by defining $g(z) := f(dz)$, where d is a positive divisor of N/M. We denote by $S_k^{\text{old}}(\Gamma_0(N), \chi)$ the subspace of $S_k(\Gamma_0(N), \chi)$ generated by all forms of type $f(dz)$ where $f \in S_k(\Gamma_0(M), \chi')$, d and M are positive integers such that $dM|N$ and $M < N$, and χ' is the Dirichlet character modulo M induced by χ. The elements in $S_k^{\text{old}}(\Gamma_0(N), \chi)$ are called *old forms*. Then the *subspace of newforms* $S_k^{\text{new}}(\Gamma_0(N), \chi)$ is defined to be the orthogonal complement of $S_k^{\text{old}}(\Gamma_0(N), \chi)$ in $S_k(\Gamma_0(N), \chi)$.

Definition 2.9. *A Hecke eigenform in* $S_k^{new}(\Gamma_0(N), \chi)$ *is called a newform of weight* k *level* N *and character* χ*. If* $f(z) = \sum_{n=1}^{\infty} a(n) q^n$ *is a newform and* $a(1) = 1$*, then* f *is said to be a normalized newform.*

Note that if $f(z) = \sum_{n=1}^{\infty} a(n) q^n$ is a newform, then $a(1) \neq 0$, so we can always find the normalized newform corresponding to f.

Next, we recall the definition of a class of newforms which are of particular interest to us and will play a major role in this thesis. Let $f(z) = \sum_{n=1}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)$ and let ϕ be a Dirichlet character modulo D. Then the twist of f by ϕ is defined by

$$f \otimes \phi(z) := \sum_{n=0}^{\infty} \phi(n) a(n) q^n.$$

It can be checked that $f \otimes \phi \in M_k(\Gamma_0(ND^2), \chi\phi^2)$, and if f is a cusp form, then so is

$f \otimes \phi$. Suppose that f is a newform. Then we say that f has complex multiplication (CM) by ϕ if $f \otimes \phi = f$. In fact, in order to conclude that f has CM by ϕ it suffices to check that $\phi(p)a_p = a_p$ for a set of primes of density 1. Moreover, if this is the case, then ϕ must be the quadratic character associated to some imaginary quadratic field K, and we also say that f has CM by K.

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$, $\mathcal{O}_K$ the ring of integers of K, Λ a nontrivial ideal in $\mathcal{O}_K$, $I(\Lambda)$ be the group of fractional ideals coprime to Λ, and let $k \geq 2$ be a positive integer. A *Hecke character* (or *Grössencharacter*) φ modulo Λ of weight k is defined to be a group homomorphism

$$\varphi : I(\Lambda) \to \mathbb{C}^\times$$

such that for every $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\Lambda}$

$$\varphi(\alpha\mathcal{O}_K) = \alpha^{k-1}$$

The ideal Λ is called the *conductor* of φ if Λ is minimal; i.e., if φ is defined modulo Λ', then $\Lambda|\Lambda'$. One can always construct a CM newform from a Hecke character of an imaginary quadratic field, and this is indeed the only way to obtain a CM newform. More precisely, we have the following result:

Theorem 2.10 (Hecke, Shimura, Ribet). *Let φ be a Hecke character of weight k with conductor Λ of the field K given above. Define $f_\varphi(z)$ by*

$$f_\varphi(z) := \sum_{\mathfrak{a}} \varphi(\mathfrak{a}) q^{N(\alpha)} = \sum_{n=1}^{\infty} a(n) q^n,$$

where $\mathfrak{a}$ runs through the integral ideals in $I(\Lambda)$, and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. Then f_φ is a newform in $S_k(\Gamma_0(D \cdot N(\Lambda)), \chi_K\omega_\varphi)$, where χ_K is the quadratic character attached to K, and ω_φ is the Dirichlet character modulo $N(\Lambda)$ given by

$$\omega_\varphi(m) = \frac{\varphi(m\mathcal{O}_K)}{m^{k-1}}, \qquad \textit{for each } m \in \mathbb{Z} \textit{ such that } (m, N(\Lambda)) = 1,$$

and f_φ has CM by K.

Furthermore, every newform which has CM by K comes from a Hecke character of K using the construction above.

If $f_\varphi(z)$ is as defined in Theorem 2.10 and the Fourier coefficients of $f_\varphi(z)$ are real, then the Dirichlet character ω_φ is either χ_K or a trivial character, depending on whether k is even or odd.

We are now in a good position to introduce the notion of Dirichlet and modular L-functions. Let χ be a primitive Dirichlet character of conductor D. Then its *Dirichlet L-series* is defined by

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which converges absolutely if $\mathrm{Re}(s) > 1$, so it is well-defined on this region. The function $L(\chi, s)$ can be analytically continued to a holomorphic function on the entire complex plane, called a *Dirichlet L-function*, and we will also denote it by $L(\chi, s)$. Recall that χ is said to be *even* if $\chi(-1) = 1$ and *odd* if $\chi(-1) = -1$. Define

$$\xi(\chi, s) := \left(\frac{\pi}{D}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(\chi, s),$$

where $a = 0$ if χ is even and $a = 1$ if χ is odd. Then we have the functional equation

$$\xi(\bar{\chi}, 1-s) = \frac{i^a D^{1/2}}{\tau(\chi)} \xi(\chi, s), \tag{2.2}$$

where $\tau(\chi) = \sum_{n=1}^{D} \chi(n) e^{2\pi i n/D}$, the *Gauss sum* of χ. If we assume further that χ is real, then the functional equation (2.2) simply reads

$$\xi(\chi, 1-s) = \xi(\chi, s). \tag{2.3}$$

Usually, we can obtain identities relating Dirichlet L-functions to their derivatives from the functional equation (2.3). For instance, if χ is an odd primitive quadratic character of conductor D, then

$$L'(\chi,-1)=\frac{D^{3/2}}{4\pi}L(\chi,2). \tag{2.4}$$

Next, we discuss basic facts about modular L-functions. One can define the L-function attached to a modular form in general, but, for our purposes, we will mainly consider the case of cusp forms. Let $f(z)=\sum_{n=1}^{\infty}a(n)q^n\in S_k(\Gamma_0(N),\chi)$, where χ here is any Dirichlet character. We define the L-series associated to f by

$$L(f,s):=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}.$$

This series is absolutely convergent in the right half-plane $\{s\in\mathbb{C}\mid \mathrm{Re}(s)>\frac{k+1}{2}\}$. Moreover, if s belongs to such region, then we have that

$$\int_0^{\infty}f(iy)y^{s-1}dy=\frac{\Lambda(f,s)}{\sqrt{N}^s},$$

where

$$\Lambda(f,s):=\left(\frac{\sqrt{N}}{2\pi}\right)^s\Gamma(s)L(f,s).$$

The integral above is known as the *Mellin transform* of f and we call the function $\Lambda(f,s)$ the *complete L-function* associated to f. The L-series of f has an analytic continuation over the whole complex plane which is an entire function called the *L-function* of f and is again denoted by $L(f,s)$. If χ is a real character and f is a Hecke eigenform with real Fourier coefficients, then $\Lambda(f,s)$ satisfies the functional equation

$$\Lambda(f,s)=\epsilon\Lambda(f,k-s), \tag{2.5}$$

where $\epsilon\in\{-1,1\}$ is called the *sign* of the functional equation. In this particular case, we can deduce from (2.5) by letting $s\to 0$ that

$$L'(f,0)=\epsilon\Lambda(f,k)=\epsilon\left(\frac{\sqrt{N}}{2\pi}\right)^k(k-1)!L(f,k). \tag{2.6}$$

We have seen L-functions attached to various types of mathematical objects. It turns out that, under certain conditions, one can draw some beautiful connections between two different objects using their L-functions. Here we state two important results which reveal relationships between L-functions attached to algebraic varieties and those attached to newforms.

Theorem 2.11 (Modularity theorem, [13, 70, 81])**.** *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Then there exists a newform $f \in S_2(\Gamma_0(N))$ with integral Fourier coefficients such that*

$$L(E, s) = L(f, s).$$

Furthermore, if E has complex multiplication, then f is a CM newform.

The statement in Theorem 2.11 is formerly known as the *Taniyama-Shimura conjecture* and is one of the main tools used by A. Wiles to prove the *Fermat's last theorem*. There is also an analogous result in a higher dimensional case, namely

Theorem 2.12 (Livné, [42])**.** *Let X be a singular $K3$ surface defined over $\mathbb{Q}$ with discriminant D. Then there exists a newform g of weight 3 with CM by $\mathbb{Q}(\sqrt{-D})$ such that*

$$L(\mathrm{T}(X), s) = L(g, s).$$

If X is a $K3$ surface over $\mathbb{Q}$ with $\rho(X) = 19$, then we have from Section 2.3 that X admits a Shioda-Inose structure, where the corresponding isogenous elliptic curves E and E' are non-CM. In this case, X is *potentially modular* in the sense that, over some number field K, the L-function $L(\mathrm{T}(X), s)$ coincides with $L(\mathrm{Sym}^2(E), s)$. (Remark that since E and E' are isogenous, their L-series are the same. In particular, we are free to consider either $L(\mathrm{Sym}^2(E), s)$ or $L(\mathrm{Sym}^2(E'), s)$.) For more details, we refer the reader to [83, Thm.4]

3. MAHLER MEASURES OF TWO-VARIABLE POLYNOMIALS*

After the remarkable conjectured formula (1.4) was discovered by Deninger, Boyd [12] extensively computed Mahler measures of several families of two-variable polynomials and found many conjectural formulas expressed in terms of rational multiples of $L'(E,0)$, where E is the elliptic curve associated to the polynomial. Motivated by Boyd's results and the Bloch-Beilinson conjectures, Rodriguez Villegas [55] constructed sixteen equivalence classes of newton polygons with one interior lattice point. To each of these polygons, we can associate a family of tempered Laurent polynomials of two variables with a hypergeometric period. Listed below are families of Laurent polynomials whose periods are associated to the four hypergeometric weight systems in Table 2.1.

$$P_t := x + x^{-1} + y + y^{-1} - t^{1/2}, \qquad Q_t := x^3 + y^3 + 1 - t^{1/3}xy,$$

$$R_t := y + xy^{-1} + (xy)^{-1} - t^{1/4}, \qquad S_t := x^2y^{-1} - yx^{-1} - (xy)^{-1} - t^{1/6},$$

(Here t is a complex parameter.) Note that these are just the chosen representatives of equivalence classes of families of polynomials; i.e., there are many more families which have the same periods (and Mahler measures) as the families above. Also, we take a root of t in each family in order to normalize their Mahler measure formulas. This will become clear later.

3.1 The family $P_t = x + x^{-1} + y + y^{-1} - t^{1/2}$

Let us first consider the family P_t. Before discussing its Mahler measure, we will explain how one can obtain a period of this in terms of a hypergeometric function using Rodriguez Villegas's arguments in [55, IV]. Let $\lambda = 1/t$. Then the zero locus of P_t

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coincides with that of $1-\lambda^{1/2}\left(x+x^{-1}+y+y^{-1}\right)$, so they define the same curve in the projective space $\mathbb{P}^2$. By the framework of Griffiths [31], one has that a period of this curve is given by

$$u_0(\lambda)=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\frac{1}{1-\lambda^{1/2}\left(x+x^{-1}+y+y^{-1}\right)}\frac{dx}{x}\frac{dy}{y},\qquad |\lambda|<\frac{1}{16},$$

where $\mathbb{T}^n$ is the n-dimensional torus. By the change of variable $x\mapsto xy, y\mapsto y/x$, we have that

$$\begin{aligned}u_0(\lambda)&=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\frac{1}{1-\lambda^{1/2}\left(x+x^{-1}\right)\left(y+y^{-1}\right)}\frac{dx}{x}\frac{dy}{y}\\&=\int_0^1\int_0^1\frac{d\theta_1 d\theta_2}{1-4\lambda^{1/2}\cos(2\pi\theta_1)\cos(2\pi\theta_2)}.\end{aligned}$$

Then we take the power series of the integrand and use the elementary formula, for $n\geq 0$,

$$\int_0^1\cos(2\pi\theta)^n d\theta=\begin{cases}0 & \text{if } n \text{ is odd,}\\ \dfrac{n!}{2^n(n/2)!^2} & \text{if } n \text{ is even,}\end{cases}$$

to deduce that

$$u_0(\lambda)=\sum_{n=0}^{\infty}\binom{2n}{n}^2\lambda^n={}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix};16\lambda\right).$$

Now by the definition of the Mahler measure and the change of variables used previously we have that, for $|\lambda|<1/16$,

$$\begin{aligned}m(P_{t(\lambda)})&=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log|\lambda^{-1/2}-\left(x+x^{-1}+y+y^{-1}\right)|\frac{dx}{x}\frac{dy}{y}\\&=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log|\lambda^{-1/2}-\left(x+x^{-1}\right)\left(y+y^{-1}\right)|\frac{dx}{x}\frac{dy}{y}\\&=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\operatorname{Re}\left(\log\left(\lambda^{-1/2}-\left(x+x^{-1}\right)\left(y+y^{-1}\right)\right)\right)\frac{dx}{x}\frac{dy}{y}\\&=\operatorname{Re}\left(-\frac{\log\lambda}{2}+\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log\left(1-\lambda^{1/2}\left(x+x^{-1}\right)\left(y+y^{-1}\right)\right)\frac{dx}{x}\frac{dy}{y}\right)\end{aligned}$$

$$= \frac{1}{2}\operatorname{Re}\left(-\log\lambda - \int_0^\lambda (u_0(s)-1)\frac{ds}{s}\right)$$

$$= \frac{1}{2}\operatorname{Re}\left(-\log\lambda - \sum_{n=1}^\infty \binom{2n}{n}^2 \frac{\lambda^n}{n}\right)$$

$$= \frac{1}{2}\operatorname{Re}\left(-\log\lambda - 4\lambda\, {}_4F_3\left(\begin{matrix}\frac{3}{2},\ \frac{3}{2},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix};16\lambda\right)\right),$$

where here and throughout we use the principal branch of the logarithm. Let $m_2(t) := 2m(P_t)$. Then, for $|t| > 16$, we have the Mahler measure formula

$$m_2(t) = \operatorname{Re}\left(\log t - \frac{4}{t}\, {}_4F_3\left(\begin{matrix}\frac{3}{2},\ \frac{3}{2},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix};\frac{16}{t}\right)\right). \tag{3.1}$$

Indeed, it was proved by Rodriguez Villegas that the formula (3.1) holds for all $t \neq 0$. As mentioned in Section 1, the first known formula of $m_2(t)$ involving a special value of L-function is

$$m_2(1) = \frac{15}{2\pi^2}L(E,2) = 2L'(E,0),$$

where E is the elliptic curve of conductor 15 defined by the projective closure of the zero locus of P_1. This formula had been conjectural for years before being proved by Rogers and Zudilin [61]. Boyd and Rodriguez Villegas then found many other formulas of the form

$$m_2(t) \stackrel{?}{=} c_t L'(E_t, 0), \tag{3.2}$$

where $c_t \in \mathbb{Q}$ and E_t is the elliptic curve associated to P_t. In particular, Rodriguez Villegas observed that the formula (3.2) seems to be true for all sufficiently large $t \in \mathbb{Z}$ and he gave a list of conjectural formulas of $m_2(t)$ in [55, Tab.4]. One of the possible reasons why one needs t to satisfy such conditions is that E_t has a Weierstrass form

$$E_t : y^2 = x^3 + \frac{t}{8}\left(\frac{t}{8} - 1\right)x^2 + \frac{t^2}{256}x,$$

which is defined over $\mathbb{Q}$ if $t \in \mathbb{Q}$. Furthermore, the Bloch-Beilinson conjectures seem to be applicable only to Néron models, which can be thought of as an integral version of elliptic curves. Although Boyd's and Rodriguez Villegas' formulas can be verified numerically to high degree of accuracy, rigorous proofs of these formulas are quite rare (see Table 3.1 below). Rodriguez Villegas [55] proved that $m_2(t)$ can be expressed in terms of Eisenstein-Kronecker series, and for certain values of t they turn out to be related to special values of L-series of elliptic curves with complex multiplication. When $t = 16$,

t	Reference(s)
$8, 18, 32$	Rodriguez Villegas, [55]
1	Rogers, Zudilin, [61, 90]
$4, 64$	Lalín, Rogers, [39]
$-4, -1, 2$	Rogers, Zudilin, [60, 90]

Table 3.1: Values of t for which Formula (3.2) is known to be true.

the curve E_t is of genus zero, and it was proved in [61, §IV.15] that

$$m_2(16) = \frac{8}{\pi} L(\chi_{-4}, 2) = 4L'(\chi_{-4}, -1),$$

where here and throughout $\chi_D = \left(\frac{D}{\cdot}\right)$, and $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. We shall briefly explain Rodriguez Villegas' ideas used to prove some formulas of the form (3.2). Recall that the period $u_0(\lambda)$ associated to the family P_t satisfies the second order differential equation

$$\lambda(16\lambda - 1)\frac{d^2u}{d\lambda^2} + (32\lambda - 1)\frac{du}{d\lambda} + 4u = 0,$$

so it is the Picard-Fuchs differential equation of this family. A non-holomorphic solution around $\lambda = 0$ is

$$u_1(\lambda) = u_0(\lambda)\log\lambda + 8\lambda + 84\lambda^2 + \frac{2960}{3}\lambda^3 + \cdots.$$

Let

$$\tau := \frac{1}{2\pi i}\frac{u_1}{u_0}(\lambda), \qquad q := e^{2\pi i\tau}.$$

Then we can write λ as a function of τ in terms of a q-series by locally inverting q above. Also, define

$$c(\tau) = u_0(\lambda(\tau)), \qquad e(\tau) = q\frac{c(\tau)}{\lambda(\tau)}\frac{d\lambda}{dq}(\tau).$$

The functions $\lambda(\tau), c(\tau)$, and $e(\tau)$ turn out to be modular forms of weight $0, 1$, and 3, respectively, under the action of the monodromy group of the Picard-Fuchs equation, which is $\Gamma_0(4)$ in this case. We have further that $t_2(\tau) := 1/\lambda(\tau)$ is a Hauptmodul for this group, and it can be written as

$$t_2(\tau) = \frac{\eta^{24}(2\tau)}{\eta^8(\tau)\eta^{16}(\tau)},$$

where $\eta(\tau)$ is the *Dedekind eta function*

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n).$$

The function $c(\tau)$ and $e(\tau)$ can be expressed as

$$c(\tau) = 1 + 4\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)q^n,$$

$$e(\tau) = 1 - 4\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2q^n.$$

In fact, the function $e(\tau)$ is an *Eisenstein series*, which is a well-known modular form defined below.

Proposition 3.1 ([35, Prop. 5.1.2]). *Let χ be a non-trivial primitive character modulo N, and let $B_{n,\chi}$ denote the n^{th} generalized Bernoulli number, defined by*

$$\sum_{n=1}^{N}\frac{\chi(n)te^{nt}}{e^{Nt}-1} = \sum_{n=0}^{\infty}B_{n,\chi}t^n, \qquad |t| < \frac{2\pi}{N}.$$

If $k \geq 2, \chi(-1) = (-1)^k$, *and* $q = q(\tau) := e^{2\pi i \tau}$, *then the function*

$$E_{k,\chi}(\tau) := 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} q^n$$

belongs to $M_k(\Gamma_0(N), \chi)$.

In particular, we have that $e(\tau) = E_{3,\chi_{-4}}(\tau)$. Note also that if χ is the trivial character modulo 1, then, for $n \geq 2$, $B_{n,\chi}$ is the usual n^{th} Bernoulli number B_n, and $B_{1,\chi} = 1/2 = -B_1$, and $E_k(\tau) := E_{k,\chi}(\tau)$ is the classical Eisenstein series of weight k. Now it is not hard to see that, for sufficiently large $|\tau|$,

$$\begin{aligned} m_2(t_2(\tau)) &= \operatorname{Re}\left(-\log \lambda(\tau) - \int_0^{\lambda(\tau)} (u_0(s) - 1) \frac{ds}{s}\right) \\ &= \operatorname{Re}\left(-2\pi i \tau + 4 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 \frac{q^n}{n}\right). \end{aligned}$$

Then Rodriguez Villegas used some calculation of the Fourier series of the the series above to establish the following result:

Proposition 3.2 ([56, §6]). *Let* $\mathcal{F}$ *be the fundamental domain of* $\Gamma_0(4)$ *formed by the geodesic triangle in* $\{x + iy \mid x, y \geq 0\}$ *with vertices* $i\infty, 0,$ *and* $1/2$ *and its reflection along the* y*-axis. Then* $t_2(\tau)$ *is a surjective map from* $\mathcal{F}$ *to* $\mathbb{C} \cup \{\infty\}$*. Furthermore, for every* $\tau \in \mathcal{F}$*, we have that*

$$m_2(t_2(\tau)) = \frac{32 \operatorname{Im} \tau}{\pi^2} \sideset{}{'}\sum_{m,n \in \mathbb{Z}} \frac{\chi_{-4}(m)(m + 4n \operatorname{Re}(\tau))}{[(m + 4n\tau)(m + 4n\bar{\tau})]^2}, \tag{3.3}$$

where $\sideset{}{'}\sum_{m,n}$ *means that* $(m, n) = (0, 0)$ *is excluded from the summation.*

The series in the formula (3.3) is known as an *Eisenstein-Kronecker series*, whose general definition shall be given later. If τ is a CM point, then the corresponding elliptic curve $E_{t_2(\tau)}$ has complex multiplication by an order in some imaginary quadratic field K, in which case the L-function $L(E_{t_2(\tau)}, s)$ is the L-series attached to some weight 2

Hecke character of K. In certain cases, the latter L-series can be rewritten as Eisenstein-Kronecker series, so we can derive formulas of the form (3.2) using these facts. For instance, choosing $\tau_1 = \sqrt{-1}/2$, and $\tau_2 = (1+\sqrt{-1})/4$ we have $t_2(\tau_1) = 32$, $t_2(\tau_2) = 8$, and

$$m_2(t_2(\tau_1)) = 2L'(E_{32}, 0) = 2L'(f_{64}, 0), \tag{3.4}$$

$$m_2(t_2(\tau_2)) = 2L'(E_8, 0) = 2L'(f_{32}, 0), \tag{3.5}$$

where $f_{64}(\tau) = \dfrac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)} \in S_2(\Gamma_0(64))$ and $f_{32}(\tau) = \eta^2(4\tau)\eta^2(8\tau) \in S_2(\Gamma_0(32))$ are the newforms corresponding to E_{32} and E_8, respectively, via the modularity theorem. (Throughout, f_N denotes a normalized newform of weight 2 and level N with rational Fourier coefficients.)

Kurokawa and Ochiai [38] and Lalín and Rogers [39] showed that $m_2(t)$ satisfies some functional equations, which enable us to prove and conjecture new Mahler measure formulas for some $t \notin \mathbb{Z}$.

Theorem 3.3. *The following identities are true:*

$$m_2(8+6\sqrt{2}) = L'(f_{64}, 0) + L'(f_{32}, 0), \tag{3.6}$$

$$m_2(8-6\sqrt{2}) = L'(f_{64}, 0) - L'(f_{32}, 0). \tag{3.7}$$

Proof. It was proved in [38, Thm. 7] that if $k \in \mathbb{R} \setminus \{0\}$, then

$$2m_2\left(4\left(k+\frac{1}{k}\right)^2\right) = m_2(16k^4) + m_2\left(\frac{16}{k^4}\right). \tag{3.8}$$

Recall from (3.4) and (3.5) that $m_2(32) = 2L'(f_{64}, 0)$ and $m_2(8) = 2L'(f_{32}, 0)$, so we can deduce (3.6) easily by substituting $k = 2^{1/4}$ in (3.8). On the other hand, one sees from [39, Thm. 2.2] that the following functional equation holds for any k such that $0 < |k| < 1$:

$$m_2\left(4\left(k+\frac{1}{k}\right)^2\right) + m_2\left(-4\left(k-\frac{1}{k}\right)^2\right) = m_2\left(\frac{16}{k^4}\right). \tag{3.9}$$

In particular, choosing $k = 2^{-1/4}$, we obtain

$$m_2(8+6\sqrt{2}) + m_2(8-6\sqrt{2}) = m_2(32).$$

Now (3.7) follows immediately from the known information above. $\square$

Rodriguez Villegas [55, Tab. 4] verified numerically that $m_2(128) \stackrel{?}{=} \frac{1}{2}L'(f_{448}, 0)$ and $m_2(2) = \frac{1}{2}L'(f_{56}, 0)$, where $f_{448}(\tau) = q - 2q^5 - q^7 - 3q^9 + 4q^{11} - 2q^{13} - 6q^{17} - \cdots$ and $f_{56}(\tau) = q + 2q^5 - q^7 - 3q^9 - 4q^{11} + 2q^{13} - 6q^{17} + \cdots$. (The latter identity was recently proved by Zudilin [90].) Therefore, letting $k = 2^{3/4}$ in (3.8) and $k = 2^{-3/4}$ in (3.9) results in a couple of conjectured formulas similar to (3.6) and (3.7).

Conjecture 3.4. *The following identities are true:*

$$m_2\left(8 \pm 9\sqrt{2}\right) \stackrel{?}{=} \frac{1}{4}\left(L'(f_{448}, 0) \pm L'(f_{56}, 0)\right).$$

We also found via numerical computations the following conjectured formulas:

$$m_2\left(\frac{49 \pm 9\sqrt{17}}{2}\right) \stackrel{?}{=} \frac{1}{2}\left(L'(f_{289}, 0) \pm 8L'(f_{17}, 0)\right),$$

where $f_{289}(\tau) = q - q^2 - q^4 + 2q^5 - 4q^7 + 3q^8 - 3q^9 - \cdots$ and $f_{17}(\tau) = q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + \cdots$. Observe that we can again employ the identity (3.8) for $k = (1+\sqrt{17})/4$ to deduce

$$2m_2(17) = m_2\left(\frac{49 + 9\sqrt{17}}{2}\right) + m_2\left(\frac{49 - 9\sqrt{17}}{2}\right) \stackrel{?}{=} L'(f_{289}, 0),$$

which is equivalent to a conjectured formula in [55, Tab. 4]. A weaker form of these formulas, namely

$$m_2\left(\frac{49 + 9\sqrt{17}}{2}\right) - m_2(17) \stackrel{?}{=} 4L'(f_{17}, 0),$$

was also briefly discussed in [59, §4].

3.2 The family $Q_t = x^3 + y^3 + 1 - t^{1/3}xy$

For $t \neq 0, 27$, the polynomials Q_t again define a family of elliptic curves, generally known as the *Hesse family*. It has a Weierstrass form

$$E_t : y^2 = x^3 - 27t^2x^2 + 216t^3(t-27)x - 432t^4(t-27)^2.$$

We can imitate arguments in Section 3.1 to obtain Mahler measure formulas analogous to (3.1) and (3.3). More precisely, if $m_3(t) := 3m(Q_t)$, then, for $|t| \geq 27$,

$$m_3(t) = \operatorname{Re}\left(\log(t) - \frac{6}{t}\,{}_4F_3\left(\begin{matrix}\frac{4}{3}, \frac{5}{3}, 1, 1\\ 2, 2, 2\end{matrix}; \frac{27}{t}\right)\right). \tag{3.10}$$

After Rodriguez Villegas, the Weierstrass form of E_t partially suggests that if $t \in \mathbb{Z}$ is sufficiently large, then

$$m_3(t) \stackrel{?}{=} c_t L'(E_t, 0), \qquad c_t \in \mathbb{Q}. \tag{3.11}$$

By some combinatorial arguments (see [55, §IV.12]), one has that a period of the family E_t is

$$\begin{aligned} u_0(\lambda) &= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \frac{1}{1 - \lambda^{1/3}\left(x^2y^{-1} + y^2x^{-1} + x^{-1}y^{-1}\right)}\frac{dx}{x}\frac{dy}{y}, \\ &= \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\lambda^n = {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}; 27\lambda\right), \qquad \lambda = 1/t. \end{aligned}$$

Hence the Picard-Fuchs equation of this family, satisfied by $u_0(\lambda)$, is

$$\lambda(27\lambda - 1)\frac{d^2u}{d\lambda^2} + (54\lambda - 1)\frac{du}{d\lambda} + 6u = 0,$$

and it has a non-holomorphic solution

$$u_1(\lambda) = u_0(\lambda)\log\lambda + 15\lambda + \frac{513}{2}\lambda^2 + 5018\lambda^3 + \cdots.$$

Let $\tau, q, \lambda(\tau), c(\tau)$, and $e(\tau)$ be define analogously to those in Section 3.1. Then we have, for sufficiently large $|\tau|$,

$$t_3(\tau) := \frac{1}{\lambda(\tau)} = 27 + \left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12},$$

$$c(\tau) = 1 + 6\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-3}(d)q^n,$$

$$e(\tau) = 1 - 9\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-3}(d)d^2q^n = E_{3,\chi_{-3}}(\tau),$$

$$m_3(t_3(\tau)) = \operatorname{Re}\left(-2\pi i\tau + 9\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-3}(d)d^2\frac{q^n}{n}\right).$$

Note that the function $t_3(\tau)$ is a Hauptmodul for $\Gamma_0(3)$, and these formulas lead to the following result:

Proposition 3.5 (Rodriguez Villegas, [55, §IV])**.** *Let $\mathcal{F}$ be the fundamental domain for $\Gamma_0(3)$ with vertices $i\infty, 0, (1+i/\sqrt{3})/2$, and $(-1+i/\sqrt{3})/2$. If $\tau \in \mathcal{F}$, then*

$$m_3(t_3(\tau)) = \frac{81\sqrt{3}\operatorname{Im}(\tau)}{4\pi^2}\sum_{m,n\in\mathbb{Z}}{}' \frac{\chi_{-3}(m)(m+3n\operatorname{Re}(\tau))}{[(m+3n\tau)(m+3n\bar{\tau})]^2}.$$

When $t = -216$ or $t = 54$, E_t is a CM elliptic curve, and the formula (3.11) is known to be true [55, 57]. These two values of t are corresponding to $\tau = (1+\sqrt{-3})/2$ and $\tau = \sqrt{-3}/3$, respectively, via Proposition 3.5. For the degenerate case $t = 27$, we have from [55, §IV.14] that $m_3(27) = 9L'(\chi_{-3}, -1)$. By suitably choosing a CM point τ, we can also prove an interesting formula for $m_3(t)$ analogous to the formulas in Theorem 3.3.

Theorem 3.6. *If $t = 6 - 6\sqrt[3]{2} + 18\sqrt[3]{4}$, then*

$$m_3(t) = \frac{3}{2}\left(L'(f_{108}, 0) + L'(f_{36}, 0) - 3L'(f_{27}, 0)\right),$$

where $f_{36}(\tau) = \eta^4(6\tau) \in S_2(\Gamma_0(36))$, $f_{27}(\tau) = \eta^2(3\tau)\eta^2(9\tau) \in S_2(\Gamma_0(27))$, and $f_{108}(\tau) = q + 5q^7 - 7q^{13} - q^{19} - 5q^{25} - 4q^{31} - q^{37} + \cdots$, the unique normalized newform *in* $S_2(\Gamma_0(108))$.

Remark that, for t given above, the elliptic curve E_t is defined over $\mathbb{Q}\left(\sqrt[3]{2}\right)$ rather than $\mathbb{Q}$, so it is not surprising that our result is somewhat different from (3.11). To establish Theorem 3.6, we require some identities for L-values of the involved cusp forms, which will be verified in the following lemmas.

Lemma 3.7. *Let $f_{36}(\tau)$ be as defined in Theorem 3.6. Then the following equality holds:*

$$L(f_{36},2) = \frac{1}{2}\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{m\chi_{-3}(m)}{(m^2+3n^2)^2}.$$

Proof. First, note that for any τ in the upper half plane $\eta(\tau)$ satisfies the functional equation

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

Hence it is easily seen that

$$\frac{\eta\left(\frac{\sqrt{-3}}{3}\right)}{\eta\left(\sqrt{-3}\right)} = 3^{\frac{1}{4}},$$

which implies that $t_3\left(\frac{\sqrt{-3}}{3}\right) = 54$. Thus we have from Theorem 3.5 that

$$m_3(54) = \frac{81}{4\pi^2}\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{m\chi_{-3}(m)}{(m^2+3n^2)^2}.$$

On the other hand, Rogers [57, Thm. 2.1, Thm. 5.2] proved that

$$m_3(54) = \frac{81}{2\pi^2}L(f_{36},2),$$

whence the lemma follows. □

Lemma 3.8. *Let $f_{108}(\tau)$ be the unique normalized newform with rational coefficients in $S_2(\Gamma_0(108))$, and let $\mathcal{A} = \{(m,n) \in \mathbb{Z}^2 \mid (m,n) \equiv (-1,-2),(2,1),(1,0),(-2,3) \bmod 6\}$. Then*

$$L(f_{108},2) = \sum_{(m,n)\in\mathcal{A}} \frac{m+3n}{(m^2+3n^2)^2}.$$

Proof. By taking the Mellin transform of the newform, it suffices to prove that

$$f_{108}(\tau) = \sum_{(m,n)\in\mathcal{A}} (m+3n)q^{m^2+3n^2}. \tag{3.12}$$

Let $K = \mathbb{Q}(\sqrt{-3}), \mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right], \Lambda = (3+3\sqrt{-3}) \subset \mathcal{O}_K$, and $I(\Lambda)$ = the group of fractional ideals of $\mathcal{O}_K$ coprime to Λ. Since Λ can be factorized as

$$\Lambda = \left(\frac{1+\sqrt{-3}}{2}\right)(\sqrt{-3})^2(2),$$

any integral ideal $\mathfrak{a}$ is coprime to Λ if and only if $(\sqrt{-3}) \nmid \mathfrak{a}$ and $(2) \nmid \mathfrak{a}$. As a consequence, every integral ideal coprime to Λ is uniquely represented by $(m+n\sqrt{-3})$, where $m, n \in \mathbb{Z}, m > 0, 3 \nmid m$, and $m \not\equiv n \pmod{2}$. Let $P(\Lambda)$ denote the monoid of integral ideals coprime to Λ.

Define $\varphi : P(\Lambda) \to \mathbb{C}^\times$ by, for each $\mathfrak{a} = (m+n\sqrt{-3}) \in P(\Lambda)$,

$$\varphi(\mathfrak{a}) = \begin{cases} \dfrac{-\chi_{-3}(m)m + \chi_{-3}(n)(3n) - (\chi_{-3}(n)m + \chi_{-3}(m)n)\sqrt{-3}}{2} & \text{if } 3 \nmid n, \\ \chi_{-3}(m)(m+n\sqrt{-3}) & \text{if } 3|n. \end{cases}$$

Then it is not difficult to check that φ is multiplicative, and for each $(m+n\sqrt{-3}) \in P(\Lambda)$ with $m+n\sqrt{-3} \equiv 1 \pmod{\Lambda}$,

$$\varphi((m+n\sqrt{-3})) = m+n\sqrt{-3}.$$

Hence we can extend φ multiplicatively to define a Hecke Grössencharacter of weight 2 and conductor Λ on $I(\Lambda)$. Now if we let

$$\Psi(\tau) := \sum_{\mathfrak{a}\in P(\Lambda)} \varphi(\mathfrak{a})q^{N(\mathfrak{a})},$$

then one sees from [50, Thm. 1.31] that $\Psi(\tau)$ is a newform in $S_2(\Gamma_0(108))$. Observe that

$$\varphi((m+n\sqrt{-3})) + \varphi((m-n\sqrt{-3})) = \begin{cases} -\chi_{-3}(m)m + \chi_{-3}(n)(3n) & \text{if } 3 \nmid n, \\ 2\chi_{-3}(m)m & \text{if } 3|n, \end{cases}$$

so we have

$$\Psi(\tau) = \sum_{\substack{m,n\in\mathbb{N}\\ 3\nmid m, 3\nmid n\\ m\not\equiv n \pmod 2}} (-\chi_{-3}(m)m+\chi_{-3}(n)(3n))q^{m^2+3n^2} + \sum_{\substack{m\in\mathbb{N},n\in\mathbb{Z}\\ 3\nmid m, 3\mid n\\ m\not\equiv n \pmod 2}} \chi_{-3}(m)mq^{m^2+3n^2}.$$

Working modulo 6, one can show that

$$\sum_{\substack{m,n\in\mathbb{N}\\ 3\nmid m, 3\nmid n\\ m\not\equiv n \pmod 2}} (-\chi_{-3}(m)m+\chi_{-3}(n)(3n))q^{m^2+3n^2} = \sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\equiv(-1,2),(2,1)\\ \pmod 6}} (m+3n)q^{m^2+3n^2},$$

and

$$\sum_{\substack{m\in\mathbb{N},n\in\mathbb{Z}\\ 3\nmid m, 3\mid n\\ m\not\equiv n \pmod 2}} \chi_{-3}(m)mq^{m^2+3n^2} = \sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\equiv(1,0),(-2,3)\\ \pmod 6}} mq^{m^2+3n^2}$$

$$= \sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\equiv(1,0),(-2,3)\\ \pmod 6}} (m+3n)q^{m^2+3n^2}.$$

Consequently, the coefficients of $\Psi(\tau)$ are rational, which implies that $\Psi(\tau) = f_{108}(\tau)$, and (3.12) holds. (One can check using, for example, `Sage` or `Magma` that there is only one normalized newform in $S_2(\Gamma_0(108))$.) □

Lemma 3.9. *Let $f_{27}(\tau)$ be as defined in Theorem 3.6, and let $\mathcal{B} = \{(m,n) \in \mathbb{Z}^2 \mid (m,n) \equiv (1,0),(-2,3),(1,-1),(-2,2),(2,-1),(-1,2) \mod 6\}$. Then*

$$L(f_{27},2) = \sum_{(m,n)\in\mathcal{B}}{}' \frac{m+3n}{(m^2+3n^2)^2}.$$

Proof. As before, we will establish a q-expansion for $f_{27}(\tau)$ first; i.e., we aim at proving that

$$f_{27}(\tau) = \sum_{(m,n)\in\mathcal{B}} (m+3n)q^{m^2+3n^2}.$$

Recall from [57, §6] that the following identity is true:

$$f_{27}(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\equiv(1,1),(-2,-2) \\ \pmod 6}} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}}. \tag{3.13}$$

Therefore, it is sufficient to prove the following claims, each of which involves only simple manipulation. (Unless otherwise stated, each ordered pair (a, b) listed beneath the sigma sign indicates all $(m, n) \in \mathbb{Z}^2$ such that $m \equiv a$ and $n \equiv b \pmod 6$.)

Claim 1.

$$\sum_{(1,1)} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}} = \sum_{(1,0),(-2,3)} (m+3n)q^{m^2+3n^2} + \sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2}.$$

Claim 2.

$$\sum_{(-2,-2)} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}} = \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2} + \sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2}.$$

Proof of Claim 1. It is clear that

$$\begin{aligned}\sum_{(1,0),(-2,3)} (m+3n)q^{m^2+3n^2} &= \sum_{(1,0),(-2,3)} mq^{m^2+3n^2} \\ &= \sum_{(1,0),(-2,3)} \left(\frac{(m+3n)+3(m-n)}{4}\right) q^{\frac{(m+3n)^2+3(m-n)^2}{4}}, \text{ and}\end{aligned}$$

$$\sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2} = \sum_{(2,-1),(-1,2)} \left(\frac{(3n-m)+3(m+n)}{4}\right) q^{\frac{(3n-m)^2+3(m+n)^2}{4}}.$$

Also, it can be verified in a straightforward manner that

$$\begin{aligned}\{(m,n) \mid m \equiv n \equiv 1 \pmod 6\} = \{(k+3l, k-l) \mid (k,l) \equiv (1,0),(-2,3) \pmod 6\} \\ \sqcup \{(3l-k, k+l) \mid (k,l) \equiv (2,-1),(-1,2) \pmod 6\},\end{aligned}$$

where $\sqcup$ denotes disjoint union, so we obtain Claim 1.

Proof of Claim 2. Let us make some observation first that, by symmetry,

$$\sum_{(1,-1),(-2,2)} (3m+3n)q^{m^2+3n^2} = 0,$$

so we have that

$$\sum_{(1,-1),(-2,2)} (-2m)q^{m^2+3n^2} = \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2}.$$

It follows that

$$\begin{aligned}\sum_{(-1,-1),(2,2)} (m+3n)q^{m^2+3n^2} &= \sum_{(1,-1),(-2,2)} (-m+3n)q^{m^2+3n^2} \\ &= \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2} + \sum_{(1,-1),(-2,2)} (-2m)q^{m^2+3n^2} \\ &= 2\sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2}.\end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned}\sum_{(-2,-2)} \left(\frac{m+3n}{4}\right) q^{\frac{m^2+3n^2}{4}} &= \sum_{(-1,-1) \pmod 3} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2} \\ &= \sum_{\substack{(-1,-1),(2,2)\\(2,-1),(-1,2)}} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2} \\ &= \sum_{(1,-1),(-2,2)} (m+3n)q^{m^2+3n^2} \\ &\qquad + \sum_{(2,-1),(-1,2)} \left(\frac{m+3n}{2}\right) q^{m^2+3n^2},\end{aligned}$$

where the last equality comes from (3.14).

□

Lemma 3.10. *The following equality is true:*

$$L(f_{108},2)-\frac{3}{4}L(f_{27},2)=\frac{3}{2}\sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}}\frac{m\chi_{-3}(m)}{(3m^2+n^2)^2}.$$

Proof. Taking the Mellin transform of $f_{27}(\tau)$ in (3.13) yields

$$L(f_{27},2)=4\sum_{(1,1),(-2,-2)}\frac{m+3n}{(3m^2+n^2)^2}. \tag{3.15}$$

Since $\chi_{-3}(n)=j$ iff $n\equiv j \pmod 3$, where $j\in\{-1,0,1\}$, we have that

$$\sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}}\frac{m\chi_{-3}(m)}{(3m^2+n^2)^2}=\sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid m}}\frac{n\chi_{-3}(n)}{(m^2+3n^2)^2}$$

$$=\sum_{\substack{n\equiv -1 \pmod 3\\ 3\nmid m}}\frac{-2n}{(m^2+3n^2)^2}.$$

Also, it is obvious that the symmetry of the summation yields

$$\sum_{\substack{n\equiv -1 \pmod 3\\ 3\nmid m}}\frac{m}{(m^2+3n^2)^2}=0.$$

Hence, using Lemma 3.8, one sees that

$$L(f_{108},2)-\frac{3}{2}\sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}}\frac{m\chi_{-3}(m)}{(3m^2+n^2)^2}=\sum_{\substack{(-1,-2),(2,1)\\ (1,0),(-2,3)}}\frac{m+3n}{(m^2+3n^2)^2}+\sum_{\substack{n\equiv -1 \pmod 3\\ 3\nmid m}}\frac{3n}{(m^2+3n^2)^2}$$

$$=\sum_{\substack{(-1,-2),(2,1)\\ (1,0),(-2,3)}}\frac{m+3n}{(m^2+3n^2)^2}+\sum_{\substack{n\equiv -1 \pmod 3\\ 3\nmid m}}\frac{m+3n}{(m^2+3n^2)^2}$$

$$=\sum_{\substack{(-1,-2),(2,1)\\ (1,0),(-2,3)}}\frac{m+3n}{(m^2+3n^2)^2}+\sum_{\substack{(-2,2),(-2,-1)\\ (-1,2),(-1,-1)\\ (1,2),(1,-1)\\ (2,2),(2,-1)}}\frac{m+3n}{(m^2+3n^2)^2}$$

$$=\sum_{\substack{(1,0),(-2,3)\\ (1,-1),(-2,2)\\ (2,-1),(-1,2)}}\frac{m+3n}{(m^2+3n^2)^2}-\sum_{(1,1),(-2,-2)}\frac{m+3n}{(m^2+3n^2)^2}$$

$$= L(f_{27}, 2) - \frac{1}{4} L(f_{27}, 2),$$

where we have applied Lemma 3.9 and (3.15) in the last equality. □

Putting the previous lemmas together, we are now ready to complete a proof of Theorem 3.6.

Proof of Theorem 3.6. Let $\tau_0 = \sqrt{-3}/9$. Then $t_3(\tau_0) = 6 - 6\sqrt[3]{2} + 18\sqrt[3]{4}$. This can be verified by considering numerical approximation of $t_3(\tau_0)$ and using the following identities:

$$j(\tau) = j(-1/\tau), \qquad \mathfrak{f}^3(\sqrt{-27}) = 2(1 + \sqrt[3]{2} + \sqrt[3]{4}),$$
$$j(\tau) = \frac{(\mathfrak{f}^{24}(\tau) - 16)^3}{\mathfrak{f}^{24}(\tau)} = \frac{t_3(\tau)(t_3(\tau) + 216)^3}{(t_3(\tau) - 27)^3},$$

where $j(\tau)$ is the j-invariant, and $\mathfrak{f}(\tau)$ is a Weber modular function defined by

$$\mathfrak{f}(\tau) = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}.$$

(For references to these identities, see [19, §1], [79, Tab. VI], and [84, §1].) Then we see from Proposition 3.5 that

$$\begin{aligned}
m_3(t_3(\tau_0)) &= \frac{27}{4\pi^2} {\sum_{m,n\in\mathbb{Z}}}' \frac{m\chi_{-3}(m)}{(m^2 + \frac{n^2}{3})^2} \\
&= \frac{3}{2} \left(\frac{81}{2\pi^2} {\sum_{m,n\in\mathbb{Z}}}' \frac{m\chi_{-3}(m)}{(3m^2 + n^2)^2} \right) \\
&= \frac{3}{2} \left(\frac{81}{2\pi^2} {\sum_{\substack{m,n\in\mathbb{Z}\\ 3\mid n}}}' \frac{m\chi_{-3}(m)}{(3m^2 + n^2)^2} + \frac{81}{2\pi^2} \sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}} \frac{m\chi_{-3}(m)}{(3m^2 + n^2)^2} \right) \\
&= \frac{3}{2} \left(\frac{9}{2\pi^2} {\sum_{m,n\in\mathbb{Z}}}' \frac{m\chi_{-3}(m)}{(m^2 + 3n^2)^2} + \frac{81}{2\pi^2} \sum_{\substack{m,n\in\mathbb{Z}\\ 3\nmid n}} \frac{m\chi_{-3}(m)}{(3m^2 + n^2)^2} \right).
\end{aligned}$$

Now we can deduce using Lemma 3.7 and Lemma 3.10 that

$$m_3(t_3(\tau_0)) = \frac{3}{2}\left(\frac{27}{\pi^2}L(f_{108},2) + \frac{9}{\pi^2}L(f_{36},2) - \frac{81}{4\pi^2}L(f_{27},2)\right). \tag{3.16}$$

Finally, the formula stated in the theorem is merely a simple consequence of (3.16) and the functional equation (2.6). (The signs of the functional equations for the newforms f_{27}, f_{36}, and f_{108} are all 1.) $\square$

In addition to the formula stated in Theorem 3.6, we discovered some other conjectured formulas of similar type using numerical values of the hypergeometric representation of $m_3(t)$ given by (3.10):

$$m_3\left(17766 + 14094\sqrt[3]{2} + 11178\sqrt[3]{4}\right) \stackrel{?}{=} \frac{3}{2}(L'(f_{108},0) + 3L'(f_{36},0) + 3L'(f_{27},0)),$$

$$m_3(\alpha \pm \beta i) \stackrel{?}{=} \frac{3}{2}(L'(f_{108},0) + 3L'(f_{36},0) - 6L'(f_{27},0)),$$

$$m_3\left(\frac{(7+\sqrt{5})^3}{4}\right) \stackrel{?}{=} \frac{1}{8}\left(9L'(f_{100},0) + 38L'(f_{20},0)\right),$$

$$m_3\left(\frac{(7-\sqrt{5})^3}{4}\right) \stackrel{?}{=} \frac{1}{4}\left(9L'(f_{100},0) - 38L'(f_{20},0)\right),$$

where $\alpha = 17766 - 7047\sqrt[3]{2} - 5589\sqrt[3]{4}$, $\beta = 27\sqrt{3}(261\sqrt[3]{2} - 207\sqrt[3]{4})$, $f_{100}(\tau) = q + 2q^3 - 2q^7 + q^9 - 2q^{13} + 6q^{17} - 4q^{19} - \cdots$, and $f_{20}(\tau) = \eta^2(2\tau)\eta^2(10\tau)$.

It is worth mentioning that the last two Mahler measures above also appear in [32, Thm. 6] and [60, §4]. More precisely, it was shown that

$$19m_3(32) = 16m_3\left(\frac{(7+\sqrt{5})^3}{4}\right) - 8m_3\left(\frac{(7-\sqrt{5})^3}{4}\right), \tag{3.17}$$

$$m_3(32) = 8L'(f_{20},0). \tag{3.18}$$

Many of the identities like (3.17) can be proved using the elliptic dilogarithm evaluated at some torsion points on the corresponding elliptic curve. However, to our knowledge, no rigorous proof of the conjectured formulas for the individual terms on the right seems to appear in the literature.

3.3 The families $R_t = y + xy^{-1} + (xy)^{-1} - t^{1/4}$ and $S_t = x^2y^{-1} - yx^{-1} - (xy)^{-1} - t^{1/6}$

The family S_t was discussed briefly in [55, §IV.16]. It was shown that a period of this family is

$$u_0(\lambda) = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!(2n)!n!}\lambda^n = {}_2F_1\left(\begin{matrix}\frac{1}{6}, \ \frac{5}{6} \\ 1\end{matrix}; 432\lambda\right), \qquad \lambda = 1/t.$$

Hence we can easily derive the Mahler measure formula

$$m_6(t) := 6m(S_t) = \operatorname{Re}\left(\log t - \frac{60}{t}{}_4F_3\left(\begin{matrix}\frac{7}{6}, \ \frac{11}{6}, \ 1, \ 1 \\ 2, \ 2, \ 2\end{matrix}; \frac{432}{t}\right)\right), \qquad |t| \geq 432.$$

In this case, the (meromorphic) modular forms $t_6(\tau) := 1/\lambda(\mu), c(\tau)$, and $e(\tau)$ obtained from the same procedures above have somewhat complicated expressions in terms of Eisenstein series, namely,

$$t_6(\tau) = 864\left(1 - \frac{E_6(\tau)}{E_4^{3/2}(\tau)}\right)^{-1},$$

$$c(\tau) = E_4(\tau)^{1/4},$$

$$e(\tau) = \frac{1}{2}\left(E_4^{3/4}(\tau) + \frac{E_6(\tau)}{E_4(\tau)^{3/4}}\right).$$

In contrast to previous cases, to our knowledge, the function $t_6(\tau)$ is not known to be a Hauptmodul of a genus zero congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Moreover, we cannot deduce a Mahler measure formula $m_6(t)$ in terms of Eisenstein-Kronecker series using the Fourier development trick since $e(\tau)$ is not an Eisenstein series. One can, however, formulate some conjectures of $m_6(t)$ analogous to (3.2) and (3.11). (Rodriguez Villegas gave a conjectural formula for $t = 864$.) For $t = 432$, the curve E_t defined by the zero locus of S_t has genus zero, in which case we found numerically that

$$m_6(432) \stackrel{?}{=} 10L'(\chi_{-4}, -1),$$

which still remains unproved.

On the other hand, the family R_t was not studied directly in [55], though we can apply similar methods to show that it has a period

$$u_0(\lambda) = \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!n!^2}\lambda^n = {}_2F_1\left(\begin{matrix}\frac{1}{4}, \frac{3}{4} \\ 1\end{matrix}; 64\lambda\right), \qquad \lambda = 1/t.$$

With the aid of the package `DEtools` in `Maple`, we found that $u_0(\lambda)$ satisfies the Picard-Fuchs equation

$$\lambda(64\lambda - 1)\frac{d^2u}{d\lambda^2} + (128\lambda - 1)\frac{du}{d\lambda} + 12u = 0,$$

and the differential equation has a second solution

$$u_1(\lambda) = u_0(\lambda)\log\lambda + 40\lambda + 1556\lambda^2 + \frac{213232}{3}\lambda^3 + \cdots$$

around $\lambda = 0$. Then we can express the functions $t_4(\tau) := 1/\lambda(\tau)$, $c(\tau)$, and $e(\tau)$ in terms of q-series as follows:

$$t_4(\tau) = q^{-1} + 40 + 276q - 2048q^2 + 11202q^3 + \cdots \stackrel{?}{=} 64 + \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24},$$

$$c(\tau) = 1 + 12q - 60q^2 + 768q^3 - 11004q^4 + 178200q^5 + \cdots \stackrel{?}{=} (2E_2(2\tau) - E_2(\tau))^{1/2},$$

$$\begin{aligned} e(\tau) &= 1 - 28q + 508q^2 - 8922q^3 + 172028q^4 + \cdots \\ &\stackrel{?}{=} (2E_2(2\tau) - E_2(\tau))^{3/2}\left(1 + 64\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}\right), \end{aligned}$$

where the conjectural equalities $\stackrel{?}{=}$ above were found by Stienstra [76, §2]. Remark that the function $64 + (\eta(\tau)/\eta(2\tau))^{24}$ is a Hauptmodul for $\Gamma_0(2)$ (see, for example, [33, Tab.1]).

Let $m_4(t) := 4m(R_t)$. Then we have that

$$m_4(t) = \operatorname{Re}\left(\log t - \frac{12}{t}\,{}_4F_3\left(\begin{matrix}\frac{5}{4},\ \frac{7}{4},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix};\frac{64}{t}\right)\right), \qquad |t| \geq 64.$$

Rogers and Zudilin [62] investigated Mahler measures of another family of polynomials:

$$w(t) := 2m\left(\left(x + x^{-1}\right)^2\left(y + y^{-1}\right) + t^{1/2}\right),$$

which turns out to equal $m_4(t)$ for all $|t| \geq 64$. They also discovered some conjectural formulas of the form

$$w(t) = c_t L'(E_t, 0),$$

where $t \in \mathbb{Z}$, $c_t \in \mathbb{Q}$ and E_t is the curve

$$E_t : y^2 = x^3 - \frac{t(t-48)}{3}x + \frac{2t^2(t-72)}{27}.$$

By numerical computation, we also found that

$$m_4(64) \stackrel{?}{=} 2L'(\chi_{-8}, -1),$$

which corresponds to a degenerate case of E_t, since E_{64} is of genus zero. Again, since the function $e(\tau)$ does not seem to be an Eisenstein series or a linear combination of Eisenstein series, we still see no way of proving these formulas by means of Eisenstein-Kronecker series.

4. THREE-VARIABLE MAHLER MEASURES AND SPECIAL VALUES OF MODULAR AND DIRICHLET L-SERIES*

In this section, we consider the Mahler measures of four families of three-variable Laurent polynomials which were introduced by Rogers [58]. These families include

$$A_s := (x + x^{-1})(y + y^{-1})(z + z^{-1}) + s^{1/2},$$

$$B_s := (x + x^{-1})^2(y + y^{-1})^2(1 + z)^3 z^{-2} - s,$$

$$C_s := x^4 + y^4 + z^4 + 1 + s^{1/4}xyz,$$

$$D_s := (x + x^{-1})^2(1 + y)^3 y^{-2}(z + z^{-1})^6 - s,$$

where s in each family is a complex parameter. Using Rodriguez Villegas' method (see Section 3.1), Rogers proved the following formulas:

Theorem 4.1 ([58, Prop.2.2]). *Let* $n_2(s) := 2m(A_s), n_3(s) := m(B_s), n_4(s) := 4m(C_s)$, *and* $n_6(s) := m(D_s)$. *For* $|s|$ *sufficiently large,*

$$n_2(s) = \operatorname{Re}\left(\log(s) - \frac{8}{s}\,{}_5F_4\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1\\ 2, 2, 2, 2\end{matrix}; \frac{64}{s}\right)\right),$$

$$n_3(s) = \operatorname{Re}\left(\log(s) - \frac{12}{s}\,{}_5F_4\left(\begin{matrix}\frac{4}{3}, \frac{3}{2}, \frac{5}{3}, 1, 1\\ 2, 2, 2, 2\end{matrix}; \frac{108}{s}\right)\right),$$

$$n_4(s) = \operatorname{Re}\left(\log(s) - \frac{24}{s}\,{}_5F_4\left(\begin{matrix}\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 1\\ 2, 2, 2, 2\end{matrix}; \frac{256}{s}\right)\right).$$

The main goal of this section is to establish some results for the Mahler measures $n_j(s)$, $j = 2, 3, 4$, which are analogous to known results in the two-variable case. We will also

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deduce that, for certain values of s, $n_j(s)$ can be written as a rational linear combination of modular and Dirichlet L-values. However, the methods used in our proofs seem not applicable to the family D_s. We will discuss this obstruction at the end of this section.

4.1 Expressing Mahler measures as Eisenstein-Kronecker series

Throughout this section, q will be a function of $\tau \in \mathcal{H}$ given by $q := q(\tau) = e^{2\pi i\tau}$, and, as usual, we denote

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \quad \text{and} \quad \Delta(\tau) := \eta^{24}(\tau).$$

We will prove first that when s is parameterized properly, $n_2(s), n_3(s),$ and $n_4(s)$ can be expressed as Eisenstein-Kronecker series, which is analogous to Proposition 3.2 and Proposition 3.5. This is a crucial result which will be used to deduce other important results later. As we have not yet formally defined Eisenstein-Kronecker series, the reader may think of it as a function on $\mathcal{H}$ expressed in terms of a two-dimensional series. The relationship between the these Mahler measures and Eisenstein-Kronecker series will be clarified explicitly in Section 5.

Proposition 4.2. *Denote*

$$s_2(\tau) = s_2(q(\tau)) := -\frac{\Delta\left(\frac{2\tau+1}{2}\right)}{\Delta(2\tau+1)},$$

$$s_3(\tau) = s_3(q(\tau)) := \left(27\left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^6 + \left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^6\right)^2,$$

$$s_4(\tau) = s_4(q(\tau)) := \frac{\Delta(2\tau)}{\Delta(\tau)}\left(16\left(\frac{\eta(\tau)\eta(4\tau)^2}{\eta(2\tau)^3}\right)^4 + \left(\frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)^2}\right)^4\right)^4.$$

(i) If $\tau = y_1 i$ or $\tau = 1/2 + y_2 i$, where $y_1 \in [1/2, \infty)$ and $y_2 \in (0,\infty)$, then

$$n_2(s_2(q)) = \frac{2\,\mathrm{Im}(\tau)}{\pi^3} \sum_{m,n\in\mathbb{Z}}{}' \Bigg(-\left(\frac{4(m\,\mathrm{Re}(\tau)+n)^2}{[(m\tau+n)(m\bar{\tau}+n)]^3} - \frac{1}{[(m\tau+n)(m\bar{\tau}+n)]^2}\right)$$

$$+\,16\left(\frac{4(4m\,\mathrm{Re}(\tau)+n)^2}{[(4m\tau+n)(4m\bar\tau+n)]^3}-\frac{1}{[(4m\tau+n)(4m\bar\tau+n)]^2}\right)\Bigg).$$

(ii) If $\tau = yi$, where $y \in [1/\sqrt{3}, \infty)$ then

$$n_3(s_3(q)) = \frac{15\,\mathrm{Im}(\tau)}{4\pi^3}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\Bigg(-\left(\frac{4(m\,\mathrm{Re}(\tau)+n)^2}{[(m\tau+n)(m\bar\tau+n)]^3}-\frac{1}{[(m\tau+n)(m\bar\tau+n)]^2}\right)$$
$$+\,9\left(\frac{4(3m\,\mathrm{Re}(\tau)+n)^2}{[(3m\tau+n)(3m\bar\tau+n)]^3}-\frac{1}{[(3m\tau+n)(3m\bar\tau+n)]^2}\right)\Bigg).$$

(iii) If $\tau = y_1 i$ or $\tau = 1/2 + y_2 i$, where $y_1 \in [1/\sqrt{2}, \infty)$ and $y_2 \in (1/2, \infty)$, then

$$n_4(s_4(q)) = \frac{10\,\mathrm{Im}(\tau)}{\pi^3}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\Bigg(-\left(\frac{4(m\,\mathrm{Re}(\tau)+n)^2}{[(m\tau+n)(m\bar\tau+n)]^3}-\frac{1}{[(m\tau+n)(m\bar\tau+n)]^2}\right)$$
$$+\,4\left(\frac{4(2m\,\mathrm{Re}(\tau)+n)^2}{[(2m\tau+n)(2m\bar\tau+n)]^3}-\frac{1}{[(2m\tau+n)(2m\bar\tau+n)]^2}\right)\Bigg).$$

Proof. We prove this proposition mainly using a method due to Bertin [6]. Recall from [58, Thm. 2.3] that for sufficiently small $|q|$

$$n_2(s_2(q)) = -\frac{2}{15}G(q) - \frac{1}{15}G(-q) + \frac{3}{5}G(q^2), \tag{4.1}$$

where

$$G(q) = \mathrm{Re}\left(-\log(q) + 240\sum_{n=1}^{\infty} n^2 \log(1-q^n)\right).$$

We have from [88, §8] that $s_2(q) = j_4^*(\tau) + 24$,, where

$$j_4^*(\tau) = \left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8 + 8 + 4^4\left(\frac{\eta(4\tau)}{\eta(\tau)}\right)^8,$$

and $j_4^*(\tau)$ is a Hauptmodul associated to the genus zero subgroup $\Gamma_0(4)^*$ of $\mathrm{GL}_2(\mathbb{R})$ generated by $\Gamma_0(4)$ and the Atkin-Lehner involutions W_2 and W_4. Therefore, we can apply similar arguments in [56, §6] to show that the formula (4.1) holds for every τ given in (i) by choosing a suitable fundamental domain for $\Gamma_0(4)^*$. It was also shown in [58, Thm.

2.3] that

$$G(-q) = 9G(q^2) - 4G(q^4) - G(q). \tag{4.2}$$

Substituting (4.2) into (4.1) yields

$$n_2(s_2(q)) = -\frac{1}{15}G(q) + \frac{4}{15}G(q^4). \tag{4.3}$$

From now on we let $\sigma_3(n) = \sum_{d|n} d^3$, $E_4(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$, the Eisenstein series of weight 4 for $\Gamma(1)$, $D = q\frac{d}{dq}$, and $\operatorname{Li}_k(z) = \sum_{m=1}^{\infty}\frac{z^m}{m^k}$, the usual polylogarithm function.

It follows by taking differentials in (4.3) that

$$\begin{aligned} dn_2(s_2(q)) &= \left(\frac{1}{15}E_4(q) - \frac{16}{15}E_4(q^4)\right)\frac{dq}{q} \\ &= -\frac{1}{q} + \sum_{n\geq 1}\sigma_3(n)(16q^{n-1} - 256q^{4n-1})dq. \end{aligned}$$

Then we integrate both sides and use the identity

$$D^2\left(\operatorname{Li}_3\left(q^{jd}\right)\right) = (jd)^2\operatorname{Li}_1\left(q^{jd}\right), \qquad j, d \in \mathbb{N},$$

to recover

$$\begin{aligned} n_2(s_2(q)) &= \operatorname{Re}\left(-2\pi i\tau + \sum_{n\geq 1}\sigma_3(n)\left(16\frac{q^n}{n} - 64\frac{q^{4n}}{n}\right)\right) \\ &= \operatorname{Re}\left(-2\pi i\tau + 16D^2\left(\sum_{d\geq 1}\operatorname{Li}_3(q^d) - \frac{1}{4}\operatorname{Li}_3(q^{4d})\right)\right). \end{aligned} \tag{4.4}$$

For $j = 1, 4$ let

$$F_j(\xi) = \sum_{d\geq 1}\operatorname{Li}_3(q^{jd+\xi}) = \sum_{d\geq 1}\sum_{m\geq 1}\frac{e^{2\pi i\tau m(jd+\xi)}}{m^3}.$$

It is not hard to see that $F_j(\xi)$ is differentiable at $\xi = 0$. Indeed, for any $\xi \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

$$\left|\operatorname{Li}_3(q^{jd+\xi})\right| = \left|\sum_{m\geq 1}\frac{e^{2\pi i\tau m(jd+\xi)}}{m^3}\right| = \sum_{m\geq 1}\frac{e^{-2\pi tm(jd+\xi)}}{m^3} \leq \sum_{m\geq 1}\frac{e^{-2\pi t(jd+\xi)}}{m^3} = e^{-2\pi t(jd+\xi)}\zeta(3),$$

where ζ is the Riemann zeta function. Since $e^{2\pi t} > 1$, it is immediate that $\sum_{d\geq 1} e^{-2\pi t(jd+\xi)}\zeta(3)$ converges. Therefore, it follows by the Weierstrass M-test that $\sum_{d\geq 1} \mathrm{Li}_3(q^{jd+\xi})$ converges uniformly on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. It is easily seen that $\mathrm{Li}_3(q^{jd+\xi})$ is differentiable at $\xi = 0$ and hence so is $F_j(\xi)$. As a consequence, we have from a basic fact in Fourier analysis (cf. [75, Thm. 3.2.1]) that the Fourier series of $F_j(\xi)$ converges pointwise to $F_j(\xi)$ at $\xi = 0$; i.e.,

$$F_j(0) = \sum_{n\in\mathbb{Z}} \hat{F}_j(n),$$

where $\hat{F}_j(n)$ denote the Fourier coefficients of F_j. Following similar computations to those in [6], one sees that

$$\hat{F}_j(n) = \begin{cases} -\dfrac{1}{2\pi i}\displaystyle\sum_{m\geq 1} \frac{1}{m^3(jm\tau - \frac{n}{4})} & \text{if } 4|n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $F_j(0) = \sum_{d\geq 1} \mathrm{Li}_3(q^{jd})$ and $D^2 = -\dfrac{1}{4\pi^2}\dfrac{d^2}{d\tau^2}$, we have from (4.4) that

$$\begin{aligned}
n_2(s_2(q)) &= \mathrm{Re}\left(-2\pi i\tau + 16D^2\left(F_1(0) - \frac{1}{4}F_4(0)\right)\right) \\
&= \mathrm{Re}\left(-2\pi i\tau + \frac{8i}{\pi}D^2\left(\sum_{n\in\mathbb{Z}}\sum_{m\geq 1}\frac{1}{m^3}\left(\frac{1}{m\tau+n} - \frac{1}{4(4m\tau+n)}\right)\right)\right) \\
&= \mathrm{Re}\left(-2\pi i\tau - \frac{4i}{\pi^3}\sum_{n\in\mathbb{Z}}\sum_{m\geq 1}\frac{1}{m}\left(\frac{1}{(m\tau+n)^3} - \frac{4}{(4m\tau+n)^3}\right)\right) \\
&= \mathrm{Re}\left(-i\left(2\pi\tau + \frac{2}{\pi^3}\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{1}{(m\tau+n)^3} - \frac{4}{(4m\tau+n)^3}\right)\right)\right) \\
&= \mathrm{Im}\left(2\pi\tau + \frac{2}{\pi^3}\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{1}{(m\tau+n)^3} - \frac{4}{(4m\tau+n)^3}\right)\right) \\
&= \frac{2\,\mathrm{Im}(\tau)}{\pi^3}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\left(-\left(2\,\mathrm{Re}\left(\frac{1}{(m\tau+n)^3(m\bar{\tau}+n)}\right) + \frac{1}{[(m\tau+n)(m\bar{\tau}+n)]^2}\right)\right.
\end{aligned}$$

$$+ 16\left(2\operatorname{Re}\left(\frac{1}{(4m\tau+n)^3(4m\bar{\tau}+n)}\right)+\frac{1}{[(4m\tau+n)(4m\bar{\tau}+n)]^2}\right)\Bigg),$$

where we have applied the same tricks from [6] to obtain the last equality. We then use the fact that $2\operatorname{Re}(z) = z + \bar{z}$ for any $z \in \mathbb{C}$ to finish the proof of (i).

One can prove (ii) and (iii) in a similar fashion using the fact that if $|q|$ is sufficiently small, then

$$n_3(s_3(q)) = -\frac{1}{8}G(q) + \frac{3}{8}G(q^3),$$

$$n_4(s_4(q)) = -\frac{1}{3}G(q) + \frac{2}{3}G(q^2)$$

[58, Thm. 2.3], and $s_3(q)$ and $s_4(q)$ are Hauptmoduls for $\Gamma_0(3)^*$ and $\Gamma_0(2)^*$, respectively [88, §8]. □

4.2 Expressing Mahler measures as linear combinations of L-values

For $j = 2, 3$, and 4, if we choose a CM point τ properly, then we have that $s_j(\tau) \in \mathbb{Z}$, and $n_j(s_j(\tau))$ has a simple formula of the form

$$n_j(s_j(\tau)) = c_1 L'(g, 0) + c_2 L'(\chi, -1), \tag{4.5}$$

where g is a CM newform of weight 3 with rational Fourier coefficients, χ is an odd quadratic character , and $c_1, c_2 \in \mathbb{Q}$. We will use Proposition 4.2 to prove some formulas of this type.

Theorem 4.3. *The following equalities hold:*

$$n_2(64) = 8L'(g_{16}, 0), \tag{4.6}$$

$$n_2(256) = \frac{4}{3}(L'(g_{48}, 0) + 2L'(\chi_{-4}, -1)), \tag{4.7}$$

$$n_3(216) = \frac{15}{4}(L'(g_{24}^{(1)}, 0) + L'(\chi_{-3}, -1)), \tag{4.8}$$

$$n_3(1458) = \frac{15}{8}(9L'(g_{12}, 0) + 2L'(\chi_{-4}, -1)), \tag{4.9}$$

$$n_4(648) = \frac{5}{2}(4L'(g_{16},0) + L'(\chi_{-4},-1)), \tag{4.10}$$

$$n_4(2304) = \frac{20}{3}(L'(g_{24}^{(2)},0) + L'(\chi_{-3},-1)), \tag{4.11}$$

$$n_4(20736) = \frac{4}{5}(5L'(g_{40},0) + 2L'(\chi_{-8},-1)), \tag{4.12}$$

$$n_4(614656) = \frac{40}{3}(5L'(g_8,0) + L'(\chi_{-3},-1)), \tag{4.13}$$

where

$$g_8(\tau) = \eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2, \quad g_{12}(\tau) = \eta(2\tau)^3\eta(6\tau)^3,$$

$$g_{16}(\tau) = \eta(4\tau)^6, \qquad g_{48}(\tau) = \frac{\eta(4\tau)^9\eta(12\tau)^9}{\eta(2\tau)^3\eta(6\tau)^3\eta(8\tau)^3\eta(24\tau)^3},$$

$$g_{24}^{(1)}(\tau) = q + 2q^2 - 3q^3 + 4q^4 - 2q^5 - 6q^6 - 10q^7 + 8q^8 + 9q^9 - 4q^{10} + \cdots,$$

$$g_{24}^{(2)}(\tau) = q - 2q^2 + 3q^3 + 4q^4 + 2q^5 - 6q^6 - 10q^7 - 8q^8 + 9q^9 - 4q^{10} - \cdots,$$

$$g_{40}(\tau) = q - 2q^2 + 4q^4 + 5q^5 + 6q^7 - 8q^8 + 9q^9 - 10q^{10} - 18q^{11} - 6q^{13} - \cdots.$$

We see from [27] that g_8, g_{12}, and g_{16} defined above are CM newforms with complex multiplication in $S_3(\Gamma_0(8),\chi_{-8}), S_3(\Gamma_0(12),\chi_{-3})$, and $S_3(\Gamma_0(16),\chi_{-4})$, respectively. We will see in the next section that $g_{24}^{(1)}, g_{24}^{(2)} \in S_3(\Gamma_0(24),\chi_{-24})$ and $g_{40} \in S_3(\Gamma_0(40),\chi_{-40})$ are both newforms of CM type. It also follows immediately by [50, Thm. 1.64] that $g_{48} \in S_3(\Gamma_0(48),\chi_{-3})$. Computing some first Fourier coefficients yields

$$g_{48}(\tau) = q + 3q^3 - 2q^7 + 9q^9 - 22q^{13} - 26q^{19} - 6q^{21} + \cdots,$$

$$g(\tau) = q - 3q^3 + 2q^7 + 9q^9 - 22q^{13} + 26q^{19} - 6q^{21} + \cdots;$$

that is, g_{48} is a twist of g by χ_{-4}, so g_{48} is also a CM newform. Throughout, we will use g_N to denote a normalized CM newform with rational Fourier coefficients in $S_3(\Gamma_0(N),\chi_{-D_N})$, where $-D_N$ is the discriminant of $\mathbb{Q}(\sqrt{-N})$, and we use superscripts if there is more than one such newform. It might be worth pointing out that although $g_{24}^{(1)}$ and $g_{24}^{(2)}$ cannot be represented by eta quotients, we can write them as linear combinations of eta quotients which form a basis for $S_3(\Gamma_0(24), \chi_{-24})$. However, this fact will not be used to prove (4.8) and (4.11). Applying Proposition 4.1 together with Theorem 4.3 one can easily deduce many formulas similar to [58, Cor.2.6].

Corollary 4.4. *Let $g_8, g_{12}, g_{16}, g_{48}, g_{24}^{(1)}, g_{24}^{(2)}$, and g_{40} be as defined in Theorem 4.3. Then the following formulas hold:*

$$
\begin{aligned}
{}_5F_4\left(\begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; 1\right) &= 48\log(2) - 64L'(g_{16}, 0), \\
{}_5F_4\left(\begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; \frac{1}{4}\right) &= 256\log(2) - \frac{128}{3}\left(L'(g_{48}, 0) + 2L'(\chi_{-4}, -1)\right), \\
{}_5F_4\left(\begin{matrix} \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; \frac{1}{2}\right) &= 54\log(6) - \frac{135}{2}\left(L'(g_{24}^{(1)}, 0) + L'(\chi_{-3}, -1)\right), \\
{}_5F_4\left(\begin{matrix} \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; \frac{2}{27}\right) &= \frac{243}{2}\log(2) + 729\log(3) \\
&\quad - \frac{3645}{16}\left(9L'(g_{12}, 0) + 2L'(\chi_{-4}, -1)\right), \\
{}_5F_4\left(\begin{matrix} \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; \frac{32}{81}\right) &= 81\log(2) + 108\log(3) \\
&\quad - \frac{135}{2}\left(4L'(g_{16}, 0) + L'(\chi_{-4}, -1)\right), \\
{}_5F_4\left(\begin{matrix} \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; \frac{1}{9}\right) &= 768\log(2) + 192\log(3) \\
&\quad - 640\left(L'(g_{24}^{(2)}, 0) + L'(\chi_{-3}, -1)\right), \\
{}_5F_4\left(\begin{matrix} \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; \frac{1}{81}\right) &= 6912\log(2) + 3456\log(3) \\
&\quad - \frac{3456}{5}\left(5L'(g_{40}, 0) + 2L'(\chi_{-8}, -1)\right), \\
{}_5F_4\left(\begin{matrix} \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1, 1 \\ 2, 2, 2, 2 \end{matrix}; \frac{1}{2401}\right) &= \frac{614656}{3}\log(2) + \frac{307328}{3}\log(7)
\end{aligned}
$$

$$-\frac{3073280}{9}\left(5L'(g_8,0)+L'(\chi_{-3},-1)\right).$$

To prove Theorem 4.3, we require evaluations of $s_j(\tau)$ at some CM points.

Lemma 4.5. *Let $s_2(q), s_3(q)$, and $s_4(q)$ be as defined in Proposition 4.2. Then*

$$s_2\left(q\left(\frac{\sqrt{-1}}{2}\right)\right)=64,\qquad s_2\left(q\left(\frac{\sqrt{-3}}{2}\right)\right)=256,$$

$$s_3\left(q\left(\frac{\sqrt{-3}}{3}\right)\right)=108,\qquad s_3\left(q\left(\frac{\sqrt{-6}}{3}\right)\right)=216,\qquad s_3\left(q\left(\frac{\sqrt{-12}}{3}\right)\right)=1458,$$

$$s_4\left(q\left(\frac{\sqrt{-2}}{2}\right)\right)=256,\qquad s_4\left(q\left(\frac{\sqrt{-4}}{2}\right)\right)=648,\qquad s_4\left(q\left(\frac{\sqrt{-6}}{2}\right)\right)=2304,$$

$$s_4\left(q\left(\frac{\sqrt{-10}}{2}\right)\right)=20736,\quad s_4\left(q\left(\frac{\sqrt{-18}}{2}\right)\right)=614656.$$

Proof. Let us consider the following two Weber modular functions:

$$\mathfrak{f}(\tau):=e^{-\frac{\pi i}{24}}\frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)},\qquad \mathfrak{f}_1(\tau):=\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}.$$

Weber listed a number of special values of these functions in [79, p. 721], including

$$\mathfrak{f}\left(\sqrt{-1}\right)=2^{\frac{1}{4}},\qquad \mathfrak{f}\left(\sqrt{-3}\right)=2^{\frac{1}{3}},$$

$$\mathfrak{f}_1\left(\sqrt{-2}\right)=2^{\frac{1}{4}},\qquad \mathfrak{f}_1\left(\sqrt{-4}\right)=8^{\frac{1}{8}},$$

$$\mathfrak{f}_1\left(\sqrt{-6}\right)^6=4+2\sqrt{2},\qquad \mathfrak{f}_1\left(\sqrt{-8}\right)^8=8+8\sqrt{2},$$

$$\sqrt{2}\mathfrak{f}_1\left(\sqrt{-10}\right)^2=1+\sqrt{5},\qquad \mathfrak{f}_1\left(\sqrt{-12}\right)^4=2^{\frac{7}{6}}\left(1+\sqrt{3}\right),$$

$$\mathfrak{f}_1\left(\sqrt{-16}\right)^4=2^{\frac{7}{4}}(1+\sqrt{2}),\qquad \mathfrak{f}_1\left(\sqrt{-18}\right)^3=2^{\frac{3}{4}}\left(\sqrt{2}+\sqrt{3}\right),$$

$$\mathfrak{f}_1\left(\sqrt{-24}\right)^{24}=2^9\left(1+\sqrt{2}\right)^2\left(2+\sqrt{3}\right)^3\left(\sqrt{2}+\sqrt{3}\right)^3,$$

$$\mathfrak{f}_1\left(\sqrt{-40}\right)^8=2\left(1+\sqrt{5}\right)^2\left(1+\sqrt{2}\right)^2\left(3+\sqrt{10}\right),$$

$$\mathfrak{f}_1\left(\sqrt{-72}\right)^{24}=2^7\left(2+\sqrt{6}\right)^4\left(1+\sqrt{2}\right)^9\left(2+\sqrt{3}\right)^6.$$

(Actually, there are some typographical errors in the original table containing these values, which were corrected later by Brillhart and Morton [14].)

Since $\Delta(\tau)$ is a modular form for the full modular group $\Gamma(1)$, we have immediately that

$$s_2(q) = \mathfrak{f}(2\tau)^{24},$$

so the first two equalities in the lemma follow easily. Note also that

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau).$$

Hence

$$\frac{\eta\left(\sqrt{-3}\right)}{\eta\left(\frac{\sqrt{-3}}{3}\right)} = \frac{1}{3^{\frac{1}{4}}}, \qquad \frac{\eta\left(\sqrt{-6}\right)}{\eta\left(\frac{\sqrt{-6}}{3}\right)} = \left(\frac{2}{3}\right)^{\frac{1}{4}} \frac{\eta\left(\sqrt{-6}\right)}{\eta\left(\frac{\sqrt{-6}}{2}\right)} = \left(\frac{2}{3}\right)^{\frac{1}{4}} \frac{1}{\mathfrak{f}_1\left(\sqrt{-6}\right)},$$

and

$$\frac{\eta\left(\sqrt{-12}\right)}{\eta\left(\frac{\sqrt{-12}}{3}\right)} = \left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}} \frac{\eta\left(\sqrt{-12}\right)}{\eta\left(\frac{\sqrt{-12}}{4}\right)} = \left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}} \frac{1}{\mathfrak{f}_1\left(\sqrt{-12}\right)\mathfrak{f}_1\left(\sqrt{-3}\right)}$$

$$= \left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{2}} \frac{\mathfrak{f}\left(\sqrt{-3}\right)}{\mathfrak{f}_1\left(\sqrt{-12}\right)^2},$$

where the last equality follows from the relation

$$\mathfrak{f}_1(2\tau) = \mathfrak{f}(\tau)\mathfrak{f}_1(\tau).$$

These enable us to evaluate $s_3\left(q(\tau)\right)$ for $\tau \in \left\{\frac{\sqrt{-3}}{3}, \frac{\sqrt{-6}}{3}, \frac{\sqrt{-12}}{3}\right\}$.

Finally, observe that for every $m \in \mathbb{N}$

$$s_4\left(q\left(\frac{\sqrt{-m}}{2}\right)\right) = \frac{1}{\mathfrak{f}_1(\sqrt{-m})^{24}}\left(16\frac{\mathfrak{f}_1(\sqrt{-m})^4}{\mathfrak{f}_1(\sqrt{-4m})^8} + \frac{\mathfrak{f}_1(\sqrt{-4m})^8}{\mathfrak{f}_1(\sqrt{-m})^4}\right)^4.$$

Using Weber's results above, one can check in a straightforward manner that the evaluations of $s_4(q)$ in the lemma hold. □

Let us prove a few more lemmas before establishing Theorem 4.3.

Lemma 4.6. *If g_8, g_{12}, and g_{16} are as defined in Theorem 4.3, then*

$$g_8(\tau) = \sum_{m,n\in\mathbb{Z}} \frac{m^2-2n^2}{2} q^{m^2+2n^2}, \tag{4.14}$$

$$g_{12}(\tau) = \sum_{m,n\in\mathbb{Z}} \frac{m^2-3n^2}{2} q^{m^2+3n^2}, \tag{4.15}$$

$$g_{16}(\tau) = \sum_{m,n\in\mathbb{Z}} \frac{m^2-4n^2}{2} q^{m^2+4n^2}. \tag{4.16}$$

Proof. We will show (4.16) first. Let $K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}[i]$, $\Lambda = (2) \subset \mathcal{O}_K$, and $I(\Lambda)$ = the group of fractional ideals of $\mathcal{O}_K$ coprime to Λ. Then we define the Hecke Grössencharacter $\phi : I(\Lambda) \to \mathbb{C}^\times$ of conductor Λ by

$$\phi((m+in)) = (m+in)^2$$

for any $m, n \in \mathbb{Z}$ such that m is odd and n is even, and let

$$\Psi(\tau) := \sum_{\mathfrak{a}\subseteq\mathcal{O}_K} \phi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{k=1}^{\infty} a(k) q^k,$$

where the sum runs through the integral ideals of $\mathcal{O}_K$ coprime to Λ and $N(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$. It then follows from [50, Thm. 1.31] that $\Psi(\tau)$ is a newform in $S_3(\Gamma_0(16), \chi_{-4})$. Moreover, by [50, Ex. 1.33], we have that $a(p) = 0$ for every prime $p \equiv 3 \pmod 4$, and if p is a prime such that $p = (m+in_0)(m-in_0) = m^2+n_0^2$ for some $m, n_0 \in \mathbb{Z}$ with m odd and $n_0 = 2n$, then $a(p) = 2(m^2-4n^2)$. Also, it is clear by the definition of Ψ that $a(k) = 0$ for every $k \in \mathbb{N}_{\text{even}}$. Next, we shall examine $a(k)$ explicitly for each $k \in \mathbb{N}_{\text{odd}}$.

Recall first that since $\Psi(\tau)$ is a Hecke eigenform in $S_3(\Gamma_0(16), \chi_{-4})$,

$$a(k)a(l) = \sum_{d|(k,l)} \chi_{-4}(d) d^2 a\left(\frac{kl}{d^2}\right) \tag{4.17}$$

holds for all $k, l \in \mathbb{N}$ (cf. [36, Ch. 6]). If k is odd and all prime factors of k are congruent to 1 modulo 4, then it is easily seen by induction that $k = m^2 + 4n^2$ for some $m, n \in \mathbb{Z}$

with m odd. Now suppose k is odd and k has a prime factor congruent to 3 modulo 4, say

$$k = \prod_{p_i \equiv 1 \pmod 4} p_i \cdot \prod_{r_j \equiv 3 \pmod 4} r_j$$

for some primes p_i and r_j. If $\displaystyle\prod_{r_j \equiv 3 \pmod 4} r_j$ is a perfect square, then k is again of the form $k = m^2 + 4n^2$ with m odd. Otherwise, there exists a prime factor $r \equiv 3 \pmod 4$ of k such that $r^l \| k$ for some odd l. But then it can be shown inductively using (4.17) that $a(r^l) = 0$, so $a(k)$ vanishes in this case. Note that for any $k = m^2 + 4n^2$ with m odd

$$a(k) = \begin{cases} \phi((m + 2in)) + \phi((m - 2in)) & \text{if } n \neq 0, \\ \phi((m)) & \text{if } n = 0, \end{cases}$$

$$= \begin{cases} 2(m^2 - 4n^2) & \text{if } n \neq 0, \\ m^2 & \text{if } n = 0. \end{cases}$$

Consequently, we may express $\Psi(\tau)$ as

$$\Psi(\tau) = \sum_{k=1}^{\infty} a(k) q^k = \sum_{\substack{m,n \in \mathbb{Z} \\ m \text{ odd}}} \frac{m^2 - 4n^2}{2} q^{m^2+4n^2} = \sum_{m,n \in \mathbb{Z}} \frac{m^2 - 4n^2}{2} q^{m^2+4n^2},$$

since

$$\sum_{\substack{m,n \in \mathbb{Z} \\ m \text{ even}}} \frac{m^2 - 4n^2}{2} q^{m^2+4n^2} = 0.$$

Computing the first few Fourier coefficients of $\Psi(\tau)$ we see that

$$\Psi(\tau) = q - 6q^5 + 9q^9 + \cdots .$$

On the other hand, we know from [27] that

$$\eta(4\tau)^6 = q - 6q^5 + 9q^9 + \cdots \in S_3(\Gamma_0(16), \chi_{-4}).$$

Hence

$$g_{16}(\tau) = \eta(4\tau)^6 = \Psi(\tau)$$

by Sturm's theorem (cf. [50, Thm. 2.58]).

Equalities (4.14) and (4.15) can be established in a similar way. Indeed, we see from [5] and [6] that $g_8(\tau)$ and $g_{12}(\tau)$ are the inverse Mellin transforms of the Hecke L-series with respect to some weight 3 Hecke Grössencharacters defined for the rings $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[2\sqrt{-3}]$, respectively. $\square$

Lemma 4.7. *If g_{48} and g_{12} are as defined in Theorem 4.3, then the following identities hold:*

$$g_{48}(\tau) = \sum_{m,n\in\mathbb{Z}} \left(\left(\frac{m^2 - 12n^2}{2} \right) q^{m^2+12n^2} + \left(\frac{3m^2 - 4n^2}{2} \right) q^{3m^2+4n^2} \right), \tag{4.18}$$

$$g_{12}(\tau) + 8g_{12}(4\tau) = \sum_{m,n\in\mathbb{Z}} \left(\left(\frac{m^2 - 12n^2}{2} \right) q^{m^2+12n^2} + \left(\frac{4n^2 - 3m^2}{2} \right) q^{3m^2+4n^2} \right). \tag{4.19}$$

Proof. Let $\mathfrak{h}_1(\tau) := \sum_{m,n\in\mathbb{Z}} \left(\left(\frac{m^2 - 12n^2}{2} \right) q^{m^2+12n^2} + \left(\frac{3m^2 - 4n^2}{2} \right) q^{3m^2+4n^2} \right)$. Note that by the symmetry of the summation we have

$$\sum_{\substack{m,n\in\mathbb{Z} \\ m \text{ even}}} \left(\left(\frac{m^2 - 12n^2}{2} \right) q^{m^2+12n^2} + \left(\frac{3m^2 - 4n^2}{2} \right) q^{3m^2+4n^2} \right) = 0.$$

Also, it is obvious that for all $x, y \in \mathbb{Z}$

$$x^2 + 3y^2 = \frac{3(x-y)^2 + (x+3y)^2}{4} = \frac{3(x+y)^2 + (x-3y)^2}{4}.$$

Hence

$$\begin{aligned}\mathfrak{h}_1(\tau) &= \sum_{\substack{m\in\mathbb{Z} \\ m \text{ odd}}} \sum_{\substack{n\in\mathbb{Z} \\ n \text{ even}}} \left(\left(\frac{m^2 - 3n^2}{2} \right) q^{m^2+3n^2} + \left(\frac{3m^2 - n^2}{2} \right) q^{3m^2+n^2} \right) \\ &= \sum_{\substack{m\in\mathbb{Z} \\ m \text{ odd}}} \sum_{\substack{n\in\mathbb{Z} \\ n \text{ even}}} \left(\left(\frac{m^2 - 3n^2}{4} \right) q^{\frac{3(m-n)^2+(m+3n)^2}{4}} + \left(\frac{m^2 - 3n^2}{4} \right) q^{\frac{3(m+n)^2+(m-3n)^2}{4}} \right.\end{aligned}$$

$$+ \left(\frac{3m^2-n^2}{4}\right) q^{\frac{3(m-n)^2+(3m+n)^2}{4}} + \left(\frac{3m^2-n^2}{4}\right) q^{\frac{3(m+n)^2+(3m-n)^2}{4}}\Bigg)$$

$$= \sum_{\substack{m>0 \\ m \text{ odd}}} \sum_{\substack{n\in\mathbb{Z} \\ n \text{ even}}} \Bigg(\frac{(m-n)(m+3n)}{2} q^{\frac{3(m-n)^2+(m+3n)^2}{4}} + \frac{(m+n)(m-3n)}{2} q^{\frac{3(m+n)^2+(m-3n)^2}{4}}$$

$$+ \frac{(m-n)(3m+n)}{2} q^{\frac{3(m-n)^2+(3m+n)^2}{4}} + \frac{(m+n)(3m-n)}{2} q^{\frac{3(m+n)^2+(3m-n)^2}{4}} \Bigg)$$

$$= \sum_{\substack{m>0 \\ m \text{ odd}}} \sum_{\substack{n>0 \\ n \text{ even}}} \Bigg((m-n)(m+3n) q^{\frac{3(m-n)^2+(m+3n)^2}{4}} + (m+n)(m-3n) q^{\frac{3(m+n)^2+(m-3n)^2}{4}}$$

$$+ (m-n)(3m+n) q^{\frac{3(m-n)^2+(3m+n)^2}{4}} + (m+n)(3m-n) q^{\frac{3(m+n)^2+(3m-n)^2}{4}} \Bigg)$$

$$+ \sum_{\substack{m>0 \\ m \text{ odd}}} \left(m^2 q^{m^2} + 3m^2 q^{3m^2} \right).$$

Let $\mathcal{A} = \{(k,l) \in \mathbb{N}_{\text{odd}}^2 \mid l \neq k \text{ and } l \neq 3k\}$ and $\mathcal{B} = \mathbb{N}_{\text{odd}} \times \mathbb{N}_{\text{even}}$. Recall that for any $k \in \mathbb{N}$

$$\chi_{-8}(k) = \begin{cases} 1 & \text{if } k \equiv 1,3 \pmod 8, \\ -1 & \text{if } k \equiv 5,7 \pmod 8, \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Thus it is easy to verify that for all $(m,n) \in \mathcal{B}$ the following equalities are true:

$$m - n = \chi_{-8}(|m-n|(m+3n))|m-n| = \chi_{-8}(|m-n|(3m+n))|m-n|,$$

$$m - 3n = \chi_{-8}((m+n)|m-3n|)|m-3n|,$$

$$3m - n = \chi_{-8}((m+n)|3m-n|)|3m-n|.$$

Let $(k,l) \in \mathcal{A}$. Then it is obvious that $(3k^2+l^2)/4 \in \mathbb{N}_{\text{odd}}$.

If $(3k^2+l^2)/4 \equiv 1 \pmod 4$, then either $8|(k-l)$ or $8|(k+l)$, so letting

$$(m,n) = \begin{cases} \left(\frac{3k+l}{4}, \frac{|k-l|}{4}\right) & \text{if } 8|(k-l), \\ \left(\frac{|3k-l|}{4}, \frac{k+l}{4}\right) & \text{if } 8|(k+l), \end{cases}$$

yields $(m,n) \in \mathcal{B}$. Consequently, we have the equality

$$\left\{(k,l) \in \mathcal{A} \mid \frac{3k^2+l^2}{4} \equiv 1 \pmod 4\right\} = \{(|m-n|, m+3n) \mid (m,n) \in \mathcal{B}\}$$

$$\sqcup \{(m+n, |m-3n|) \mid (m,n) \in \mathcal{B}\}$$

since the inclusion $\supseteq$ is obvious.

If $(3k^2+l^2)/4 \equiv 3 \pmod 4$, then either $8|(3k-l)$ or $8|(3k+l)$. Hence, if we let

$$(m,n) = \begin{cases} \left(\frac{k+l}{4}, \frac{|3k-l|}{4}\right) & \text{if } 8|(3k-l), \\ \left(\frac{|k-l|}{4}, \frac{3k+l}{4}\right) & \text{if } 8|(3k+l), \end{cases}$$

then $(m,n) \in \mathcal{B}$, so

$$\left\{(k,l) \in \mathcal{A} \mid \frac{3k^2+l^2}{4} \equiv 3 \pmod 4\right\} = \{(|m-n|, 3m+n) \mid (m,n) \in \mathcal{B}\}$$

$$\sqcup \{(m+n, |3m-n|) \mid (m,n) \in \mathcal{B}\}.$$

Therefore, we can simplify the last expression of $\mathfrak{h}_1(\tau)$ above to obtain

$$\mathfrak{h}_1(\tau) = \sum_{(k,l)\in\mathcal{A}} \chi_{-8}(kl)klq^{\frac{3k^2+l^2}{4}} + \sum_{\substack{m>0 \\ m \text{ odd}}} \left(m^2 q^{m^2} + 3m^2 q^{3m^2}\right) = \sum_{m,n\in\mathbb{N}} \chi_{-8}(mn)mnq^{\frac{3m^2+n^2}{4}}.$$

Then (4.18) follows easily since

$$g_{48}(\tau) = \left(\frac{\eta(4\tau)^9}{\eta(2\tau)^3\eta(8\tau)^3}\right)\left(\frac{\eta(12\tau)^9}{\eta(6\tau)^3\eta(24\tau)^3}\right)$$

and the following identity holds [37, Prop. 1.6]:

$$\frac{\eta(2\tau)^9}{\eta(\tau)^3\eta(4\tau)^3} = \sum_{n\in\mathbb{N}} \chi_{-8}(n) n q^{\frac{n^2}{8}}.$$

Now, let

$$\mathfrak{h}_2(\tau) := \sum_{m,n\in\mathbb{Z}} \left(\left(\frac{m^2-12n^2}{2}\right) q^{m^2+12n^2} + \left(\frac{4n^2-3m^2}{2}\right) q^{3m^2+4n^2} \right.$$

$$\left. - (4m^2-12n^2) q^{4m^2+12n^2} \right).$$

Then it is easy to see that

$$\mathfrak{h}_2(\tau) = \sum_{\substack{m\in\mathbb{Z} \\ m \text{ odd}}} \sum_{\substack{n\in\mathbb{Z} \\ n \text{ even}}} \left(\left(\frac{m^2-3n^2}{2}\right) q^{m^2+3n^2} + \left(\frac{n^2-3m^2}{2}\right) q^{3m^2+n^2} \right).$$

Repeating the arguments above and using the fact that for every $(m,n) \in \mathcal{B}$

$$m - n = \chi_{-4}\left(|m-n|(m+3n)\right)|m-n|,$$

$$n - m = \chi_{-4}\left(|n-m|(3m+n)\right)|n-m|,$$

$$m - 3n = \chi_{-4}\left((m+n)|m-3n|\right)|m-3n|,$$

$$n - 3m = \chi_{-4}\left((m+n)|n-3m|\right)|n-3m|,$$

we can deduce that

$$\mathfrak{h}_2(\tau) = \sum_{m,n\in\mathbb{N}} \chi_{-4}(mn) mn q^{\frac{3m^2+n^2}{4}}.$$

We then employ the q-series identity [37, Cor. 1.4]

$$\eta(\tau)^3 = \sum_{n\in\mathbb{N}} \chi_{-4}(n) n q^{\frac{n^2}{8}}$$

to conclude that

$$g_{12}(\tau) = \eta(2\tau)^3\eta(6\tau)^3 = \mathfrak{h}_2(\tau).$$

By (4.15), we see that

$$\sum_{m,n\in\mathbb{Z}} (4m^2 - 12n^2)q^{4m^2+12n^2} = 8g_{12}(4\tau),$$

so (4.19) follows. □

Lemma 4.8. *If $g_{24}^{(1)}, g_{24}^{(2)}$ and g_{40} are as defined in Theorem 4.3 and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$, then the following identities hold:*

$$L(g_{24}^{(1)}, s) = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2 - 6n^2}{(m^2 + 6n^2)^s} + \frac{2m^2 - 3n^2}{(2m^2 + 3n^2)^s} \right), \tag{4.20}$$

$$L(g_{24}^{(2)}, s) = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2 - 6n^2}{(m^2 + 6n^2)^s} + \frac{3m^2 - 2n^2}{(3m^2 + 2n^2)^s} \right), \tag{4.21}$$

$$L(g_{40}, s) = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2 - 10n^2}{(m^2 + 10n^2)^s} + \frac{5m^2 - 2n^2}{(5m^2 + 2n^2)^s} \right). \tag{4.22}$$

Proof. We have immediately from the proof of [6, Thm. 4.1] that

$$L_{\mathbb{Q}(\sqrt{-6})}(\phi, s) = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2 - 6n^2}{(m^2 + 6n^2)^s} + \frac{3m^2 - 2n^2}{(3m^2 + 2n^2)^s} \right),$$

where ϕ is the Hecke Grössencharacter given by

$$\phi((m + n\sqrt{-6})) = (m + n\sqrt{-6})^2,$$

$$\phi((2, \sqrt{-6})) = -2,$$

for any $m, n \in \mathbb{Z}$. Considering the first terms of this Hecke L-series, one sees that its inverse Mellin transform is exactly $g_{24}^{(2)}(\tau)$ by Sturm's theorem. Similarly, if we define the Hecke Grössencharacter ψ by

$$\psi((m + n\sqrt{-6})) = (m + n\sqrt{-6})^2,$$

$$\psi((2, \sqrt{-6})) = 2,$$

then we obtain the Hecke L-series

$$L_{\mathbb{Q}(\sqrt{-6})}(\psi, s) = \frac{1}{2}{\sum_{m,n\in\mathbb{Z}}}' \left(\frac{m^2 - 6n^2}{(m^2+6n^2)^s} + \frac{2m^2-3n^2}{(2m^2+3n^2)^s} \right),$$

whose inverse Mellin transform is $g_{24}^{(1)}(\tau)$. Consequently, (4.20) and (4.21) follow.

To show (4.22) we shall imitate the proof of [6, Thm. 4.1]. Recall that in $\mathbb{Z}\left[\sqrt{-10}\right]$ there are two classes of ideals, namely

$$\mathcal{A}_0 = \left\{(m+n\sqrt{-10}) \mid m,n\in\mathbb{Z}\right\} \text{ and } \mathcal{A}_1 = \left\{(m+n\sqrt{-10})\mathcal{P} \mid m,n\in\mathbb{Z}\right\},$$

where $\mathcal{P} = (2, \sqrt{-10})$. Defining the Hecke character

$$\phi((m+n\sqrt{-10})) = (m+n\sqrt{-10})^2, \quad \phi(\mathcal{P}) = -2$$

and applying the formula

$$L_F(\phi, s) = \sum_{cl(P)} \frac{\phi(P)}{N(P)^{2-s}} \left(\frac{1}{2}{\sum_{\lambda\in P}}' \frac{\bar{\lambda}^2}{(\lambda\bar{\lambda})^s} \right),$$

we have

$$L_{\mathbb{Q}(\sqrt{-10})}(\phi, s) = \frac{1}{2}{\sum_{m,n\in\mathbb{Z}}}' \left(\frac{m^2 - 10n^2}{(m^2+10n^2)^s} + \frac{5m^2-2n^2}{(5m^2+2n^2)^s} \right),$$

and the inverse Mellin transform of this Hecke L-series equals $g_{40}(\tau)$. Since the conductors of the Hecke characters defined above are trivial and the discriminants of $\mathbb{Q}(\sqrt{-6})$ and $\mathbb{Q}(\sqrt{-10})$ are -24 and -40, respectively, we have that g_N are newforms of weight 3 and level N having CM by χ_{-N} (cf. [67, §1]). □

Lemma 4.9. *Let $t\in\mathbb{C}$ be such that $\mathrm{Re}(t) > 1$. Then the following equalities hold:*

$$2\left(1 - \frac{3}{2^t} + \frac{2}{2^{2t}}\right)\zeta(t)L(\chi_{-4}, t) = {\sum_{m,n\in\mathbb{Z}}}' \left(\frac{1}{(m^2+4n^2)^t} - \frac{1}{(2m^2+2n^2)^t} \right), \tag{4.23}$$

$$2L(\chi_8, t)L(\chi_{-3}, t) = {\sum_{m,n\in\mathbb{Z}}}' \left(\frac{1}{(m^2+6n^2)^t} - \frac{1}{(2m^2+3n^2)^t} \right), \tag{4.24}$$

$$2L(\chi_5,t)L(\chi_{-8},t) = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+10n^2)^t} - \frac{1}{(2m^2+5n^2)^t}\right), \quad (4.25)$$

$$2L(\chi_{12},t)L(\chi_{-4},t) = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+12n^2)^t} - \frac{1}{(3m^2+4n^2)^t}\right), \quad (4.26)$$

$$2L(\chi_{24},t)L(\chi_{-3},t) = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+18n^2)^t} - \frac{1}{(2m^2+9n^2)^t}\right). \quad (4.27)$$

Proof. First, recall from [29, §IV] that if we set

$$S(a,b,c;t) := \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(am^2+bmn+cn^2)^t},$$

then the following equalities hold:

$$S(1,0,1;t) = 4\zeta(t)L(\chi_{-4},t),$$

$$S(1,0,4;t) = 2\left(1-2^{-t}+2^{1-2t}\right)\zeta(t)L(\chi_{-4},t),$$

$$S(1,0,6;t) = \zeta(t)L(\chi_{-24},t) + L(\chi_8,t)L(\chi_{-3},t),$$

$$S(1,0,10;t) = \zeta(t)L(\chi_{-40},t) + L(\chi_5,t)L(\chi_{-8},t),$$

$$S(1,0,12;t) = \left(1+2^{-2t}+2^{2-4t}\right)\zeta(t)L(\chi_{-3},t) + L(\chi_{12},t)L(\chi_{-4},t),$$

$$S(1,0,18;t) = \left(1-2\cdot 3^{-t}+3^{1-2t}\right)\zeta(t)L(\chi_{-8},t) + L(\chi_{24},t)L(\chi_{-3},t).$$

We will exhibit how to prove (4.24) only, since the other identities can be shown similarly. Let Q_1 and Q_2 be the quadratic forms of discriminant -24 given by

$$Q_1(m,n) = m^2+6n^2, \ \ Q_2(m,n) = 2m^2+3n^2,$$

and, for each $j \in \{1,2\}$ and $k \in \mathbb{N}$, let $R_{Q_j}(k) = \#\left\{(m,n)\in\mathbb{Z}^2 \mid Q_j(m,n)=k\right\}$. By the formulas above, we see that

$$\sum_{k=1}^{\infty} \frac{R_{Q_1}(k)}{k^t} = \zeta(t)L(\chi_{-24},t) + L(\chi_8,t)L(\chi_{-3},t). \quad (4.28)$$

Notice that, for any given $l \in \mathbb{N}$, $2m^2 + 3n^2 = 2l$ is equivalent to $m^2 + 6b^2 = l$, where $n = 2b$. This implies that $R_{Q_2}(2l) = R_{Q_1}(l)$. Similarly, it can be checked that $R_{Q_2}(3l) = R_{Q_1}(l)$ and $R_{Q_2}(6l) = R_{Q_2}(l)$. As a result, we have

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(2m^2+3n^2)^t} &= \sum_{k=1}^{\infty} \frac{R_{Q_2}(k)}{k^t} \\ &= \sum_{\substack{k=1\\(k,6)=1}}^{\infty} \frac{R_{Q_2}(k)}{k^t} + \sum_{\substack{k=1\\2|k}}^{\infty} \frac{R_{Q_2}(k)}{k^t} + \sum_{\substack{k=1\\3|k}}^{\infty} \frac{R_{Q_2}(k)}{k^t} - \sum_{\substack{k=1\\6|k}}^{\infty} \frac{R_{Q_2}(k)}{k^t} \\ &= \sum_{\substack{k=1\\(k,6)=1}}^{\infty} \frac{R_{Q_2}(k)}{k^t} + \left(\frac{1}{2^t} + \frac{1}{3^t}\right) \sum_{k=1}^{\infty} \frac{R_{Q_1}(k)}{k^t} - \frac{1}{6^t} \sum_{k=1}^{\infty} \frac{R_{Q_2}(k)}{k^t}.\end{aligned} \tag{4.29}$$

If $(k, 6) = 1$, then

$$k \equiv \begin{cases} 1 \pmod 3 & \text{if } k = Q_1(m,n), \\ -1 \pmod 3 & \text{if } k = Q_2(m,n). \end{cases}$$

Hence we find from the well-known formula due to Dirichlet [25, p. 229] that

$$R_{Q_2}(k) = (1 - \chi_{-3}(k)) \sum_{l|k} \chi_{-24}(l) = \sum_{l|k} \chi_{-24}(l) - \sum_{l|k} \chi_{-3}\left(\frac{k}{l}\right) \chi_8(l).$$

It follows that

$$\begin{aligned}\sum_{\substack{k=1\\(k,6)=1}}^{\infty} \frac{R_{Q_2}(k)}{k^t} &= \sum_{\substack{k=1\\(k,6)=1}}^{\infty} \frac{(\mathbf{1} * \chi_{-24})(k)}{k^t} - \sum_{\substack{k=1\\(k,6)=1}}^{\infty} \frac{(\chi_{-3} * \chi_8)(k)}{k^t} \\ &= \left(1 - \frac{1}{2^t}\right)\left(1 - \frac{1}{3^t}\right) \zeta(t) L(\chi_{-24}, t) \\ &\quad - \left(1 + \frac{1}{2^t}\right)\left(1 + \frac{1}{3^t}\right) L(\chi_{-3}, t) L(\chi_8, t),\end{aligned} \tag{4.30}$$

where $*$ denotes the Dirichlet convolution. Then (4.24) can be derived easily using (4.28), (4.29), and (4.30). □

We are now in a good position to prove our main theorem.

Proof of Theorem 4.3. Applying Lemma 4.5, Proposition 4.2(i) for $\tau \in \{\frac{\sqrt{-1}}{2}, \frac{\sqrt{-3}}{2}\}$, Lemma 4.6, Lemma 4.7, and Lemma 4.9, we have immediately that

$$\begin{aligned}
n_2(64) &= \frac{1}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \Bigg(-\left(\frac{256n^2}{(m^2+4n^2)^3} - \frac{16}{(m^2+4n^2)^2}\right) \\
&\qquad + 16\left(\frac{4n^2}{(4m^2+n^2)^3} - \frac{1}{(4m^2+n^2)^2}\right)\Bigg) \\
&= \frac{128}{\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{m^2-4n^2}{(m^2+4n^2)^3}\right) = \frac{128}{\pi^3}L(g_{16},3), \\
n_2(256) &= \frac{\sqrt{3}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \Bigg(-\left(\frac{256n^2}{(3m^2+4n^2)^3} - \frac{16}{(3m^2+4n^2)^2}\right) \\
&\qquad + 16\left(\frac{4n^2}{(12m^2+n^2)^3} - \frac{1}{(12m^2+n^2)^2}\right)\Bigg) \\
&= \frac{64\sqrt{3}}{\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-12n^2}{(m^2+12n^2)^3} + \frac{3m^2-4n^2}{(3m^2+4n^2)^3}\right)\right) \\
&\qquad + \frac{16\sqrt{3}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+12n^2)^2} - \frac{1}{(3m^2+4n^2)^2}\right) \\
&= \frac{64\sqrt{3}}{\pi^3}L(g_{48},3) + \frac{16}{3\pi}L(\chi_{-4},2),
\end{aligned}$$

where we have used the fact that $L(\chi_{12},2) = \frac{\pi^2}{6\sqrt{3}}$ to get the last equality.

Similarly, using Proposition 4.2 and the lemmas in this section properly, we get

$$\begin{aligned}
n_3(216) &= \frac{45\sqrt{6}}{\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-6n^2}{(m^2+6n^2)^3} + \frac{2m^2-3n^2}{(2m^2+3n^2)^3}\right)\right) \\
&\qquad + \frac{45\sqrt{6}}{4\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+6n^2)^2} - \frac{1}{(2m^2+3n^2)^2}\right) \\
&= \frac{45\sqrt{6}}{\pi^3}L(g_{24}^{(1)},3) + \frac{45\sqrt{3}}{16\pi}L(\chi_{-3},2),
\end{aligned}$$

$$n_3(1458) = \frac{90\sqrt{3}}{\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-12n^2}{(m^2+12n^2)^3} + \frac{4m^2-3n^2}{(4m^2+3n^2)^3}\right)\right)$$

$$+ \frac{45\sqrt{3}}{2\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+12n^2)^2} - \frac{1}{(4m^2+3n^2)^2}\right)$$

$$= \frac{810\sqrt{3}}{8\pi^3}L(g_{12},3) + \frac{15}{2\pi}L(\chi_{-4},2),$$

$$n_4(648) = \frac{160}{\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{m^2-4n^2}{(m^2+4n^2)^3}\right) + \frac{40}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+4n^2)^2} - \frac{1}{(2m^2+2n^2)^2}\right)$$

$$= \frac{160}{\pi^3}L(g_{16},3) + \frac{5}{\pi}L(\chi_{-4},2),$$

$$n_4(2304) = \frac{80\sqrt{6}}{\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-6n^2}{(m^2+6n^2)^3} + \frac{3m^2-2n^2}{(3m^2+2n^2)^3}\right)\right)$$

$$+ \frac{20\sqrt{6}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+6n^2)^2} - \frac{1}{(2m^2+3n^2)^2}\right)$$

$$= \frac{80\sqrt{6}}{\pi^3}L(g_{24}^{(2)},3) + \frac{5\sqrt{3}}{\pi}L(\chi_{-3},2),$$

$$n_4(20736) = \frac{80\sqrt{10}}{\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-10n^2}{(m^2+10n^2)^3} + \frac{5m^2-2n^2}{(5m^2+2n^2)^3}\right)\right)$$

$$+ \frac{20\sqrt{10}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+10n^2)^2} - \frac{1}{(5m^2+2n^2)^2}\right)$$

$$= \frac{80\sqrt{10}}{\pi^3}L(g_{40},3) + \frac{32\sqrt{2}}{5\pi}L(\chi_{-8},2),$$

$$n_4(614656) = \frac{800\sqrt{2}}{3\pi^3}\left(\frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{m^2-2n^2}{(m^2+2n^2)^3}\right)$$

$$+ \frac{60\sqrt{2}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+18n^2)^2} - \frac{1}{(2m^2+9n^2)^2}\right)$$

$$= \frac{800\sqrt{2}}{3\pi^3}L(g_8,3) + \frac{10\sqrt{3}}{\pi}L(\chi_{-3},2),$$

since $\zeta(2) = \frac{\pi^2}{6}$, $L(\chi_8, 2) = \frac{\pi^2}{8\sqrt{2}}$, $L(\chi_5, 2) = \frac{4\pi^2}{25\sqrt{5}}$, and $L(\chi_{24}, 2) = \frac{\pi^2}{4\sqrt{6}}$. Then equalities (4.6)-(4.13) can be deduced using functional equations (2.4) and (2.6). (For each newform given in Theorem 4.3, the sign of the functional equation (2.6) is '$+$' by numerical approximation.) □

We also found some formulas of $n_j(s)$, when s are algebraic integers in some number fields, which look quite similar to those in Theorem 3.3 and Theorem 3.6. While most of them are still conjectural (see Section 4.3), we give a rigorous proof of two such formulas below.

Theorem 4.10. *The following identities are true:*

$$n_4(26856 + 15300\sqrt{3}) = \frac{5}{12}\left(20L'(g_{12}, 0) + 4L'(g_{48}, 0) + 11L'(\chi_{-3}, -1) + 8L'(\chi_{-4}, -1)\right),$$

$$n_4(26856 - 15300\sqrt{3}) = \frac{5}{6}\left(-20L'(g_{12}, 0) + 4L'(g_{48}, 0) - 11L'(\chi_{-3}, -1) + 8L'(\chi_{-4}, -1)\right),$$

where g_{12} and g_{48} are as defined in Theorem 4.3.

Proof. Observe that the function $s_4(\tau)$ defined in Theorem 4.3 can be rewritten in the form

$$s_4(\tau) = \frac{1}{\mathfrak{f}_1^8(2\tau)}\left(\frac{16}{\mathfrak{f}_1^8(4\tau)} + \frac{\mathfrak{f}_1^8(4\tau)}{\mathfrak{f}_1^8(2\tau)}\right)^4,$$

where $\mathfrak{f}_1(\tau) = \eta\left(\frac{\tau}{2}\right)/\eta(\tau)$. We obtain from [79, Tab. VI] that

$$\mathfrak{f}_1^4(\sqrt{-12}) = 2^{\frac{7}{6}}(1+\sqrt{3}), \qquad \mathfrak{f}_1^8(\sqrt{-48}) = 2^{\frac{19}{6}}(1+\sqrt{3})(\sqrt{2}+\sqrt{3})^2(1+\sqrt{2})^2.$$

Therefore, after simplifying, we have $s_4\left(\sqrt{-3}\right) = 26856 + 15300\sqrt{3}$, and substituting

$\tau = \sqrt{-3}$ in Proposition 4.2(iii) yields

$$\begin{aligned} n_4(26856 + 15300\sqrt{3}) &= \frac{10\sqrt{3}}{\pi^3} \sum_{m,n\in\mathbb{Z}}{}' \Bigg(-\left(\frac{4n^2}{(3m^2+n^2)^3} - \frac{1}{(3m^2+n^2)^2} \right) \\ &\qquad + 4\left(\frac{4n^2}{(12m^2+n^2)^3} - \frac{1}{(12m^2+n^2)^2} \right) \Bigg) \\ &= \frac{10\sqrt{3}}{\pi^3} \sum_{m,n\in\mathbb{Z}}{}' \Bigg(\frac{2(3n^2-m^2)}{(m^2+3n^2)^3} + \frac{8(m^2-12n^2)}{(m^2+12n^2)^3} \\ &\qquad + \frac{4}{(m^2+12n^2)^2} - \frac{1}{(m^2+3n^2)^2} \Bigg). \end{aligned} \tag{4.31}$$

It was proved in [6, Cor. 4.4] that the following identity holds:

$$\frac{9}{8} \sum_{m,n\in\mathbb{Z}}{}' \frac{m^2-3n^2}{(m^2+3n^2)^3} = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-12n^2}{(m^2+12n^2)^3} + \frac{4n^2-3m^2}{(3m^2+4n^2)^3} \right). \tag{4.32}$$

Equivalently, one has that

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{2(3n^2-m^2)}{(m^2+3n^2)^3} + \frac{8(m^2-12n^2)}{(m^2+12n^2)^3} \right) &= \frac{5}{2} \sum_{m,n\in\mathbb{Z}}{}' \frac{m^2-3n^2}{(m^2+3n^2)^3} \\ &\quad + 4 \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{m^2-12n^2}{(m^2+12n^2)^3} + \frac{3m^2-4n^2}{(3m^2+4n^2)^3} \right) \\ &= 5L(g_{12},3) + 8L(g_{48},3), \end{aligned} \tag{4.33}$$

where the last equality is a direct consequence of Lemma 4.6 and Lemma 4.7.

Recall from Glasser and Zucker's results on lattice sums [29, Tab. VI] that

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+3n^2)^2} &= \frac{9}{4}\zeta(2)L(\chi_{-3},2) = \frac{3\pi^2}{8}L(\chi_{-3},2), \\ \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+12n^2)^2} &= \frac{69}{64}\zeta(2)L(\chi_{-3},2) + L(\chi_{12},2)L(\chi_{-4},2) \\ &= \frac{23\pi^2}{128}L(\chi_{-3},2) + \frac{\pi^2}{6\sqrt{3}}L(\chi_{-4},2). \end{aligned} \tag{4.34}$$

Then we substitute (4.33) and (4.34) in (4.31) to get

$$n_4(26856+15300\sqrt{3}) = \frac{50\sqrt{3}}{\pi^3}L(g_{12},3)+\frac{80\sqrt{3}}{\pi^3}L(g_{48},3)+\frac{55\sqrt{3}}{16\pi}L(\chi_{-3},2)+\frac{20}{3\pi}L(\chi_{-4},2).$$

Finally, the derivative expression follows directly from the functional equations for the involved L-functions.

The second formula can be shown in a similar manner by choosing $\tau_0 = \sqrt{-3}/2$. Although Weber did not list an explicit value of $\mathfrak{f}_1(\sqrt{-3})$ in his book, one can find it easily using the identity $\mathfrak{f}_1(2\tau) = \mathfrak{f}(\tau)\mathfrak{f}_1(\tau)$ and the fact that $\mathfrak{f}(\sqrt{-3}) = 2^{\frac{1}{3}}$. Therefore, we have $s_4(\tau_0) = 26856 - 15300\sqrt{3}$, and

$$\begin{aligned} n_4(s_4(\tau_0)) &= \frac{20\sqrt{3}}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \left(\frac{8(3m^2-4n^2)}{(3m^2+4n^2)^3}+\frac{2(m^2-3n^2)}{(m^2+3n^2)^3}+\frac{1}{(m^2+3n^2)^2}-\frac{4}{(3m^2+4n^2)^2}\right) \\ &= \frac{20\sqrt{3}}{\pi^3}\left(-5L(g_{12},3)+8L(g_{48},3)-\frac{11\pi^2}{32}L(\chi_{-3},2)+\frac{2\pi^2}{3\sqrt{3}}L(\chi_{-4},2)\right), \end{aligned}$$

where we again use (4.32), (4.34), and the identity

$$2L(\chi_{12},2)L(\chi_{-4},2) = \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{1}{(m^2+12n^2)^2}-\frac{1}{(3m^2+4n^2)^2}\right)$$

(see Lemma 4.9). □

4.3 Arithmetic of the associated $K3$ surfaces

We have seen that many three-variable Mahler measures can be expressed as special L-values, so it might be interesting to understand the geometric and arithmetic interpretation of these formulas. We shall denote by X_s, Y_s, and Z_s the projective hypersurfaces corresponding to the one-parameter families A_s, B_s, and C_s, respectively. The family Z_s is sometimes called the *Dwork family* and is known to be $K3$ surfaces (see; e.g., [34]). To see that, for all but finitely many s, X_s is a $K3$ surface, it suffices to show that it is birational to an elliptic surface which has a minimal Weierstrass form

$$y^2 = x^3 + A_4(z)x + A_6(z),$$

where $A_4(z), A_6(z) \in \mathbb{Z}[s, z]$ with $\deg(A_i) \leq 2i$ for all i and $\deg(A_i) > i$ for some i [69, §4]. Indeed, one can manipulate this using `Maple` and find that

$$A_4(z) = -768\left(z^2+1\right)^4 + 48sz^2\left(z^2+1\right)^2 - 3s^2z^4,$$

$$A_6(z) = 8192\left(z^2+1\right)^6 - 768sz^2\left(z^2+1\right)^4 - 48s^2z^4\left(z^2+1\right)^2 + 2s^3z^6.$$

Since $A_4(z)$ and $A_6(z)$ satisfy the conditions above, it follows that X_s is generically a family of $K3$ surfaces. Also, using the Weierstrass model above, we have that X_s is defined over $\mathbb{Q}$ if $s \in \mathbb{Q}$. Letting $s = 1/\mu$, we have that a period of $X_{s(\mu)}$ is

$$\begin{aligned} u_0(\mu) :=& \frac{1}{(2\pi i)^3}\int_{\mathbb{T}^3} \frac{1}{1-\mu^{1/2}\left(x+x^{-1}\right)\left(y+y^{-1}\right)\left(z+z^{-1}\right)}\frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \\ =& {}_3F_2\left(\begin{matrix}\frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2} \\ 1, \ 1\end{matrix}; 64\mu\right). \end{aligned}$$

One can observe from the definition of the Mahler measure that in this case, for $s > 64$,

$$\frac{dn_2(s)}{ds} = 2\mu^{\frac{1}{2}}u_0(\mu).$$

Furthermore, it can be checked easily that u_0 is a holomorphic solution around $\mu = 0$ of the third-order differential equation

$$\mu^2(64\mu-1)\frac{d^3u}{d\mu^3} + \mu(288\mu-3)\frac{d^2u}{d\mu^2} + (208\mu-1)\frac{du}{d\mu} + 8u = 0.$$

Therefore, the differential equation is the Picard-Fuchs equation of $X_{s(\mu)}$. Since the order of the Picard-Fuchs equation equals the rank of the transcendental lattice $\mathrm{T}(X)$, the generic Picard number of X_s must be 19, and we have from Morrison's result [49, Cor. 6.4] that X_s admits a Shida-Inose structure for every nonzero s. Hence there are isogenous elliptic curves E_s and E'_s together with the following diagram:

$$\begin{array}{ccc} X_s & & E_s \times E'_s \\ & \searrow \quad \swarrow & \\ & \mathrm{Km}(E_s \times E'_s) & \end{array}$$

Here $\mathrm{Km}(E_s \times E_s')$ is the Kummer surface for E_s and E_s', and the dashed arrows denote rational maps of degree 2. In addition, E_s is a CM elliptic curve if and only if X_s is singular. It is known from the results due to Ahlgren, Ono, and Penniston [1] and Long [44, 45] that $u_0\left(-\frac{\mu}{64}\right)$ is a holomorphic solution around $\mu = 0$ of the Picard-Fuchs equation of the family of $K3$ surfaces given by the equation

$$\tilde{X}_\mu : z^2 = xy(x+1)(y+1)(x+\mu y).$$

In particular, they proved that the family of elliptic curves associated to $\tilde{X}_\mu$ via a Shioda-Inose structure is

$$\tilde{E}_\mu : y^2 = (x-1)\left(x^2 - \frac{1}{1+\mu}\right).$$

Hence, by simple reparametrization, the family of elliptic curves

$$E_s : y^2 = (x-1)\left(x^2 - \frac{s}{s-64}\right)$$

gives rise to the Shioda-Inose structure of X_s, and the j-function of E_s is

$$j(E_s) = \frac{(s-16)^3}{s}.$$

Recall from [72, §A.3] that if E_s is defined over $\mathbb{Q}$, then E_s has complex multiplication if and only if

$$\begin{aligned} j(E_s) \in &\{-640320^3, -5280^3, -960^3, -3\cdot 160^3, -96^3, -32^3, -15^3,\\ &0, 12^3, 20^3, 2\cdot 30^3, 66^3, 255^3\} =: \mathcal{C}_1. \end{aligned}$$

Furthermore, with the aid of `Sage`, we find that the set of the CM j-invariants in $\mathbb{Q}(\sqrt{2})$ is

$$\begin{aligned} \mathcal{C}_1 \cup \{&41113158120 \pm 29071392966\sqrt{2}, 26125000 \pm 18473000\sqrt{2}, 2417472 \pm 1707264\sqrt{2},\\ &3147421320000 \pm 2225561184000\sqrt{2}\} =: \mathcal{C}_2. \end{aligned}$$

As a consequence, we can explicitly determine the values of s such that E_s has a CM j-invariant in $\mathcal{C}_2$. Some of these values are given below, together with $j(E_s)$, the discriminant D, and the conductor f of the order of the complex multiplication.

s	$j(E_s)$	D	f
16	0	-3	1
$256, -104 \pm 60\sqrt{3}$	$2 \cdot 30^3$	-3	2
-8	12^3	-4	1
$-512, 280 \pm 198\sqrt{2}$	66^3	-4	2
$1, \frac{47 \pm 45\sqrt{-7}}{2}$	-15^3	-7	1
$4096, -2024 \pm 765\sqrt{7}$	-15^3	-7	2
$-64, 56 \pm 40\sqrt{2}$	20^3	-8	1
$-1088 \pm 768\sqrt{2}$	$2417472 \mp 1707264\sqrt{2}$	-24	1
$568 + 384\sqrt{2} \pm 336\sqrt{3} \pm 216\sqrt{6}$	$2417472 + 1707264\sqrt{2}$	-24	1
$568 \pm 384\sqrt{2} + 336\sqrt{3} \pm 216\sqrt{6}$	$2417472 - 1707264\sqrt{2}$	-24	1

Table 4.1: Some values of s for which E_s is CM.

For each value of s in Table 4.1, we can verify, at least numerically, that $n_2(s)$ equals rational linear combinations of L-values of CM weight three newforms and those of Dirichlet characters, as listed in Table 4.3. We also hypothesize that for $s \in \mathbb{Q}$ the newforms are associated to the corresponding singular $K3$ surfaces X_s via Livné's theorem (see Theorem 2.12). On the other hand, it is unclear what is the role of the Dirichlet L-values which appear in the formulas. A possible explanation might be that they arise from the Mahler measures of two-dimensional faces of the Newton polytopes associated to the polynomials, as suggested by Bertin et al. [9, §1]. Note also that there are several algebraic values of s other than those in Table 4.1 which yield CM elliptic curves E_s, but we have not been able to determine whether the corresponding $n_2(s)$ are related to L-values. For example, if $s = 16 + 1600\sqrt[3]{2} - 1280\sqrt[3]{4}$, then $j(E_s) = -3 \cdot 160^3$, so E_s is CM by an order in $\mathbb{Q}(\sqrt{-3})$. We predict from the known examples that $n_2(s)$ should involve exactly three modular L-values, though no such conjectural formula has been found.

Now let us consider the family Z_s of quartic surfaces defined by $C_s = 0$. Let $\mu = 1/s$. Consider the integral

$$w_0(\mu) := \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3} \frac{1}{1 - \mu^{1/4}\left(\frac{x^4+y^4+z^4+1}{xyz}\right)} \frac{dx}{x}\frac{dy}{y}\frac{dz}{z},$$

which can be realized as a formal period $\int_\gamma \omega_s$, where ω_s is a holomorphic 2-form and γ is a 2-cycle on Z_s. Then using the Taylor series expansion and combinatorial arguments, one can find easily that for $|\mu|$ sufficiently small

$$w_0(\mu) = {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix}; 256\mu\right).$$

Therefore, $w_0(\mu)$ satisfies the differential equation

$$\mu^2(256\mu - 1)\frac{d^3w}{d\mu^3} + \mu(1152\mu - 3)\frac{d^2w}{d\mu^2} + (816\mu - 1)\frac{dw}{d\mu} + 24w = 0. \tag{4.35}$$

In other words, (4.35) is the Picard-Fuchs equation of the quartic surfaces. By direct calculation, one sees that (4.35) is the symmetric square of the second-order differential equation

$$\mu(256\mu - 1)\frac{d^2w}{d\mu^2} + (384\mu - 1)\frac{dw}{d\mu} + 12w = 0, \tag{4.36}$$

whose non-holomorphic solution around $\mu = 0$ is

$${}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{matrix}; \frac{1+\sqrt{1-256\mu}}{2}\right).$$

It was obtained in the proof of [47, Cor. 2.2] that if E_λ denotes the *Clausen form* elliptic curves

$$y^2 = (x-1)(x^2+\lambda), \qquad \lambda \notin \{0, -1\},$$

then the real period $\Omega(E_\lambda)$ of E_λ is

$$\Omega(E_\lambda) = \pi {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{matrix}; -\lambda\right).$$

Hence (4.36) is the Picard-Fuchs equation of the family

$$G_s : y^2 = (x-1)(x-r')(x+r'), \qquad r' = \sqrt{\frac{1+\sqrt{1-\frac{256}{s}}}{2}},$$

and this family of elliptic curves is associated to the $K3$ surfaces Z_s by a Shioda-Inose structure (see Theorem 2.6). It is also worth mentioning that if we set $s = -2^{10}u^4/(u^4-1)^2$, then the j-function of the family G_s is given by

$$j(G_{s(u)}) = \frac{64(u^2+3)^3(3u^2+1)^3}{(u^2-1)^4(u^2+1)^2},$$

which coincides with Long's result [43, 5.15]. Thus it can be shown in a similar manner that if s is an algebraic number in the first column of Table 4.2, then Z_s is a singular $K3$ surface, and $n_4(s)$ relates to modular and Dirichlet L-values (see Table 4.5 below).

s	$j(G_s)$	D	f
$-144, 26856-15300\sqrt{3}$	$2\cdot 30^3$	-3	2
$26856+15300\sqrt{3}$	$1417905000+818626500\sqrt{3}$	-3	4
$648, 143208-101574\sqrt{2}$	66^3	-4	2
-12288	$76771008+44330496\sqrt{3}$	-4	3
$143208+101574\sqrt{2}$	$41113158120+29071392966\sqrt{2}$	-4	4
81	-15^3	-7	1
$-3969, 8292456-3132675\sqrt{7}$	255^3	-7	2
$8292456+3132675\sqrt{7}$	$137458661985000+51954490735875\sqrt{7}$	-7	4
$256, 3656-2600\sqrt{2}$	20^3	-8	1
$3656+2600\sqrt{2}$	$26125000+18473000\sqrt{2}$	-8	2
614656	$188837384000+77092288000\sqrt{6}$	-8	3
$\frac{-192303\pm 85995\sqrt{5}}{2}$	$\frac{37018076625\mp 16554983445\sqrt{5}}{2}$	-15	2
-1024	$632000+282880\sqrt{5}$	-20	1
$2304, 1207368+853632\sqrt{2}-697680\sqrt{3}-493272\sqrt{6}$	$2417472+1707264\sqrt{2}$	-24	1
$1207368-853632\sqrt{2}-697680\sqrt{3}+493272\sqrt{6}$	$2417472-1707264\sqrt{2}$	-24	1
$1207368\pm 853632\sqrt{2}+697680\sqrt{3}\pm 493272\sqrt{6}$	$5835036074184\pm 4125993565824\sqrt{2}+3368859648336\sqrt{3}\pm 2382143496408\sqrt{6}$	-24	2
20736	$212846400+95178240\sqrt{5}$	-40	1
-82944	$3448440000+956448000\sqrt{13}$	-52	1
$-893952\pm 516096\sqrt{3}$	$799200236736\mp 461418467328\sqrt{3}+302069634048\sqrt{7}\mp 174399982848\sqrt{21}$	-84	1
$347648256\pm 141926400\sqrt{6}$	$120858928019208000\pm 49340450750976000\sqrt{6}\pm 32300907105600000\sqrt{14}+26373580212672000\sqrt{21}$	-168	1

Table 4.2: Some values of s for which G_s is CM.

We conjecture that the hypersurfaces Y_s are also $K3$ surfaces, though this assertion has not yet been verified. By similar calculation above, if $\mu = 1/s$, we find that the formal period $v_0(\mu)$, of Y_s is

$$\begin{aligned} v_0(\mu) &:= \frac{1}{(2\pi i)^3}\int_{\mathbb{T}^3} \frac{1}{1-\mu(x+x^{-1})^2(y+y^{-1})^2(1+z)^3z^{-2}}\frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \\ &= {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix}; 108\mu\right), \end{aligned}$$

which is a solution of

$$\mu^2(108\mu-1)\frac{d^3v}{d\mu^3} + 3\mu(162\mu-1)\frac{d^2v}{d\mu^2} + (348\mu-1)\frac{dv}{d\mu} + 12v = 0.$$

This third-order differential equation is the symmetric square of

$$\mu(108\mu-1)\frac{d^2v}{d\mu^2} + (162\mu-1)\frac{dv}{d\mu} + 6v = 0,$$

which is the Picard-Fuchs equation of the reparameterized Hesse family

$$F_s : x^3+y^3+1-rxy = 0, \qquad r = \sqrt[3]{\frac{s+\sqrt{s(s-108)}}{2}}.$$

Hence it is reasonable to guess that Y_s is a $K3$ hypersurface associated to F_s via a Shioda-Inose structure. In Section 5, we will derive some interesting formulas of $n_j(s), j = 2,3,4$, which involve torsion points on the curves E_s, F_s, and G_s, respectively.

One has from Livné's theorem that a singular $K3$ surface defined over $\mathbb{Q}$ is always modular. Nevertheless, the modularity of singular $K3$ surfaces defined over arbitrary number fields is not known. The numerical evidences of relationships between the three-variable Mahler measures and L-values listed in Table 4.3-Table 4.5 might give us some clues about modularity of the corresponding $K3$ surfaces defined over some number fields. One certainly requires further investigation to gain more insight into this subject. It would also be highly desirable to find all possible Mahler measure formulas $n_j(s)$ which are expressible in terms of special L-values.

We end this section by tabulating all three-variable Mahler measure formulas that we found from numerical computations. Some of these formulas have been proved previously in this section. In Table 4.3-Table 4.5, we use the following shorthand notations:

$$d_k := L'(\chi_{-k}, -1), \qquad M_N := L'(g_N, 0), \qquad M_{N\otimes D} := L'(g_N \otimes \chi_D, 0),$$

where g_N is a normalized CM newform in $S_3(\Gamma_0(N), \chi_{-D_N})$ with rational Fourier coefficients, and $g_N \otimes \chi_D$ is the quadratic twist of g_N by χ_D. Each value of τ in the first column of each table can be determined as follows: Recall from the proof of [58, Thm.2.3] that if

$$q_j(\alpha) = \exp\left(-\frac{\pi}{\sin(\pi/j)} \frac{{}_2F_1\left(\begin{matrix}\frac{1}{j}, \frac{j-1}{j} \\ 1\end{matrix}; 1-\alpha\right)}{{}_2F_1\left(\begin{matrix}\frac{1}{j}, \frac{j-1}{j} \\ 1\end{matrix}; \alpha\right)}\right),$$

then $s_2(q_2(\alpha)) = \frac{16}{\alpha(1-\alpha)}$, $s_3(q_3(\alpha)) = \frac{27}{\alpha(1-\alpha)}$, and $s_4(q_4(\alpha)) = \frac{64}{\alpha(1-\alpha)}$. Hence, for instance, each τ in Table 4.3 is given by

$$\tau = \frac{i}{2} \frac{{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; 1 - \frac{1+\sqrt{1-\frac{64}{s_2(\tau)}}}{2}\right)}{{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}; \frac{1+\sqrt{1-\frac{64}{s_2(\tau)}}}{2}\right)}.$$

τ	$s_2(\tau)$	$n_2(s_2(\tau))$
$\frac{\sqrt{-1}}{2}$	64	$8M_{16}$
$\frac{1+\sqrt{-1}}{2}$	-8	$4M_{16}+d_4$
$\frac{\sqrt{-4}}{2}$	$280+198\sqrt{2}$	$\frac{1}{8}\left(36M_{16}+4M_{16\otimes 8}+13d_4+4d_8\right)$
$\frac{2+\sqrt{-1}}{4}$	$280-198\sqrt{2}$	$\frac{1}{2}\left(36M_{16}-4M_{16\otimes 8}-13d_4+4d_8\right)$
$\frac{1+\sqrt{-4}}{2}$	-512	$M_{64}+d_8$
$\frac{\sqrt{-2}}{2}$	$56+40\sqrt{2}$	$\frac{1}{4}\left(60M_8+4M_{8\otimes 8}+4d_4+d_8\right)$
$\frac{2+\sqrt{-2}}{4}$	$56-40\sqrt{2}$	$\frac{1}{2}\left(60M_8-4M_{8\otimes 8}+4d_4-d_8\right)$
$\frac{1+\sqrt{-2}}{2}$	-64	$2\left(M_{8\otimes 8}+d_4\right)$
$\frac{\sqrt{-3}}{2}$	256	$\frac{4}{3}\left(M_{12\otimes(-4)}+2d_4\right)$
$\frac{1+\sqrt{-3}}{4}$	16	$8M_{12}$
$\frac{3+\sqrt{-3}}{6}$	$-104+60\sqrt{3}$	$\frac{1}{2}\left(4M_{12\otimes(-4)}-36M_{12}+15d_3-8d_4\right)$
$\frac{1+\sqrt{-3}}{2}$	$-104-60\sqrt{3}$	$\frac{1}{6}\left(4M_{12\otimes(-4)}+36M_{12}+15d_3+8d_4\right)$
$\frac{\sqrt{-6}}{2}$	$568+384\sqrt{2}$ $+336\sqrt{3}+216\sqrt{6}$	$\frac{1}{24}\big(60M_{24}^{(1)}+12M_{24}^{(2)}+4M_{24\otimes(-8)}^{(1)}+4M_{24\otimes(-8)}^{(2)}$ $+60d_3+24d_4+8d_8+d_{24}\big)$
$\frac{6+\sqrt{-6}}{12}$	$568+384\sqrt{2}$ $-336\sqrt{3}-216\sqrt{6}$	$\frac{1}{4}\big(60M_{24}^{(1)}+12M_{24}^{(2)}-4M_{24\otimes(-8)}^{(1)}-4M_{24\otimes(-8)}^{(2)}$ $-60d_3+24d_4+8d_8-d_{24}\big)$
$\frac{\sqrt{-6}}{6}$	$568-384\sqrt{2}$ $+336\sqrt{3}-216\sqrt{6}$	$\frac{1}{12}\big(60M_{24}^{(1)}-12M_{24}^{(2)}+4M_{24\otimes(-8)}^{(1)}-4M_{24\otimes(-8)}^{(2)}$ $+60d_3+24d_4-8d_8-d_{24}\big)$
$\frac{-2+\sqrt{-6}}{10}$	$568-384\sqrt{2}$ $-336\sqrt{3}+216\sqrt{6}$	$\frac{1}{12}\big(60M_{24}^{(1)}-12M_{24}^{(2)}-4M_{24\otimes(-8)}^{(1)}+4M_{24\otimes(-8)}^{(2)}$ $+60d_3-24d_4+8d_8-d_{24}\big)$
$\frac{3+\sqrt{-6}}{6}$	$-1088+768\sqrt{2}$	$M_{24\otimes(-8)}^{(1)}-M_{24\otimes(-8)}^{(2)}-6d_4+2d_8$
$\frac{1+\sqrt{-6}}{2}$	$-1088-768\sqrt{2}$	$\frac{1}{3}\left(M_{24\otimes(-8)}^{(1)}+M_{24\otimes(-8)}^{(2)}+6d_4+2d_8\right)$
$\frac{\sqrt{-7}}{2}$	4096	$\frac{4}{7}\left(M_{7\otimes(-4)}+8d_4\right)$
$\frac{3+\sqrt{-7}}{8}$	1	$8M_7$
$\frac{\pm 1+\sqrt{-7}}{8}$	$\frac{47\pm 45\sqrt{-7}}{2}$	$\frac{4}{7}\left(54M_7+d7\right)$
$\frac{7+\sqrt{-7}}{14}$	$-2024+765\sqrt{7}$	$\frac{1}{2}\left(4M_{7\otimes(-4)}-384M_7-32d_4+11d_7\right)$
$\frac{1+\sqrt{-7}}{2}$	$-2024-765\sqrt{7}$	$\frac{1}{14}\left(4M_{7\otimes(-4)}+384M_7+32d_4+11d_7\right)$

Table 4.3: Some L-value expressions of $n_2(s)$

τ	$s_3(\tau)$	$n_3(s_3(\tau))$
$\frac{1+\sqrt{-2}}{3}$	8	$15M_8$
$\frac{\sqrt{-3}}{3}$	108	$15M_{12}$
$\frac{\sqrt{-6}}{3}$	216	$\frac{15}{4}\left(M_{24}^{(2)}+d_3\right)$
$\frac{\sqrt{-9}}{3}$	$288+168\sqrt{3}$	$\frac{5}{12}\left(3M_{36}^{(2)}+3M_{36}^{(1)}+6d_3+4d_4\right)$
$\frac{1+\sqrt{-1}}{2}$	$288-168\sqrt{3}$	$\frac{5}{6}\left(3M_{36}^{(2)}-3M_{36}^{(1)}-6d_3+4d_4\right)$
$\frac{\sqrt{-12}}{3}$	1458	$\frac{15}{8}\left(9M_{12}+2d_4\right)$
$\frac{\sqrt{-15}}{3}$	3375	$\frac{3}{5}\left(20M_{15}^{(2)}+13d_3\right)$
$\frac{\sqrt{-18}}{3}$	$3704+1456\sqrt{6}$	$\frac{5}{24}\left(3M_{8\otimes(-3)}+72M_8+18d_3+4d_8\right)$
$\frac{\sqrt{-2}}{2}$	$3704-1456\sqrt{6}$	$\frac{5}{12}\left(3M_{8\otimes(-3)}-72M_8-18d_3+4d_8\right)$
$\frac{\sqrt{-21}}{3}$	$7344+2808\sqrt{7}$	$\frac{15}{28}\left(M_{84}^{(2)}+M_{84}^{(4)}+4d_4+2d_7\right)$
$\frac{3+\sqrt{-21}}{6}$	$7344-2808\sqrt{7}$	$\frac{15}{14}\left(M_{84}^{(2)}-M_{84}^{(4)}-4d_4+2d_7\right)$
$\frac{\sqrt{-24}}{3}$	$14310+8262\sqrt{3}$	$\frac{15}{32}\left(7M_{24}^{(2)}+M_{24\otimes(-8)}^{(2)}+11d_3+6d_4\right)$
$\frac{-3+\sqrt{-6}}{2}$	$14310-8262\sqrt{3}$	$\frac{15}{8}\left(7M_{24}^{(2)}-M_{24\otimes(-8)}^{(2)}+11d_3-6d_4\right)$
$\frac{\sqrt{-30}}{3}$	$48168+15120\sqrt{10}$	$\frac{3}{40}\left(5M_{120}^{(2)}+5M_{120}^{(4)}+5d_{15}+2d_{24}\right)$
$\frac{6+\sqrt{-30}}{6}$	$48168-15120\sqrt{10}$	$\frac{3}{20}\left(5M_{120}^{(2)}-5M_{120}^{(4)}+5d_{15}-2d_{24}\right)$

Table 4.4: Some L-value expressions of $n_3(s)$

τ	$s_4(\tau)$	$n_4(s_4(\tau))$
$\frac{\sqrt{-2}}{2}$	256	$40M_8$
$\frac{\sqrt{-8}}{2}$	$3656+2600\sqrt{2}$	$\frac{5}{8}\left(4M_{8\otimes 8}+28M_8+4d_4+d_8\right)$
$\frac{1+\sqrt{-2}}{2}$	$3656-2600\sqrt{2}$	$\frac{5}{4}\left(4M_{8\otimes 8}-28M_8+4d_4-d_8\right)$
$\frac{\sqrt{-12}}{2}$	$26856+15300\sqrt{3}$	$\frac{5}{12}\left(4M_{12\otimes(-4)}+20M_{12}+11d_3+8d_4\right)$
$\frac{\sqrt{-3}}{2}$	$26856-15300\sqrt{3}$	$\frac{5}{6}\left(4M_{12\otimes(-4)}-20M_{12}-11d_3+8d_4\right)$
$\frac{1+\sqrt{-3}}{2}$	-144	$\frac{10}{3}\left(4M_{12}+d_3\right)$
$\frac{\sqrt{-4}}{2}$	648	$\frac{5}{2}\left(4M_{16}+d_4\right)$
$\frac{\sqrt{-16}}{2}$	$143208+101574\sqrt{2}$	$\frac{5}{16}\left(4M_{16\otimes 8}+20M_{16}+9d_4+4d_8\right)$
$\frac{1+\sqrt{-4}}{2}$	$143208-101574\sqrt{2}$	$\frac{5}{8}\left(4M_{16\otimes 8}-20M_{16}-9d_4+4d_8\right)$
$\frac{1+\sqrt{-5}}{2}$	-1024	$\frac{8}{5}\left(5M_{20}^{(1)}+2d_4\right)$
$\frac{\sqrt{-6}}{2}$	2304	$\frac{20}{3}\left(M_{24}^{(1)}+d_3\right)$
$\frac{\sqrt{-24}}{2}$	$1207368+853632\sqrt{2}$ $+697680\sqrt{3}+493272\sqrt{6}$	$\frac{5}{48}\big(4M_{24\otimes(-8)}^{(1)}+4M_{24\otimes(-8)}^{(2)}+28M_{24}^{(1)}+12M_{24}^{(2)}$ $+28d_3+24d_4+8d_8+d_{24}\big)$
$\frac{1+\sqrt{-6}}{2}$	$1207368+853632\sqrt{2}$ $-697680\sqrt{3}-493272\sqrt{6}$	$\frac{5}{24}\big(4M_{24\otimes(-8)}^{(1)}+4M_{24\otimes(-8)}^{(2)}-28M_{24}^{(1)}-12M_{24}^{(2)}$ $-28d_3+24d_4+8d_8-d_{24}\big)$
$\frac{\sqrt{-6}}{4}$	$1207368-853632\sqrt{2}$ $+697680\sqrt{3}-493272\sqrt{6}$	$\frac{5}{16}\big(4M_{24\otimes(-8)}^{(1)}-4M_{24\otimes(-8)}^{(2)}+28M_{24}^{(1)}-12M_{24}^{(2)}$ $-28d_3-24d_4+8d_8+d_{24}\big)$
$\frac{2+\sqrt{-6}}{4}$	$1207368-853632\sqrt{2}$ $-697680\sqrt{3}+493272\sqrt{6}$	$\frac{5}{12}\big(-4M_{24\otimes(-8)}^{(1)}+4M_{24\otimes(-8)}^{(2)}+28M_{24}^{(1)}-12M_{24}^{(2)}$ $+28d_3-24d_4+8d_8-d_{24}\big)$
$\frac{\sqrt{-28}}{2}$	$8292456+3132675\sqrt{7}$	$\frac{5}{28}\left(4M_{7\otimes(-4)}+224M_7+32d_4+7d_7\right)$
$\frac{\sqrt{-7}}{2}$	$8292456-3132675\sqrt{7}$	$\frac{5}{14}\left(4M_{7\otimes(-4)}-224M_7+32d_4-7d_7\right)$
$\frac{\sqrt{14}+\sqrt{-28}}{8}$	81	$40M_7$
$\frac{1+\sqrt{-7}}{2}$	-3969	$\frac{10}{7}\left(40M_7+d_7\right)$
$\frac{1+\sqrt{-9}}{2}$	-12288	$\frac{40}{9}\left(M_{36}^{(1)}+2d_3\right)$
$\frac{\sqrt{-10}}{2}$	20736	$\frac{4}{5}\left(5M_{40}^{(1)}+2d_8\right)$
$\frac{1+\sqrt{-13}}{2}$	-82944	$\frac{40}{13}\left(M_{52}^{(1)}+2d_4\right)$
$\frac{3+\sqrt{-15}}{6}$	$\frac{-192303+85995\sqrt{5}}{2}$	$\frac{1}{5}\left(160M_{15}^{(1)}-120M_{15}^{(2)}-88d_3+5d_{15}\right)$
$\frac{1+\sqrt{-15}}{2}$	$\frac{-192303-85995\sqrt{5}}{2}$	$\frac{1}{15}\left(160M_{15}^{(1)}+120M_{15}^{(2)}+88d_3+5d_{15}\right)$
$\frac{\sqrt{-18}}{2}$	614656	$\frac{40}{3}\left(5M_8+d_3\right)$
$\frac{3+\sqrt{-21}}{6}$	$-893952+516096\sqrt{3}$	$\frac{20}{7}\left(M_{84}^{(3)}-M_{84}^{(4)}+8d_3-4d_4\right)$
$\frac{1+\sqrt{-21}}{2}$	$-893952-516096\sqrt{3}$	$\frac{20}{21}\left(M_{84}^{(3)}+M_{84}^{(4)}+8d_3+4d_4\right)$
$\frac{\sqrt{-42}}{42}$	$347648256+141926400\sqrt{6}$	$\frac{10}{21}\left(M_{168}^{(3)}+M_{168}^{(4)}+20d_3+4d_8\right)$
$\frac{\sqrt{-42}}{14}$	$347648256-141926400\sqrt{6}$	$\frac{10}{7}\left(M_{168}^{(3)}-M_{168}^{(4)}-20d_3+4d_8\right)$

Table 4.5: Some L-value expressions of $n_4(s)$

4.4 The family $D_s = (x+x^{-1})^2(1+y)^3y^{-2}(z+z^{-1})^6 - s$

Rogers mentioned the family D_s in [58, §4] and regarded its Mahler measure $n_6(s)$ as a special function arising from Ramanujan's theory of signature 6. However, no known results about $n_6(s)$ seem to exist in the literature. Note that the functions $s_j(\tau), j = 2, 3, 4,$ and linear combinations of weight 4 Eisenstein series appearing in the proof of Proposition 4.2 are the analogues of the functions $t_j(\tau)$ and $e(\tau)$ in Section 3. Thus one might try to study the Mahler measure of this family using Rodriguez Villegas' approach. First, we have by direct calculation that, if $\mu = 1/s$, a period of this family is

$$\begin{aligned} u_0(\mu) :&= \frac{1}{(2\pi i)^3}\int_{\mathbb{T}^3}\frac{1}{1-\mu(x+x^{-1})^2(1+y)^3y^{-2}(z+z^{-1})^6}\frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \\ &= {}_3F_2\left(\begin{matrix}\frac{1}{6}, \ \frac{1}{2}, \ \frac{5}{6} \\ 1, \ 1\end{matrix}; 1728\mu\right), \end{aligned}$$

and the Mahler measure $n_6(s)$ can be expressed as

$$n_6(s) = \operatorname{Re}\left(\log(s) - \frac{120}{s}{}_5F_4\left(\begin{matrix}\frac{7}{6}, \ \frac{9}{6}, \ \frac{11}{6}, \ 1, \ 1 \\ 2, \ 2, \ 2, \ 2\end{matrix}; \frac{1728}{s}\right)\right), \qquad |s| \geq 1728.$$

The function $u_0(\mu)$ satisfies the differential equation

$$\mu^2(1728\mu-1)\frac{d^3u}{d\mu^3} + \mu(7776\mu-3)\frac{d^2u}{d\mu^2} + (5424\mu-1)\frac{du}{d\mu} + 120u = 0, \tag{4.37}$$

which is the symmetric square of the second-order differential equation

$$\mu(1728\mu-1)\frac{d^2u}{d\mu^2} + (2592\mu-1)\frac{du}{d\mu} + 60u = 0.$$

Then we found using `Maple` that a second solution of (4.37) is

$$u_1(\mu) = u_0(\mu)\log\mu + 744\mu + 562932\mu^2 + 570443360\mu^3 + \cdots .$$

Let $\tau = \frac{1}{2\pi i}\frac{u_1}{u_0}(\mu)$ and $q = e^{2\pi i\tau}$. Then we can write the following functions as q-series:

$$\mu(\tau) = q - 744q^2 + 356652q^3 - 140361152q^4 + 49336682190q^5 + \cdots ,$$

$$s_6(\tau) := 1/\mu(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots \stackrel{?}{=} j(\tau),$$

$$c(\tau) := u_0(\mu(\tau)) = 1 + 120q - 6120q^2 + 737760q^3 - 107249640q^4 + \cdots \stackrel{?}{=} E_4^{1/2}(\tau),$$

$$e(\tau) := \frac{c}{\mu}(\tau)q\frac{d\mu}{dq} = 1 - 624q + 64368q^2 - 12403776q^3 + 2449464432q^4 + \cdots .$$

Note that we get two conjectural equalities above by comparing the first few coefficients of the q-series. If these equalities hold, then we can deduce an interesting formula for $e(\tau)$. Recall that the *Ramanujan's theta operator* is defined by

$$\Theta := q\frac{d}{dq}.$$

Hence, if $1/\mu(\tau)) = j(\tau)$ and $c(\tau) = E_4^{1/2}(\tau)$, we have immediately from [15, (1.9)] and [51, Cor.2.3] that

$$e(\tau) = -\frac{E_4^{1/2}(\tau)}{j(\tau)}\Theta(j(\tau)) = \frac{E_6(\tau)}{E_4(\tau)^{1/2}}.$$

Since $e(\tau)$ does not appear to be a linear combination of Eisenstein series, we cannot directly use the Fourier development technique to prove a formula similar to those in Proposition 4.2. Despite the lack of such general formula, it is worth noting that we were able to find a conjectural formula relating $n_6(s)$ to a modular L-value via computational experiments.

Conjecture 4.11. *Let g_{16} be a weight 3 newform as defined in Theorem 4.3. Then the following equality holds:*

$$n_6(1728) \stackrel{?}{=} \frac{1}{2} L'(g_{16} \otimes \chi_{12}, 0). \tag{4.38}$$

In fact, the twist $g_{16} \otimes \chi_{12}$ is a CM newform in $S_3(\Gamma_0(144), \chi_{-12})$. After extensively searching for similar formulas, we ended up finding nothing other than (4.38), and it remains a challenge to verify (4.38) rigorously.

5. THE ELLIPTIC TRILOGARITHM AND MAHLER MEASURES OF $K3$ SURFACES

5.1 Background

Let us first give a brief introduction to polylogarithms and higher polylogarithms, which will play a major role in this section. For $m \in \mathbb{N}$, the classical m^{th} polylogarithm function is defined by

$$\mathrm{Li}_m(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^m}, \qquad |z| < 1.$$

One can obtain a multivalued function on $\mathbb{C} \setminus \{0,1\}$ from $\mathrm{Li}_m(z)$ by extending it analytically. There are many versions of higher polylogarithms in the literature, including the following single-valued function defined in [89, §2]

$$\mathcal{L}_m(z) = \mathfrak{R}_m\left(\sum_{k=0}^{m-1} \frac{2^k B_k}{k!} \log^k |z| \,\mathrm{Li}_{m-k}(z)\right), \qquad |z| \le 1,$$

where

$$\mathfrak{R}_m = \begin{cases} \mathrm{Re} & \text{if } m \text{ is odd,} \\ \mathrm{Im} & \text{if } m \text{ is even,} \end{cases}$$

and B_k is the k^{th} Bernoulli number $(B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, \ldots)$. It can be extended to a continuous function on $\mathbb{P}^1(\mathbb{C})$ by the functional equation

$$\mathcal{L}_m\left(z^{-1}\right) = (-1)^{m-1}\mathcal{L}_m(z).$$

For $m = 2$, $\mathcal{L}_m(z)$ becomes the *Bloch-Wigner dilogarithm*

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log|z|\log(1-z)).$$

The dilogarithm function and the function $D(z)$ have been studied extensively and are known to satisfy several interesting properties. They are also found to have fruitful applications in algebraic K-theory and other related areas (see, for example, [87]). Another version of higher polylogarithm functions constructed by Ramakrishnan [53] and formulated in terms of the polylogarithms by Zagier [86] is

$$D_m(z) = \Re_m \left(\sum_{k=0}^{m-1} \frac{(-1)^k}{k!} \log^k |z| \operatorname{Li}_{m-k}(z) - \frac{(-1)^m}{2m!} \log^m |z| \right),$$

where $|z| \leq 1$.

Now let us consider an "averaged" version of the function $D(z)$, which will be described below. Recall that for every elliptic curve $E/\mathbb{C}$, there exist $\tau \in \mathcal{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$ and isomorphisms

$$\begin{array}{ccccc} E(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{C}/\Lambda & \xrightarrow{\sim} & \mathbb{C}^\times/q^{\mathbb{Z}} \\ (\wp_\Lambda(u), \wp'_\Lambda(u)) & \longmapsto & u \pmod{\Lambda} & \longmapsto & e^{2\pi i u}, \end{array} \tag{5.1}$$

where $\wp_\Lambda$ denotes the Weierstrass $\wp$-function, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, and $q = e^{2\pi i \tau}$. Using the transformations above, Bloch [10] defined the *elliptic dilogarithm* $D^E : E(\mathbb{C}) \to \mathbb{R}$ by

$$D^E(P) := D^E(x) = \sum_{n=-\infty}^{\infty} D(q^n x),$$

where $q = e^{2\pi i \tau}$ and $x = e^{2\pi i u}$ is the image of P in $\mathbb{C}^\times/q^{\mathbb{Z}}$. (Note that the series above converges absolutely with exponential rapidity and is invariant under $x \mapsto qx$ [86, §2].) Recall from [80] that for $a, b \in \mathbb{N}$ the series

$$K_{a,b}(\tau; u) = \sum_{m,n \in \mathbb{Z}}{}' \frac{e^{2\pi i (n\xi - m\eta)}}{(m\tau + n)^a (m\bar{\tau} + n)^b},$$

where $u = \xi\tau + \eta$ and $\xi, \eta \in \mathbb{R}/\mathbb{Z}$, is called the *Eisenstein-Kronecker series.* Then Bloch introduced the regulator function

$$R^E(e^{2\pi i u}) = \frac{\mathrm{Im}(\tau)^2}{\pi} K_{2,1}(\tau; u).$$

It can be shown that $\mathrm{Re}(R^E) = D^E$, and that $-\mathrm{Im}(R^E)$ is the real-valued function given by

$$J^E(x) = \sum_{n=0}^{\infty} J(q^n x) - \sum_{n=1}^{\infty} J(q^n x^{-1}) + \frac{1}{3}\log^2|q| B_3\left(\frac{\log|x|}{\log|q|}\right),$$

where $J(x) = \log|x| \log|1-x|$, and $B_n(X)$ denotes the n^{th} Bernoulli polynomial. Similar to D^E, the function J^E is well-defined and invariant under $x \mapsto qx$. We can also extend R^E, D^E, and J^E by linearity to the group of divisors on $E(\mathbb{C})$.

It turns out that the function D^E is related to special L-values of elliptic curves by the following result of Beilinson, originally conjectured by Bloch:

Theorem 5.1. *If E is a modular elliptic curve over $\mathbb{Q}$, then there exists a divisor $\xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})_{tors}]$ such that $\pi D^E(\xi) \sim_{\mathbb{Q}^\times} L(E, 2)$, where $x \sim_{\mathbb{Q}^\times} y$ means $y = cx$ for some $c \in \mathbb{Q}^\times$.*

Furthermore, Bloch and Grayson [11] hypothesized from computational experiments that:

Conjecture 5.2. *Let E be an elliptic curve over $\mathbb{Q}$ with discriminant $\Delta < 0$, so that $E(\mathbb{Q})_{tors}$ is cyclic. Suppose $d = \#E(\mathbb{Q})_{tors} > 2$ and write Σ for the number of fibers of type I_v with $v \geq 3$ in the Néron model of E. If $l := \lfloor \frac{d-1}{2} \rfloor - \Sigma - 1 > 0$, then there are at least l exotic relations*

$$\sum_{r=1}^{\lfloor \frac{d-1}{2} \rfloor} a_r D^E(rP) = 0,$$

where $a_r \in \mathbb{Z}$ and P is a generator of $E(\mathbb{Q})_{tors}$.

It has been revealed that the two-variable Mahler measures $m_2(t)$ and $m_3(t)$ can be written in terms of the elliptic dilogarithm. For instance, Guillera and Rogers [32, Thm. 5] showed that if $E(k,l)$ is the elliptic curve given by the equation

$$y^2 = 4x^3 - 27(k^4 - 16k^2 + 16)l^2 x + 27(k^6 - 24k^4 + 120k^2 + 64)l^3,$$

and $t = 18, 25, 64$, and 256, then $m_2(t)$ can be expressed as

$$m_2(t) = \frac{8}{\pi} D^{E(\sqrt{t},l)}(P),$$

where $l \in \{1/2, 1, 2\}$ and P is a 4-torsion point on the corresponding elliptic curve. They also proved that if E is the elliptic curve $y^2 = 4x^3 - 432x + 1188$ and $P = (-6, 54)$, then the Mahler measure identity

$$16 m_3\left(\frac{(7+\sqrt{5})^3}{4}\right) - 8 m_3\left(\frac{(7-\sqrt{5})^3}{4}\right) = 19 m_3(32)$$

is equivalent to the exotic relation

$$16 D^E(P) - 11 D^E(2P) = 0,$$

verified by Bertin in [7]. On the other hand, Lalín and Rogers[39] verified that for every $t \in \mathbb{C}$, if E is the elliptic curve defined by $x + x^{-1} + y + y^{-1} + t^{1/2} = 0$ and $\tau = \omega_2/\omega_1$, where ω_1 and ω_2 are the real and complex periods of E, respectively, then

$$m_2\left(t^2\right) = -\frac{2}{\pi \operatorname{Im}\left(\frac{1}{4\tau}\right)} J^{\tilde{E}}(e^{-\pi i/2\tau}),$$

where $\tilde{E} \cong \mathbb{C}/(\mathbb{Z} + (-1/\tau)\mathbb{Z})$. This result then implies formula (3.3) immediately.

In the present section, we will investigate the elliptic version of

$$\mathcal{L}_3(z) = \mathrm{Re}\left(\mathrm{Li}_3(z) - \log|z|\,\mathrm{Li}_2(z) + \frac{1}{3}\log^2|z|\,\mathrm{Li}_1(z)\right),$$

namely *the elliptic trilogarithm*

$$\mathcal{L}^E_{3,1}(P) := \mathcal{L}^E_{3,1}(x) = \sum_{n=-\infty}^{\infty} \mathcal{L}_3(q^n x),$$

where E is again identified with $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, u is the image of P in $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, $q = e^{2\pi i\tau}$, and $x = e^{2\pi i u}$. This function was first defined by Zagier in [89, §10]. He also introduced a companion of $\mathcal{L}^E_{3,1}$, which is the following single-valued function:

$$\mathcal{L}^E_{3,2}(P) := \mathcal{L}^E_{3,2}(x) = \sum_{n=0}^{\infty} J_3(q^n x) + \sum_{n=1}^{\infty} J_3(q^n x^{-1}) + \frac{\log^2|x|\log^2|qx^{-1}|}{4\log|q|},$$

where $J_3(x) = \log^2|x|\log|1-x|$. Again, one can extend $\mathcal{L}^E_{3,j}$, $j = 1, 2$, to all divisors on $E(\mathbb{C})$ by linearity. Zagier claimed that $\mathcal{L}^E_{3,1}$ and $\mathcal{L}^E_{3,2}$ are linear combinations of the Eisenstein-Kronecker series $K_{1,3}$ and $K_{2,2}$, and we shall derive this result explicitly in Section 5.2. Furthermore, we will deduce some integer relations satisfied by $\mathcal{L}^E_{3,1}$ evaluated at the torsion points of order $2, 3$, and 4 on elliptic curves. These relations may be considered as a higher dimensional analogue of exotic relations of the elliptic trilogarithm. More importantly, we aim at establishing some connections between the Mahler measures

$$n_2(s) = 2m(A_s) = 2m((x+x^{-1})(y+y^{-1})(z+z^{-1}) + s^{1/2}),$$

$$n_3(s) = m(B_s) = m((x+x^{-1})^2(y+y^{-1})^2(1+z)^3 z^{-2} - s),$$

$$n_4(s) = 4m(C_s) = 4m(x^4+y^4+z^4+1+s^{1/4}xyz),$$

and the families E_s, F_s, and G_s of elliptic curves given in Section 4.3. To be precise, we will prove the result below in Section 5.3:

Theorem 5.3. *Let E_s, F_s, and G_s be the families of elliptic curves given by*

$$E_s : y^2 = (x-1)\left(x^2 - \frac{s}{s-64}\right),$$

$$F_s : x^3 + y^3 + 1 - rxy = 0, \qquad r = \sqrt[3]{\frac{s+\sqrt{s(s-108)}}{2}},$$

$$G_s : y^2 = (x-1)(x-r')(x+r'), \qquad r' = \sqrt{\frac{1+\sqrt{1-\frac{256}{s}}}{2}}.$$

(i) If $s \in \mathbb{R} \setminus [0,64]$, $r := \sqrt{\frac{s}{s-64}}$, $P := (-r,0)$, and $Q := (r,0)$, then

$$n_2(s) = \frac{8}{3\pi^2}\left(6\mathcal{L}_{3,1}^{E_s}((P)-(Q)) - \mathcal{L}_{3,2}^{E_s}((P)-(Q))\right). \tag{5.2}$$

(ii) Let O, P and Q be the points on F_s corresponding to $1, 1/3$ and $\tau/3$, respectively, via the isomorphism $F_s \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau), \tau \in \mathcal{H}$. Then, for every $s \in [108,\infty)$,

$$\begin{aligned} n_3(s) = \frac{3}{4\pi^2}\big(&15\mathcal{L}_{3,1}^{F_s}((Q)-3(P)-6(P+Q)) \\ &+ \mathcal{L}_{3,2}^{F_s}(3(P)+6(P+Q)-7(Q)-2(O))\big). \end{aligned} \tag{5.3}$$

(iii) Let P and Q be the points on G_s corresponding to $\tau/2$ and $3/4$, respectively, via the isomorphism $G_s \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. Then, for every $s \geq 256$,

$$\begin{aligned} n_4(s) = \frac{16}{9\pi^2}\big(&15\mathcal{L}_{3,1}^{G_s}(2(P)-(2Q)+2(P+2Q)) \\ &+ \mathcal{L}_{3,2}^{G_s}(4(Q)-5(P)+2(2Q)+4(P+Q)-5(P+2Q))\big). \end{aligned} \tag{5.4}$$

On the other hand, if $s < 0$, then

$$\begin{aligned} n_4(s) = \frac{8}{9\pi^2}\big(&30\mathcal{L}_{3,1}^{G_s}(2(2Q) - (P+2Q) + 2(P)) \\ &+ \mathcal{L}_{3,2}^{G_s}(5(P+2Q) + 8(Q) + 8(P+Q) - 11(2Q) - 10(P))\big). \end{aligned} \tag{5.5}$$

We have shown that the families of elliptic curves E_s and G_s are indeed related to the $K3$ surfaces defined by the zero loci of A_s and C_s via Shioda-Inose structures. Hence Theorem 5.3 might give us some interesting arithmetic interpretation of Mahler measures of polynomials defining $K3$ surfaces, especially those with large Picard numbers. We have also verified in Section 4.2 that for some values of s, $n_j(s), j = 2, 3, 4$, are rational linear combinations of $L'(g, 0)$ and $L'(\chi, -1)$, where g is a CM newform of weight 3 and χ is a Dirichlet character. Therefore, we obtain immediately explicit relations between the elliptic trilogarithm and those L-values. We shall also list some conjectural formulas of evaluations of $\mathcal{L}_{3,j}^{E_s}, j = 1, 2$, in terms of special values of L-functions discovered via our numerical computations in Section 5.4.

5.2 The functions $\mathcal{L}_{3,1}^{E}$ and $\mathcal{L}_{3,2}^{E}$

Most components of the results in this section are deduced from Zagier's results in [86]. Thus let us first recall some notations and facts obtained from that paper. For any $a, b, l, m \in \mathbb{N}$ with $1 \leq a, m \leq l$ and $x, q \in \mathbb{C}$ with $|q| < 1$, Zagier defined

$$c_{a,m}^{(l)} = \sum_{h=1}^{a} (-1)^{h-1} \binom{m-1}{h-1} \binom{l-m}{a-h},$$

$$D_m^*(x) = \begin{cases} D_m(x) & \text{if } m \text{ is odd,} \\ iD_m(x) & \text{if } m \text{ is even,} \end{cases}$$

$$D_{a,b}(x) = 2\sum_{m=1}^{r} c_{a,m}^{(r)} D_m^*(x)\frac{(-\log|x|)^{r-m}}{(r-m)!} + \frac{(-2\log|x|)^r}{2\cdot r!}, \qquad r = a+b-1,$$

$$D_{a,b}(q;x) = \sum_{n=0}^{\infty} D_{a,b}(q^n x) + (-1)^{r-1}\sum_{n=1}^{\infty} D_{a,b}(q^n x^{-1}) + \frac{(-2\log|q|)^r}{(r+1)!}B_{r+1}\left(\frac{\log|x|}{\log|q|}\right).$$

Proposition 5.4 (Zagier [86, §2])**.** *Unless otherwise stated, let $a, b \in \mathbb{N}$ and $r = a+b-1$.*

(i) $D_{a,b}$ is a single-valued real-analytic function on $\mathbb{C}\setminus[1,\infty)$ and satisfies the functional equation

$$D_{a,b}(x^{-1}) = (-1)^{r-1}D_{a,b}(x) + \frac{(2\log|x|)^r}{r!}.$$

(ii) For any $x \in \mathbb{C}, m \geq 1$, and $n \geq 0$, we have the inversion formula

$$D_m^*(x)\frac{(-\log|x|)^n}{n!} = \sum_{\substack{a,b\geq 1\\ a+b=r+1}} c_{m,a}^{(r)}\left(\frac{D_{a,b}(x)}{2^r} - \frac{(-\log|x|)^r}{2\cdot r!}\right), \quad \text{where } r = m+n.$$

(iii) Let $q = e^{2\pi i\tau}$ and $x = e^{2\pi i u}$, where $\tau \in \mathcal{H}$ and $u = \xi\tau + \eta, \xi, \eta \in \mathbb{R}$. Then

$$D_{a,b}(q;x) = \frac{(\tau-\bar{\tau})^r}{2\pi i}K_{a,b}(\tau;u).$$

It was pointed out in [89, §10] that $\mathcal{L}_{3,1}^E$ and $\mathcal{L}_{3,2}^E$ are linear combinations of $K_{1,3}$ and $K_{2,2}$, but this fact does not seem to be shown in the literature. Therefore, we will give an account of it here before applying it to prove other results.

Proposition 5.5. *Suppose that $E \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with $\tau \in \mathcal{H}$. Let $q = e^{2\pi i\tau}$ and $x \in \mathbb{C}$.*

Then the following identities hold:

$$6\mathcal{L}^E_{3,1}(x) - \mathcal{L}^E_{3,2}(x) = \frac{3}{4}\left(2\,\mathrm{Re}\left(D_{1,3}(q;x)\right) - D_{2,2}(q;x)\right) - \frac{\log^3|q|}{120}, \tag{5.6}$$

$$\mathcal{L}^E_{3,1}(x) = \frac{1}{6}\left(\mathrm{Re}\left(D_{1,3}(q;x)\right) - D_{2,2}(q;x)\right), \tag{5.7}$$

$$\mathcal{L}^E_{3,2}(x) = -\frac{1}{4}\left(2\,\mathrm{Re}\left(D_{1,3}(q;x)\right) + D_{2,2}(q;x)\right) + \frac{\log^3|q|}{120}. \tag{5.8}$$

Proof. It suffices to prove any two equalities of the above, so we will show (5.6) and (5.7) only. First, using the identity (34) in [85], one has

$$\mathcal{L}_3(x) = D_3(x) - \frac{\log^2|x| D_1(x)}{6}. \tag{5.9}$$

It then follows from Proposition 5.4(ii) that

$$D_3(x) = \frac{1}{8}\left(D_{1,3}(x) + D_{3,1}(x) - D_{2,2}(x)\right) + \frac{\log^3|x|}{12}. \tag{5.10}$$

Since $D_1(x) = -\log\left|x^{1/2} - x^{-1/2}\right|$, we can deduce

$$\mathcal{L}_3(x) = \frac{1}{8}\left(D_{1,3}(x) + D_{3,1}(x) - D_{2,2}(x)\right) + \frac{J_3(x)}{6}.$$

Next, by simple manipulations and Proposition 5.4(i), we have that

$$\begin{aligned}
\mathcal{L}^E_{3,1}(x) &= \sum_{n\in\mathbb{Z}} \mathcal{L}_3(q^n x) \\
&= \frac{1}{8}\sum_{n\in\mathbb{Z}}\left(D_{1,3}(q^n x) + D_{3,1}(q^n x) - D_{2,2}(q^n x)\right) + \frac{1}{6}\sum_{n\in\mathbb{Z}} J_3(q^n x) \\
&= \frac{1}{8}\sum_{n\geq 0}\left(D_{1,3}(q^n x) + D_{3,1}(q^n x) - D_{2,2}(q^n x)\right) + \frac{1}{6}\sum_{n\geq 0} J_3(q^n x)
\end{aligned}$$

$$+\frac{1}{8}\sum_{n\geq 1}\big(D_{1,3}(q^nx^{-1})+D_{3,1}(q^nx^{-1})-D_{2,2}(q^nx^{-1})\big)+\frac{1}{6}\sum_{n\geq 1}J_3(q^nx^{-1})$$

$$=\frac{1}{8}\left(D_{1,3}(q;x)+D_{3,1}(q;x)-D_{2,2}(q;x)\right)+\frac{\mathcal{L}^E_{3,2}(x)}{6}-\frac{\log^3|q|}{720}.$$

Now we obtain (5.6) by using Proposition 5.4(iii) and the fact that $K_{3,1}(\tau;u)=\overline{K_{1,3}(\tau;u)}$. To prove (5.7), we again start with the equation (5.9). Applying Proposition 5.4(ii) with $m=1$ and $n=2$, we get

$$D_1(x)\frac{\log^2|x|}{2}=\frac{1}{8}\left(D_{1,3}(x)+D_{3,1}(x)+D_{2,2}(x)\right)+\frac{\log^3|x|}{4}.$$

Therefore, by (5.10),

$$\mathcal{L}_3(x)=\frac{1}{12}\left(D_{1,3}(x)+D_{3,1}(x)-2D_{2,2}(x)\right).$$

Then one can prove (5.7) easily using similar arguments above. □

Corollary 5.6. *With the same settings in Proposition 5.5, if $x=e^{2\pi i(\xi\tau+\eta)}$, where $\xi,\eta\in\mathbb{R}/\mathbb{Z}$, then the following identities hold:*

$$\mathcal{L}^E_{3,1}(x)=\frac{2\,\mathrm{Im}(\tau)^3}{3\pi}\left[{\sum_{m,n\in\mathbb{Z}}}'\frac{e^{2\pi i(n\xi-m\eta)}}{|m\tau+n|^4}-\mathrm{Re}\left({\sum_{m,n\in\mathbb{Z}}}'e^{2\pi i(n\xi-m\eta)}\frac{(m\tau+n)^2}{|m\tau+n|^6}\right)\right],\quad (5.11)$$

$$\mathcal{L}^E_{3,2}(x)=\frac{\mathrm{Im}(\tau)^3}{\pi}\left[{\sum_{m,n\in\mathbb{Z}}}'\frac{e^{2\pi i(n\xi-m\eta)}}{|m\tau+n|^4}+2\,\mathrm{Re}\left({\sum_{m,n\in\mathbb{Z}}}'e^{2\pi i(n\xi-m\eta)}\frac{(m\tau+n)^2}{|m\tau+n|^6}\right)\right]+\frac{\log^3|q|}{120}.\quad (5.12)$$

Proof. Use (5.7), (5.8), and Proposition 5.4(iii). □

As an easy consequence of (5.11), we have the following result:

Theorem 5.7. *(i) Let E be an elliptic curve given by the equation*

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3),$$

where $e_j \in \mathbb{C}$ are pairwise distinct, and denote by P_j and O the point $(e_j, 0)$ and the point at infinity, respectively. Then

$$\mathcal{L}_{3,1}^{E}(4(P_1) + 4(P_2) + 4(P_3) + 3(O)) = 0. \tag{5.13}$$

(ii) Let E be an elliptic curve isomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. If $P, Q, R, S,$ and O are the points on E corresponding to $\tau/3, 1/3, \tau/2, 3/4$ and 1, respectively, via the isomorphism above, then

$$\mathcal{L}_{3,1}^{E}(8(S) + 8(R + S) - (2S)) = 0. \tag{5.14}$$

Moreover, if τ is purely imaginary, then

$$\mathcal{L}_{3,1}^{E}(9(P) + 9(Q) + 18(P + Q) + 4(O)) = 0, \tag{5.15}$$

Proof. We shall prove (i) first. Denote by ω_1 and ω_2 the real and complex periods of E and let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Then it follows from a well-known fact about evaluations of $\wp_\Lambda$ and $\wp'_\Lambda$ at the half-periods of Λ that

$$\left\{(\wp_\Lambda(u), \wp'_\Lambda(u)) \mid u = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}\right\} = \{P_1, P_2, P_3\}.$$

Let $\tau = \omega_2/\omega_1$. Then, by using (5.11), we find that

$$\begin{aligned}
\mathcal{L}^E_{3,1}((P_1)+(P_2)+(P_3)) &= \frac{2\operatorname{Im}(\tau)^3}{3\pi}\Bigg[\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{(-1)^n+(-1)^m+(-1)^{m+n}}{|m\tau+n|^4} \\
&\qquad - \operatorname{Re}\left(\sideset{}{'}\sum_{m,n\in\mathbb{Z}} \left((-1)^n+(-1)^m+(-1)^{m+n}\right)\frac{(m\tau+n)^2}{|m\tau+n|^6}\right)\Bigg] \\
&= \frac{2\operatorname{Im}(\tau)^3}{3\pi}\Bigg[\operatorname{Re}\left(\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{(m\tau+n)^2}{|m\tau+n|^6} - 4\sideset{}{'}\sum_{\substack{m \text{ even}\\ n \text{ even}}}\frac{(m\tau+n)^2}{|m\tau+n|^6}\right) \\
&\qquad - \left(\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{1}{|m\tau+n|^4} - 4\sideset{}{'}\sum_{\substack{m \text{ even}\\ n \text{ even}}}\frac{1}{|m\tau+n|^4}\right)\Bigg] \\
&= \frac{\operatorname{Im}(\tau)^3}{2\pi}\left[\operatorname{Re}\left(\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{(m\tau+n)^2}{|m\tau+n|^6}\right) - \sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{1}{|m\tau+n|^4}\right] \\
&= -\frac{3}{4}\mathcal{L}^E_{3,1}(O),
\end{aligned}$$

so (5.13) follows. Next, suppose that $E \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. Using (5.11) and the fact that $\mathcal{L}^E_{3,1}$ is a real-valued function, one has that

$$\begin{aligned}
\mathcal{L}^E_{3,1}(8(S)+8(R+S)) &= \frac{16y^3}{3\pi}\Bigg[\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{i^m\left(1+(-1)^n\right)}{|m\tau+n|^4} \\
&\qquad - \operatorname{Re}\left(\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\left(i^m\left(1+(-1)^n\right)\right)\frac{(m\tau+n)^2}{|m\tau+n|^6}\right)\Bigg] \\
&= \frac{16y^3}{3\pi}\Bigg[\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{(-1)^m\left(1+(-1)^n\right)}{|2m\tau+n|^4} \\
&\qquad - \sideset{}{'}\sum_{m,n\in\mathbb{Z}}\left((-1)^m\left(1+(-1)^n\right)\right)\frac{n^2-4y^2m^2}{|2m\tau+n|^6}\Bigg]
\end{aligned}$$

$$= \frac{2y^3}{3\pi} \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{(-1)^m}{|m\tau+n|^4} - \frac{(-1)^m(n^2-y^2m^2)}{|m\tau+n|^6} \right)$$

$$= \mathcal{L}^E_{3,1}(2S),$$

which yields (5.14). On the other hand, if $\omega = e^{2\pi i/3}$ and $\tau = yi$, where $y \in \mathbb{R}$, then

$$\begin{aligned}
&\mathcal{L}^E_{3,1}(9(P)+9(Q)+18(P+Q)+4(O)) \\
&\quad = \frac{2y^3}{3\pi}\Bigg[\sum_{m,n\in\mathbb{Z}}{}' \frac{9\omega^n+9\omega^{-m}+18\omega^{n-m}+4}{|m\tau+n|^4} \\
&\qquad - \operatorname{Re}\left(\sum_{m,n\in\mathbb{Z}}{}' \left(9\omega^n+9\omega^{-m}+18\omega^{n-m}+4\right) \frac{(m\tau+n)^2}{|m\tau+n|^6} \right) \Bigg] \\
&\quad = \frac{2y^5}{3\pi} \sum_{m,n\in\mathbb{Z}}{}' \frac{(80-27|\chi(n)|-27|\chi(m)|-54|\chi(n-m)|)\, m^2}{(n^2+y^2m^2)^3},
\end{aligned}$$

where χ is the quadratic character of conductor 3. To obtain the last equality, we use the identity

$$\omega^n = -\frac{3}{2}|\chi(n)| + 1 + \chi(n)\frac{\sqrt{3}}{2}i, \qquad n \in \mathbb{Z}. \tag{5.16}$$

Now it can be shown that the last series above vanishes by considering $80 - 27|\chi(n)| - 27|\chi(m)| - 54|\chi(n-m)|$ for each m and n modulo 3. □

5.3 Connection with Mahler measures

The main goal of this section is to give a proof of Theorem 5.3. The key idea of the proof is to use Proposition 4.2; i.e., when s is properly parametrized, the Mahler measures $n_j(s), j = 2, 3, 4$ can be expressed as Eisenstein-Kronecker series, which turns out to equal the series obtained from the right-hand sides of (5.2), (5.3), (5.4), and (5.5). The results below are also essentially required in the proof of our main theorem.

Lemma 5.8. *For $t = 2, 3,$ and 4, we denote*

$$\mathrm{F}_t(z) := {}_2F_1\left(\begin{matrix}\frac{1}{t}, & \frac{t-1}{t} \\ & 1\end{matrix}; z\right).$$

(i) Let $a, b, c \in \mathbb{R}$ be such that $a > b > c$ and let E be the elliptic curve $y^2 = (x - a)(x - b)(x - c)$. Then E is isomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where

$$\tau = \frac{\mathrm{F}_2\left(\frac{a-b}{a-c}\right)}{\mathrm{F}_2\left(\frac{b-c}{a-c}\right)} i.$$

(ii) Let $k \in (3, \infty)$ and let E be the elliptic curve $x^3 + y^3 + 1 - kxy = 0$. Then E is isomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where

$$\tau = \sqrt{3}\frac{\mathrm{F}_3\left(\frac{27}{k^3}\right)}{\mathrm{F}_3\left(1 - \frac{27}{k^3}\right)} i.$$

Proof. First, consider the elliptic curve $E : y^2 = (x - a)(x - b)(x - c)$, where $a > b > c$. Let ω_1 and ω_2 be the real and complex periods of E, respectively, and let $\tau_1 = \omega_2/\omega_1$. Then we can find ω_1 and ω_2 using the following formulas (see, for example, [22, Algorithm 7.4.7]):

$$\omega_1 = \frac{\pi}{\mathrm{AGM}(\sqrt{a-c}, \sqrt{a-b})}, \qquad \omega_2 = \frac{i\pi}{\mathrm{AGM}(\sqrt{a-c}, \sqrt{b-c})},$$

where $\mathrm{AGM}(\alpha, \beta)$ denotes the arithmetic-geometric mean of α and β, defined as follows: Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences given by $a_1 = \alpha, b_1 = \beta$, and $a_{n+1} = (a_n + b_n)/2, b_{n+1} = \sqrt{a_n b_n}$ for $n \geq 1$. Then these two sequences converge to the same number, and we call this number $\mathrm{AGM}(\alpha, \beta)$.

By [47, (3.5)], one has that the AGM can be represented by a ${}_2F_1$-hypergeometric series, namely,

$$\mathrm{AGM}(\alpha,\beta)=\frac{\alpha}{\mathrm{F}_2\left(1-\left(\frac{\beta}{\alpha}\right)^2\right)}. \tag{5.17}$$

Therefore, we have immediately that

$$\tau_1=\frac{\mathrm{AGM}(\sqrt{a-c},\sqrt{a-b})}{\mathrm{AGM}(\sqrt{a-c},\sqrt{b-c})}i=\frac{\mathrm{F}_2\left(\frac{a-b}{a-c}\right)}{\mathrm{F}_2\left(\frac{b-c}{a-c}\right)}i,$$

and (i) is proved.

Now for a given $k\in(3,\infty)$, let $\tau_2=\left(\sqrt{3}\,\mathrm{F}_3\left(\frac{27}{k^3}\right)/\,\mathrm{F}_3\left(1-\frac{27}{k^3}\right)\right)i$, and let E be the elliptic curve defined by $x^3+y^3+1-kxy=0$. To establish (ii), we will show that $j(E)=j(\tau_2)$, where the latter j is the usual j-invariant. Let us first introduce a generalized Weber function

$$\mathfrak{g}_3(\tau)=\sqrt{3}\frac{\eta(3\tau)}{\eta(\tau)}.$$

It is a classical result due to Weber that for any $\tau\in\mathcal{H}$, $\mathfrak{g}_3^{12}(\tau)$ is a zero of the polynomial $x^4+36x^3+270x^2+(756-j(\tau))x+3^6$ (see, for example, [77, Thm. 5]). Consequently, we can write $j(\tau)$ as a rational function of $\mathfrak{g}_3(\tau)$, namely,

$$j(\tau)=\frac{\left(\mathfrak{g}_3^{12}(\tau)+3\right)^3\left(\mathfrak{g}_3^{12}(\tau)+27\right)}{\mathfrak{g}_3^{12}(\tau)}. \tag{5.18}$$

Observe that we can also rewrite the function $s_3(q)$, defined in Proposition 4.2, as

$$s_3(q)=\left(\mathfrak{g}_3^{6}(\tau)+27\mathfrak{g}_3^{-6}(\tau)\right)^2.$$

Recall from Ramanujan's theory of elliptic functions of signature 3 that if $q_t(\alpha)$ is the *elliptic nome*

$$q_t(\alpha)=\exp\left(-\frac{\pi}{\sin\left(\frac{\pi}{t}\right)}\frac{\mathrm{F}_t(1-\alpha)}{\mathrm{F}_t(\alpha)}\right), \tag{5.19}$$

then $s_3(q_3(\alpha)) = \frac{27}{\alpha(1-\alpha)}$ for any α which makes both $\mathrm{F}_3(1-\alpha)$ and $\mathrm{F}_3(\alpha)$ convergent. Hence it follows that

$$\left(\mathfrak{g}_3^6\left(-\frac{1}{\tau_2}\right) + 27\mathfrak{g}_3^{-6}\left(-\frac{1}{\tau_2}\right)\right)^2 = s_3\left(q\left(-\frac{1}{\tau_2}\right)\right)$$

$$= s_3\left(q_3\left(\frac{27}{k^3}\right)\right)$$

$$= \frac{k^6}{k^3-27},$$

which implies that $\mathfrak{g}_3^{12}\left(-\frac{1}{\tau_2}\right)$ can possibly be either k^3-27 or $729/(k^3-27)$. However, as a function of k, $\mathfrak{g}_3^{12}\left(-\frac{1}{\tau_2}\right)$ is decreasing on $(3,\infty)$, so it must equal $729/(k^3-27)$ on this interval. Therefore, we have by (5.18) that

$$j(\tau_2) = j\left(-\frac{1}{\tau_2}\right) = \left(\frac{k(k^3+216)}{k^3-27}\right)^3.$$

On the other hand, it can be found using standard computer algebra systems such as `Maple` that if $k^3-27 \neq 0$, then $j(E)$ coincides with $j(\tau_2)$ obtained above. □

Proof of Theorem 5.3. (i) Let $s \in \mathbb{R} \setminus [0,64]$, $r = \sqrt{\frac{s}{s-64}}$, $P = (-r, 0)$, and $Q = (r, 0)$. Then the equation representing E_s can be rewritten as

$$E_s : y^2 = (x-1)(x-r)(x+r). \tag{5.20}$$

We will divide the proof of into two cases, whose arguments are somewhat parallel.

Case 1. $s > 64$

In this case, we have $r > 1$. Then using Lemma 5.8(i) it follows that $E_s \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, where

$$\tau = \frac{\mathrm{F}_2\left(\frac{r-1}{2r}\right)}{\mathrm{F}_2\left(\frac{r+1}{2r}\right)} i.$$

By a result from Ramanujan's theory of elliptic functions of signature 2, we have that $s_2(q_2(\alpha)) = \frac{16}{\alpha(1-\alpha)}$, where $q_2(\alpha)$ is as defined in (5.19). As a consequence, we can easily deduce that

$$s_2\left(q\left(-\frac{1}{2\tau}\right)\right) = s_2\left(\exp\left(-\pi\frac{\mathrm{F}_2\left(\frac{r+1}{2r}\right)}{\mathrm{F}_2\left(\frac{r-1}{2r}\right)}\right)\right) = s_2\left(q_2\left(\frac{r-1}{2r}\right)\right) = \frac{64r^2}{r^2-1} = s.$$

Since τ is purely imaginary and $0 < \mathrm{Im}(\tau) < 1$, it follows that $-\frac{1}{2\tau}$ is also purely imaginary, and

$$\left|-\frac{1}{2\tau}\right| = \mathrm{Im}\left(-\frac{1}{2\tau}\right) = \frac{1}{2\,\mathrm{Im}(\tau)} > \frac{1}{2}.$$

Let $y = \mathrm{Im}(\tau)$. Applying Proposition 4.2(i), with m and n switched, one sees that

$$\begin{aligned}
n_2\left(s_2\left(q\left(-\frac{1}{2\tau}\right)\right)\right) &= \frac{1}{y\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \Bigg(16\left(\frac{4m^2}{((4n^2/y^2)+m^2)^3} - \frac{1}{((4n^2/y^2)+m^2)^2}\right) \\
&\qquad - \left(\frac{4m^2}{((n^2/4y^2)+m^2)^3} - \frac{1}{((n^2/4y^2)+m^2)^2}\right)\Bigg) \\
&= \frac{16y^3}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}' \Bigg(\left(\frac{4y^2m^2}{(4n^2+y^2m^2)^3} - \frac{1}{(4n^2+y^2m^2)^2}\right) \\
&\qquad - \left(\frac{16y^2m^2}{(n^2+4y^2m^2)^3} - \frac{1}{(n^2+4y^2m^2)^2}\right)\Bigg) \\
&= \frac{16y^3}{\pi^3}\Bigg[\sum_{\substack{m\in\mathbb{Z}\\ n \text{ even}}}{}' \left(\frac{4y^2m^2}{(n^2+y^2m^2)^3} - \frac{1}{(n^2+y^2m^2)^2}\right) \\
&\qquad - \sum_{\substack{m \text{ even}\\ n\in\mathbb{Z}}}{}' \left(\frac{4y^2m^2}{(n^2+y^2m^2)^3} - \frac{1}{(n^2+y^2m^2)^2}\right)\Bigg]
\end{aligned}$$

$$= \frac{16y^3}{\pi^3} \left[\sum_{\substack{m \text{ even} \\ n \text{ odd}}} \frac{n^2 - 3y^2m^2}{(n^2 + y^2m^2)^3} - \sum_{\substack{m \text{ odd} \\ n \text{ even}}} \frac{n^2 - 3y^2m^2}{(n^2 + y^2m^2)^3} \right].$$

On the other hand, recall that if $\tilde{E}$ is the elliptic curve

$$Y^2 = 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3),$$

where $e_1, e_2, e_3 \in \mathbb{R}$ with $e_3 < e_2 < e_1$, and $\tilde{\omega_1}$ and $\tilde{\omega_2}$ are the real and complex periods of $\tilde{E}$, then

$$(\wp_\Lambda(\tilde{\omega_1}/2), \wp'_\Lambda(\tilde{\omega_1}/2)) = (e_1, 0),$$

$$(\wp_\Lambda((\tilde{\omega_1} + \tilde{\omega_2})/2), \wp'_\Lambda((\tilde{\omega_1} + \tilde{\omega_2})/2)) = (e_2, 0),$$

$$(\wp_\Lambda(\tilde{\omega_2}/2), \wp'_\Lambda(\tilde{\omega_2}/2)) = (e_3, 0),$$

where $\Lambda = \mathbb{Z}\tilde{\omega_1} + \mathbb{Z}\tilde{\omega_2}$. It can be checked in a straightforward manner that the birational map

$$(x, y) \mapsto (36x - 12, 432y)$$

gives an isomorphism between $E_s(\mathbb{R})$ and

$$\tilde{E}_s(\mathbb{R}) : y^2 = 4x^3 - (5184r^2 + 1728)x - (13824 - 124416r^2)$$

$$= 4(x - 24)(x - (36r - 12))(x - (-36r - 12)).$$

Since $-r < 1 < r$, one finds immediately that the isomorphism $E_s \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ sends P to $\tau/2$ and Q to $1/2$. Let $\xi = (P) - (Q)$ and $q = e^{2\pi i\tau}$. Then it follows from Corollary 5.6

that

$$\begin{aligned}
&\frac{8}{3\pi^2}\left(6\mathcal{L}^{E_s}_{3,1}-\mathcal{L}^{E_s}_{3,2}\right)(\xi) \\
&\quad=\frac{8\operatorname{Im}(\tau)^3}{\pi^3}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\Bigg(\frac{((-1)^m-(-1)^n)\left(2(m\operatorname{Re}(\tau)+n)^2-2m^2\operatorname{Im}(\tau)^2\right)}{|m\tau+n|^6} \\
&\qquad\qquad-\frac{((-1)^m-(-1)^n)}{|m\tau+n|^4}\Bigg) \\
&\quad=\frac{16y^3}{\pi^3}\left[\sum_{\substack{m \text{ even} \\ n \text{ odd}}}\left(\frac{2(n^2-y^2m^2)}{(n^2+y^2m^2)^3}-\frac{1}{(n^2+y^2m^2)^2}\right)\right. \\
&\qquad\qquad\left.-\sum_{\substack{m \text{ odd} \\ n \text{ even}}}\left(\frac{2(n^2-y^2m^2)}{(n^2+y^2m^2)^3}-\frac{1}{(n^2+y^2m^2)^2}\right)\right] \\
&\quad=\frac{16y^3}{\pi^3}\left[\sum_{\substack{m \text{ even} \\ n \text{ odd}}}\frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3}-\sum_{\substack{m \text{ odd} \\ n \text{ even}}}\frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3}\right],
\end{aligned}$$

where we use the fact that $\operatorname{Re}\left((m\tau+n)^2\right)=(m\operatorname{Re}(\tau)+n)^2-m^2\operatorname{Im}(\tau)^2$ in the second equality. Therefore, the first case of the theorem is proved.

Case 2. $s<0$

In this case, we have $0<r<1$. Then by Lemma 5.8(i) one finds that $E_s\cong\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ where

$$\tau=\frac{\mathrm{F}_2\left(\frac{1-r}{1+r}\right)}{\mathrm{F}_2\left(\frac{2r}{1+r}\right)}i.$$

Thus τ is again purely imaginary and $\operatorname{Im}(\tau)\in(0,\infty)$. Let $\alpha=\frac{1+r}{2r}$. Then $\alpha>1$, and

$$\tau=\frac{\mathrm{F}_2\left(\frac{\alpha-1}{\alpha}\right)}{\mathrm{F}_2\left(\frac{1}{\alpha}\right)}i. \tag{5.21}$$

Now we apply the hypergeometric transformations [2, Thm. 2.2.5] and [82] to deduce the following identities:

$$\mathrm{F}_2\left(\frac{\alpha-1}{\alpha}\right) = \alpha^{\frac{1}{2}}\,\mathrm{F}_2(1-\alpha)$$
$$\mathrm{F}_2\left(\frac{1}{\alpha}\right) = \alpha^{\frac{1}{2}}\left(\mathrm{F}_2(\alpha)+\mathrm{F}_2(1-\alpha)i\right).$$

Plugging the expressions above into (5.21), one has immediately that

$$\frac{\tau-1}{2\tau} = \frac{\mathrm{F}_2(\alpha)}{2\,\mathrm{F}_2(1-\alpha)}i.$$

By the same argument in Case 1, we then obtain

$$s_2\left(q\left(\frac{\tau-1}{2\tau}\right)\right) = s_2\left(\exp\left(-\pi\frac{\mathrm{F}_2(\alpha)}{\mathrm{F}_2(1-\alpha)}\right)\right) = \frac{16}{\alpha(1-\alpha)} = \frac{64r^2}{r^2-1} = s.$$

Again, we let $y = \mathrm{Im}(\tau)$. It is easily seen that

$$\frac{\tau-1}{2\tau} = \frac{1}{2} + \frac{1}{2y}i.$$

Hence it follows by Proposition 4.2(i) that

$$\begin{aligned} n_2(s) &= n_2\left(s_2\left(q\left(\frac{\tau-1}{2\tau}\right)\right)\right) \\ &= \frac{16y^3}{\pi^3}\sum_{m,n\in\mathbb{Z}}{}^{\prime}\left(\left(\frac{4y^2(2n+m)^2}{\left(y^2(2n+m)^2+(2n)^2\right)^3} - \frac{1}{\left(y^2(2n+m)^2+(2n)^2\right)^2}\right)\right. \\ &\qquad\left. - \left(\frac{4y^2(n+2m)^2}{\left(y^2(n+2m)^2+n^2\right)^3} - \frac{1}{\left(y^2(n+2m)^2+n^2\right)^2}\right)\right) \end{aligned}$$

$$= \frac{16y^3}{\pi^3}\left[\sum_{\substack{m \text{ odd} \\ n \text{ even}}}\left(\frac{4y^2(m+n)^2}{(n^2+y^2(m+n)^2)^3} - \frac{1}{(n^2+y^2(m+n)^2)^2}\right)\right.$$

$$\left. - \sum_{\substack{m \text{ even} \\ n \text{ odd}}}\left(\frac{4y^2(m+n)^2}{(n^2+y^2(m+n)^2)^3} - \frac{1}{(n^2+y^2(m+n)^2)^2}\right)\right]$$

$$= \frac{16y^3}{\pi^3}\left[\sum_{\substack{m \text{ odd} \\ n \text{ even}}}\left(\frac{4y^2m^2}{(n^2+y^2m^2)^3} - \frac{1}{(n^2+y^2m^2)^2}\right)\right.$$

$$\left. - \sum_{\substack{m \text{ odd} \\ n \text{ odd}}}\left(\frac{4y^2m^2}{(n^2+y^2m^2)^3} - \frac{1}{(n^2+y^2m^2)^2}\right)\right]$$

$$= \frac{16y^3}{\pi^3}\left[\sum_{\substack{m \text{ odd} \\ n \text{ odd}}} \frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3} - \sum_{\substack{m \text{ odd} \\ n \text{ even}}} \frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3}\right].$$

To evaluate $\mathcal{L}_{3,1}^{E_s}$ and $\mathcal{L}_{3,2}^{E_s}$ at $\xi := (P)-(Q)$, we first use the fact that $-r < r < 1$ and the argument in Case 1 to find that P and Q are mapped to $\tau/2$ and $(1+\tau)/2$, respectively, via the isomorphism $E_s \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. This therefore yields

$$\frac{8}{3\pi^2}\left(6\mathcal{L}_{3,1}^{E_s} - \mathcal{L}_{3,2}^{E_s}\right)(\xi)$$

$$= \frac{8\,\mathrm{Im}(\tau)^3}{\pi^3}{\sum_{m,n\in\mathbb{Z}}}'\left(\frac{\left((-1)^{n-m}-(-1)^n\right)\left(2(m\,\mathrm{Re}(\tau)+n)^2 - 2m^2\,\mathrm{Im}(\tau)^2\right)}{|m\tau+n|^6}\right.$$

$$\left. - \frac{\left((-1)^{n-m}-(-1)^n\right)}{|m\tau+n|^4}\right)$$

$$= \frac{8y^3}{\pi^3}{\sum_{m,n\in\mathbb{Z}}}'(-1)^n\left((-1)^m-1\right)\left(\frac{(2n^2-2y^2m^2)}{|m\tau+n|^6} - \frac{1}{|m\tau+n|^4}\right)$$

$$= \frac{16y^3}{\pi^3}\left[\sum_{\substack{m \text{ odd} \\ n \text{ odd}}}\left(\frac{2(n^2-y^2m^2)}{(n^2+y^2m^2)^3} - \frac{1}{(n^2+y^2m^2)^2}\right)\right.$$

$$- \sum_{\substack{m \text{ odd} \\ n \text{ even}}} \left(\frac{2(n^2 - y^2m^2)}{(n^2 + y^2m^2)^3} - \frac{1}{(n^2 + y^2m^2)^2} \right) \Bigg]$$

$$= \frac{16y^3}{\pi^3} \left[\sum_{\substack{m \text{ odd} \\ n \text{ odd}}} \frac{n^2 - 3y^2m^2}{(n^2 + y^2m^2)^3} - \sum_{\substack{m \text{ odd} \\ n \text{ even}}} \frac{n^2 - 3y^2m^2}{(n^2 + y^2m^2)^3} \right] = n_2(s),$$

as desired.

(ii) Let $s \in [108, \infty)$. By Lemma 5.8(ii), one has $F_s \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where

$$\tau = \sqrt{3} \frac{\mathrm{F}_3\left(\frac{27}{r^3}\right)}{\mathrm{F}_3\left(1 - \frac{27}{r^3}\right)} i.$$

Since $s = r^6/(r^3 - 27)$, we see immediately from the proof of Lemma 5.8(ii) that $s_3(q(-1/\tau)) = s$. Also, letting $y = \mathrm{Im}(\tau)$, we have $\mathrm{Im}(-1/\tau) = 1/y \geq 1/\sqrt{3}$. Hence we can apply Proposition 4.2(ii) to show that

$$n_3(s) = n_3\left(s_3\left(q\left(-\frac{1}{\tau}\right)\right)\right) = \frac{15y^3}{4\pi^3} \left(\sum_{\substack{m,n \in \mathbb{Z} \\ 3 \nmid n}}{}' \frac{n^2 - 3y^2m^2}{(n^2 + y^2m^2)^3} - 8 \sum_{\substack{m,n \in \mathbb{Z} \\ 3 \mid n}}{}' \frac{n^2 - 3y^2m^2}{(n^2 + y^2m^2)^3} \right).$$

Let RHS denote the right-hand side of (5.3). Then it can be seen using Corollary 5.6 that if $\omega = e^{2\pi i/3}$, then

$$RHS = \frac{3y^3}{4\pi^3} \sum_{m,n \in \mathbb{Z}}{}' \Bigg(\frac{3\omega^n - 27\omega^{-m} - 54\omega^{n-m} - 2}{|m\tau + n|^4}$$

$$+ 4 \,\mathrm{Re}\left(\frac{(9\omega^{-m} - 6\omega^n + 18\omega^{n-m} - 1)(m\tau + n)^2}{|m\tau + n|^6} \right) \Bigg).$$

It then can be shown that $RHS = n_3(s)$ by using (5.16) and properly rearranging the terms inside the summation above.

(iii) Similar to Theorem 5.3(i), the proof can be divided into two cases, depending on the value of s. Assume first that $s \geq 256$. Then $r' \in [1/\sqrt{2}, 1)$. Hence by Lemma 5.8(i) we have that $G_s \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where

$$\tau = \frac{\mathrm{F}_2\left(\frac{1-r'}{1+r'}\right)}{\mathrm{F}_2\left(\frac{2r'}{1+r'}\right)} i,$$

so that $\mathrm{Im}(-1/2\tau) \geq 1/\sqrt{2}$. Then using [4, Thm. 9.1,Thm. 9.2] and the fact that $s_4(q_4(\alpha)) = \frac{64}{\alpha(1-\alpha)}$ we can deduce that

$$\begin{aligned} s_4\left(q\left(-\frac{1}{2\tau}\right)\right) &= s_4\left(\exp\left(-\pi\frac{\mathrm{F}_2\left(\frac{2r'}{1+r'}\right)}{\mathrm{F}_2\left(\frac{1-r'}{1+r'}\right)}\right)\right) \\ &= s_4\left(\exp\left(-\sqrt{2}\pi\frac{\mathrm{F}_4(r'^2)}{\mathrm{F}_4(1-r'^2)}\right)\right) \\ &= s_4\left(q_4(1-r'^2)\right) = s. \end{aligned}$$

By elementary but tedious calculations analogous to those in the proof of Theorem 5.3(i), if $y = \mathrm{Im}(\tau)$, then

$$\begin{aligned} n_4\left(s_4\left(q\left(-\frac{1}{2\tau}\right)\right)\right) &= \frac{20y^3}{\pi^3}\left(4\sum_{\substack{m,n\in\mathbb{Z}\\ m \text{ even}}}{}' \frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3} - \sum_{m,n\in\mathbb{Z}}{}' \frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3}\right) \\ &= \frac{16}{9\pi^2}\big(15\mathcal{L}_{3,1}^{G_s}(2(P)-(2Q)+2(P+2Q)) \\ &\qquad + \mathcal{L}_{3,2}^{G_s}(4(Q)-5(P)+2(2Q)+4(P+Q)-5(P+2Q))\big). \end{aligned}$$

Next, if $s < 0$, then $r' > 1$, so the normalized period lattice of G_s is generated by 1 and

$$\tau := \frac{\mathrm{F}_2\left(\frac{r'-1}{2r'}\right)}{\mathrm{F}_2\left(\frac{r'+1}{2r'}\right)} i.$$

We employ the hypergeometric transformations [2, Thm. 2.2.5] and [82] one more time to deduce that

$$\frac{\tau-1}{2\tau} = \frac{\mathrm{F}_2\left(\frac{2r'}{r'+1}\right)}{\mathrm{F}_2\left(\frac{1-r'}{r'+1}\right)} i.$$

Hence, in this case,

$$s_4\left(q\left(\frac{\tau-1}{2\tau}\right)\right) = s,$$

by the same argument used for the case $s \geq 256$. If $y = \mathrm{Im}(\tau)$, then $\mathrm{Re}((\tau-1)/2\tau) = 1/2$ and $\mathrm{Im}((\tau-1)/2\tau) = 1/2y > 1/2$. Finally, it remains to show that

$$\begin{aligned} n_4\left(s_4\left(q\left(\frac{\tau-1}{2\tau}\right)\right)\right) &= \frac{20y^3}{\pi^3}\left(4\sum_{\substack{m \text{ odd} \\ n \text{ even}}}{}' \frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3} - \frac{3}{4}\sum_{m,n\in\mathbb{Z}}{}' \frac{n^2-3y^2m^2}{(n^2+y^2m^2)^3}\right) \\ &= \frac{8}{9\pi^2}\big(30\mathcal{L}_{3,1}^{G_s}(2(2Q)-(P+2Q)+2(P)) \\ &\quad + \mathcal{L}_{3,2}^{G_s}(5(P+2Q)+8(Q)+8(P+Q)-11(2Q)-10(P))\big), \end{aligned}$$

which, again, requires only Proposition 4.2(iii), Corollary 5.6, and some laborious work.

□

It is also interesting to consider $n_j(s)$ for the real values of s omitted from the results in Theorem 5.3. For instance, if $s \in \{0, 64\}$, then E_s is singular; i.e., it is no longer an elliptic curve, so $\mathcal{L}_{3,j}^{E_s}, j = 1, 2$ are not defined. If $s \in (0, 64)$; i.e., $(x-1)\left(x^2 - \frac{s}{s-64}\right)$ has only one real root, the story turns out to be quite different. Indeed, we will see that Formula (5.2) is not true in this case by the following observation:

Proposition 5.9. *Let $s \in (0, 64)$ and let E_s, r, P, and Q be as defined in Theorem 5.3(i). Then*

$$\mathcal{L}_{3,j}^{E_s}(P) = \mathcal{L}_{3,j}^{E_s}(Q), \qquad j = 1, 2.$$

Our proof of this proposition relies on the following facts:

Lemma 5.10. *Let $\tau = 1/2 + yi$, where $y \in \mathbb{R}$. Then the following identities hold:*

$$\sum_{\substack{m \text{ odd} \\ n \in \mathbb{Z}}} \frac{(-1)^n m^2}{|m\tau + n|^6} = 0, \tag{5.22}$$

$$\sum_{\substack{m \text{ odd} \\ n \in \mathbb{Z}}} \frac{(-1)^n (m/2 + n)^2}{|m\tau + n|^6} = 0. \tag{5.23}$$

Proof. Using the fact that $|z| = |-z| = |\bar{z}|$ for any $z \in \mathbb{C}$ and simple substitution, we find that

$$\begin{aligned}
\sum_{\substack{m \text{ odd} \\ n \in \mathbb{Z}}} \frac{(-1)^n m^2}{|m\tau + n|^6} &= \sum_{m,n \in \mathbb{Z}} \frac{(-1)^n (2m+1)^2}{|(2m+1)(-1/2 - yi) - n|^6} \\
&= \sum_{m,n \in \mathbb{Z}} \frac{(-1)^n (2m+1)^2}{|(2m+1)(1/2 - yi) - (n + 2m + 1)|^6} \\
&= \sum_{m,n \in \mathbb{Z}} \frac{(-1)^{-n-1} (2m+1)^2}{|(2m+1)(1/2 - yi) + n|^6} \\
&= -\sum_{\substack{m \text{ odd} \\ n \in \mathbb{Z}}} \frac{(-1)^n m^2}{|m\tau + n|^6}.
\end{aligned}$$

Hence (5.22) follows. Then we apply (5.22) to show that

$$\sum_{\substack{m \text{ odd} \\ n \in \mathbb{Z}}} \frac{(-1)^n (m/2 + n)^2}{|m\tau + n|^6} = \sum_{\substack{m \text{ odd} \\ n \in \mathbb{Z}}} \left(\frac{(-1)^n m^2}{4|m\tau + n|^6} + \frac{(-1)^n n(m+n)}{|m\tau + n|^6} \right)$$

$$= \sum_{\substack{m \text{ odd} \\ n \text{ even}}} \frac{nm}{|(m-n)\tau + n|^6} - \sum_{\substack{m \text{ even} \\ n \text{ odd}}} \frac{nm}{|(m-n)\tau + n|^6}.$$

Next, we use similar tricks from the proof of (5.22) to argue that

$$\begin{aligned}
\sum_{\substack{m \text{ even} \\ n \text{ odd}}} \frac{nm}{|(m-n)\tau + n|^6} &= \sum_{\substack{m \text{ even} \\ n \text{ odd}}} \frac{nm}{|(m-n)(-1/2 - yi) - n|^6} \\
&= \sum_{\substack{m \text{ even} \\ n \text{ odd}}} \frac{nm}{|(m-n)(1/2 - yi) - m|^6} \\
&= \sum_{\substack{m \text{ even} \\ n \text{ odd}}} \frac{nm}{|(n-m)(1/2 - yi) + m|^6} \\
&= \sum_{\substack{m \text{ odd} \\ n \text{ even}}} \frac{nm}{|(m-n)\tau + n|^6}.
\end{aligned}$$

Thus we have (5.23). □

Proof of Proposition 5.9. Applying the transformation $y \mapsto y/2$, we instead consider the family

$$\tilde{E}_s : y^2 = 4(x-1)(x-r)(x+r) = 4x^3 - 4x^2 - 4r^2x + 4r^2.$$

Then we again find the period lattice of $\tilde{E}_s$ using [22, Algorithm 7.4.7]. Indeed, one has immediately that

$$E_s \cong \tilde{E}_s \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

where

$$\tau = \frac{1}{2} + \frac{\operatorname{AGM}\left(2\sqrt[4]{1-r^2}, \sqrt{2\left(\sqrt{1-r^2}+1\right)}\right)}{2\operatorname{AGM}\left(2\sqrt[4]{1-r^2}, \sqrt{2\left(\sqrt{1-r^2}-1\right)}\right)} i.$$

Note that the arguments in the AGM above are all positive, since $r^2 = \frac{s}{s-64} < 0$. By similar analysis in the proof of Theorem 5.3(i), it can be shown that P and Q are mapped to $(1+\tau)/2$ and $\tau/2$ via the isomorphism above. Now by (5.11) one sees that

$$\mathcal{L}_{3,1}^{E_s}(P) = \frac{2y^3}{3\pi} \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{(-1)^{n-m}}{|m\tau+n|^4} - (-1)^{n-m} \frac{(m/2+n)^2 - m^2y^2}{|m\tau+n|^6} \right),$$

$$\mathcal{L}_{3,1}^{E_s}(Q) = \frac{2y^3}{3\pi} \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{(-1)^{n}}{|m\tau+n|^4} - (-1)^{n} \frac{(m/2+n)^2 - m^2y^2}{|m\tau+n|^6} \right).$$

Therefore, by (5.22)

$$\begin{aligned}\mathcal{L}_{3,1}^{E_s}((Q)-(P)) &= \frac{2y^3}{3\pi} \sum_{\substack{m \text{ odd} \\ n\in\mathbb{Z}}} \left(\frac{(-1)^{n}}{|m\tau+n|^4} - (-1)^{n} \frac{(m/2+n)^2 - m^2y^2}{|m\tau+n|^6} \right) \\ &= \frac{4y^3}{3\pi} \sum_{\substack{m \text{ odd} \\ n\in\mathbb{Z}}} \frac{(-1)^n m^2 y^2}{|m\tau+n|^6} = 0.\end{aligned}$$

Similarly, by (5.12),

$$\mathcal{L}_{3,2}^{E_s}(P) = \frac{y^3}{\pi} \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{(-1)^{n-m}}{|m\tau+n|^4} + 2(-1)^{n-m} \frac{(m/2+n)^2 - m^2y^2}{|m\tau+n|^6} \right),$$

$$\mathcal{L}_{3,2}^{E_s}(Q) = \frac{y^3}{\pi} \sum_{m,n\in\mathbb{Z}}{}' \left(\frac{(-1)^{n}}{|m\tau+n|^4} + 2(-1)^{n} \frac{(m/2+n)^2 - m^2y^2}{|m\tau+n|^6} \right);$$

whence

$$\begin{aligned}\mathcal{L}_{3,2}^{E_s}((Q)-(P)) &= \frac{y^3}{\pi} \sum_{\substack{m \text{ odd} \\ n\in\mathbb{Z}}} \left(\frac{(-1)^{n}}{|m\tau+n|^4} + 2(-1)^{n} \frac{(m/2+n)^2 - m^2y^2}{|m\tau+n|^6} \right) \\ &= \frac{y^3}{\pi} \sum_{\substack{m \text{ odd} \\ n\in\mathbb{Z}}} (-1)^n \frac{3(m/2+n)^2 - m^2y^2}{|m\tau+n|^6} = 0,\end{aligned}$$

where the last equality follows from (5.22) and (5.23). □

Although no general formula for $n_2(s)$ where $s \in (0, 64)$ has been found, we were able to verify the following formulas numerically in `PARI` and `Maple`:

$$n_2(1) \stackrel{?}{=} -\frac{12}{7\pi^2}\mathcal{L}_{3,1}^{E_1}(4(R)-(O)),$$

$$n_2(16) \stackrel{?}{=} -\frac{12}{\pi^2}\mathcal{L}_{3,1}^{E_{16}}(4(R)+(O)),$$

where $R=(1,0)$ and $\stackrel{?}{=}$ means that they are equal to at least 75 decimal places. (Note that E_1 and E_{16} are both CM elliptic curves.)

5.4 Connection with special values of L-functions

In this section, we investigate relationships between evaluations of $\mathcal{L}_{3,j}^{E}, j=1,2$ and some special values of L-functions, the first evidence of which is the symmetric square L-function of E. It was verified numerically in [48, §3] that for some non-CM elliptic curves $E \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ there exist degree zero divisors ξ_1 and ξ_2 on E such that

$$\begin{vmatrix} \operatorname{Re}\left(K_{1,3}(\tau;\xi_1)\right) & K_{2,2}(\tau;\xi_1) \\ \operatorname{Re}\left(K_{1,3}(\tau;\xi_2)\right) & K_{2,2}(\tau;\xi_2) \end{vmatrix} \stackrel{?}{\sim}_{\mathbb{Q}^\times} \frac{\pi^6}{\operatorname{Im}(\tau)^4} L''(\operatorname{Sym}^2 E,0). \tag{5.24}$$

Indeed, the above relation can be rephrased in terms of the determinant of $\mathcal{L}_{3,j}^{E}, j=1,2$.

Proposition 5.11. *Let E be an elliptic curve over $\mathbb{C}$ and suppose that $E \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. If ξ_1 and ξ_2 are divisors of degree zero on E, then*

$$\begin{vmatrix} \mathcal{L}_{3,1}^{E}(\xi_1) & \mathcal{L}_{3,2}^{E}(\xi_1) \\ \mathcal{L}_{3,1}^{E}(\xi_2) & \mathcal{L}_{3,2}^{E}(\xi_2) \end{vmatrix} = -\frac{2\operatorname{Im}(\tau)^6}{\pi^2} \begin{vmatrix} \operatorname{Re}\left(K_{1,3}(\tau;\xi_1)\right) & K_{2,2}(\tau;\xi_1) \\ \operatorname{Re}\left(K_{1,3}(\tau;\xi_2)\right) & K_{2,2}(\tau;\xi_2) \end{vmatrix},$$

where $K_{a,b}(\tau;\xi)=\sum_{P\in E} n_P K_{a,b}(\tau;u_P)$ if $\xi=\sum_{P\in E} n_P(P)$ and u_P is the image of P in $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$.

Proof. For any degree zero divisor $\xi=\sum_{P\in E} n_P(P)$, it follows directly from (5.7) and

(5.8) that

$$\operatorname{Re}(D_{1,3}(q;\xi)) = 2\mathcal{L}^E_{3,1}(\xi) - \frac{4}{3}\mathcal{L}^E_{3,2}(\xi),$$
$$D_{2,2}(q;\xi) = -4\mathcal{L}^E_{3,1}(\xi) - \frac{4}{3}\mathcal{L}^E_{3,2}(\xi),$$

where $q = e^{2\pi i\tau}$ and $D_{a,b}(q;\xi) = \sum_{P\in E} n_P D_{a,b}(q; e^{2\pi i u_P})$. Then by Proposition 5.4(iii) and the two equations above one has that

$$\left(\frac{4\operatorname{Im}(\tau)^3}{\pi}\right)^2 \begin{vmatrix} \operatorname{Re}(K_{1,3}(\tau;\xi_1)) & K_{2,2}(\tau;\xi_1) \\ \operatorname{Re}(K_{1,3}(\tau;\xi_2)) & K_{2,2}(\tau;\xi_2) \end{vmatrix} = \begin{vmatrix} \operatorname{Re}(D_{1,3}(q;\xi_1)) & D_{2,2}(q;\xi_1) \\ \operatorname{Re}(D_{1,3}(q;\xi_2)) & D_{2,2}(q;\xi_2) \end{vmatrix}$$
$$= -8 \begin{vmatrix} \mathcal{L}^E_{3,1}(\xi_1) & \mathcal{L}^E_{3,2}(\xi_1) \\ \mathcal{L}^E_{3,1}(\xi_2) & \mathcal{L}^E_{3,2}(\xi_2) \end{vmatrix}.$$

□

More generally, a conjecture relating $L(\operatorname{Sym}^n E, n+1)$ to determinants of Eisenstein-Kronecker series was formulated by Goncharov in [30, §6]. By the functional equation for $L(\operatorname{Sym}^2 E, s)$, the relation (5.24) can be seen as a special case of this conjecture when $n = 2$.

On the other hand, we observe from our computational experiments that if E_s is CM, then the functions $\mathcal{L}^{E_s}_{3,1}$ and $\mathcal{L}^{E_s}_{3,2}$ evaluated at some torsion divisors are individually related to lower degree L-values; i.e., the ones associated to Dirichlet characters and elliptic modular forms. These results are listed at the end of this section. In particular, some weaker results below are immediate consequences of Theorem 5.3 and Theorem 4.3. Recall that M_N and d_k are as defined in Section 4.2.

Proposition 5.12. *Let E_s, F_s, and G_s be the families of elliptic curves as defined in Theorem 5.3.*

(i) Let E denote $E_{256}: y^2 = (x-1)\left(x^2 - \frac{4}{3}\right)$, $P = \left(-\frac{2}{\sqrt{3}}, 0\right)$, and $Q = \left(\frac{2}{\sqrt{3}}, 0\right)$. Then we have

$$\left(6\mathcal{L}_{3,1}^{E} - \mathcal{L}_{3,2}^{E}\right)((P) - (Q)) = \frac{\pi^2}{2}(M_{48} + 2d_4).$$

(ii) Let O, P and Q are the points on F_s corresponding to $1, 1/3$ and $\tau/3$, respectively, via the isomorphism $F_s \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \tau \in \mathcal{H}$.
If $\mathcal{T}(s) := 15\mathcal{L}_{3,1}^{F_s}((Q) - 3(P) - 6(P+Q)) + \mathcal{L}_{3,2}^{F_s}(3(P) + 6(P+Q) - 7(Q) - 2(O))$, then the following formulas are true:

$$\mathcal{T}(108) = 20\pi^2 M_{12}, \qquad \mathcal{T}(216) = 5\pi^2\left(M_{24}^{(2)} + d_3\right),$$

$$\mathcal{T}(1458) = \frac{3\pi^2}{2}\left(9M_{12} + 2d_4\right).$$

(iii) Let P and Q denote the points on G_s corresponding to $\tau/2$ and $3/4$, respectively, via the isomorphism $G_s \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. If we set

$$\begin{aligned}\mathcal{U}(s) &:= 15\mathcal{L}_{3,1}^{G_s}(2(P) - (2Q) + 2(P+2Q)) \\ &\quad + \mathcal{L}_{3,2}^{G_s}(4(Q) - 5(P) + 2(2Q) + 4(P+Q) - 5(P+2Q)), \\ \mathcal{V}(s) &:= 30\mathcal{L}_{3,1}^{G_s}(2(2Q) - (P+2Q) + 2(P)) \\ &\quad + \mathcal{L}_{3,2}^{G_s}(5(P+2Q) + 8(Q) + 8(P+Q) - 11(2Q) - 10(P)),\end{aligned}$$

then the following formulas are true:

$$\mathcal{U}(256) = \frac{45\pi^2}{2}M_8, \qquad \mathcal{U}(648) = \frac{45\pi^2}{32}\left(4M_{16} + d_4\right),$$

$$\mathcal{U}(2304) = \frac{15\pi^2}{4}\left(M_{24}^{(1)} + d_3\right), \quad \mathcal{U}(20736) = \frac{9\pi^2}{20}\left(5M_{40}^{(1)} + 2d_3\right),$$

$$\mathcal{U}(614656) = \frac{15\pi^2}{2}\left(5M_8 + d_3\right),$$

$$\mathcal{U}(3656 + 2600\sqrt{2}) = \frac{45\pi^2}{128}\left(4M_{32} + 28M_8 + 4d_4 + d_8\right),$$

$$\mathcal{V}(3656 - 2600\sqrt{2}) = \frac{45\pi^2}{64}\left(44M_{32} - 28M_8 + 4d_4 - d_8\right).$$

We will conclude this section by listing some conjectural formulas for $\mathcal{L}_{3,j}^{E_s}$ evaluations at torsion points when E_s is a CM elliptic curve over $\mathbb{Q}$. For each fixed s, we let $L_j = \frac{1}{\pi^2}\mathcal{L}_{3,j}^{E_s}$, and O, P, and Q denote the points on E_s corresponding to $1, \tau/2$, and $3/4$, respectively, via the isomorphism $E_s \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. We firmly believe that some of these formulas could be verified rigorously using Corollary 5.6 and double series expressions of L-series established in Section 4.

One of the remarkable features of these formulas is that they appear to support the conjectural relation (5.24). For instance, consider the CM elliptic curve $E := E_{-8}$ of conductor 576 with the corresponding $\tau = \sqrt{-1}$. One can verify, at least numerically, that

$$L''(\operatorname{Sym}^2 E, 0) \stackrel{?}{=} 2d_4M_{16}.$$

Then, choosing $\xi_1 = (Q)+(P+Q)-2(O)$ and $\xi_2 = (2Q)-(P)$, we obtain the following identity directly from our formulas in Table 5.1 and Table 5.2:

$$\begin{vmatrix} \mathcal{L}_{3,1}^{E}(\xi_1) & \mathcal{L}_{3,2}^{E}(\xi_1) \\ \mathcal{L}_{3,1}^{E}(\xi_2) & \mathcal{L}_{3,2}^{E}(\xi_2) \end{vmatrix} \stackrel{?}{=} -\frac{43\pi^4}{64}d_4M_{16} \stackrel{?}{=} -\frac{43\pi^4}{128}\operatorname{Im}(\tau)^2L''(\operatorname{Sym}^2 E, 0).$$

To our knowledge, this particular example does not seem to appear in the literature, though it is exactly analogous to the numerical result due to Zagier which involves a non-CM elliptic curve of conductor 37 [89, §10]. As a possible continuing research project, it would be interesting to find and prove this type of relations for some CM elliptic curves, which we will not pursue here.

$s = -512$	$L_2((P) - (P + 2Q)) \stackrel{?}{=} \frac{M_{64} - d_8}{8}, \quad L_2(3(P + Q) - 4(Q) + (O)) \stackrel{?}{=} \frac{3M_{16} - d_4}{4}$
$s = -64$	$L_2((P) - (P + 2Q)) \stackrel{?}{=} -\left(\frac{M_{32} - d_4}{4}\right), \quad L_2((Q) - (P + Q)) \stackrel{?}{=} \frac{M_{32} + d_4}{16}$
$s = -8$	$L_2((2Q) - (P)) \stackrel{?}{=} M_{16}, \quad L_2((Q) + (P + Q) - 2(O)) \stackrel{?}{=} \frac{4M_{16} - 43d_4}{64}$
$s = 1$	$L_2((2Q) - (P)) \stackrel{?}{=} \frac{54M_7 + d_7}{8}, \quad L_2((Q) - (P + Q)) \stackrel{?}{=} \frac{M_{112} + 8d_4}{16}$
$s = 16$	$L_2((2Q) - (P)) \stackrel{?}{=} \frac{3M_{12}}{4}, \quad L_2((Q) - (P + Q)) \stackrel{?}{=} \frac{M_{48} + 2d_4}{16}$
$s = 256$	$L_2((2Q) - (P)) \stackrel{?}{=} \frac{M_{48} - 2d_4}{6}, \quad L_2((Q) + (P + Q) - 2(O)) \stackrel{?}{=} \frac{508M_{12} + 4M_{48} - 385d_3 - 8d_4}{384}$
$s = 4096$	$L_2((2Q) - (P)) \stackrel{?}{=} \frac{M_{112} - 8d_4}{14}, \quad L_2((Q) + (P + Q) - 2(O)) \stackrel{?}{=} \frac{6112M_7 + 4M_{112} - 32d_4 - 213d_7}{896}$

Table 5.1: Conjectured formulas of L_2

	$L_1(2Q)$	$L_1(P)$	$L_1(P+2Q)$	$L_1(Q)$	$L_1(P+Q)$	$L_1(O)$
$s=-512$	$-\frac{(6M_{16}+5d_4)}{36}$	$\frac{1}{288}(6M_{16}+6M_{64} -d_4+12d_8)$	$\frac{1}{288}(6M_{16}-6M_{64} -d_4-12d_8)$	$-\frac{(6M_{16}+d_4)}{288}$	$-\frac{d_4}{72}$	$\frac{6M_{16}+7d_4}{36}$
$s=-64$	$-\frac{(18M_8+d_8)}{36}$	$\frac{1}{144}(18M_8+6M_{32} +12d_4-d_8)$	$\frac{1}{144}(18M_8-6M_{32} -12d_4-d_8)$	$-\frac{1}{576}(18M_8+6M_{32} -12d_4+d_8)$	$-\frac{1}{576}(18M_8-6M_{32} +12d_4+d_8)$	$\frac{(6M_8+d_8)}{18}$
$s=-8$	$-\frac{(6M_{16}+d_4)}{36}$	$\frac{6M_{16}-d_4}{36}$	$-\frac{d_4}{9}$	$-\frac{1}{576}(6M_{16}+6M_{64} +d_4-12d_8)$	$-\frac{1}{576}(6M_{16}-6M_{64} +d_4+12d_8)$	$\frac{2d_4}{9}$
$s=1$	$-\frac{(48M_7-d_7)}{36}$	$\frac{33M_7-2d_7}{36}$	$\frac{33M_7-2d_7}{36}$	$-\frac{1}{576}(48M_7+6M_{112} -96d_4-d_7)$	$-\frac{1}{576}(48M_7-6M_{112} +96d_4-d_7)$	$-\frac{(6M_7-d_7)}{9}$
$s=16$	$-\frac{(2M_{12}+d_3)}{12}$	$\frac{M_{12}-d_3}{12}$	$\frac{M_{12}-d_3}{12}$	$-\frac{1}{192}(2M_{12}+2M_{48} +d_3-8d_4)$	$-\frac{1}{192}(2M_{12}-2M_{48} +d_3+8d_4)$	$\frac{d_3}{3}$
$s=256$	$\frac{1}{72}(2M_{12}-2M_{48} -d_3-8d_4)$	$\frac{1}{72}(2M_{12}+2M_{48} -d_3+8d_4)$	$-\frac{(2M_{12}+2d_3)}{9}$	(*)[a]	(*)	$\frac{2M_{12}+3d_3}{9}$
$s=4096$	$\frac{1}{504}(48M_7-6M_{112} -96d_4-d_7)$	$\frac{1}{504}(48M_7+6M_{112} +96d_4+d_7)$	$-\frac{66M_7+4d_7}{63}$	(*)	(*)	$\frac{72M_7+5d_7}{63}$

Table 5.2: Conjectured formulas of L_1

[a] When $s=256$ and $s=4096$, no individual conjectural formulas of $L_1(Q)$ and $L_1(P+Q)$ were detected in our numerical computations. However, we found that

$$s=256: \qquad L_1((Q)+(P+Q)) \stackrel{?}{=} \frac{2M_{12}-2M_{48}-d_3-8d_4}{576},$$

$$s=4096: \qquad L_1((Q)+(P+Q)) \stackrel{?}{=} \frac{48M_7-6M_{112}-96d_4+d_7}{4032}.$$

6. GENERALIZATIONS AND CONCLUSIONS

We have seen examples of Mahler measures of polynomials of two and three variables which have connections with the hypergeometric weight systems in Table 2.1 and Table 2.2. Therefore, one way to generalize the previous results is to consider Mahler measures of some families of four-variable polynomials associated to the fourteen hypergeometric weight systems in Table 2.3. Here we systematically construct fourteen families of polynomials, each of whose formal period is respectively the ${}_4F_3$-hypergeometric series listed in Table 2.3:

$$W_1(k) := (x+x^{-1})(y+y^{-1})(z+z^{-1})(w+w^{-1}) - k^{1/2},$$

$$W_2(k) := (x+x^{-1})^2(y+y^{-1})^2(1+z)^3z^{-2}(w+w^{-1})^4 - k,$$

$$W_3(k) := (x+x^{-1})(y+y^{-1})(z+z^{-1})(w+w^{-1})^2 - k^{1/2},$$

$$W_4(k) := x^5+y^5+z^5+w^5+1-k^{1/5}xyzw,$$

$$W_5(k) := (x+x^{-1})^2(y+y^{-1})^2(1+z)^3z^{-2}(1+w)^3w^{-2} - k,$$

$$W_6(k) := (x+x^{-1})(y+y^{-1})(z+z^{-1})^2(w+w^{-1})^2 - k^{1/2},$$

$$W_7(k) := (x+x^{-1})^2(y+y^{-1})^2(z+z^{-1})^2(1+w)^3w^{-2} - k,$$

$$W_8(k) := (x+x^{-1})^2(y+y^{-1})^2(1+z)^3z^{-2}(w+w^{-1})^6 - k,$$

$$W_9(k) := (x+x^{-1})^2(1+y)^3y^{-2}(1+z)^3z^{2}(w+w^{-1})^6 - k,$$

$$W_{10}(k) := (x+x^{-1})(y+y^{-1})(z+z^{-1})^2(w+w^{-1})^4 - k^{1/2},$$

$$W_{11}(k) := (x+x^{-1})^2(1+y)^3y^{-2}(z+z^{-1})^4(w+w^{-1})^6 - k,$$

$$W_{12}(k) := (x+x^{-1})^2(1+y)^3y^{-2}(1+z)^5z^{-3}(w+w^{-1})^{10}-k,$$

$$W_{13}(k) := (1+x)^3x^{-2}(1+y)^3y^{-2}(z+z^{-1})^6(w+w^{-1})^6-k,$$

$$W_{14}(k) := (x+x^{-1})(y+y^{-1})(1+z)^3z^{-2}(w+w^{-1})^6-k^{1/2}.$$

We can obtain Mahler measures of these polynomials by means of the calculation used in the two and three-variable cases. Explicitly, if

$$p_l(k) := \begin{cases} m(W_l(k)), & l = 2,5,7,8,9,11,12,13, \\ 2m(W_l(k)), & l = 1,3,6,10,14, \\ 5m(W_l(k)), & l = 4, \end{cases}$$

then, for $|k|$ sufficiently large,

$$p_l(k) = \mathrm{Re}\left(\log(k) - \frac{a_1a_2a_3a_4\lambda_0^{-1}}{k}\,{}_6F_5\left(\begin{matrix} a_1+1,\ a_2+1,\ a_3+1,\ a_4+1,\ 1,\ 1 \\ 2,\ 2,\ 2,\ 2,\ 2 \end{matrix};\frac{\lambda_0^{-1}}{k}\right)\right),$$

where a_1, a_2, a_3, a_4, and λ_0^{-1} are given accordingly in Table 2.3. It might be interesting to interpret the polynomials $W_l(k)$ geometrically, though this direction will not be pursued here. For instance, the quintic polynomial $W_4(k)$ defines a family of Calabi-Yau varieties in $\mathbb{P}^4$, and their geometry and arithmetic are very well-studied [16, 17, 66]. Instead, we shall conclude this thesis by giving some comments about the general framework of what we have done so far.

For the two-variable case, Rodriguez Villegas derived Mahler measures from certain functions $e(\tau)$ constructed out of two solutions of Picard-Fuchs equations, and the parameter t can be chosen to be the functions $t_j(\tau)$, which also arise from the same differential equations. These functions are meromorphic modular forms of weight 3 and 0, respectively, under the action of the monodromy groups associated to the differential equations. Usually, these groups are genus zero congruence subgroups of $\mathrm{SL}_2(\mathbb{R})$ with Hauptmoduls $t_j(\tau)$. Similarly, if we repeat the same procedure for the three-variable case, we obtain modular forms of weight 0 and weight 4, where the latter could be used to derive Mahler measure formulas. Note that in this case the monodromy groups of Picard-Fuchs equations would belong to $\mathrm{SL}_3(\mathbb{R})$. However, since these Picard-Fuchs equations are the symmetric squares of some second-order differential equations, the monodromy groups are isomorphic to congruence subgroups of $\mathrm{SL}_2(\mathbb{R})$ [78, Prop.5.5]. Also, the modular functions obtained from the construction above; i.e., the functions $s_j(\tau)$, become Haupmoduls for these congruence subgroups. Therefore, to generalize this idea to the four-variable case, one might need to identify the monodromy group of each family first. In fact, it is an important result due to Chen, Yang, Yui, and Erdenberger [18] that the monodromy groups of the Picard-Fuchs equations associated to the fourteen hypergeometric families of Calabi-Yau varieties are contained in congruence subgroups of $\mathrm{Sp}(4,\mathbb{Z})$. Hence modular forms of more variables, such as Siegel modular forms, might come into play in order to formulate Mahler measures of Calabi-Yau threefolds in terms of special L-values. On the other hand, we have found an interesting formula of the Mahler measure of one of the polynomials above, which involves a weight 4 classical modular form.

Theorem 6.1 (Papanikolas, Rogers, Samart)**.** *The following formula is true:*

$$p_1(256) = 16L'(h,0) - 56\zeta'(-2), \tag{6.1}$$

where $h(\tau) = \eta^4(2\tau)\eta^4(4\tau) \in S_4(\Gamma_0(8))$*, and* $\zeta(s)$ *is the Riemann zeta function.*

Note that the modular form h defined above is also known to have a connection with the first hypergeometric weight system in Table 2.3. More specifically, its Fourier coefficients appear in the zeta function of the corresponding (singular) Calabi-Yau threefold [54, §3].

Furthermore, h is a non-CM newform, so we require arguments which completely differ from the CM case to prove Theorem 6.1. We refer the reader to [52] for the proof of this formula.

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