THE DUFFING OSCILLATOR AND LINEARIZATION TECHNIQUES FOR ITS

MOTION CONSTANT

A Thesis

by

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Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

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May 2014

Major Subject: Aerospace Engineering

ABSTRACT

Analyzing the characteristics of higher order nonlinear dynamic systems is really difficult. This can involve giving solutions with respect to time. Motion constants are another way of studying the behavior of the dynamic system. If the motion constant is known, solving the system is no longer needed to analyze the characteristics of the system. Motion constants are time independent integrals that are hard to find for nonlinear dynamic systems. We chose the Duffing Oscillator as a higher order nonlinear dynamic system to have its motion constants investigated. The Duffing Oscillator was chosen because studying it gives a better view of how rigid bodies act. It forms a clear dynamic analog of the general torque-free motion of an arbitrary rigid body, meaning it covers most of the arbitrary rigid body dynamics.

Investigating the motion constants for a finite dimensional nonlinear system, such as the Duffing Oscillator (can be quite difficult) but finding the motion constants for a linear autonomous system, regardless of its dimension, is easier and has recently been found. In this study we propose finding the motion constants of the Duffing Oscillator through the motion constant of a linear representation. A linear representation is found through Carleman Linearization. This is a technique used to linearize a finite dimensional nonlinear system of differential equations to an infinite dimensional, linear, autonomous system of differential equations. Using Carleman Linearization, the Duffing equation is linearized; the motion constant was found, and compared to the true known value of the real system.

DEDICATION

To my family and friends, and the people who believed in me.

ACKNOWLEDGEMENTS

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1.   INTRODUCTION

1.1 Motion Constants

In mechanics, a motion constant, or constant of motion, is a quantity that is conserved throughout the motion. It forces a constraint on the motion, which is a time independent algebraic or transcendental equation involving the system states. Motion constants give us information about the system and properties of the motion without solving equations of motion. Known examples include energy, linear momentum, and angular momentum.

Motion constants solve the dynamic system; they give a description of the shape of the space-state, and provide a big picture of the motion irrespective of time. In some cases, the trajectory of the motion can be derived as the intersection of surfaces corresponding to the constants of motion.

For finite dimensional nonlinear systems, independent motion constants are generally hard to find, yet they have been found by linearizing the nonlinear system to a number of linear systems. This process includes a state transformation to form a canonical, uncoupled real system. After that, internal integrals for each subsystem are found, and comparison integrals that connect subsystems to each other are computed. For n linear systems, a complete set of n-1 motion constants, or time independent integrals, can be found with a single exception—when a system is composed of

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undamped oscillators whose frequency ratio is irrational when two subsystems are compared. Such systems lack a number of analytical time-independent integrals.

Usually, the solutions of dynamical systems are solved in terms of collections of integrals. The integrals can be time independent or dependent. Here, the method presented by Drs. Sinclair and Hurtado, in their paper written in 2013 [1], specifically deals with time independent integrals, which are referred to as motion constants . Classical reviews and solutions for the time independent integrals of nth order linear autonomous systems are presented.

1.2 Integrals of Linear Autonomous Dynamical Systems

As discussed earlier, Sinclair and Hurtado (2013) [1] proposes a method for evaluating a complete set of time independent integrals for arbitrary linear autonomous systems. An nth order autonomous system gives a description of the evolution of the state vector x (t):

$$\dot{X} = f(x) \tag{1}$$

If possible, the system can be solved by finding n functions of state and time, which are constant over the trajectory associated with an initial condition x_0 . These are called integrals [1] :

$$\emptyset(x(t), t) = \emptyset_k(x_0, 0) \ for \ t \ \in R, k = 1, \dots, n \tag{2}$$

Evaluating these integrals might require knowledge of both current states and times, but here we are only interested in evaluating the time independent integrals. The

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methods presented involve time independent integrals for linear autonomous systems, which depend on the order of the system. The focus here is on finding the time independent integrals:

$$\dot{X} = AX \qquad (3)$$

where an *n*th order autonomous system can have up to *n*-1 time independent integrals [1]:

$$\psi\big(x(t)\big) = \psi_k(x_0) \ \ for \ t \ \in R, k = 1, \dots, n-1 \qquad (4)$$

1.2.1 Second Order Systems

Time independent integrals of second order systems have a solution that has been found by Burns and Palmore.[1, 2]

General Second order systems can be written in first order form:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \dot{x} = Ax; \ \ x(t_0) = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} \qquad (5)$$

There are three possibilities that exist for the A matrix:

1. Diagonalizable with real eigenvalues λ_1 and λ_2 .

2. Diagonalizable with complex eigenvalues λ_1 , λ_2 =a$\pm ib$.

3. Non-Diagonalizable with real eigenvalues λ_1 , λ_2 =λ.

1.2.1.1 Diagonalizable With Real Eigenvalues

A is Diagonalizable using its eigenvalues and eigenvectors**:**

$$A = W \Lambda W^{-1}; \; \Lambda = \begin{bmatrix} \lambda 1 & 0 \\ 0 & \lambda 2 \end{bmatrix} \tag{6}$$

Where Λ and W are matrices of eigenvalues and eigenvectors of A respectively. Using a coordinate transformation using W a canonical form of independent subsystems can be defined:

$$\boldsymbol{X} = W\boldsymbol{Y}; \; \dot{\boldsymbol{Y}} = \Lambda \boldsymbol{Y} \tag{7}$$

A time independent integral for new states y was presented by Burns & Palmore [2] :

$$\psi = tan^{-1}(\frac{y1^{\frac{-1}{\lambda 1}}}{y2^{\frac{-1}{\lambda 2}}}) \tag{8}$$

For simplicity an alternative expression is suggested:

$$\psi = \; \lambda_1 \; \ln|y_2| - \lambda_2 \; \ln|y_1| \tag{9}$$

For the specific case that λ_1 or λ_2 equals zero, the time independent integral is the associated canonical state. The y_1 and y_2 states are individual first order systems; separately they don't have any time independent integrals.

1.2.1.2 Complex Conjugate Eigenvalues

If A contains two complex eigenvalues λ_1 , λ_2 =a$\pm ib$ and eigenvectors w_1 ,$w_{2=}$ u$\pm iv$, to avoid dealing with complex an alternate transformation is done [3]:

$$A = P \, R \, P^{-1} \; ; R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}; \quad P = [\, v \; u \,] \tag{10}$$

$$\boldsymbol{X} = P \, \boldsymbol{Y} \; ; \; \dot{\boldsymbol{Y}} = R\boldsymbol{Y} \tag{11}$$

A simple time independent integral for this case is presented below:

$$\psi = \tan[\frac{b}{2a}\lambda_1 \ln(y_1{}^2 + y_2{}^2) - \tan^{-1}\frac{y_2}{y_1}] \tag{12}$$

For the special case of imaginary eigenvalues, a=0, the time independent integral or the

motion constant is: $\psi = y_1{}^2 + y_2{}^2$.

1.2.1.3 Non-Diagonalizable With Real Eigenvalues

The case with one eigenvalue that has only one unique eigenvector can be put in

Jordan form: (where J and M are matrices of eigenvalues and eigenvectors of A

respectively):

$$A = M\,J\,M^{-1}\;;\; J = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \tag{13}$$

Using a coordinate transformation using M and J, a canonical form of independent

subsystems can be defined:

$$X = MY;\; \dot{Y} = JY \tag{14}$$

This system is similar to the two cascaded first order systems: y_1 is an independent first

order system, and y_2 is driven by y_1. A simple form of the time independent integral is:

$$\psi = \lambda\frac{y_2}{y_1} - \ln|y_1| \tag{15}$$

In the special case that λ=0, the time independent integral (motion constant) equals

$\psi = y_1$.

1.2.2 Higher Order Systems

For an *n*th order system, an *n*-1 time independent integrals can be found. The

process has three steps: step one is to transform the systems in to a canonical block-

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diagonal form, then to find the internal integrals of each block, and finally to find the

comparison integral of two neighboring blocks.

1.2.2.1 Motion-Constant Canonical Form

The first step in finding the motion constants is to transform the system into a

canonical form, in order to get the first order independent subsystems that form the

dynamics of the system. In $X = MY$; $\dot{Y} = JY$, J and M are matrices of eigenvalues and

eigenvectors of A respectively. Using a coordinate transformation using M and J, a

canonical form of independent subsystems can be defined, and A is taken to a block

diagonal form:

$$
\text{A= M J M}^{-1}; J =
\begin{bmatrix}
J_1 & & & & & 0 \\
& \ddots & & & & \\
& & J_p & & & \\
& & & H_1 & & \\
& & & & \ddots & \\
0 & & & & & H_q
\end{bmatrix}
\tag{16}
$$

The canonical form is similar to Jordan form, but we have ones in the sub-diagonal.

Each J_k for k=1,..,p, has one real eigenvalue, one real eigenvector and ones on the sub-

diagonal if the eigenvalue is repeated as shown below:

$$
J_k =
\begin{bmatrix}
\lambda_k & & & 0 \\
1 & \ddots & & \\
& \ddots & \lambda_k & \\
0 & & 1 & \lambda_k
\end{bmatrix}
\tag{17}
$$

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The columns of M associated with the repeated are the generalized eigenvectors (they will be discussed later on). For the case of non-repeated eigenvectors $J_k=\lambda$, and the columns of M are the eigenvectors. The dimension of J_k can go from one to n. H_k deals with the complex eigenvalues, each H_k for k=1,..,q, has a pair of complex-conjugate eigenvectors, and one pair of complex-conjugate eigenvalues λ_1, $\lambda_2 = a \pm ib$, a and b are real numbers, $b \neq 0$. Also, each H_k is made up of R (2x2 block) on the diagonal , and 2x2 I identity matrix on the sub-diagonal as shown below:

$$H_k = \begin{bmatrix} R & & & 0 \\ I & \ddots & & \\ & \ddots & R & \\ 0 & & I & R \end{bmatrix} \quad ; R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ (18)

The dimension of H_k is even and could be two or greater. The columns of M associated with it are the imaginary and the real parts of the generalized eigenvectors associated with $\lambda_1 = a+ib$, P = [v u] .

1.2.2.2 Internal Integrals

Each H_k , or J_k block gives a representation of an independent subsystem, and the time independent of these subsystems can be evaluated. Each subsystem will have a one less time independent integral than the dimension of the subsystem. These time independent integrals include the states related to a single independent subsystem, and they are called internal integrals.

Any J_k nonzero block that has a dimension of 1x1 has no internal integral. For the independent subsystem in J_k, the state equations are written below:

$$\dot{y}_1 = \lambda y_1 \tag{19}$$

$$\dot{y}_2 = \lambda y_2 + y_1$$

$$\vdots$$

$$\dot{y}_m = \lambda y_m + y_{m-1}$$

All states are driven by the previous state except for the first state. After solving for y in the above equations and rearranging to eliminate time, the first motion constant (time independent integral) associated with J_k is $\psi = \lambda \frac{y_2}{y_1} - \ln|y_1|$ which is eq. (15).

In the same manner, the time independent integrals are calculated for the subsystem by solving the equations then eliminating time. As for H_k, the states also form an independent subsystem, and the state equations for it is below:

$$\dot{y}_1 = ay_1 - by_2 \tag{20}$$

$$\dot{y}_2 = by_2 + ay_1$$

$$\dot{y}_{2m-1} = ay_{2m-1} - by_{2m} + y_{2m-3}$$

$$\dot{y}_{2m} = by_{2m-1} + ay_{2m} + y_{2m-2}$$

As shown above, the first two states are an independent subsystem; the other pairs are driven by the previous pair. After solving for y in the above equations and rearranging to eliminate time, the first motion constant (time independent integral) associated with H_k or of a second order system with complex-conjugate eigenvalues is:

$\psi = \tan \ [\ \frac{b}{2a}\lambda_1 \ln(y_1{}^2 + y_2{}^2) - tan^{-1}\frac{y_2}{y_1} \ ]$ which is eq. (12) as discussed earlier.

For the special case of imaginary eigenvalues , a=0, the time independent integral or the motion constant is $\psi = y_1{}^2 + y_2{}^2$. Since H_k has even dimensions (for

dimensions greater than two) additional pairs of integrals are found by investigating the

solutions of eq. (20), then rearranging and eliminating time.

1.2.2.3 Comparison Integrals

An *n*th order linear independent system is represented with p+q independent

subsystems; the number of these motion constants is one less than the order of the

subsystem, so this leaves p+q-1 additional time independent integrals to be found to

complete the *n*-1 integrals for the system [1]. The comparison integral is formed from

comparing the motion constants of two independent subsystems, so the comparison

integrals can be found from comparing neighboring blocks.

For higher order systems, there are a greater number of subsystems that have to

be used in forming comparison integrals, a method for finding comparison integrals is

described and discussed in the following paragraphs. After examining the behavior of

each subsystem with respect to time, time can be written as a function of the states.

Subsystems with nonzero real part as an eigenvalue have a representation of an

exponential behavior in time. The first state of the subsystem is an independent first

order system as shown in eq. (19):

$$\dot{y}_1 = \lambda y_1 \; ; \; y_1 = e^{\lambda t} y_{1,0} \tag{21}$$

$$t = \frac{1}{\lambda} \left(ln|y_1| - ln|y_{1,0}| \right)$$

Subsystems with zero eigenvalue have a linear behavior in time, so they have a

time dependent integral. If the dimension of the subsystem is one, then it doesn't need to

be used in forming comparison integrals. When the subsystem has a dimension greater than one, the permanence of the first state will be found by finding the internal integrals as shown in eq. (19): [1]

$$y_2 = y_1 t + y_{2,0} \tag{22}$$

$$t = \frac{y_2}{y_1} - \frac{y_{2,0}}{y_{1,0}}$$

Subsystems with imaginary eigenvalues have a periodic behavior in time. The first two states of the subsystem make an independent second order system as shown below:

$$\dot{y}_1 = -by_2 \; ; \; \dot{y}_2 = by_1 \tag{23}$$

The motion is periodic:

$$y_1 = y_{1,0} \cos bt - y_{2,0} \sin bt \tag{24}$$

$$y_2 = y_{1,0} \sin bt + y_{2,0} \cos bt$$

$$t = \frac{1}{b} (\tan^{-1} \frac{y_2}{y_1} - \tan^{-1} \frac{y_{2,0}}{y_{1,0}} + \pi k) \tag{25}$$

The integer k in the above equations is the number of complete oscillations since the initial time.

If the eigenvalues are complex $a \pm ib$ with a, b$\neq$ 0, then the subsystem shows both exponential and periodic behaviors. In eq. (21) Exponential behavior is associated with the magnitude of the states y_1 and y_2:

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$$u = \sqrt{y_1{}^2 + y_2{}^2} \tag{26}$$

$$\dot{u} = a\,u \ ; u = e^{at}$$

$$t = \frac{1}{a}(ln|u| - ln|u_0|)$$

The periodic behavior is associated with the phase solution in the y1, y2 plane:

$$y_1 = e^{at}(y_{1,0}\cos bt - y_{2,0}\sin bt) \tag{27}$$

$$y_2 = e^{at}(y_{1,0}\sin bt + y_{2,0}\cos bt\,)$$

$$t = \frac{1}{b}(\tan^{-1}\tfrac{y_2}{y_1} - \tan^{-1}\tfrac{y_{2,0}}{y_{1,0}} + \pi k)$$

Again the integer k is the number of complete oscillations since the initial time. Both

equations (26) and (27) are applicable, but to avoid integer ambiguity, it's easier to use

eq. (26).

Comparison integrals can be constructed for pairs of subsystems. With

appropriate selection of the equations (22), (23), and (26), the comparison integral can

be found by setting them equal to each other depending on the sub-systems. That can be

done by separating the current and initial values of the states. As an example, take a third

order system that has a nonzero real eigenvalue and a pair of imaginary eigenvalues:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{28}$$

Equations (22) and (24) reveal the following:

$$\frac{1}{\lambda}(ln|y_1| - ln|y_{1,0}|) \quad = t = \frac{1}{b}(\tan^{-1}\tfrac{y_3}{y_2} - \tan^{-1}\tfrac{y_{3,0}}{y_{2,0}} + \pi k) \tag{29}$$

$$\frac{b}{\lambda}\ ln|y_1| - \tan^{-1}\frac{y_3}{y_2} = \frac{b}{\lambda}ln|y_{1,0}| - \tan^{-1}\frac{y_{3,0}}{y_{2,0}} + \pi k \tag{30}$$

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So the comparison integral is:

$$\psi = \ tan\ [\tfrac{b}{\lambda} \ \ ln|y_1| - tan^{-1} \tfrac{y_3}{y_2} \ ] \qquad (31)$$

1.2.3   Incommensurate Oscillators

Dealing with two blocks; two pairs of imaginary eigenvalues $\pm ib_1$ and $\pm ib_2$

depends on the value of the ratio b_1/b_2. If the value of the ratio is rational there is a

comparison integral, but if the value of the ratio is irrational there is no comparison

integral and it can't be formed. As an example, consider a fourth order system of two

pairs of imaginary eigenvalues $\pm ib_1$ and $\pm ib_2$:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & -b_1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2 \\ 0 & 0 & b_2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad (32)$$

Eq. (26) shows the following:

$$t = \frac{1}{b_1}(tan^{-1} \tfrac{y_2}{y_1} - tan^{-1} \tfrac{y_{2,0}}{y_{1,0}} + \pi k_1) \quad = t = \frac{1}{b_2}(tan^{-1} \tfrac{y_4}{y_3} - tan^{-1} \tfrac{y_{4,0}}{y_{3,0}} + \pi k_2) \qquad (33)$$

$$b_2 \tan^{-1} \tfrac{y_2}{y_1} - b_1 \tan^{-1} \tfrac{y_4}{y_3} = b_2 tan^{-1} \tfrac{y_{2,0}}{y_{1,0}} - b_1 tan^{-1} \tfrac{y_{4,0}}{y_{3,0}} - \pi(b_2 k_1 - b_1 k_2) \qquad (34)$$

The right hand side of eq. (34) is not constant since k_1 and k_2 increase in integer

values. By applying a doubly-periodic function to both sides of eq. (34) using periods of

πb_2 and πb_1, a time independent integral can be created. If the ratio b_1/b_2 is rational,

then the frequencies can be written as $b_1 = a\, m_1$ and $b_2 = a\, m_2$, where m_1, m_2 are integers, and the comparison integral can be formed :

$$\psi = \ tan(\, m_2 \tan^{-1} \frac{y_2}{y_1} - m_1 \tan^{-1} \frac{y_3}{y_4}\,) \qquad (35)$$

If the ratio b_1/b_2 is irrational, global comparison integral can't be formed [1].

2. CARLEMAN LINEARIZATION AND EMBEDDING

The systems that we are dealing with are nonlinear dynamic systems, but the methods presented earlier for getting the motion constants are for linear autonomous systems. This is why we need to linearize the nonlinear system to a linear system using a method called Carleman Linearization.

The idea of Carleman is carried out by taking any nonlinear dynamic system and linearizing it. This is done by defining new linear states. Then by taking the derivative of each one, the system ends up being $\dot{X} = AX$. New states are introduced each time, resulting in an infinite number of states [4]. To summarize, Carleman is a technique used to transform a finite dimensional nonlinear system to an infinite set of linear equations. An example is presented below:

$$\dot{x} = -x + x^2 \tag{36}$$

Here we introduce the new states: $= x_1$; $\quad x_2 = \dot{x}_1 = \dot{x}$:

$$x_{(ij)} = x_1^i x_2^j \quad i,j = 1,\dots\dots,\infty \tag{37}$$

$$\dot{x}_{(ij)} = i\, x_{(i-1\ j+1)} - j x_{(i+1\ j-1)} - j\, x_{(i+2\ j-1)}$$

$$\dot{x}_{10} = \dot{x}_1 = x_{01} = x_2$$

$$\dot{x}_{01} = \dot{x}_2 = -x_{10} - x_{20} = -x_1 - x_1{}^2$$

$$\dot{x}_{11} = \dot{x}_3 = x_{01} - x_{10} - x_{20} = x_2 - x_1 - x_1{}^2$$

This process continues and will result in an infinite linear system of this form:

$$\dot{X} = AX. \tag{38}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}$$

2.1 Carleman Embedding

As discussed earlier, motion constants can give a great deal of information about the states of the dynamic system. For a dynamic nonlinear system, Carleman Linearization will result in an infinite number of linear autonomous equations. Finding the motion constants of the linear representation is easier and known as discussed earlier. In this case, an nth order system will produce an infinite order system that will result in infinity-1 motion constants, instead of n-1 real motion constants, which are the result of the true system. The nonlinear system was transferred to an infinite collection of linear autonomous systems by mapping.

We hope to find the motion constants of the real nonlinear systems through motion constants of the Carleman representation, since we don't know where the real motion constant of the real nonlinear system is in our new representation $\dot{X} = AX$. Truncating the A matrix into a certain square matrix value would be a useful way to see if the n-1 motion constants of the true system can be found up to a certain level of truncation of matrix A . Depending on the nonlinearity of the original system, the truncation of the A matrix could go higher or lower. The characteristics of the new

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infinite dimension linear system and the original finite dimension nonlinear system should be the same .[5]

Studies were done on some sample problems to evaluate the motion constants of the real nonlinear system using Carleman representation. The method was done similarly to the steps discussed above, the motion constants of the true system were found. Of course, the real motion constants are already known and have been found in experiments. But we are using Carleman representation to compare them with the known value and find them using a new technique. Work has been done on using Carleman Linearization which includes a study done by Drs. Hurtado and Sinclair in 2012 [5] on a tumbling rigid body with quadratic nonlinear terms.

3. PREVIOUS WORK

3.1 Tumbling Rigid Body With Quadratic Nonlinear Terms

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The torque-free rotational dynamics of a rigid body is used as an example. Here, for a rigid body of a third order system that has quadratic nonlinear terms, motion constants were derived and found using Carleman embedding. [5]

It's a classic nonlinear autonomous system with the equation:

$$\dot{\omega} = f(\omega) \rightarrow \quad \dot{\omega}_1 = a_1\omega_2\omega_3 \;;\; \dot{\omega}_2 = a_2\omega_1\omega_3 \;;\; \dot{\omega}_3 = a_3\omega_1\omega_2 \tag{39}$$

Where, ω is the body-fixed components of the angular velocity vector and a_k is the rigid body inertia parameters:

$$a_1 = (I_2 - I_3)/ I_1; \; a_2 = (I_3 - I_1;)/ I_2 \;;\; a_3 = (I_1 - I_2)/ I_3 \tag{40}$$

The system above has two time-independent integrals (motion constants), which are the rotational kinetic energy and the magnitude of the angular momentum:

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 \;;\; h^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \tag{41}$$

To transfer the three-dimensional nonlinear system to an infinite dimensional, over parameterized, the definition of new coordinates is introduced.

Here to linearize using Carleman representation, a pattern is found in which each nonlinear coordinate is mapped into a linear coordinate:

$$x_{(ijk)} = \omega_1^i \omega_2^j \omega_3^k \quad i,j,k = 1, \ldots \ldots, \infty \tag{42}$$

Based on the definition above, there are three x coordinates that have linear ω states; and six x coordinates that have quadratic ω states in them, and ten coordinates that are cubic ω states. So we have $(p + 1)(p + 2)/2$ coordinates of x that have the order p in the ω states.

Below is the system of linear autonomous equations for the new coordinates:

$$\dot{x}_{(ijk)} = i\, a_1 x_{(i-1\ j+1\ k+1)} + j\, a_2 x_{(i+1\ j-1\ k+1)} + k\, a_1 x_{(i+1\ j+1\ k-1)} \qquad (43)$$

The steps for finding and discovering the two motion constants for the original nonlinear system are: [5]

1.   Truncate the length of the new coordinates at some number, for example n. The truncated states are given as $\boldsymbol{x}_*$.

2.   Take the upper $n \times n$ block of A and call the truncated state matrix A_*.

3.   Jordan canonical form can be created using $A_* = \text{MJM}^{-1}$ where M and J are the eigenvectors and eigenvalues of A_* respectively. J has eigenvalues on the diagonal and sometimes it has ones on the sub-diagonal.

4.   Jordan coordinates are defined by $\mathbf{Y} = \text{M}^{-1}\mathbf{x_*}$ which are governed by $\dot{\mathbf{Y}} = J\,\mathbf{Y}$.

5.   Determine the motion constants using the methods discussed earlier in (integrals of linear autonomous dynamic systems section) $\psi = \psi(y)$.

6.   After getting the motion constants (which will be in terms of $\mathbf{Y}$ coordinates) use $\mathbf{Y} = \text{M}^{-1}\mathbf{x_*}$ to get the motion constants in terms of the truncated linear states $\mathbf{x_*}$.

7.   Then use $x = x\,(\omega) \approx \mathbf{x_*}$ write the motion constants in terms of the original coordinates ω using the original relationship $x = x(\omega) \approx \mathbf{x_*}$.

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To facilitate the computations, certain inertia values are used; in the work done in the paper a tri-inertial case was selected $I_1 = 1$, $I_2 = 2$, and $I_3 = 3$. Whatever is the truncation limit, an eigen analysis of the matrix $A*$ reveals that all of its eigenvalues equal zero. Some of the zero eigenvalues create an independent unique eigenvectors, however others generate an eigenvector that is linearly dependent on the others.

Sinclair and Hurtado (2013) [1] has shown that a 1×1 Jordan block with a zero eigenvalue creates a motion constant which is equal to the Jordan coordinate (state). After that using $\mathbf{Y} = M^{-1} \mathbf{x}*$ for each of these special Jordan coordinates expresses the motion constant in terms of the truncated coordinates $\mathbf{x}*$.

In the example being discussed, using these special Jordan coordinates, the true motion constants of the original nonlinear system are what we are aiming for. So in this example, to get the nonlinearity up to the quadratic terms in the ω states a truncation is done to the linear representation to the coordinates of x. This results in a nine element vector $\mathbf{x}*$: x_{*100}, x_{*010}, x_{*001}, x_{*110}, x_{*101}, x_{*011}, x_{*200}, x_{*020}, x_{*002}

A 9×9 matrix $A*$ results from the truncation above. Noticing that it has three 1×1 Jordan blocks; each one of them has a zero eigenvalue. The Jordan form is arranged (J and M) such that y_1, y_2, and y_3 are the special Jordan coordinates that remain constant at all times. These three Jordan coordinates are called motion constants, and using M as discussed earlier these special coordinates or motion constants can be mapped back to be in terms of $\mathbf{x}*$. After that, using eq.(42), which is the Carleman coordinate definitions, the motion constants are transferred to be in terms of ω:

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$$\psi_1 = y_1 = -x_{*101} + x_{*200} = -\omega_1\omega_3 + \omega_1^2 \tag{44}$$

$$\psi_2 = y_2 = -x_{*200} + x_{*020} = -\omega_1^2 + \omega_2^2 \tag{45}$$

$$\psi_3 = y_3 = -x_{*020} + x_{*002} = -\omega_1^2 + \omega_2^2 \tag{46}$$

Usually computing the derivative of a constant results with a zero, but here after computing the derivatives of the above equations, the answers aren't zeros. So this means that these expressions in ω are not the motion constants for the true nonlinear system; this truncation level doesn't capture the real motion constant, and taking the truncation into a higher level is needed. Also at this truncation level none of the motion constants that we calculated have the constraints that are in the over parameterized Carleman coordinate definition in eq. (42), that can be read such as:

$$x_{(ijk)} = x_{100}^i \, x_{010}^j \, x_{001}^k.$$

Truncating the linear representation of the coordinates x until the fifth order (quintic) in the ω states results in a 55×55 matrix $A*$ that has a total of fifty-five elements for x$*$. The $A*$ consists of fourteen 1×1 Jordan blocks that each have a zero eigenvalue. y_1 through y_{14} are the special Jordan coordinates that remain constant at all times. These fourteen Jordan coordinates are called motion constants, and using M as discussed earlier these special coordinates or motion constants can be mapped back to be in terms of x$*$. After that, using eq. (42) which is the Carleman coordinate definitions, the motion constants are transferred to be in terms of ω. Two of them are expressed below:

$$\psi_1 = -x_{*020} - 3\, x_{*002} + 3x_{*004} - x_{*002} - \frac{1}{3}\, x_{*220} \tag{47}$$

$$= -\omega_2^2 - 3\, \omega_3^2 + 3\omega_3^4 - \omega_3^2 - \frac{1}{3}\omega_1^2\, \omega_2^2$$

$$\psi_2 = -x_{*200} + 3\, x_{*002} + x_{*040} + 2x_{*220} + x_{*400} \tag{48}$$

$$= -\omega_1^2 + 3\, \omega_3^2 + \omega_2^4 + 2\omega_1^2\, \omega_2^2 + \omega_2^4$$

The derivatives of the motion constants above equal zero, which confirms that the expressions that are in ω are the true motion constants of the real nonlinear system. Also, the rotational kinetic energy, T, and magnitude of angular momentum, h^2, are related to ψ_1 and ψ_2 and not independent from them. The real exact motion constants of the true nonlinear system have been found, but still at this truncation level none of the motion constants that we calculated have the constraints that are in the over parameterized Carleman coordinate definitions. This is because this truncation level neglects the effect of higher order states in the governing differential equations.

4. A COMPARISON

The Duffing Oscillator's motion constants are investigated using two linearization techniques. Motion constants are obtained using Taylor Series Linearization (the traditional way of linearizing) and Carleman Linearization. Later the results are compared to each other and to the true value of the motion constant (which is well known and has been found before). This allows us to see which method provides a closer, more accurate value to the true motion constant.

4.1 Duffing Oscillator

Duffing oscillator, named after Georg Duffing, is a second order non-linear differential equation, it is considered a periodically forced damped oscillator with a nonlinear elasticity, which can be written as [6] :

$$k\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma cos\omega t \tag{49}$$

The Duffing equation is an example of a dynamic system that shows chaotic behavior. Where the parameters are:

- δ is the damping coefficient.

- β is the stiffness coefficient.

- α is the non-linearity coefficient . If $\alpha = 0$, the Duffing equation becomes a damped and driven simple harmonic oscillator.

- γ is the amplitude of the periodic driving force. If $\gamma = 0$ we have an unforced system (system driven without force).

- ω is the frequency of the periodic driving force. [6, 7]

The Duffing equation doesn't have a symbolic exact solution, but there are other approximate solutions that can help solve it like: using Fourier series expansion, or using some numeric methods such as Euler's method and Runge-Kutta. In our study and investigation of the Duffing problem, we will study the case with no driving force and no damping where $\delta = 0$ and $= 0$. Taking $k=1$, the Duffing equation becomes:

$$\ddot{x} + Ax + Bx^3 = 0 \qquad (50)$$

where A and B are the stiffness coefficient and the non-linearity coefficient respectively.[8]

This equation has three uncoupled nonlinear representations depending on the value of A and B, because studying it gives a better view of how rigid bodies act. The three uncoupled nonlinear oscillators form a clear dynamic analog of the general torque-free motion of an arbitrary rigid body. So, the analysis of the nonlinear Duffing Oscillator can lead to the analysis of rigid body dynamics.

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The uncoupled nonlinear oscillators are three represented below:

$$\ddot{x} + A_i x + B_i x^3 = 0 \qquad i = 1, 2, 3 \tag{51}$$

Values for A_i and B_i are determined by the values of the moment of Inertia, I, kinetic energy, T, and the magnitude of the angular momentum, h^2. This results with three uncoupled nonlinear equations. Values for A_i and B_i for the three equations are usually between:

$$A_1 \gtrless 0 \quad , B_1 \geq 0 \tag{52}$$

$$A_2 \geq 0 \quad , B_2 \leq 0$$

$$A_3 \gtrless 0 \quad , B_3 \geq 0$$

From the above, we conclude that A_1 and A_3 can produce a negative spring effect, destabilizing force. B_1 and B_3 are always positive, so they produce a spring restoring force. Since B_1 and B_3 are the cubic coefficients, they will override the effect of the linear coefficients. So, the equations 1 and 3 will produce a closed phase trajectory. As for B_2, which is negative and produces a negative force, destabilizing force will also override the positive linear spring , so the phase trajectories will be open [8].

We will consider cases for equations 1 and 3 which mean $B_i \geq 0$ that gives closed phase trajectories. Since the solutions for this equation, for velocities and accelerations are periodic, and these values are only possible with closed phase trajectories, that's why we depict that closed trajectories are physically possible [8].

4.1.1 Generalized Eigenvectors

While analyzing the Duffing problem, I noticed that some repeated eigenvalues don't have unique eigenvectors; they have the same eigenvectors. In this case, we have to generate for each repeated eigenvalue, an eigenvector. This eigenvector is called a generalized eigenvector which is independent from other eigenvectors. [9]

The process of finding the generalized eigenvector starts with knowing the repeated eigenvalue that has a repeated eigenvector. Let's call the eigenvalue λ_i and the eigenvector $\boldsymbol{v_i}$. The generalized eigenvector should satisfy the equation:

$$(A - \lambda_i I)^k \boldsymbol{v_i} = 0 \tag{53}$$

k is a positive integer, which refers to the order of the eigenvector, which is also how many times the eigenvector is repeated. If the order of the eigenvector is one, this means it is the real eigenvector. Numbers larger than one would be generalized eigenvectors.

Calculating the generalized eigenvectors using another generalized eigenvector would be using this equation:

$$\boldsymbol{v_{k-1}} = (A - \lambda I)\boldsymbol{v_k} \tag{54}$$

$\boldsymbol{v_k}$ is an eigenvector with index or order k associated to the eigenvalue $\boldsymbol{\lambda}$. And k is the order of the eigenvector starts from 1 and ends at k.

Claiming that $\boldsymbol{v_{k-1}}$ is a generalized eigenvector of index k-1 associated to the eigenvalue λ. [10]

Finding the generalized eigenvectors will change the J for the repeated eigenvalues. There will be a one in the super-diagonal between the repeated eigenvalues.

A= M J M^{-1}, M are the eigenvectors , J are the eigenvalues on the diagonal, so for repeated eigenvalues after finding the generalized eigenvectors, the J matrix would be a little different. For an example for a 2x2 matrix the J matrix would look:

$$J=\begin{bmatrix} \lambda1 & 1 \\ 0 & \lambda2 \end{bmatrix} \tag{55}$$

M would be the generalized eigenvectors as columns.

In our case while studying the motion constants, the J that we are using would be slightly different as mentioned earlier in the time independent integrals section. Thus, the J that is used in calculating the motion constant will have the ones for the repeated eigenvalues in the sub-diagonal. For a 2x2 matrix example the J matrix would be:

$$J=\begin{bmatrix} \lambda1 & 0 \\ 1 & \lambda2 \end{bmatrix} \tag{56}$$

M stays the same as before; M would be the generalized eigenvectors as columns.

4.2 Linearizing Using Carleman Linearization

Our Duffing Equation that is being considered is the one that is associated with the closed phase trajectories. Using Carleman embedding, motion constants are calculated. Let's start by taking a look at the main form of the Duffing Oscillator equations that we are considering:

$$\ddot{x} + A_i x + B_i x^3 = 0 \quad \text{Where} \quad A_i \underset{<}{\overset{>}{}} 0 \ , \ B_i \geq 0 \tag{57}$$

The system above has one time-independent integral, which involves kinetic and potential energy, and it's well known:

$$\psi = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{B}{4}x_1^4 \tag{58}$$

$$x = x_1 \; ; \quad x_2 = \dot{x}_1 = \dot{x}$$

Here to linearize using Carleman representation, a pattern is found, in which each nonlinear coordinate is mapped into a linear coordinate. Taking the case A=1 and B= ε ; our Duffing equation becomes:

$$\ddot{x} + x + \varepsilon x^3 = 0 \tag{59}$$

$$x = x_1 \; ; \quad x_2 = \dot{x}_1 = \dot{x}$$

$$\pmb{x}_{(ij)} = x_1^i x_2^j \quad \text{i,j} = 1,\dots\dots,\infty \tag{60}$$

Based on the definition above, there are two $\pmb{x}_{(ij)}$ coordinates that have linear x states, and three $\pmb{x}_{(ij)}$ coordinates that have quadratic x states in them, and four $\pmb{x}_{(ij)}$ coordinates that are cubic x states. This indicates that we have (p + 1) coordinates of $\pmb{x}_{(ij)}$ that have the order p in the x states. The table below elaborates on how many coordinates are needed:

Order of x	Number of new $\pmb{x}_{(ij)}$	Total Coordinates
1	2	2
2	3	5
3	4	9
4	5	14
5	6	20
6	7	27
7	8	35

Table 1: Total coordinates needed.

Linear in x $\longrightarrow x_{10}, x_{01}$

Quadratic in x $\longrightarrow x_{11}, x_{02}, x_{20}$

Cubic in x $\longrightarrow x_{12}, x_{21}, x_{30}x_{03},$

Quart in x $\longrightarrow x_{13}, x_{22}, x_{31}, x_{40}, x_{04}$

Below is the system of linear autonomous equations for the new coordinates:

$$\dot{x}_{(ij)} = i\, x_{(i-1\ j+1)} - j x_{(i+1\ j-1)} - \varepsilon\, j\, x_{(i+3\ j-1)} \tag{61}$$

The steps for finding and discovering the motion constant for the original nonlinear system are:

1. Truncate the length of the new coordinates at some number, for example n. The truncated states are given as $\boldsymbol{x}_*$.

2. Take the upper $n \times n$ block of A and call the truncated state matrix A_*.

3. Create Jordan canonical form using $A_* = \mathrm{MJM}^{-1}$ where M and J are the eigenvectors and eigenvalues of A_* respectively. J has eigenvalues on the diagonal and sometimes it has ones on the sub-diagonal.

4. Define Jordan coordinates by $\mathbf{Y} = \mathrm{M}^{-1}\mathbf{x}*$ which are governed by $\dot{\mathbf{Y}} = \mathrm{J}\mathbf{Y}$.

5. Determine the motion constants using the methods discussed earlier (in integrals of linear autonomous dynamic systems section) $\psi = \psi(y)$.

6. After getting the motion constants, which will be in terms of Y coordinates, use $\mathbf{Y} = \mathrm{M}^{-1}\mathbf{x}*$ to get the motion constants in terms of the truncated linear states $\mathbf{x}*$.

7. Then use $x_{(ij)} = x_1^i x_2^j$ to write the motion constants in terms of the original coordinates x.

When any truncation is done, an eigen analysis of the matrix $A*$ reveals that some of its eigenvalues equal zero, and others are complex conjugate eigenvalues. Some of the zero eigenvalues create independent, unique eigenvectors; however, others generate an eigenvector that is linearly dependent on the others.

Using different truncation levels (starting from linear up to the third order) didn't capture the real known motion constant of the system. Taking the truncation level up to the fourth order, which produces 14 coordinates, the $A*$ matrix becomes a 14x14 matrix. At this level the real motion constant was captured.

The eigen analysis for 14x14 $A*$ matrix, as mentioned before, reveals that some of its eigenvalues are zeros. We get two unique zero eigenvalues. Afterwards, using $\mathbf{Y} = \mathbf{M}^{-1}\mathbf{x}*$ for each of these special Jordan coordinates expresses the motion constant in terms of the truncated coordinates $\mathbf{x}*$.

Using these special Jordan coordinates, we are seeking the true motion constant of the original nonlinear system. In this example, to get the nonlinearity up to the 4th order terms in the x states, a truncation is done to the linear representation to the coordinates of x, also taking $\varepsilon = .1$. This results in a fourteen element vector $\mathbf{x}*$:

$$x_{10}, x_{01}, x_{11}, x_{02}, x_{20}, x_{12}, x_{21}, x_{30}, x_{03}, x_{13}, x_{22}, x_{31}, x_{40}, x_{04}$$

A 14×14 matrix $A*$ results from the truncation above, noticing that it has two 1×1 Jordan blocks that each have a zero eigenvalue. Considering the 1×1 Jordan blocks; the Jordan form is arranged (J and M) such that y_7 and y_{14} are the motion constants. Using M as discussed earlier, these special coordinates or motion constants can be mapped

back in terms of **x∗**, and afterwards, using eq. (60) which is the Carleman coordinate

definitions, the motion constants are transferred in terms of x:

$$\psi_1 = y_7 = -8.6022\,x_{*20} - 8.6022\,x_{*02} - .5806\,x_{*22} - .7204\,x_{*40} \qquad (62)$$

$$- .2903\,x_{*04}$$

$$= -8.6022\,x_1^2 - 8.6022\,x_2^2 - .5806\,x_1^2 x_2^2 - .7204\,x_1^4 - .2903\,x_2^4$$

$$\psi_2 = y_{14} = 6.6667 x_{*20} + 6.6667\,x_{*02} + 2\,x_{*22} + 1.333\,x_{*40} + x_{*04} \qquad (63)$$

$$= 6.6667\,x_1^2 + 6.6667\,x_2^2 + 2\,x_1^2 x_2^2 + 1.333\,x_1^4 + x_2^4$$

Usually computing the derivative of a constant results with a zero, computing the

derivatives of the above equations, the answers are not zeros. But noticing that adding a

ratio of ψ_1 to ψ_2, which will make some states cancel out, and the result of this addition

would be a ratio of the true motion constant. This means that we can find the true motion

constant, not only by taking the derivative of Carleman motion constant, but also by

adding Carleman motion constants:

$$\psi_3 = \alpha\psi_1 + \beta\psi_2 \qquad (64)$$

Then taking the derivative of ψ_3, which results with a zero. So, the true motion constant

of the real nonlinear system can be found also by taking the derivative of a combination

of two Carleman motion constants or more:

$$\dot{\psi}_3 = \alpha\,\dot{\psi}_1 + \beta\dot{\psi}_2 = zero \qquad (65)$$

Consequently, there are an α and a β that will make $\dot{\psi}_1$ and $\dot{\psi}_2$ when added together

zero.   Here in the Duffing Oscillator problem the above method was used. If we take the

derivatives ψ_1 and ψ_2 that we get from the 14x14 A matrix eq. (62) and eq.(63), then

the result is:

$$\dot\psi_1 = -8.6022 \; \dot{x}_{*20} - 8.6022 \; \dot{x}_{*02} - .5806 \; \dot{x}_{*22} - .7204 \; \dot{x}_{*40} \tag{66}$$

$$- .2903 \; \dot{x}_{*04}$$

Use eq.(60) then take the derivative in terms of x_1 and x_2 :

$$\dot\psi_1 = -8.6022 \, (2x_1 x_2) - 8.6022\big(2x_2(-x_1 - \varepsilon x_1{}^3)\big) - .5806 \, (2x_1 x_2{}^3) \tag{67}$$

$$- .5806 * 2x_1{}^2 x_2(-x_1 - \varepsilon x_1{}^3) - .7204(4x_1{}^3 x_2)$$

$$- .2903(4 \, x_2{}^3(-x_1 - \varepsilon x_1{}^3))$$

Taking $\varepsilon = .1$ for this case, gathering terms, and taking each state by itself:

$$\dot\psi_1 = (-17.2044 + 17.2044)x_1 x_2 + (1.72044 - 2.8816 + 1.1612)x_1{}^3 x_2 \tag{68}$$

$$+ (-1.1612 + 1.1612)x_1 x_2{}^3 + .11612 \, x_1{}^5 x_2$$

$$+ .11612 x_1{}^3 x_2{}^3$$

All terms cancel except:

$$\dot\psi_1 = .11612 \, x_1{}^5 x_2 + .11612 x_1{}^3 x_2{}^3 \tag{69}$$

Doing the same procedure for ψ_2:

$$\dot\psi_2 = 6.6667 \; \dot{x}_{*20} + 6.6667 \; \dot{x}_{*02} + 2 \, \dot{x}_{*22} + 1.3333 \; \dot{x}_{*40} + \dot{x}_{*04} \tag{70}$$

All terms cancel except:

$$\dot\psi_2 = -.4 \, x_1{}^5 x_2 - .4 \, x_1{}^3 x_2{}^3 \tag{71}$$

Here adding $\dot\psi_1$ and $\dot\psi_2$ together, we notice that there are an α and a β that makes:

$$\dot\psi_3 = \alpha \, \dot\psi_1 + \beta \dot\psi_2 = zero$$

Noticing the relationship between $\dot\psi_1$ and $\dot\psi_2$ $\frac{.4}{.11612} = 3.4443$

Multiply $\dot\psi_1$ by $\alpha = 3.4443$ and $\dot\psi_2$ by $\beta = 1$ so we have:

$$\dot{\psi}_3 = \alpha \, \dot{\psi}_1 + \beta \dot{\psi}_2 = (.4 - .4)x_1{}^5 x_2 + (.4 - .4)\, x_1{}^3 x_2{}^3 = \text{zero}. \tag{72}$$

To see what the motion constant is, eq. (62) and eq. (63) are examined:

$$\psi_1 \;=\; y_7 \;=\; -8.6022\, x_{*20} - 8.6022\, x_{*02} - .5806\, x_{*22} - .7204\, x_{*40} - .2903\, x_{*04}$$

$$\psi_2 \;=\; y_{14} = 6.6667 x_{*20} + 6.6667\, x_{*02} + 2\, x_{*22} + 1.333\, x_{*40} + x_{*04}$$

Noticing that $\dfrac{2}{.5806} = \dfrac{1}{.2903} = 3.4443 = \alpha$

Multiply ψ_1 by 3.4443 so we have:

$$\psi_1 \times 3.4443 = \psi_{1*} = -29.6320\, x_{*4} - 29.6320\, x_{*5} - 2\, x_{*11} - 2.4816\, x_{*13} - \tag{73}$$

$$x_{14}$$

Adding $\psi_{1*} + \psi_2 = \psi_3$

$$\psi_3 = -22.9630\, x_{*4} - 22.9630\, x_{*5} - 1.1481\, x_{*13} \tag{74}$$

$$\psi_3 = \; -22.9630\, x_1^2 - 22.9630\, x_2^2 - 1.1481\, x_1^4$$

Looking at ψ_3 here, we find out that it is a ratio of the true motion constant of the real nonlinear system.

Simplifying eq. (74) further we end up with:

$$\psi_{3*} = \; \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{.1}{4}x_1^4 \tag{75}$$

Comparing this result to the true motion constant, we find out that they are exactly the same when ε=B=.1. Same results, meaning that the exact motion constant was found, even if we change the values of $\varepsilon = B_i$ and A_i in the main Duffing oscillator that we are studying, $\ddot{x} + A_i x + B_i x^3 = 0$, where $A_i \gtrless 0$, $B_i \geq 0$. These calculations and equations prove that the true motion constant can be found, also here we notice that a combination of ψ_1 and ψ_2 would result in the same exact true motion constant in them.

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A code was developed for any form of the Duffing Oscillator $\ddot{x} + A_i x + B_i x^3 = 0$, where $A_i \underset{<}{\overset{>}{}} 0$, $B_i \geq 0$, in which M^{-1} is calculated, so $\mathbf{Y} = M^{-1} \mathbf{x}*$ is easily known. After that the motion constant is derived by hand. In all of the cases of the Duffing Oscillator, the motion constant is at first captured at the 14th level coordinates; $A*$ is 14 x 14.

Truncating the A matrix to some coordinates, which is just before or just after the 14x14 A matrix also captures the motion constants. Looking at Table 1, we notice that 14 coordinates capture nonlinearities up to the 4th order, and 20 coordinates capture the nonlinearities up to the 5th order. So, to see if the motion constant can be found when we have 13 coordinates or 15 coordinates, which are not the full coordinates at those levels, as an example, calculations were done after truncating the A matrix into 13 and 15 coordinates. In these two trials, motion constants were captured, results and calculations for these two approaches are below:

For the13x13 A matrix, when ε=B=.1, which has only one zero eigenvalue, we have a 1x1 Jordan block at y_7:

$$\psi_1 = y7 = -6.6667\, x_{4*} - 6.6667 x_{5*} - .3333\, x_{13*} \tag{76}$$

$$\psi_1 = -6.6667\, x_1^2 - 6.6667\, x_2^2 - .3333\, x_1^4$$

Since we already know what the true motion constant is, we notice that ψ_1 has the same states as the true motion constant, doing calculations and simplifying ψ_1, so the result is:

$$\psi_1 = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{.1}{4}x_1^4 \qquad (77)$$

The result above is exactly the same as the true motion constant of the real nonlinear

system.

For the15x15 A matrix, when ε=B=.1, which has only three zero eigenvalues,

we have three 1x1 Jordan block at y_7, y_{14} and y_{15}. Since we already know what the true

motion constant is, the approach using $\psi_N = \alpha\psi_1 + \beta\psi_2$ is used to see if ψ_N is a ratio of

the true motion constant like the 14x14 case, but here we have an extra $\psi_3 = y_{15}=x_{*15}$,

so this forces us to add ψ_3, and take it into consideration so eq. (64) changes to:

$$\psi_N = \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3 \qquad (78)$$

$$\psi_1 = y_7 = -8.6022\, x_{*20} - 8.6022\, x_{*02} - .5806\, x_{*22} - .7204\, x_{*40} - .2903\, x_{*04}$$

$$\psi_2 = y_{14} = 6.6667 x_{*20} + 6.6667\, x_{*02} + 2\, x_{*22} + 1.333\, x_{*40} + x_{*04}$$

$$\text{Noticing that } \frac{2}{.5806} = \frac{1}{.2903} = 3.4443 = \alpha$$

$$\psi_3 = y_{15}=x_{*15}$$

Multiply ψ_1 by $\alpha = 3.4443$ and $\beta = 1$ so we have:

$$\psi_1 \times 3.4443 = \psi_{1*} = -29.6320\, x_{*4} - 29.6320\, x_{*5} - 2\, x_{*11} - \qquad (79)$$

$$2.4816\, x_{*13} - x_{14}$$

Noticing that the real motion constant does not have x_{*15}, so γ must be equal to zero.

Adding them together: $\psi_N = \alpha\psi_1 + \beta\psi_2 + \gamma\psi_3$

$$\psi_N = -22.9630 \; x_{*4} - 22.9630 \; x_{*5} - 1.1481 \; x_{*13} \tag{80}$$

$$\psi_N = -22.9630 \; x_1^2 - 22.9630 \; x_2^2 - 1.1481 \; x_1^4$$

Simplifying eq. (80) further we end up with ψ_N equals the exact value of the true motion

constant of the nonlinear system when $\varepsilon = B_i = .1$ and $A_i = 1$.

$$\psi_N = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{.1}{4} x_1^4 \tag{81}$$

4.3 Linearizing Using Taylor Series

Taylor series expansion is the most general known way of linearizing, where an

equilibrium point is calculated and then the linearization is done around the equilibrium

point. For a function $f(x)$ and equilibrium point a, the Taylor series would be :

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \tag{82}$$

where $n!$ is the factorial of n and $f^{(n)}(a)$ is the nth derivative of f evaluated at the

point a. This power series is also called a Maclaurin series.

So f(x) at point a would be:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \tag{83}$$

For an example let's take the taylor expansion for e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \qquad -\infty < x < \infty \tag{84}$$

For our Duffing equation , Taylor Linearization for the case $A_i = 1$ using only first order terms for simplicity would be as discussed below:

$$\ddot{x} + x + \varepsilon x^3 = 0 \tag{85}$$

$$x = x_1 ; \quad x_2 = \dot{x}_1 = \dot{x}$$

$$\dot{x}_2 = -x_1 - \varepsilon x_1^3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon x_1^3 \end{bmatrix}$$

$$\dot{X} = f(x) \tag{86}$$

Finding the equilibrium point:

$$f(x) = 0 \implies \dot{x} = 0 \qquad\qquad \dot{x}_1 = \dot{x}_2 = 0 \tag{87}$$

$$-x_1 - \varepsilon x_1^3 = 0 \implies x(1 + \varepsilon x^2) = 0$$

So $x^2 = \frac{-1}{\varepsilon}$ not a real number $\varepsilon \geq 0$; so $x = 0 = x_1$

Since $x_1 = x_2 = 0$; the equilibrium point is (0,0).

$$x = x_* + \delta \quad ; \quad x_* = 0 \text{ equilibrium point } ; \ \delta \text{ is a small} \tag{88}$$

$$\text{perturbation.}$$

$$\dot{x} = \dot{\delta} ; \ \ddot{x} = \ddot{\delta} \tag{89}$$

Taylor series $\frac{df}{dx} \Big\downarrow_{x_*} . \delta + H.O.T$:

$$\ddot{\delta} + (x_* + \delta) + \varepsilon \overbrace{(x_* + \delta)}^{\substack{=0 \\ 3}} = 0 \tag{90}$$

$$\ddot{\delta} + \delta = 0$$

$$x_1 = \delta \qquad\qquad\qquad \dot{x}_1 = x_2 \tag{91}$$

$$\dot{x}_1 = \dot{\delta} = x_2 \qquad\qquad\qquad \dot{x}_2 = -x_1$$

$$\dot{x}_2 = \ddot{\delta} = -x_1$$

So this results in: $\dot{X} = Ax$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{92}$$

$$A = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

Characteristic equation: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm 1i$ $\tag{93}$

Using the results above; attempting to find the motion constant for the linear system that

has a complex eigenvalue as discussed earlier:

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \qquad\qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{94}$$

$$A = P\,R\,P^{-1}$$

$$\psi = y_1{}^2 + y_2{}^2 \tag{95}$$

$$\text{Using } Y = M^{-1}X \tag{96}$$

$$y_1 = x_1; \; y_2 = x_2 \tag{97}$$

$$\psi = x_1{}^2 + x_2{}^2 \tag{98}$$

From the above, we notice that the motion constant obtained from Taylor

Linearization, does not capture the nonlinearities up to the fourth order that appear in the

true motion constant for the real system. However, the motion constant obtained from Carleman had the higher order nonlinear terms that appear in the real motion constant.

Below are plots of the motion constants for the cases obtained from Carleman Linearization, Taylor Linearization, and the true motion constant:
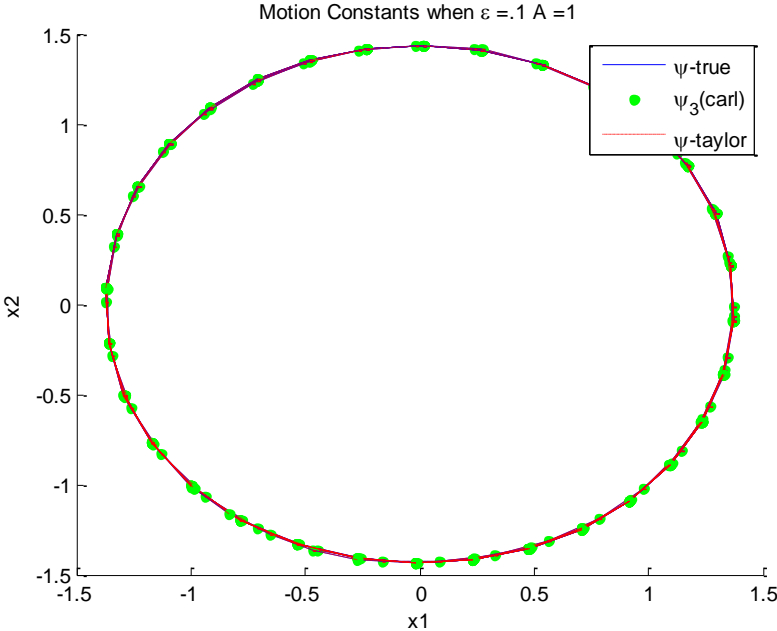

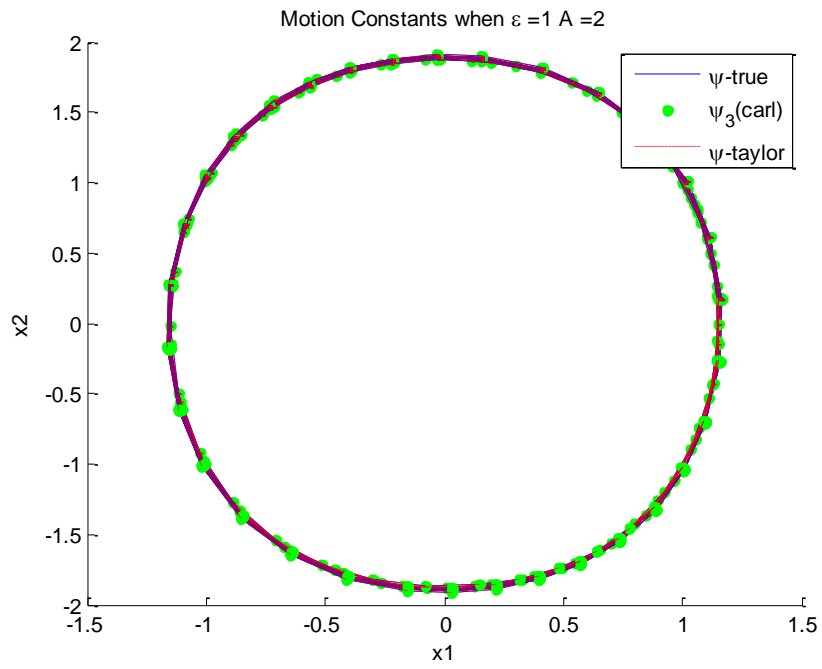Figure 1: Motion Constants When $\varepsilon=.1$ and $A=1$.
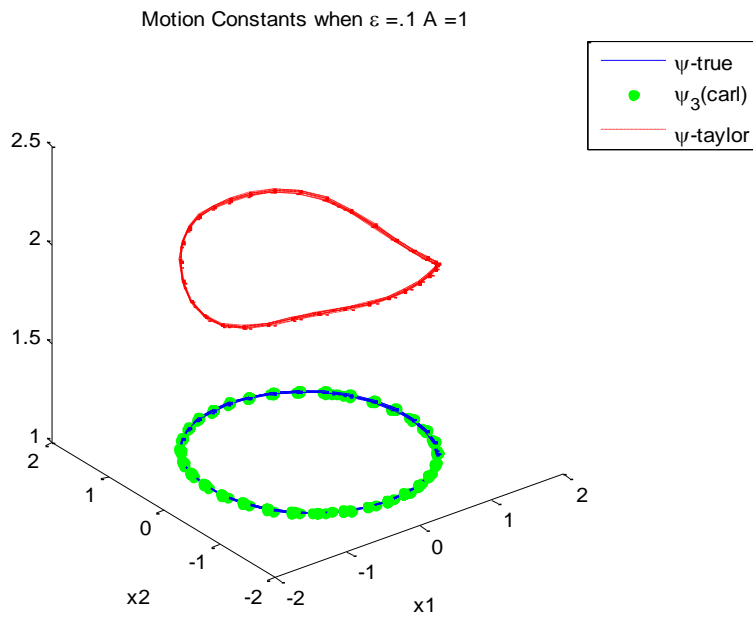
Figure 2: Motion Constants When ε=1 and **A**=2.

Figure 3: 3D Plot For Motion Constants When ε=.1 and **A** =1.

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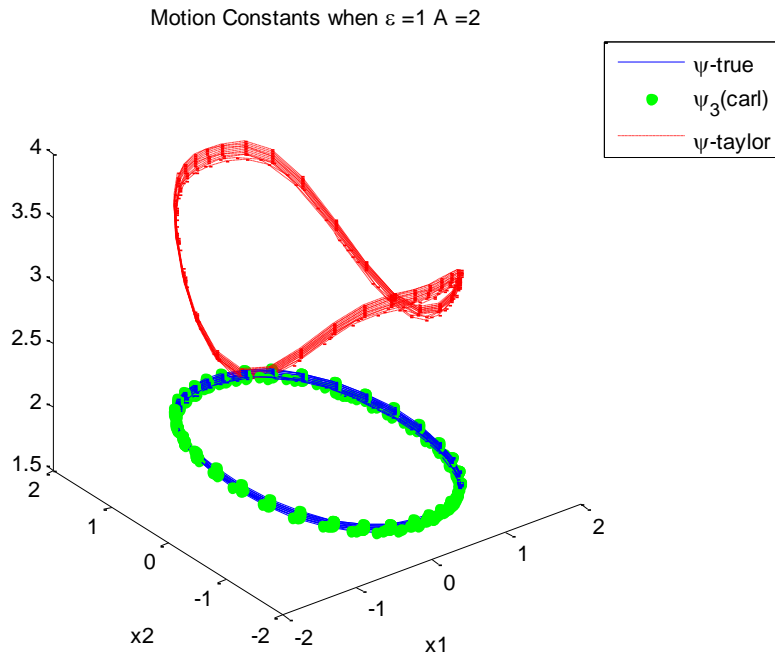Figure 4: 3D Plot For Motion Constants When ε=1 and *A* =2.

In the previous figures, all of the motion constants are plotted in the same figure for certain conditions. In Figure 1 the true motion constant is plotted along with the motion constant ψ_3 obtained from Carleman linearization and motion constant ψ obtained from Taylor Linearization when $\varepsilon = B = .1$ and $A = 1$. Where in Figure 2 the true motion constants are plotted when $\varepsilon = B = 1$ and $A = 2$. Figure 3 shows a 3D plot for all motion constants when $\varepsilon = B = .1$ and $A = 1$. Figure 4 is a 3D plot for motion constants when $\varepsilon = B = 1$ and $A = 2$. These two figures show that Carleman's motion constant is exactly the same as the true motion constant, where Taylor's motion constant is not exactly constant and it is not close to the true motion constant.
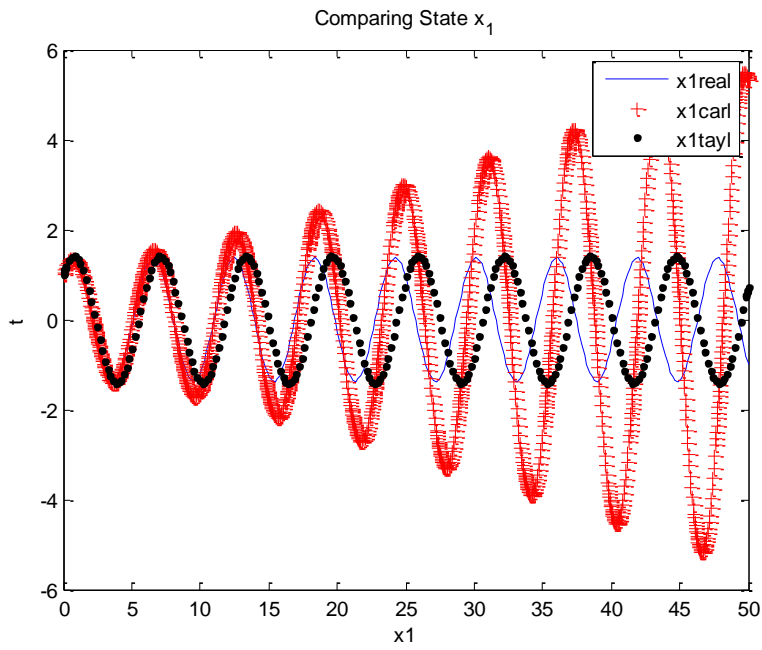
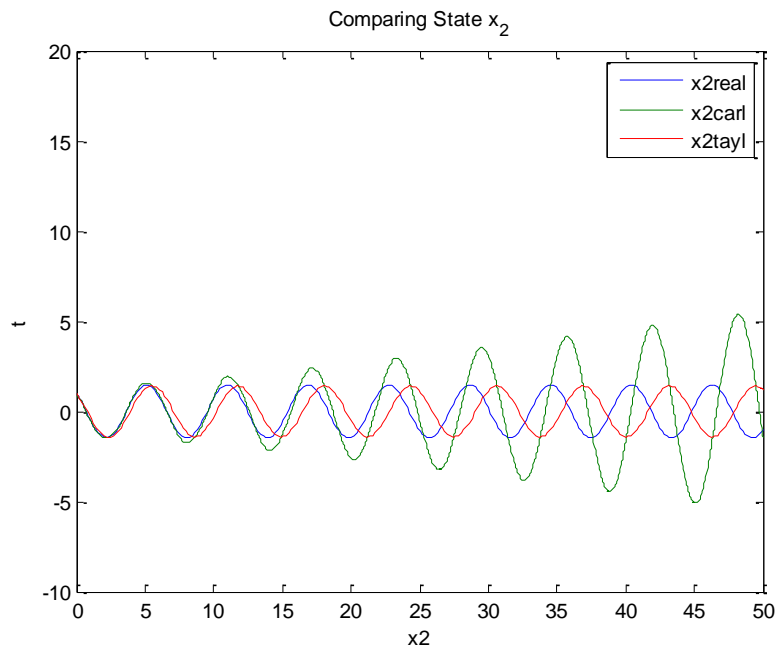Figure 5: Comparing x_1 State For The 3 Methods.

Figure 6: Comparing x_2 State For The 3 Methods.

Figures 5 and 6 compare the states x_1 and x_2 obtained from integrating three differential equations:

1) The real differential equation.

2) Carleman's differential equation.

3) Taylor's differential equation.

We notice that the states x_1 and x_2 are different for the three differential equations, that is because these are three different differential equations. Here Carleman finds the exact motion constant as the one found in the real system, even though they are two different differential equations.

5. CONCLUSIONS AND FUTURE WORK

Motion constants are time independent equations involving the system states that solve the dynamic system. They provide information about the system and properties of the motion without solving equations of motion, that's why they are useful.

For an nth order linear autonomous system, motion constants can be found. Motion constants for nth order nonlinear systems are hard to find. That's why we need to linearize the nonlinear system. If we have an nth order linear autonomous system, we can obtain up to n-1 time independent integrals (or motion constants).

Carleman Linearization is a technique that transforms a finite dimensional nonlinear system to an infinite set of linear equations. Carleman Linearization results with an infinite differential equations in the form of $\dot{X} = AX.$ The A matrix resulting from Carleman can be truncated up to a certain level as discussed in Table 1, where the real motion constants of the true nonlinear system can be found.

The Duffing Oscillator was taken as an example, because it forms a clear dynamic analog of the general torque-free motion of an arbitrary rigid body. For the Duffing Oscillator the exact motion constant was found at the 14x14 $A*$ matrix 4th order level , but it was first found at the 13 x13 $A*$ matrix .

The true motion constant of the real nonlinear system can be found, not only by taking the derivative of Carleman motion constant, but also by taking the derivative of a combination of two Carleman motion constants or more:

$$\dot{\psi}_3 = \alpha\,\dot{\psi}_1 + \beta\dot{\psi}_2 = zero \tag{99}$$

Consequently, there are an α and a β that will make $\dot{\psi}_1$ and $\dot{\psi}_2$ when added together zero.

Based on work done earlier, and the study that was done on the Duffing Oscillator, I believe that any arbitrary nonlinear dynamic system can be linearized using Carleman Linearization. From there, the real motion constants can be derived. Moreover, after Carleman Linearization is done, a MATLAB function can be developed, in which the truncated $A*$ matrix resulting from that linearization can be taken, and the function would calculate the derivative of the motion constants. If the answer is zero that means that the true motion constant was found. If the answer is a number, then the $A*$ matrix needed to be truncated to a higher level in order to find the motion constant.

REFERENCES

[1]   Sinclair, A. J., and Hurtado, J. E. "The Motion Constants of Linear Autonomous Dynamical Systems," *Applied Mechanics Reviews* Vol. 65, No. 4, 2013, pp. 041002-041002. doi: 10.1115/1.4024769

[2]   Burns, S. A., and Palmore, J. I. "The Newton Transform: an Operational Method for Constructing Integrals of Dynamical Systems," *Physica D* Vol. 37, 1989, pp. 83-90.

[3]   Perko, L. *Differential Equations and Dynamical Systems*. New York Springer, 2001.

[4]   Kowalski, K., and Steeb, W.-H. *Nonlinear Dynamical Systems and Carleman Linearization*. Singapore ; Teaneck, NJ: World Scientific 1991.

[5]   Hurtado, J.E., and Sinclair, A.J., "Using Carleman Embedding to Discover a System's Motion Constants", AAS 12-629, Jer-Nan Juang Astrodynamics Symposium, College Station, TX 24-26 June 2012, p. 8.

[6]   Kanamaru, T. "Duffing Oscillator," *Scholarpedia* Vol. 3, 2008, p. 6327. doi: doi::10.4249/scholarpedia.6327

[7]   Weisstein, Eric W. "Duffing Differential Equation." From *MathWorld*--A Wolfram WebResource. http://mathworld.wolfram.com/DuffingDifferentialEquation.html

[8]   Junkins, J. L., Jacobson, I. D., and Blanton, J. N. " A Nonlinear Oscillator

        Analog of Rigid Body Dynamics," *Celestial Mechanics* Vol. 7, No. 4, 1973, p.

        10.

[9]   Strang, G. *Introduction to Linear Algebra*. Wellesley, MA: Wellesley-Cambridge

        Press, 2009.

[10]   Weintraub, S. H. *Jordan Canonical Form: Application to Differential Equations*.

        San Rafael, CA: Morgan & Claypool Publishers 2008.