

SUPERSYMMETRIC CURVATURE SQUARED INVARIANTS IN FIVE AND  
SIX DIMENSIONS

A Dissertation

by

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## ABSTRACT

In this dissertation, we investigate the supersymmetric completion of curvature squared invariants in five and six dimensions as well as the construction of off-shell Poincaré supergravities and their matter couplings.

We use superconformal calculus in five and six dimensions, which are an off-shell formalisms. In five dimensions, there are two inequivalent Weyl multiplets: the standard Weyl multiplet and the dilaton Weyl multiplet. The main difference between these two Weyl multiplets is that the dilaton Weyl multiplet contains a graviphoton in its field content whereas the standard Weyl multiplet does not. A supergravity theory based on the standard Weyl multiplet requires coupling to an external vector multiplet.

In five dimensions, we construct two new formulations for 2-derivative off-shell Poincaré supergravity theories and present the internally gauged models.

We also construct supersymmetric completions of all curvature squared terms in five dimensional supergravity with eight supercharges. Adopting the dilaton Weyl multiplet, we construct a Weyl squared invariant, the supersymmetric combination of Gauss-Bonnet combination and the Ricci scalar squared invariant as well as all vector multiplets coupled curvature squared invariants. Since the minimal off-shell supersymmetric Riemann tensor squared invariant has been obtained before, both the minimal off-shell and the vector multiplets coupled curvature squared invariants in the dilation Weyl multiplet are complete. We also constructed an off-shell Ricci scalar squared invariant utilizing the standard Weyl multiplet. The supersymmetric Ricci scalar squared in the standard Weyl multiplet is coupled to  $n$  number of vector multiplets by construction, and it deforms the very special geometry. We found

that in the supersymmetric  $AdS_5$  vacuum, the very special geometry defined on the moduli space is modified in a simple way. We study the vacuum solutions with  $AdS_2 \times S^3$  and  $AdS_3 \times S^2$  structures. We also analyze the spectrum around a maximally supersymmetric Minkowski<sub>5</sub>, and study the magnetic string and electric black hole.

Finally, we generalize our procedure for the construction of an off-shell Ricci scalar squared invariant in five dimensions to  $\mathcal{N} = (1, 0)$ ,  $D = 6$  supergravity.

To My Family

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## 1. INTRODUCTION, MOTIVATION AND BACKGROUND\*

Extensive investigation on the quantum mechanical behaviour of gravity has led us to conclude that a consistent quantum gravity may not be understood from a classical point particle viewpoint, but the dynamical theory of one dimensional extended objects, the string theory, is needed [1] - [5]. This viewpoint has numerous advantages including

1. Strings can vibrate in space-time, and different vibrational modes of strings lead to very rich structures
2. String theory only has one meaningful parameter, the string length  $l_s$ . The string tension is given by  $T = (2\pi\alpha')^{-1}$  where  $\alpha' = l_s^2$ ,
3. Open strings can join to form closed strings, but closed strings can exist without open strings. Thus, a realistic open string theory would contain closed-string states,
4. The spectrum of the closed string contains a massless spin-2 particle, the graviton,
5. At low energies, string theory reduce to the known conventional field theories; the Yang Mills theory, and general relativity. When supersymmetry is in presence as in superstring theory, then the low energy limit is given by supergravity [6]

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\*Portions of this chapter are reprinted from *An off-shell formulation for internally gauged D=5, N=2 supergravity from superconformal methods* by Frederik Coomans and Mehmet Ozkan, 2013. JHEP **1301**, 099 (2013), Copyright 2013, with permission from SISSA; *Supersymmetric Completion of Gauss-Bonnet Combination in Five Dimensions* by Mehmet Ozkan and Yi Pang, 2013. JHEP **1303**, 158 (2013), Copyright 2013, with permission from SISSA.

Aside from Einstein supergravities, effective superstring actions also originate higher curvature terms in all orders in  $\alpha'$ , the small slope parameter. The first higher derivative term in a supergravity theory in 10 dimensions was introduced by Green and Schwarz [7] as a Lorentz - Chern - Simons 3-form for the cancellation of anomalies in Yang-Mills gauge currents and gravitational gauge currents in 10 dimensional supergravity coupled to Yang-Mills matter

$$\omega_{3L} = tr(d\omega \wedge \omega - \frac{1}{3}\omega \wedge \omega \wedge \omega). \quad (1.1)$$

Because of that anomaly cancellation mechanism, the 10 dimensional supergravity Lagrangian [7] was modified by adding (1.1) to the definition of the field strength of the two form  $B$

$$\hat{H} = dB + tr(A \wedge F - \frac{1}{3}A \wedge A \wedge A) - tr(d\omega \wedge \omega - \frac{1}{3}\omega \wedge \omega \wedge \omega) \quad (1.2)$$

As this modification is desirable from the field theory side, it is highly undesirable from the supersymmetry side since it breaks the supersymmetry of the model. Also, since the addition of such a higher curvature term is lacking its supersymmetric completions., it is not clear how the modified theory is a low energy approximation of a superstring theory. It is therefore necessary to understand how to restore supersymmetry, and what the supersymmetric completion terms are. To this end, different methods from both string theory side (string amplitudes in tree level [8, 9] and at one loop level [10, 11, 12]) and supergravity side (superspace methods [13, 14, 15] and Noether procedure [16]) was employed in order to obtain the supersymmetric completion of the modified model.

This interplay between anomaly cancellation and supersymmetric completion had led to new bosonic terms in the Lagrangian, and most importantly the square of Riemann tensor [17] - [31]

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad (1.3)$$

There is, however, an obvious problem with this modification. The modified Lagrangian contains ghost particles due to addition of Riemann squared term, whereas string theory itself is ghost-free. In [32], Zweibach showed that such ghost states can be avoided when the curvature square terms take the form of Gauss-Bonnet combination

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \quad (1.4)$$

However, there are ambiguities in the argument on the inclusion of Riemann tensor square and the ghost freedom as the 10 dimensional model is on-shell, thus, one can add or remove terms proportional to Ricci tensor and Ricci scalar by a field redefinition

$$g'_{\mu\nu} = g_{\mu\nu} + aR_{\mu\nu} + b g_{\mu\nu}R. \quad (1.5)$$

Also, the terms proportional to Ricci tensor and Ricci scalar cannot be derived from the string theory side as no off-shell formulation of string theory is known. As far as the supersymmetric completion of the modified Lagrangian is concerned, both Riemann tensor squared model and the Gauss-Bonnet extended model are compatible with the modification of  $\hat{H}$ , (1.2), since neither the supersymmetric completion of Ricci tensor square nor supersymmetrization of Ricci scalar square contains such a

Chern-Simons term.

The investigation of the supersymmetric completion in the 10 dimensional model takes place up to lowest order in  $\alpha'$  as the model is on-shell. The situation becomes more interesting when exact supersymmetry is considered. In  $N = 1, D = 4$  supergravity, the full answer to the exact supersymmetric model was achieved since the off-shell formulation of supergravity is known [33, 34]. In the minimal off-shell field formulation, it was shown that in the presence of the higher derivatives, the auxiliary fields of the model do not give rise to algebraic equations, but results in differential equations instead. However, in the form of Gauss-Bonnet combination, the dynamical terms in the higher derivatives are removed and the equations become first order, thus, the theory contains no ghost in its spectrum. This fact is essential in the construction of a supersymmetric Gauss-Bonnet combination.

Similar to the 4 dimensional  $N = 1$  case, 6 dimensional  $N = (1, 0)$  [35] and 5 dimensional  $N = 2$  supergravities are also known off-shell [36, 37, 38]. Therefore, exact supersymmetric  $R^2$  supergravity models were constructed in 6 dimensions as the supersymmetric completion of Riemann tensor square [39], and 5 dimensions as the supersymmetric completion of Weyl tensor square [40].

Since the supersymmetric completion of curvature square terms requires many other bosonic terms, these supersymmetric constructions are also important to discuss compactification as they may modify the allowed vacuum solutions to the two-derivative theory. Possible modifications to maximally symmetric vacuum solutions in 5 dimensions for the Weyl tensor squared extended model, and 6 dimensions for the Riemann tensor squared extended model were investigated in [41] and [42] in the respective order.

In addition to modifications to vacuum solutions and finding the supersymmetric complete structures for higher curvature extended models, 5 and 6 dimensional  $N = 2$

models are of special interest due to various reasons.

There has been lots of interest in the study of five-dimensional  $\mathcal{N} = 2$  supergravity in past years. On one hand, the solutions in this theory have rich structures including black holes, black rings and black strings [43]-[50]. On the other hand, this theory can come from string/M theory via Calabi-Yau compactifications [51, 52] which provides a platform for a detailed comparison between the microscopic and macroscopic descriptions of black holes in string theory [53, 54]. A further compactification of the 5D theory on a circle gives rise to 4D  $\mathcal{N} = 2$  supergravity which is important for the study of string triality [55, 56].

In 6 dimensions, Weyl multiplets, Yang-Mills multiplet and linear multiplet are known off-shell formulation, therefore construction of exact curvature square invariants is possible. Also, the  $D = 6$  Yang-Mills coupled supergravity resembles the Yang-Mills coupled  $D = 10$  supergravity. Also, the six dimensional model may be the low energy limit of a heterotic string theory [57].

In this thesis, we consider the supersymmetric completion of curvature squared terms in 5 and 6 dimensional supergravities with 8 real supercharges. The rest of this thesis is organised as follows. In the next section we review the systematic approach that we use in the construction of curvature squared invariants: Superconformal tensor calculus. In Section 3, we introduce the superconformal multiplets of the  $\mathcal{N} = 2$ ,  $D = 5$  superconformal theory, which includes the Weyl multiplets of the theory as well as Yang-Mills and linear multiplets that will be used as compensating multiplets when constructing Poincaré supergravity theories.

In Section 4, we introduce the superconformal actions for the linear and the vector multiplets. The construction procedures are based on superconformal tensor calculus, and we use these superconformal actions to construct Poincaré supergravities upon gauge fixing procedure.

In Section 5 we construct different off-shell formulations of  $\mathcal{N} = 2, D = 5$  Poincaré supergravity. This is a consequence of our Weyl multiplet choices and our gauge choices. We first consider standard Weyl multiplet and introduce a superconformal supergravity for that choice by combining superconformal vector and linear multiplet actions. We then gauge fix the redundant symmetries and obtain a Poincaré supergravity for standard Weyl multiplet choice. In the final step, we introduce the R-symmetry gauging of the theory and show that our model indeed give rise to the known minimal on-shell gauged model. Upon completing our construction for the standard Weyl multiplet, we construct a superconformal supergravity for the dilaton Weyl multiplet. In this case, a superconformal theory can be achieved by only using the linear multiplet action. We then introduce two different gauge fixing conditions for the dilaton Weyl multiplet. The first choice give rise to canonical Einstein-Hilbert term, whereas the second choice leads us to a Poincaré theory that is of Brans-Dicke type. We introduce a detailed analysis of both cases, and investigate their R-symmetry gauging.

In Section 6, we introduce all curvature squared invariants of five dimensional  $N = 2$  supergravity for the dilaton Weyl multiplet and the dilaton Weyl coupled to n-number of vector multiplet choices. In order to do so, we first construct the Riemann squared invariant using the Yang-Mills coupled supergravity action, and a map between dilaton Weyl multiplet and the Yang-Mills multiplet. Then, we consider the construction of the supersymmetric completion of Gauss-Bonnet combination. The crucial observation in our construction of supersymmetric Gauss-Bonnet combination is that such a construction might be possible with only two independent curvature squared super-invariants. This observation is based on the fact that the Riemann squared invariant obtained contains an ordinary kinetic term for the auxiliary vector field  $V_\mu^{ij}$ . Thus the theory consisting of the Einstein-Hilbert action and



a Riemann squared invariant contains a dynamical massive auxiliary vector in its spectrum which forms the same multiplet with the massive graviton generated by the Riemann squared term. By counting degrees of freedom, we notice that it might always be the case (except for the pure Ricci scalar squared invariant) that when formulated in terms of dilaton Weyl multiplet, the curvature squared super-invariant includes an ordinary kinetic term for the auxiliary vector field  $V_\mu^{ij}$ . Therefore, if there exist two independent curvature squared super-invariants, a particular combination of them can be formed in which the kinetic term for the auxiliary vector vanishes. This implies that there is no massive graviton since the massive vector and massive graviton fall into the same multiplet, suggesting that the curvature squared terms comprise Gauss-Bonnet combination. Finally, we present the supersymmetric completion of Ricci scalar square in 5 dimensions, thus, completing all the off-shell curvature squared actions in five dimensions and complete our analysis on the dilaton Weyl multiplet.

We then extend our analysis to the standard Weyl multiplet and introduce an off-shell Weyl squared invariant, and construct an off-shell Ricci scalar squared invariant for that choice. We observe that an important contribution of the Ricci scalar squared invariant is that compared with the Weyl squared invariant, the supersymmetric Ricci scalar squared invariant modifies the very special geometry defined on the moduli space of  $n$  vector multiplets.

In Section 7, we investigate the vacuum solutions of the curvature squared extended Poincaré supergravity for the dilaton Weyl multiplet, and show that the spectrum of Gauss-Bonnet combination is ghost-free as expected. In Section 8, we investigate supersymmetric magnetic string and electric black hole solutions for the gauged Ricci scalar squared model for the standard Weyl multiplet.

In Section 8, we discuss the superconformal multiplets of  $D = 6, N = (1, 0)$  model, possible off-shell Poincaré supergravities and their gauging. Our construction in that section resembles the five dimensional construction. Finally, we discuss the Ricci scalar square invariant extension of the six dimensional model. We summarize in Section 10.

This thesis is based on the works [58], [59] and [60] in collaboration with Frederik Coomans and Yi Pang.

## 2. INTRODUCTION TO CONFORMAL TENSOR CALCULUS

As discussed in the previous section, there are various methods and techniques for the construction of off-shell supergravity theories. In 5 and 6 dimensions, such constructions most conveniently take place via superconformal tensor calculus [61, 62] since the superconformal multiplets are known. In this section, we explain the basic idea of the superconformal calculus as we shall use the method extensively in the rest of this thesis.

### 2.1 Conformal Symmetry and Conformal Algebra

Conformal transformations are defined as general coordinate transformations that leave the geometry invariant up to a scaling factor

$$g_{\mu\nu} \longrightarrow g'_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x). \quad (2.1)$$

When considering infinitesimal transformations, the infinitesimal replacement,  $k_\mu(x)$  satisfies the conformal Killing equation

$$\delta_{gct}(k)g_{\mu\nu} = 2\nabla_{(\mu}k_{\nu)}(x) = e^{2\omega(x)}g_{\mu\nu}(x), \quad (2.2)$$

where we have defined the covariant derivative with respect to Levi-Civita connection. If we are to restrict ourselves a D-dimensional flat geometry, the conformal Killing equation (2.2) implies

$$\partial_{(\mu}k_{\nu)}(x) - \frac{1}{D}\eta_{\mu\nu}\partial_\rho k^\rho = 0. \quad (2.3)$$

Excluding the special case  $D = 2$ , this equation has the following general solution

$$k^\mu(x) = \xi^\mu + \lambda_M^{\mu\nu} x_\nu + \lambda_D x^\mu + (x^2 \Lambda_K^\mu - 2x^\mu x \cdot \Lambda_K). \quad (2.4)$$

In the equation above, the right hand side terms describe translations, Lorentz rotations, dilatations and conformal boost in the respective order. The corresponding generators are  $P_\mu, M_{\mu\nu}, D$  and  $K$ . Therefore, a conformal transformation is expressed as

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \lambda_K^\mu K_\mu. \quad (2.5)$$

In  $D$  dimensions, we have the following simple representations for the conformal generators

$$\begin{aligned} P_\mu &= \partial_\mu, & M_{\mu\nu} &= x_{[\mu} \partial_{\nu]} \\ D &= x^\mu \partial_\mu, & K_\mu &= x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu. \end{aligned} \quad (2.6)$$

It follows that they form an  $SO(D, 2)$  algebra

$$\begin{aligned} [P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]}, & [M_{\mu\nu}, M^{\rho\sigma}] &= -2\delta_{[\mu}^{\rho} M_{\nu]}^{\sigma]}, \\ [K_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} K_{\rho]}, & [P_\mu, K_\nu] &= 2(\eta_{\mu\nu} D + 2M_{\mu\nu}), \end{aligned} \quad (2.7)$$

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad (2.8)$$

which includes the Poincaré symmetry as a subalgebra. This fact is the essential ingredient to go from a conformal model to a Poincaré model. An explicit embedding of the  $D$ -dimensional generators into  $(D+2)$ -dimensional objects satisfying  $SO(D, 2)$

can be found as

$$\begin{pmatrix} M_{\mu\nu} & \frac{1}{4}(P_\mu - K_\mu) & \frac{1}{4}(P_\mu + K_\mu) \\ -\frac{1}{4}(P_\mu - K_\mu) & 0 & -\frac{1}{2}D \\ -\frac{1}{4}(P_\mu + K_\mu) & \frac{1}{2}D & 0 \end{pmatrix} \quad (2.9)$$

In order to specify for each field,  $\phi^i$  its transformations under conformal group one has to specify the followings

- **Transformations under the Lorentz group:** The explicit form for Lorentz transformation matrices is for vectors should satisfy

$$M_{\mu\nu}{}^\rho{}_\sigma = -\delta_{[\mu}^\rho \eta_{\nu]\sigma}, \quad (2.10)$$

while for spinors,

$$M_{\mu\nu} = -\frac{1}{4}\gamma_{\mu\nu}. \quad (2.11)$$

- **The dilatational transformation:** Excluding the scalars, we have

$$k_D^i = w\phi^i, \quad (2.12)$$

where  $w$  is the Weyl weight of the field  $\phi^i$ . However, for scalars in a non-trivial manifold with affine connection  $\Gamma_{ij}{}^k$ , these are the solutions of

$$\partial_i k_D^j + \Gamma_{ik}{}^j k_D^k = w\delta_i^j \quad (2.13)$$

For  $\Gamma_{ij}{}^k = 0$ , the situation reduces to (2.12).

- **Special conformal transformations:** Possible extra parts apart from those in (2.4) connected to translations, rotations and dilatations. These are denoted as  $(k_\mu\phi)^i$ . The commutator between the special conformal transformations and dilations gives the restriction

$$(k_\mu\phi)^j\partial_j k_D^i - k_D^j\partial_j(k_\mu\phi)^i = k_\mu\phi^i \quad (2.14)$$

For the simple form of the dilatations (2.12) this means that, this means that the Weyl weight of  $k_\mu\phi^i$  should be one less than that of  $\phi^i$ .

In this way, the algebra (2.8) is realized on the fields as

$$[\delta_C(\xi_1), \delta_C(\xi_2)] = \delta_C(\xi^\mu = \xi_2^\nu\partial_\nu\xi_1^\mu - \xi_1^\nu\partial_\nu\xi_2^\mu) \quad (2.15)$$

After this brief introduction on conformal symmetry, we are now ready to proceed with superconformal tensor calculus.

## 2.2 Superconformal Tensor Calculus

The purpose of superconformal tensor calculus is to construct supersymmetric Poincaré invariant theories by using superconformal symmetries. Although such (super)conformal symmetries are not realized in nature, such extra symmetries are extremely helpful in theoretical constructions. The advantage of having extra symmetries is that those redundant symmetries restricts the system in such a way that one simply avoids all complications faced in direct off-shell constructions. The procedure ends with gauge-fixing the unwanted symmetries. The superconformal tensor calculus, therefore, runs according to the following algorithm

1. Introduce as many symmetries as possible,

2. Introduce compensating fields (multiplets) to compensate new degrees of freedom,
3. Construct superconformally invariant actions,
4. Gauge fix the redundant symmetries.

In the rest of the thesis, we will apply this algorithm to construct off-shell 2-derivative Poincaré supergravity theories as well as their off-shell supersymmetric curvature squared extensions.

### 3. SUPERCONFORMAL MULTIPLETS OF $\mathcal{N} = 2, D = 5$ SUPERGRAVITY\*

In this section, we introduce the superconformal multiplets of  $\mathcal{N} = 2, D = 5$  supergravity. In the first two subsections, we discuss the Weyl multiplets of the theory. A superconformal Weyl multiplet contains all the gauge fields of the superconformal algebra as well as proper matter fields. The latter is included in order to balance the bosonic and fermionic degrees of freedom and to implement the off-shell closure of the algebra. In the  $\mathcal{N} = 2, D = 5$  supergravity, there are two possible Weyl multiplet choices: the standard Weyl multiplet and the dilaton Weyl multiplet. We investigate these two multiplets in detail in the first two subsections. The last two subsections of this section are devoted to the introduction of the superconformal matter multiplets including the vector multiplet and the linear multiplet, which will be utilized as compensating multiplets in the construction of off-shell Poincaré theories.

#### 3.1 Standard Weyl Multiplet

The  $\mathcal{N} = 2, D = 5$  superconformal tensor calculus is based on the superconformal algebra<sup>1</sup>  $F^2(4)$  with the generators

$$P_a, \quad M_{ab}, \quad D, \quad K_a, \quad U_{ij}, \quad Q_{\alpha i}, \quad S_{\alpha i}, \quad (3.1)$$

where  $a, b, \dots$  are Lorentz indices. Here  $M_{ab}$  and  $P_a$  are the usual Poincaré generators,  $D$  is the generator for dilatations,  $K_a$  generates special conformal boosts,  $U_{ij}$  is the  $SU(2)$  generator and  $Q_{\alpha i}$  and  $S_{\alpha i}$  are the supersymmetry and conformal

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\*Portions of this chapter are reprinted from *An off-shell formulation for internally gauged  $D=5, N=2$  supergravity from superconformal methods* by Frederik Coomans and Mehmet Ozkan, 2013. JHEP **1301**, 099 (2013), Copyright 2013, with permission from SISSA.

<sup>1</sup>The notation  $F^p(4)$  refers to a compact form of  $F(4)$  with bosonic subalgebra  $SO(7-p, p)$ .



supersymmetry generators respectively.

For each of the generators above we now introduce the following gauge fields

$$h_\mu^A \equiv \{e_\mu^a, \omega_\mu^{ab}, b_\mu, f_\mu^a, V_\mu^{ij}, \psi_\mu^i, \phi_\mu^i\}, \quad (3.2)$$

where  $\mu, \nu, \dots$  are world vector indices. Using the structure constants  $f_{AB}^C$  of the superconformal algebra (given e.g. in appendix B of [37]) and the basic rules

$$\begin{aligned} \delta h_\mu^A &= \partial_\mu \epsilon^A + \epsilon^C h_\mu^B f_{BC}^A, \\ R_{\mu\nu}^A &= 2\partial_{[\mu} h_{\nu]}^A + h_\nu^C h_\mu^B f_{BC}^A, \end{aligned} \quad (3.3)$$

one can easily write down the linear transformation rules and the linear curvatures  $R_{\mu\nu}^A$  of the superconformal gauge fields given in (3.2). The linear transformations given in [37] satisfy the  $F^2(4)$  superalgebra, thus resulting in a gauge theory of  $F^2(4)$  since we have not related the generators  $P_a, M_{ab}$  to the diffeomorphisms of space-time. This problem can be solved by imposing the so-called curvature constraints [37]. These constraints determine the gauge fields  $\omega_\mu^{ab}, \phi_\mu^i$  and  $f_\mu^a$  in terms of the independent gauge fields  $e_\mu^a, \psi_\mu^i, b_\mu, V_\mu^{ij}$  and, in addition, achieve maximal irreducibility of the superconformal gauge field configuration.

A simple counting argument shows that the superconformal gauge fields, after imposing the conventional constraints, represent  $21 + 24$  off-shell degrees of freedom and therefore cannot represent a supersymmetric theory. Additional matter fields  $T_{\mu\nu}(10), D(1)$  and  $\chi^i(8)$  must be added to the gauge fields in order to obtain an off-shell closed multiplet [36, 37]. This multiplet is known as the standard Weyl multiplet.

Starting from the linear transformation rules of the superconformal fields, the

linear curvatures  $R_{\mu\nu}{}^A$  and the matter fields  $T_{\mu\nu}$ ,  $D$ , and  $\chi^i$  we can construct the full nonlinear  $\mathcal{N} = 2, D = 5$  Weyl Multiplet by applying an iterative procedure (described in detail for 6 dimensions in [35]). The results are [37] (we only give  $Q$ ,  $S$  and  $K$  transformations)

$$\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu^i &= (\partial_\mu + \frac{1}{2}b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\epsilon^i - V_\mu^{ij}\epsilon_j + i\gamma \cdot T\gamma_\mu\epsilon^i - i\gamma_\mu\eta^i, \\
\delta V_\mu{}^{ij} &= -\frac{3}{2}i\bar{\epsilon}^{(i}\phi_\mu^{j)} + 4\bar{\epsilon}^{(i}\gamma_\mu\chi^{j)} + i\bar{\epsilon}^{(i}\gamma \cdot T\psi_\mu^{j)} + \frac{3}{2}i\bar{\eta}^{(i}\psi_\mu^{j)}, \\
\delta T_{ab} &= \frac{1}{2}i\bar{\epsilon}\gamma_{ab}\chi - \frac{3}{32}i\bar{\epsilon}\widehat{R}_{ab}(Q), \\
\delta\chi^i &= \frac{1}{4}\epsilon^i D - \frac{1}{64}\gamma \cdot \widehat{R}^{ij}(V)\epsilon_j + \frac{1}{8}i\gamma^{ab}\mathcal{D}T_{ab}\epsilon^i - \frac{1}{8}i\gamma^a\mathcal{D}^b T_{ab}\epsilon^i \\
&\quad - \frac{1}{4}\gamma^{abcd}T_{ab}T_{cd}\epsilon^i + \frac{1}{6}T^2\epsilon^i + \frac{1}{4}\gamma \cdot T\eta^i, \\
\delta D &= \bar{\epsilon}\mathcal{D}\chi - \frac{5}{3}i\bar{\epsilon}\gamma \cdot T\chi - i\bar{\eta}\chi, \\
\delta b_\mu &= \frac{1}{2}i\bar{\epsilon}\phi_\mu - 2\bar{\epsilon}\gamma_\mu\chi + \frac{1}{2}i\bar{\eta}\psi_\mu + 2\Lambda_{K\mu}, \tag{3.4}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}_\mu\chi^i &= (\partial_\mu - \frac{7}{2}b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\chi^i - V_\mu^{ij}\chi_j - \frac{1}{4}\psi_\mu^i D + \frac{1}{64}\gamma \cdot \widehat{R}^{ij}(V)\psi_{\mu j} \\
&\quad - \frac{1}{8}i\gamma^{ab}\mathcal{D}T_{ab}\psi_\mu^i + \frac{1}{8}i\gamma^a\mathcal{D}^b T_{ab}\psi_\mu^i + \frac{1}{4}\gamma^{abcd}T_{ab}T_{cd}\psi_\mu^i - \frac{1}{6}T^2\psi_\mu^i - \frac{1}{4}\gamma \cdot T\phi_\mu^i, \\
\mathcal{D}_\mu T_{ab} &= \partial_\mu T_{ab} - b_\mu T_{ab} - 2\omega_\mu{}^c{}_{[a}T_{b]c} - \frac{1}{2}i\bar{\psi}_\mu\gamma_{ab}\chi + \frac{3}{32}i\bar{\psi}_\mu\widehat{R}_{ab}(Q). \tag{3.5}
\end{aligned}$$

The relevant modified curvatures are

$$\begin{aligned}
\widehat{R}_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b + 8f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + i\bar{\psi}_{[\mu}\gamma^{ab}\psi_{\nu]} + i\bar{\psi}_{[\mu}\gamma^{[a}\gamma \cdot T\gamma^{b]}\psi_{\nu]} \\
&\quad + \bar{\psi}_{[\mu}\gamma^{[a}\widehat{R}_{\nu]}{}^{b]}(Q) + \frac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]}\widehat{R}^{ab}(Q) - 8\bar{\psi}_{[\mu}e_{\nu]}{}^{[a}\gamma^{b]}\chi + i\bar{\phi}_{[\mu}\gamma^{ab}\psi_{\nu]},
\end{aligned}$$

$$\begin{aligned}
\widehat{R}_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}{}^{ij} - 2V_{[\mu}{}^k(i)V_{\nu]k}{}^j) - 3i\bar{\phi}_{[\mu}{}^i\psi_{\nu]}{}^j) - 8\bar{\psi}_{[\mu}{}^i\gamma_{\nu]}\chi^j) - i\bar{\psi}_{[\mu}{}^i\gamma \cdot T\psi_{\nu]}{}^j), \quad (3.6) \\
\widehat{R}_{\mu\nu}{}^i(Q) &= 2\partial_{[\mu}\psi_{\nu]}{}^i + \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]}{}^i + b_{[\mu}\psi_{\nu]}{}^i - 2V_{[\mu}{}^{ij}\psi_{\nu]j} - 2i\gamma_{[\mu}\phi_{\nu]}{}^i + 2i\gamma \cdot T\gamma_{[\mu}\psi_{\nu]}{}^i.
\end{aligned}$$

As mentioned before, the dependent fields, which relate the generators  $P_a, M_{ab}$  to the diffeomorphisms of spacetime, are completely determined by the following curvature constraints

$$\begin{aligned}
R_{\mu\nu}{}^a(P) &= 0, \\
e^\nu{}_b\widehat{R}_{\mu\nu}{}^{ab}(M) &= 0, \\
\gamma^\mu\widehat{R}_{\mu\nu}{}^i(Q) &= 0. \quad (3.7)
\end{aligned}$$

Notice that our choices for the above constraints are not unique, i.e. one can impose different constraints by adding further terms to (3.7). However such additional terms only amount to redefinitions of the dependent fields defined below. Using the curvature constraints we identify  $\omega_\mu{}^{ab}$ ,  $\phi_\mu{}^i$  and  $f_\mu{}^a$  in terms of the other gauge fields and matter fields

$$\begin{aligned}
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c + 2e_\mu{}^{[ab]} - \frac{1}{2}\bar{\psi}^{[b}\gamma^a]\psi_\mu - \frac{1}{4}\bar{\psi}^b\gamma_\mu\psi^a, \\
\phi_\mu{}^i &= \frac{1}{3}i\gamma^a\widehat{R}'_{\mu a}{}^i(Q) - \frac{1}{24}i\gamma_\mu\gamma^{ab}\widehat{R}'_{ab}{}^i(Q), \quad (3.8) \\
f_\mu{}^a &= -\frac{1}{6}\mathcal{R}_\mu{}^a + \frac{1}{48}e_\mu{}^a\mathcal{R},
\end{aligned}$$

where  $\mathcal{R}_{\mu\nu} \equiv \widehat{R}'_{\mu\rho}{}^{ab}(M)e_b{}^\rho e_{\nu a}$  and  $\mathcal{R} \equiv \mathcal{R}_\mu{}^\mu$ . The notation  $\widehat{R}'(M)$  and  $\widehat{R}'(Q)$  indicates that we have omitted the  $f_\mu{}^a$  dependent term in  $\widehat{R}(M)$  and the  $\phi_\mu{}^i$  dependent term in  $\widehat{R}(Q)$ . The constraints imply through Bianchi identities further relations

between the curvatures. The Bianchi identities for  $R(P)$  imply [37]

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\nu\mu}, \quad e_{[\mu}{}^a \widehat{R}_{\nu\rho]}(D) = \widehat{R}_{[\mu\nu\rho]}{}^a(M), \quad \widehat{R}_{\mu\nu}(D) = 0. \quad (3.9)$$

The full commutator of two supersymmetry transformations is

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{cgt}(\xi_3^\mu) + \delta_M(\lambda_3^{ab}) + \delta_S(\eta_3) + \delta_U(\lambda_3^{ij}) + \delta_K(\Lambda_{K3}^a), \quad (3.10)$$

where  $\delta_{cgt}$  represents a covariant general coordinate transformation<sup>2</sup>. The parameters appearing in (3.10) are

$$\begin{aligned} \xi_3^\mu &= \frac{1}{2} \bar{\epsilon}_2 \gamma^\mu \epsilon_1, \\ \lambda_3^{ab} &= -i \bar{\epsilon}_2 \gamma^{[a} \gamma \cdot T \gamma^{b]} \epsilon_1, \\ \lambda_3^{ij} &= i \bar{\epsilon}_2^{(i} \gamma \cdot T \epsilon_1^{j)}, \\ \eta_3^i &= -\frac{9}{4} i \bar{\epsilon}_2 \epsilon_1 \chi^i + \frac{7}{4} i \bar{\epsilon}_2 \gamma_c \epsilon_1 \gamma^c \chi^i \\ &\quad + \frac{1}{4} i \bar{\epsilon}_2^{(i} \gamma_{cd} \epsilon_1^{j)} \left( \gamma^{cd} \chi_j + \frac{1}{4} \widehat{R}^{cd}{}_j(Q) \right), \\ \Lambda_{K3}^a &= -\frac{1}{2} \bar{\epsilon}_2 \gamma^a \epsilon_1 D + \frac{1}{96} \bar{\epsilon}_2^i \gamma^{abc} \epsilon_1^j \widehat{R}_{bcij}(V) \\ &\quad + \frac{1}{12} i \bar{\epsilon}_2 \left( -5 \gamma^{abcd} D_b T_{cd} + 9 D_b T^{ba} \right) \epsilon_1 \\ &\quad + \bar{\epsilon}_2 \left( \gamma^{abcde} T_{bc} T_{de} - 4 \gamma^c T_{cd} T^{ad} + \frac{2}{3} \gamma^a T^2 \right) \epsilon_1. \end{aligned} \quad (3.11)$$

For the  $Q, S$  commutators we find the following algebra

$$\begin{aligned} [\delta_S(\eta), \delta_Q(\epsilon)] &= \delta_D(\frac{1}{2} i \bar{\epsilon} \eta) + \delta_M(\frac{1}{2} i \bar{\epsilon} \gamma^{ab} \eta) + \delta_U(-\frac{3}{2} i \bar{\epsilon}^{(i} \eta^{j)}) + \delta_K(\Lambda_{3K}^a), \\ [\delta_S(\eta_1), \delta_S(\eta_2)] &= \delta_K(\frac{1}{2} \bar{\eta}_2 \gamma^a \eta_1), \end{aligned} \quad (3.12)$$

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<sup>2</sup>The covariant general coordinate transformations are defined as  $\delta_{cgt}(\xi) = \delta_{gct}(\xi) - \delta_I(\xi^\mu h_\mu^I)$ , where the index  $I$  runs over all transformations except the general coordinate transformations and the  $h_\mu^I$  represent the corresponding gauge fields.

with

$$\Lambda_{3K}^a = \frac{1}{6}\bar{\epsilon}(\gamma \cdot T\gamma^a - \frac{1}{2}\gamma^a\gamma \cdot T)\eta. \quad (3.13)$$

### 3.2 Dilaton Weyl Multiplet

In [37], it was established that there exist another Weyl multiplets for  $\mathcal{N} = 2$  conformal supergravity in five dimensions: the dilaton Weyl multiplet. This multiplet has the same contents of gauge fields as the standard Weyl multiplet, but different in matter fields. However, the matter fields of the standard multiplet can be built from the fundamental fields in the dilaton Weyl multiplet as composite fields. The gauge sector of the dilaton Weyl multiplet consists of a fünfbein  $e_\mu^a$ , a gravitino  $\psi_\mu^i$ , the dilatation gauge field  $b_\mu$ , and the  $SU(2)$  gauge field  $V_\mu^{ij}$ . Matter. For the dilaton Weyl multiplet, the matter sector consists of a physical vector  $C_\mu$ , an antisymmetric two-form gauge field  $B_{\mu\nu}$ , a dilaton field  $\sigma$  and a dilatino  $\psi^i$ . The Q, S and K transformation rules for the dilaton Weyl Multipet are given by [37]

$$\begin{aligned} \delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\ \delta\psi_\mu^i &= (\partial_\mu + \frac{1}{2}b_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab})\epsilon^i - V_\mu^{ij}\epsilon_j + i\gamma \cdot \underline{T}\gamma_\mu\epsilon^i - i\gamma_\mu\eta^i, \\ \delta V_\mu^{ij} &= -\frac{3}{2}i\bar{\epsilon}^{(i}\phi_\mu^{j)} + 4\bar{\epsilon}^{(i}\gamma_\mu\underline{\chi}^{j)} + i\bar{\epsilon}^{(i}\gamma \cdot \underline{T}\psi_\mu^{j)} + \frac{3}{2}i\bar{\eta}^{(i}\psi_\mu^{j)}, \\ \delta C_\mu &= -\frac{1}{2}i\sigma\bar{\epsilon}\psi_\mu + \frac{1}{2}\bar{\epsilon}\gamma_\mu\psi, \\ \delta B_{\mu\nu} &= \frac{1}{2}\sigma^2\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + \frac{1}{2}i\sigma\bar{\epsilon}\gamma_{\mu\nu}\psi + C_{[\mu}\delta(\epsilon)C_{\nu]}, \\ \delta\psi^i &= -\frac{1}{4}\gamma \cdot \widehat{G}\epsilon^i - \frac{1}{2}i\mathcal{D}\sigma\epsilon^i + \sigma\gamma \cdot \underline{T}\epsilon^i - \frac{1}{4}i\sigma^{-1}\epsilon_j\bar{\psi}^i\psi^j + \sigma\eta^i, \\ \delta\sigma &= \frac{1}{2}i\bar{\epsilon}\psi, \\ \delta b_\mu &= \frac{1}{2}i\bar{\epsilon}\phi_\mu - 2\bar{\epsilon}\gamma_\mu\underline{\chi} + \frac{1}{2}i\bar{\eta}\psi_\mu + 2\Lambda_{K\mu}, \end{aligned} \quad (3.14)$$

where

$$\mathcal{D}_\mu \sigma = (\partial_\mu - b_\mu)\sigma - \frac{1}{2}i\bar{\psi}_\mu\psi, \quad (3.15)$$

$$\begin{aligned} \mathcal{D}_\mu \psi^i &= (\partial_\mu - \frac{3}{2}b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi^i - V_\mu{}^{ij}\psi_j + \frac{1}{4}\gamma \cdot \widehat{G}\psi_\mu^i \\ &\quad + \frac{1}{2}i\mathcal{D}\sigma\psi_\mu^i + \frac{1}{4}i\sigma^{-1}\psi_{\mu j}\bar{\psi}^i\psi^j - \sigma\gamma \cdot \underline{T}\psi_\mu^i - \sigma\phi_\mu^i, \end{aligned} \quad (3.16)$$

and the supercovariant curvatures are defined according to

$$\begin{aligned} \widehat{G}_{\mu\nu} &= G_{\mu\nu} - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi + \frac{1}{2}i\sigma\bar{\psi}_{[\mu}\psi_{\nu]}, \\ \widehat{H}_{\mu\nu\rho} &= H_{\mu\nu\rho} - \frac{3}{4}\sigma^2\bar{\psi}_{[\mu}\gamma_{\nu}\psi_{\rho]} - \frac{3}{2}i\sigma\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi. \end{aligned} \quad (3.17)$$

In above expressions,  $G_{\mu\nu} = 2\partial_{[\mu}C_{\nu]}$  and  $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + \frac{3}{2}C_{[\mu}G_{\nu\rho]}$ . Note that  $\widehat{G}_{\mu\nu}$  and  $\widehat{H}_{\mu\nu\rho}$  are invariant under following gauge transformations

$$\delta C_\mu = \partial_\mu\Lambda, \quad \delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]} - \frac{1}{2}\Lambda G_{\mu\nu}. \quad (3.18)$$

The commutator of Q-transformations picks up the following modifications

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \dots + \delta_{U(1)}(\Lambda_3 = -\frac{1}{2}i\sigma\bar{\epsilon}_2\epsilon_1) + \delta_B(-\frac{1}{4}\sigma^2\bar{\epsilon}_2\gamma_\mu\epsilon_1 - \frac{1}{2}C_\mu\Lambda_3), \quad (3.19)$$

where the ellipses refer to the standard commutation rule and  $\delta_B$  is a vector gauge transformation for the field  $B_{\mu\nu}$ . The Q- and S- transformations of the field strengths  $\widehat{G}_{\mu\nu}$  and  $\widehat{H}_{abc}$  are presented in [37]

$$\begin{aligned} \delta\widehat{G}_{ab} &= -\frac{1}{2}i\sigma\bar{\epsilon}\widehat{R}_{ab}(Q) - \bar{\epsilon}\gamma_a\mathcal{D}_b\psi + i\bar{\epsilon}\gamma_{[a}\gamma \cdot \underline{T}\gamma_{b]}\psi + i\bar{\eta}\gamma_{ab}\psi, \\ \delta\widehat{H}_{abc} &= -\frac{3}{4}\sigma^2\bar{\epsilon}\gamma_{[a}\widehat{R}_{bc]}(Q) + \frac{3}{2}i\bar{\epsilon}\gamma_{[ab}\mathcal{D}_c\psi + \frac{3}{2}i\bar{\epsilon}\gamma_{[ab}\psi\mathcal{D}_c\sigma \\ &\quad - \frac{3}{2}\sigma\bar{\epsilon}\gamma_{[a}\gamma \cdot \underline{T}\gamma_{bc]}\psi - \frac{3}{2}\bar{\epsilon}\gamma_{[a}\widehat{G}_{bc]}\psi - \frac{3}{2}\sigma\bar{\eta}\gamma_{abc}\psi. \end{aligned} \quad (3.20)$$

The underlined expressions  $\underline{T}^{ab}$ ,  $\underline{\chi}^i$  and  $\underline{D}$ , which are the fundamental auxiliary fields in the standard Weyl multiplet, become composite expressions in the dilaton Weyl multiplet [37]

$$\begin{aligned}
\underline{T}^{ab} &= \frac{1}{8}\sigma^{-2}\left(\sigma\widehat{G}^{ab} + \frac{1}{6}\epsilon^{abcde}\widehat{H}_{cde} + \frac{1}{4}\text{i}\bar{\psi}\gamma^{ab}\psi\right), \\
\underline{\chi}^i &= \frac{1}{8}\text{i}\sigma^{-1}\mathcal{D}\psi^i + \frac{1}{16}\text{i}\sigma^{-2}\mathcal{D}\sigma\psi^i - \frac{1}{32}\sigma^{-2}\gamma\cdot\widehat{G}\psi^i + \frac{1}{4}\sigma^{-1}\gamma\cdot\underline{T}\psi^i \\
&\quad + \frac{1}{32}\text{i}\sigma^{-3}\psi_j\bar{\psi}^i\psi^j, \\
\underline{D} &= \frac{1}{4}\sigma^{-1}\square^c\sigma + \frac{1}{8}\sigma^{-2}(\mathcal{D}_a\sigma)(\mathcal{D}^a\sigma) - \frac{1}{16}\sigma^{-2}\widehat{G}_{\mu\nu}\widehat{G}^{\mu\nu} \\
&\quad - \frac{1}{8}\sigma^{-2}\bar{\psi}\mathcal{D}\psi - \frac{1}{64}\sigma^{-4}\bar{\psi}^i\psi^j\bar{\psi}_i\psi_j - 4\text{i}\sigma^{-1}\psi\underline{\chi} \\
&\quad + \left(-\frac{26}{3}\underline{T}_{ab} + 2\sigma^{-1}\widehat{G}_{ab} + \frac{1}{4}\text{i}\sigma^{-2}\bar{\psi}\gamma_{ab}\psi\right)\underline{T}^{ab}, \tag{3.21}
\end{aligned}$$

where the superconformal d'Alembertian for  $\sigma$  is given by

$$\begin{aligned}
\square^c\sigma &= (\partial^a - 2b^a + \omega_b{}^{ba})\mathcal{D}_a\sigma - \frac{1}{2}\text{i}\bar{\psi}_a\mathcal{D}^a\psi - 2\sigma\bar{\psi}_a\gamma^a\chi \\
&\quad + \frac{1}{2}\bar{\psi}_a\gamma^a\gamma\cdot\underline{T}\psi + \frac{1}{2}\bar{\phi}_a\gamma^a\psi + 2f_a{}^a\sigma. \tag{3.22}
\end{aligned}$$

These composite expressions define a map from the dilaton Weyl multiplet to the standard Weyl multiplet.

### 3.3 Yang-Mills Multiplet

The off-shell non-abelian  $D = 5$ ,  $\mathcal{N} = 2$  vector multiplet consists of  $8n$  (bosonic) +  $8n$  (fermionic) degrees of freedom (where  $n$  is the dimension of the gauge group). Denoting the Yang-Mills index by  $\Sigma$  ( $\Sigma = 1, \dots, n$ ), the bosonic sector consists of vector fields  $A_\mu^\Sigma$ , scalar fields  $\rho^\Sigma$  and  $SU(2)$ -triplet auxiliary fields  $Y^{ij\Sigma} = Y^{(ij)\Sigma}$ .  $SU(2)$ -doublet fields  $\lambda^{i\Sigma}$  constitute the fermionic sector.

In the background of the standard Weyl multiplet, the  $Q$ - and  $S$ -transformations

of the fields in the vector multiplet are given by [38]

$$\begin{aligned}
\delta A_\mu^\Sigma &= -\frac{1}{2}i\rho^\Sigma\bar{\epsilon}\psi_\mu + \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^\Sigma, \\
\delta Y^{ij\Sigma} &= -\frac{1}{2}\bar{\epsilon}^{(i}\mathcal{D}\lambda^{j)\Sigma} + \frac{1}{2}i\bar{\epsilon}^{(i}\gamma\cdot T\lambda^{j)\Sigma} - 4i\rho^\Sigma\bar{\epsilon}^{(i}\chi^{j)} + \frac{1}{2}i\bar{\eta}^{(i}\lambda^{j)\Sigma} - \frac{1}{2}ig\bar{\epsilon}^{(i}f_{\Lambda\Upsilon}{}^\Sigma\rho^\Lambda\lambda^{j)\Upsilon}, \\
\delta\lambda^{i\Sigma} &= -\frac{1}{4}\gamma\cdot\widehat{F}^\Sigma\epsilon^i - \frac{1}{2}i\mathcal{D}\rho^\Sigma\epsilon^i + \rho^\Sigma\gamma\cdot T\epsilon^i - Y^{ij\Sigma}\epsilon_j + \rho^\Sigma\eta^i, \\
\delta\rho^\Sigma &= \frac{1}{2}i\bar{\epsilon}\lambda^\Sigma.
\end{aligned} \tag{3.23}$$

The superconformally covariant derivatives used here are

$$\mathcal{D}_\mu\rho^\Sigma = (\partial_\mu - b_\mu)\rho^\Sigma + gf_{\Lambda\Upsilon}{}^\Sigma A_\mu^\Lambda\rho^\Upsilon - \frac{1}{2}i\bar{\psi}_\mu\lambda^\Sigma, \tag{3.24}$$

$$\begin{aligned}
\mathcal{D}_\mu\lambda^{i\Sigma} &= (\partial_\mu - \frac{3}{2}b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\lambda^{i\Sigma} - V_\mu^{ij}\lambda_j^\Sigma + gf_{\Lambda\Upsilon}{}^\Sigma A_\mu^\Lambda\lambda^{i\Upsilon} \\
&\quad + \frac{1}{4}\gamma\cdot\widehat{F}^\Sigma\psi_\mu^i + \frac{1}{2}i\widehat{\mathcal{D}}\rho^\Sigma\psi_\mu^i + Y^{ij\Sigma}\psi_{\mu j} - \rho^\Sigma\gamma\cdot T\psi_\mu^i - \rho^\Sigma\phi_\mu^i,
\end{aligned} \tag{3.25}$$

where the supercovariant Yang-Mills curvature is given as

$$\widehat{F}_{\mu\nu}^\Sigma = 2\partial_{[\mu}A_{\nu]}^\Sigma + gf_{\Lambda\Upsilon}{}^\Sigma A_\mu^\Lambda A_\nu^\Upsilon - \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^\Sigma + \frac{1}{2}i\rho^\Sigma\bar{\psi}_{[\mu}\psi_{\nu]}. \tag{3.26}$$

The local supersymmetry transformation rules given in (3.23) are obtained by coupling the rigid supersymmetric theory to a Weyl multiplet [38]. In the above transformation rules, we utilized the standard Weyl multiplet. If the dilaton Weyl multiplet is considered, the supersymmetry transformation rules can be obtained straightforwardly by replacing  $T_{ab}$ ,  $D$  and  $\chi^i$  by their composite expressions according to (3.21).



### 3.4 Linear Multiplet

The off-shell  $D = 5$ ,  $\mathcal{N} = 2$  linear multiplet contains 8 (bosonic)+8 (fermionic) degrees of freedom carried by the following fields

$$(L^{ij}, E^a, N, \varphi^i). \quad (3.27)$$

The bosonic fields are an  $SU(2)$  triplet  $L^{ij} = L^{(ij)}$ , a constrained vector  $E_a$  and a scalar  $N$ . The fermionic fields are given by an  $SU(2)$  doublet  $\varphi^i$ . In the background of the standard Weyl multiplet, the  $Q$ - and  $S$ - transformations of the fields in the linear multiplet are given by [58]

$$\begin{aligned} \delta L^{ij} &= i\bar{\epsilon}^{(i}\varphi^{j)}, \\ \delta \varphi^i &= -\frac{1}{2}i\mathcal{D}L^{ij}\epsilon_j - \frac{1}{2}i\gamma^a E_a \epsilon^i + \frac{1}{2}N\epsilon^i - \gamma \cdot TL^{ij}\epsilon_j + 3L^{ij}\eta_j, \\ \delta E_a &= -\frac{1}{2}i\bar{\epsilon}\gamma_{ab}\mathcal{D}^b\varphi - 2\bar{\epsilon}\gamma^b\varphi T_{ba} - 2\bar{\eta}\gamma_a\varphi, \\ \delta N &= \frac{1}{2}\bar{\epsilon}\mathcal{D}\varphi + \frac{3}{2}i\bar{\epsilon}\gamma \cdot T\varphi + 4i\bar{\epsilon}^i\chi^j L_{ij} + \frac{3}{2}i\bar{\eta}\varphi, \end{aligned} \quad (3.28)$$

where the super-covariant derivatives are defined as

$$\begin{aligned} \mathcal{D}_\mu L^{ij} &= (\partial_\mu - 3b_\mu)L^{ij} + 2V_\mu^{(i}L^{j)k} - i\bar{\psi}_\mu^{(i}\varphi^{j)}, \\ \mathcal{D}_\mu \varphi^i &= (\partial_\mu - \frac{7}{2}b_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab})\varphi^i - V_\mu^{ij}\varphi_j + \frac{1}{2}i\mathcal{D}L^{ij}\psi_{\mu j} + \frac{1}{2}i\gamma^a E_a\psi_\mu^i \\ &\quad - \frac{1}{2}N\psi_\mu^i + \gamma \cdot TL^{ij}\psi_{\mu j} - 3L^{ij}\phi_{\mu j}, \\ \mathcal{D}_\mu E_a &= (\partial_\mu - 4b_\mu)E_a + \omega_{\mu ab}E^b + \frac{1}{2}i\bar{\psi}_\mu\gamma_{ab}\mathcal{D}^b\varphi + 2\bar{\psi}_\mu\gamma^b\varphi T_{ba} + 2\bar{\phi}_\mu\gamma_a\varphi. \end{aligned} \quad (3.29)$$

The closure of the superconformal algebra requires that the following constraint must be satisfied

$$\mathcal{D}^a E_a = 0. \quad (3.30)$$

Thus  $E_a$  can be solved in terms of a 3-form  $E_{\mu\nu\rho}$  as

$$E^\mu = -\frac{1}{12}\epsilon^{\mu\nu\rho\sigma\lambda}\mathcal{D}_\nu E_{\rho\sigma\lambda}, \quad (3.31)$$

where  $E_{\mu\nu\rho}$  is invariant under the following gauge transformation

$$\delta_\Lambda E_{\mu\nu\rho} = 3\partial_{[\mu}\Lambda_{\nu\rho]}. \quad (3.32)$$

We can also express  $E^\mu$  and  $E_{\mu\nu\rho}$  in terms of a 2-form potential according to [58]

$$E^\mu = \mathcal{D}_\nu E^{\mu\nu}, \quad E_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma\lambda} E^{\sigma\lambda}. \quad (3.33)$$

The supersymmetry transformations of the 2-form gauge field  $E_{\mu\nu}$  and 3-form gauge field  $E_{\mu\nu\rho}$  are given in [58]

$$\begin{aligned} \delta E^{\mu\nu} &= -\frac{1}{2}\bar{\epsilon}\gamma^{\mu\nu}\varphi - \frac{1}{2}\bar{\psi}_\rho^i\gamma^{\mu\nu\rho}\epsilon^j L_{ij} - \partial_\rho\tilde{\Lambda}^{\mu\nu\rho}, \\ \delta E_{\mu\nu\rho} &= -\bar{\epsilon}\gamma_{\mu\nu\rho}\varphi + \bar{\psi}_{[\mu}^i\gamma_{\nu\rho]}\epsilon^j L_{ij}. \end{aligned} \quad (3.34)$$

With these results in hand, we conclude our discussion on the superconformal multiplets of  $N = 2, D = 5$  superconformal theory. Similar to the Yang-Mills multiplet, if the dilaton Weyl multiplet is adopted, the supersymmetry transformation rules for the linear multiplet can be obtained by using the map (3.21).

## 4. SUPERCONFORMAL ACTIONS\*

In this section, we construct superconformal actions for the linear multiplet and for the vector multiplet coupled to Weyl multiplet. Our starting point is the density formula for the product of a vector multiplet and a linear multiplet. This will be presented in subsection 4.1. In subsection 4.2 we will use this formula, after embedding the linear multiplet into a vector multiplet to construct the superconformal action for the linear multiplet. In the last subsection 4.3 we construct the elements of linear multiplet in terms of the fields in vector multiplet, thus obtain a superconformal action for vector multiplet.

### 4.1 Density Formula

We need an expression constructed from the components of the linear and vector multiplet that can be used as a superconformal action. In [63] a density formula is given for the product of a Vector Multiplet and a linear multiplet

$$\begin{aligned}
 e^{-1}\mathcal{L}_{VL} = & Y^{ij}L_{ij} + i\bar{\psi}\varphi - \frac{1}{2}\bar{\psi}_{\mu}^i\gamma^{\mu}\psi^jL_{ij} + C_{\mu}P^{\mu} \\
 & +\sigma(N + \frac{1}{2}\bar{\psi}_{\mu}\gamma^{\mu}\varphi + \frac{1}{4}i\bar{\psi}_{\mu}^i\gamma^{\mu\nu}\psi_{\nu}^jL_{ij}),
 \end{aligned}
 \tag{4.1}$$

where  $P_{\mu}$ , the pure bosonic part of the supercovariant field  $E_{\mu}$ , is defined as

$$P^a = E^a + \frac{1}{2}i\bar{\psi}_b\gamma^{ba}\varphi + \frac{1}{4}\bar{\psi}_b^i\gamma^{abc}\psi_c^jL_{ij}.
 \tag{4.2}$$

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\*Portions of this chapter are reprinted from *An off-shell formulation for internally gauged D=5, N=2 supergravity from superconformal methods* by Frederik Coomans and Mehmet Ozkan, 2013. JHEP **1301**, 099 (2013), Copyright 2013, with permission from SISSA; *Supersymmetric Completion of Gauss-Bonnet Combination in Five Dimensions* by Mehmet Ozkan and Yi Pang, 2013. JHEP **1303**, 158 (2013), Copyright 2013, with permission from SISSA.

Using (3.31), we can express  $P^a$  as

$$P^a = -\frac{1}{12}e_\mu{}^a e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda}\partial_\nu E_{\rho\sigma\lambda}. \quad (4.3)$$

Using (4.3) and (3.34), one can rewrite  $\mathcal{L}_{VL}$  as

$$\begin{aligned} e^{-1}\mathcal{L}_{VL} &= Y^{ij}L_{ij} + i\bar{\psi}\varphi - \frac{1}{2}\bar{\psi}_\mu^i\gamma^\mu\psi^j L_{ij} + \frac{1}{2}G_{\mu\nu}E^{\mu\nu} \\ &+ \sigma(N + \frac{1}{2}\bar{\psi}_\mu\gamma^\mu\varphi + \frac{1}{4}i\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j L_{ij}). \end{aligned} \quad (4.4)$$

## 4.2 Linear Multiplet Action

We want to use the density formula (4.1) to construct an action for the linear multiplet. Hence, we start with embedding the components of the linear multiplet  $(L_{ij}, \varphi^i, E_a, N)$  into the components of the vector multiplet  $(Y^{ij}, A_\mu, \rho, \lambda^i)$ . Such embeddings are already considered in 4 and 6 dimensions [64, 35] and here we will follow the same procedure.

As mentioned, the linear multiplet consists of a triplet of scalars  $L_{ij}$ , a constrained vector  $E_a$ , a doublet of Majorana spinors  $\varphi^i$  and a scalar  $N$ . One starts the construction of the vector multiplet with the identification  $\rho = N$ , where  $\rho$  is the scalar of the vector multiplet. There is, however, a mismatch between the Weyl weights of these fields. Therefore one needs another scalar field to compensate for this mismatch. For this we will use

$$L^2 = L_{ij}L^{ij}. \quad (4.5)$$

We can then identify the scalar of the vector multiplet as  $\rho = 2L^{-1}N + i\bar{\varphi}_i\varphi_j L^{ij}L^{-3}$ . This identification has the correct Weyl weight, and the second term is the supersym-

metric completion that is determined by the  $S$ -invariance of  $\rho$ . Using this expression and applying a sequence of supersymmetry transformations, we obtain the full expressions for the components of vector multiplets in terms of elements in the linear multiplet [58]

$$\begin{aligned}
\rho &= 2L^{-1}N + iL^{-3}\bar{\varphi}^i\varphi^j L_{ij}, \\
\lambda_i &= -2i\mathcal{D}\varphi_i L^{-1} + (16L_{ij}\chi^j + 4\gamma \cdot T\varphi_i)L^{-1} - 2NL_{ij}\varphi^j L^{-3} \\
&\quad + 2i(\mathcal{D}L_{ij}L^{jk}\varphi_k - \mathcal{E}L_{ij}\varphi^j)L^{-3} + 2i\varphi^j\bar{\varphi}_i\varphi_j L^{-3} \\
&\quad - 6i\varphi^j\bar{\varphi}_k\varphi_l L^{kl}L_{ij}L^{-5}, \\
Y_{ij} &= L^{-1}\square^C L_{ij} - \mathcal{D}_a L_{k(i}\mathcal{D}^a L_{j)m}L^{km}L^{-3} - N^2 L_{ij}L^{-3} - E_\mu E^\mu L_{ij}L^{-3} \\
&\quad + \frac{8}{3}L^{-1}T^2 L_{ij} + 4L^{-1}DL_{ij} + 2E_\mu L_{k(i}\mathcal{D}^\mu L_{j)}^k L^{-3} - \frac{1}{2}iNL^{-3}\bar{\varphi}_{(i}\varphi_{j)} \\
&\quad - \frac{4}{3}L^{-5}NL_{k(i}L_{j)m}\bar{\varphi}^k\varphi^m - \frac{2}{3}L^{-3}\bar{\varphi}_{(i}\mathcal{E}\varphi_{j)} - \frac{1}{3}L^{-5}L_{k(i}L_{j)m}\bar{\varphi}^k\mathcal{E}\varphi^m \\
&\quad - 8iL^{-1}\bar{\chi}_{(i}\varphi_{j)} + 16iL^{-3}L_{k(i}L_{j)m}\bar{\chi}^k\varphi^m + 2L^{-3}L_{k(i}\bar{\varphi}^k\mathcal{D}\varphi_{j)} \\
&\quad + 2iL^{-3}L_{ij}\bar{\varphi}\gamma \cdot T\varphi - \frac{2}{3}L^{-3}\bar{\varphi}_{(i}\mathcal{D}L_{j)k}\varphi^k - L^{-5}L_{mn}L^k{}_{(i}\bar{\varphi}^m\mathcal{D}L_{j)k}\varphi^n \\
&\quad - \frac{1}{6}L^{-5}L_{km}\bar{\varphi}_i\gamma^a\varphi_j\bar{\varphi}^k\gamma_a\varphi^m + \frac{1}{12}L^{-7}L_{ij}L_{km}L^{pq}\bar{\varphi}^k\gamma_a\varphi^m\bar{\varphi}_p\gamma^a\varphi_q, \\
\widehat{F}_{\mu\nu} &= 4\mathcal{D}_{[\mu}(L^{-1}E_{\nu]}) + 2L^{-1}\widehat{R}_{\mu\nu}{}^{ij}(V)L_{ij} - 2L^{-3}L_k^l\mathcal{D}_{[\mu}L^{kp}\mathcal{D}_{\nu]}L_{lp} \\
&\quad - 2\mathcal{D}_{[\mu}(L^{-3}\bar{\varphi}^i\gamma_{\nu]}\varphi^j L_{ij}) - iL^{-1}\bar{\varphi}\widehat{R}_{\mu\nu}(Q). \tag{4.6}
\end{aligned}$$

Substituting above composite expressions into the vector-linear Lagrangian (4.4), one obtains the superconformal action for the linear multiplet [58]

$$\begin{aligned}
e^{-1}\mathcal{L}_L^S &= L^{-1}L_{ij}\square^c L^{ij} - L^{ij}\mathcal{D}_\mu L_{k(i}\mathcal{D}^\mu L_{j)m}L^{km}L^{-3} + N^2 L^{-1} \\
&\quad - E_\mu E^\mu L^{-1} + \frac{8}{3}LT^2 + 4DL - \frac{1}{2}L^{-3}E^{\mu\nu}L_k^l\partial_\mu L^{kp}\partial_\nu L_{lp} \\
&\quad + 2E^{\mu\nu}\partial_\mu(L^{-1}E_\nu + V_\nu^{ij}L_{ij}L^{-1}), \tag{4.7}
\end{aligned}$$

where the complete expression for the superconformal d'Alembertian is defined as

$$\begin{aligned}
L_{ij}\square^c L^{ij} &= L_{ij}(\partial^a - 4b^a + \omega_b^{ba})\mathcal{D}_a L^{ij} + 2L_{ij}V_a^i{}_k \mathcal{D}^a L^{jk} + 6L^2 f_a^a \\
&\quad - iL_{ij}\bar{\psi}^{ai}\mathcal{D}_a \varphi^j - 6L^2 \bar{\psi}^a \gamma_a \chi - L_{ij}\bar{\varphi}^i \gamma \cdot T \gamma^a \psi_a^j + L_{ij}\bar{\varphi}^i \gamma^a \phi_a^j. \quad (4.8)
\end{aligned}$$

Fermionic contribution to above action can be straightforwardly read from the formulae given in (4.6). The linear multiplet action (4.7) can be transferred to an action describing the linear multiplet coupled to the dilaton Weyl multiplet by replacing the  $T_{ab}$ ,  $D$  and  $\chi^i$  by their composite expressions (3.21) in the action (4.7).

### 4.3 Vector Multiplet Action

The elements of linear multiplet can be written in terms of the elements of a vector multiplet and a Weyl multiplet [36]

$$\begin{aligned}
L_{ij} &= 2\rho Y_{ij} - \frac{1}{2}i\bar{\lambda}_i \lambda_j \\
\varphi_i &= i\rho \not{D}\lambda_i + 2\rho\gamma \cdot T\lambda_i - 8\rho^2 \chi_i - \frac{1}{4}\gamma \cdot \widehat{F}\lambda_i + \frac{1}{2}\not{D}\rho\lambda_i - Y_{ij}\lambda^j \\
E^a &= \mathcal{D}_b \left( -\rho F^{ab} + 8\rho^2 T^{ab} - \frac{1}{4}i\bar{\lambda}\gamma_{ab}\lambda \right) - \frac{1}{8}\epsilon^{abcde} F_{bc}F_{de} \\
N &= \rho\square^C \rho + \frac{1}{2}\mathcal{D}_a \rho \mathcal{D}^a \rho - \frac{1}{4}\widehat{F}_{ab}\widehat{F}^{ab} + Y^{ij}Y_{ij} + 8\widehat{F}_{ab}T^{ab} \\
&\quad - 4\rho^2 \left( D + \frac{26}{3}T^2 \right) - \frac{1}{2}\bar{\lambda}\not{D}\lambda + i\bar{\lambda}\gamma \cdot T\lambda + 16i\rho\bar{\chi}\lambda \quad (4.9)
\end{aligned}$$

where, once again we have utilized the standard Weyl multiplet. Adopting the standard Weyl multiplet, an action for  $n$  vector multiplets is given by [36, 38]

$$\begin{aligned}
e^{-1}\mathcal{L}_V^S &= C_{IJK} \left( -\frac{1}{4}\rho^I F_{\mu\nu}^J F^{K\mu\nu} + \frac{1}{3}\rho^I \rho^J \square^C \rho^K + \frac{1}{6}\rho^I \mathcal{D}_\mu \rho^J \mathcal{D}^\mu \rho^K \right. \\
&\quad \left. + \rho^I Y^J{}_{ij} Y_{ij}^K - \frac{4}{3}\rho^I \rho^J \rho^K \left( D + \frac{26}{3}T_{\mu\nu} T^{\mu\nu} \right) \right. \\
&\quad \left. + 4\rho^I \rho^J F_{\mu\nu}^K T^{\mu\nu} - \frac{1}{24}\epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^I F_{\nu\rho}^J F_{\sigma\lambda}^K \right), \quad (4.10)
\end{aligned}$$

where we have generalized the single vector multiplet action to  $n$ -vector multiplets action,  $I = 1, \dots, n$ . The coefficient  $C_{IJK}$  is symmetric in  $I, J, K$  and determines the coupling of  $n$  vector multiplets. The complete expression of the superconformal d'Alembertian for  $\rho^I$  is [38]

$$\begin{aligned} \square^c \rho^I &= (\partial^a - 2b^a + \omega_b^{ba}) \mathcal{D}_a \rho^I - \frac{1}{2} i \bar{\psi}_a \mathcal{D}^a \lambda^I - 2 \rho^I \bar{\psi}_a \gamma^a \chi \\ &\quad + \frac{1}{2} \bar{\psi}_a \gamma^a \gamma \cdot T \lambda^I + \frac{1}{2} \bar{\phi}_a \gamma^a \lambda^I + 2 f_a^a \rho^I. \end{aligned} \quad (4.11)$$

Similar to the linear multiplet action, the fermionic contribution to above action can be straightforwardly read from the composite expressions given in (4.9), and the vector multiplet action (4.10) can be transferred to an action describing vector multiplets coupled to the dilaton Weyl multiplet by replacing the  $T_{ab}, D$  and  $\chi^i$  with their composite expressions according to (3.21) in the action (4.10).

#### 4.4 Intermezzo: The Map Between Weyl Multiplets

In the previous section, we discussed that the matter fields of the standard Weyl multiplet can be written in terms of the elements of the dilaton Weyl multiplet, hence, and action that is written in standard Weyl multiplet language can easily be translated to the dilaton Weyl multiplet language (3.21). In this section, we shall investigate how to derive such a map. In order to do so, let us consider a vector multiplet with the following fields

$$(\sigma, C_\mu, Y^{ij}, \psi^i) \quad (4.12)$$

In that case, an action for a single abelian vector multiplet coupled to the standard Weyl multiplet can be given up to 4-fermion terms as [38]

$$\begin{aligned}
e^{-1}\mathcal{L}_{AV} &= -\frac{1}{4}\sigma G_{\mu\nu}G^{\mu\nu} + \frac{1}{3}\sigma^2\Box^c\sigma + \frac{1}{6}\sigma\mathcal{D}_\mu\sigma\mathcal{D}^\mu\sigma + \sigma Y_{ij}Y^{ij} \\
&\quad -\frac{4}{3}\sigma^3\left(D + \frac{26}{3}T^2\right) + 4\sigma^2G_{\mu\nu}T^{\mu\nu} - \frac{1}{24}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda}C_\mu G_{\nu\rho}G_{\sigma\lambda} \\
&\quad -\frac{1}{2}\sigma\bar{\psi}\mathcal{D}\psi - \frac{1}{8}\mathbf{i}\bar{\psi}\gamma\cdot G\psi - \frac{1}{2}\mathbf{i}\bar{\psi}^i\psi^jY_{ij} + \mathbf{i}\sigma\bar{\psi}\gamma\cdot T\psi - 8\mathbf{i}\sigma^2\bar{\psi}\chi \\
&\quad +\frac{1}{6}\sigma\bar{\psi}_\mu\gamma^\mu(\mathbf{i}\sigma\mathcal{D}\psi + \frac{1}{2}\mathbf{i}\mathcal{D}\sigma\psi - \frac{1}{4}\gamma\cdot G\psi + 2\sigma\gamma\cdot T\psi - 8\sigma^2\chi) \\
&\quad -\frac{1}{6}\bar{\psi}_\mu\gamma_\nu\psi(-2\sigma G^{\mu\nu} - 8\sigma^2T^{\mu\nu}) - \frac{1}{12}\sigma\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi G_{\mu\nu} \\
&\quad +\frac{1}{12}\mathbf{i}\sigma\bar{\psi}_\mu\psi_\nu(-2\sigma G^{\mu\nu} - 8\sigma^2T^{\mu\nu}) + \frac{1}{48}\mathbf{i}\sigma^2\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho G_{\mu\nu} \\
&\quad -\frac{1}{2}\sigma\bar{\psi}_\mu^i\gamma^\mu\psi^jY_{ij} + \frac{1}{6}\mathbf{i}\sigma^2\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^jY_{ij}, \tag{4.13}
\end{aligned}$$

The equation of motion for the auxiliary  $Y^{ij}$ , and the field equations for  $\sigma$  and  $\psi^i$  allow us to express the Standard Weyl matter fields  $Y^{ij}$ ,  $D$  and  $\chi^i$  in terms of the fields of the vector multiplet [37].

$$\begin{aligned}
\underline{Y}^{ij} &= \frac{1}{4}\mathbf{i}\sigma^{-1}\bar{\psi}^i\psi^j, \\
\underline{\chi}^i &= \frac{1}{8}\mathbf{i}\sigma^{-1}\mathcal{D}\psi^i + \frac{1}{16}\mathbf{i}\sigma^{-2}\mathcal{D}\sigma\psi^i - \frac{1}{32}\sigma^{-2}\gamma\cdot\widehat{G}\psi^i + \frac{1}{4}\sigma^{-1}\gamma\cdot\underline{T}\psi^i \\
&\quad +\frac{1}{32}\mathbf{i}\sigma^{-3}\psi_j\bar{\psi}^i\psi^j, \\
\underline{D} &= \frac{1}{4}\sigma^{-1}\Box^c\sigma + \frac{1}{8}\sigma^{-2}(\mathcal{D}_a\sigma)(\mathcal{D}^a\sigma) - \frac{1}{16}\sigma^{-2}\widehat{G}_{\mu\nu}\widehat{G}^{\mu\nu} \\
&\quad -\frac{1}{8}\sigma^{-2}\bar{\psi}\mathcal{D}\psi - \frac{1}{64}\sigma^{-4}\bar{\psi}^i\psi^j\bar{\psi}_i\psi_j - 4\mathbf{i}\sigma^{-1}\psi\underline{\chi} \\
&\quad +\left(-\frac{26}{3}\underline{T}_{ab} + 2\sigma^{-1}\widehat{G}_{ab} + \frac{1}{4}\mathbf{i}\sigma^{-2}\bar{\psi}\gamma_{ab}\psi\right)\underline{T}^{ab}, \tag{4.14}
\end{aligned}$$

where we have underlined the fields to indicate that they are now composite expressions. The equation of motion for  $C_\mu$  implies a Bianchi identity for an antisymmetric



2-form tensor gauge field  $B_{\mu\nu}$

$$\mathcal{D}_{[a}\widehat{H}_{bcd]} = \frac{3}{4}\widehat{G}_{[ab}\widehat{G}_{cd]}, \quad (4.15)$$

where the 3-form curvature tensor  $\widehat{H}_{abc}$  is defined as

$$\frac{1}{6}\varepsilon_{abcde}\widehat{H}^{cde} = 8\sigma^2\underline{T}_{ab} - \sigma\widehat{G}_{ab} - \frac{1}{4}\mathrm{i}\bar{\psi}\gamma_{ab}\psi. \quad (4.16)$$

This equation allows us to identify  $T_{ab}$  in terms of the elements of the vector multiplet and the supercovariant field strength  $\widehat{H}_{\mu\nu\rho}$ . Therefore, using the fields  $\sigma, C_\mu, B_{\mu\nu}$  and  $\psi^i$  along with the gauge fields  $e_\mu^a, \psi_\mu^i, b_\mu, V_\mu^{ij}$  we form an alternative Weyl multiplet: the dilaton Weyl multiplet, and the equations (4.14) and (4.16) present a map between the standard Weyl multiplet and the dilaton Weyl multiplet (3.21).

## 5. $\mathcal{N} = 2$ , $D = 5$ OFF-SHELL POINCARÉ SUPERGRAVITIES\*

In the previous section, we have constructed superconformal actions for the linear multiplet and for  $n$  number of vector multiplets. In this section, we concentrate on the construction of off-shell Poincaré supergravity theories for the standard Weyl multiplet and the dilaton Weyl multiplet. In order to do so, we shall first find appropriate superconformal supergravities such that when the redundant symmetries are fixed, they give rise to off-shell Poincaré supergravity theories. Once we construct an off-shell Poincaré supergravity, we discuss its R-symmetry gauging and show that it corresponds to the conventional minimal gauged on-shell  $\mathcal{N} = 2$ ,  $D = 5$  Poincaré supergravity.

### 5.1 Poincaré Supergravity in the Standard Weyl Multiplet

In [40, 65], a Poincaré supergravity was constructed by coupling the standard Weyl multiplet to a hypermultiplet and  $n$  number of vector multiplets. However, as our choices for the compensating multiplets are vector and linear multiplets, we shall devote this section to the construction of an off-shell Poincaré supergravity in the standard Weyl multiplet with vector and linear multiplet compensators.

A consistent superconformal supergravity is given by combining the linear multiplet action and vector multiplet action

$$\mathcal{L}_R^S = -\mathcal{L}_L^S - 3\mathcal{L}_V^S, \quad (5.1)$$

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\*Portions of this chapter are reprinted from *An off-shell formulation for internally gauged  $D=5$ ,  $N=2$  supergravity from superconformal methods* by Frederik Coomans and Mehmet Ozkan, 2013. JHEP **1301**, 099 (2013), Copyright 2013, with permission from SISSA; *Supersymmetric Completion of Gauss-Bonnet Combination in Five Dimensions* by Mehmet Ozkan and Yi Pang, 2013. JHEP **1303**, 158 (2013), Copyright 2013, with permission from SISSA.

where  $\mathcal{L}_L^S$  is given in (4.7) and  $\mathcal{L}_V^S$  is given in (4.10), and the superscript  $S$  refers to the fact that this action is designed for the standard Weyl multiplet. This action has redundant superconformal symmetries needed be fixed in order to obtain an off-shell Poincaré supergravity. The gauge fixing conditions adopted in this section are

$$L_{ij} = \frac{1}{\sqrt{2}}\delta_{ij}, \quad b_\mu = 0, \quad \varphi^i = 0, \quad (5.2)$$

where the first one breaks  $SU(2)_R$  to  $U(1)_R$  and fixes dilatation by setting  $L = 1$ . The second one fixes the special conformal symmetry, and the last choice fixes the  $S$ -supersymmetry. To maintain the gauge (5.2), the compensating transformations are required. Here we only present the compensating special supersymmetry and the compensating conformal boost with parameters

$$\eta_k = \frac{1}{3} \left( \gamma \cdot T \epsilon_k - \frac{1}{\sqrt{2}} N \delta_{ik} \epsilon^i + \frac{1}{\sqrt{2}} i \not{E} \delta_{ik} \epsilon^i + i \gamma^a V_a'^{(i} \delta^{j)l} \delta_{ik} \epsilon_j \right), \quad (5.3)$$

$$\Lambda_{K\mu} = -\frac{1}{4} i \bar{\epsilon} \phi_\mu - \frac{1}{4} i \bar{\eta} \psi_\mu + \bar{\epsilon} \gamma_\mu \chi. \quad (5.4)$$

Using the gauge fixing conditions, the bosonic part of the corresponding off-shell Poincaré supergravity is given by

$$\begin{aligned} e^{-1} \mathcal{L}_R^S &= \frac{1}{8} (\mathcal{C} + 3) R + \frac{1}{3} (104\mathcal{C} - 8) T^2 + 4(\mathcal{C} - 1) D - N^2 - P_\mu P^\mu + V_\mu'^{ij} V_{ij}'^\mu \\ &\quad - \sqrt{2} V_\mu P^\mu + \frac{3}{4} C_{IJK} \rho^I F_{\mu\nu}^J F^{\mu\nu K} + \frac{3}{2} C_{IJK} \rho^I \partial_\mu \rho^J \partial^\mu \rho^K - 3 C_{IJK} \rho^I Y_{ij}^J Y^{ij K} \\ &\quad - 12 C_{IJK} \rho^I \rho^J F_{\mu\nu}^K T^{\mu\nu} + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma\lambda} C_{IJK} A_\mu^I F_{\nu\rho}^J F_{\sigma\lambda}^K, \end{aligned} \quad (5.5)$$

where we have defined  $\mathcal{C} = C_{IJK} \rho^I \rho^J \rho^K$ . In the context of M-theory, the theory of five-dimensional  $\mathcal{N} = 2$  supergravity coupled to Abelian vector supermultiplets arise by compactifying eleven-dimensional supergravity, the low-energy theory of M-

theory, on a Calabi-Yau three-folds [51, 52].  $STU$  model corresponds to  $\mathcal{C} = STU$ , where  $S$ ,  $T$  and  $U$  are three vector moduli.

### 5.1.1 Gauged Model

As a result of our gauge choices (5.2), the  $U(1)_R$  symmetry of the off-shell Poincare theory (5.5) is gauged by the auxiliary vector  $V_\mu$ , i.e. the full  $U(1)_R$  covariant derivative for gravitino is given by,

$$\nabla_\mu \psi_\nu^i = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \psi_\nu^i - \frac{1}{2} V_\mu \delta^{ij} \psi_{\nu j}. \quad (5.6)$$

where  $V_\mu'^{ij}$ , the traceless part of  $V_\mu^{ij}$  does not appear in the  $U(1)_R$  covariant derivative for gravitino as a consequence of our gauge fixing choices (5.2). In this section, we discuss the  $U(1)_R$  gauging of the Poincaré the theory by physical vectors  $A_\mu^I$ . In the rest of the paper, we use the following notation

$$\mathcal{C} = C_{IJK} \rho^I \rho^J \rho^K, \quad \mathcal{C}_I = 3C_{IJK} \rho^I \rho^K, \quad \mathcal{C}_{IJ} = 6C_{IJK} \rho^K. \quad (5.7)$$

The off-shell gauged model is given by

$$\begin{aligned} e^{-1} \mathcal{L}_{gR}^S &= e^{-1} (\mathcal{L}_R^S - 3g_I \mathcal{L}_{VL}^I) |_{L=1} \\ &= \frac{1}{8} (\mathcal{C} + 3) R + \frac{1}{3} (104\mathcal{C} - 8) T^2 + 4(\mathcal{C} - 1) D - N^2 - P_\mu P^\mu + V_\mu'^{ij} V_{ij}'^\mu \\ &\quad - \sqrt{2} P_\mu V^\mu + \frac{1}{8} \mathcal{C}_{IJ} F_{\mu\nu}^I F^{\mu\nu J} + \frac{1}{4} \mathcal{C}_{IJ} \partial_\mu \rho^I \partial^\mu \rho^J - \frac{1}{2} \mathcal{C}_{IJ} Y_{ij}^I Y^{ij J} - 4\mathcal{C}_I F_{\mu\nu}^I T^{\mu\nu} \\ &\quad + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma\lambda} C_{IJK} A_\mu^I F_{\nu\rho}^J F_{\sigma\lambda}^K - \frac{3}{\sqrt{2}} g_I Y_{ij}^I \delta^{ij} - 3g_I P^\mu A_\mu^I - 3g_I \rho^I N. \end{aligned} \quad (5.8)$$

where  $L = 1$  indicates the gauge fixing condition (5.2). As the field equation of  $V_\mu$  implies that  $P_\mu = 0$ , we can immediately see that the  $P_\mu$  equation implies that

$$V_\mu = -\frac{3}{\sqrt{2}}g_I A_\mu^I, \quad (5.9)$$

hence, the auxiliary vector  $V_\mu$  is replaced by a linear combination of physical vectors  $A_\mu^I$

$$\nabla_\mu \psi_\nu^i = \left( \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} \right) \psi_\nu^i + \frac{3}{2\sqrt{2}}g_I A_\mu^I \delta^{ij} \psi_{\nu j}. \quad (5.10)$$

Therefore, the  $U(1)_R$  symmetry is gauged by the physical vectors. The equations of motion for  $D, T_{ab}, N$  and  $Y_{ij}^I$  lead to

$$\begin{aligned} 0 &= \mathcal{C} - 1, \\ 0 &= \frac{2}{3}(104\mathcal{C} - 8)T_{ab} - 4\mathcal{C}_I F_{ab}^I, \\ 0 &= 2N + 3g_I \rho^I, \\ 0 &= \mathcal{C}_{IJ} Y_{ij}^J + \frac{3}{\sqrt{2}}g_I \delta_{ij}. \end{aligned} \quad (5.11)$$

The field equation for  $D$  implies the constraint for very special geometry

$$C_{IJK} \rho^I \rho^J \rho^K = 1. \quad (5.12)$$

Eliminating  $T_{ab}, N$  and  $Y_{ij}^I$  according to their field equations gives rise to the following on-shell action

$$\begin{aligned} e^{-1} \mathcal{L}_{gR}^S|_{\text{on-shell}} &= \frac{1}{2}R + \frac{1}{8}(\mathcal{C}_{IJ} - \mathcal{C}_I \mathcal{C}_J) F_{\mu\nu}^I F^{\mu\nu J} + \frac{1}{4}\mathcal{C}_{IJ} \partial^\mu \rho^I \partial^\mu \rho^J \\ &\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda} C_{IJK} A_\mu^I F_{\nu\rho}^J F_{\sigma\lambda}^K + \Lambda(\rho), \end{aligned} \quad (5.13)$$

with

$$\Lambda(\rho) = \frac{9}{4}(g_I \rho^I)^2 + \frac{9}{2} \mathcal{C}^{IJ} g_I g_J, \quad (5.14)$$

where  $\mathcal{C}^{IJ}$  is the inverse of  $\mathcal{C}_{IJ}$  since  $\rho^I$  satisfies  $\rho^I \rho_I = 1$ . One can proceed further and truncate the on-shell vector multiplets to obtain the minimal gauged supergravity. In order to do so, we consider a single graviphoton via

$$\rho^I = \bar{\rho}^I, \quad A_\mu^I = \bar{\rho}^I A_\mu, \quad g_I = \bar{\rho}_I g, \quad (5.15)$$

where  $\bar{\rho}^I$  is VEV of the scalar at the critical value of the scalar potential (5.14). The truncation conditions in (5.15) are consistent with the supersymmetry transformation rules and lead to

$$e^{-1} \mathcal{L}_{gR}^{\min} = \frac{1}{2} R - \frac{3}{8} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma\lambda} A_\mu F_{\nu\rho} F_{\sigma\lambda} + 3g^2, \quad (5.16)$$

which reproduces the conventional minimal on-shell supergravity in five dimensions.

## 5.2 Poincaré Supergravity in the Dilaton Weyl Multiplet

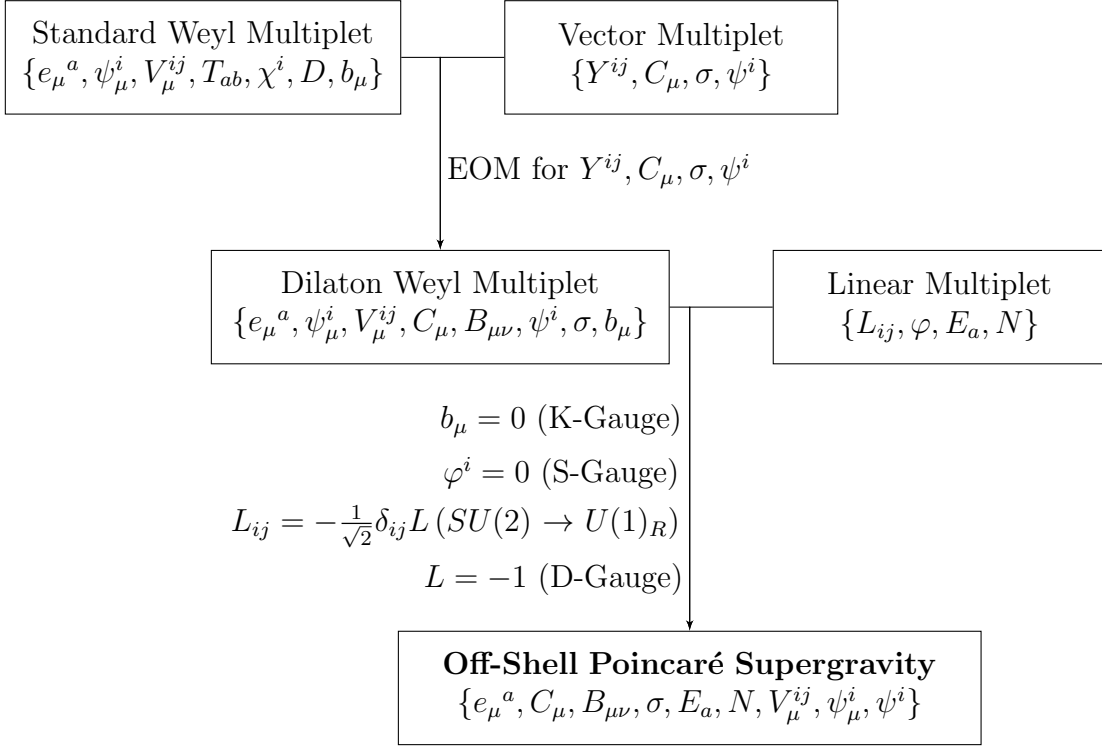
As opposed to the construction of an off-shell Poincaré supergravity in standard Weyl multiplet, where both vector and linear multiplets are required to compensate the redundant superconformal symmetries, a single compensating multiplet, is sufficient for constructing an off-shell Poincaré supergravity upon gauge fixing

$$\mathcal{L}_R^D = \mathcal{L}_L^D \quad (5.17)$$

where the superscript  $D$  emphasise the fact that this action is written in dilaton Weyl multiplet language. However, as discussed in the section 4.4, and described

FIGURE 5.1

The schematic description of how to obtain off-shell  $\mathcal{N} = 2, D = 5$  Poincaré Supergravity



schematically in Figure 5.1, one can find a map between the Weyl multiplets by considering the field equations for the elements of the vector multiplet in the vector multiplet action (4.13). Therefore, a Poincaré theory in dilaton Weyl multiplet can also be given as

$$\mathcal{L}_R^D = \mathcal{L}_L^S - 3\mathcal{L}_{AV} \quad (5.18)$$

where  $\mathcal{L}_{AV}$  is given in (4.13). Thus, once the field equations for the elements of the vector multiplets are solved, one goes back to (5.17). Therefore, we give the

superconformal supergravity for the dilaton Weyl multiplet as

$$\begin{aligned}
e^{-1}\mathcal{L}_R^D &= L^{-1}L_{ij}\square^c L^{ij} - L^{ij}\mathcal{D}_\mu L_{k(i}\mathcal{D}^\mu L_{j)m}L^{km}L^{-3} + N^2L^{-1} \\
&\quad - E_\mu E^\mu L^{-1} + \frac{8}{3}L\underline{T}^2 + 4\underline{D}L - \frac{1}{2}L^{-3}E^{\mu\nu}L_k^l\partial_\mu L^{kp}\partial_\nu L_{pl} \\
&\quad + 2E^{\mu\nu}\partial_\mu(L^{-1}E_\nu + V_\nu^{ij}L_{ij}L^{-1}), \tag{5.19}
\end{aligned}$$

where the underlined composite equations are given in (3.21). At this stage, one may choose to apply two different gauge fixing conditions, as we shall discuss in detail in the following subsections.

### 5.2.1 $L = -1$ Gauge Fixing

The first gauge fixing choice we adopt here is

$$L_{ij} = -\frac{1}{\sqrt{2}}\delta_{ij}, \quad b_\mu = 0, \quad \varphi^i = 0. \tag{5.20}$$

The first gauge condition breaks the  $SU(2)$  symmetry down to  $U(1)_R$ , and fixes the dilatation symmetry by the choice  $L = -1$ . The second condition fixes the special conformal symmetry whereas the last one fixes the S-supersymmetry. In order for these gauge conditions to be invariant under supersymmetry, one needs to add compensating conformal boost transformations with parameter

$$\Lambda_{K\mu} = -\frac{1}{4}\mathrm{i}\bar{\epsilon}\phi_\mu - \frac{1}{4}\mathrm{i}\bar{\eta}\psi_\mu + \bar{\epsilon}\gamma_\mu\chi \tag{5.21}$$

and compensating S-supersymmetry transformations with parameter

$$\eta_k = \frac{1}{3}\left(\gamma \cdot T\epsilon_k + \frac{1}{\sqrt{2}}N\delta_{ik}\epsilon^i - \frac{1}{\sqrt{2}}\mathrm{i}\not{E}\delta_{ik}\epsilon^i - \mathrm{i}\gamma^\mu V_\mu^{(i}{}^j)\delta_{ik}\epsilon_j\right). \tag{5.22}$$



Performing all the steps of gauge fixing, and using the expressions for the dependent gauge fields (3.8) into the Lagrangian (5.19), we end up with the following off-shell Lagrangian for  $\mathcal{N} = 2$ ,  $D = 5$  Poincaré supergravity (up to 4-fermion terms)

$$\begin{aligned}
e^{-1}\mathcal{L}_{0,DW}|_{L=-1} &= \frac{1}{2}R - \frac{1}{4}\sigma^{-2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{6}\sigma^{-4}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{3}{2}\sigma^{-2}\partial_\mu\sigma\partial^\mu\sigma \\
&\quad - N^2 - P_\mu P^\mu + \sqrt{2}P_\mu V^\mu + V_\mu^{ij}V_j'^\mu \\
&\quad - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{3}{2}\sigma^{-2}\bar{\psi}\not{D}'\psi - \frac{3}{2}i\sigma^{-2}\bar{\psi}\gamma^\mu\gamma^\rho\psi_\mu\partial_\rho\sigma \\
&\quad - \frac{1}{8}i\sigma^{-1}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu G_{\rho\sigma} - \frac{1}{4}i\sigma^{-1}\bar{\psi}_\mu\psi_\nu G^{\mu\nu} - \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi G_{\nu\rho} \\
&\quad + \frac{1}{2}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi G^{\mu\nu} - \frac{1}{8}i\sigma^{-3}\bar{\psi}\gamma\cdot G\psi + \frac{1}{2\sqrt{2}}\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}\psi_\nu^j P_\rho\delta_{ij} \\
&\quad - \frac{1}{24}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma\lambda}\psi_\nu H_{\rho\sigma\lambda} + \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi_\rho H^{\mu\nu\rho} \\
&\quad - \frac{1}{6}i\sigma^{-3}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi H_{\nu\rho\sigma} + \frac{1}{2}i\sigma^{-3}\bar{\psi}_\mu\gamma_{\nu\rho}\psi H^{\mu\nu\rho} - \frac{5}{24}\sigma^{-4}\bar{\psi}\gamma\cdot H\psi,
\end{aligned} \tag{5.23}$$

where the subscript  $L = -1$  is shorthand for the gauge fixing described in (5.20). Notice that we have decomposed the field  $V_\mu^{ij}$  into its trace and traceless part, i.e.  $V_\mu^{ij} = V_\mu'^{ij} + \frac{1}{2}\delta^{ij}V_\mu$  with  $V_\mu'^{ij}\delta_{ij} = 0$ . The 2- and 3-form field strengths are defined as

$$\begin{aligned}
G_{\mu\nu} &= 2\partial_{[\mu}C_{\nu]}, \\
H_{\mu\nu\rho} &= B_{\mu\nu\rho} + \frac{3}{2}C_{[\mu}G_{\nu\rho]},
\end{aligned} \tag{5.24}$$

where  $B_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ , and the  $U(1)_R$  covariant derivative  $D_\mu\psi_\nu^i$  and full  $SU(2)$  covariant derivative  $D'_\mu\psi^i$  are defined as

$$\begin{aligned}
D_\mu\psi_\nu^i &= \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\psi_\nu^i - \frac{1}{2}V_\mu\delta^{ij}\psi_{\nu j}, \\
D'_\mu\psi^i &= \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\psi^i + V_\mu^i{}_j\psi^j - \frac{1}{2}V_\mu\delta^{ij}\psi_j.
\end{aligned} \tag{5.25}$$

The  $\mathcal{N} = 2$  off-shell supergravity that we constructed above by means of superconformal tensor calculus has the following field content

$$(e_\mu^a, \psi_\mu^i, C_\mu, B_{\mu\nu}, \psi^i, \sigma, E_\mu, N, V_\mu, V_\mu'^{ij}) \quad (5.26)$$

with (10, 32, 4, 6, 8, 1, 4, 1, 4, 10) off-shell degrees of freedoms respectively. Therefore our off-shell Poincaré multiplet has 40 + 40 off-shell degrees of freedom. The supersymmetry transformations, up to 3-fermions, are

$$\begin{aligned} \delta e_\mu^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu, \\ \delta \psi_\mu^i &= (\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}) \epsilon^i - V_\mu'^{ij} \epsilon_j + i \gamma \cdot \underline{T} \gamma_\mu \epsilon^i - i \gamma_\mu \eta^i, \\ \delta V_\mu &= -\frac{3}{2} i \bar{\epsilon}^i \phi_\mu^j \delta_{ij} + 4 \bar{\epsilon}^i \gamma_\mu \underline{\chi}^j \delta_{ij} + i \bar{\epsilon}^i \gamma \cdot \underline{T} \psi_\mu^j \delta_{ij} + \frac{3}{2} i \bar{\eta}^i \psi_\mu^j \delta_{ij}, \\ \delta V_\mu'^{ij} &= -\frac{3}{2} i \bar{\epsilon}^i \phi_\mu^j + 4 \bar{\epsilon}^i \gamma_\mu \underline{\chi}^j + i \bar{\epsilon}^i \gamma \cdot \underline{T} \psi_\mu^j + \frac{3}{2} i \bar{\eta}^i \psi_\mu^j + \frac{3}{2} i \bar{\epsilon}^k \phi_\mu^l \delta_{kl} \delta^{ij} \\ &\quad - 2 \bar{\epsilon}^k \gamma_\mu \underline{\chi}^l \delta_{kl} \delta^{ij} - \frac{1}{2} i \bar{\epsilon}^k \gamma \cdot \underline{T} \psi_\mu^l \delta_{kl} \delta^{ij} - \frac{3}{4} i \bar{\eta}^k \psi_\mu^l \delta_{kl} \delta^{ij}, \\ \delta C_\mu &= -\frac{1}{2} i \sigma \bar{\epsilon} \psi_\mu + \frac{1}{2} \bar{\epsilon} \gamma_\mu \psi, \\ \delta B_{\mu\nu} &= \frac{1}{2} \sigma^2 \bar{\epsilon} \gamma_{[\mu} \psi_{\nu]} + \frac{1}{2} i \sigma \bar{\epsilon} \gamma_{\mu\nu} \psi + C_{[\mu} \delta(\epsilon) C_{\nu]}, \\ \delta \psi^i &= -\frac{1}{4} \gamma \cdot G \epsilon^i - \frac{1}{2} i \not{\partial} \sigma \epsilon^i + \sigma \gamma \cdot \underline{T} \epsilon^i - \frac{1}{4} i \sigma^{-1} \epsilon_j \bar{\psi}^i \psi^j + \sigma \eta^i, \\ \delta \sigma &= \frac{1}{2} i \bar{\epsilon} \psi, \\ \delta E_a &= -\frac{1}{2\sqrt{2}} \bar{\epsilon}^i \gamma_{ab} \gamma^c V_{c(i}^k \delta_{j)k} \psi^{bj} + \frac{1}{4} \bar{\epsilon} \gamma_{ab} \gamma^c E_c \psi^b + \frac{1}{4} i \bar{\epsilon} \gamma_{ab} N \psi^b \\ &\quad - \frac{1}{2\sqrt{2}} i \bar{\epsilon}^i \gamma_{ab} \gamma \cdot \underline{T} \psi^{bj} \delta_{ij} + \frac{3}{2\sqrt{2}} i \bar{\epsilon}^i \gamma_{ab} \phi^{bj} \delta_{ij}, \\ \delta N &= -\frac{1}{2\sqrt{2}} i \bar{\epsilon}^i \gamma^a \gamma^b V_{b(i}^k \delta_{j)k} \psi_a^j + \frac{1}{4} \bar{\epsilon} \gamma^a \gamma^b E_b \psi_a - \frac{1}{4} \bar{\epsilon} \gamma_a N \psi^a \\ &\quad + \frac{1}{2\sqrt{2}} \bar{\epsilon}^i \gamma^a \gamma \cdot \underline{T} \psi_a^j \delta_{ij} - \frac{3}{2\sqrt{2}} \bar{\epsilon}^i \gamma^a \phi_a^j \delta_{ij} - 2\sqrt{2} i \bar{\epsilon}^i \underline{\chi}^j \delta_{ij}, \end{aligned} \quad (5.27)$$

where the parameter  $\eta^i$  is as described in (5.22), and the composite expression for  $\underline{Y}^{ij}$  can be found for the ungauged scenario in (4.14). Note that the  $U(1)_R$  symmetry

of the off-shell supergravity is gauged via the auxiliary  $V_\mu$

$$\delta_\lambda V_\mu = \partial_\mu \lambda, \quad \delta_\lambda \psi_\mu^i = \frac{1}{2} \delta^{ij} \lambda \psi_{\mu j}, \quad \delta_\lambda \psi^i = \frac{1}{2} \delta^{ij} \lambda \psi_j, \quad (5.28)$$

where  $\lambda$  is the parameter of the  $U(1)_R$  symmetry. Also note that the CS term, which is characteristic of the  $\mathcal{N} = 2, D = 5$  formulation is hidden inside the term  $H_{\mu\nu\rho} H^{\mu\nu\rho}$ , and it becomes manifest in the action in the on-shell formalism due to the dualization of  $H_{\mu\nu\rho}$  as we shall discuss in the following section.

### 5.2.2 Gauged Model

Our starting point for the construction of the internally gauged supergravity is the following Lagrangian

$$\mathcal{L}_g^D = \mathcal{L}_L^V - 3\mathcal{L}_{AV} - 3g\mathcal{L}_{VL}, \quad (5.29)$$

The field equations for  $Y^{ij}, \sigma, \psi^i$  and  $C_\mu$  give rise to the following map between the standard Weyl multiplet and the dilaton Weyl multiplet

$$\underline{Y}_g^{ij} = \frac{1}{4} i \sigma^{-1} \bar{\psi}^i \psi^j - \frac{1}{2} g \sigma^{-1} L^{ij}, \quad (5.30)$$

$$\begin{aligned} \underline{\chi}_g^i &= \frac{1}{8} i \sigma^{-1} \mathcal{D} \psi^i + \frac{1}{16} i \sigma^{-2} \mathcal{D} \sigma \psi^i - \frac{1}{32} \sigma^{-2} \gamma \cdot G \psi^i \\ &\quad + \frac{1}{4} \sigma^{-1} \gamma \cdot T \psi^i + \frac{1}{8} \sigma^{-2} Y^{ij} \psi_j + \frac{1}{8} g \sigma^{-2} \varphi^i, \end{aligned} \quad (5.31)$$

$$\begin{aligned} \underline{D}_g &= \frac{1}{4} \sigma^{-1} \square^c \sigma + \frac{1}{8} \sigma^{-2} (\mathcal{D}_a \sigma) (\mathcal{D}^a \sigma) - \frac{1}{16} \sigma^{-2} G_{\mu\nu} G^{\mu\nu} \\ &\quad - \frac{1}{8} \sigma^{-2} \bar{\psi} \mathcal{D} \psi + \frac{1}{4} \sigma^{-2} Y^{ij} Y_{ij} - 4i \sigma^{-1} \bar{\psi} \chi + \frac{1}{8} \sigma^{-2} \bar{\psi}_\mu \gamma_\nu \psi G^{\mu\nu} \\ &\quad - \frac{1}{16} i \sigma^{-1} \bar{\psi}_\mu \psi_\mu G^{\mu\nu} + \left( -\frac{26}{3} T_{ab} + 2\sigma^{-1} \widehat{G}_{ab} + \frac{1}{4} i \sigma^{-2} \bar{\psi} \gamma_{ab} \psi \right) T^{ab} \\ &\quad + \frac{1}{4} g \sigma^{-2} N, \end{aligned}$$

$$\mathcal{D}_{[a} \widehat{\mathcal{H}}_{bcd]} = \frac{3}{4} \widehat{G}_{[ab} \widehat{G}_{cd]} + \frac{1}{2} g \mathcal{D}_{[a} E_{bcd]}, \quad (5.32)$$

where

$$-\frac{1}{6}\varepsilon_{abcde}\widehat{\mathcal{H}}^{edc} = 8\sigma^2\underline{T}_{ab} - \sigma\widehat{G}_{ab} - \frac{1}{4}i\bar{\psi}\gamma_{ab}\psi. \quad (5.33)$$

The subscript  $g$  in the equations (5.30), (5.31) and (5.32) indicates that the expressions for  $Y^{ij}$ ,  $\chi^i$  and  $D$  now pick up  $g$  dependent terms. Note that the expressions for  $\chi_g^i$  and  $D_g$  are given up to 3- and 4-fermion terms respectively. Comparing the above map with the one in the ungauged case, (3.21), it is clear that the map gets deformed by the gauging. The Bianchi identity (5.32) implies that

$$\widehat{\mathcal{H}}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + \frac{3}{2}C_{[\mu}G_{\nu\rho]} + \frac{1}{2}gE_{\mu\nu\rho} - \frac{3}{4}\sigma^2\bar{\psi}_{[\mu}\gamma_{\nu}\psi_{\rho]} - \frac{3}{2}i\sigma\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi. \quad (5.34)$$

The above equation for  $\widehat{\mathcal{H}}_{\mu\nu\rho}$  is clearly not gauge invariant since  $E_{\mu\nu\rho}$  has the gauge invariance  $\delta_\Lambda E_{\mu\nu\rho} = 3\partial_{[\mu}\Lambda_{\nu\rho]}$ . In order to balance that out,  $B_{\mu\nu}$  needs to have the additional gauge invariance,  $\delta_\Lambda B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]} - \frac{1}{2}\Lambda G_{\mu\nu} - \frac{1}{2}g\Lambda_{\mu\nu}$ .

Using the above expressions for  $Y^{ij}$ ,  $D$ ,  $T_{ab}$  and  $\chi^i$  in the Lagrangian (5.29) and imposing the gauge fixing conditions (5.20) we obtain the following off-shell Poincaré Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_g^D &= \frac{1}{2}R - \frac{1}{4}\sigma^{-2}G_{\mu\nu}G^{\mu\nu} - 2gC_\mu P^\mu - \frac{1}{6}\sigma^{-4}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} \\ &\quad - N^2 - gN(\sigma^{-2} + 2\sigma) - g^2(\frac{1}{4}\sigma^{-4} - \sigma^{-1}) \\ &\quad - P^\mu P_\mu + \sqrt{2}P^\mu V_\mu + V_\mu{}^{ij}V'_{\mu ij} - \frac{3}{2}\sigma^{-2}\partial_\mu\sigma\partial^\mu\sigma \\ &\quad - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{3}{2}\sigma^{-2}\bar{\psi}\not{D}'\psi - \frac{3}{2}i\sigma^{-2}\bar{\psi}\gamma^\mu\gamma^\rho\psi_\mu\partial_\rho\sigma \\ &\quad - \frac{1}{8}i\sigma^{-1}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu G_{\rho\sigma} - \frac{1}{4}i\sigma^{-1}\bar{\psi}_\mu\psi_\nu G^{\mu\nu} - \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi G_{\nu\rho} \\ &\quad + \frac{1}{2}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi G^{\mu\nu} - \frac{1}{8}i\sigma^{-3}\bar{\psi}\gamma \cdot G\psi + \frac{1}{2\sqrt{2}}\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}\psi_\nu^j P_\rho\delta_{ij} \\ &\quad - \frac{1}{24}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma\lambda}\psi_\nu\mathcal{H}_{\rho\sigma\lambda} + \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi_\rho\mathcal{H}^{\mu\nu\rho} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{6}i\sigma^{-3}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi\mathcal{H}_{\nu\rho\sigma} + \frac{1}{2}i\sigma^{-3}\bar{\psi}_\mu\gamma_{\nu\rho}\psi\mathcal{H}^{\mu\nu\rho} - \frac{5}{24}\sigma^{-4}\bar{\psi}\gamma\cdot\mathcal{H}\psi \\
& + \frac{1}{4\sqrt{2}}ig\sigma^{-2}\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} + \frac{1}{2\sqrt{2}}ig\sigma\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} - \frac{1}{\sqrt{2}}g\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} \\
& + \frac{1}{\sqrt{2}}g\sigma^{-3}\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} - \frac{1}{4\sqrt{2}}ig\sigma^{-1}\bar{\psi}^i\psi^j\delta_{ij} - \frac{5}{4\sqrt{2}}ig\sigma^{-4}\bar{\psi}^i\psi^j\delta_{ij}, \tag{5.35}
\end{aligned}$$

where  $\mathcal{H}_{\mu\nu\rho}$  is defined as

$$\mathcal{H}_{\mu\nu\rho} = B_{\mu\nu\rho} + \frac{3}{2}C_{[\mu}G_{\nu\rho]} + \frac{1}{2}gE_{\mu\nu\rho}. \tag{5.36}$$

This Lagrangian is invariant under the transformation rules given in (5.27), where the underlined fields are now to be evaluated using the deformed expressions, (5.30) to (5.33).

This theory has a  $U(1)_V \times U(1)_C$  gauge group parametrized by  $\lambda$  and  $\eta$  respectively. The fermion covariant derivatives are defined in (5.25) and contain  $V_\mu$ . The gauge transformations of the relevant fields are given by

$$\begin{aligned}
\delta_\lambda V_\mu &= \partial_\mu \lambda, & \delta_\eta C_\mu &= \partial_\mu \eta, \\
\delta_\lambda \psi_\mu^i &= \frac{1}{2}\lambda\delta^{ij}\psi_{\mu j}, & \delta_\lambda \psi^i &= \frac{1}{2}\lambda\delta^{ij}\psi_j. \tag{5.37}
\end{aligned}$$

Let's now eliminate the auxiliaries  $V_\mu, V_\mu'^{ij}, N$  and  $P_\mu$ , and present the dualization of the 2-form gauge field  $B_{\mu\nu}$  to a vector field  $\tilde{C}_\mu$  and discuss the resulting on-shell theory. We will show that the on-shell theory describes Einstein-Maxwell supergravity as constructed in [66, 67].

Let us start with the field equations for  $N$  and  $V_\mu'^{ij}$

$$0 = 2N + g(\sigma^{-2} + 2\sigma), \tag{5.38}$$

$$0 = V_\mu'^{ij} - \frac{3}{4}\sigma^{-2}\bar{\psi}^i\gamma_\mu\psi^j. \tag{5.39}$$

Using these two field equations in (5.35), we obtain the following Lagrangian (up to 4-fermion terms)

$$\begin{aligned}
e^{-1}\mathcal{L}_2 = & \frac{1}{2}R + g^2(2\sigma^{-1} + \sigma^2) - \frac{3}{2}\sigma^{-2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4}\sigma^{-2}G_{\mu\nu}G^{\mu\nu} \\
& - \frac{1}{6}\sigma^{-4}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} - P^\mu P_\mu + \sqrt{2}P^\mu V_\mu - 2gC_\mu P^\mu \\
& - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{3}{2}\sigma^{-2}\bar{\psi}\not{D}'\psi - \frac{3}{2}i\sigma^{-2}\bar{\psi}\gamma^\mu\gamma^\rho\psi_\mu\partial_\rho\sigma \\
& - \frac{1}{8}i\sigma^{-1}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu G_{\rho\sigma} - \frac{1}{4}i\sigma^{-1}\bar{\psi}_\mu\psi_\nu G^{\mu\nu} - \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi G_{\nu\rho} \\
& + \frac{1}{2}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi G^{\mu\nu} - \frac{1}{8}i\sigma^{-3}\bar{\psi}\gamma\cdot G\psi + \frac{1}{2\sqrt{2}}\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}\psi_\nu^j P_\rho\delta_{ij} \\
& - \frac{1}{24}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma\lambda}\psi_\nu\mathcal{H}_{\rho\sigma\lambda} + \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi_\rho\mathcal{H}^{\mu\nu\rho} \\
& - \frac{1}{6}i\sigma^{-3}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi\mathcal{H}_{\nu\rho\sigma} + \frac{1}{2}i\sigma^{-3}\bar{\psi}_\mu\gamma_{\nu\rho}\psi\mathcal{H}^{\mu\nu\rho} - \frac{5}{24}\sigma^{-4}\bar{\psi}\gamma\cdot\mathcal{H}\psi \\
& + \frac{1}{4\sqrt{2}}ig\sigma^{-2}\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} + \frac{1}{2\sqrt{2}}ig\sigma\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} - \frac{1}{\sqrt{2}}g\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} \\
& + \frac{1}{\sqrt{2}}g\sigma^{-3}\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} - \frac{1}{4\sqrt{2}}ig\sigma^{-1}\bar{\psi}^i\psi^j\delta_{ij} - \frac{5}{4\sqrt{2}}ig\sigma^{-4}\bar{\psi}^i\psi^j\delta_{ij}, \quad (5.40)
\end{aligned}$$

where

$$D_\mu\psi^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\psi^i - \frac{1}{2}V_\mu\delta^{ij}\psi_j. \quad (5.41)$$

Before proceeding it is useful to dualize the 2-form  $B_{\mu\nu}$  to a vector  $\tilde{C}_\mu$ . We do this by adding a Lagrange multiplier term

$$\mathcal{L}' = -\frac{1}{6}\varepsilon^{\mu\nu\rho\sigma\lambda}B_{\mu\nu\rho}\partial_\sigma\tilde{C}_\lambda. \quad (5.42)$$

This Lagrange multiplier introduces another  $U(1)$  symmetry in the theory which we will denote  $U(1)_{\tilde{C}}$  and which is parametrized by  $\tilde{\eta}$ . Since the Bianchi identity  $\partial^{[\mu}B^{\nu\rho\sigma]} = 0$  is now imposed by the field equation for  $\tilde{C}_\mu$ , we can treat  $B_{\mu\nu\rho}$  as an independent field and compute its field equation. Taking both the Lagrangian (5.40)

and the Lagrange multiplier term into account, the field equation for  $B_{\mu\nu\rho}$  reads

$$\begin{aligned}\mathcal{H}^{\mu\nu\rho} &= -\frac{1}{2}\sigma^4\epsilon^{\mu\nu\rho\sigma\lambda}\partial_\sigma\tilde{C}_\lambda - \frac{1}{8}\sigma^2\bar{\psi}_\sigma\gamma^{\mu\nu\rho\sigma\lambda}\psi_\lambda + \frac{3}{4}\sigma^2\bar{\psi}^{[\mu}\gamma^\nu\psi^{\rho]} \\ &\quad + \frac{1}{2}i\sigma^{-3}\bar{\psi}_\sigma\gamma^{\mu\nu\rho\sigma}\psi + \frac{3}{2}i\sigma\bar{\psi}^\mu\gamma^{\nu\rho}\psi - \frac{5}{8}\bar{\psi}\gamma^{\mu\nu\rho}\psi\end{aligned}\quad (5.43)$$

or, using (5.36),

$$\begin{aligned}B^{\mu\nu\rho} &= \mathcal{H}^{\mu\nu\rho} - \frac{3}{2}C^{[\mu}G^{\nu\rho]} - \frac{1}{2}gE^{\mu\nu\rho} \\ &= -\frac{1}{2}\sigma^4\epsilon^{\mu\nu\rho\sigma\lambda}\partial_\sigma\tilde{C}_\lambda - \frac{3}{2}C^{[\mu}G^{\nu\rho]} - \frac{1}{2}gE^{\mu\nu\rho} - \frac{1}{8}\sigma^2\bar{\psi}_\sigma\gamma^{\mu\nu\rho\sigma\lambda}\psi_\lambda \\ &\quad + \frac{3}{4}\sigma^2\bar{\psi}^{[\mu}\gamma^\nu\psi^{\rho]} + \frac{1}{2}i\sigma^{-3}\bar{\psi}_\sigma\gamma^{\mu\nu\rho\sigma}\psi + \frac{3}{2}i\sigma\bar{\psi}^\mu\gamma^{\nu\rho}\psi - \frac{5}{8}\bar{\psi}\gamma^{\mu\nu\rho}\psi.\end{aligned}\quad (5.44)$$

From (5.43) we can read of the transformation rule for  $\tilde{C}_\mu$  can be given as  $\delta\tilde{C}_\mu = -\frac{1}{2}i\sigma^{-2}\bar{\epsilon}\psi_\mu + \frac{1}{2}\sigma^{-3}\bar{\epsilon}\gamma_\mu\psi$ . Using (5.44), the Lagrangian now reads (up to 4-fermion terms)

$$\begin{aligned}e^{-1}(\mathcal{L}_2 + \mathcal{L}') &= \frac{1}{2}R + g^2(2\sigma^{-1} + \sigma^2) - P^\mu P_\mu + \sqrt{2}P^\mu V_\mu - \frac{3}{2}\sigma^{-2}\partial_\mu\sigma\partial^\mu\sigma \\ &\quad - \frac{1}{4}\sigma^{-2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\sigma^4\partial_{[\mu}\tilde{C}_{\nu]} \partial^\mu\tilde{C}^\nu - 2gC_\mu P^\mu + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda}G_{\mu\nu}C_\rho\partial_\sigma\tilde{C}_\lambda \\ &\quad + \frac{1}{12}g\epsilon^{\mu\nu\rho\sigma\lambda}E_{\mu\nu\rho}\partial_\sigma\tilde{C}_\lambda - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{3}{2}\sigma^{-2}\bar{\psi}\not{D}\psi \\ &\quad - \frac{3}{2}i\sigma^{-2}\bar{\psi}\gamma^\mu\gamma^\rho\psi_\mu\partial_\rho\sigma - \frac{1}{8}i\sigma^{-1}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu G_{\rho\sigma} - \frac{1}{4}i\sigma^{-1}\bar{\psi}_\mu\psi_\nu G^{\mu\nu} \\ &\quad - \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi G_{\nu\rho} + \frac{1}{2}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi G^{\mu\nu} - \frac{1}{8}i\sigma^{-3}\bar{\psi}\gamma \cdot G\psi \\ &\quad + \frac{1}{2\sqrt{2}}\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}\psi_\nu^j P_\rho\delta_{ij} - \frac{1}{4}i\sigma^2\bar{\psi}_{[\mu}\psi_{\nu]}\partial^\mu\tilde{C}^\nu - \frac{1}{8}i\sigma^2\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu\partial_\rho\tilde{C}_\sigma \\ &\quad - \sigma\bar{\psi}_{[\mu}\gamma_{\nu]}\psi\partial^\mu\tilde{C}^\nu + \frac{1}{2}\sigma\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi\partial_\nu\tilde{C}_\rho + \frac{5}{8}i\bar{\psi}\gamma_{\mu\nu}\psi\partial^\mu\tilde{C}^\nu \\ &\quad + \frac{1}{4\sqrt{2}}ig\sigma^{-2}\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} + \frac{1}{2\sqrt{2}}ig\sigma\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} - \frac{1}{\sqrt{2}}g\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} \\ &\quad + \frac{1}{\sqrt{2}}g\sigma^{-3}\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} - \frac{1}{4\sqrt{2}}ig\sigma^{-1}\bar{\psi}^i\psi^j\delta_{ij} - \frac{5}{4\sqrt{2}}ig\sigma^{-4}\bar{\psi}^i\psi^j\delta_{ij}.\end{aligned}\quad (5.45)$$

Let us now consider the field equations for  $V_\mu$  and  $E_{\mu\nu\rho}$

$$0 = P^\mu + \frac{1}{4\sqrt{2}}\bar{\psi}_\nu^i\gamma^{\mu\nu\rho}\psi_\rho^j\delta_{ij} - \frac{3}{4\sqrt{2}}\sigma^{-2}\bar{\psi}_i\gamma^\mu\psi_j\delta^{ij}, \quad (5.46)$$

$$0 = \varepsilon^{\mu\nu\rho\sigma\lambda}\partial_\mu\left(-P_\nu + \frac{1}{\sqrt{2}}V_\nu - gC_\nu - \frac{1}{2}g\tilde{C}_\nu + \frac{1}{4\sqrt{2}}\bar{\psi}_i^\tau\gamma_{\nu\tau\xi}\psi_j^\xi\delta^{ij}\right). \quad (5.47)$$

The latter equation implies that

$$P_\mu = \partial_\mu\phi + \frac{1}{\sqrt{2}}V_\mu - gC_\mu - \frac{1}{2}g\tilde{C}_\mu + \frac{1}{4\sqrt{2}}\bar{\psi}_i^\tau\gamma_{\mu\tau\xi}\psi_j^\xi\delta^{ij}, \quad (5.48)$$

where  $\phi$  is a Stueckelberg scalar that transforms under  $U(1)_V \times U(1)_C \times U(1)_{\tilde{C}}$  as

$$\delta_g\phi = -\frac{1}{\sqrt{2}}\lambda + g\eta + \frac{1}{2}g\tilde{\eta}. \quad (5.49)$$

We can break the gauge group down to  $U(1)^2$  by fixing the Stueckelberg scalar to a constant  $\phi = \phi_0$ . If we now use (5.48) in combination with (5.46) we obtain

$$V_\mu = -\sqrt{2}g(C_\mu + \frac{1}{2}\tilde{C}_\mu) + \frac{3}{4}\sigma^{-2}\bar{\psi}^i\gamma_\mu\psi^j\delta_{ij} \quad (5.50)$$

and find a decomposition law for the  $U(1)$  parameters

$$\lambda = -\sqrt{2}g(\eta + \frac{1}{2}\tilde{\eta}). \quad (5.51)$$

Using this in the Lagrangian given in (5.45) we find that the on-shell theory is given, up to 4-fermion terms, by



$$\begin{aligned}
e^{-1}\mathcal{L}_{EM} &= \frac{1}{2}R + g^2(2\sigma^{-1} + \sigma^2) - \frac{3}{2}\sigma^{-2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4}\sigma^{-2}G_{\mu\nu}G^{\mu\nu} \\
&\quad - \frac{1}{2}\sigma^4\partial_{[\mu}\tilde{C}_{\nu]}\partial^\mu\tilde{C}^\nu + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda}G_{\mu\nu}C_\rho\partial_\sigma\tilde{C}_\lambda \\
&\quad - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\tilde{\nabla}_\nu\psi_\rho - \frac{3}{2}\sigma^{-2}\bar{\psi}\tilde{\nabla}\psi \\
&\quad - \frac{3}{2}i\sigma^{-2}\bar{\psi}\gamma^\mu\gamma^\rho\psi_\mu\partial_\rho\sigma - \frac{1}{8}i\sigma^{-1}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu G_{\rho\sigma} - \frac{1}{4}i\sigma^{-1}\bar{\psi}_\mu\psi_\nu G^{\mu\nu} \\
&\quad - \frac{1}{4}\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi G_{\nu\rho} + \frac{1}{2}\sigma^{-2}\bar{\psi}_\mu\gamma_\nu\psi G^{\mu\nu} - \frac{1}{8}i\sigma^{-3}\bar{\psi}\gamma\cdot G\psi \\
&\quad - \frac{1}{4}i\sigma^2\bar{\psi}_{[\mu}\psi_{\nu]}\partial^\mu\tilde{C}^\nu - \frac{1}{8}i\sigma^2\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu\partial_\rho\tilde{C}_\sigma - \sigma\bar{\psi}_{[\mu}\gamma_{\nu]}\psi\partial^\mu\tilde{C}^\nu \\
&\quad + \frac{1}{2}\sigma\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi\partial_\nu\tilde{C}_\rho + \frac{5}{8}i\bar{\psi}\gamma_{\mu\nu}\psi\partial^\mu\tilde{C}^\nu \\
&\quad + \frac{1}{4\sqrt{2}}ig\sigma^{-2}\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} + \frac{1}{2\sqrt{2}}ig\sigma\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} - \frac{1}{\sqrt{2}}g\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} \\
&\quad + \frac{1}{\sqrt{2}}g\sigma^{-3}\bar{\psi}_\mu^i\gamma^\mu\psi^j\delta_{ij} - \frac{1}{4\sqrt{2}}ig\sigma^{-1}\bar{\psi}^i\psi^j\delta_{ij} - \frac{5}{4\sqrt{2}}ig\sigma^{-4}\bar{\psi}^i\psi^j\delta_{ij}, \quad (5.52)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\nabla}_\mu\psi_\nu^i &= \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\psi_\nu^i + \frac{1}{\sqrt{2}}g(C_\mu + \frac{1}{2}\tilde{C}_\mu)\delta^{ij}\psi_{\nu j}, \\
\tilde{\nabla}_\mu\psi^i &= \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\psi^i + \frac{1}{\sqrt{2}}g(C_\mu + \frac{1}{2}\tilde{C}_\mu)\delta^{ij}\psi_j. \quad (5.53)
\end{aligned}$$

From (5.37) we find that  $\psi_\mu^i$  transforms under the remaining gauge symmetry as

$$\begin{aligned}
\delta_g\psi_\mu^i &= -\frac{1}{\sqrt{2}}g(\eta + \frac{1}{2}\tilde{\eta})\delta^{ij}\psi_{\mu j}, \\
\delta_g\psi^i &= -\frac{1}{\sqrt{2}}g(\eta + \frac{1}{2}\tilde{\eta})\delta^{ij}\psi_j. \quad (5.54)
\end{aligned}$$

The theory given in (5.52) describes Einstein-Maxwell Supergravity constructed in [66]. It consists of the fields  $\{e_\mu^a, C_\mu, \psi_\mu^i, \sigma, \tilde{C}_\mu, \psi^i\}$  accounting for 20 + 20 on-shell degrees of freedom. The supersymmetry transformation rules, up to 3-fermion terms,

are

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu^i &= (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab})\epsilon^i + \frac{1}{\sqrt{2}}g(C_\mu + \frac{1}{2}\tilde{C}_\mu)\delta^{ij}\epsilon_j - \frac{1}{6\sqrt{2}}ig(\sigma^{-2} + 2\sigma)\gamma_\mu\delta^{ij}\epsilon_j \\
&\quad + \frac{1}{12}i\sigma^{-1}(\gamma_\mu^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)(G_{\nu\rho} + \sigma^3\partial_{[\nu}\tilde{C}_{\rho]})\epsilon^i, \\
\delta C_\mu &= -\frac{1}{2}i\sigma\bar{\epsilon}\psi_\mu + \frac{1}{2}\bar{\epsilon}\gamma_\mu\psi, \\
\delta\sigma &= \frac{1}{2}i\bar{\epsilon}\psi, \\
\delta\tilde{C}_\mu &= -\frac{1}{2}i\sigma^{-2}\bar{\epsilon}\psi_\mu + \frac{1}{2}\sigma^{-3}\bar{\epsilon}\gamma_\mu\psi, \\
\delta\psi^i &= -\frac{1}{2}i\not{\partial}\sigma\epsilon^i - \frac{1}{12}\gamma^{\mu\nu}(G_{\mu\nu} - 2\sigma^3\partial_\mu\tilde{C}_\nu)\epsilon^i + \frac{1}{3\sqrt{2}}g(\sigma^2 - \sigma^{-1})\delta^{ij}\epsilon_j. \tag{5.55}
\end{aligned}$$

The theory has a  $U(1) \times U(1)$  gauge symmetry parametrized by  $\eta$  and  $\tilde{\eta}$ . The gauge transformation rules for the gauge vectors and the fermions are given in (5.37) and (5.54) respectively.

### 5.2.2.1 Truncation to Minimal Gauged On-Shell Supergravity

In this subsection we show that we can consistently truncate the fields  $(\sigma, \psi^i, \tilde{C}_\mu)$  to obtain on-shell pure gauged  $D = 5, \mathcal{N} = 2$  supergravity [66].

Consider the field equation for  $\sigma$

$$\begin{aligned}
0 &= 3\sigma^{-2}\square\sigma - 3\sigma^{-3}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\sigma^{-3}G_{\mu\nu}G^{\mu\nu} - 2\sigma^3\partial_{[\mu}\tilde{C}_{\nu]}\partial^\mu\tilde{C}^\nu \\
&\quad + \frac{1}{8}i\sigma^{-2}\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu G_{\rho\sigma} + \frac{1}{4}i\sigma^{-2}\bar{\psi}_\mu\psi_\nu G^{\mu\nu} - \frac{1}{2}i\sigma\bar{\psi}_{[\mu}\psi_{\nu]}\partial^\mu\tilde{C}^\nu \\
&\quad - \frac{1}{4}i\sigma\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu\partial_\rho\tilde{C}_\sigma + 2g^2(\sigma - \sigma^{-2}) - \frac{1}{2\sqrt{2}}ig\sigma^{-3}\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} \\
&\quad + \frac{1}{2\sqrt{2}}ig\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij} + (\psi^i\text{-terms}) \tag{5.56}
\end{aligned}$$

and the field equation for  $\psi^i$

$$\begin{aligned}
0 = & -\frac{3}{2}i\sigma^{-2}\gamma^\mu\gamma^\nu\psi_\mu^i\partial_\nu\sigma - \frac{1}{4}\sigma^{-2}\gamma^{\mu\nu\rho}\psi_\mu^i G_{\nu\rho} + \frac{1}{2}\sigma^{-2}\gamma_\mu\psi_\nu^i G^{\mu\nu} \\
& + \frac{1}{2}\sigma\gamma^{\mu\nu\rho}\psi_\mu^i\partial_\nu\tilde{C}_\rho - \sigma\gamma_{[\mu}\psi_{\nu]}^i\partial^\mu\tilde{C}^\nu + \frac{1}{\sqrt{2}}g\gamma_\mu\psi_j^\mu\delta^{ij} \\
& - \frac{1}{\sqrt{2}}g\sigma^{-3}\gamma_\mu\psi_j^\mu\delta^{ij} + (\psi^i\text{-terms}), \tag{5.57}
\end{aligned}$$

both up to 4-fermion terms. From these equations and from the transformation rules of  $\sigma$  and  $\psi^i$  in (5.55), we observe that one can consistently eliminate the matter fields  $(\sigma, \psi^i, \tilde{C}_\mu)$  by setting  $\sigma = 1$ ,  $\psi^i = 0$  and  $\tilde{C}_\mu - C_\mu = \partial_\mu a$ , where  $a$  is a Stueckelberg scalar. The gauge transformation of  $a$  is given by

$$\delta_g a = \eta - \tilde{\eta}. \tag{5.58}$$

We can break the  $U(1) \times U(1)$  gauge symmetry down to  $U(1)$  by setting  $a$  to a constant  $a = a_0$ . This implies

$$C_\mu = \tilde{C}_\mu, \quad \eta = \tilde{\eta}. \tag{5.59}$$

Performing this truncation in (5.52) we obtain the on-shell Lagrangian for pure gauged  $D = 5$ ,  $\mathcal{N} = 2$  supergravity

$$\begin{aligned}
e^{-1}\mathcal{L}_{EM}|_{\sigma=1,\psi=0} = & \frac{1}{2}R + 3g^2 - \frac{3}{8}G_{\mu\nu}G^{\mu\nu} + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}C_\mu G_{\nu\rho}G_{\sigma\lambda} \\
& - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho - \frac{3}{8}i\bar{\psi}_\mu\psi_\nu G^{\mu\nu} - \frac{3}{16}i\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma}\psi_\nu G_{\rho\sigma} \\
& + \frac{3}{4\sqrt{2}}ig\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j\delta_{ij}, \tag{5.60}
\end{aligned}$$

where we defined

$$\nabla_\mu \psi_\nu^i = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \psi_\nu^i + \frac{3}{2\sqrt{2}} g C_\mu \delta^{ij} \psi_{\nu j}. \quad (5.61)$$

This Lagrangian is invariant under the transformation rules for  $e_\mu^a$ ,  $\psi_\mu^i$  and  $C_\mu$  given in (5.55) with  $\sigma = 1$ ,  $\psi^i = 0$  and  $\tilde{C}_\mu = C_\mu$ . The Lagrangian (5.60) agrees completely with the result obtained in [66]

### 5.2.3 $\sigma = 1$ Gauge Fixing

If we do not insist on the canonical Einstein-Hilbert term in the action, there exists a set of gauge choices facilitating the derivation of curvature squared invariant. These gauge choices are

$$L_{ij} = \frac{1}{\sqrt{2}} \delta_{ij} L, \quad \sigma = 1, \quad \psi^i = 0, \quad b_\mu = 0. \quad (5.62)$$

The first gauge choice breaks the  $SU(2)_R$  down to  $U(1)_R$  whereas the second one fixes dilatations, the third one fixes special supersymmetry transformations and the last one fixes conformal boosts. After fixing the gauge, the remaining fields are

$$e_\mu^a(10), \psi_\mu^i(32), C_\mu(4), B_{\mu\nu}(6), \varphi^i(8), L(1), E_{\mu\nu\rho}(4), N(1), V_\mu(4), V_\mu'^{ij}(10). \quad (5.63)$$

To maintain the gauge (5.62), the compensating transformations are required including a compensating  $SU(2)$ , a compensating special supersymmetry and a compensating conformal boost with parameters (up to cubic fermion terms)

$$\begin{aligned} \lambda^{ij} &= -\frac{1}{\sqrt{2}L} \left( S^{k(i} \delta^{j)l} \epsilon_{kl} \right), & S^{ij} &= \bar{\epsilon}^{(i} \varphi^{j)} - \frac{1}{2} \delta^{ij} \bar{\epsilon}^k \varphi^l \delta_{kl}, \\ \eta^i &= \left( -\gamma \cdot T + \frac{1}{4} \gamma \cdot \widehat{G} \right) \epsilon^i, & \Lambda_{K\mu} &= -\frac{1}{4} i \bar{\epsilon} \phi_\mu - \frac{1}{4} i \bar{\eta} \psi_\mu + \bar{\epsilon} \gamma_\mu \chi. \end{aligned} \quad (5.64)$$

Imposing the gauge fixing conditions (5.62) in the linear multiplet action (5.19), one can obtain a consistent Poincaré supergravity whose action is given by<sup>1</sup>

$$\begin{aligned}
e^{-1}\mathcal{L}_{LR}^D &= \frac{1}{2}LR + \frac{1}{2}L^{-1}\partial_\mu L\partial^\mu L - \frac{1}{4}LG_{\mu\nu}G^{\mu\nu} - \frac{1}{6}LH_{\mu\nu\rho}H^{\mu\nu\rho} \\
&\quad - L^{-1}N^2 - L^{-1}P_\mu P^\mu - \sqrt{2}P_\mu V^\mu + LV'_\mu{}^{ij}V'_{ij}{}^\mu
\end{aligned} \tag{5.65}$$

The Poincaré supergravity presented above is invariant under the following supersymmetry transformation rules (up to cubic fermion terms)

$$\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu{}^i &= \mathcal{D}_\mu(\omega_-)\epsilon^i - \frac{1}{2}i\widehat{G}_{\mu\nu}\gamma^\nu\epsilon^i, \\
\delta V_\mu{}^{ij} &= \frac{1}{2}\bar{\epsilon}^{(i}\gamma^\nu\widehat{\psi}_{\mu\nu}^{j)} - \frac{1}{6}\bar{\epsilon}^{(i}\gamma\cdot\widehat{H}\psi_\mu^{j)} - \frac{1}{4}i\bar{\epsilon}^{(i}\gamma\cdot\widehat{G}\psi_\mu^{j)} + \partial_\mu\lambda^{ij} + \lambda^{(i}{}^k{}_{\mu}V^{j)k}, \\
\delta C_\mu &= -\frac{1}{2}i\bar{\epsilon}\psi_\mu, \\
\delta B_{\mu\nu} &= \frac{1}{2}\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + C_{[\mu}\delta(\epsilon)C_{\nu]}, \\
\delta L &= \frac{1}{\sqrt{2}}i\bar{\epsilon}^i\varphi^j\delta_{ij}, \\
\delta\varphi^i &= -\frac{1}{2\sqrt{2}}i\not{\partial}L\delta^{ij}\epsilon_j - \frac{1}{\sqrt{2}}iV_\mu{}^{(i}{}^k\delta^{j)k}L\epsilon_j - \frac{1}{2}i\not{E}\epsilon^i + \frac{1}{2}N\epsilon^i + \frac{1}{4\sqrt{2}}L\gamma\cdot\widehat{G}\delta^{ij}\epsilon_j \\
&\quad - \frac{1}{6\sqrt{2}}iL\gamma\cdot\widehat{H}\delta^{ij}\epsilon_j, \\
\delta E_{\mu\nu\rho} &= -\bar{\epsilon}\gamma_{\mu\nu\rho}\varphi + \frac{1}{\sqrt{2}}iL\bar{\psi}_{[\mu}{}^i\gamma_{\nu\rho]}\epsilon^j\delta_{ij}, \\
\delta N &= \frac{1}{2}\bar{\epsilon}\gamma^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\right)\varphi + \frac{1}{2}\bar{\epsilon}^i\gamma^a V_{a\,ij}\varphi^j - \frac{1}{4\sqrt{2}}i\bar{\epsilon}^i\gamma^a\not{\partial}L\psi_a^j\delta_{ij} \\
&\quad + \frac{1}{4\sqrt{2}}i\bar{\epsilon}^i\gamma^a\gamma^b V'_{b(i}{}^k\delta_{j)k}\psi_a^j + \frac{1}{4}i\bar{\epsilon}\gamma^a\not{E}\psi_a - \frac{1}{4}N\bar{\epsilon}\gamma^a\psi_a + \frac{1}{8\sqrt{2}}L\bar{\epsilon}^i\gamma^a\gamma\cdot\widehat{G}\psi_a^j\delta_{ij} \\
&\quad - \sqrt{2}L\bar{\epsilon}^i\gamma^a\phi_a^j\delta_{ij} + \frac{1}{8}i\bar{\epsilon}\gamma\cdot\widehat{H}\varphi,
\end{aligned} \tag{5.66}$$

where we have used the torsionful spin connection, defined as  $\omega_{\mu\pm}{}^{ab} = \omega_\mu{}^{ab} \pm \widehat{H}_\mu{}^{ab}$ , and the supercovariant curvatures under the gauge (5.62) are [68]

<sup>1</sup>The action directly coming from (5.19) by imposing (5.62) is equal to  $-e^{-1}\mathcal{L}_{LR}$ .

$$\widehat{\psi}_{\mu\nu} = 2D_{[\mu}(\omega_{-})\psi_{\nu]} + i\gamma^\lambda \widehat{G}_{\lambda[\mu}\psi_{\nu]} , \quad (5.67)$$

$$\widehat{G}_{\mu\nu} = 2\partial_{[\mu}C_{\nu]} + \frac{1}{2}i\bar{\psi}_{[\mu}\psi_{\nu]} , \quad (5.68)$$

$$\widehat{H}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} - \frac{3}{4}\bar{\psi}_{[\mu}\gamma_{\nu}\psi_{\rho]} + \frac{3}{2}C_{[\mu}G_{\nu\rho]} . \quad (5.69)$$

Finally, we note that the gauging procedure described for the  $L = 1$  case can be repeated exactly to gauge the  $\sigma = 1$  gauge fixing choice.

## 6. CURVATURE SQUARED INVARIANTS IN FIVE DIMENSIONAL

### $\mathcal{N} = 2$ SUPERGRAVITY\*

In this section, we use five dimensional to construct off-shell curvature squared invariants in five dimensions. As mentioned before, in five dimensions, there are two inequivalent Weyl multiplets: the standard Weyl multiplet and the dilaton Weyl multiplet. The main difference between these two Weyl multiplets is that the dilaton Weyl multiplet contains a graviphoton in its field content whereas the standard Weyl multiplet does not. A supergravity theory based on the standard Weyl multiplet requires coupling to an external vector multiplet.

In section 6.1 we first review the previously constructed minimal off-shell curvature squared invariant purely based on the dilaton Weyl multiplet: the supersymmetric Riemann squared action. Then, we construct the supersymmetric completion of Gauss-Bonnet combination and minimal off-shell Ricci scalar squared invariant. In section 6.2, we derive the vector multiplets coupled Riemann tensor squared and Ricci scalar squared invariants and review the vector multiplets coupled Weyl tensor squared for a complete discussion. Starting from section 6.3 we begin to use the standard Weyl multiplet, and after a brief discussion about the supersymmetric Weyl tensor squared, we construct an off-shell vector multiplets coupled Ricci scalar squared invariant.

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## 6.1 Minimal Curvature Squared Actions in Dilaton Weyl Multiplet

The five dimensional minimal off-shell Poincaré supergravity multiplet consists of the fields

$$e_\mu^a(10), \psi_\mu^i(32), C_\mu(4), B_{\mu\nu}(6), \varphi^i(8), L(1), E_{\mu\nu\rho}(4), N(1), V_\mu(4), V_\mu'^{ij}(10), \quad (6.1)$$

where the number in the bracket denotes the off-shell degrees of freedom carried by the fields. The map from the dilaton Weyl multiplet to the standard Weyl multiplet (3.21) plays a crucial role in the construction of curvature squared actions. In particular, the composite expression for  $D$  contains a curvature term. Thus, the existence of a  $D^2$  term in a curvature squared action means the curvature terms get an extra  $R^2$  contribution from the composite expression of  $D$ . As we shall see, this fact is essential in the construction of supersymmetric completion of Gauss-Bonnet combination [59].

### 6.1.1 Riemann Squared Action

In this section, we construct the supersymmetric Riemann squared action. To begin with, we shall review a map between the Yang-Mills super-multiplet and a set of fields in the Poincaré multiplet (6.1).

In establishing the map between Yang-Mills and Poincaré multiplets, it is important to consider the full supersymmetry transformations, including the cubic fermion terms which have been omitted so far. In the following, we shall need the full supersymmetry transformation rules for the fields  $(e_\mu^a, \psi_\mu^i, V_\mu'^{ij}, C_\mu, B_{\mu\nu})$ . Up to cubic fermions, the transformation rules of  $(e_\mu^a, \psi_\mu^i, V_\mu'^{ij}, C_\mu, B_{\mu\nu})$  are already given in (5.66). In this section, we will, however, keep the complete  $SU(2)$  symmetry, i.e. we do not impose  $L^{ij} = \frac{1}{\sqrt{2}}L\delta^{ij}$ . In this way we do not need to accommodate for



the compensating  $SU(2)$  transformations proportional to  $\lambda^{ij1}$ . The full version of the supersymmetry transformations, in the  $\sigma = 1$  gauge fixing (5.62), are given by [68]

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu^i &= D_\mu(\omega_-)\epsilon^i - \frac{1}{2}i\widehat{G}_{\mu\nu}\gamma^\nu\epsilon^i, \\
\delta V_\mu^{ij} &= \frac{1}{2}\bar{\epsilon}^{(i}\gamma^\nu\psi_{\mu\nu}^{j)} - \frac{1}{6}\bar{\epsilon}^{(i}\gamma\cdot\widehat{H}\psi_\mu^{j)} - \frac{1}{4}i\bar{\epsilon}^{(i}\gamma\cdot\widehat{G}\psi_\mu^{j)}, \\
\delta C_\mu &= -\frac{1}{2}i\bar{\epsilon}\psi_\mu, \\
\delta B_{\mu\nu} &= \frac{1}{2}\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + C_{[\mu}\delta(\epsilon)C_{\nu]}.
\end{aligned} \tag{6.2}$$

Next, we consider the following supersymmetry transformations [68]

$$\begin{aligned}
\delta\omega_{\mu+}^{ab} &= -\frac{1}{2}i\widehat{G}^{ab}\bar{\epsilon}\psi_\mu - \frac{1}{2}\bar{\epsilon}\gamma_\mu\widehat{\psi}^{ab}, \\
\delta\widehat{\psi}_{ab}^i &= \frac{1}{4}\gamma^{cd}\widehat{R}_{cdab}(\omega_+)\epsilon^i - \widehat{V}_{ab}^{ij}\epsilon_j + \frac{1}{2}i\gamma^\mu\mathcal{D}_\mu(\omega_+)\widehat{G}_{ab}\epsilon^i - \frac{1}{4}\widehat{G}_{ab}\gamma\cdot\widehat{G}\epsilon^i, \\
\delta\widehat{G}_{ab} &= -\frac{1}{2}i\bar{\epsilon}\widehat{\psi}_{ab}, \\
\delta\widehat{V}_{ab}^{ij} &= -\frac{1}{2}\bar{\epsilon}^{(i}\widehat{\mathcal{D}}(\omega, \omega_-)\widehat{\psi}_{ab}^{j)} - \frac{1}{24}\bar{\epsilon}^{(i}\gamma\cdot\widehat{H}\psi_{ab}^{j)} - i\bar{\epsilon}^{(i}\widehat{G}_{[a}^d\widehat{\psi}_{b]d}^{j)},
\end{aligned} \tag{6.3}$$

where  $\widehat{R}_{abcd}(\omega_+)$  denotes the super-covariant curvature of the torsionful connection  $\omega_+$ . In  $\mathcal{D}_\mu(\omega_+)\widehat{G}_{ab}$ , the connection  $\omega_+$  rotates both the indices  $a$  and  $b$ , and in  $\mathcal{D}_\mu(\omega, \omega_-)\widehat{\psi}_{ab}^j$  the connection  $\omega$  rotates the spinor index, while the connection  $\omega_-$  rotates the Lorentz vector indices.  $\widehat{V}_{\mu\nu}^{ij}$  is the supercovariant curvature of  $V_\mu^{ij}$  under the gauge choices (5.62)

$$\widehat{V}_{\mu\nu}^{ij} = V_{\mu\nu}^{ij} - \bar{\psi}_{[\mu}^{(i}\gamma^\rho\widehat{\psi}_{\nu]\rho}^{j)} + \frac{1}{6}\bar{\psi}_\mu^{(i}\gamma\cdot\widehat{H}\psi_\nu^{j)} + \frac{1}{4}i\bar{\psi}_\mu^{(i}\gamma\cdot\widehat{G}\psi_\nu^{j)}. \tag{6.4}$$

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<sup>1</sup>After we construct the action, we can still impose the gauge  $L^{ij} = \frac{1}{\sqrt{2}}L\delta^{ij}$ . This will not affect the Riemann squared invariant.

We now compare the above transformation rules with those of the  $D = 5$ ,  $\mathcal{N} = 2$  Yang-Mills multiplet in  $\sigma = 1$  gauge fixing (5.62) [68]

$$\begin{aligned}
\delta A_\mu^\Sigma &= -\frac{1}{2}i\rho^\Sigma \bar{\epsilon} \psi_\mu + \frac{1}{2}\bar{\epsilon} \gamma_\mu \lambda^\Sigma, \\
\delta Y^{ij\Sigma} &= -\frac{1}{2}\bar{\epsilon}^{(i} \not{D} \lambda^{j)\Sigma} - \frac{1}{24}\bar{\epsilon}^{(i} \gamma \cdot \widehat{H} \lambda^{j)\Sigma} - \frac{1}{2}ig\bar{\epsilon}^{(i} f_{\Lambda\Upsilon}^\Sigma \rho^\Lambda \lambda^{j)\Upsilon}, \\
\delta \lambda^{i\Sigma} &= -\frac{1}{4}\left(\gamma \cdot \widehat{F}^\Sigma - \rho^\Sigma \gamma \cdot \widehat{G}\right) \epsilon^i - \frac{1}{2}i\not{D} \rho^\Sigma \epsilon^i - Y^{ij\Sigma} \epsilon_j, \\
\delta \rho^\Sigma &= \frac{1}{2}i\bar{\epsilon} \lambda^\Sigma,
\end{aligned} \tag{6.5}$$

where  $\widehat{F}_{\mu\nu}^\Sigma$  and  $\mathcal{D}_\mu \rho^\Sigma$  can be found in (3.26) and (3.25) by imposing the gauge choices (5.62)

$$\begin{aligned}
\mathcal{D}_\mu \lambda^{i\Sigma} &= (\partial_\mu + \frac{1}{4}\omega_\mu^{ab} \gamma_{ab}) \lambda^{i\Sigma} - V_\mu^{ij} \lambda_j^\Sigma + g f_{\Lambda\Upsilon}^\Sigma A_\mu^\Lambda \lambda^{i\Upsilon} \\
&\quad + \frac{1}{4}\left(\gamma \cdot \widehat{F}^\Sigma - \rho^\Sigma \gamma \cdot \widehat{G}\right) \psi_\mu^i + \frac{1}{2}i\not{D} \rho^\Sigma \psi_\mu^i + Y^{ij\Sigma} \psi_{\mu j}.
\end{aligned} \tag{6.6}$$

We observe that the transformations (6.3) and (6.5) become identical by making the following identifications [68]

$$(A_\mu^\Sigma, Y_\Sigma^{ij}, \lambda_\Sigma^i, \rho_\Sigma) \longleftrightarrow (\omega_{\mu+}^{ab}, -\widehat{V}_{ab}^{ij}, -\widehat{\psi}_{ab}^i, \widehat{G}_{ab}). \tag{6.7}$$

Therefore, upon constructing a Yang-Mills action, and gauge fixing by using the conditions (5.62), one can construct a Riemann tensor squared invariant.

At this stage, we have three options leading to the same Yang-Mills action: the superconformal tensor calculus, a special choice of very special geometry (4.10), and dimensional reduction from the six dimensional theory [68].

From the superconformal tensor calculus viewpoint, we start from the following

identification

$$L_{ij} = Y_{ij}. \quad (6.8)$$

This identification, again, has the wrong Weyl weight and fails to satisfy the  $S$ -invariance of  $L_{ij}$ . The one with the right Weyl weight and invariant under the  $S$ -transformation can be given by

$$L_{ij} = \sigma Y_{ij} + \frac{1}{4}i\rho\sigma^{-1}\bar{\psi}_{(i}\psi_{j)} - \frac{1}{2}i\bar{\lambda}_{(i}\psi_{j)}. \quad (6.9)$$

since this time we use the dilaton Weyl multiplet, thus can utilize the scalar of the dilaton Weyl multiplet,  $\sigma$ . After employing a sequence of  $Q$ -and  $S$ -transformations to (6.9), we obtain the full expressions for the components of linear multiplet in terms of the fields in the vector multiplet and dilaton Weyl multiplet

$$\begin{aligned} L_{ij} &= \sigma Y_{ij} + \frac{1}{4}i\rho\sigma^{-1}\bar{\psi}_{(i}\psi_{j)} - \frac{1}{2}i\bar{\lambda}_{(i}\psi_{j)}, \\ \varphi_i &= \frac{1}{2}i\sigma\mathcal{D}\lambda_i + \frac{1}{2}i\rho\mathcal{D}\psi_i + \rho\gamma \cdot T\psi_i + \sigma\gamma \cdot T\lambda_i - 8\sigma\rho\chi_i - \frac{1}{8}\gamma \cdot \widehat{G}\lambda_i \\ &\quad - \frac{1}{8}\gamma \cdot \widehat{F}\psi_i + \frac{1}{4}\mathcal{D}\sigma\lambda_i + \frac{1}{4}\mathcal{D}\rho\psi_i - \frac{1}{2}Y_{ij}\psi^j - \frac{1}{8}i\sigma^{-1}\lambda^j\bar{\psi}_i\psi_j, \\ E^a &= \mathcal{D}_b(-\frac{1}{2}\sigma\widehat{F}^{ab} - \frac{1}{2}\rho\widehat{G}^{ab} + 8\sigma\rho T^{ab} - \frac{1}{8}i\bar{\lambda}\gamma_{ab}\psi) - \frac{1}{8}\epsilon^{abcde}G_{bc}F_{de}, \\ N &= \frac{1}{2}\rho\Box^C\sigma + \frac{1}{2}\sigma\Box^C\rho + \frac{1}{2}\mathcal{D}_a\rho\mathcal{D}^a\sigma - \frac{1}{4}\widehat{G}_{ab}\widehat{F}^{ab} - 4\rho\sigma\left(D + \frac{26}{3}T^2\right) \\ &\quad + 4\sigma\widehat{F}^{ab}T_{ab} + 4\rho\widehat{G}^{ab}T_{ab} + 8i\sigma\bar{\chi}\lambda + 8i\rho\bar{\chi}\psi - \frac{1}{4}\bar{\lambda}\mathcal{D}\psi - \frac{1}{4}\bar{\psi}\mathcal{D}\lambda \\ &\quad + i\bar{\psi}\gamma \cdot T\lambda. \end{aligned} \quad (6.10)$$

Inserting above expressions into density formula (4.1), we derive an action for an abelian vector multiplet coupled to a dilaton Weyl multiplet. Generalization of the action for abelian vector multiplet to that for Yang-Mills multiplet is straightforward.

The result is given by

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{YM}} &= a_{\Sigma\Lambda}\left(\sigma Y_{ij}^\Sigma Y^{ij\Lambda} - \frac{1}{4}\sigma F_{\mu\nu}^\Sigma F^{\mu\nu\Lambda} - \frac{1}{2}\rho^\Sigma F_{\mu\nu}^\Lambda G^{\mu\nu} + 8\sigma\rho^\Sigma F_{\mu\nu}^\Lambda T^{\mu\nu} + \frac{1}{2}\rho^\Sigma\rho^\Lambda\Box^C\sigma\right. \\
&\quad + \frac{1}{2}\sigma\rho^\Sigma\Box^C\rho^\Lambda + \frac{1}{2}\rho^\Sigma\mathcal{D}_a\rho^\Lambda\mathcal{D}^a\sigma - 4\sigma\rho^\Sigma\rho^\Lambda(D + \frac{26}{3}T^2) + 4\rho^\Sigma\rho^\Lambda G_{\mu\nu}T^{\mu\nu} \\
&\quad \left. - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^\Sigma F_{\rho\sigma}^\Lambda C_\lambda\right). \tag{6.11}
\end{aligned}$$

Then, if we apply the gauge fixing conditions (5.62), we obtain

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{YM}}|_{\sigma=1} &= Y_{ij}^\Sigma Y^{ij\Sigma} - \frac{1}{2}D_\mu\rho^\Sigma D^\mu\rho^\Sigma - \frac{1}{4}(F_{ab}^\Sigma - \rho^\Sigma G_{ab})(F^{ab\Sigma} - \rho^\Sigma G^{ab}) \\
&\quad - \frac{1}{8}\epsilon^{abcde}(F_{ab}^\Sigma - \rho^\Sigma G_{ab})(F_{cd}^\Sigma - \rho^\Sigma G_{cd})C_e \\
&\quad - \frac{1}{2}\epsilon^{abcde}(F_{ab}^\Sigma - \rho^\Sigma G_{ab})B_{cd}D_e\rho^\Sigma. \tag{6.12}
\end{aligned}$$

One can also start from the vector multiplet action (4.10), and use the map between the Weyl multiplets (3.21). In that case, if we make the following choice for  $C_{IJK}$ ,

$$C_{IJK} = \begin{cases} C_{1IJ} = a_{IJ} \\ 0 & \text{otherwise.} \end{cases} \tag{6.13}$$

we obtain the same action given in (6.12) upon gauge fixing (5.62). Finally, using the map (6.7) in above action (6.12), we obtain the supersymmetric Riemann squared action. Its purely bosonic part is given as

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{Riem}^2}^D &= -\frac{1}{4}\left(R_{\mu\nu ab}(\omega_+) - G_{\mu\nu}G_{ab}\right)\left(R^{\mu\nu ab}(\omega_+) - G^{\mu\nu}G^{ab}\right) \\
&\quad - \frac{1}{2}D_\mu(\omega_+)G^{ab}D^\mu(\omega_+)G_{ab} + V_{\mu\nu}{}^{ij}V^{\mu\nu}{}_{ij} \\
&\quad - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}\left(R_{\mu\nu ab}(\omega_+) - G_{\mu\nu}G_{ab}\right)\left(R_{\rho\sigma}{}^{ab}(\omega_+) - G_{\rho\sigma}G^{ab}\right)C_\lambda \\
&\quad - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\lambda}B_{\rho\sigma}\left(R_{\mu\nu ab}(\omega_+) - G_{\mu\nu}G_{ab}\right)D_\lambda(\omega_+)G^{ab}. \tag{6.14}
\end{aligned}$$

Finally, we notice that the actions (6.12) and (6.14) obtained via superconformal tensor calculus match with those derived through the circle reduction of six-dimensional actions [68].

### 6.1.2 *Supersymmetric Gauss-Bonnet Combination*

In this section, we shall construct the supersymmetric completion of Weyl squared invariant in order to obtain the supersymmetric completion of the Gauss-Bonnet combination

$$e^{-1}\mathcal{L}_{\text{GB}} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \quad (6.15)$$

According to the usual routine, one may think of constructing three independent curvature squared super-invariants first, then combining them with proper coefficients to form a supersymmetric Gauss-Bonnet combination. However, as we mentioned before, two independent curvature squared invariants may be enough to obtain the supersymmetric completion of Gauss-Bonnet combination based on counting the degrees of freedom and the cancelation of the kinetic term for the auxiliary vector  $V_{\mu}^{ij}$ . This section is devoted to construct another curvature squared invariant.

We start from the conventional constraint imposed on the supercovariant curvature of  $\omega_{\mu}^{ab}$  [36, 37]

$$e^{\nu}{}_{\hat{b}}\widehat{R}_{\mu\nu}{}^{ab}(M) = 0, \quad (6.16)$$

where  $\widehat{R}_{\mu\nu}{}^{ab}(M)$  is defined in (3.7). The conventional constraint (6.16) implies that the supercovariant curvature of  $\omega_{\mu}^{ab}$  gives the Weyl Tensor, which is defined as

$$\begin{aligned}
C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \frac{1}{3}(g_{\mu\rho}R_{\nu\sigma} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho}) \\
&\quad + \frac{1}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R.
\end{aligned} \tag{6.17}$$

Its square is

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{4}{3}R_{\mu\nu}R^{\mu\nu} + \frac{1}{6}R^2. \tag{6.18}$$

In the rest of this paper, we use  $\widehat{C}_{\mu\nu\rho\sigma}$  to denote the superconformally covariant Weyl tensor instead of  $\widehat{R}_{\mu\nu}{}^{ab}(M)$ . Because the off-shell supersymmetric Riemann squared invariant is known, the Gauss-Bonnet super-invariant can be obtained by combining the Riemann squared invariant with another curvature squared invariant in which the curvature squared terms take the form

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{6}R^2. \tag{6.19}$$

Although, none of the terms in (6.19) is a supercovariant quantity, we can replace (6.19) by the following supercovariant expression

$$\widehat{C}_{\mu\nu\rho\sigma}\widehat{C}^{\mu\nu\rho\sigma} + \frac{512}{3}D^2, \tag{6.20}$$

since the composite field  $D$  (3.21) under the gauge choices (5.62) reads

$$D = -\frac{1}{32}R - \frac{1}{16}G^{ab}G_{ab} - \frac{26}{3}T^{ab}T_{ab} + 2T^{ab}G_{ab} + \text{fermions}. \tag{6.21}$$

Therefore, if (6.20) can be supersymmetrized, we will get the desired the curvature

squared terms in (6.19). When carrying out the supersymmetrization of (6.20), we find that in fact, the  $D^2$  term is indispensable to the supersymmetrization of the Weyl tensor squared term, moreover, the relative coefficient between the Weyl squared term and the  $D^2$  exactly matches with the one in (6.20), the magical  $\frac{512}{3}$ .

Let us first supersymmetrize the square of Weyl tensor by using (4.1) in which the fields of linear multiplet are expressed as composites in terms of fields in dilaton Weyl multiplet. We notice that to obtain the Weyl tensor squared term,  $N$  should begin with  $\widehat{C}_{\mu\nu\rho\sigma}\widehat{C}^{\mu\nu\rho\sigma}$ . The complete expression for  $N$  include a term  $\frac{512}{3}D^2$ . After expanding  $D$  in terms of independent fields, we find that the curvature squared terms take the form of  $C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \frac{1}{6}R^2$ , which is different from those in the supersymmetric completion of  $\widehat{C}_{\mu\nu\rho\sigma}\widehat{C}^{\mu\nu\rho\sigma}$  considered in [40] by using standard Weyl multiplet where  $D$  is merely an auxiliary field. We obtain full composite expressions for the fields of linear multiplet in terms of fields in the dilation Weyl multiplet as

$$\begin{aligned}
L^{ij} &= \frac{1}{4}i\widehat{R}_{ab}^{(i}(Q)\widehat{R}^{j)ab}(Q) + \frac{256}{3}i\widehat{\chi}^{(i}\widehat{\chi}^{j)} + \frac{16}{3}\widehat{R}_{ab}{}^{ij}(V)\underline{T}^{ab}, \\
\varphi^i &= -\frac{1}{8}\gamma_{cd}\widehat{R}_{ab}^i(Q)\widehat{C}^{abcd} - 4i\gamma_c\widehat{R}_{ab}^i(Q)\mathcal{D}^a\underline{T}^{bc} + \frac{128}{3}\widehat{\chi}^i\underline{D} \\
&\quad + 8i\gamma_c\mathcal{D}^c\widehat{R}_{ab}^i(Q)\underline{T}^{ab} + 8i\gamma_a\mathcal{D}^c\widehat{R}_{bc}^i(Q)\underline{T}^{ab} - \frac{64}{3}i\gamma^{ab}\gamma^c\mathcal{D}^a\underline{T}_{bc}\widehat{\chi}^i + \frac{1024}{9}\underline{T}^2\widehat{\chi}^i \\
&\quad + \frac{128}{3}i\gamma_a\mathcal{D}_b\widehat{\chi}^i\underline{T}^{ab} + \frac{16}{3}\gamma_{ab}\widehat{R}_{cd}^i(Q)\underline{T}^{ab}\underline{T}^{cd} + \frac{1}{2}\widehat{R}^{abi}{}_j(V)\widehat{R}_{ab}^j(Q) \\
&\quad - \frac{8}{3}\widehat{R}^{abi}{}_j(V)\gamma_{ab}\widehat{\chi}^j, \\
E_a &= \frac{1}{16}\epsilon_{abcde}C^{bcfg}C^{de}{}_{fg} - \frac{1}{12}\epsilon_{abcde}V^{bc}{}_{ij}V^{de}{}^{ij} \\
&\quad + \mathcal{D}^b\left(4C_{abcd}\underline{T}^{cd} - \frac{64}{3}\underline{D}\underline{T}_{ab} - \frac{128}{9}\underline{T}_{ab}\underline{T}^2 - \frac{512}{3}\underline{T}_{ac}\underline{T}^{cd}\underline{T}_{bd}\right) \\
&\quad - 32\epsilon_{abcde}\mathcal{D}^b\left(\frac{2}{3}\underline{T}^{cf}\mathcal{D}_f\underline{T}^{de} + \underline{T}^c{}_f\mathcal{D}^d\underline{T}^{ef}\right) + \text{fermions}, \\
N &= \frac{1}{8}C^{abcd}C_{abcd} + \frac{64}{3}\underline{D}^2 + \frac{1024}{9}\underline{T}^2\underline{D} - \frac{16}{3}C_{abcd}\underline{T}^{ab}\underline{T}^{cd} - \frac{1}{3}V_{ab}{}^{ij}V^{ab}{}_{ij} \\
&\quad - \frac{64}{3}\mathcal{D}_a\underline{T}_{bc}\mathcal{D}^a\underline{T}^{bc} + \frac{64}{3}\mathcal{D}_b\underline{T}_{ac}\mathcal{D}^a\underline{T}^{bc} - \frac{128}{3}\underline{T}_{ab}\mathcal{D}^b\mathcal{D}_c\underline{T}^{ac} \\
&\quad - \frac{128}{3}\epsilon_{abcde}\underline{T}^{ab}\underline{T}^{cd}\mathcal{D}_f\underline{T}^{ef} + 1024\underline{T}^4 - \frac{2816}{27}(\underline{T}^2)^2 + \text{fermions}. \tag{6.22}
\end{aligned}$$

where the following notations are introduced for simplicity

$$T^4 \equiv T_{ab}T^{bc}T_{cd}T^{da}, \quad (T^2)^2 \equiv (T_{ab}T^{ab})^2. \quad (6.23)$$

Under the gauge choices (5.62)  $T_{ab}\mathcal{D}^b\mathcal{D}_cT^{ac}$  is given by

$$T_{ab}\mathcal{D}^b\mathcal{D}_cT^{ac} = T_{ab}\nabla^b\nabla_cT^{ac} + \frac{2}{3}R^{bc}T_{ab}T^a{}_c - \frac{1}{12}T^2R + \text{fermions}, \quad (6.24)$$

where  $\nabla_\mu$  only contains the usual spin connection

$$\nabla_\mu T_{ab} = \partial_\mu T_{ab} - 2\omega_\mu{}^c{}_{[a}T_{b]c}. \quad (6.25)$$

To obtain (6.22) we have used the  $Q$ - and  $S$ - transformations of supercovariant curvatures which can be found in [37]. Substituting the composite expressions (6.22) into the density formula (4.1), we obtain the following action

$$\begin{aligned} e^{-1}\mathcal{L}_{\rho R^2} &= \frac{1}{8}\rho C^{abcd}C_{abcd} + \frac{64}{3}\rho\underline{D}^2 + \frac{1024}{9}\rho\underline{T}^2\underline{D} - \frac{32}{3}\underline{D}T_{ab}F^{ab} \\ &\quad - \frac{16}{3}\rho C_{abcd}\underline{T}^{ab}\underline{T}^{cd} + 2C_{abcd}\underline{T}^{cd}F^{ab} + \frac{1}{16}\epsilon_{abcde}A^aC^{bcfg}C^{de}{}_{fg} \\ &\quad - \frac{1}{12}\epsilon_{abcde}A^aV^{bc}{}_{ij}V^{de}{}^{ij} + \frac{16}{3}Y_{ij}V_{ab}{}^{ij}\underline{T}^{ab} - \frac{1}{3}\rho V_{ab}{}^{ij}V^{ab}{}_{ij} + \frac{64}{3}\rho\mathcal{D}_b\underline{T}_{ac}\mathcal{D}^a\underline{T}^{bc} \\ &\quad - \frac{128}{3}\rho\underline{T}_{ab}\mathcal{D}^b\mathcal{D}_c\underline{T}^{ac} - \frac{64}{3}\rho\mathcal{D}_a\underline{T}_{bc}\mathcal{D}^a\underline{T}^{bc} + 1024\rho\underline{T}^4 - \frac{2816}{27}\rho(\underline{T}^2)^2 \\ &\quad - \frac{64}{9}\underline{T}_{ab}F^{ab}\underline{T}^2 - \frac{256}{3}\underline{T}_{ac}\underline{T}^{cd}\underline{T}_{bd}F^{ab} - \frac{32}{3}\epsilon_{abcde}\underline{T}^{cf}\mathcal{D}_f\underline{T}^{de}F^{ab} \\ &\quad - 16\epsilon_{abcde}\underline{T}^c{}_f\mathcal{D}^d\underline{T}^{ef}F^{ab} - \frac{128}{3}\rho\epsilon_{abcde}\underline{T}^{ab}\underline{T}^{cd}\mathcal{D}_f\underline{T}^{ef}, \end{aligned} \quad (6.26)$$

where

$$V_{\mu\nu}{}^{ij} \equiv 2\partial_{[\mu}V_{\nu]}{}^{ij} - 2V_{[\mu}{}^{k(i}V_{\nu]k}{}^{j)}. \quad (6.27)$$



This action (6.26) describes the coupling between an external vector multiplet and dilaton Weyl multiplet. If we simply combine above action with the Riemann tensor squared invariant, we are not able to obtain the supersymmetric Gauss-Bonnet combination since the curvature squared terms in (6.26) is multiplied by  $\rho$  which stays the same after imposing the gauge choices (5.62). By comparing the superconformal transformation rules of vector multiplet

$$\begin{aligned}
\delta\rho &= \frac{1}{2}\mathrm{i}\bar{\epsilon}\lambda, \\
\delta A_\mu &= -\frac{1}{2}\sigma\bar{\epsilon}\psi_\mu + \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda, \\
\delta\lambda^i &= -\frac{1}{4}\gamma\cdot\widehat{F}\epsilon^i - \frac{1}{2}\mathrm{i}\mathcal{D}\rho\epsilon^i + \rho\gamma\cdot T\epsilon^i - Y^{ij}\epsilon_j + \rho\eta^i, \\
\delta Y^{ij} &= -\frac{1}{2}\bar{\epsilon}^{(i}\mathcal{D}\lambda^{j)} + \frac{1}{2}\mathrm{i}\bar{\epsilon}^{(i}\gamma\cdot T\lambda^{j)} - 4\mathrm{i}\rho\bar{\epsilon}^{(i}\chi^{j)} + \frac{1}{2}\mathrm{i}\bar{\eta}^{(i}\lambda^{j)},
\end{aligned} \tag{6.28}$$

with those of  $(\sigma, C_\mu, \psi^i)$  in the dilaton Weyl multiplet

$$\begin{aligned}
\delta\sigma &= \frac{1}{2}\mathrm{i}\bar{\epsilon}\psi, \\
\delta C_\mu &= -\frac{1}{2}\sigma\bar{\epsilon}\psi_\mu + \frac{1}{2}\bar{\epsilon}\gamma_\mu\psi, \\
\delta\psi^i &= -\frac{1}{4}\gamma\cdot\widehat{G}\epsilon^i - \frac{1}{2}\mathrm{i}\mathcal{D}\sigma\epsilon^i + \sigma\gamma\cdot\underline{T}\epsilon^i - \frac{1}{4}\mathrm{i}\sigma^{-1}\epsilon_j\bar{\psi}^i\psi^j + \sigma\eta^i,
\end{aligned} \tag{6.29}$$

we notice that there exists a map from vector multiplet to  $(\sigma, C_\mu, \psi^i)$

$$\rho \rightarrow \sigma, \quad A_a \rightarrow C_a, \quad \lambda^i \rightarrow \psi^i, \quad Y^{ij} \rightarrow \frac{1}{4}\mathrm{i}\sigma^{-1}\bar{\psi}^{(i}\psi^{j)}, \tag{6.30}$$

since

$$\delta\left(\frac{1}{4}\mathrm{i}\sigma^{-1}\bar{\psi}^{(i}\psi^{j)}\right) = -\frac{1}{2}\bar{\epsilon}^{(i}\mathcal{D}\psi^{j)} + \frac{1}{2}\mathrm{i}\bar{\epsilon}^{(i}\gamma\cdot\underline{T}\psi^{j)} - 4\mathrm{i}\sigma\bar{\epsilon}^{(i}\underline{\chi}^{j)} + \frac{1}{2}\mathrm{i}\bar{\eta}^{(i}\psi^{j)}. \tag{6.31}$$

Using (6.30), we obtain the supersymmetrization of  $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$  purely based on the fields of dilaton Weyl multiplet

$$\begin{aligned}
e^{-1}\mathcal{L}_{\sigma C^2} &= \frac{1}{8}\sigma C^{abcd}C_{abcd} + \frac{64}{3}\sigma \underline{D}^2 + \frac{1024}{9}\sigma \underline{T}^2 \underline{D} - \frac{32}{3}\underline{D} \underline{T}_{ab} G^{ab} \\
&\quad - \frac{16}{3}\sigma C_{abcd} \underline{T}^{ab} \underline{T}^{cd} + 2C_{abcd} \underline{T}^{cd} G^{ab} + \frac{1}{16}\epsilon_{abcde} C^a C^{bcfg} C^{de}{}_{fg} \\
&\quad - \frac{1}{12}\epsilon_{abcde} C^a V^{bc}{}_{ij} V^{de}{}_{ij} + \frac{16}{3} Y_{ij} V_{ab}{}^{ij} \underline{T}^{ab} - \frac{1}{3}\sigma V_{ab}{}^{ij} V^{ab}{}_{ij} + \frac{64}{3}\sigma \mathcal{D}_b \underline{T}_{ac} \mathcal{D}^a \underline{T}^{bc} \\
&\quad - \frac{128}{3}\sigma \underline{T}_{ab} \mathcal{D}^b \mathcal{D}_c \underline{T}^{ac} - \frac{64}{3}\sigma \mathcal{D}_a \underline{T}_{bc} \mathcal{D}^a \underline{T}^{bc} + 1024\sigma \underline{T}^4 - \frac{2816}{27}\sigma (\underline{T}^2)^2 \\
&\quad - \frac{64}{9}\underline{T}_{ab} G^{ab} \underline{T}^2 - \frac{256}{3}\underline{T}_{ac} \underline{T}^{cd} \underline{T}_{bd} G^{ab} - \frac{32}{3}\epsilon_{abcde} \underline{T}^{cf} \mathcal{D}_f \underline{T}^{de} G^{ab} \\
&\quad - 16\epsilon_{abcde} \underline{T}^c{}_f \mathcal{D}^d \underline{T}^{ef} G^{ab} - \frac{128}{3}\sigma \epsilon_{abcde} \underline{T}^{ab} \underline{T}^{cd} \mathcal{D}_f \underline{T}^{ef}. \tag{6.32}
\end{aligned}$$

Imposing the gauge fixing conditions (5.62), we obtain

$$\begin{aligned}
e^{-1}\mathcal{L}_{\sigma C^2}^D|_{\sigma=1} &= \frac{1}{8}R_{abcd}R^{abcd} - \frac{1}{6}R_{ab}R^{ab} + \frac{1}{48}R^2 + \frac{64}{3}\underline{D}^2 + \frac{1024}{9}\underline{T}^2 \underline{D} \\
&\quad - \frac{16}{3}R_{abcd} \underline{T}^{ab} \underline{T}^{cd} + 2R_{abcd} \underline{T}^{cd} G^{ab} + \frac{1}{3}R \underline{T}_{ab} G^{ab} - \frac{8}{3}R_{bd} G_c{}^b \underline{T}^{cd} \\
&\quad - \frac{64}{3}R^{bc} \underline{T}_{ab} \underline{T}^a{}_c + \frac{8}{3}R \underline{T}^2 - \frac{32}{3}\underline{D} \underline{T}_{ab} G^{ab} + \frac{1}{16}\epsilon_{abcde} C^a R^{bcfg} R^{de}{}_{fg} \\
&\quad - \frac{1}{12}\epsilon_{abcde} C^a V^{bc}{}_{ij} V^{de}{}_{ij} - \frac{1}{3}V_{ab}{}^{ij} V^{ab}{}_{ij} - \frac{64}{3}\nabla_a \underline{T}_{bc} \nabla^a \underline{T}^{bc} \\
&\quad + \frac{64}{3}\nabla_b \underline{T}_{ac} \nabla^a \underline{T}^{bc} - \frac{128}{3}\underline{T}_{ab} \nabla^b \nabla_c \underline{T}^{ac} - \frac{128}{3}\epsilon_{abcde} \underline{T}^{ab} \underline{T}^{cd} \nabla_f \underline{T}^{ef} \\
&\quad + 1024 \underline{T}^4 - \frac{2816}{27}(\underline{T}^2)^2 - \frac{64}{9}\underline{T}_{ab} G^{ab} \underline{T}^2 - \frac{256}{3}\underline{T}_{ac} \underline{T}^{cd} \underline{T}_{bd} G^{ab} \\
&\quad - \frac{32}{3}\epsilon_{abcde} \underline{T}^{cf} \nabla_f \underline{T}^{de} G^{ab} - 16\epsilon_{abcde} \underline{T}^c{}_f \nabla^d \underline{T}^{ef} G^{ab}, \tag{6.33}
\end{aligned}$$

where

$$\begin{aligned}
D &\equiv -\frac{1}{32}R - \frac{1}{16}G^{ab}G_{ab} - \frac{26}{3}T^{ab}T_{ab} + 2T^{ab}G_{ab} + \text{fermions}, \\
T_{ab} &\equiv \frac{1}{8}G_{ab} + \frac{1}{48}\epsilon_{abcde}H^{cde} + \text{fermions}. \tag{6.34}
\end{aligned}$$

So far, we obtained the supersymmetric completion of Einstein-Hilbert, Riemann tensor squared and Weyl tensor squared actions. Because of the off-shell nature of these invariants, we can combine them to form a more general theory with two free parameters

$$\mathcal{L} = \mathcal{L}_{LR}^D + \alpha \mathcal{L}_{\text{Riem}^2}^D + \beta \mathcal{L}_{\sigma C^2}^D \Big|_{\sigma=1} . \quad (6.35)$$

The Gauss-Bonnet combination corresponds to case with  $\beta = 3\alpha$  in which the kinetic term of auxiliary vector  $V_\mu^{ij}$  vanishes. Using  $\beta = 3\alpha$ , the purely bosonic part of Lagrangian (6.35) takes the form

$$\begin{aligned} e^{-1} \left( \mathcal{L}_{LR}^D + \alpha \mathcal{L}_{\text{GB}} \right) = & \frac{1}{2} L R + \frac{1}{2} L^{-1} \partial_\mu L \partial^\mu L - \frac{1}{4} L G_{\mu\nu} G^{\mu\nu} - \frac{1}{6} L H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & - L^{-1} N^2 - L^{-1} P_\mu P^\mu - \sqrt{2} P_\mu V^\mu + L V_\mu'^{ij} V_{ij}'^\mu \\ & + \alpha \left[ -\frac{1}{4} \left( R_{\mu\nu ab}(\omega_+) - G_{\mu\nu} G_{ab} \right) \left( R^{\mu\nu ab}(\omega_+) - G^{\mu\nu} G^{ab} \right) \right. \\ & + \frac{3}{8} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{2} R_{\mu\nu} R^{\mu\nu} + \frac{1}{16} R^2 + 64 \underline{D}^2 - \frac{64}{3} \underline{T}_{\mu\nu} G^{\mu\nu} \underline{T}^2 \\ & - \frac{1}{8} \epsilon^{\mu\nu\rho\sigma\lambda} \left( R_{\mu\nu ab}(\omega_+) - G_{\mu\nu} G_{ab} \right) \left( R_{\rho\sigma}{}^{ab}(\omega_+) - G_{\rho\sigma} G^{ab} \right) C_\lambda \\ & - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma\lambda} B_{\rho\sigma} \left( R_{\mu\nu ab}(\omega_+) - G_{\mu\nu} G_{ab} \right) \nabla_\lambda(\omega_+) G^{ab} \\ & + \frac{3}{16} \epsilon_{\mu\nu\rho\sigma\lambda} C^\mu R^{\nu\rho\tau\delta} R^{\sigma\lambda}{}_{\tau\delta} - \frac{1}{4} \epsilon_{\mu\nu\rho\sigma\lambda} C^\mu V^{\nu\rho}{}_{ij} V^{\sigma\lambda}{}^{ij} \\ & - 16 R_{\mu\nu\rho\sigma} \underline{T}^{\mu\nu} \underline{T}^{\rho\sigma} + 6 R_{\mu\nu\rho\sigma} G^{\mu\nu} \underline{T}^{\rho\sigma} + R \underline{T}_{\mu\nu} G^{\mu\nu} + 8 R \underline{T}^2 \\ & - 64 R^{\mu\nu} \underline{T}_{\sigma\mu} \underline{T}^\sigma{}_\nu - 8 R_{\mu\nu} G_{\sigma}{}^\mu \underline{T}^{\sigma\nu} - 32 \underline{D} \underline{T}_{\mu\nu} G^{\mu\nu} + \frac{1024}{3} \underline{T}^2 \underline{D} \\ & - 64 \nabla_\mu \underline{T}_{\nu\rho} \nabla^\mu \underline{T}^{\nu\rho} + 64 \nabla^\mu \underline{T}^{\nu\rho} \nabla_\nu \underline{T}_{\mu\rho} - 128 \underline{T}_{\mu\nu} \nabla^\nu \nabla_\sigma \underline{T}^{\mu\sigma} \\ & - \frac{1}{2} \nabla_\mu(\omega_+) G^{ab} \nabla^\mu(\omega_+) G_{ab} + 3072 \underline{T}^4 - \frac{2816}{9} (\underline{T}^2)^2 \\ & - 256 \underline{T}_{\mu\sigma} \underline{T}^{\sigma\rho} \underline{T}_{\rho\nu} G^{\nu\mu} - 128 \epsilon_{\mu\nu\rho\sigma\lambda} \underline{T}^{\mu\nu} \underline{T}^{\rho\sigma} \nabla_\tau \underline{T}^{\lambda\tau} \\ & \left. - 32 \epsilon_{\mu\nu\rho\sigma\lambda} G^{\mu\nu} \underline{T}^{\rho\tau} \nabla_\tau \underline{T}^{\sigma\lambda} - 48 \epsilon_{\mu\nu\rho\sigma\lambda} G^{\mu\nu} \underline{T}^\rho{}_\tau \nabla^\sigma \underline{T}^{\lambda\tau} \right] . \quad (6.36) \end{aligned}$$

We notice that the ratio of the coefficients in front of the Gauss-Bonnet combination and the Chern-Simons coupling  $\epsilon_{\mu\nu\rho\sigma\lambda}C^\mu R^{\nu\rho\delta\tau}R^{\sigma\lambda}{}_{\delta\tau}$  is  $\frac{1}{2}$  which is consistent with the value resulting from the circle reduction of the partial results given in [39, 31] on the six-dimensional supersymmetric Gauss-Bonnet combination.

### 6.1.2.1 On-Shell Theory

In this subsection, we shall study the on-shell theory of the Gauss-Bonnet extended supergravity to first order in  $\alpha$  upon eliminating the auxiliary fields. In order to do so, we first present the minimal ungauged on-shell Poincaré supergravity by eliminating the the auxiliary fields  $(E_{\mu\nu\rho}, V_\mu, N, V_\mu{}^{ij})$  and truncating the matter multiplet  $(B_{\mu\nu}, L, \varphi^i)$ . We then obtain the on-shell Gauss-Bonnet extended Einstein-Maxwell supergravity to first order in  $\alpha$  by using the equations derived from the two-derivative Lagrangian that is zeroth order in  $\alpha$ .

To eliminate the auxiliary fields  $(N, P_a, V_\mu, V_\mu{}^{ij})$ , we use their equations of motion

$$0 = N, \quad 0 = P_\mu, \quad 0 = V_\mu{}^{ij}, \quad (6.37)$$

$$0 = \epsilon^{\mu\nu\rho\sigma\lambda}\partial_\mu(-L^{-1}P_\nu - \frac{1}{\sqrt{2}}V_\nu). \quad (6.38)$$

Equation (6.38) implies that locally

$$-L^{-1}P_\mu - \frac{1}{\sqrt{2}}V_\mu = \partial_\mu\phi, \quad (6.39)$$

where  $\phi$  is a Stueckelberg scalar. Eliminating this scalar by using the shift symmetry transformation and using the second equation in (6.37), we obtain <sup>2</sup>

$$V_\mu = 0. \quad (6.40)$$

---

<sup>2</sup>In the original Poincaré theory (5.65)  $U(1)_R$  symmetry is gauged by the auxiliary vector  $V_\mu$ . However, in the on-shell theory, the  $U(1)_R$  symmetry becomes global due to the elimination of  $V_\mu$ .

It follows that the corresponding on-shell theory is given by

$$e^{-1}\mathcal{L}'_{\text{EM}} = \frac{1}{2}LR + \frac{1}{2}L^{-1}\partial_\mu L\partial^\mu L - \frac{1}{4}LG_{\mu\nu}G^{\mu\nu} - \frac{1}{6}LH_{\mu\nu\rho}H^{\mu\nu\rho}. \quad (6.41)$$

To truncate out the matter multiplet  $(B_{\mu\nu}, L, \varphi^i)$ , we first dualize  $B_{\mu\nu}$  to a vector field  $\tilde{C}_\mu$  by adding the following Lagrange multiplier to (6.41)

$$\Delta\mathcal{L} = -\frac{1}{12}\epsilon^{\mu\nu\rho\sigma\lambda}B_{\mu\nu\rho}\tilde{G}_{\sigma\lambda}, \quad \tilde{G}_{\mu\nu} \equiv 2\partial_{[\mu}\tilde{C}_{\nu]}, \quad (6.42)$$

and replacing  $H_{\mu\nu\rho}$  by  $B_{\mu\nu\rho} + \frac{3}{2}C_{[\mu}G_{\nu\rho]}$ . The field equations of  $\tilde{C}_\mu$  and  $B_{\mu\nu\rho}$  imply that

$$B_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}, \quad H^{\mu\nu\rho} = -\frac{1}{4}L^{-1}\epsilon^{\mu\nu\rho\sigma\lambda}\tilde{G}_{\sigma\lambda}. \quad (6.43)$$

Substituting (6.43) to (6.41), we obtain the on-shell ungauged Einstein-Maxwell supergravity

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{EM}} &= \frac{1}{2}LR + \frac{1}{2}L^{-1}\partial_\mu L\partial^\mu L - \frac{1}{4}LG_{\mu\nu}G^{\mu\nu} - \frac{1}{8}L^{-1}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} \\ &\quad + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}C_\mu G_{\nu\rho}\tilde{G}_{\sigma\lambda}, \end{aligned} \quad (6.44)$$

where  $(e_\mu^a, \psi_\mu^i, C_\mu)$  constitute the supergravity multiplet while  $(\tilde{C}_\mu, \varphi^i, L)$  comprise the Maxwell multiplet.

Truncation of the Einstein-Maxwell theory to the minimal on-shell theory can be implemented by imposing

$$L = 1, \quad \tilde{C}_\mu = C_\mu, \quad \varphi^i = 0, \quad (6.45)$$

which is consistent with the equations of motion

$$\begin{aligned}
R &= 2L^{-1}\square L - L^{-2}\partial_\mu L\partial^\mu L + \frac{1}{2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{4}L^{-2}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}, \\
LR_{\mu\nu} &= \nabla_\mu\nabla_\nu L - L^{-1}\partial_\mu L\partial_\nu L + LG_\mu{}^\sigma G_{\nu\sigma} + \frac{1}{2}L^{-1}\tilde{G}_\mu{}^\sigma\tilde{G}_{\nu\sigma} \\
&\quad - \frac{1}{4}g_{\mu\nu}L^{-1}\tilde{G}_{\rho\sigma}\tilde{G}^{\rho\sigma}, \\
0 &= \nabla^\nu(LG_{\nu\mu}) + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma\lambda}G^{\nu\rho}\tilde{G}^{\sigma\lambda}, \\
0 &= \nabla^\nu(L^{-1}\tilde{G}_{\nu\mu}) + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma\lambda}G^{\nu\rho}G^{\sigma\lambda},
\end{aligned} \tag{6.46}$$

and leads to the following transformation

$$\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu^i &= (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\epsilon^i + \frac{1}{8}\text{i}(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)G_{\nu\rho}, \\
\delta C_\mu &= -\frac{1}{2}\text{i}\bar{\epsilon}\psi_\mu.
\end{aligned} \tag{6.47}$$

The resulting action coincides with the minimal on-shell supergravity in five dimensions [69, 70]

$$e^{-1}\mathcal{L}_{\text{EH}}^{\text{min}} = \frac{1}{2}R - \frac{3}{8}G_{\mu\nu}G^{\mu\nu} + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}C_\mu G_{\nu\rho}G_{\sigma\lambda}. \tag{6.48}$$

The canonical kinetic term of  $C_\mu$  can be recovered by a scaling  $C_\mu \rightarrow \frac{2}{\sqrt{6}}C_\mu$ .

With the Gauss-Bonnet combination added, the duality relation (6.43) and truncation condition must receive corrections proportional to the powers of  $\alpha$ , if we consider a perturbative expansion valid when the energy scale  $\Lambda$  satisfies  $\Lambda^2 \ll 1/|\alpha|$ . We follow the procedure of [71]. Schematically, the off-shell action (6.36) takes the form

$$S_{\text{off-shell}}[\phi] = S_0[\phi] + \alpha S_1[\phi]. \quad (6.49)$$

It follows that the auxiliary field equations (6.37) - (6.38), the field equation for  $B_{\mu\nu\rho}$  (6.43) as well as the truncation equation (6.45) must receive corrections proportional to  $\alpha$ . The solution to those equations can be expressed in terms of a series expansion in  $\alpha$ ,  $\phi = \phi_0 + \alpha\phi_1 + \alpha^2\phi_2 + \dots$ , where  $\phi_0$  is the solution to the zeroth order equation given in previous section. As a consequence, the on-shell action possesses the form  $S_{\text{on-shell}}[\phi] = S_0[\phi_0] + \alpha(S_1[\phi_0] + \phi_1 S'_0[\phi_0]) + \dots$ . In this equation,  $S'_0[\phi_0] = 0$  when  $\phi_0$  is an auxiliary field or a Lagrangian multiplier. We eliminate the auxiliary fields and Lagrangian multiplier  $B_{\mu\nu\rho}$  by plugging their zeroth order solutions to the action

$$\begin{aligned}
e^{-1}(\mathcal{L}_{\text{EM}} + \alpha\mathcal{L}_{\text{GB}}) &= \frac{1}{2}LR + \frac{1}{2}L^{-1}\partial_\mu L\partial^\mu L - \frac{1}{4}LG_{\mu\nu}G^{\mu\nu} - \frac{1}{8}L^{-1}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} \\
&+ \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}C_\mu G_{\nu\rho}\tilde{G}_{\sigma\lambda} + \alpha\left[\frac{3}{8}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{2}R_{\mu\nu}R^{\mu\nu} + \frac{1}{16}R^2\right. \\
&+ 64\underline{D}^2 - \frac{1}{4}\left(R_{\mu\nu ab}(\omega_+) - G_{\mu\nu}G_{ab}\right)\left(R^{\mu\nu ab}(\omega_+) - G^{\mu\nu}G^{ab}\right) \\
&- \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}\left(R_{\mu\nu ab}(\omega_+) - G_{\mu\nu}G_{ab}\right)\left(R_{\rho\sigma}{}^{ab}(\omega_+) - G_{\rho\sigma}G^{ab}\right)C_\lambda \\
&- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\lambda}B_{\rho\sigma}\left(R_{\mu\nu ab}(\omega_+) - G_{\mu\nu}G_{ab}\right)\nabla_\lambda(\omega_+)G^{ab} + \frac{3}{16}\epsilon_{\mu\nu\rho\sigma\lambda}C^\mu R^{\nu\rho\tau\delta}R^{\sigma\lambda}{}_{\tau\delta} \\
&- 16R_{\mu\nu\rho\sigma}\underline{T}^{\mu\nu}\underline{T}^{\rho\sigma} + 6R_{\mu\nu\rho\sigma}G^{\mu\nu}\underline{T}^{\rho\sigma} + R\underline{T}_{\mu\nu}G^{\mu\nu} - 8R_{\mu\nu}G_\sigma{}^\mu\underline{T}^{\sigma\nu} \\
&- 64R^{\mu\nu}\underline{T}_{\sigma\mu}\underline{T}^\sigma{}_\nu + 8R\underline{T}^2 - 32\underline{D}\underline{T}_{\mu\nu}G^{\mu\nu} + \frac{1024}{3}\underline{T}^2\underline{D} - 64\nabla_\mu\underline{T}_{\nu\rho}\nabla^\mu\underline{T}^{\nu\rho} \\
&+ 64\nabla^\mu\underline{T}^{\nu\rho}\nabla_\nu\underline{T}_{\mu\rho} - 128\underline{T}_{\mu\nu}\nabla^\nu\nabla_\sigma\underline{T}^{\mu\sigma} - \frac{1}{2}\nabla_\mu(\omega_+)G^{ab}\nabla^\mu(\omega_+)G_{ab} \\
&+ 3072\underline{T}^4 - \frac{2816}{9}(\underline{T}^2)^2 - \frac{64}{3}\underline{T}_{\mu\nu}G^{\mu\nu}\underline{T}^2 - 256\underline{T}_{\mu\sigma}\underline{T}^{\sigma\rho}\underline{T}_{\rho\nu}G^{\nu\mu} \\
&- 128\epsilon_{\mu\nu\rho\sigma\lambda}\underline{T}^{\mu\nu}\underline{T}^{\rho\sigma}\nabla_\tau\underline{T}^{\lambda\tau} - 32\epsilon_{\mu\nu\rho\sigma\lambda}G^{\mu\nu}\underline{T}^{\rho\tau}\nabla_\tau\underline{T}^{\sigma\lambda} \\
&- 48\epsilon_{\mu\nu\rho\sigma\lambda}G^{\mu\nu}\underline{T}^\rho{}_\tau\nabla^\sigma\underline{T}^{\lambda\tau}] + \mathcal{O}(\alpha^2), \quad (6.50)
\end{aligned}$$

where  $T_{\mu\nu}$  and  $\omega_{+\mu}{}^{ab}$  are now given by

$$T_{\mu\nu} = \frac{1}{16}(2G_{\mu\nu} + L^{-1}\tilde{G}_{\mu\nu}), \quad \omega_{+\mu}{}^{ab} = \omega_{\mu}{}^{ab} - \frac{1}{4}L^{-1}e_{\mu f}\epsilon^{fabcd}\tilde{G}_{cd}. \quad (6.51)$$

### 6.1.3 Supersymmetric Ricci Scalar Squared Action

In this section, we construct the supersymmetric completion of Ricci scalar squared action using the dilaton Weyl multiplet. The key observation behind the construction is that the composite expression of  $Y^{ij}$  (4.6) contains the Ricci scalar implicitly in the superconformal d'Alembertian of  $L^{ij}$ . Therefore, the supersymmetric Ricci scalar squared action can be obtained by substituting the composite expressions (4.6) in the vector multiplet action given in (6.12) since the off-shell vector multiplet action has a  $Y^{ij}Y_{ij}$  term. The construction of supersymmetric Ricci scalar squared action completes the off-shell curvature squared actions based on the dilaton Weyl multiplet in  $\mathcal{N} = 2$ ,  $D = 5$  supergravity.

For the construction of Ricci scalar squared invariant, we first rewrite the vector multiplet action (6.12) for a single vector multiplet under the gauge fixing conditions (5.62)

$$\begin{aligned} e^{-1}\mathcal{L}_V^D|_{\sigma=1} &= Y_{ij}Y^{ij} - \frac{1}{2}\nabla_{\mu}\rho\nabla^{\mu}\rho - \frac{1}{4}(F_{\mu\nu} - \rho G_{\mu\nu})(F^{\mu\nu} - \rho G^{\mu\nu}) \\ &\quad - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu} - \rho G_{\mu\nu})(F_{\rho\sigma} - \rho G_{\rho\sigma})C_{\lambda} \\ &\quad - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu} - \rho G_{\mu\nu})B_{\rho\sigma}\nabla_{\lambda}\rho. \end{aligned} \quad (6.52)$$

Using the same gauge fixing, the composite expressions (4.6) for the elements of vector multiplet can be recast into



$$\begin{aligned}
\rho|_{\sigma=1} &= 2NL^{-1}, \\
\underline{Y}_{ij}|_{\sigma=1} &= \frac{1}{\sqrt{2}}\delta_{ij}\left(-\frac{1}{2}R + \frac{1}{4}G_{ab}G^{ab} + \frac{1}{6}H_{abc}H^{abc} - L^{-2}N^2 - L^{-2}P_aP^a - V_a{}^{kl}V'_{kl}{}^a\right. \\
&\quad \left.+ L^{-1}\square L - \frac{1}{2}L^{-2}\partial_a L\partial^a L\right) + 2L^{-1}P^aV'_{a ij} - \sqrt{2}L^{-1}\nabla^a(LV'_a{}^m{}_{(i}\delta_{j)m}), \\
\widehat{F}_{ab}|_{\sigma=1} &= 2\sqrt{2}\partial_{[a}(V_{b]} + \sqrt{2}L^{-1}P_{b]). \tag{6.53}
\end{aligned}$$

The fermionic terms in the composite expressions of vector multiplet can be straightforwardly figured out by using the complete results given in (4.6). Using the above formulas in (6.12), we obtain the supersymmetric Ricci scalar squared action in the dilaton Weyl multiplet whose bosonic part reads

$$\begin{aligned}
e^{-1}\mathcal{L}_{R^2}^D &= \frac{1}{4}\left(R - \frac{1}{2}G_{\mu\nu}G^{\mu\nu} - \frac{1}{3}H_{\mu\nu\rho}H^{\mu\nu\rho} + 2L^{-2}N^2 + 2L^{-2}P_\mu P^\mu - 4Z_\mu\bar{Z}^\mu - 2L^{-1}\square L\right. \\
&\quad \left.+ L^{-2}\partial_\mu L\partial^\mu L\right)^2 - L^{-2}\left|2\nabla^\mu(LZ_\mu) + 2\sqrt{2}iP^\mu Z_\mu\right|^2 - 2\nabla_\mu(L^{-1}N)\nabla^\mu(L^{-1}N) \\
&\quad - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\lambda}\left(\partial_\mu\tilde{C}_\nu - NL^{-1}G_{\mu\nu}\right)\left(\partial_\rho\tilde{C}_\sigma - NL^{-1}G_{\rho\sigma}\right)C_\lambda \\
&\quad - 2\epsilon^{\mu\nu\rho\sigma\lambda}\left(\partial_\mu\tilde{C}_\nu - NL^{-1}G_{\mu\nu}\right)B_{\rho\sigma}\nabla_\lambda(L^{-1}N) \\
&\quad - \left(\partial_{[\mu}\tilde{C}_{\nu]} - NL^{-1}G_{\mu\nu}\right)\left(\partial^\mu\tilde{C}^\nu - NL^{-1}G^{\mu\nu}\right), \tag{6.54}
\end{aligned}$$

where for simplicity, we have defined

$$Z_\mu = V_\mu{}^{12} + iV_\mu{}^{11}, \quad \tilde{C}_\mu = \sqrt{2}V_\mu + 2L^{-1}P_\mu. \tag{6.55}$$

The general  $R + R^2$  action in the dilation Weyl multiplet can therefore be written as

$$\left(\mathcal{L}_{LR}^D + \alpha\mathcal{L}_{\text{Riem}^2}^D + \beta\mathcal{L}_{\sigma C^2}^D + \gamma\mathcal{L}_{R^2}^D\right)|_{\sigma=1}. \tag{6.56}$$

## 6.2 Vector Multiplets Coupled Curvature Squared Invariants in the Dilaton Weyl Multiplet

In the previous section, we have completed the curvature squared invariants purely based on the off-shell Poincaré multiplet (6.1). In this section, we couple the external vector multiplets to the curvature squared invariants. The inclusion of the external vector multiplet gives rise to a mixed Chern-Simons term in the supersymmetric Riemann squared action

$$A \wedge R \wedge R, \tag{6.57}$$

where the vector  $A_\mu$  belongs to a vector multiplet, as opposed to the case of minimal off-shell curvature squared invariants in the dilaton Weyl multiplet where the Chern-Simons term is purely gravitational (6.14)

$$C \wedge R \wedge R, \tag{6.58}$$

where  $C_\mu$  is the vector in the Poincaré multiplet. In the following, we directly present the results for the vector multiplets coupled curvature squared term which can be straightforwardly obtained from the results for single vector multiplet coupled curvature squared term.

### 6.2.1 Vector Multiplets Coupled Riemann Squared Action

In this subsection, we generalize the Riemann squared action purely based on the off-shell Poincaré multiplet to the vector multiplets coupled Riemann squared action in which the Chern-Simons term takes the form of  $A \wedge R \wedge R$ . In order to construct the vector multiplets coupled Riemann squared action, we consider the following Yang-

Mills action in the dilaton Weyl multiplet. This action is the Yang-Mills analogue of the  $n$  Abelian vector action

$$\begin{aligned}
e^{-1}\mathcal{L}'_{\text{YM}}{}^D &= \rho Y_{ij}^\Sigma Y^{ij\Sigma} + 2\rho^\Sigma Y_{ij}^\Sigma Y^{ij} + \rho\rho^\Sigma \nabla_\mu \nabla^\mu \rho^\Sigma + \frac{1}{2}\rho \nabla_\mu \rho^\Sigma \nabla^\mu \rho^\Sigma \\
&\quad - \frac{1}{4}\rho(F_{\mu\nu}^\Sigma - \rho^\Sigma G_{\mu\nu})(F^{\mu\nu\Sigma} - 3\rho^\Sigma G^{\mu\nu}) - \frac{1}{2}(F_{\mu\nu}^\Sigma - \rho^\Sigma G_{\mu\nu})\rho^\Sigma F^{\mu\nu} \\
&\quad + \frac{1}{12}\rho^\Sigma \rho^\Sigma \epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu} - 2\rho G_{\mu\nu})H_{\rho\sigma\lambda} + \frac{1}{6}\rho\rho^\Sigma \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^\Sigma H_{\rho\sigma\lambda} \\
&\quad - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^\Sigma F_{\rho\sigma}^\Sigma A_\lambda, \tag{6.59}
\end{aligned}$$

where  $\Sigma$  is the Yang-Mills group index. The construction procedure of the vector multiplets coupled Riemann squared action is the same as before. Upon applying the map between the Yang-Mills multiplet and the dilaton Weyl multiplet, we obtain the vector multiplets coupled Riemann squared action

$$\begin{aligned}
e^{-1}\mathcal{L}'_{\text{Riem}^2}{}^D &= \alpha_I \left[ -\frac{1}{4}\rho^I (R_{\mu\nu ab}(\omega_+) - G_{\mu\nu} G_{ab})(R^{\mu\nu ab}(\omega_+) - 3G^{\mu\nu} G^{ab}) \right. \\
&\quad - \frac{1}{2}(R^{\mu\nu ab}(\omega_+) - G^{\mu\nu} G^{ab})F_{\mu\nu}^I G_{ab} + \rho^I V_{ij}{}^{\mu\nu} V^{ij}{}_{\mu\nu} - 2G_{\mu\nu} V_{ij}{}^{\mu\nu} Y^{ijI} \\
&\quad + \rho^I G_{ab} \nabla_\mu(\omega_+) \nabla^\mu(\omega_+) G^{ab} + \frac{1}{2}\rho^I \nabla_\mu(\omega_+) G^{ab} \nabla^\mu(\omega_+) G_{ab} \\
&\quad + \frac{1}{12}\epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu}^I - 2\rho^I G_{\mu\nu})H_{\rho\sigma\lambda} G_{ab} G^{ab} + \frac{1}{6}\rho^I \epsilon^{\mu\nu\rho\sigma\lambda} R_{\mu\nu ab}(\omega_+) G^{ab} H_{\rho\sigma\lambda} \\
&\quad \left. - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda} R_{\mu\nu ab}(\omega_+) R_{\rho\sigma}{}^{ab}(\omega_+) A_\lambda^I \right]. \tag{6.60}
\end{aligned}$$

Note that this action recovers the Riemann squared invariant (6.14) upon considering a single vector multiplet,  $I = 1$ , and applying the map from the dilaton Weyl multiplet to the vector multiplet (6.30).

### 6.2.2 Vector Multiplets Coupled $C_{\mu\nu\rho\lambda}^2 + \frac{1}{6}R^2$ Action

The  $n$ -vector multiplets coupled  $C_{\mu\nu\rho\lambda}^2 + \frac{1}{6}R^2$  action can be straightforwardly obtained from the Weyl squared action [40] in standard Weyl multiplet by underlining

$D$ ,  $T_{ab}$  and  $\chi_i$

$$\begin{aligned}
e^{-1}\mathcal{L}'_{C^2+\frac{1}{6}R^2} &= \beta_I \left[ \frac{1}{8}\rho^I C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \frac{64}{3}\rho^I \underline{D}^2 + \frac{1024}{9}\rho^I \underline{T}^2 \underline{D} - \frac{32}{3}\underline{D} \underline{T}_{\mu\nu} F^{\mu\nu I} \right. \\
&\quad - \frac{16}{3}\rho^I C_{\mu\nu\rho\sigma} \underline{T}^{\mu\nu} \underline{T}^{\rho\sigma} + 2C_{\mu\nu\rho\sigma} \underline{T}^{\mu\nu} F^{\rho\sigma I} + \frac{1}{16}\epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^I C_{\nu\rho\tau\delta} C_{\sigma\lambda}{}^{\tau\delta} \\
&\quad - \frac{1}{12}\epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^I V_{\nu\rho}{}^{ij} V_{\sigma\lambda}{}^{ij} + \frac{16}{3}Y_{ij}^I V_{\mu\nu}{}^{ij} \underline{T}^{\mu\nu} - \frac{1}{3}\rho^I V_{\mu\nu}{}^{ij} V^{\mu\nu}{}_{ij} \\
&\quad + \frac{64}{3}\rho^I \nabla_\nu \underline{T}_{\mu\rho} \nabla^\mu \underline{T}^{\nu\rho} - \frac{128}{3}\rho^I \underline{T}_{\mu\nu} \nabla^\nu \nabla_\rho \underline{T}^{\mu\rho} - \frac{256}{9}\rho^I R^{\nu\rho} \underline{T}_{\mu\nu} \underline{T}^\mu{}_\rho \\
&\quad + \frac{32}{9}\rho^I R \underline{T}^2 - \frac{64}{3}\rho^I \nabla_\mu \underline{T}_{\nu\rho} \mathcal{D}^\mu \underline{T}^{\nu\rho} + 1024\rho^I \underline{T}^4 - \frac{2816}{27}\rho^I (\underline{T}^2)^2 \\
&\quad - \frac{64}{9}\underline{T}_{\mu\nu} F^{\mu\nu I} \underline{T}^2 - \frac{256}{3}\underline{T}_{\mu\rho} \underline{T}^{\rho\lambda} \underline{T}_{\nu\lambda} F^{\mu\nu I} - \frac{32}{3}\epsilon_{\mu\nu\rho\sigma\lambda} \underline{T}^{\rho\tau} \nabla_\tau \underline{T}^{\sigma\lambda} F^{\mu\nu I} \\
&\quad \left. - 16\epsilon_{\mu\nu\rho\sigma\lambda} \underline{T}^\rho{}_\tau \nabla^\sigma \underline{T}^{\lambda\tau} F^{\mu\nu I} - \frac{128}{3}\rho^I \epsilon_{\mu\nu\rho\sigma\lambda} \underline{T}^{\mu\nu} \underline{T}^{\rho\sigma} \mathcal{D}_\tau \underline{T}^{\lambda\tau} \right]. \quad (6.61)
\end{aligned}$$

where the composite expressions for  $D$  and  $T_{ab}$  in  $\sigma = 1$  gauge fixing are given in (6.34).

### 6.2.3 Vector Multiplets Coupled Ricci Scalar Squared Action

To obtain an off-shell Ricci scalar squared invariant coupled to vector multiplets, we use the same strategy as we construct the minimal Ricci scalar squared invariant. The starting point is the vector multiplet action (6.59). By choosing the nonvanishing components of  $C_{IJK}$  to be  $C_{I11} = \alpha_I$  and replacing  $D$ ,  $T_{ab}$  by their composite expressions (3.21), we obtain the  $n$  vector coupled Ricci scalar squared action

$$\begin{aligned}
e^{-1}\mathcal{L}'_{R^2} &= \gamma_I \left( \rho^I \underline{Y}_{ij} \underline{Y}^{ij} + 2\rho \underline{Y}^{ij} Y_{ij}^I - \frac{1}{4}\rho^I \underline{F}_{\mu\nu} \underline{F}^{\mu\nu} - \frac{1}{2}\rho \underline{F}^{\mu\nu} (F_{\mu\nu}^I - 2\rho^I G_{\mu\nu}) \right. \\
&\quad + \frac{1}{2}\rho^2 (F_{\mu\nu}^I - \frac{3}{2}\rho^I G_{\mu\nu}) G^{\mu\nu} + \frac{1}{12}\rho^2 \epsilon^{\mu\nu\rho\sigma\lambda} (F_{\mu\nu}^I - 2\rho^I G_{\mu\nu}) H_{\rho\sigma\lambda} \\
&\quad + \frac{1}{6}\rho^I \underline{\rho} \epsilon_{\mu\nu\rho\sigma\lambda} \underline{F}^{\mu\nu} H^{\rho\sigma\lambda} - \frac{1}{8}\epsilon_{\mu\nu\rho\sigma\lambda} A^{\mu I} \underline{F}^{\nu\rho} \underline{F}^{\sigma\lambda} + \frac{1}{2}\rho^I \nabla_\mu \underline{\rho} \nabla^\mu \underline{\rho} \\
&\quad \left. + \rho^I \underline{\rho} \square \underline{\rho} \right). \quad (6.62)
\end{aligned}$$

The composite expressions for the elements of a vector multiplet are given as

$$\begin{aligned}
\rho|_{\sigma=1} &= 2NL^{-1}, \\
\underline{Y}_{ij}|_{\sigma=1} &= \frac{1}{\sqrt{2}}\delta_{ij}\left(-\frac{1}{2}R + \frac{1}{4}G_{ab}G^{ab} + \frac{1}{6}H_{abc}H^{abc} - L^{-2}N^2 - L^{-2}P_aP^a - V_a{}^{kl}V'_{kl}{}^a\right. \\
&\quad \left.+ L^{-1}\square L - \frac{1}{2}L^{-2}\partial_a L\partial^a L\right) + 2L^{-1}P^aV'_{aij} - \sqrt{2}L^{-1}\nabla^a(LV'_a{}^m{}_{(i}\delta_{j)m}), \\
\widehat{F}_{ab}|_{\sigma=1} &= 2\sqrt{2}\partial_{[a}(V_{b]} + \sqrt{2}L^{-1}P_{b]}. \tag{6.63}
\end{aligned}$$

At this moment, we have also completed the vector multiplets coupled curvature squared terms. The general vector multiplets coupled  $R + R^2$  theory is given by

$$\left(\mathcal{L}_{LR}^D + \mathcal{L}_V^D + \mathcal{L}_{\text{Riem}^2}^D + \mathcal{L}_{C^2 + \frac{1}{6}R^2}^D + \mathcal{L}_{R^2}^D\right)|_{\sigma=1}, \tag{6.64}$$

in which the vector multiplet action in  $\sigma = 1$  gauge is given as

$$\begin{aligned}
e^{-1}\mathcal{L}'_V{}^D|_{\sigma=1} &= a_{IJ}\left(\rho Y_{ij}^I Y^{ijJ} + 2\rho^I Y_{ij}^J Y^{ij} + \rho\rho^I\nabla_\mu\nabla^\mu\rho^J + \frac{1}{2}\rho\nabla_\mu\rho^I\nabla^\mu\rho^J\right. \\
&\quad - \frac{1}{4}\rho(F_{\mu\nu}^I - \rho^I G_{\mu\nu})(F^{\mu\nu J} - 3\rho^J G^{\mu\nu}) - \frac{1}{2}(F_{\mu\nu}^I - \rho^I G_{\mu\nu})\rho^J F^{\mu\nu} \\
&\quad + \frac{1}{12}\rho^I\rho^J\epsilon^{\mu\nu\rho\sigma\lambda}(F_{\mu\nu} - 2\rho G_{\mu\nu})H_{\rho\sigma\lambda} + \frac{1}{6}\rho\rho^I\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^J H_{\rho\sigma\lambda} \\
&\quad \left. - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda\right). \tag{6.65}
\end{aligned}$$

The supersymmetric completion of vector multiplets coupled Gauss-Bonnet combination can be achieved by setting  $\gamma_I = 0$ , and choosing the and choosing the free parameters of  $\mathcal{L}'_{\text{Riem}^2}{}^D$  and  $\mathcal{L}'_{C^2 + \frac{1}{6}R^2}{}^D$  to be related to each other according to  $\beta_I = 3\alpha_I$ .

### 6.3 Supersymmetric Curvature Squared Actions in the Standard Weyl Multiplet

In this section, we quickly review the Weyl squared action derived in [40] and construct an off-shell Ricci scalar squared action based on the standard Weyl multiplet.

The procedure for constructing the Ricci scalar squared action in the standard Weyl multiplet is similar to the one used in the dilaton Weyl multiplet. In the standard Weyl multiplet, the Ricci scalar squared term can be coupled to  $n$  vector multiplets and alter the very special geometry.

### 6.3.1 Supersymmetric Weyl Squared Action

Using superconformal tensor calculus, an off-shell Weyl squared action in the standard Weyl multiplet was constructed in [40], and its bosonic part reads

$$\begin{aligned}
e^{-1}\mathcal{L}_{C^2}^S &= c_I \left[ \frac{1}{8}\rho^I C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \frac{64}{3}\rho^I D^2 + \frac{1024}{9}\rho^I T^2 D - \frac{32}{3}D T_{\mu\nu} F^{\mu\nu I} \right. \\
&\quad - \frac{16}{3}\rho^I C_{\mu\nu\rho\sigma} T^{\mu\nu} T^{\rho\sigma} + 2C_{\mu\nu\rho\sigma} T^{\mu\nu} F^{\rho\sigma I} + \frac{1}{16}\epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^I C_{\nu\rho\tau\delta} C_{\sigma\lambda}{}^{\tau\delta} \\
&\quad - \frac{1}{12}\epsilon^{\mu\nu\rho\sigma\lambda} A_\mu^I V_{\nu\rho}{}^{ij} V_{\sigma\lambda}{}_{ij} + \frac{16}{3}Y_{ij}^I V_{\mu\nu}{}^{ij} T^{\mu\nu} - \frac{1}{3}\rho^I V_{\mu\nu}{}^{ij} V^{\mu\nu}{}_{ij} \\
&\quad + \frac{64}{3}\rho^I \nabla_\nu T_{\mu\rho} \nabla^\mu T^{\nu\rho} - \frac{128}{3}\rho^I T_{\mu\nu} \nabla^\nu \nabla_\rho T^{\mu\rho} - \frac{256}{9}\rho^I R^{\nu\rho} T_{\mu\nu} T^\mu{}_\rho \\
&\quad + \frac{32}{9}\rho^I R T^2 - \frac{64}{3}\rho^I \nabla_\mu T_{\nu\rho} \mathcal{D}^\mu T^{\nu\rho} + 1024\rho^I T^4 - \frac{2816}{27}\rho^I (T^2)^2 \\
&\quad - \frac{64}{9}T_{\mu\nu} F^{\mu\nu I} T^2 - \frac{256}{3}T_{\mu\rho} T^{\rho\lambda} T_{\nu\lambda} F^{\mu\nu I} - \frac{32}{3}\epsilon_{\mu\nu\rho\sigma\lambda} T^{\rho\tau} \nabla_\tau T^{\sigma\lambda} F^{\mu\nu I} \\
&\quad \left. - 16\epsilon_{\mu\nu\rho\sigma\lambda} T^\rho{}_\tau \nabla^\sigma T^{\lambda\tau} F^{\mu\nu I} - \frac{128}{3}\rho^I \epsilon_{\mu\nu\rho\sigma\lambda} T^{\mu\nu} T^{\rho\sigma} \mathcal{D}_\tau T^{\lambda\tau} \right], \quad (6.66)
\end{aligned}$$

where the five dimensional Weyl tensor reads

$$\begin{aligned}
C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \frac{1}{3}(g_{\mu\rho}R_{\nu\sigma} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho}) \\
&\quad + \frac{1}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \quad (6.67)
\end{aligned}$$

Now, we would like to comment more rigorously on the difference between the Weyl squared invariant in the standard Weyl multiplet (6.66) and its counterpart in the dilaton Weyl multiplet (6.33). As mentioned before, one of the main differences between these actions relies on the definition of  $D$  which is an independent field in the

standard multiplet but a composite field in the dilaton Weyl multiplet. As a composite field in the dilation Weyl multiplet,  $D$  contains a curvature term (3.21). However, simply replacing  $D, T_{ab}$  and  $\chi^i$  by their composite expressions does not produce an action solely based on the dilation Weyl multiplet. The resulting action also depends on the fields in the vector multiplet. We recall that neither two-derivative Poincaré supergravity (5.65) nor the Riemann squared action (6.14) has any dependence on the vector multiplet in the minimal off-shell supersymmetric model. To obtain the Weyl squared actions solely constructed in terms of the dilaton Weyl multiplet, the map (6.30) from the dilaton Weyl multiplet to the vector multiplet is indispensable.

The Weyl squared action in (6.66) is invariant under the transformation rules given in (3.4) and (3.23) with  $\eta^i$  and  $\Lambda_{K\mu}$  being replaced according to (5.3) and (5.4).

### 6.3.2 Supersymmetric Ricci Scalar Squared Action

To obtain the Ricci scalar squared invariant in the standard Weyl multiplet, we begin with the composite expressions given in (4.6) after fixing the redundant symmetries,

$$\begin{aligned}
\underline{\rho}|_{L=1} &= 2N, \\
\underline{Y}^{ij}|_{L=1} &= \frac{1}{\sqrt{2}}\delta^{ij}\left(-\frac{3}{8}R - N^2 - P^2 + \frac{8}{3}T^2 + 4D - V_a{}^{kl}V_{kl}{}^a\right) \\
&\quad + 2P^a V_a{}^{ij} - \sqrt{2}\nabla^a V_a{}^{tm(i}\delta^j)_{m}, \\
\underline{F}^{ab}|_{L=1} &= 2\sqrt{2}\partial^{[a}(V^{b]} + \sqrt{2}P^{b]}. \tag{6.68}
\end{aligned}$$

From the above expressions, one sees that the Ricci scalar squared can come from  $Y_{ij}Y^{ij}$  term in the vector action. By choosing  $C_{I11} = a_I$  and all other possibilities to

zero in (4.10), we obtain the following Ricci scalar squared action

$$\begin{aligned}
e^{-1}\mathcal{L}_{R^2}^S &= a_I \left( \rho^I \underline{Y}_{ij} \underline{Y}^{ij} + 2\rho \underline{Y}^{ij} Y_{ij}^I - \frac{1}{8} \rho^I \underline{\rho}^2 R - \frac{1}{4} \rho^I \underline{F}_{\mu\nu} \underline{F}^{\mu\nu} - \frac{1}{2} \rho \underline{F}^{\mu\nu} F_{\mu\nu}^I \right. \\
&\quad + \frac{1}{2} \rho^I \partial_\mu \underline{\rho} \partial^\mu \underline{\rho} + \rho^I \underline{\rho} \square \underline{\rho} - 4\rho^I \underline{\rho}^2 (D + \frac{26}{3} T^2) + 4\rho^2 F_{\mu\nu}^I T^{\mu\nu} \\
&\quad \left. + 8\rho^I \underline{\rho} \underline{F}_{\mu\nu} T^{\mu\nu} - \frac{1}{8} \epsilon_{\mu\nu\rho\sigma\lambda} A^{\mu I} \underline{F}^{\nu\rho} \underline{F}^{\sigma\lambda} \right). \tag{6.69}
\end{aligned}$$

As we will extensively work on this action in the following section, here we postpone to plug in the composite expressions given in (6.68). To study the solutions with vanishing auxiliary fields, we can use a truncated version of above action as we shall see in the next section. Off-shell supersymmetry allows us to combine the Ricci scalar squared action with the two-derivative Poincaré supergravity (5.5) and the Weyl squared action (6.66) to form a more general supergravity theory

$$\mathcal{L}_R^S + \mathcal{L}_{C^2}^S + \mathcal{L}_{R^2}^S \tag{6.70}$$

where  $\mathcal{L}_R^S$  is given in (5.5),  $\mathcal{L}_{C^2}^S$  is given in (6.66) and  $\mathcal{L}_{R^2}^S$  is given in (6.69).

### 6.3.3 Ricci Scalar Squared Extended Gauged Model and Corrected Very Special Geometry

In this section, we consider the off-shell Ricci scalar squared extended gauged model

$$\mathcal{L}_R^S + \mathcal{L}_{R^2}^S + g_I \mathcal{L}_{VL}^I. \tag{6.71}$$



We consider the maximal supersymmetric  $AdS_5$  solutions. The ansatz preserving  $SO(4, 2)$  symmetry takes the form

$$\begin{aligned} R_{\mu\nu\rho\lambda} &= -\frac{\mathcal{R}}{2}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}), \quad A_\mu^I = 0, \quad T_{\mu\nu} = 0, \\ \rho &= \bar{\rho}, \quad N = \bar{N}, \quad D = \bar{D}, \end{aligned} \quad (6.72)$$

where  $\mathcal{R}$ ,  $\bar{\rho}$ ,  $\bar{N}$  and  $\bar{D}$  are some constants. The maximal supersymmetry requires that

$$\mathcal{R} = \frac{4}{9}\bar{N}^2, \quad Y_{ij}^I = \frac{1}{3\sqrt{2}}\bar{\rho}^I\bar{N}\delta_{ij}, \quad \bar{D} = 0. \quad (6.73)$$

Employing  $\rho^I$  equation and  $N$  equation for the Lagrangian (6.71), we obtain

$$2\bar{N}C_{IJK}\bar{\rho}^J\bar{\rho}^K + 3g_I - \frac{8}{3}a_I\bar{N}^3 = 0, \quad 2\bar{N} + 3g_I\bar{\rho}^I = 0. \quad (6.74)$$

These two equations imply

$$\bar{C} = 1 + \frac{4}{3}a_I\bar{\rho}^I\bar{N}^2, \quad (6.75)$$

which is consistent with  $D$  field equation,  $Y^{Iij}$  equation and Einstein equation. Therefore, in the presence of Ricci scalar squared term,  $AdS_5$  maintains to be the maximally supersymmetric solution. However, in this case, the very special geometry is modified according to (6.75). Inserting  $N = -\frac{3}{2}g_I\bar{\rho}^I$  into (6.75), the quantum corrected very special geometry on the moduli space of  $AdS_5$  vacuum can be written as

$$\tilde{C}_{IJK}\bar{\rho}^I\bar{\rho}^J\bar{\rho}^K = 1, \quad \tilde{C}_{IJK} = C_{IJK} + 3a_{(I}g_{J)K}. \quad (6.76)$$

We emphasize that the inclusion of the Weyl squared action (6.66) also modifies the definition of very special geometry, however the modification vanishes on the maximally supersymmetric  $AdS_5$  background (6.73).

## 7. VACUUM SOLUTIONS AND SPECTRUM ANALYSIS\*

In this section, we investigate the vacuum solutions and spectrum to the general theory (6.35). The results for Poincaré supergravity extended by Gauss-Bonnet combination can be obtained as special case when  $\beta = 3\alpha$ .

### 7.1 Vacuum Solutions with 2-form and 3-form Fluxes

We first consider solutions with  $AdS_3 \times S^2$  structure. To solve the equation of motion, we make the following ansatz where Greek indices denote the coordinates on Lorentzian  $AdS_3$ , while latin indices stand for the coordinates on  $S^2$

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -a(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), & R_{pqrs} &= b(g_{pr}g_{qs} - g_{ps}g_{qr}), \\ L &= L_0, & G_{pq} &= c\epsilon_{pq}, & H_{\mu\nu\rho} &= d\epsilon_{\mu\nu\rho}. \end{aligned} \quad (7.1)$$

In above equation,  $\epsilon_{\mu\nu\rho}$  and  $\epsilon_{rs}$  are the Levi-Civita tensors on  $AdS_3$  and  $S^2$  respectively. The full set of equations of motion are solved provided that the following equations are satisfied

$$\begin{aligned} 6a - 2b + c^2 - 2d^2 &= 0, \\ \frac{1}{2}L_0(-a + d^2) + \frac{\alpha}{2}(-a^2 + b^2 - 2bc^2 + c^4 - 4acd + 10ad^2 + 4cd^3 - 9d^4) \\ &\quad + \frac{\beta}{6}(a^2 + ab - b^2 + 2bc^2 - c^4 + 2acd - 10ad^2 - bd^2 - 2cd^3 + 9d^4) = 0, \\ \frac{1}{4}L_0(b - c^2) + \frac{\alpha}{2}(3a^2 - b^2 + 4bc^2 - 3c^4 - 4bcd + 4c^3d - 6ad^2 + 3d^4) \\ &\quad + \frac{\beta}{6}(-3a^2 + b^2 - 4bc^2 + 3c^4 + 4bcd - 4c^3d + 6ad^2 - 3d^4) = 0. \end{aligned} \quad (7.2)$$

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The integrability conditions for the Killing spinor equations  $\delta_\epsilon \psi_\mu^i = 0$  and  $\delta_\epsilon \varphi^i = 0$  are

$$\left(R_{\hat{\mu}\hat{\nu}\hat{a}\hat{b}}(\omega_-) - 2G_{\hat{\rho}\hat{a}}G_{\hat{\nu}\hat{a}}\right)\gamma^{\hat{a}\hat{b}}\epsilon = 0, \quad \left(\frac{3}{2}G_{\hat{\mu}\hat{\nu}} - iH_{\hat{\mu}\hat{\nu}\hat{\lambda}}\gamma^{\hat{\lambda}}\right)\gamma^{\hat{\mu}\hat{\nu}}\epsilon = 0, \quad (7.3)$$

where  $\hat{\mu}, \hat{a} = 0, 1, \dots, 4$ . Substituting the ansatz (7.1) into the integrability conditions (7.3), we find that when

$$a = d^2, \quad b = c^2, \quad c = -2d, \quad (7.4)$$

the integrability conditions are satisfied automatically without imposing any projection condition on the  $Q$  transformation parameter  $\epsilon$ . Therefore, this solution possesses maximum supersymmetry. Remarkably, this solution exists for arbitrary values of  $L_0, \alpha, \beta$ . Thus it seems that the higher derivative correction will not affect the supersymmetric solutions. A similar phenomenon happens in 6D chiral gauged supergravity extended by Riemann squared invariant [42]. Next we investigate solutions with  $AdS_2 \times S^3$  structure. We make similar ansatz as previous case except that Greek indices denote the coordinates on Lorentzian  $AdS_2$ , while latin indices are used for the coordinates on  $S^3$

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -b(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), & R_{pqrs} &= a(g_{pr}g_{qs} - g_{ps}g_{qr}), \\ L &= L_0, & G_{\mu\nu} &= c\epsilon_{\mu\nu}, & H_{pqr} &= d\epsilon_{pqr}. \end{aligned} \quad (7.5)$$

In this case, the solutions of equation of motion are determined by

$$\begin{aligned} 6a - 2b + c^2 - 2d^2 &= 0, \\ \frac{1}{2}L_0(a - d^2) + \frac{\alpha}{2}(-a^2 + b^2 - 2bc^2 + c^4 - 4acd + 10ad^2 + 4cd^3 - 9d^4) &= 0 \end{aligned}$$

$$\begin{aligned}
& + \frac{\beta}{6}(a^2 + ab - b^2 + 2bc^2 - c^4 - 2acd - 10ad^2 - bd^2 + 2cd^3 + 9d^4) = 0, \\
\frac{1}{4}L_0(-b + c^2) + \frac{\alpha}{2}(3a^2 - b^2 + 4bc^2 - 3c^4 - 4bcd + 4c^3d - 6ad^2 + 3d^4) \\
& + \frac{\beta}{6}(-3a^2 + b^2 - 4bc^2 + 3c^4 - 4bcd + 4c^3d + 6ad^2 - 3d^4) = 0. \quad (7.6)
\end{aligned}$$

By examining the integrability conditions (7.3), we find that solution with maximum supersymmetry is given by

$$a = d^2, \quad b = c^2, \quad c = 2d, \quad (7.7)$$

for arbitrary values of  $L_0$ ,  $\alpha$ ,  $\beta$ .

## 7.2 Vacuum Solutions without Fluxes

If we set  $c = d = 0$ , the solutions are simply

$$\begin{aligned}
1) \quad & AdS_3 \times S^2 : b = 3a, \quad \beta = 6\alpha, \quad a = -\frac{L_0}{2\alpha}, \\
2) \quad & AdS_2 \times S^3 : b = 3a, \quad \beta = 6\alpha, \quad a = \frac{L_0}{2\alpha}, \\
3) \quad & Minkowski_5 \quad (7.8)
\end{aligned}$$

In this case, the maximally supersymmetric vacuum solution is just Minkowski<sub>5</sub>. Following the procedure carried out in the spectrum analysis of six-dimensional higher derivative chiral supergravity [42, 72], we study the bosonic spectrum of the perturbations around the maximally supersymmetric Minkowski<sub>5</sub> vacuum. We define the linearized fluctuations,

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, & L &= L_0 + \phi, & C_\mu &= c_\mu, \\
V_\mu^{ij} &= v_\mu^{ij}, & B_{\mu\nu} &= b_{\mu\nu}. \quad (7.9)
\end{aligned}$$

The linearized Einstein equation and  $L$  field equation take the following form

$$\begin{aligned} \left(L_0 + \frac{2}{3}(\beta - 3\alpha)\square\right)R_{\mu\nu}^{(L)} &= \frac{1}{3}(\beta - 3\alpha)\partial_\mu\partial_\nu R^{(L)} + \frac{L_0}{2}\eta_{\mu\nu}R^{(L)} - \eta_{\mu\nu}\square\phi \\ &\quad + \partial_\mu\partial_\nu\phi, \end{aligned} \tag{7.10}$$

$$L_0 R^{(L)} = 2\square\phi, \tag{7.11}$$

where  $R_{\mu\nu}^{(L)}$  and  $R^{(L)}$  are the linearized Ricci tensor and Ricci scalar. Inserting (7.11) into the trace of linearized Einstein equation, we get

$$\left(L_0 + \frac{2}{3}(\beta - 3\alpha)\square\right)\square\phi = 0. \tag{7.12}$$

This equation describes a massless scalar and a massive scalar with mass squared

$$m^2 = \frac{3L_0}{2(3\alpha - \beta)}. \tag{7.13}$$

To simplify the linearized Einstein equation, we choose the usual De Donder gauge in which,

$$R_{\mu\nu}^{(L)} = -\frac{1}{2}\square h_{\mu\nu}. \tag{7.14}$$

Then using the (7.11) and (7.12), we find

$$(\square - m^2)\square h_{\mu\nu} = -2L_0^{-1}(\square - m^2)\partial_\mu\partial_\nu\phi. \tag{7.15}$$

Since  $\phi$  can be solved from (7.12), the right hand side of above equation is known function. The homogeneous solutions of above equation describe a massless graviton and a massive graviton with a mass squared the same as that of the massive scalar.

Equations of motion for the remaining fields can be straightforwardly obtained by choosing the Lorentz gauge for the gauge fields

$$\left(L_0 + \frac{2}{3}(\beta - 3\alpha)\square\right)\square \begin{pmatrix} c_\mu \\ b_{\mu\nu} \end{pmatrix} = 0, \quad \left(L_0 + \frac{2}{3}(\beta - 3\alpha)\square\right)v_\mu^{ij} = 0. \quad (7.16)$$

In summary, for generic  $\alpha, \beta$ , the full spectrum consists of the (reducible) massless 12+12 supergravity multiplet with fields  $(h_{\mu\nu}, b_{\mu\nu}, c_\mu, \phi, \psi_\mu^i, \varphi^i)$  and a massive 32+32 supergravity multiplet with ghost fields  $(h_{\mu\nu}, b_{\mu\nu}, c_\mu, \phi, v_\mu^{ij}, \psi_\mu^i, \varphi^i)$ . At the special point where  $\beta = 3\alpha$ , the curvature squared terms in the action furnish the Gauss-Bonnet combination, massive particles become infinitely heavy and decouple from the spectrum leaving only the massless excitations as expected from the ghost-free feature of Gauss-Bonnet combination.

Finally we note that the inclusion of the Ricci scalar squared action does not affect the existence of maximally supersymmetric Minkowski<sub>5</sub> vacuum, however it brings a massive vector multiplet with  $m^2 = \frac{L_0}{2\gamma}$ . The 8+8 degrees of freedom in the massive vector multiplet are carried by  $(L, N, \partial^\mu Z_\mu, \tilde{C}_\mu, \varphi^i)$ .

## 8. SUPERSYMMETRIC SOLUTIONS WITH $AdS_3 \times S^2$ AND $AdS_2 \times S^3$ NEAR HORIZON GEOMETRIES\*

The strategy for finding regular solutions in higher derivative theory is to first write an ansatz consistent with the assumed symmetries, and then demand unbroken supersymmetry. The supersymmetric magnetic strings and electric black holes preserving one half of the supersymmetries have been studied in [43, 44] for the case of  $n$  vector multiplets coupled to Poincaré supergravity and in [73] for the higher derivative case where only the off-shell Weyl squared invariant is taken into account. In the presence of Weyl squared, the magnetic strings and electric black holes receive corrections. In the following, we consider the simplest curvature squared extended theory adding the Ricci scalar invariant (6.69) into the ungauged two-derivative action (5.5). Explicitly, in this section we study the theory

$$\mathcal{L} = \mathcal{L}_R^S + \mathcal{L}_{R^2}^S, \tag{8.1}$$

where  $\mathcal{L}_R^S$  and  $\mathcal{L}_{R^2}^S$  are given by (5.5) and (6.69) respectively. We are interested in solutions with vanishing auxiliary fields. It can be checked that  $Y_{ij}^I = N = E_a = V_a = V_a'^{ij} = 0$ , is a consistent truncation of (8.1) leading to a simpler effective action

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describing the very special geometry extended by Ricci scalar squared invariant

$$\begin{aligned}
e^{-1}\mathcal{L} &= \frac{1}{8}(\mathcal{C} + 3)R + \frac{1}{3}(104\mathcal{C} - 8)T^2 + 4(\mathcal{C} - 1)D + \frac{3}{4}C_{IJK}\rho^I F_{ab}^J F^{abK} \\
&+ \frac{3}{2}C_{IJK}\rho^I \partial_\mu \rho^J \partial^\mu \rho^K - 12C_{IJK}\rho^I \rho^J F_{ab}^K T^{ab} + \frac{1}{8}\epsilon^{abcde}C_{IJK}A_a^I F_{bc}^J F_{de}^K \\
&+ a_I \rho^I \left( \frac{9}{64}R^2 - 3DR - 2RT^2 + 16D^2 + \frac{64}{3}DT^2 + \frac{64}{9}(T^2)^2 \right). \quad (8.2)
\end{aligned}$$

The supersymmetry transformations for the fermionic fields take the following forms when the auxiliary fields vanish

$$\begin{aligned}
\delta\psi_\mu^i &= \left( \nabla_\mu - 4i\gamma^a T_{\mu a} + \frac{2}{3}i\gamma_\mu \gamma \cdot T \right) \epsilon^i, \\
\delta\chi^i &= \left( \frac{1}{4}D + \frac{1}{8}i\gamma^{ab}\not{\nabla}T_{ab} - \frac{1}{8}i\gamma^a \nabla^b T_{ab} - \frac{1}{6}\gamma^{abcd}T_{ab}T_{cd} \right) \epsilon^i, \\
\delta\lambda_i^I &= \left( -\frac{1}{4}\gamma \cdot \widehat{F}^I - \frac{1}{2}i\not{\nabla}\rho^I + \frac{4}{3}\rho^I \gamma \cdot T \right) \epsilon_i. \quad (8.3)
\end{aligned}$$

## 8.1 Magnetic String Solutions

The metric preserving the symmetry of a static string takes the form

$$ds^2 = e^{2U_1(r)}(-dt^2 + dx_1^2) + e^{-4U_2(r)}dx^i dx^i, \quad dx^i dx^i = dr^2 + r^2 d\Omega_2^2, \quad (8.4)$$

where  $i = 2, 3, 4$ .  $F_{ab}^I$  and  $T_{ab}$  are chosen to be proportional to the volume form of  $S^2$ . A natural choice for the veilbein is given by

$$e^{\hat{a}} = e^{U_1} dx^a, \quad a = 0, 1, \quad e^{\hat{i}} = e^{-2U_2} dx^i, \quad i = 2, 3, 4. \quad (8.5)$$

Accordingly, the non-vanishing components of the spin connections are

$$\omega_a^{\hat{a}\hat{i}} = e^{U_1+2U_2} \partial_i U_1, \quad \omega_k^{\hat{i}\hat{j}} = -2\delta_k^i \partial_j U_2 + 2\delta_k^j \partial_i U_2. \quad (8.6)$$

Similar to [73], the supersymmetry parameter  $\epsilon^i$  is constant along the string and obeys the projection condition which breaks half of the supersymmetries

$$\gamma_{\hat{t}\hat{1}}\epsilon = -\epsilon. \quad (8.7)$$

We first study the gravitino variation which fixes  $U_1 = U_2$ .

$$\delta\psi_\mu = \left( \nabla_\mu - 4i\gamma^a T_{\mu a} + \frac{2}{3}i\gamma_\mu \gamma \cdot T \right) \epsilon. \quad (8.8)$$

The covariant derivative is

$$\begin{aligned} \nabla_a &= \partial_a + \frac{1}{2}e^{U_1+2U_2}\partial_i U_1 \gamma_{\hat{a}\hat{i}}, \\ \nabla_i &= \partial_i + \partial_j U_2 \gamma_{\hat{j}\hat{i}}. \end{aligned} \quad (8.9)$$

Along the string direction, we have

$$\left[ \frac{1}{2}e^{U_1+2U_2}\partial_i U_1 \gamma_{\hat{a}\hat{i}} + \frac{2}{3}ie^{U_1}\gamma_{\hat{a}\hat{i}\hat{j}}, T^{\hat{i}\hat{j}} \right] \epsilon = 0. \quad (8.10)$$

We use the convention that  $\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = i\epsilon^{01234}$  with  $\epsilon^{01234} = 1$ . Therefore (8.7) implies

$$\gamma_{\hat{i}\hat{j}\hat{k}}\epsilon = i\epsilon_{\hat{i}\hat{j}\hat{k}}\epsilon, \quad \gamma_{\hat{i}\hat{j}}\epsilon = \epsilon_{\hat{i}\hat{j}\hat{k}}\gamma_{\hat{k}}\epsilon. \quad (8.11)$$

where  $\epsilon_{234} = 1$ . Using the above conditions, it can be obtained that

$$\left[ \frac{1}{2}e^{U_1+2U_2}\partial_k U_1 - \frac{2}{3}e^{U_1}T^{\hat{i}\hat{j}}\epsilon_{\hat{i}\hat{j}\hat{k}} \right] \gamma_{\hat{a}\hat{k}}\epsilon = 0. \quad (8.12)$$

The auxiliary field  $T_{ab}$  can be solved as

$$T_{\hat{i}\hat{j}} = \frac{3}{8}e^{2U_2}\epsilon_{\hat{i}\hat{j}\hat{k}}\partial_{\hat{k}}U_1. \quad (8.13)$$

The gravitino variatoin along  $x^i$  direction leads to

$$\left[\partial_k - i\epsilon_{\hat{i}\hat{j}\hat{k}}\partial_i U_2 \gamma_{\hat{j}} - \frac{8}{3}i\gamma^{\hat{i}}T_{k\hat{i}} - \frac{2}{3}\epsilon_{\hat{i}\hat{j}\hat{l}}e^{\hat{l}}{}_k T_{\hat{i}\hat{j}}\right]\epsilon = 0. \quad (8.14)$$

The ‘‘radial’’ part and ‘‘angular’’ part result in two conditions

$$\begin{aligned} 0 &= \left[\partial_k - \frac{2}{3}\epsilon_{\hat{i}\hat{j}\hat{l}}e^{\hat{l}}{}_k T_{\hat{i}\hat{j}}\right]\epsilon, \\ 0 &= \left[-\epsilon_{\hat{i}\hat{k}\hat{j}}\partial_i U_2 + \frac{8}{3}T_{k\hat{j}}\right]\gamma_{\hat{j}}\epsilon. \end{aligned} \quad (8.15)$$

The second equation restricts

$$U_1 = U_2, \quad (8.16)$$

then the first equation implies Killing spinor is

$$\epsilon = e^{U/2}\epsilon_0, \quad (8.17)$$

where  $\epsilon_0$  is some constant spinor. In cylindrical coordinates,  $T_{ab}$  can be expressed as

$$T_{\theta\phi} = \frac{3}{8}e^{-2U}r^2 \sin\theta\partial_r U, \quad T_{\hat{\theta}\hat{\phi}} = \frac{3}{8}e^{2U}\partial_r U. \quad (8.18)$$

The projection in cylindrical coordinates can be written as

$$\gamma_{\hat{r}\hat{\theta}\hat{\phi}}\epsilon = i\epsilon. \quad (8.19)$$

The gaugino variation  $\delta\lambda_i^I$  on the magnetic background gives

$$\left(-\frac{1}{2}\gamma_{\hat{\theta}\hat{\phi}}F^{I\hat{\theta}\hat{\phi}} - \frac{1}{2}i\gamma^{\hat{r}}e^{\hat{r}}\partial_r\rho^I + \frac{8}{3}\rho^I\gamma_{\hat{\theta}\hat{\phi}}T^{\hat{\theta}\hat{\phi}}\right)\epsilon = 0. \quad (8.20)$$

Then

$$F_{\hat{\theta}\hat{\phi}}^I = -e^{2U_1}\partial_r\rho^I + \frac{16}{3}\rho^IT_{\hat{\theta}\hat{\phi}} = -\partial_r(\rho^Ie^{-2U})e^{4U}. \quad (8.21)$$

The supersymmetry variation of  $\chi^i$  leads to

$$\left(\frac{1}{4}D + \frac{1}{8}i\gamma^{ab}\nabla T_{ab} - \frac{1}{8}i\gamma^a\nabla^bT_{ab} - \frac{1}{6}\gamma^{abcd}T_{ab}T_{cd}\right)\epsilon = 0. \quad (8.22)$$

Explicit computation shows that auxiliary field  $D$  can be solved from the above equation as

$$D = \frac{3}{8}e^{6U}r^{-2}\partial_r(e^{-3U}r^2\partial_rU) = -\frac{3}{16}e^{6U}\nabla^2e^{-2U}. \quad (8.23)$$

So far we have exhausted the constraints which can be derived from the variations of fermions. In the following, we have to use the equations of motion. For the magnetic string configuration, the equations of motion of gauge potential are automatically satisfied, however, the Bianchi identity gives rise to

$$\partial_rF_{\theta\phi}^I = -\partial_r\left(r^2\partial_r(\rho^Ie^{-2U})\right)\sin\theta = 0. \quad (8.24)$$

The solution to the above equation is given by [43]

$$\rho^Ie^{-2U} = H^I = \rho_\infty^I + \frac{p^I}{2r}, \quad F^I = \frac{p^I}{2}\epsilon_2, \quad (8.25)$$

where  $\rho_\infty^I$  is the value of  $\rho^I$  in the asymptotically flat region where  $U = 0$ .

The equation of  $D$  derived from action (8.2) is

$$\mathcal{C} = 1 + a_I \rho^I \left( \frac{3}{4} R - 8D - \frac{16}{3} T^2 \right). \quad (8.26)$$

After substituting  $T_{ab}$ ,  $D$  and  $R$  according to

$$T_{\hat{t}\hat{\phi}} = \frac{3}{8} e^{2U} \partial_r U, \quad D = -\frac{3}{16} e^{6U} \nabla^2 e^{-2U}, \quad R = \frac{2e^{4U}}{r} (4U' - 3rU'^2 + 2rU''), \quad (8.27)$$

where “prime” means derivative with respect to  $r$ . We find that the higher derivative corrections to the  $D$  equation of motion vanishes. Similarly, there are no higher derivative corrections to the equations of motion of  $T_{ab}$ ,  $g_{\mu\nu}$  and  $\rho^I$ . Therefore, the magnetic strings do not receive corrections from the Ricci scalar squared invariant. This result seems to be compatible with the expectation from string theory. From string theory point of view, the Ricci scalar squared invariant has no effects on the on-shell quantities since it can just come from a field redefinition of the two derivative action. This result also suggests that it is the supersymmetrization of curvature squared terms which captures the correct feature of quantum corrections of  $\mathcal{N} = 2$  string vacua, because an arbitrary combination of  $R^2$ ,  $D$  and  $T_{ab}$  will modify the equations of motion in general.

## 8.2 Electric Black Holes

Finding electric black holes follow the procedure as [73]. Given the ansatz

$$ds^2 = -e^{4U_1(r)} dt^2 + e^{-2U_2(r)} dx^i dx^i, \quad dx^i dx^i = dr^2 + r^2 d\Omega_3^2. \quad (8.28)$$

Supersymmetry requires that

$$\begin{aligned}
 U_1 = U_2, \quad T_{ti} = \frac{3}{8}e^{2U}\partial_i U, \quad A_t^I = -e^{2U}\rho^I, \\
 D = \frac{3}{16}e^{2U}(3r^{-1}U' + U'' - 2r^{-2}U'^2).
 \end{aligned}
 \tag{8.29}$$

In this case,

$$R = \frac{2e^{2U}}{r}(3U' - 3rU'^2 + rU'').
 \tag{8.30}$$

Again, it can be checked that the higher derivative corrections to the equations of motion vanish. Therefore, the electric black holes are not modified by the inclusion of Ricci scalar invariant.

## 9. RICCI SCALAR SQUARED EXTENDED $D = 6, N = (1, 0)$ OFF-SHELL SUPERGRAVITY

In this section, we construct an off-shell Ricci scalar squared action for  $D = 6, N = (1, 0)$  supergravity. The construction of the six dimensional model is exactly the same as the construction of the five dimensional model as we also use superconformal tensor calculus in  $D = 6$  for the off-shell constructions.

In the first subsection, we very briefly review the superconformal multiplets of  $D = 6, N = (1, 0)$  supergravity. Then, we discuss the superconformal actions and present the composite expressions for describing one multiplet in terms of the others. Finally, as in 5 dimensional case, we use the superconformal vector multiplet action and construct the Ricci scalar square extended off-shell Poincaré supergravity in 6 dimensions.

### 9.1 Superconformal Multiplets of $D = 6, N = (1, 0)$ Theory

The purpose of this section is to introduce the superconformal multiplets of six dimensional  $\mathcal{N} = (1, 0)$  theory that are required for the construction of superconformal Ricci scalar square invariant.

#### *9.1.1 Dilaton Weyl Multiplet*

The gauge sector of dilaton Weyl multiplet consists of a sechshein  $e_\mu^a$ , a gravitino  $\psi_\mu^i$ , a dilatation gauge field  $b_\mu$  and an  $SU(2)$  gauge field  $V_\mu^{ij}$ . The matter sector of the multiplet contains of a dilaton field  $\sigma$ , a 2-form gauge field  $B_{\mu\nu}$  and an  $SU(2)$  Majorana spinor of negative chirality  $\psi^i$ . The full Q, S and K transformation rules

are given by [35]

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu^i &= \partial_\mu\epsilon^i + \frac{1}{2}b_\mu\epsilon^i + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon^i + V_\mu^i{}_j\epsilon^j + \frac{1}{48}\sigma^{-1}\gamma\cdot\widehat{H}\gamma_\mu\epsilon^i + \gamma_\mu\eta^i, \\
\delta b_\mu &= -\frac{1}{2}\bar{\epsilon}\phi_\mu - \frac{1}{24}\bar{\epsilon}\gamma_\mu\chi + \frac{1}{2}\bar{\eta}\psi_\mu - 2\Lambda_{K\mu}, \\
\delta V_\mu^{ij} &= 2\bar{\epsilon}^{(i}\psi_\mu^{j)} + 2\bar{\eta}^{(i}\psi_\mu^{j)} + \frac{1}{6}\bar{\epsilon}^{(i}\gamma_\mu\chi^{j)}, \\
\delta B_{\mu\nu} &= -\sigma\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} - \bar{\epsilon}\gamma_{\mu\nu}\psi, \\
\delta\psi^i &= \frac{1}{48}\gamma\cdot\widehat{H}\epsilon^i + \frac{1}{2}\mathcal{D}\sigma\epsilon^i - \sigma\eta^i, \\
\delta\sigma &= \bar{\epsilon}\psi.
\end{aligned} \tag{9.1}$$

where

$$\begin{aligned}
\mathcal{D}_\mu\sigma &= (\partial_\mu - 2b_\mu)\sigma - \bar{\psi}_\mu\psi, \\
\mathcal{D}_\mu\psi^i &= \left(\partial_\mu - \frac{5}{2}b_\mu + \frac{1}{4}\omega_\mu^{ab}\right)\psi^i + V_\mu^i{}_j\psi^j - \frac{1}{48}\gamma\cdot\widehat{H}\psi_\mu^i - \frac{1}{4}\mathcal{D}\sigma\psi_\mu^i.
\end{aligned} \tag{9.2}$$

and

$$\widehat{H}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + 3\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi + \frac{3}{2}\sigma\bar{\psi}_{[\mu}\gamma_\nu\psi_{\rho]}. \tag{9.3}$$

Finally, the composite field  $\chi^i$  is defined as

$$\chi^i = \frac{15}{4}\sigma^{-1}\mathcal{D}\psi^i + \frac{3}{8}\gamma^{ab}\widehat{R}'^i{}_{ab}(Q) - \frac{5}{32}\sigma^{-2}\gamma\cdot\widehat{H}\psi^i. \tag{9.4}$$

where

$$\widehat{R}'^i{}_{ab}(Q) = 2\left(\partial_{[\mu} + \frac{1}{2}b_{[\mu} + \frac{1}{4}\omega_{[\mu}{}^{ab}\gamma_{ab}]\right)\psi_{\nu]} + 2V_{[\mu}{}^i{}_j\psi_{\nu]}^j + \frac{1}{24}\sigma^{-1}\gamma\cdot\widehat{H}\gamma_{[\mu}\psi_{\nu]}^i \tag{9.5}$$



### 9.1.2 Linear Multiplet

The linear multiplet consists of a triplet of scalars  $L^{ij}$ , an  $SU(2)$  Majorana spinor  $\varphi^i$  of negative chirality, and a constrained vector  $E_a$ . The full Q and S transformations are given by [35]

$$\begin{aligned}
\delta L^{ij} &= \bar{\epsilon}^{(i} \varphi^{j)}, \\
\delta \varphi^i &= \frac{1}{2} \not{D} L^{ij} \epsilon_j - \frac{1}{4} \not{E} \epsilon^i - 4 L^{ij} \eta_j, \\
\delta E_a &= \bar{\epsilon} \gamma_{ab} \mathcal{D}^b \varphi + \frac{1}{48} \sigma^{-1} \bar{\epsilon} \gamma_a \gamma \cdot \widehat{H} \varphi - \frac{1}{3} \bar{\epsilon}^{(i} \gamma_a \chi^{j)} L_{ij} - 5 \bar{\eta} \gamma_a \varphi
\end{aligned} \tag{9.6}$$

where the covariant derivatives are defined as

$$\begin{aligned}
\mathcal{D}_\mu L^{ij} &= (\partial_\mu - 4 b_\mu) L^{ij} + 2 V_\mu^{(i} L^{j)k} - \bar{\psi}_\mu^{(i} \varphi^{j)}, \\
\mathcal{D}_\mu \varphi^i &= (\partial_\mu - \frac{9}{2} b_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}) \varphi^i - V_\mu^{ij} \varphi_j - \frac{1}{2} \not{D} L^{ij} \psi_{\mu j} + \frac{1}{4} \not{E} \psi_\mu^i + 4 L^{ij} \phi_{\mu j}
\end{aligned} \tag{9.7}$$

### 9.1.3 Vector Multiplet

The vector multiplet consists of a real vector field  $W_\mu$ , an  $SU(2)$  Majorana spinor  $\Omega^i$  of positive chirality, and a triplet of scalar fields  $Y^{ij}$ . The full Q and S transformation rules are given by [35]

$$\begin{aligned}
\delta W_\mu &= -\bar{\epsilon} \gamma_\mu \Omega, \\
\delta \Omega^i &= \frac{1}{8} \gamma \cdot \widehat{F} \epsilon^i - \frac{1}{2} Y^{ij} \epsilon_j, \\
\delta Y^{ij} &= -\bar{\epsilon}^{(i} \not{D} \Omega^{j)} + 2 \bar{\eta}^{(i} \Omega^{j)}.
\end{aligned} \tag{9.8}$$

where

$$\begin{aligned}\widehat{F}_{\mu\nu} &= 2\partial_{[\mu}W_{\nu]} + 2\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega, \\ \mathcal{D}_\mu\Omega^i &= \partial_\mu\Omega^i - \frac{3}{2}b_\mu\Omega^i + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\Omega^i - \frac{1}{2}V_\mu{}^i{}_j\Omega^j - \frac{1}{8}\gamma \cdot \widehat{F}\psi_\mu^i + \frac{1}{2}Y^{ij}\psi_{\mu j}\end{aligned}\quad (9.9)$$

## 9.2 Construction of a Superconformal $R^2$ Invariant

For the construction of an off-shell Ricci scalar square invariant, our starting point is the following vector multiplet action [35]

$$e^{-1}\mathcal{L}_V = \sigma Y^{ij}Y_{ij} - \frac{1}{4}\sigma F_{ab}F^{ab} - \frac{1}{16}\epsilon_{abcdef}B^{ab}F^{cd}F^{ef}\quad (9.10)$$

The idea of the construction is to find composite expressions for the fields of vector multiplet. Our only demand is

$$Y^{ij} = RL^{ij} + \dots\quad (9.11)$$

so that when we plug in the composite expressions, the vector multiplet action gives rise to a superconformal  $R^2$  action. Such composite expressions are already found, and here we only present the bosonic parts of those superconformal expressions [35]

$$\begin{aligned}Y^{ij} &= -L^{-1}\square^C L^{ij} + L^{-3}L_{kl}\mathcal{D}^a L^{k(i}\mathcal{D}_a L^{j)l} + \frac{1}{4}L^{-3}E_a E^a L^{ij} \\ &\quad - L^{-3}E^a L^{k(i}\mathcal{D}_a L^{j)k} - \frac{1}{3}L^{-1}L^{ij}D, \\ F_{ab} &= 2L^{-1}L^{ij}\widehat{R}_{abij}(V) - 2\mathcal{D}_{[a}(L^{-1}E_{b]}) - 2L^{-3}L^l{}_k\mathcal{D}_{[a}L^{kp}\mathcal{D}_{b]}L_{lp}\end{aligned}\quad (9.12)$$

where the bosonic part of the conformal box  $\square^C L_{ij}$  and the composite expression for  $D$  read

$$\begin{aligned}\square^C L^{ij} &= (\partial^a - 5b^a + \omega_b^{ba})\mathcal{D}_a L^{ij} + 2V^{a(i}{}_k \mathcal{D}_a L^{j)k} - 8f_a{}^a L^{ij}, \\ D &= \frac{15}{4}\sigma^{-1}\square\sigma - \frac{3}{4}R + \frac{5}{16}\sigma^{-2}H_{abc}H^{abc},\end{aligned}\tag{9.13}$$

where the curvature of  $V_\mu^{ij}$  reads

$$\widehat{R}_{ab}{}^{ij}(V) = 2\partial_{[a}V_{b]}{}^{ij} - 2V_{[a}{}^k(i)V_{b]}{}^{j)k} - 4\bar{\psi}_{[a}{}^{(i}\phi_{b]}{}^{j)} - \frac{1}{3}\bar{\psi}_{[a}{}^{(i}\gamma_{b]}\chi^{j)}.\tag{9.14}$$

Notice that in the above expressions we have omitted all the fermionic terms. Finally, the bosonic part of  $f_a{}^a$  is given by

$$8f_a{}^a = \frac{2}{5}R + \frac{1}{5}D\tag{9.15}$$

Using the above definitions, one can read that  $Y^{ij} = \frac{1}{2}L^{-1}L^{ij}R + \dots$  as expected. Therefore, upon plugging in the composite expressions given in (9.12) into the vector multiplet action (9.10), one obtains a superconformal  $R^2$  action.

### 9.3 Gauge Fixing and the Off-Shell Action

Our gauge fixing choices to fix the redundant superconformal symmetries are [42]

$$L^{ij} = \frac{1}{\sqrt{2}}\delta^{ij}L, \quad b_\mu = 0, \quad \sigma = 1, \quad \psi^i = 0.\tag{9.16}$$

Decomposing  $SU(2)$  gauge field  $V_\mu^{ij}$  into

$$V_a{}^{ij} = V_a{}'^{ij} + \frac{1}{2}V_a\delta^{ij}, \quad V_a{}'^{ij}\delta_{ij} = 0,\tag{9.17}$$

we define the complex vector  $Z_\mu$

$$Z_\mu = V_\mu^{12} + iV_\mu^{11}. \quad (9.18)$$

Under the above gauge choices, the vector multiplet action is given by

$$e^{-1}\mathcal{L}_V|_{\sigma=1} = Y_{ij}Y^{ij} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}\epsilon_{\mu\nu\lambda\rho\sigma\delta}B^{\mu\nu}F^{\lambda\rho}F^{\sigma\delta}, \quad (9.19)$$

where upon using the composite expression, we obtain the supersymmetric Ricci scalar squared action

$$\begin{aligned} Y^{ij}Y_{ij} &= \frac{1}{4}\Upsilon^2 - \Xi^2, \\ \Upsilon &= R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - 2L^{-1}\square L + L^{-2}\partial_\mu L\partial^\mu L - 4Z\bar{Z} + \frac{1}{2}L^{-2}E^\mu E_\mu, \\ \Xi &= 2L^{-1}\nabla^\mu(LZ_\mu) - i\sqrt{2}L^{-1}E^\mu Z_\mu, \\ F_{\mu\nu} &= 2\sqrt{2}\partial_{[\mu}(V_{\nu]} - \frac{1}{\sqrt{2}}L^{-1}E_{\nu]}). \end{aligned} \quad (9.20)$$

Using, the 2-derivative action is given in [42], the Ricci scalar square extended model is given by [42]

$$\begin{aligned} e^{-1}\mathcal{L} &= \frac{1}{2}LR + \frac{1}{2}L^{-1}\partial_\mu L\partial^\mu L - \frac{1}{24}LH_{\mu\nu\rho}H^{\mu\nu\rho} - 2LZ_\mu\bar{Z}^\mu \\ &\quad - \frac{1}{4}L^{-1}E_\mu E^\mu + \frac{1}{\sqrt{2}}E^\mu V_\mu \\ &\quad + \alpha' \left[ \frac{1}{4}(R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - 2L^{-1}\square L + L^{-2}\partial_\mu L\partial^\mu L - 4Z\bar{Z} + \frac{1}{2}L^{-2}E^\mu E_\mu)^2 \right. \\ &\quad - \left( 2L^{-1}\nabla^\mu(LZ_\mu) - i\sqrt{2}L^{-1}E^\mu Z_\mu \right) \left( 2L^{-1}\nabla^\mu(L\bar{Z}_\mu) + i\sqrt{2}L^{-1}E^\mu \bar{Z}_\mu \right) \\ &\quad - \frac{1}{2}\epsilon_{\mu\nu\lambda\rho\sigma\delta}B^{\mu\nu}\partial^{[\lambda}(V^{\rho]} - \frac{1}{\sqrt{2}}L^{-1}E^{\rho]})\partial^\sigma(V^\delta - \frac{1}{\sqrt{2}}L^{-1}E^\delta) \\ &\quad \left. - 2\partial_{[\mu}(V_{\nu]} - \frac{1}{\sqrt{2}}L^{-1}E_{\nu]})\partial^\mu(V^\nu - \frac{1}{\sqrt{2}}L^{-1}E^\nu) \right]. \end{aligned} \quad (9.21)$$

This Lagrangian respects the following transformation rules

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu , \\
\delta \psi_\mu^i &= (\partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^{ab}) \epsilon^i + V_\mu^i{}_j \epsilon^j + \frac{1}{8} \widehat{H}_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon^i , \\
\delta B_{\mu\nu} &= -\bar{\epsilon} \gamma_{[\mu} \psi_{\nu]} , \\
\delta \varphi^i &= \frac{1}{2\sqrt{2}} \gamma^\mu \delta^{ij} \partial_\mu L \epsilon_j - \frac{1}{4} \gamma^\mu E_\mu \epsilon^i + \frac{1}{\sqrt{2}} \gamma^\mu V'^{(i}{}_\mu{}^k \delta^{j)k} L \epsilon_j - \frac{1}{12\sqrt{2}} L \delta^{ij} \gamma \cdot \widehat{H} \epsilon_j , \\
\delta L &= \frac{1}{\sqrt{2}} \bar{\epsilon}^i \varphi^j \delta_{ij} , \\
\delta E_{\mu\nu\rho\sigma} &= L \bar{\epsilon}^i \gamma_{[\mu\nu\rho} \psi_{\sigma]}^j \delta_{ij} - \frac{1}{2\sqrt{2}} \bar{\epsilon} \gamma_{\mu\nu\rho\sigma} \varphi , \\
\delta V_\mu &= \frac{1}{2} \bar{\epsilon}^i \gamma^\nu \widehat{\psi}_{\mu\nu}{}^j \delta_{ij} + \frac{1}{12} \bar{\epsilon}^i \gamma \cdot \widehat{H} \psi_\mu^j \delta_{ij} - 2 \lambda'^i{}_k V_\mu'^{jk} \delta_{ij} , \\
\delta V_\mu'^{ij} &= \frac{1}{2} \bar{\epsilon}^{(i} \gamma^\nu \widehat{\psi}_{\mu\nu}{}^{j)} + \frac{1}{12} \bar{\epsilon}^{(i} \gamma \cdot \widehat{H} \psi_\mu^{j)} - \frac{1}{4} \bar{\epsilon}^k \gamma^\nu \widehat{\psi}_{\mu\nu}{}^\ell \delta_{k\ell} \delta^{ij} \\
&\quad - \frac{1}{24} \bar{\epsilon}^k \gamma \cdot \widehat{H} \psi_\mu{}^\ell \delta_{k\ell} \delta^{ij} + \partial_\mu \lambda'^{ij} - \lambda'^{(i}{}_{k} \delta^{j)k} V_\mu , 
\end{aligned} \tag{9.22}$$

where

$$\begin{aligned}
\widehat{\psi}_{\mu\nu}{}^i &= 2D_{[\mu} \psi_{\nu]}^i - 2V'_{[\mu}{}^{ij} \psi_{\nu]j} + \frac{1}{4} \gamma^{ab} \psi_{[\nu} \widehat{H}_{\mu]ab} , \\
E_a &= \frac{1}{24} \epsilon_{abcdef} \mathcal{D}^b E^{cdef} .
\end{aligned} \tag{9.23}$$

## 10. CONCLUSIONS AND OUTLOOK

In this thesis, we have constructed Poincaré supergravity theories and completed all off-shell curvature squared invariants in  $D = 5$ ,  $\mathcal{N} = 2$  supergravity based on the dilaton Weyl multiplet for both the minimal and the vector multiplets coupled curvature squared invariants, namely the complete minimal curvature squared invariants consist of

$$\alpha \mathcal{L}_{Riem^2}^D + \beta \mathcal{L}_{C^2 + \frac{1}{6}R^2}^D + \gamma \mathcal{L}_{R^2}^D, \quad (10.1)$$

and the complete vector multiplets coupled curvature squared invariants take the form

$$\mathcal{L}'^D_{Riem^2} + \mathcal{L}'^D_{C^2 + \frac{1}{6}R^2} + \mathcal{L}'^D_{R^2}. \quad (10.2)$$

A particularly important combination of curvature squared terms, the supersymmetric Gauss-Bonnet extended Poincaré theory corresponds to the case where  $\gamma = 0$  and  $\beta = 3\alpha$  in (10.1), and  $\gamma^I = 0$  and  $\beta^I = 3\alpha^I$  in (10.2). Although the auxiliary fields do not propagate in these models, they can be eliminated order by order in  $\alpha$ . We obtain the on-shell theory of this model to first order in  $\alpha$ . The maximally supersymmetric solutions to the ordinary 2-derivative Einstein-Maxwell supergravity are known including  $Minkowski_5$ ,  $AdS_3 \times S^2$  and  $AdS_3 \times S^2$ . We found that these solutions are not modified by the inclusion of the higher-derivative interactions proportional to  $\alpha$  and  $\beta$  for arbitrary values. The spectrum of this theory around the maximally supersymmetric  $Minkowski_5$  is determined. We show that the spectrum has a ghostly massive spin two multiplet in addition to a massless supergravity and a Maxwell vector mul-

triplet. However, when  $\beta = 3\alpha$  corresponding to the Gauss-Bonnet combination, the massive spin-2 multiplet decouples.

Adopting the standard Weyl multiplet, we also constructed an off-shell Poincaré supergravity by using the linear and vector multiplets as compensators and a supersymmetric Ricci scalar squared coupled to  $n$  vector multiplets. In the standard Weyl multiplet, the curvature squared extended model is generalized to take the form

$$\mathcal{L}_R^S + \mathcal{L}_{C^2}^S + \mathcal{L}_{R^2}^S. \quad (10.3)$$

It is known that the gauged two-derivative Poincaré supergravity possesses an maximally supersymmetric  $AdS_5$  vacuum solution. When the Ricci scalar squared is included, we found that the very special geometry defined on the moduli space gets modified. We then study the effects of Ricci scalar squared to the supersymmetric magnetic string and electric black hole solutions which are the 1/2 BPS solutions of the ungauged two-derivative theory. It is found that neither the magnetic string nor the electric black hole solutions gets modified by the supersymmetric completion of the Ricci scalar squared.

A comparison between the results in the dilaton Weyl multiplet and the standard Weyl multiplet leads to a natural question that what is the analogue of supersymmetric Riemann squared in the standard Weyl multiplet. At this moment, we do not know the exact answer. However, if such an invariant exists, one should be able to recover the Riemann squared invariant based on the dilaton Weyl multiplet from the Riemann squared invariant based on the standard Weyl multiplet by using the map between the Weyl multiplets. This argument constrains the form of the supersymmetric Riemann squared in the standard Weyl multiplet.

The modified very special geometry around the  $AdS_5$  vacuum by the Ricci scalar

squared invariant is very intriguing unlike the supersymmetric completion of the Weyl tensor squared which does not affect the definition of very special geometry in the  $AdS_5$  vacuum. Interpretation of the modified very special geometry from string/M theory and its application in the context of AdS/CFT correspondence [73] - [76] deserve future investigation.

In the final section, we repeated the construction procedure presented in 5 dimensions, and obtained the 6 dimensional Ricci scalar squared invariant. Since the Gauss-Bonnet combination in six dimensions is not known, the curvature squared invariants in six dimensions are not complete. The construction of Gauss-Bonnet invariant is technically more difficult as the compensating multiplets, the vector and the linear multiplets, do not contain a scalar field.



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## A. NOTATIONS FOR FIVE DIMENSIONAL MODEL

In this thesis, we use the conventions of [37]. The signature of the metric is  $\text{diag}(-, +, +, +, +)$ . The SU(2) indices are lowered or raised according to NW-SE convention

$$A^i = \varepsilon^{ij} A_j, \quad A_i = A^j \varepsilon_{ji}, \quad (\text{A.1})$$

where  $\varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = 1$ . When SU(2) indices on spinors are suppressed, NW-SE contraction is understood.

$$\bar{\psi} \gamma^{a_1 \dots a_n} \chi = \bar{\psi}^i \gamma^{a_1 \dots a_n} \chi_i, \quad (\text{A.2})$$

where  $\gamma^{a_1 \dots a_n}$  is defined as

$$\gamma^{a_1 \dots a_n} = \gamma^{[a_1} \gamma^{a_2} \dots \gamma^{a_n]}. \quad (\text{A.3})$$

Changing the order of spinors in a bilinear leads to the following signs

$$\bar{\psi}^i \gamma_{(n)} \chi^j = t_n \bar{\chi}^j \gamma_{(n)} \psi^i, \quad (\text{A.4})$$

where  $t_0 = t_1 = -t_2 = -t_3 = 1$ . We also used the following Fierz identity

$$\psi_j \bar{\chi}^i = -\frac{1}{4} \bar{\chi}^i \psi_j - \frac{1}{4} \bar{\chi}^i \gamma^a \psi_j \gamma_a + \frac{1}{8} \bar{\chi}^i \gamma^{ab} \psi_j \gamma_{ab}. \quad (\text{A.5})$$

The Levi-Civita tensor is real and satisfies

$$\epsilon_{p_1 \dots p_n q_1 \dots q_m} e^{p_1 \dots p_n r_1 \dots r_m} = -n!m! \delta_{[q_1 \dots q_m]}^{[r_1 \dots r_m]}. \quad (\text{A.6})$$

Finally, the product of all gamma matrices is proportional to the unit matrix, and we use

$$\gamma^{abcde} = i\epsilon^{abcde}. \quad (\text{A.7})$$

## B. MULTIPLETS OF $\mathcal{N} = 2, D = 5$ SUPERCONFORMAL THEORY

In this appendix, we give the  $SU(2)$  representations and Weyl weights of the fields appearing in this paper.

TABLE B.1  
Multiplets of  $\mathcal{N} = 2, D = 5$  Superconformal Theory

| Multiplet              | Field        | $SU(2)$ reps. | Weyl weight    |
|------------------------|--------------|---------------|----------------|
| Dilaton Weyl Multiplet | $e_\mu^a$    | 1             | -1             |
|                        | $\psi_\mu^i$ | 2             | $-\frac{1}{2}$ |
|                        | $b_\mu$      | 1             | 0              |
|                        | $V_\mu^{ij}$ | 3             | 0              |
|                        | $C_\mu$      | 1             | 0              |
|                        | $B_{\mu\nu}$ | 1             | 0              |
|                        | $\sigma$     | 1             | 1              |
|                        | $\psi^i$     | 2             | $\frac{3}{2}$  |
| Vector Multiplet       | $A_\mu$      | 1             | 0              |
|                        | $\lambda^i$  | 2             | $\frac{3}{2}$  |
|                        | $\rho$       | 1             | 1              |
|                        | $Y^{ij}$     | 3             | 2              |
| Linear Multiplet       | $L^{ij}$     | 3             | 3              |
|                        | $\varphi^i$  | 2             | $\frac{7}{2}$  |
|                        | $E_a$        | 1             | 4              |
|                        | $N$          | 1             | 4              |

### C. MULTIPLETS OF $\mathcal{N} = (1, 0)$ , $D = 6$ SUPERCONFORMAL THEORY

In this appendix, we give the  $SU(2)$  representations and Weyl weights of the fields appearing in this paper.

TABLE C.1  
Multiplets of  $\mathcal{N} = (1, 0)$ ,  $D = 6$  Superconformal Theory

| Multiplet              | Field        | $SU(2)$ reps. | Weyl weight    |
|------------------------|--------------|---------------|----------------|
| Dilaton Weyl Multiplet | $e_\mu^a$    | 1             | -1             |
|                        | $\psi_\mu^i$ | 2             | $-\frac{1}{2}$ |
|                        | $b_\mu$      | 1             | 0              |
|                        | $V_\mu^{ij}$ | 3             | 0              |
|                        | $B_{\mu\nu}$ | 1             | 0              |
|                        | $\sigma$     | 1             | 2              |
|                        | $\psi^i$     | 2             | $\frac{3}{2}$  |
| Vector Multiplet       | $W_\mu$      | 1             | 0              |
|                        | $\Omega^i$   | 2             | $\frac{3}{2}$  |
|                        | $Y^{ij}$     | 3             | 2              |
| Linear Multiplet       | $L^{ij}$     | 3             | 4              |
|                        | $\varphi^i$  | 2             | $\frac{9}{2}$  |
|                        | $E_a$        | 1             | 5              |