

A-DISCRIMINANT VARIETIES AND AMOEBAE

A Dissertation

by

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ABSTRACT

The motivating question behind this body of research is Smale's 17th problem:

Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?

While certain aspects and viewpoints of this problem have been solved, the analog over the real numbers largely remains open. This is an important question with applications in celestial mechanics, kinematics, polynomial optimization, and many others.

Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_{n+k}\} \subset \mathbb{Z}^n$. The \mathcal{A} -discriminant variety is, among other things, a tool that can be used to categorize polynomials based on the topology of their real solution set. This fact has made it useful in solving aspects and special cases of Smale's 17th problem. In this thesis, we take a closer look at the structure of the \mathcal{A} -discriminant with an eye toward furthering progress on analogs of Smale's 17th problem. We examine a mostly ignored form called the Horn uniformization. This represents the discriminant in a compact form. We study properties of the Horn uniformization to find structural properties that can be used to better understand the \mathcal{A} -discriminant variety. In particular, we use a little known theorem of Kapranov limiting the normals of the \mathcal{A} -discriminant amoeba.

We give new $O(n^2)$ bounds on the number of components in the complement of the real \mathcal{A} -discriminant when $k = 3$, where previous bounds had been $O(n^6)$ or even exponential before that. We introduce new tools that can be used in discovering various types of extremal \mathcal{A} -discriminants as well as examples found with these tools: a family of \mathcal{A} -discriminant varieties with the maximal number of cusps and a family

that appears to asymptotically admit the maximal number of chambers. Finally we give sage code that efficiently plots the \mathcal{A} -discriminant amoeba for $k = 3$.

Then we switch to a non-Archimedean point of view. Here we also give $O(n^2)$ bounds for the number of connected components in the complement of the non-Archimedean \mathcal{A} -discriminant amoeba when $k = 3$, but we also get a bound of $O(n^{2(k-1)(k-2)})$ when $k > 3$. As in the real case, we also give a family exhibiting $O(n^2)$ connected components asymptotically. Finally we give code that efficiently plots the p -adic \mathcal{A} -discriminant amoeba for all $k \geq 3$.

These results help us understand the structure of the \mathcal{A} -discriminant to a degree, as yet, unknown. This can ultimately help in solving Smale's 17th problem as it gives a better understanding of how complicated the solution set can be.

DEDICATION

When I was a very small boy, my mother, Deborah O’Nan, was a home health nurse. While driving from one patient’s house to the next, she taught me to read and basic arithmetic. Who would have ever dreamed that those lessons would culminate in the work presented here? More than any other individual, she has played a vital role in me becoming the man I am today. Raising two young boys while putting herself through school, she still found time to put utmost emphasis in my own education and development. My mother has instilled a love of education and learning in me. But she is not satisfied stopping there, she is encouraging and supportive without end. I have never heard her give a discouraging word about my ambitions or goals. I have honestly come to the conclusion that she really does believe that I could be and do *anything*.

I also want to dedicate this work to my late stepfather, Rodney Bryant. I’ve spent my entire life being educated – nearly half of that at universities – and I don’t know if I have ever met anyone as excited about knowledge and learning. He saw that I was forming intellectual interests in areas that he may not have loved or had experience in, but he was always excited to encourage my curiosities. I remember several times he was excited to give me “surprise gifts” – calculus books, programming books, books, books, books. Furthermore, although we shared no blood relation, Rodney loved me like one of his own children. I am saddened that he cannot share in this life changing event with me, but I will forever cherish the time I had with him.

You both were and are willing to give everything for me. I cannot adequately express my gratitude for the innumerable ways you’ve molded and shaped me. From marathons to PhDs, without you I could never have accomplished a third of what I

have done. I love you and I thank God for both of you.

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I could never have arrived at the point I am now without the guidance and work of my graduate advisor, J. Maurice Rojas. The stereotypical graduate advisor is a crotchety old slave driver to be feared. Maurice does not follow the stereotype at all. He is down to earth and friendly. He realizes that new information is difficult and takes time to assimilate. He understands that my career is part of his job and he is happy and willing to work with me and help me through difficult work. But beyond those things that make him a good advisor and mentor, Maurice is also a friend. I want to thank him for his guidance these past several years and, of equal importance, for his friendship.

NOMENCLATURE

$\Delta_{\mathcal{A}}$	The \mathcal{A} -discriminant polynomial
$\nabla_{\mathcal{A}}$	The \mathcal{A} -discriminant variety
$\overline{\nabla}_{\mathcal{A}}$	The reduced \mathcal{A} -discriminant variety
$\#S$	The cardinality (size) of the set S
\mathbb{C}	The field of complex numbers
\mathbb{C}_p	The complex p -adic numbers
$\mathrm{GL}_k(\mathbb{F})$	The set of invertible $k \times k$ matrices with elements in \mathbb{F}
$H(m, n)$	The number of connected components in the complement of m real hyperplanes in \mathbb{R}^n
$H_T(m, n)$	The number of connected components in the complement of m tropical hyperplanes in \mathbb{R}^n
\mathbb{K}	An algebraically closed field of characteristic 0
\mathbb{F}^\times	The non-zero elements of the field \mathbb{F}
\mathbb{N}	The natural numbers, $1, 2, \dots$
$\mathbb{P}(\mathbb{F})$	Projective space over the field \mathbb{F}
\mathbb{Q}	The field of rational numbers
\mathbb{Q}_p	The p -adic numbers
\mathbb{R}	The field of real numbers
\mathbb{R}^+	The positive real numbers
$\mathrm{Res}_x(f, g)$	The elimination resultant of $f(x)$ and $g(x)$
\mathbb{Z}	The ring of integers
$Z(F)$	The zeroset of the collection of equations F
φ	The map of the reduced \mathcal{A} -discriminant amoeba

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1. INTRODUCTION

At its heart, Algebraic Geometry is the science of solutions of equations. When one studies solutions of equations there are several, related questions that can be asked about such a solution set, S . Among these questions are the following:

- How many connected components does S have?
- How many isolated points does S have?
- How “hard” is it to computationally approximate some or all of S ?

Before we can even begin to answer such questions we must decide where our solution set lies. That is, where we are looking for our solutions. To illustrate, let us analyze the very simple polynomial

$$f(x) = (x^2 - 1)(x^2 + 1) = x^4 - 1.$$

This polynomial, $f(x)$, is the product of two other polynomials. The first polynomial, $(x^2 - 1)$, is zero at 1 and -1 , whereas the other polynomial, $(x^2 + 1)$, is zero at the purely complex numbers $-i$ and i . That is, the polynomial $f(x)$ has two real solutions and two imaginary solutions. This means that if we are only looking in the real numbers ($S \subset \mathbb{R}$) then the answer to the first and second question is 2, but if our search includes the complex numbers ($S \subset \mathbb{C}$) then our answer to those same questions is 4. This illustrates an important fact of number sets (i.e. \mathbb{R} and \mathbb{C}). The particular field we are using plays its own unique role in answering these questions. This thesis will begin with a discussion of tools and their particular application to the reals. But \mathbb{R} and \mathbb{C} are not the only fields of interest. In the second half of this

thesis we will introduce special non-Archimedean fields called the p -adic numbers. For each prime, p , we have a different field, \mathbb{Q}_p , with its own special characteristics. For example, the $f(x)$ defined above has 4 roots in \mathbb{Q}_p when 4 divides $p - 1$ and 2 roots when it does not.

The last question above is a generalization of Smale's 17th problem:

Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm? [15]

In our modern computational world Smale's 17th problem, and its generalizations, are of increasing interest. When searching for real solutions, the popular approach is to use methods designed to answer Smale's 17th question in its original form. That is, to find all complex solutions and afterwards filter these solutions, keeping only the real solutions. While this approach has been quite successful it has obvious complexity theoretic and practical drawbacks. For example, in 2002 Bertrand Haas [8] found a special polynomial system:

$$\begin{aligned} x^{106} + \frac{11}{10}y^{53} - \frac{11}{10}y \\ y^{106} + \frac{11}{10}x^{53} - \frac{11}{10}x. \end{aligned}$$

This is special because it has the *maximum* number of positive real solutions possible with two trinomials: 5. Despite this fact, a mixed volume calculation shows that the system has 11,235 non-zero complex solutions. Therefore, a traditional solver would have to find 11,235 solutions and then hopefully be able to successfully filter out the 11,230 extra solutions to save the 5 real solutions. This is not an isolated case. It is

very common for polynomial systems to have large numbers of complex solutions, yet admit very few real solutions. This results in a large amount of extra computation when one only desires the real solutions. For this reason, we study tools that have been used in recent work to focus on real solutions from the beginning.

This thesis will focus on the \mathcal{A} -discriminant variety, which is a tool that can be used to help answer these questions [9, 12]. It is a tool of primary importance in Gel'fand, Kapranov, and Zelevinsky's book [7]. It can categorize families of equations according to the topology of their real solutions. First, in Chapter 2, we will introduce the \mathcal{A} -discriminant variety in the general setting. Then Chapter 3 will focus on the special properties and uses of the \mathcal{A} -discriminant variety over \mathbb{R} . We will discuss ways it is used to approach real polynomial solving. Then we will present new results and tools discovered by this author. In particular, we improve upon many of the results found in [5]. For example, [5] has a $O(n^6)$ upper bound on the number of connected components in the complement of the reduced real \mathcal{A} -discriminant variety when in 2 dimensions. We reduce this to a likely asymptotically tight bound of $O(n^2)$. This author and Rojas are in the midst of finding the first polynomial bounds in the 3-dimensional case. A key tool used to reach this bound is an extension of a result of this author (as well as Shakalli and Sottile) in [14]. The result used is a polynomial bound on the number of solutions to mixed Gale dual systems. That is, solutions to systems of the form

$$f_j = \prod_{i=1}^{\ell} (y_i)^{b_{i,j}} \prod_{i=1}^n (p_i(y))^{c_{i,j}} = 1,$$

for $j \in \{1, \dots, \ell\}$ and $p_i \in \mathbb{R}[y_1, \dots, y_\ell]$ with $\deg p_i = d$.

In Chapter 4, we move to the non-Archimedean setting. The general non-Archimedean \mathcal{A} -discriminant variety is less commonly studied than its real counterpart. The non-Archimedean setting generalizes tropical results of Dickenstein, Rincón,

Sturmfels, et al [4, 13]. Again, the non-Archimedean \mathcal{A} -discriminant variety is less commonly studied so has been used with less success than its real counterpart. Nevertheless, tools and constructions related to this have been used with success. To illustrate, let \mathbb{F} be a field and \mathcal{F} a family of n -variate polynomial systems. Define the feasibility problem, $\text{Feas}_{\mathbb{F}}(\mathcal{F})$, as follows: given $F \in \mathcal{F}$ does F have a solution in \mathbb{F}^n ?

This is related Smale's 17th problem.

When $\mathcal{A} = \{0, 1, 2\}$ and $f \in \mathcal{F}_{\mathcal{A}}$ the sign of $\Delta_{\mathcal{A}}(f)$ tells us whether f has a solution in \mathbb{R} .

In [2], Bihan, Rojas, and Stella proved that $\text{Feas}_{\mathbb{R}}(\mathcal{F}_{1,3}) \in \text{P}$. The \mathcal{A} -discriminant was a key tool in their work. Using tools related to the information on non-Archimedean \mathcal{A} -discriminants presented here and linear forms in logs the current author along with Avendaño, Ibrahim, and Rojas partially extended this result to \mathbb{Q}_p , proving that $\text{Feas}_{\mathbb{Q}_p}(\mathcal{F}_{1,3}) \in \text{NP}$ [1]. Previously the best known bound was EXP .

With Chapter 4, we give a terse discussion and introduction of various non-Archimedean fields. Finally, Chapter 5 will present new results analogous to the ones discovered and presented in the real setting, as well as some generalizations that have yet to be found in the real setting.

Lastly, in the appendices, we include Sage code that has been developed to plot various \mathcal{A} -discriminant amoebae. Appendix A has code to plot the real \mathcal{A} -discriminant amoeba for $k = 3$. Next, Appendix B gives code to plot the p -adic \mathcal{A} -discriminant amoeba for $k = 3$ and Appendix C gives code to plot the p -adic \mathcal{A} -discriminant for $k > 3$. All three appendices include sample code explaining how to run them as well as links to download the code online.

2. BACKGROUND ON DISCRIMINANTS

Although the name may not be immediately recognizable, the concept of the discriminant is already well known. Before we get into any finer details, we begin with an example from high school algebra.

Example 2.1. We look at the most well known example. Suppose we want to investigate the family of quadratic polynomials

$$\mathcal{F} = \{ x^2 + bx + c \mid b, c \in \mathbb{R} \}.$$

We know that any given polynomial $f(x) = x^2 + bx + c$ in \mathcal{F} has roots

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

If we consider the coefficients of a given $f \in \mathcal{F}$ as indeterminates, then the polynomial $\Delta := b^2 - 4c$ gives important information about its roots. That is, we identify $f = x^2 + bx + c$ with the point (b, c) , and $\Delta(b, c)$ will tell us information about the polynomial f . Firstly, if $\Delta = 0$, then we have only one degenerate real root, $x = \frac{-b}{2}$, with multiplicity 2. When $\Delta > 0$ then there are two distinct real roots. On the other hand, when $\Delta < 0$ there are no real roots. Again considering b as the horizontal axis and c as the vertical, the location of f on the graph in Figure 2.1 tells us how many real roots we should expect.

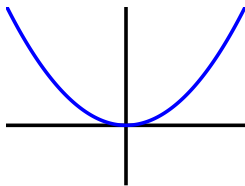


Figure 2.1: The discriminant variety for $\mathcal{F} = \{x^2 + bx + c \mid b, c \in \mathbb{R}\}$.

In general, the \mathcal{A} -discriminant can be used in a similar way with more complicated polynomials. If we let \mathbb{K} be an algebraically closed field of characteristic 0 and $\mathcal{A} = \{\alpha_1, \alpha_2, \dots, \alpha_{n+k}\} \subset \mathbb{Z}^n$, then we consider the family of equations

$$\mathcal{F}_{\mathcal{A}} = \left\{ \sum_{i=1}^{n+k} c_i x^{\alpha_i} \mid c_i \in \mathbb{K} \right\},$$

where $x = (x_1, \dots, x_n)$ and $x^{\alpha_i} = x_1^{\alpha_{i,1}} \dots x_n^{\alpha_{i,n}}$. We call \mathcal{A} the *support* of $\mathcal{F}_{\mathcal{A}}$. We identify points in $\mathcal{F}_{\mathcal{A}}$ with points in $\mathbb{P}(\mathbb{K})^{n+k-1}$. The discriminant for a quadratic polynomial is zero when the polynomial has a degenerate root ($f(x) = 0$ and $f'(x) = 0$ for some $x \in \mathbb{C}$). We want to define the \mathcal{A} -discriminant similarly. We call a root of a polynomial degenerate if it is also zero for all partial derivatives. Thus we define the \mathcal{A} -discriminant variety to be

$$\nabla_{\mathcal{A}} = \overline{\left\{ f \in \mathcal{F}_{\mathcal{A}} \mid \exists z \in (\mathbb{K}^{\times})^n, f(z) = 0, \frac{\partial f}{\partial x_i}(z) = 0, \forall i \in \{1, \dots, n\} \right\}}.$$

In words we have: $\nabla_{\mathcal{A}}$ is the closure of the locus of polynomials with a non-zero degenerate root.

2.1 The Cayley Trick and the Discriminant of Systems of Polynomials

The discussion so far has centered around individual polynomials in one or more variables. We are also interested (in practice, perhaps more so) in the discriminant of a system of polynomials. We say that a system of polynomials $F = \{f_1, \dots, f_m\}$ in the variables x_1, \dots, x_n has a degenerate root if the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ has rank smaller than m at that root. We can also define a discriminant polynomial for a system of equations by an extension called the Cayley Trick. We introduce new variables and define a new polynomial $G(x; y) = y_1 f_1(x) + \dots + y_m f_m(x)$. We can see that $G(x; y)$ has a degenerate root in $\mathbb{K}^n \times \mathbb{P}(\mathbb{K})^{m-1}$ (or $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$) if and only if F has a degenerate root in \mathbb{K}^n .

Theorem 2.2. *Let $F = \{f_1, \dots, f_m\}$ be a system of polynomials in $\mathbb{K}[x_1, \dots, x_n]$. Introduce new, independent, variables t_1, \dots, t_m . The Cayley polynomial*

$$G(x; t) = t_1 f_1(x) + t_2 f_2(x) + \dots + t_{m-1} f_{m-1}(x) + t_m f_m(x)$$

has a degenerate root $(x; t) \in \mathbb{K}^n \times \mathbb{P}(\mathbb{K})^{m-1}$ if and only if F has degenerate root x .

Proof. Let

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_{m-1}}{\partial x_1} & \frac{\partial f_m}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_{m-1}}{\partial x_2} & \frac{\partial f_m}{\partial x_2} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \dots & \frac{\partial f_{m-1}}{\partial x_n} & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

be the Jacobian matrix of the system F . For x to be a degenerate root of F we need $f_i(x) = 0$ for all i and we need J to have rank less than m . On the other hand, for (x, t) to be a degenerate root of G , we need G and all of its partial derivatives to be

zero on (x, t) . Now we have

$$J[t_1, t_2, \dots, t_{m-1}, t_m]^T = \left[\frac{\partial G(x, t)}{\partial x_1}, \frac{\partial G(x, t)}{\partial x_2}, \dots, \frac{\partial G(x, t)}{\partial x_n} \right]^T.$$

Therefore J is rank smaller than m at x if and only if $\frac{\partial G(x, t)}{\partial x_i} = 0$ for each i for some $t \in \mathbb{P}(\mathbb{K})^{m-1}$. That is, J having small rank is equivalent to the partials of G with respect to each of the x_i being zero. Next, we have $\frac{\partial G(x, t)}{\partial t_i} = f_i(x)$ for all $i \in \{1, \dots, m\}$. That is, the partials of G with respect to the t_i being zero is equivalent to the f_i being zero. Finally, if $f_i(x) = 0$ for all $i \in \{1, \dots, m\}$ then $G(x, t) = 0$. Therefore F is degenerate at some point $x \in \mathbb{K}^n$ if and only if there is a $t \in \mathbb{P}(\mathbb{K})^{m-1}$ such that G is degenerate at $(x; t)$.

□

2.2 Discriminant Equations

We will present two ways to calculate \mathcal{A} -discriminants. The first, and classical way is via resultants and elimination theory. This method was first explored by Sylvester, but under a different framework. Resultants have the advantage that they can be used to calculate discriminant polynomials, which are interesting and useful in their own right. Most of the time, in this paper, when we make comments about \mathcal{A} -discriminant polynomials the information was derived using resultants. The primary disadvantage is their computational complexity, in both space and time.

The other tool we will introduce is the Horn uniformization. This method gives a compact and simple parametric form of the \mathcal{A} -discriminant variety. This has the distinct advantage that it is much smaller and easier to represent, as well as the fact that it can be calculated comparatively quite easily in many instances.

2.2.1 Resultants

In its most basic form the resultant is a determinantal method to decide whether two polynomials have common roots in an algebraically closed field.

Definition 2.3 (See, e.g. [7]). Let D be an integral domain. Suppose $f(x) = a_0 + a_1x + \cdots + a_nx^n \in D[x]$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m \in D[x]$, with $a_n, b_m \neq 0$. Let $d = n + m$ and we define their *Sylvester matrix* to be the $d \times d$ matrix

$$\text{Syl}_x(f, g) = \begin{array}{cccccc} \left[\begin{array}{cccccc} a_0 & \cdots & a_n & 0 & \cdots & 0 \\ & & \ddots & & & \ddots \\ 0 & \cdots & 0 & a_0 & \cdots & a_n \\ b_0 & \cdots & b_m & 0 & \cdots & 0 \\ & & \ddots & & & \ddots \\ 0 & \cdots & 0 & b_0 & \cdots & b_m \end{array} \right] & \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} & \begin{array}{l} m \text{ rows} \\ \\ n \text{ rows} \end{array} \end{array}$$

and their *Sylvester resultant* to be $\text{Res}_x(f, g) = \det \text{Syl}_x(f, g)$.

The following theorem is key in using resultants to find \mathcal{A} -discriminant polynomials. It is a classical result. A proof can be found in many references, for example [7]. We will not include the proof, as the resultant will not play an important role in this thesis. The statement of the theorem is included for completeness and since resultants are used in the background to discuss the size of $\Delta_{\mathcal{A}}$ on occasion.

Theorem 2.4. *Let D be an integral domain. Suppose $f(x), g(x) \in D[x]$, with $\deg(f) = n$ and $\deg(g) = m$. Then $\text{Res}(f, g) = 0$ if and only if $f(x) = g(x) = 0$ for some x in the algebraic closure of D .*

Using this theorem, we can use resultants to find \mathcal{A} -discriminants. Here we return to the quadratic polynomial.

Example 2.5. Let $f(x) = x^2 + bx + c$. Then the resultant, $\text{Res}_x(f, f')$ is

$$\begin{vmatrix} 1 & b & c \\ 2 & b & 0 \\ 0 & 2 & b \end{vmatrix} = -(b^2 - 4c).$$

That is, $\text{Res}_x(f, f')$ is the negative of the discriminant polynomial.

Computationally one of the biggest problems with using resultants to find \mathcal{A} -discriminant polynomials is the size of the polynomials. This directly results in fairly slow computation and difficulty using them effectively in practice. We have seen that the \mathcal{A} -discriminant polynomial of the quadratic is not very complicated, but you don't have to go far to get more complicated equations. For example, the discriminant of the quintic is degree 8 with 59 terms and the discriminant of the degree 6 support has degree 8 with 246 terms.

Resultants can also be used to find the \mathcal{A} -discriminant polynomial for multivariate polynomials (and by the Cayley trick systems). However the polynomial can then become even more unwieldy. We will not go into detail, but the following example illustrates this with an example from [5].

Example 2.6. Suppose we want to examine the system

$$\begin{aligned} f_1 &= x^6 + ay^3 - y \\ f_2 &= y^6 + bx^3 - x. \end{aligned}$$

Let

$$J = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial y}.$$

The discriminant of the family of $\{f_1, f_2\}$ is the collection of a and b such that $\{f_1, f_2, J\}$ admit a simultaneous root. We have three equations and two variables (x and y) we need to eliminate. We start by eliminating the x variable. This gives us

$$J_1 = \text{Res}_x(f_1, J), \text{ and } J_2 = \text{Res}_x(f_2, J).$$

Now J_1 and J_2 are both polynomials in $\mathbb{Z}[y, a, b]$. They are quite large. They are both degree 45 in y and 6 in b . J_1 is degree 8 in a and J_2 is degree 3 in b . Next we want to eliminate the y and so we have

$$R = \text{Res}_y(J_1, J_2).$$

Now this R is a multiple of the final discriminant, but here multiple means that

$$p(a, b)\Delta_{\mathcal{A}}(1, a, -1, 1, b, -1) = R(a, b),$$

for some $p \in \mathbb{R}[a, b]$. While the total degree of R in a and b is 513, the final \mathcal{A} -discriminant polynomial is a factor of R with total degree 90, degree 47 with respect to each variable, and exactly 58 monomial terms with coefficients on the order of 10^{43} [5].

Although the size in the example is quite large, it is actually significantly smaller than $\Delta_{\mathcal{A}}(a, b, c, d, e, f)$ would be, which is the full discriminant polynomial.

2.2.2 Horn Uniformization

We concluded the previous subsection with a discussion of the size and computational complexity of resultants. This is not simply a failing of the resultant, but more fundamentally a failing of \mathcal{A} -discriminant polynomials themselves. Algorithms

to compute the polynomials are limited in complexity by the polynomials themselves. That is, if you desire to compute the polynomial efficiently you are limited by the sheer size of the polynomials themselves. Therefore, we desire to find a way to express the \mathcal{A} -discriminant in way that is more compact, both formulaically and computationally. This is where the Horn uniformization comes into play.

The *Horn uniformization* is a simple parametrization of the \mathcal{A} -discriminant variety. Again let $\mathcal{A} = \{\alpha_1, \dots, \alpha_{n+k}\} \in \mathbb{Z}^n$ be the support of our polynomial family $\mathcal{F}_{\mathcal{A}}$, with cardinality $n+k$. Now, define the matrix

$$\hat{\mathcal{A}} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_{n+k} \\ 1 & \cdots & 1 \end{bmatrix}$$

Then the \mathcal{A} -discriminant variety is parametrized as the Zariski closure of

$$\left\{ [u_1 t^{a_1} : \cdots : u_{n+k} t^{a_{n+k}}] \mid u \in \mathbb{P}(\mathbb{K})^{n+k-1}, \hat{\mathcal{A}}u = 0, t \in (\mathbb{K}^*)^n \right\}.$$

Theorem 2.7. *Let \mathbb{K} be an algebraically closed field. Let $\mathcal{A} = \{a_1, \dots, a_{n+k}\} \subset \mathbb{Z}^n$ be generic. Then $\nabla_{\mathcal{A}}$ is the Zariski closure of*

$$\left\{ [u_1 t^{a_1} : \cdots : u_{n+k} t^{a_{n+k}}] \mid u = (u_1, \dots, u_{n+k}) \in \mathbb{P}(\mathbb{K})^{n+k-1}, \hat{\mathcal{A}}u = 0, t \in (\mathbb{K}^*)^{n+k} \right\}.$$

Proof. Let N be the set

$$\{[u_1 t^{a_1} : \cdots : u_{n+k} t^{a_{n+k}}] \mid u = (u_1, \dots, u_{n+k}) \in \mathbb{P}(\mathbb{K})^{n+k-1}, \hat{\mathcal{A}}u = 0, t \in (\mathbb{K}^*)^{n+k}\}.$$

Pick any such u and t . Let $z = (t_1^{-1}, \dots, t_m^{-1})$ and $f(x) = \sum_{i=1}^m u_i t^{a_i} x^{a_i}$. We will show that z is a degenerate root of $f(x)$. First of all $f(z) = \sum u_i = 0$, since the all

ones vector is a row of \hat{A} . Similarly since $\hat{A}u = 0$ then $a_i \cdot u = 0$ for all i . This tells us that

$$\begin{aligned} z_j \frac{\partial f(z)}{\partial x_j} &= \sum_{i=1}^m a_{j,i} u_i t^{a_i} z^{a_j} \\ &= \sum_{i=1}^m a_{j,i} u_i = a_j \cdot u = 0. \end{aligned}$$

Hence z is a root of $f(x)$ and its derivatives and so a degenerate root of $f(x)$. Hence $N \subset \nabla_A$ and since ∇_A is closed, we have $\overline{N} \subset \nabla_A$.

Now suppose that $f(x) = \sum_{i=1}^m c_i x^{a_i}$ is a polynomial with a degenerate root $z = (z_1, \dots, z_m) \in (\mathbb{K}^*)^m$ and $c_i \neq 0$ for all i . Let $t = (z_1^{-1}, \dots, z_m^{-1})$ and define $u = (c_1 z^{a_1}, \dots, c_m z^{a_m})$. We will see that $\hat{A}u = 0$. Of course $\sum_{i=1}^m u_i = f(z) = \sum_{i=1}^m c_i z^{a_i} = 0$. Similarly, $u \cdot a_j = z_j \frac{\partial f(z)}{\partial x_j} = 0$. Now since $c_j = u_j t^{a_j}$ then c has the desired form. This shows that a dense subset of ∇_A is contained in N . Hence $\nabla_A \subset \overline{N}$ and $\overline{N} = \nabla_A$. \square

2.3 Reduced Discriminant and the B Matrix

The \mathcal{A} -discriminant is a $k-1$ dimensional surface stretched over $n+k$ dimensions. We can take a slice of the \mathcal{A} -discriminant variety called the reduced \mathcal{A} -discriminant variety [7]. We will denote this by $\overline{\nabla}_A$. Again, we assume $\mathcal{A} = \{\alpha_1, \dots, \alpha_{n+k}\} \subset \mathbb{Z}^n$ and we define the matrix

$$\hat{\mathcal{A}} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n+k} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{Z}^{(n+1) \times (n+k)}.$$

We suppose $\hat{\mathcal{A}}$ affinely generates \mathbb{Z}^n and that the rank of $\hat{\mathcal{A}}$ is $(n+1)$. Then we let $B \in \mathbb{Z}^{(n+k) \times (k-1)}$ be an integer matrix whose columns form a basis of the null space of $\hat{\mathcal{A}}$. We require that the maximal minors of B be relatively prime. Now we define

a rational map $(\cdot)^{B^T} : \mathbb{P}(\mathbb{K})^{k-2} \dashrightarrow \mathbb{K}^{k-1}$ as

$$\lambda \rightarrow \left(\prod_{j=1}^{n+k} \lambda_i^{\beta_{j,i}} \right)_{i=1}^{k-1}.$$

It is quick to show that this map is well defined. First we observe that for any i we have $\sum_{j=1}^{n+k} \beta_{j,i} = 0$, as the columns of B are orthogonal to the rows of $\hat{\mathcal{A}}$ (in particular, the last row of $\hat{\mathcal{A}}$). Then if $\lambda' = (c\lambda_1, \dots, c\lambda_{k-1})$, with $c \neq 0$, looking at the i^{th} coordinate, we have

$$\prod_{j=1}^{n+k} (c\lambda_i)^{\beta_{j,i}} = c^{\sum_{j=0}^{n+k} \beta_{j,i}} \prod_{j=0}^{n+k} (\lambda_i)^{\beta_{j,i}} = c^0 \prod_{j=1}^{n+k} (\lambda_i)^{\beta_{j,i}}.$$

With the exponentiation well defined, we can now write the reduced Horn uniformization.

Definition 2.8. Let $\overline{\nabla}_{\mathcal{A}}$ be the reduced \mathcal{A} -discriminant variety. Then it is parametrized by $\mathbb{P}(\mathbb{K})^{k-2} \dashrightarrow \mathbb{K}^{k-1}$,

$$\lambda \mapsto (B \cdot \lambda)^{B^T}.$$

Throughout this thesis we will use φ to denote reduced \mathcal{A} -discriminant amoeba maps with various domains.

To make certain proofs simpler, we will make a few assumptions on the B matrix. First we will state our list of assumptions for easy reference, then we will justify them.

- B will have no rows of all zeros.
- B will have no rows that are multiples of each other.

We may assume that no rows of B are the zero vectors, as removing the vector

would not change the final product. We may also assume that rows of B are not multiples of each other. Indeed, suppose that $B = \{ \beta_1, \dots, \beta_{n+k} \}$ with $c\beta_1 = \beta_2$ with $c \neq -1, 0$. Let $B' = \{ \beta_1 + \beta_2, \beta_3, \dots, \beta_{n+k} \}$ and let $\varphi'(\lambda)$ be the associated reduced horn parametrization. Remember that $\langle a\beta_i, \lambda \rangle^d = a^d \langle \beta_i, \lambda \rangle^d$ for any $a \in \mathbb{Q}$ and $d \in \mathbb{Z}$. For notational simplicity, we let $P_i = \prod_{j=3}^{n+k} \langle \beta_j, \lambda \rangle^{\beta_{j,i}}$. We then have

$$\begin{aligned}
\varphi_i(\lambda) &= \langle \beta_1, \lambda \rangle^{\beta_{1,i}} \langle \beta_2, \lambda \rangle^{\beta_{2,i}} \prod_{j=3}^{n+k} \langle \beta_j, \lambda \rangle^{\beta_{j,i}} \\
&= \langle \beta_1, \lambda \rangle^{\beta_{1,i}} \langle c\beta_1, \lambda \rangle^{c\beta_{1,i}} P_i \\
&= c^{c\beta_{1,i}} \langle \beta_1, \lambda \rangle^{\beta_{1,i}} \langle \beta_1, \lambda \rangle^{c\beta_{1,i}} P_i \\
&= c^{c\beta_{1,i}} \langle \beta_1, \lambda \rangle^{(1+c)\beta_{1,i}} P_i \\
&= c^{c\beta_{1,i}} \frac{(1+c)^{(1+c)\beta_{1,i}}}{(1+c)^{(1+c)\beta_{1,i}}} \langle \beta_1, \lambda \rangle^{(1+c)\beta_{1,i}} P_i \\
&= \frac{c^{c\beta_{1,i}}}{(1+c)^{(1+c)\beta_{1,i}}} \langle (1+c)\beta_1, \lambda \rangle^{(1+c)\beta_{1,i}} \prod_{j=3}^{n+k} \langle \beta_j, \lambda \rangle^{\beta_{j,i}} \\
&= \frac{c^{c\beta_{1,i}}}{(1+c)^{(1+c)\beta_{1,i}}} \varphi'_i(\lambda) \\
&= \frac{c^{\beta_{2,i}}}{(1+c)^{\beta_{1,i} + \beta_{2,i}}} \varphi'_i(\lambda).
\end{aligned}$$

Since B is an integer matrix then $c \in \mathbb{Q}$ and $\beta_{1,i} + \beta_{2,i} \in \mathbb{Z}$. Hence $\varphi(\lambda)$ and $\varphi'(\lambda)$ differ by component-wise rational constants. On the other hand, if $\beta_1 = -\beta_2$, then

$$\langle \beta_1, \lambda \rangle^{\beta_{1,i}} \langle \beta_2, \lambda \rangle^{\beta_{2,i}} = \langle \beta_1, \lambda \rangle^{\beta_{1,i}} \langle -\beta_1, \lambda \rangle^{-\beta_{1,i}} = (-1)^{\beta_{1,i}}$$

So we can take $B' = \{ \beta_3, \beta_4, \dots, \beta_{n+k} \}$. In either case, $\varphi(\lambda)$ is a component-wise constant multiple of $\varphi'(\lambda)$. Therefore it suffices to assume rows of B are not multiples of each other.

By construction B is an integer matrix in $\mathbb{Z}^{(n+k) \times (k-1)}$ of rank $k-1$. We will oc-

asionally assume that integer multiples of the standard basis vectors, $\{e_1, \dots, e_{k-1}\}$, are among the rows of B . We may satisfy this assumption, by simply choosing the Smith normal form of B .

3. DISCRIMINANTS AND CONTOURS

Throughout this chapter, we will assume that $\mathbb{K} = \mathbb{C}$. Hence $\overline{\nabla}_{\mathcal{A}}$ will be the complex reduced \mathcal{A} -discriminant variety. We will let $\overline{\nabla}_{\mathcal{A}}(\mathbb{R})$ be the the real part of the reduced \mathcal{A} -discriminant variety. That is $\overline{\nabla}_{\mathcal{A}}(\mathbb{R}) = \overline{\nabla}_{\mathcal{A}} \cap \mathbb{R}^{k-1}$. Now let $X \subset (\mathbb{C}^{\times})^n$ be an algebraic hypersurface and let $\text{Log}: (\mathbb{C}^{\times})^n \rightarrow \mathbb{R}^n$ be the component-wise absolute value followed by logarithm. This map gives us the reduced \mathcal{A} -discriminant amoeba,

$$\{ \text{Log}(z) \mid z \in \overline{\nabla}_{\mathcal{A}} \cap (\mathbb{C}^{\times})^{k-1} \}.$$

When we perform a similar procedure on $\overline{\nabla}_{\mathcal{A}}(\mathbb{R})$ we get a special object called the *contour*, $\mathcal{C}_{\mathcal{A}}$, of the \mathcal{A} -discriminant amoeba.

Definition 3.1. The *contour*, $\mathcal{C}_{\mathcal{A}}$, of the \mathcal{A} -discriminant amoeba is the amoeba of $\overline{\nabla}_{\mathcal{A}}(\mathbb{R})$. That is,

$$\mathcal{C}_{\mathcal{A}} = \{ \text{Log}(x) \mid x \in \overline{\nabla}_{\mathcal{A}} \cap (\mathbb{R}^{\times})^{k-1} \}.$$

The contour is called such because it is the collection of critical points of the amoeba and it contains the boundary [11]. Usually it also contains points in the interior. Figure 3.1 shows the amoeba when $\mathcal{A} = [0, 1, 2, 3]$ and its contour.

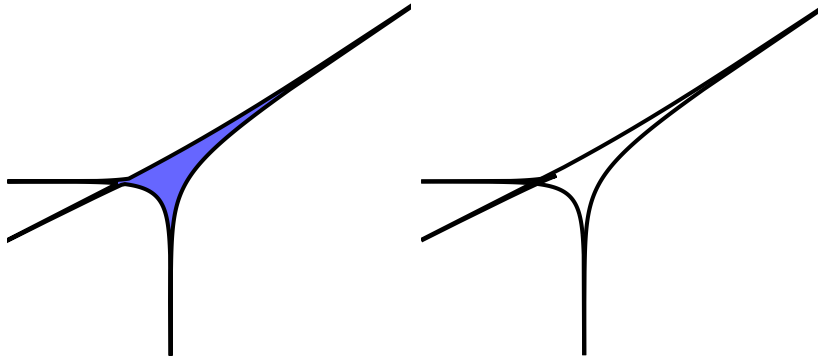


Figure 3.1: An \mathcal{A} -discriminant amoeba and its contour.

The reduced real \mathcal{A} -discriminant variety and the contour of the amoeba are both quite powerful tools in their own right. In the introduction, we mentioned the discriminant of the quadratic, $\Delta = b^2 - 4ac$. We observed that the sign of Δ tells us how many real roots a polynomial has. More generally, one could say it tells us about the topology of the zero set. This idea holds with these more general \mathcal{A} -discriminants. All the polynomials inside a given connected component of the complement of the contour have diffeotopic zero sets [7]. (See Figure 3.2.) Our primary result in this

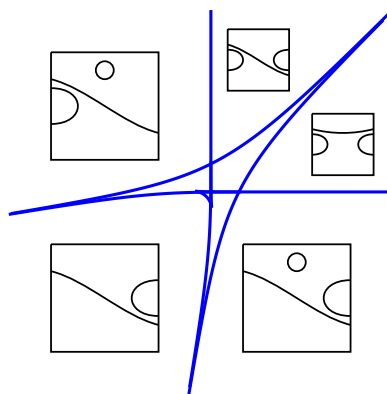


Figure 3.2: Distinct chambers have the diffeotopic zero sets.

chapter is a new bound on the number of components in the complement of $\mathcal{C}_{\mathcal{A}}$ that extends to the reduced real \mathcal{A} -discriminant variety when $k = 3$. We will prove

Theorem 3.7. *When $k = 3$, the reduced real \mathcal{A} -discriminant variety, $\overline{\nabla}_{\mathcal{A}}$ has no more than*

$$2n^2 + 4n + 4$$

connected components in its complement.

Previously, the best known bound was $O(n^6)$ in a paper by Dickenstein, et al [5]. Prior to that Gabrielov, et al wrote a paper on Pfaffian functions and quantifier elimination for fewnomials that appears to imply an upper bound of $2^{O(n^6)}$ [6]. The contour and its structure plays a primary role. Along the way we will introduce a simple construction that we call a collection of signed points on \mathbb{P}^1 that will be useful to understand the structure of the contour. For example, Theorem 3.4 will show that the contour can have no more than n cusps and Theorem 3.10 will use the concept to create a family of \mathcal{A} -discriminant amoebae whose contours have n cusps for any even $n \geq 2$. Suppose we have an n -variate polynomial, f , with $n+k$ distinct monomials. We can construct the \mathcal{A} -discriminate amoeba contour which will be a $k-2$ dimensional surface in \mathbb{R}^{k-1} . Let $\mathcal{A} = \{\alpha_1, \alpha_2, \dots, \alpha_{n+k}\} \in \mathbb{Z}^n$. Again, we are examining polynomials of the form

$$\sum_{i=1}^{n+k} c_{\alpha_i} x^{\alpha_i},$$

where $c_{\alpha_i} \in \mathbb{R}^{\times}$, $x = (x_1, x_2, \dots, x_n)$ and $x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdot x_2^{\alpha_{i,2}} \cdots x_n^{\alpha_{i,n}}$. To this end we

define an intermediate matrix

$$\hat{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n+k} \end{bmatrix}$$

and let $B \in \mathbb{Z}^{(n+k) \times (k-1)}$ be an integer matrix whose columns form a basis of the right null space of \hat{A} . We will label the rows of B as $\{\beta_1, \beta_2, \dots, \beta_{n+k}\}$. As previously discussed the Horn uniformization (cf. Section 2.2.2) of the contour of the \mathcal{A} -discriminant amoeba is the rational map $\varphi : \mathbb{P}\mathbb{R}^{k-2} \dashrightarrow \mathbb{R}^{k-1}$ given by

$$\lambda \mapsto B^T \log(|B \cdot \lambda|), \quad (3.1)$$

where $\log(|\cdot|) : \mathbb{R}^{n+k} \dashrightarrow \mathbb{R}^{n+k}$ is the component-wise absolute value followed by logarithm. The map φ can equivalently be written as

$$\varphi(\lambda) = \sum_{i=1}^{n+k} \beta_i \log(|\langle \beta_i, \lambda \rangle|), \quad (3.2)$$

where $\langle \beta_i, \lambda \rangle := \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2 + \cdots + \beta_{i,k-1}\lambda_{k-1}$ is the inner product. When it simplifies the argument we will resort to this slightly less compact form.

Many of the results of this chapter hinge on an important fact proved by Mikhail Kapranov in 1991 [10]. He proved that at the points where the Gauss map is well-defined, the Gauss map is φ^{-1} . This simple fact has been overlooked in large part in the literature, but it is, in fact, quite powerful and leads to many strong results. We include a statement of a special case of his theorem (translated into the language of this work) and a proof for completeness.

Theorem 3.2 (See [10] Theorem 2.1). *Let B and $\varphi : \mathbb{P}\mathbb{R}^{k-2} \dashrightarrow \mathbb{R}^{k-1}$ be described as above. Then the Gauss map $N : \bar{\nabla}_{\mathcal{A}} : \mathbb{P}\mathbb{R}^{k-2} \dashrightarrow \mathbb{P}\mathbb{R}^{k-1}$ is $N(\varphi(\lambda)) = \lambda$ at all*

points where φ is differentiable.

Proof. For this proof, it simplifies things to look at φ in the form of equation 3.2.

Thus we have

$$\varphi(\lambda) = \sum_{i=1}^{n+k} \beta_i \log(|\langle \beta_i, \lambda \rangle|).$$

Then the projection of φ to the j^{th} coordinate, φ_j , can be written as

$$\varphi_j(\lambda) = \sum_{i=1}^{n+k} \beta_{i,j} \log(|\langle \beta_i, \lambda \rangle|).$$

Next the ℓ^{th} derivative of φ_j is

$$\frac{\partial \varphi_j}{\partial \lambda_\ell}(\lambda) = \sum_{i=1}^{n+k} \frac{\beta_{i,j} \beta_{i,\ell}}{\langle \beta_i, \lambda \rangle}.$$

The full derivative of φ_j is then simply the sum over all ℓ , or

$$d\varphi_j(\lambda) = \sum_{\ell=1}^{k-1} \sum_{i=1}^{n+k} \frac{\beta_{i,j} \beta_{i,\ell}}{\langle \beta_i, \lambda \rangle}.$$

If we define $d\varphi = (d\varphi_1, d\varphi_2, \dots, d\varphi_{k-1})$ then to say that $N(\lambda) = \lambda$, for some $\lambda \in \mathbb{P}\mathbb{R}^{k-2}$, is to say that $\langle \lambda, d\varphi(\lambda) \rangle = 0$, provided that the derivative $d\varphi(\lambda)$, is well

defined. This is not difficult to show

$$\begin{aligned}
\lambda \cdot d\varphi(\lambda) &= \sum_{j=1}^{k-1} \lambda_j \sum_{\ell=1}^{k-1} \sum_{i=1}^{n+k} \frac{\beta_{i,j} \beta_{i,\ell}}{\langle \beta_i, \lambda \rangle} \\
&= \sum_{\ell=1}^{k-1} \sum_{i=1}^{n+k} \frac{\beta_{i,\ell} \sum_{j=1}^{k-1} \lambda_j \beta_{i,j}}{\langle \beta_i, \lambda \rangle} \\
&= \sum_{\ell=1}^{k-1} \sum_{i=1}^{n+k} \beta_{i,\ell} \frac{\sum_{j=1}^{k-1} \lambda_j \beta_{i,j}}{\langle \beta_i, \lambda \rangle} \\
&= \sum_{\ell=1}^{k-1} \sum_{i=1}^{n+k} \beta_{i,\ell} \frac{\langle \beta_i, \lambda \rangle}{\langle \beta_i, \lambda \rangle} \\
&= \sum_{\ell=1}^{k-1} \sum_{i=1}^{n+k} \beta_{i,\ell} = 0.
\end{aligned}$$

The final term is equal to zero because it is the sum of all the elements of the matrix B , and the vector $(1, 1, \dots, 1)$ is in the row span of \hat{A} . Therefore, we have the desired result. \square

3.1 Domain

Before we illustrate Theorem 3.2, we will discuss the domain of φ and the freedom we have. Traditionally we consider φ to be a rational map with domain $\mathbb{P}\mathbb{R}^{k-2}$, but we can get the same map using different domains. For example, if we let $C^{k-2} = \{(x_0, x_1, \dots, x_{k-2}) \in S^{k-2} \mid x_{k-2} > 0\}$ be the upper half of S^{k-2} . There is a natural isomorphism between C^{k-2} and \mathbb{R}^{k-2} using stereographic projection. Given the right assumptions, either of these can be viewed as the domain of φ . Now $\mathbb{P}\mathbb{R}^{k-2} \setminus \{[x_0 : x_1 : \dots : x_{k-2}] \mid x_{k-2} \neq 0\}$ is also naturally isomorphic to both C^{k-2} and \mathbb{R}^{k-2} . In the former case we choose the length 1 representative point with $x_{k-2} > 0$. In the latter case, we choose the first $k - 2$ coordinates of the representative point with $x_{k-2} = 1$

(affine coordinates). Now φ is undefined on the solutions to $\langle \beta_i, \lambda \rangle = 0$. That implies that if we assume that for some i , we have $\beta_i = (0, \dots, 0, t)$ for some $t \neq 0$ then we can freely move between viewing φ as any of the rational functions $\varphi: \mathbb{P}\mathbb{R}^{k-2} \dashrightarrow \mathbb{R}^{k-1}$, $\varphi: C^{k-2} \dashrightarrow \mathbb{R}^{k-1}$, and $\varphi: \mathbb{R}^{k-2} \dashrightarrow \mathbb{R}^{k-1}$ in a well-defined fashion. As discussed in the introduction, the requirement of such a β_i is an assumption that we may freely make by using the Smith Normal form. Now Example 3.3 will illustrate both Theorem 3.2 and the utility of using C^1 as our domain.

Example 3.3. Let $\mathcal{A} = [0, 1, 2, 3]$. In this case, $n = 1$ and $k = 3$ and the image of φ will be in \mathbb{R}^2 . Now forming \hat{A} and finding the right null space, we have

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3 & -2 \\ 2 & 1 \end{bmatrix}.$$

The map is a linear combination of logarithms of the linear forms defined by the rows of B . Therefore it is only undefined where the logarithms blow up. That is, it is undefined at the projective points $[0 : 1]$, $[1 : 0]$, $[-2 : 3]$, and $[-1 : 2]$. As mentioned in the paragraphs before this example, we can visualize the domain (a subset of \mathbb{P}^1) as a subset of the upper half of S^1 with the two endpoints removed. Figure 3.3 shows the upper half of S^1 , removing the points where φ is undefined.

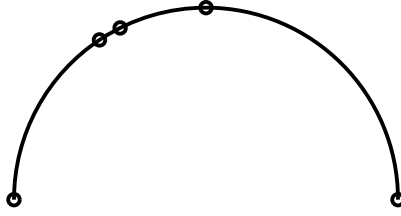


Figure 3.3: Upper half of S^1 as the domain.

The logarithm function is continuous on \mathbb{R}^\times , therefore between these undefined points, φ is continuous. Furthermore since logarithm blows up as it approaches 0, then φ blows up as it approaches these undefined points. This means a subset of S^1 between two consecutive undefined regions maps to an arc in the image between two asymptotes. Furthermore, Theorem 3.2 tells us that the slope of S^1 at a given point, x , is the same as the slope of the image at $\varphi(x)$. Figure 3.4 illustrates this.

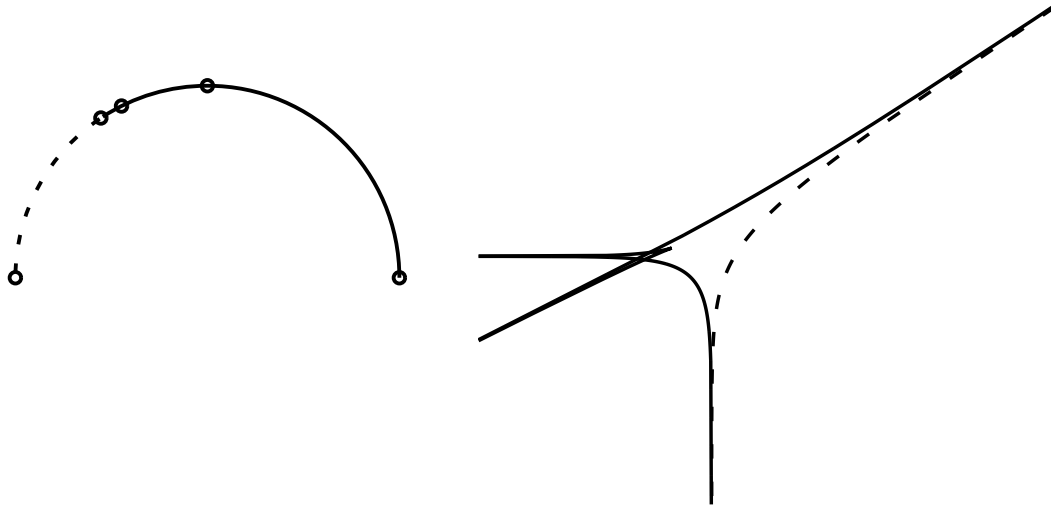


Figure 3.4: Slope is preserved.

On the left we have the domain and on the right is the image. In both images we have set apart a section by making it dashed. These two dashed parts correspond. That is, the image of the dashed part of the domain is dashed in the image. Visually, we can see that slopes are preserved. The image is simply stretched out. Viewing this figure more closely, one can tell which arcs correspond to the other segments of the image.

For the rest of our discussion we will restrict to the case $k = 3$. In this setting, the image of $\varphi : \mathbb{P}\mathbb{R} \dashrightarrow \mathbb{R}^2$ is a 1-dimensional curve in \mathbb{R}^2 . For notational simplicity, we will write $\beta_i = (a_i, b_i)$, rather than the more cumbersome $\beta_i = (\beta_{i,1}, \beta_{i,2})$. We can

then write our map as

$$\varphi(\lambda) = \left(\sum_{i=1}^{n+3} \frac{a_i}{a_i \lambda_1 + b_i \lambda_2}, \sum_{i=1}^{n+3} \frac{b_i}{a_i \lambda_1 + b_i \lambda_2} \right).$$

Furthermore, when we use affine coordinates we will write our map as

$$\varphi(x) = \left(\sum_{i=1}^{n+3} \frac{a_i}{a_i x + b_i}, \sum_{i=1}^{n+3} \frac{b_i}{a_i x + b_i} \right).$$

Both of these viewpoints are valid and prove useful depending on the setting. This will prove useful in the next section, as we will only have one variable to deal with.

3.2 Cusps

Now in reference to Theorem 3.2, we naturally ask where this map is differentiable. To answer this question we will switch to affine coordinates. That is our map becomes $\varphi : \mathbb{R} \dashrightarrow \mathbb{R}^2$,

$$\varphi(x) = \left(\sum_{i=1}^{n+3} a_i \log(a_i x + b_i), \sum_{i=1}^{n+3} b_i \log(a_i x + b_i) \right).$$

Our map is neither differentiable where φ is undefined nor where there is a cusp. The map is undefined at the zeros of $-\frac{b_i}{a_i}$, for $a_i \neq 0$. On the other hand, the component-wise derivatives of our map are smooth outside of these undefined points. Therefore, having at least one of $d\varphi_j(x) = 0$ is a necessary condition to have a cusp at x . In projective coordinates, we showed that $\lambda_1 d\varphi_1 + \lambda_2 d\varphi_2 = 0$ (Theorem 3.2), and the same argument shows us that $x d\varphi_1(x) + d\varphi_2(x) = 0$ in affine coordinates, where $d\varphi(x) = \frac{d\varphi(x)}{dx}$. Hence when $x \neq 0$ we have $d\varphi_1(x) = 0$ if and only if $d\varphi_2(x) = 0$. We can assume $x \neq 0$, by using the Smith normal form of B . These facts tell us that we can concentrate on either $d\varphi_1(x) = 0$ or $d\varphi_2(x) = 0$. Without loss of generality we

look at $d\varphi_1(x)$ and we have

$$\frac{d\varphi_1}{dx} = \sum_{i=1}^{n+3} \frac{a_i^2}{a_i x + b_i}.$$

After clearing denominators, finding the solutions here reduces to solving a polynomial of degree no more than $n + 2$ and hence no more than $n + 2$ real solutions. We will show that this polynomial actually has degree at most n and hence has no more than n real solutions.

Theorem 3.4. *Let $B = (\beta_i)_{i=1}^{n+3}$ as defined above. Then the graph of φ has at most n cusps.*

Proof. We will use affine coordinates. Therefore we seek to find the maximum number of roots of the equation

$$\sum_{i=1}^{n+3} \frac{a_i^2}{a_i x + a_i}.$$

Clearing denominators this produces a polynomial of degree at most $n + 2$. We will show that the coefficients of x^{n+2} and x^{n+1} are both zero. Let $P = \prod_i a_i$, $P_j = \prod_{i \neq j} a_i = \frac{P}{a_j}$, and $P_{j,k} = \prod_{i \neq j,k} a_i = \frac{P}{a_j a_k}$. Recall that $\sum_{i=1}^{n+3} a_i = 0$, then the coefficient of x^{n+2} is

$$\sum_{i=1}^{n+3} a_i^2 P_i = \sum_{i=1}^{n+3} a_i P = 0.$$

Next we look at the coefficient of x^{n+1} . We have the sum

$$\sum_{i=1}^{n+3} b_i \sum_{j \neq i} a_j^2 P_{i,j}.$$

This equals

$$\begin{aligned} \sum_{i=1}^{n+3} b_i P_i \sum_{j \neq i} a_j &= \sum_{i=1}^{n+3} b_i P_i (-a_i) \\ &= -P \sum_{i=1}^{n+3} b_i = 0. \end{aligned}$$

Therefore the coefficients in front of x^{n+1} and x^{n+2} are both zero. Hence our derivative has degree at most n and so we can have no more than n cusps. \square

At the close of this chapter, we will present various extremal examples. There we will present a family of examples exhibiting this bound exactly. That is, this bound is tight.

3.3 Connected Components of the Complement

We now have the necessary facts to bound the number of connected components in the complement of $\overline{\nabla}_{\mathcal{A}}(\mathbb{R})$ when $k = 3$.

In 1826, Jakob Steiner studied arrangements of lines in \mathbb{R}^2 [16]. He proved that for m lines in \mathbb{R}^2 there are no more than

$$\binom{m}{2} + m + 1$$

connected components in their complement. Since arcs do not have repeated slopes, we have a collection of arcs that act very similarly to a line arrangement. We would like to prove a similar bound on the number of components in the complement of the contour of the \mathcal{A} -discriminant amoeba. We will begin by proving a more general theorem on the number of components in the complement of arrangements of particular parametric curves. Steiner's bound can be viewed as the maximum number of intersections, $\binom{m}{2}$, plus the number of lines, m , plus 1. Our bound is very

similar, but for a different class of curves.

Suppose we have a collection, $H = \{c_1, \dots, c_m\}$, of continuous parametric curves in \mathbb{R}^2 such that all intersections are transverse. Furthermore for each i and for all $r \in \mathbb{R}$ we require an $\epsilon > 0$ such that $I = (r - \epsilon, r + \epsilon)$ gives us $c_i(r) \notin c_i(I \setminus \{r\})$. That is, each map needs to be locally injective.

We also want to define a special type of intersection, which we will call a *parametric intersection*. First, for notational simplicity we will define a map $G(i, r): \{1, \dots, m\} \times \mathcal{P}(\mathbb{R})$ as

$$G(i, r) = c_1(\mathbb{R}) \cup c_2(\mathbb{R}) \cup \dots \cup c_{i-1}(\mathbb{R}) \cup c_i((-\infty, r)).$$

This is the image of all the c_j up until the point $c_i(r)$. Now we define parametric intersections.

Definition 3.5. Let c_1, \dots, c_m be a collection of parametric curves. A parametric intersection is a tuple $(i, r) \in \{1, \dots, m\} \times \mathbb{R}$ such that $c_m(i) \cap G(i, r) \neq \emptyset$.

If our collection of parametric curves have a degenerate intersection, this gives us a clean way to “count” the degeneracy. We now continue with our bound on these parametric curves.

Theorem 3.6. *Let $H = \{c_1, \dots, c_m\}$ be a collection of continuous parametric curves, $c_i: \mathbb{R} \rightarrow \mathbb{R}^2$, such that all intersections are transverse. If there are N total parametric intersections, then the complement of the graph of all the c_i has no more than $m + N + 1$ total components.*

Proof. We prove this by induction. When there are no curves, then there are also no intersections, hence $m = N = 0$ and so there is $0 + 0 + 1 = 1$ chamber in the complement. Suppose we have proven the bound for $m - 1$ total curves and that

the number of complement components is at most $(m - 1) + M + 1$, where M is the number of parametric intersections of the $m - 1$ curves. Now we need to add in the m^{th} curve. The total number of connected components in the complement of this appended image should not increase by more than 1 more than the number of new parametric intersections introduced by c_m . Call the final curve $c = c_m$. Let

$$C = \bigcup \{c_1(\mathbb{R}), c_2(\mathbb{R}), \dots, c_{m-1}(\mathbb{R})\}$$

be the image of the first $m - 1$ curves. Let M be the number of connected components in the complement of C and let

$$R = \{r \in \mathbb{R} \mid c_m(r) \in G(m, r)\}$$

be the subset of \mathbb{R} where c_m introduces new parametric intersections. Of course, $\#R$ is the number of new parametric intersections from c_m . Let $T(r): \mathbb{R} \rightarrow \mathbb{N}$ be the number of components in the complement of $C \cup c_m((-\infty, r])$. T is an increasing function, as adding curves to a graph cannot decrease the number of components in the complement. Furthermore, T can only change when $c(r)$ intersects something in the graph $G(i, r)$. Indeed if

$$r = \begin{cases} \min R & R \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

is the first intersection (when $R \neq \emptyset$) then $C \cup c(-\infty, r')$ is contractable to C for any $r' < r$, as in Figure 3.5.

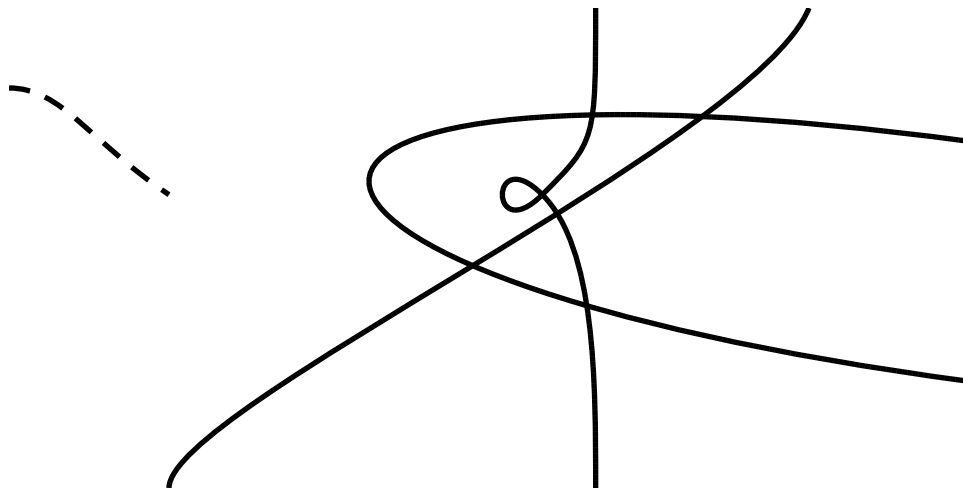


Figure 3.5: Contractable: $c_m((-\infty, r'])$ is dashed. C is solid.

On the other hand, if r and s are two consecutive intersections (or $s = \infty$ when $r = \max R$) with $r < s$ then $C \cup c(-\infty, s')$ is contractable to $C \cup c_m((-\infty, r])$, for any $r < s' < s$. That is, the topology can only change when $c_m(r)$ intersects $G(m, r)$.

Now we need to quantify what can happen to $T(r)$ as r increases. Each time $c_m(r)$ hits $G(m, r)$ then, at worst, we have sliced a component (the one that $c_m(r - \epsilon)$ was in) into two components. This means at worst, we add one more region. Thus, at worst, $T(r) = N + \#R$ for $r \in \mathbb{R}$. Finally when we finish c_m , again, at worst, we split one final component into two. In either case we have new components on either side of the segment. Thus we have that the number of connected components in the complement of $C \cup c_m((-\infty, \infty))$ is at most $M + \#R + 1$. Hence we have our result. \square

Now we would like to apply this theorem to our collection of arcs.

Theorem 3.7. *Let $\#\mathcal{A} = n + 3$, be a support with a real \mathcal{A} -discriminant containing*

ℓ cusps. Then the real \mathcal{A} -discriminant has no more than

$$\binom{n + \ell + 3}{2} - \ell + 1$$

connected components in its complement. In general, regardless of ℓ , there are no more than

$$2n^2 + 4n + 4$$

connected components in the complement.

Proof. We can break the domain into $n + 3$ arcs that map to continuous parametric curves. We assume all intersections are non-degenerate (this will only serve to increase the bound). Theorem 3.2 tells us that these curves do not repeat normals, so intersections are necessarily transverse. Hence the hypotheses of Theorem 3.6 are satisfied, so we have no more than $(n + 3) + N + 1$ connected components in the complement of the contour of the \mathcal{A} -discriminant amoeba, where N is the number of intersections. We need to bound N . Now any two arcs without cusps cannot intersect more than once, because they do not repeat normals. This fact also implies they cannot intersect themselves. Cusps make it so that some arcs can intersect other arcs multiple times (See Figure 3.6). Now, if an arc has ℓ cusps it can intersect

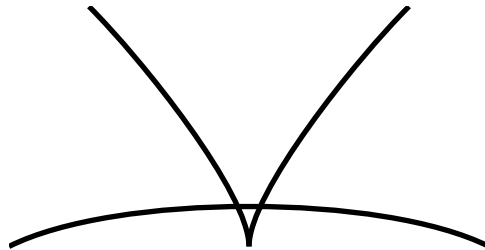


Figure 3.6: Cusps allow more intersections.

another arc without cusps at most $\ell + 1$ times. We can efficiently bound the total number of possible intersections by breaking arcs into smaller sub-arcs at the cusps. That is, an arc with ℓ cusps has $\ell + 1$ sub-arcs that can intersect any other sub-arc (if an arc has no cusps, it is its own sub-arc) at most once. If there are ℓ ($\leq n$) cusps then there are at most $n + 3 + \ell$ sub-arcs. Each sub-arc is smooth and can intersect another at most once. Therefore we cannot have more than $\binom{n+\ell+3}{2}$ total intersections.

We can actually improve this bound a little more. It just so happens that any two consecutive sub-arcs cannot intersect at all. Suppose they did have an extra intersection, then a slight perturbation would not remove the intersection as all intersections are transverse. On the other hand, they meet at infinity or at a cusp, and both of those cases do not count as an intersection, BUT if we slightly perturb the sub-arcs individually then the cusp or asymptote will create an intersection. This would give two intersections for two sub-arcs, which is not possible as normals do not repeat. Hence consecutive sub-arcs do not intersect, which removes $n + \ell + 3$ possible intersections. Therefore our bound becomes

$$\binom{n + \ell + 3}{2} - (n + \ell + 3) + (n + 3) + 1 = \binom{n + \ell + 3}{2} - \ell + 1.$$

For the general bound, simple calculus shows that this is an increasing function of ℓ , the number of cusps. There can be no more than n cusps. Hence we have no more than

$$2n^2 + 4n + 4$$

total connected components in the complement of the contour of the reduced \mathcal{A} -discriminant amoeba, when $\#\mathcal{A} = n + 3$. □

Theorem 3.8. *The $\overline{\nabla}_{\mathcal{A}}(\mathbb{R})$ has no more than*

$$2n^2 + 4n + 4$$

connected components in its complement.

Proof. The actual reduced \mathcal{A} -discriminant variety is also a collection of parametric curves. We claim that this bound still holds on these curves. Anywhere that these intersect either produces an intersection in the contour, or it goes through the origin. In the first case, the intersection is accounted for in the contour case. In the latter case, it is not counted in the intersection count, but it is counted in the line count. That is each time $\overline{\nabla}_{\mathcal{A}}(\mathbb{R})$ crosses an axis, the contour gains an extra curve. Therefore extra parametric intersections across the origin are multiply accounted for in the previous count. Hence we are done. \square

3.4 Extremal Examples and Signed Projective Points

In this section we will present and define visual aids to help understand the structure of the contour from the B matrix. First we look at a simple example to better understand. Let $\mathcal{A} = [0, 1, 2, 3]$. Then we see that

$$B = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

The contour is shown in Figure 3.7.

Now we can view B as a collection of points in $\mathbb{P}\mathbb{R}$. That is, $(2, 1)$, $(-3, -2)$, $(0, 1)$, and $(1, 0)$. Now if we plot the negative of these on a circle, we see that the tendrils of the contour are asymptotic to these directions. This is shown in Figure 3.8.

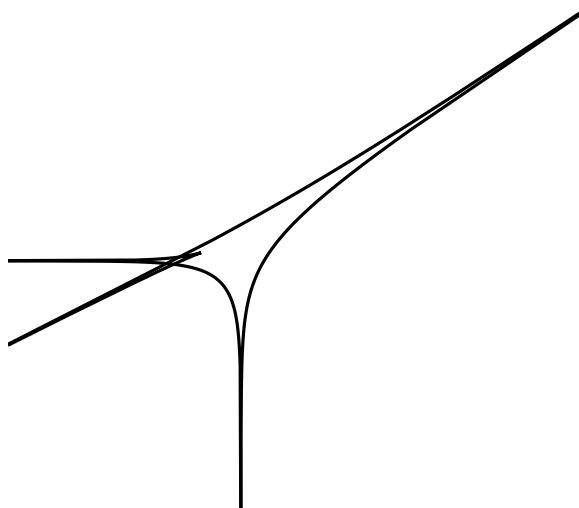


Figure 3.7: Reduced discriminant contour of the cubic.

The fact that the tendrils are asymptotic to the negatives of the rows of the B matrix is a fact of Tropical geometry (see [4]).

We previously had used points on S^1 as our parameter to our map φ . But this new view suggests a slightly modified way to view the domain. Rather than have the point on S^1 represent the parameter for φ , we use the tangent as the parameter (or simply a 90 degree phase shift) as in Figure 3.9. Therefore we can label these undefined points on S^1 and sub-arcs between them still have well-defined image and are smooth. This also lets us easily view the direction of the asymptotes at a glance. Furthermore, we distinguish the points on this new S^1 that correspond to the rows of the B matrix. If the y coordinate is positive we use a circle. If the y coordinate is negative we use a filled in circle. This is shown in Figure 3.9. We call this a collection of *signed points in \mathbb{P}^1* . This sign will visually tell us which direction the asymptote goes. If filled then it goes in the direction portrayed on S^1 . Otherwise

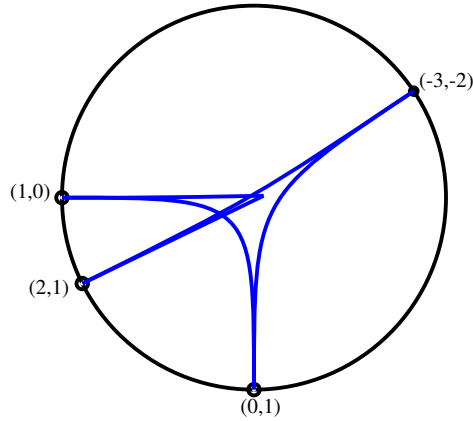


Figure 3.8: The cubic \mathcal{A} -discriminant and its rays.

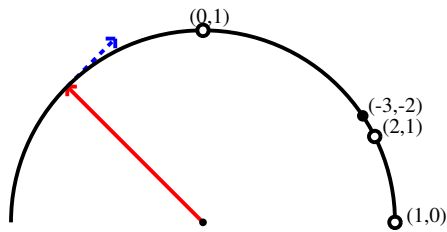


Figure 3.9: The parameter is dashed (blue) and the tangent is solid (red).

the asymptote goes the opposite direction. We will also call two sequential points on the set, *adjacent* points.

Now observe that the image of the section of S^1 between the points $(1,0)$ and $(2,1)$ has a cusp and the corresponding signed points have the same “sign”. This is not a coincidence. That this happens in general is proven in Lemma 3.9. This tool will prove useful in understanding the structure of the \mathcal{A} -discriminant contour.

3.4.1 Many Real Cusps

We begin with a very simple lemma that follows from the discussion above. This uses the observations from the signed point sets to see now to guarantee cusps.

Lemma 3.9. *If two adjacent points on the signed null fan are separated by less than 90 degrees and have the same signs then there is a cusp between their respective asymptotes.*

Proof. Suppose two signed points have the same sign and are separated by less than 90 degrees. As the parameter moves along S^1 clockwise, as it hits an asymptote the directional derivative necessarily heads toward the direction of the asymptote. When it moves past the asymptote the directional derivative goes the opposite direction. This means that as you move along the parameters between two asymptotes of the same sign then the directional derivatives differ by greater than 90 degrees. Therefore by Kapranov's Theorem (Theorem 3.2) it cannot be smooth. Thus there must be a cusp. \square

This gives us a sufficient (it is *not* necessary) condition guaranteeing a cusp between two asymptotes. We can use this condition to build amoebae with contours with many cusps.

Theorem 3.10. *Pick $m, \ell \in \mathbb{Z}$ with $\ell, m > 1$. Then we define $B \in \mathbb{Z}^{2 \times (2\ell+2)}$, by $B_{1,2i} = m^{i-1}$ and $B_{1,2i+1} = -m^{i-1}$ for $i = 1, \dots, \ell$ and $B_{2,2} = B_{2,1} = -\ell$ and $B_{2,j} = 1$ for $j > 2$. The B matrix has the following form:*

$$B = \begin{bmatrix} -1 & 1 & -m & m & -m^2 & m^2 & \cdots & -m^\ell & m^\ell \\ -\ell & -\ell & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}^T.$$

The B defines an amoeba whose contour has $2\ell - 1$ cusps.

Proof. It is easy to see that $(m^i, 1)$ and $(m^{i+1}, 1)$ are adjacent for $i = 1, \dots, \ell - 1$. Similarly $(-m^i, 1)$ and $(-m^{i+1}, 1)$ are adjacent. These also have the same sign and are clearly separated by less than 90 degrees. Hence each pair has a cusp. This gives us $2n - 2$ cusps. Finally, for the same reason, $(-1, -\ell)$ and $(1, -\ell)$ will also have a cusp. This gives us at least $2\ell - 1$ cusps. This B represents a $(2\ell - 1)$ -variate, $(2\ell + 2)$ -nomial. Hence it cannot have more than $2\ell - 1$ cusps. Thus we have exactly $2\ell - 1$ cusps. \square

For completeness and illustrative purposes, we include the null fan and reduced discriminant for $\ell, m = 3, 2$. These are in Figure 3.10.

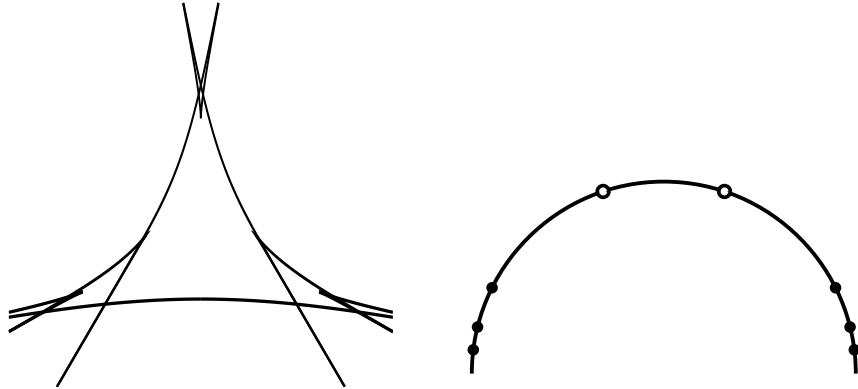


Figure 3.10: Reduced discriminant and signed point set for $(\ell, m) = (3, 2)$.

This construction gives us an odd number of cusps. A slight modification gives us a maximal even number of cusps. If we replace the first two rows of the B matrix by $(0, -2\ell)$ then we get a B matrix in $\mathbb{Z}^{2 \times 2\ell+1}$ that produces $2\ell - 2$ cusps for the same reason as that in the example. It is easy to see that this max is attained here as well. Hence we see that the maximum number of possible cusps can be attained.

3.4.2 Many Positive Roots

While it is easy to produce a B matrix with several (maximally many, even) cusps, it becomes more difficult if one tries to concentrate cusps between two asymptotes. As an example, we look at the discriminant in which the so-called Rusek-Shih system lives [5]. Using the Cayley trick (cf. Section 2.1), we get the reduced discriminant defined by

$$B = \begin{bmatrix} 0 & 1 & -3 & 6 & -6 & 2 \\ -2 & 6 & -6 & 3 & -1 & 0 \end{bmatrix}.$$

The system has the \mathcal{A} -discriminant contour in Figure 3.11.

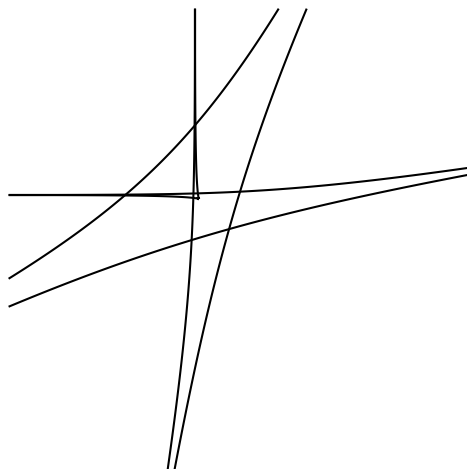


Figure 3.11: Rusek-Shih reduced \mathcal{A} -discriminant contour.

There appears to be a single cusp in the center of the image, but when one zooms into that section we find three cusps making a small chamber (Figure 3.12). The Rusek-Shih system is special because it is a pair of bivariate trinomials with 5 positive roots. This is a rare, extremal occurrence. The contour holds information

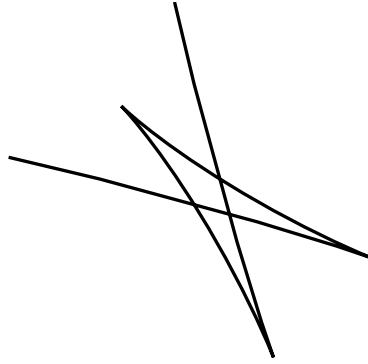


Figure 3.12: Rusek-Shih: special small chamber.

for all systems with the same support, and the special systems with 5 roots in $\mathbb{R}_{\geq 0}^2$ are in this very small chamber and one can easily verify that this chamber is the result of three cusps between two adjacent asymptotes. Furthermore, if one looks at the null vector fan of this B matrix (Figure 3.13) one sees that Lemma 3.9 would not guarantee any cusps.

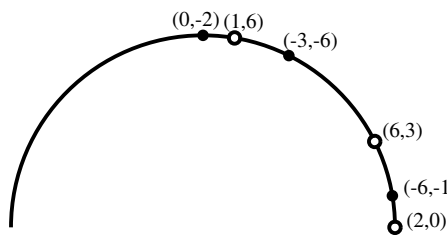


Figure 3.13: Rusek-Shih signed points.

This suggests that the special systems lie inside discriminants with many cusps nearby (between two adjacent asymptotes). The tools presented here quickly produce

systems with lots of nodes, but they are spread out. Nevertheless the tools can be used to *help* find systems with concentrated cusps.

In [5], the Rusek-Shih example was found by attempting to generalize the Haas example found in [8]. They examined systems with supports of the form:

$$\mathcal{A} = \begin{bmatrix} 2d & 0 & 0 & 0 & d & 1 \\ 0 & d & 1 & 2d & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now for $d = 3, \dots, 52$, the \mathcal{A} -discriminant amoeba contours give small chambers, bounded by cusps, and containing examples similar to the Rusek-Shih example. Furthermore, they have B matrices of the form:

$$B = \begin{bmatrix} 0 & -2d & 2d & d-1 & 1 & -d \\ 1-d & -1 & d & 0 & 2d & -2d \end{bmatrix}.$$

They each have signed point sets in $\mathbb{P}(\mathbb{R})$ similar to that found in Figure 3.13 for the Rusek-Shih example. All the cusps are concentrated on the left half of the domain. In affine coordinates this is where $x > 0$. This suggest searching for special B matrices rather than special supports. Using this approach, we searched for B matrices of the form in Figure 3.13. That is, all the asymptotes on the right have alternating signs. We also want the affine derivative to have 3 positive roots, so that the cusps are all on the left arc. In this case it is easy to through out many bad choices by checking the sign changes on the derivative. In the end we found two interesting systems with

5 positive roots. The first one is Equation 3.3.

$$\begin{aligned} x^5 + \frac{32}{23}y^2 - y \\ y^5 + \frac{32}{23}x^2 - x \end{aligned} \tag{3.3}$$

It is interesting because it is degree 5. It is the lowest degree system of two bivariate trinomials known with so many positive roots. The other system we found is Equation 3.4.

$$\begin{aligned} y^5 - \frac{49}{95}xy^3 + x^6 \\ x^5 - \frac{49}{95}yx^3 + y^6 \end{aligned} \tag{3.4}$$

This system is interesting because it only has 16 total solutions in $(\mathbb{C}^\times)^2$. This is easily verified via a mixed volume calculation. This is the system with 5 positive roots with the least known total non-zero solutions. These two systems complement each other nicely. The first has low degree, but 25 total non-zero solutions. The second has few non-zero solutions but slightly higher degree. Both were verified to have 5 positive solutions via Groebner bases and via resultant methods.

3.4.3 Many Real Chambers

Finally, it is easy to generate families that appear to have $O(n^2)$ connected components in their complement. If you look at the signed point set from a B matrix, when two adjacent points have *opposite* sign the image of the arc between them crosses the entire plane. Thus a carefully constructed sign arrangement will produce \mathcal{A} -discriminants where every arc intersects all but two others.

For example, pick $m > 0$ and we let

$$B = \begin{bmatrix} -1 & 1 & -1 & \cdots & (-1)^m & 0 & (-1)^{m+1} & (-1)^{m+2} & \cdots & 1 \\ 1 & -2 & 3 & \cdots & (-1)^{m+1}m & \ell & (-1)^{m+1}m & (-1)^{m+2}(m-1) & \cdots & 1 \end{bmatrix}^T$$

Where $\ell = 2 \sum_{i=1}^m (-1)^i i$, so that the rows add up to zero. The signed point set for $m = 3$ is in Figure 3.14. It is easy to see that any non-adjacent arc should intersect.

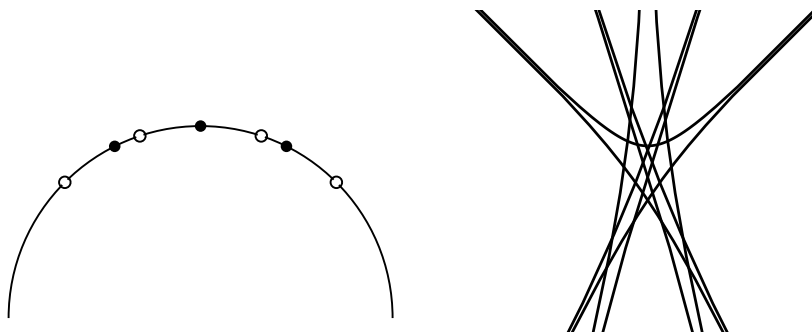


Figure 3.14: Many chambers for $m = 3$.

If there are no degenerate intersections then this family should have $m^2 + m + 1$ total chambers. It appears this is the case, but we are unable to prove it. If so, then this is $O(n^2)$ and the bound is asymptotically tight.

4. TROPICAL GEOMETRY

Let R be a ring of characteristic 0. A norm is a map $\|\cdot\|$ that assigns a non-zero, real “size” to the elements of the ring. Norms satisfy the following criteria for $x, y \in R$:

1. $\|x\| \geq 0$,
2. $\|x\| = 0$ if and only if $x = 0$,
3. $\|xy\| = \|x\| \|y\|$, and
4. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The last requirement is called the triangle inequality. If the ring satisfies the stronger requirement,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\},$$

then it is a non-Archimedean ring. This stronger requirement is called the ultrametric inequality. Tropical geometry focuses on these kinds of spaces. Furthermore we assume that our rings have an equivalent map called a valuation. A valuation is a map $\text{val} : \mathbb{F} \rightarrow \mathbb{R} \cup \{\infty\}$ that obeys similar criteria to the norm for $x, y \in R$:

1. $\text{val}(x) = \infty$ if and only if $x = 0$,
2. $\text{val}(xy) = \text{val}(x) + \text{val}(y)$, and
3. $\text{val}(x + y) \geq \min\{\text{val}(x), \text{val}(y)\}$.

The relationship between $\|\cdot\|$ and val is analogous to the relationship between $|\cdot|$ and \log in the complex case. There is one more important classical fact about the valuation (the equivalent statement also holds for $\|\cdot\|$) that we will use.

Lemma 4.1. *If R is a normed non-Archimedean ring and $x, y \in R$ then*

$$\text{val}(x + y) = \min\{\text{val}(x), \text{val}(y)\}$$

when $\text{val}(x) \neq \text{val}(y)$.

Proof. Suppose that $\text{val}(x) < \text{val}(y)$. Then $\text{val}(x) = \min\{\text{val}(x), \text{val}(y)\}$ and we have $\text{val}(x + y) \geq \text{val}(x)$ by the ultrametric inequality. Furthermore the ultrametric inequality tells us

$$\text{val}(x) = \text{val}((x + y) - y) \geq \min\{\text{val}(x - y), \text{val}(y)\}. \quad (4.1)$$

Now if $\min\{\text{val}(x - y), \text{val}(y)\} = \text{val}(y)$ then $\text{val}(x) \geq \text{val}(y)$ which is a contradiction. Therefore $\min\{\text{val}(x - y), \text{val}(y)\} = \text{val}(x - y)$, and equation 4.1 tells us that $\text{val}(x) \geq \text{val}(x + y)$. Thus we have both inequalities so $\text{val}(x) = \text{val}(x + y)$ as desired. \square

A valuation and a norm are equivalent. That is, given one you can create the other. For example, pick a positive integer $n > 0$, then you can define $|x| = n^{-\text{val}(x)}$ for all $x \in R$. Similarly a logarithm can give a valuation from a norm. We now move on to a discussion of two well known non-Archimedean fields.

4.1 Puisseux Series

We begin with the Puisseux series. Now a complex polynomial of degree d is of the form

$$\sum_{i=0}^d c_i t^i,$$

for $c_i \in \mathbb{C}$ and $c_d \neq 0$. Now we can define an ultrametric valuation on $\mathbb{C}[t]$. Suppose $f = \sum_{i=0}^d c_i t^i$ and then define $\text{val}: \mathbb{C}[t] \rightarrow \mathbb{Z}$ to be $\text{val}(f) = i$ where i is the smallest i such that $a_i \neq 0$, or $\text{val}(f) = \infty$ if $f = 0$. Using any norm derived from this

valuation, the completion of $\mathbb{C}[t]$ is the set of formal power series. That is, elements of the form

$$\sum_{i=\ell}^{\infty} c_i t^i,$$

for $\ell \in \mathbb{N}$ and $c_i \in \mathbb{C}$. This forms an integral domain. The fraction field is called the field of formal Laurent power series, $\mathbb{C}((t))$. Elements here are of the form

$$\sum_{i=\ell}^{\infty} c_i t^i,$$

where $\ell \in \mathbb{Z}$. Finally, we define the field of Puiseux series to be

$$\mathbb{C}\{\{t\}\} = \bigcup_{n \in \mathbb{Z}^+} \mathbb{C}((t^{1/n})).$$

Puiseux's theorem says that the algebraic closure of $\mathbb{C}((t))$ is $\mathbb{C}\{\{t\}\}$. All these fields have the property that an element is either 0 or has a non-zero term of minimal degree. In all these cases, we define our valuation to be that degree, or infinity for the zero element. It is not hard to see that this valuation satisfies the required properties.

When we choose $\mathbb{K} = \mathbb{C}\{\{t\}\}$, the \mathcal{A} -discriminant amoeba (we use val in place of \log) is a special object called the tropical \mathcal{A} -discriminant. The tropical \mathcal{A} -discriminant gives information about the complex \mathcal{A} -discriminant amoeba. For example, when $\mathcal{A} = \{0, 1, 2, 3\}$ the tropical \mathcal{A} -discriminant and complex \mathcal{A} -discriminant amoeba are shown in Figure 4.1. As one can see, in the case $k = 3$ the tropical \mathcal{A} -discriminant gives the directions of the asymptotes of the complex analog. The structure, properties, and computational aspects of it have been studied extensively [4, 13]. In the next chapter we deal with a more general collection of non-Archimedean fields, of which the field of Puiseux series is a special case.

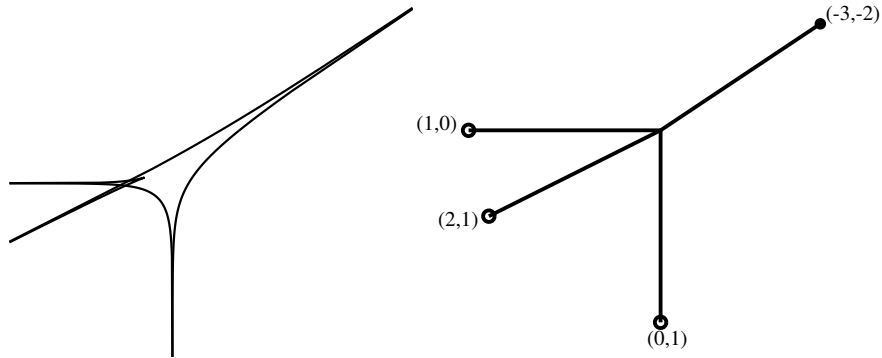


Figure 4.1: Complex \mathcal{A} -discriminant (left) and tropical \mathcal{A} -discriminant (right).

4.2 The p -adic Numbers

The construction of the p -adic numbers is very similar to the construction of the field of Puiseux series. We begin by selecting a prime number p . Now any natural number can be written in base p expansion in the form

$$\sum_{i=0}^m a_i p^i,$$

where the a_i are integers in $\{0, 1, \dots, p-1\}$. For example, the base 3 expansion of 20 is $20 = 2 \cdot 3^0 + 2 \cdot 3^2$.

For any $n \in \mathbb{N}$ we let $\text{val}_p: \mathbb{N} \rightarrow \mathbb{N}$ be the smallest i such that a_i is non-zero in the base p expansion of n . That is, $\text{val}_p(n)$ the largest power of p that divides n . Furthermore, we define $|\cdot|_p: \mathbb{N} \rightarrow \mathbb{N}$ to be $|n|_p = p^{-\text{val}_p(n)}$. For example, $\text{val}_3(45) = 2$ and $|45|_3 = 1/9$, as 9 divides 45, but 27 does not. We call $\text{val}_p(n)$ the *valuation* of n , and $|\cdot|_p$ will be extended to a norm on our p -adic fields.

Now we extend \mathbb{N} to formal infinite sums to form a new commutative ring, \mathbb{Z}_p .

That is,

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i \mid a_i \in \{0, \dots, p-1\} \right\}.$$

It is not hard to see that \mathbb{Z}_p is a commutative ring. Furthermore since $\mathbb{N} \subset \mathbb{Z}_p$ then $\mathbb{Z} \subset \mathbb{Z}_p$. Furthermore $\text{val}(\mathbb{Z}^\times) = \mathbb{Z}^+$. Then \mathbb{Q}_p is the fraction field of \mathbb{Z}_p . Elements are of the form

$$\mathbb{Q}_p = \left\{ \sum_{i=k}^{\infty} a_i p^i \mid k \in \mathbb{Z}, a_i \in \{0, \dots, p-1\} \right\}.$$

The set \mathbb{Q}_p is complete, but the algebraic closure is not. We call the completion of the algebraic closure of \mathbb{Q}_p , \mathbb{C}_p .

This is the set that will be used to form $\overline{\nabla}_{\mathcal{A}}$. It is less well-known how $\overline{\nabla}_{\mathcal{A}}$ can be used. It is worth noting that the current author, along with Avendaño, Ibrahim, and Rojas [1] did use it for computational results.

5. TROPICAL AMOEBA STRUCTURE

We can construct the non-Archimedean \mathcal{A} -discriminant amoeba for the family of polynomials with support $A = \{\alpha_1, \dots, \alpha_{n+k}\} \subset \mathbb{Z}^n$, in the same way as its complex counterpart. We will let $\hat{A} \in \mathbb{Z}^{(n+k) \times (n+1)}$ be the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n+k} \end{bmatrix}.$$

As in the introduction, we let $B \in \mathbb{Z}^{(n+k) \times (k-1)}$ be the matrix whose columns form a basis of the right null-space of \hat{A} . We will let $\beta_1, \dots, \beta_{n+k}$ be the collection of row vectors of B . Then the map, $\varphi : \mathbb{P}\mathbb{K}^{k-2} \rightarrow \mathbb{R}^{k-1}$ given by

$$\lambda \mapsto B^T \text{val}(B \cdot \lambda)$$

is a parametric form of the reduced non-Archimedean \mathcal{A} -discriminant amoeba [7]. Remember that the map val has real image, hence the outer multiplication is real matrix multiplication.

In many ways this is a simple parametric representation of the discriminant amoeba. The form is compact and the image is piecewise linear. Nevertheless, it is not entirely obvious how to get an accurate plot of φ . First of all, for many non-Archimedean fields (in particular, $\mathbb{C}\{\{t\}\}$ or \mathbb{C}_p) the valuation map is disconnected. To remedy this problem we will focus on fields where the image of the valuation map is a dense subset of \mathbb{R} containing \mathbb{Q} . Then we will take the closure of φ . Secondly, although the image is piecewise linear, it is not intuitively obvious which subsets of parameters will be sufficient to give us all the pieces. Our goal is to provide a

simplification that will give us all the linear pieces but still be an efficiently small subset of possible parameters.

We will define a couple of related maps and simplifications that will be instrumental in this work. Let $\mathrm{GL}_{\mathbb{Q}}(k-1)$ be the group of invertible matrices in $\mathbb{Q}^{(k-1) \times (k-1)}$. Given any $D \in \mathrm{GL}_{\mathbb{Q}}(k-1)$, we can define the related parametric map $\varphi_D(\lambda) = B^T \mathrm{val}(BD \cdot \lambda)$. This new parametric map, φ_D , has the same image as φ because the matrix D simply acts as a change of coordinates of a parametric map.

We will “tropicalize” these related maps. First we define a map

$$\mathbb{R}^{\ell \times m} \times (\mathbb{R}, \min, +)^m \rightarrow \mathbb{R}^{\ell},$$

$$M \cdot r = \left(\min_{1 \leq j \leq m} \{M_{1,j} + r_j\}, \min_{1 \leq j \leq m} \{M_{2,j} + r_j\}, \dots, \min_{1 \leq j \leq m} \{M_{\ell,j} + r_j\} \right)$$

This is simply adding the vector r to each row of M and then take the coordinate-wise minimum of the columns. This gives us a way to “tropicalize” our map φ_D . We have

$$\hat{\varphi}_D(r): (\mathbb{R}, \min, +)^{k-1} \rightarrow \mathbb{R}^{k-1}, \quad \hat{\varphi}_D(r) = B^T (\mathrm{val}(BD) \cdot r).$$

Now $\hat{\varphi}_D$ is easy to deal with and easily understood.

This will be a key tool in this chapter and in representing φ in a simple fashion. For now we will just state but in the next section we will fully prove the statements overviewed here. Pick $I = \{i_1, \dots, i_{k-1}\}$ with $1 \leq i_1 < \dots < i_{k-1} = n+k$ and such that the zero set of $\{\langle \beta_{i_1}, \lambda \rangle, \dots, \langle \beta_{i_{k-1}}, \lambda \rangle\}$ consists of a unique point (i.e. the β_{i_j} are linearly independent) in \mathbb{PK} . Now let $D \in \mathrm{GL}_{\mathbb{Q}}(k-1)$ be the inverse of the matrix $[\beta_{i_1}, \dots, \beta_{i_{k-1}}]$, and let T be the collection of all such D . In the next subsection, we will prove (Theorem 5.4) that

Theorem 5.4.

$$\overline{\varphi(\mathbb{P}\mathbb{K}^{k-2})} = \bigcup_{D \in T} \hat{\varphi}_D(\mathbb{R}^{k-1}).$$

This theorem is instrumental, because it gives us a way to replace φ with a collection of piecewise linear real maps. These kinds of maps are simple and easy to deal with.

5.1 Reducing Non-Archimedean Amoebae to Parametric Functions

In this section we will prove the main reduction theorem. This theorem will give a representation of the \mathcal{A} -discriminant amoeba as a collection of piecewise linear real maps.

Before we prove anything we will give an overview of the proofs in this section. For simplicity sake, we pick a map $\omega: \mathbb{K} \rightarrow \mathbb{Q}$ such that $\omega(\ell)$ is **any** element with $\text{val}(\omega(\ell)) = \ell$. This can be done, as we assume $\text{val}(\mathbb{K}^\times) \supset \mathbb{Q}$. In later sections it is important that $\omega(\ell)$ can simply be any element with the proper valuation, independent of everything else. We begin with Lemma 5.1. This will show that we can approximate $\varphi(\lambda)$ by $\varphi_D(\lambda')$ where

$$D = [e_1, \dots, e_{j-1}, \beta_{i_j}, e_{j+1}, \dots, e_{k-1}]^{-1} \in \text{GL}_{\mathbb{Q}}(k-1),$$

for some j and $\lambda' = [\lambda_1 : \dots : \lambda_{j-1} : \omega(\ell) : \lambda_{j+1} : \dots : \lambda_{k-1}]$, for some $\ell \in \mathbb{Q}$.

Next, Theorem 5.3 will apply the lemma iteratively and arrive at a $D \in \text{GL}_{\mathbb{Q}}(k-1)$ that is the inverse of a submatrix of B , and a collection of $\ell_1, \dots, \ell_{k-1} \in \mathbb{Q}$ such that $\varphi(\lambda)$ is approximated by

$$\varphi(BD[\omega(\ell_1) : \dots : \omega(\ell_{k-1})]).$$

Then by choosing distinct enough rational numbers near the ℓ_i we show that it is approximated by

$$\hat{\varphi}_D(\ell_1, \dots, \ell_{k-1}).$$

Finally, Theorem 5.4 will prove our final result.

Lemma 5.1. *Let \mathbb{K} be an algebraically closed complete valuation field with $\mathbb{Q} \subseteq \text{val}(\mathbb{K}^\times) \subseteq \mathbb{R}$. Pick any $\lambda = [\lambda_1 : \dots : \lambda_{k-1}] \in \mathbb{P}\mathbb{K}^{k-2}$ where $\varphi(\lambda)$ is well-defined. Let any $\varepsilon > 0$ and $1 \leq j \leq k-1$. If $C = [\gamma_1, \dots, \gamma_{n+k}]$ is column equivalent to B , then there are a $D_j \in \text{GL}_{k-1}(\mathbb{Q})$ and $\ell \in \mathbb{Q}$ and $1 \leq t \leq k-1$ such that*

$$\|B^T \text{val}(C\lambda) - B^T \text{val}(CD_j\lambda')\| < \varepsilon$$

for $\lambda' = [\lambda_1 : \dots : \lambda_{j-1} : \omega(\ell) : \lambda_{j+1} : \dots : \lambda_{k-1}]$ and such that $e_i D_j = e_i$ for $i \neq j$ and $\gamma_t D_j = e_j$.

Proof. Without loss of generality we may assume that $j = 1$. Let $M = \sum_{i,s} |\gamma_{i,s}|$. Pick $\lambda \in \mathbb{P}\mathbb{K}^{k-2}$ with $\varphi(\lambda)$ well-defined and choose any $0 < \varepsilon < 1$. For any $z \in \mathbb{K}$, we define $\lambda_z = (\lambda_1 + z, \lambda_2, \dots, \lambda_{k-1})$. Hence

$$\gamma_i \cdot \lambda_z = \gamma_i \cdot \lambda + \gamma_{i,1}z.$$

Now for every i with $\gamma_{i,1} \neq 0$, if we let $n_i = \text{val}(\gamma_i \cdot \lambda) - \text{val}(\gamma_{i,1})$ then by the ultrametric inequality we have

$$\begin{aligned} \text{val}(\gamma_i \cdot \lambda_z) &\geq \min\{\text{val}(\gamma_i \cdot \lambda), \text{val}(z) + \text{val}(\gamma_{i,1})\} \\ &= \min\{n_i + \text{val}(\gamma_{i,1}), \text{val}(z) + \text{val}(\gamma_{i,1})\} \\ &= \min\{n_i, \text{val}(z)\} + \gamma_{i,1} \end{aligned}$$

with equality when $n_i \neq \text{val}(z)$. Now at least one such n_i should exist, otherwise $\gamma_{i,1} = 0$ for all i and our matrix \hat{A} does not have full rank. Let N be maximal among the n_i and suppose z is chosen so that $0 < N - \text{val}(z) = \varepsilon' < \frac{\varepsilon}{M}$. We claim that

$$\|B^T \text{val}(C\lambda) - B^T \text{val}(C\lambda_z)\| < \varepsilon.$$

Indeed, we have already seen that for any row, γ_i , of C we have $\text{val}(\gamma_i \cdot \lambda) - \text{val}(\gamma_i \cdot \lambda_z) = 0$ when $n_i \neq N$, since $n_i \leq N$. Also $\text{val}(\gamma_i \cdot \lambda) - \text{val}(\gamma_i \cdot \lambda_z) = \varepsilon'$ when $n_i = N$. In either case, as N is maximal, $|\text{val}(\gamma_i \cdot \lambda) - \text{val}(\gamma_i \cdot \lambda_z)| \leq \varepsilon'$. Hence

$$\|B^T \text{val}(C\lambda) - B^T \text{val}(C\lambda_z)\| \leq M\varepsilon' < \varepsilon$$

as desired. Thus if we pick an appropriate D matrix that acts the same as adding such a z to λ_1 then we have our claim.

First we will construct this matrix D and then choose an ℓ . Pick any t such that $n_t = N$. We will pick $D \in \text{GL}_{k-1}(\mathbb{Q})$ to be

$$D = \begin{bmatrix} \frac{1}{\gamma_{1,1}} & -\frac{\gamma_{1,2}}{\gamma_{1,1}} & -\frac{\gamma_{1,3}}{\gamma_{1,1}} & \cdots & -\frac{\gamma_{1,k-2}}{\gamma_{1,1}} & -\frac{\gamma_{1,k-1}}{\gamma_{1,1}} \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Now D is constructed specifically such that $\gamma_t D = e_1$ and $e_i D = e_i$ for $i > 1$, as desired. We have shown that D satisfies the first half of the requirements.

For the other requirements, we can represent D as

$$\left[\frac{e_1}{\gamma_{1,1}} + e_1 - \frac{\gamma_1}{\gamma_{1,1}}, e_2, e_3, \dots, e_{k-1} \right]^T.$$

Now replacing λ_1 with $\omega(\ell)$, we know that $\gamma_t \cdot [\omega(\ell), \lambda_2, \dots, \lambda_{k-1}] = \gamma_t \cdot \lambda - \lambda_1 \gamma_{t,1} + \omega(\ell) \gamma_{t,1}$ and so

$$\begin{aligned} \begin{bmatrix} \frac{e_1}{\gamma_{t,1}} + e_1 - \frac{\gamma_t}{\gamma_{t,1}} \\ e_2 \\ e_3 \\ \vdots \\ e_{k-1} \end{bmatrix}^T \begin{bmatrix} \omega(\ell) \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{k-1} \end{bmatrix} &= \begin{bmatrix} \frac{\omega(\ell)}{\gamma_{t,1}} + \omega(\ell) - \frac{1}{\gamma_{t,1}}(\gamma_t \cdot \lambda - \lambda_1 \gamma_{t,1} + \omega(\ell) \gamma_{t,1}) \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 + \frac{1}{\gamma_{t,1}}(\omega(\ell) - \gamma_t \cdot \lambda) \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{k-1} \end{bmatrix}. \end{aligned}$$

Hence if $\ell < N = \text{val}(\gamma_t \cdot \lambda) - \text{val}(\gamma_{t,1})$ then $z = \frac{1}{\gamma_{t,1}}(\omega(\ell) - \gamma_t \cdot \lambda)$ satisfies our requirements. Specifically $e_i D = e_i$ for $i > 1$ and $\gamma_t D = e_1$ furthermore if $0 < N - \ell < \epsilon'$ then $\|B^T \text{val}(C\lambda) - B^T \text{val}(CD\lambda')\| < \epsilon$, where $\lambda' = (\omega(\ell), \lambda_2, \dots, \lambda_{k-1})$. \square

Definition 5.2. Let $T \subset \text{GL}_{k-1}(\mathbb{Q})$ be the set

$$T = \left\{ D \in \text{GL}_{k-1}(\mathbb{Q}) \mid \beta_{i_j} D = e_j, \text{ for some } 1 \leq i_1 < \dots < i_{k-1} = n + k \right\}.$$

In words, T is the collection of inverses of invertible submatrices of B containing β_{n+k} fixed as the last row. It is clear that an element of T is uniquely chosen by

i_1, \dots, i_{k-2} . Consequently, there are at most $\binom{n+k-1}{k-2}$ elements in T .

Theorem 5.3. *Let \mathbb{K} be an algebraically closed complete valuation field with $\mathbb{Q} \subseteq \text{val}(\mathbb{K}^\times) \subseteq \mathbb{R}$. Pick any $\lambda = [\lambda_1 : \dots : \lambda_{k-1}] \in \mathbb{P}\mathbb{K}^{k-2}$ where $\varphi(\lambda)$ is well-defined. For any $\varepsilon > 0$ there is $D \in T$ and $\ell_1, \dots, \ell_{k-1} \in \mathbb{Q}$ with $\ell_i - \ell_j \notin \mathbb{Z}$ for $i \neq j$ such that*

$$\|B^T \text{val}(B\lambda) - B^T \text{val}(BD(\ell_1, \dots, \ell_{k-1}))\| < \varepsilon.$$

Proof. We will iteratively use Lemma 5.1. Let $\lambda = (\lambda_1, \dots, \lambda_{k-1})$. Pick $0 < \varepsilon < 1$. We begin by applying the lemma to $B^T \text{val}(B\lambda)$ with $j = 1$ to get a D_1 and i_1 and an ℓ_1 such that

$$\|B^T \text{val}(B\lambda) - B^T \text{val}(BD_1\mu_1)\| < \frac{\varepsilon}{k-1},$$

where $\mu_1 = (\omega(\ell_1), \lambda_2, \dots, \lambda_{k-1})$ and $B_{i_1}D_1 = e_1$. We let $B_1 = BD_1$ and apply the lemma with $j = 2$ to get D_2 , i_2 , and ℓ_2 such that

$$\|B^T \text{val}(B_1\lambda) - B^T \text{val}(BD_2\mu_2)\| < \frac{\varepsilon}{k-1},$$

where $\mu_2 = (\omega(\ell_1), \omega(\ell_2), \lambda_3, \dots, \lambda_{k-1})$ and $\beta D_2 = e_2$, where β is the second row of B_1 . It is important to note that $i_1 \neq i_2$ since the i_1 row of B_1 is e_1 . We continue this iteratively and we have i_1, \dots, i_{k-1} and $\ell_1, \dots, \ell_{k-1}$ and $B_j = BD_1D_2 \cdots D_j$ for $j = 1, \dots, k-1$. Now if i_1, \dots, i_{k-1} are out of order then we can construct another D_k that acts to rearrange them. By the assumptions in 5.1, we see that if β_{n+k} was chosen then $D_1 \cdots D_k$ will be the desired D matrix. We claim that at some point β_{n+k} could have been chosen. Suppose we reach the last step and β_{n+k} still could not have been chosen. Let $\hat{\lambda} = (\omega(\ell_1), \omega(\ell_2), \dots, \omega(\ell_{k-2}), \lambda_{k-1})$. Now at each step j is chosen indiscriminantly among the i 's where N_i is maximal, and for any i that **could not** have yet been chosen we have $\text{val}(\beta_i \hat{\lambda}) \in \mathbb{Z}$. Hence

$\text{val}(\beta_i \hat{\lambda}) = \text{val}(\beta_{i,k-2}) + \text{val}(\lambda_{k-1})$. Therefore $n_i = \text{val}(\lambda_{k-1})$. This is the case for all i not yet chosen. Hence it is the case for $n+k$ and we can choose $n+k$ as one of our i_j . \square

Now Theorem 5.4 will follow quite easily:

Theorem 5.4. *Let $T \subset \text{GL}_{k-1}(\mathbb{Q})$ be as defined in definition 5.2. Then*

$$\overline{\varphi(\mathbb{K}^{k-1})} = \bigcup_{D \in T} \hat{\varphi}_D(\mathbb{R}^{k-1}).$$

Proof. We first make a simple observation. Suppose that $D \in T$ and pick $\ell_1, \dots, \ell_{k-1} \in \mathbb{Q}$ such that $\ell_i - \ell_j \notin \mathbb{Z}$ for $i \neq j$. We claim that $\varphi_D(\omega(\ell_1), \dots, \omega(\ell_{k-1})) = \hat{\varphi}_D(\ell_1, \dots, \ell_{k-1})$. Let $(b'_{i,j}) = BD$. Then we have

$$\hat{\varphi}_D(\ell_1, \dots, \ell_{k-1}) = B^T \begin{bmatrix} \min\{\text{val}(b_{1,j}) + \ell_j\} \\ \vdots \\ \min\{\text{val}(b_{n+k,j}) + \ell_j\} \end{bmatrix}.$$

Now when looking at φ_D we need to make two quick observations. First, since $\ell_i - \ell_j \notin \mathbb{Z}$ for $i \neq j$ and $b_{i,j} \in \mathbb{Z}$ then for each m , there is a unique minimal element among $\{\text{val}(b_{m,j}) + \ell_j \mid j \in \{1, \dots, k-1\}\}$. Secondly, for each m , we have

$$\text{val}(b_{m,1}\omega(\ell_1) + \dots + b_{m,k-1}\omega(\ell_{k-1})) = \min_j \{\text{val}(b_{m,j}) + \ell_j\},$$

since the valuation of each component in the sum is distinct and that turns the ultrametric inequality into an equality in normed vector spaces by Lemma 4.1. Therefore

we have

$$\begin{aligned} \varphi_D(\omega(\ell_1), \dots, \omega(\ell_{k-1})) &= B^T \begin{bmatrix} \text{val}(b_{1,1}\omega(\ell_1) + \dots + b_{1,k-1}\omega(\ell_{k-1})) \\ \vdots \\ \text{val}(b_{n+k,1}\omega(\ell_1) + \dots + b_{n+k,k-1}\omega(\ell_{k-1})) \end{bmatrix} \\ &= B^T \begin{bmatrix} \min\{\text{val}(b_{1,j}) + \ell_j\} \\ \vdots \\ \min\{\text{val}(b_{n+k,j}) + \ell_j\} \end{bmatrix} = \hat{\varphi}_D(\ell_1, \dots, \ell_{k-1}). \end{aligned}$$

Therefore we have our claim.

Now we prove that $\overline{\varphi(\mathbb{P}\mathbb{K}^{k-2})} \subset \bigcup \hat{\varphi}_D(\mathbb{R}^{k-1})$. Theorem 5.3 says that for any $\varepsilon > 0$ and $\lambda \in \mathbb{P}\mathbb{K}^{k-2}$ there are a $D \in T$ and an $\ell \in \mathbb{Q}^{k-1}$ with $\ell_i - \ell_j \notin \mathbb{Z}$ for $i \neq j$ such that

$$|\varphi_D(\mu) - \varphi(\lambda)| < \varepsilon,$$

for $\mu = (\omega(\ell_1), \omega(\ell_2), \dots, \omega(\ell_{k-1}))$. Now by the claim we have $\varphi_D(\mu) = \hat{\varphi}_D(\ell)$. Thus since the $\hat{\varphi}_D$ are closed maps and there are finitely many of them then $\overline{\varphi(\mathbb{P}\mathbb{K}^{k-2})} \subset \bigcup \varphi_D(\mathbb{R}^{k-2})$.

Next we prove the other inclusion. It suffices to show that $\varphi(\mathbb{P}\mathbb{K}^{k-2})$ is dense in $\hat{\varphi}_D(\mathbb{R}^{k-1})$. Pick any $D \in T$ and $\ell_1, \dots, \ell_{k-1} \in \mathbb{Q}$. We may assume that $\ell_i - \ell_j \notin \mathbb{Z}$ for $i \neq j$, as the collection of all such $\ell_1, \dots, \ell_{k-1}$ is still dense in \mathbb{R}^{k-1} . Now $\varphi_D(\mathbb{P}\mathbb{K}^{k-2}) = \varphi(\mathbb{P}\mathbb{K}^{k-2})$ since the map is parametric. So we will show that $\varphi_D(\mathbb{P}\mathbb{K}^{k-2})$ is dense in $\hat{\varphi}_D(\mathbb{R}^{k-1})$. But the claim at the beginning of this theorem tells us that

$$\varphi_D(\omega(\ell_1), \dots, \omega(\ell_{k-1})) = \hat{\varphi}_D(\ell_1, \dots, \ell_{k-1}).$$

Therefore the image of φ_D (and φ) is dense in the image of $\hat{\varphi}_D$. Hence

$\bigcup_{D \in T} \hat{\varphi}_D(\mathbb{R}^{k-1}) \subset \overline{\varphi(\mathbb{K}^{k-1})}$ and we have equality. □

5.2 Amoeba Complement Components

We want to bound the number of components in the complement of the closure of tropical discriminant amoeba. In light of our previous results, particularly theorem 5.4, this is not such a difficult task. For a given $D \in T$, we will cover the image of $\hat{\varphi}_D$ with hyperplanes. To do this, we break up the domain into polyhedra such that the image of a given polyhedron is linear. Then, in the range, we replace each of these linear pieces with a full hyperplane and count the number of connected components in the complement of this extension of the image. An example is shown in Figure 5.1.

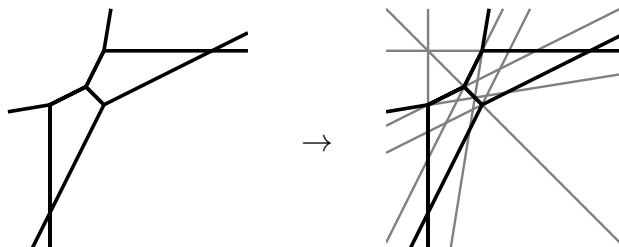


Figure 5.1: Cover the image of φ with hyperplanes.

Before we begin, we will state a bound on the number of complement components of a hyperplane arrangement. The earliest reference we can find is Buck from 1943 [3].

Theorem 5.5. *A collection of ℓ hyperplanes in \mathbb{R}^m has no more than*

$$H(\ell, m) := \binom{\ell}{m} + \binom{\ell}{m-1} + \cdots + \binom{\ell}{1} + \binom{\ell}{0},$$

connected components in its complement.

We will use this theorem repeatedly in proving our results. We now proceed with the first step in our proof bounding the number of complement components of the reduced \mathcal{A} -discriminant. This theorem gives a bound on the maximum number of hyperplanes needed to cover the image of $\hat{\varphi}_D$.

Theorem 5.6. *Let $D \in T$. The domain, \mathbb{R}^{k-2} , of $\hat{\varphi}_D$ can be broken into $H\left(\binom{k-1}{2}(n+k), k-2\right)$ or fewer polyhedra where $\hat{\varphi}_D$ is linear or constant. In particular the image of $\hat{\varphi}_D$ can be covered by that many real hyperplanes.*

Proof. Pick $D \in T$ and let $(\delta_{i,j}) = \text{val}(BD)$ be the component-wise valuation of DB . Now $\hat{\varphi}_D$ has the following form

$$\hat{\varphi}_D(r) = \sum_{i=1}^{n+k} \beta_i \min\{\delta_{i,1} + r_1, \delta_{i,2} + r_2, \dots, \delta_{i,k-2} + r_{k-2}, \delta_{i,k-1}\}.$$

For a given $i \in \{1, \dots, n+k\}$,

$$\min\{\delta_{i,1} + r_1, \delta_{i,2} + r_2, \dots, \delta_{i,k-2} + r_{k-2}, \delta_{i,k-1}\},$$

is piece-wise linear. It is linear where the derivative is well-defined. The only places the derivative can be undefined is where $\delta_{i,j} + r_j = \delta_{i,m} + r_m$ or where $\delta_{i,j} + r_j = \delta_{i,k-1}$. This gives us at most $\binom{k-1}{2}$ hyperplanes (see Figure 5.2) in the *domain* where the derivative is undefined.

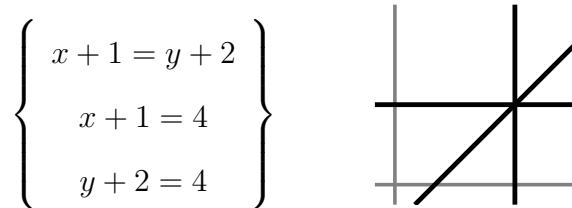


Figure 5.2: $\min\{x + 1, y + 2, 4\}$ is linear outside the lines.

Again, $\hat{\varphi}_D$ is a vector linear combination of $n + k$ of these. Hence the derivative is undefined on $(n + k) \binom{k-1}{2}$ hyperplanes. Finally since we are working in the **domain** of $\hat{\varphi}_D$, which is \mathbb{R}^{k-2} , such an arrangement has at most

$$H\left(\binom{k-1}{2}(n+k), k-2\right)$$

connected components in its complement. These components have linear (or constant) image. Hence $\hat{\varphi}_D$ can be covered by that many hyperplanes. \square

This gives us a number of planes that can cover the image of any particular $\hat{\varphi}_D$. Thus we just have to put these planes together for all D and we will get a bound for the general φ .

Corollary 5.7. *The complement of the closure of the image of φ has no more than*

$$H\left(\binom{n+k-1}{k-2} H\left(\binom{k-1}{2}(n+k), k-2\right), k-1\right)$$

connected components. In particular for fixed k this is $O(n^{2(k-1)(k-2)})$.

Proof. For a given D , the previous theorem tells us that we can replace the image of $\hat{\varphi}_D$ with $H\left(\binom{k-1}{2}(n+k), k-2\right)$ hyperplanes. Now D is uniquely defined for the

chosen $1 \leq i_1 < \cdots < i_{k-2} \leq n+k-1$. Hence there are at most $\binom{n+k-1}{k-2}$ distinct D . Therefore we can cover φ with

$$\binom{n+k-1}{k-2} H\left(\binom{k-1}{2}(n+k), k-2\right)$$

hyperplanes. Since the image is contained in \mathbb{R}^{k-1} then we have no more than

$$H\left(\binom{n+k-1}{k-2} H\left(\binom{k-1}{2}(n+k), k-2\right), k-1\right)$$

connected components in the complement of the closure of the image of φ . \square

In the next section, we will discuss a bound specifically for the case $k = 3$. We will explicitly state the bound, but for now it suffices to say that it is $O(n^2)$. On the other hand, this more general bound is only $O(n^4)$ when $k = 3$. This suggests that this general bound is not even asymptotically tight for any fixed k . But it is still worthwhile as it is the first bound on more general non-Archimedean \mathcal{A} -discriminant amoebae.

5.3 A Special Case

In the previous section we proved a general bound on the number of connected components in the complement of the non-Archimedean \mathcal{A} -discriminant amoeba for $k \geq 3$. When $k = 3$ this general bound is $O(n^4)$. When $k = 3$, there is enough overlap between the images of the various $\hat{\varphi}_D$ to reduce the bound to $O(n^2)$. Example 5.8 illustrates this. We will restrict ourselves to the p -adic case, but the methods can be extended to more general non-Archimedean fields.

Example 5.8. Suppose that

$$\mathcal{A} = \begin{bmatrix} 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $\hat{\mathcal{A}}$ is a matrix in $\mathbb{Z}^{4 \times 6}$ so B is in $\mathbb{Z}^{2 \times 6}$. Hence the Theorem 5.4 tells us that the image of ϕ is the union of 5 different $\hat{\varphi}_D$. We plot the $\hat{\varphi}_D$ iteratively on top of each other in Figure 5.3. In this example, after the first $\hat{\varphi}_D$, each consecutive

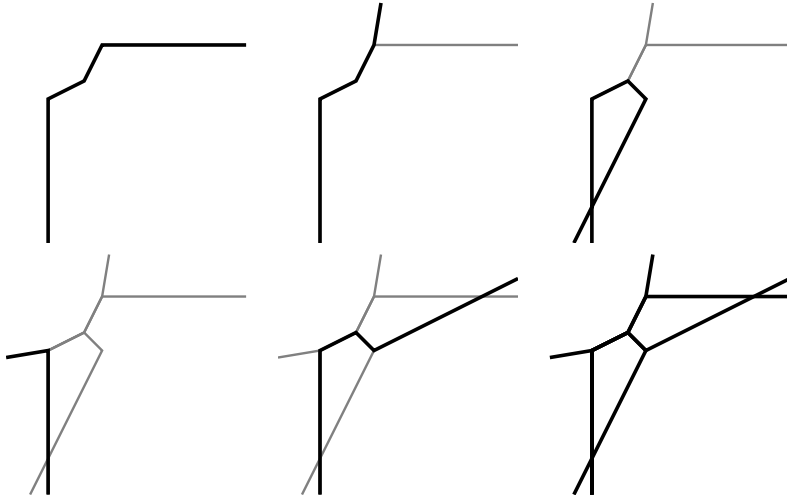


Figure 5.3: Each $\hat{\varphi}_D$ drawn iteratively over each other, followed by $\overline{\nabla}_{\mathcal{A}}$.

$\hat{\varphi}_D$ usually only introduces a new ray. In only one instance here does it introduce a segment and a ray.

The situation found in Example 5.8 is not an isolated case. In this example we will prove that this happens and explain why. Before we begin, we will make a few observations about the $\hat{\varphi}_D$.

By using the Smith normal form and rearranging rows we will assume that $\beta_{n+k} = (0, 1)$ and that no other rows are multiples of that. The fact that these assumptions can be made is discussed in the introduction. To simplify notation we will write $\beta_i = (a_i, b_i)$, also let $z_i = -\frac{b_i}{a_i}$ for $i \neq n+k$. When $i \neq n+k$ then $a_i \neq 0$. Now a given D is of the form:

$$D = \begin{bmatrix} a_i & b_i \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{a_i} \begin{bmatrix} 1 & -b_i \\ 0 & a_i \end{bmatrix}.$$

Therefore for $\lambda = [\omega(\ell) : 1]$ we have

$$D\lambda = \frac{1}{a_i} \begin{bmatrix} 1 & -b_i \\ 0 & a_i \end{bmatrix} \begin{bmatrix} \omega(\ell) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\omega(\ell)}{a_i} - z_i \\ 1 \end{bmatrix}.$$

Hence we see that

$$\varphi_D(\lambda) = \varphi \left(\left[\frac{\omega(\ell)}{a_i} - z_i : 1 \right] \right). \quad (5.1)$$

Now we define another map related to the $\hat{\varphi}_D$.

Definition 5.9. Pick $i < n+k$ and let $D = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$. Now we define $\hat{\varphi}_i : \mathbb{R} \rightarrow \mathbb{R}^2$ as $\hat{\varphi}_i(\ell) = \hat{\varphi}_D(\ell + \text{val}(a_i), 0)$.

Of course, we still have that the image of $\hat{\varphi}_i$ is the same as that of $\hat{\varphi}_D$ when D is appropriately chosen. Now we will prove a key relationship between the $\hat{\varphi}_i$.

Theorem 5.10. *Suppose that $\text{val}(z_i - z_j) > \ell$. Then $\hat{\varphi}_i(\ell) = \hat{\varphi}_j(\ell)$.*

Proof. Pick $\ell < \text{val}(z_i - z_j)$. Let D_i and D_j be the respective elements of T corresponding to $\hat{\varphi}_i$ and $\hat{\varphi}_j$ respectively. Furthermore, since $\omega(r)$ can be **any** element with valuation r , we will denote it as $\omega_i(r)$ and $\omega_j(r)$ to emphasize that they do not

need to be the same. Now by Theorem 5.4

$$\hat{\varphi}_D(r, 0) = \varphi_D([\omega_i(r) : 1]),$$

for r in a subset of \mathbb{Q} that is dense in \mathbb{R} . Hence by definition of $\hat{\varphi}_i$ we see that

$$\hat{\varphi}_i(r) = \varphi_D([\omega_i(r + \text{val}(a_i)) : 1]),$$

for a dense subset as well. Therefore by Equation 5.1, we see that

$$\hat{\varphi}_i(r) = \varphi \left(\left[\frac{\omega_i(r + \text{val}(a_i))}{a_i} - z_i : 1 \right] \right)$$

for a similar dense subset. The same argument holds for z_j . Again $\omega_i(r)$ is allowed to be **any** element of \mathbb{K} with valuation r . Hence for j , we choose $\omega_j(r + \text{val}(a_j))$ such that

$$\frac{\omega_j(r + \text{val}(a_j))}{a_j} = \frac{\omega_i(r + \text{val}(a_i))}{a_i} + z_j - z_i.$$

Now if $r < \text{val}(z_j - z_i)$ then the valuation of $\omega_j(r + \text{val}(a_j))$ is still $r + \text{val}(a_j)$. Therefore we can choose such an $\omega_j(r + \text{val}(a_j))$. Plugging this in, we see that

$$\varphi \left(\left[\frac{\omega_j(r + \text{val}(a_j))}{a_j} - z_j : 1 \right] \right) = \varphi \left(\left[\frac{\omega_i(r + \text{val}(a_i))}{a_i} - z_i : 1 \right] \right).$$

For a dense subset of $(-\infty, \text{val}(z_i - z_j))$, the left is equal to $\hat{\varphi}_i(r)$ and the right side is $\hat{\varphi}_j(r)$. Since these are piecewise linear then $\hat{\varphi}_i(r) = \hat{\varphi}_j(r)$ for all $r \in (-\infty, \text{val}(z_i - z_j))$ and in particular for ℓ .

□

Now given a collection of elements from \mathbb{Q}_p we will build a tree that collects

values where the $\hat{\varphi}_i$ are equal. Suppose we have a collection z_1, \dots, z_m of distinct elements of a value field \mathbb{K} with valuation $\text{ord}_p : \mathbb{K} \rightarrow \mathbb{R} \cup \{\infty\}$, $\text{ord}_p(z_i) \in \mathbb{Z}$, and $\text{ord}_p(z_i - z_j) \in \mathbb{Z}$ for $i \neq j$. We will construct a tree from these elements. This tree will be used to simplify the plots of non-Archimedean discriminant amoebae. Each node, N , will have two pieces of defining data, (d_N, ℓ_N) , with $d_N \in \mathbb{K}$ and $\ell_N \in \mathbb{Z} \cup \{\infty\}$. When $\ell_N < \infty$, we require $d_N = z_i \pmod{p^{\ell_N}}$ for some i . When $\ell_N = \infty$, we require $d_N = z_i$ for some i . We will call d_N the value of N and ℓ_N is the node's order. We will also define I_N to be all the i such that $d_N \equiv z_i \pmod{p^{\ell_N}}$. Each node will have p branches. The j^{th} branch of N will be the collection of all the nodes, M , such that $d_M \equiv d_N + jp^{d_N} \pmod{p^{d_N+1}}$ and $\ell_M > \ell_N$. If no M satisfies the equation then the j^{th} branch will be empty. We then construct nodes $\{(z_i \pmod{p^{\text{ord}_p(z_i - z_j)}}, \text{ord}_p(z_i - z_j)) \mid i \neq j\} \cup \{(z_i, \infty)\}_{i=1}^m$. We desire to build a tree from these elements. The first lemma shows us which nodes will represent leaves on our eventual tree.

Lemma 5.11. *Let N be a node. If $\ell_N < \infty$ then N has at least two nonempty branches. If $\ell_N = \infty$ then N has no nonempty branches.*

Proof. Suppose $\ell_N < \infty$. This means there are some i and j such that $d_N = z_i \pmod{p^w}$, where $w = \text{ord}_p(z_i - z_j)$. This means that the w^{th} coefficient of z_i and z_j are different. Suppose these coefficients are r and s respectively. Hence $z_i \equiv d_N + rp^w$ and $z_j \equiv d_N + sp^w$. Hence the r and s branches of N are nonempty.

If $\ell_N = \infty$ then it is clear that N has no nonempty branches, because the z_i are distinct and ℓ_N is infinite. □

This next lemma will be instrumental in iteratively constructing the tree. It will be used to decide which node from a branch will represent the root of a branch.

Lemma 5.12. *Suppose that M is a node with t^{th} branch nonempty. Then there is a single node, N , in the t^{th} branch with minimal order. Furthermore all other nodes in the t^{th} branch of M are in some branch of N .*

Proof. Suppose that N and L are in the t^{th} branch of M with $\ell = \ell_N = \ell_L > \ell_M$ and minimal among the elements in the branch. Now by definition there are i and j such that $d_N = z_i \pmod{p^\ell}$ and $d_M = z_j \pmod{p^\ell}$. Now since $N \neq M$, but $\ell_N = \ell_M$, then $d_N \neq d_M$. Hence $\text{ord}_p(z_i - z_j) < \ell$. Furthermore since N and L are in the t^{th} branch of M then d_N and d_L are both equal to $d_M + jp^{\ell_M}$ modulo p^{ℓ_M+1} . Hence $z_i \equiv z_j \pmod{p^{\ell_M+1}}$. Therefore since $\ell > \ell_M$ we have $\ell_M < \text{ord}_p(z_i - z_j) < \ell$. This will give us our contradicting element. Letting $w = \text{ord}_p(z_i - z_j)$ then the node $(z_i \pmod{p^w}, w)$ is in the t^{th} branch of M with $w < \ell$. Hence N and L cannot both be minimal.

Now suppose that N is the element in the t^{th} branch of M with minimal order. Let L be another element of the t^{th} branch. We want to show that L is in a branch of N . We already know that $d_L > d_N$, so we only need to show that $d_L \equiv d_N \pmod{p^{\ell_N}}$. Again both d_N and d_L are equivalent modulo p^{ℓ_M+1} to $d_M + tp^{\ell_M}$. Again $d_N \equiv z_i \pmod{p^{\ell_N}}$ and $d_L \equiv z_j \pmod{p^{\ell_L}}$. Again, if $\text{ord}_p(z_i - z_j) < \ell_N$ then N is not the minimal element in the t^{th} branch of M . \square

From this we create an undirected, simple graph. Nodes correspond to nodes of the graph. For any node with a non empty branch we create an edge from the node to the minimal element of the branch.

Theorem 5.13. *This graph makes a rooted p -ary tree.*

Proof. Let z_1, \dots, z_m be distinct elements of \mathbb{Q}_p . A simple graph is a tree if it is connected, but is not connected if any single edge is removed. First we will show that

there is an element with minimal order. Let ℓ be the smallest order among the z_i . This may be repeated. We introduce a new element to our list, $z_0 = p^{\ell-1}$. For all i , $\text{ord}_p(z_0 - z_i) = \ell - 1$. Furthermore $0 = z_0 \pmod{p^{\ell-1}}$. So if we construct nodes from z_0, z_1, \dots, z_m , we have exactly two new nodes that we wouldn't have had otherwise, N and M , with data, $(0, \ell - 1)$ and (z_0, ∞) , respectively. The first branch of N only contains M and the zeroth branch contains all of the original nodes. Lemma 5.12 then tells us that this collection has a single element with minimal order as desired.

This element with minimal order will be the root node of our tree. For a given node, each nonempty branch gives us an edge to the minimal element in the branch. We continue iteratively along every branch until we are out of elements. By Lemma 5.12 we know that all branches will have a minimal element and that all items will be used. Thus we construct a graph. Now it is clear we will not have any loops because the order of the descendants of any node are larger than the order of the original node. \square

We will show that each branch represents a linear segment of the image of φ in Theorem 5.17. For now, we will limit the number of branches.

Theorem 5.14. *There are at most $2m - 1$ nodes and $2m - 2$ edges.*

Proof. We already know there are exactly m leaves. One for each z_i . We also know from Lemma 5.11 that every non-leaf node has at least 2 child nodes. Any such tree has no more than $2m - 1$ nodes and $2m - 2$ edges. \square

Theorem 5.15. *Let e be an edge with child node, N , and parent node, M . Pick any $i, j \in I_N$. Then for any $\ell \in (\ell_M, \ell_N)$, we have $\hat{\varphi}_i(\ell) = \hat{\varphi}_j(\ell)$.*

Proof. We already showed that $\hat{\varphi}_i(\ell)$ and $\hat{\varphi}_j(\ell)$ are equal when $\ell < \text{ord}_p(z_i - z_j)$. By definition of M and N , this is the case. \square

For a given edge, e , define $\psi: E \rightarrow \mathcal{P}(\mathbb{R}^2)$ to be $\psi(e) = \varphi_i((\ell_M, \ell_N))$ for any $i \in I_N$.

Corollary 5.16. *Let E be the collection of all edges from such a tree. Then*

$$\varphi(\mathbb{PK}) = \overline{\bigcup_{e \in E} \psi(e)}.$$

Proof. Pick an $i \in \{1, \dots, n + m + 1\}$. □

Now we will show that the image of each edge is linear.

Lemma 5.17. *Let $e \in E$ be an edge. Then $\psi(e)$ is a line segment, a line, or a ray.*

Proof. Each φ_i is piecewise linear. Also, each map is differentiable everywhere except on the points $C = \{\text{val}(z_j - z_i) \mid j \neq i\}$. Thus since $\psi(e)$ is the image between two elements of C then $\psi(e)$ is linear, and hence a line segment, line, or a ray. □

We have a tree with a limited number of edges, whose image under ψ is linear and ψ gives us the reduced \mathcal{A} -discriminant amoeba. We have all the necessary pieces to bound our reduced \mathcal{A} -discriminant amoeba.

Theorem 5.18. *The closure of reduced \mathcal{A} -discriminant of $n + 3$ points in general position has no more than $2n^2 + 9n + 11$ complement components. Moreover, Section 5.4 evinces supports A_n in \mathbb{Z}^n with cardinality $n + 3$ and primes p_n such that the p_n -adic discriminant amoeba has quadratically (quadratic in n) many connected components in its complement.*

Proof. Let $G = (E, V)$ the the graph obtained from A . According to Corollary 5.16 we have

$$\overline{\varphi(\mathbb{PK})} = \bigcup_{e \in E} \psi(e).$$

By Theorem 5.14 since there are $n+3$ leaves then there are at most $2(n+3)-2 = 2n+4$ edges. By Theorem 5.17 we can replace each $\psi(e)$ by a line, which will only increase the number of complement components. Therefore we have no more than $2n^2+9n+11$ complement components. \square

We will construct a tree for the example from the beginning of this section.

Example 5.19. Consider the support

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 6 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

This is the support of the so-called Rusek-Shih example

$$f(x, y, t) = tx^6 + \frac{44}{31}ty^6 - yt + y^6 + \frac{44}{31}tx^3 - x$$

from [5]. We then choose our B matrix to be the transpose of the following:

$$\begin{bmatrix} -2 & 35 & -33 & -12 & 0 & 12 \\ -2 & 11 & -9 & -4 & -4 & 0 \end{bmatrix}$$

Here is an example tree. If our z_i are $1/3, 11/35, 1/3, 1/11,$ and 0 then we have the

following 3-adic expansions starting from index -1 .

$$\alpha_0 = \frac{1}{3} = [1, 0, 0, 0, \dots]$$

$$\alpha_1 = \frac{11}{35} = [0, 1, 2, 2, \dots]$$

$$\alpha_2 = 1 = [0, 1, 0, 0, \dots]$$

$$\alpha_3 = \frac{3}{11} = [0, 0, 2, 1, \dots]$$

$$\alpha_4 = 0 = [0, 0, 0, 0, \dots]$$

These elements would then be put into the previously described tree as shown in Figure 5.4.

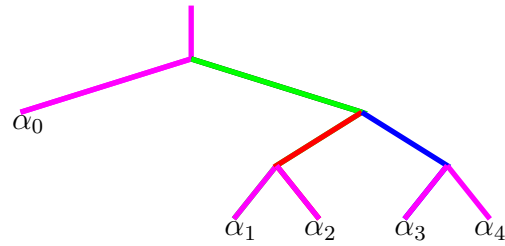


Figure 5.4: Tree for Rusek-Shih.

Each branch of this tree represent a segment (or ray, for the leaves) of constant slope. In particular, with this example, when we plot the amoeba we get Figure 5.5.

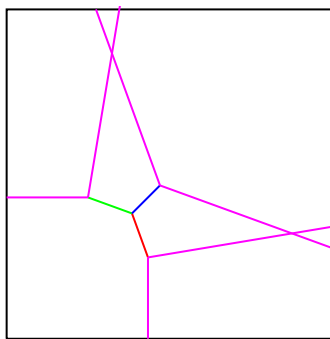


Figure 5.5: 3-adic amoeba for Rusek-Shih.

The colors are there to help indicate which branch corresponds to which segment or ray. Also, there is an extra ray. This accounts for ℓ approaching negative infinity.

5.4 Extremal Family

We will construct a family of \mathcal{A} matrices admitting quadratically many complement components. We begin by constructing a B matrix that satisfies our requirements and then work backwards from there to get the \mathcal{A} matrix. The idea is that we want to ensure that enough of our rays intersect and that these intersections are non-degenerate. Let p be any prime number and let ℓ be any integer larger than 2. We define $D \in \mathbb{Z}^{2 \times (2\ell+2)}$, by $D_{2,2i} = p^{i-1}$ and $D_{2,2i+1} = -p^{i-1}$ for $i = 1, \dots, \ell + 1$ and $D_{1,1} = D_{1,2} = -\ell$ and $D_{1,j} = 1$ for $j > 2$. That is, D has the form:

$$\begin{bmatrix} -\ell & -\ell & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & -p & p & -p^2 & p^2 & \cdots & -p^\ell & p^\ell \end{bmatrix}.$$

Our B matrix will be the transpose of D . Now the zeros of our linear forms are $\pm \frac{1}{p}, \pm p, \pm p^2, \dots, \pm p^\ell$. The p -adic order of the first two elements is less or equal to 0 and the others are their respective exponents on p . Therefore the associated tree is

rather easy to form. For example, Figure 5.6 shows the tree for $\ell = 3$ and $p = 2$.

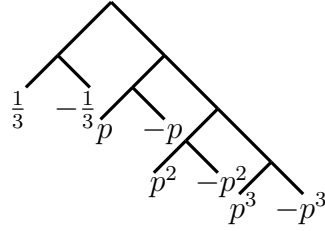


Figure 5.6: Tree for $\ell = 3$ and $p = 2$.

For larger ℓ the tree extends further to the right, whereas for larger p the leaves for p^i and $-p^i$ branch directly from the main branch on the right rather than having their own mutual branch first. This is because the i^{th} digits for p^i and $-p^i$ is the same only when $p = 2$. For example, $9 = 0 \cdot 3 + 1 \cdot 3^2, 3 = 1 \cdot 3$, and $-3 = 2 \cdot 3 + 3 \cdot 3^2 + 3 \cdot 3^3 + \dots$, while $4 = 0 \cdot 2 + 1 \cdot 2^2, 2 = 1 \cdot 2$, and $-2 = 1 \cdot 2 + 1 \cdot 2^2 + \dots$. That is, $\text{ord}_3(3 - (-3)) = 1 = \text{ord}_3(3 - 0)$, while $\text{ord}_2(2 - (-2)) = 2$.

The slope of the image of a branch is the sum of the (a_i, b_i) of the z_i associated to that branch. Therefore any non-leaf branch on the right has a slope of the form $(m, 0)$, because each p^i will cancel out with its matching $-p^i$, but the 1's in the first coordinate will add. Now at the branch point between p^i and $-p^i$ we have rays in the direction $(1, p^i)$ and $(1, -p^i)$. That is, we have a line in the positive x -direction with rays emanating with slopes $\pm p^{-i}$. Furthermore each successive ray in the direction $(1, p^i)$ is further along the x -axis than the previous one because its associated branch in the tree splits further along the main branch. Therefore, because the slope is less steep, the ray for p^i (resp. $-p^i$) intersects the ray for p^j (resp. $-p^j$) for all $j > i$. The points from which these rays are emanating are independent of p . Thus for each

ℓ , a p can be chosen assuring the intersections are non-degenerate. That is, for large enough p the p^i ray intersects the p^j with a smaller x -coordinate than the starting position of the p^{j+1} ray. Hence the rays above the x -axis give a line arrangement with at least $\binom{\ell}{2} + \binom{\ell}{1} + 1$ components. Similarly the rays below the x -axis give the same number of components, except one of these components on the right is already accounted for in the previous count. This gives us at least $\ell^2 + \ell + 1$ components in the complement of the amoeba. As a visual example, here is the relevant part of the discriminant amoeba for $p = 3$ and $\ell = 3$. Looking at Figure 5.7, you can also see the far right chamber that is not cut into two.

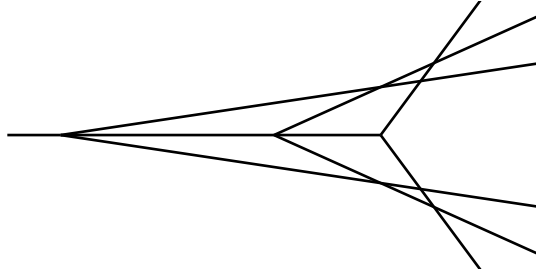


Figure 5.7: Extremal amoeba for $p = 3$ and $\ell = 4$.

Now $\ell = \frac{n+k-1}{2}$. Hence the number of components is quadratic in n .

Now constructing an \mathcal{A} matrix to accompany such a B matrix is not hard. First we find the null space, N , of D . It will be a 2ℓ by $2\ell + 2$ matrix. For $i = 1, \dots, \ell$, the odd rows will be $N(2i - 1, 1) = -\ell p^i + 1$ and $N(2i - 1, 2) = \ell p^i + 1$ and $N(2i - 1, 2i + 1) = 2\ell$, all other coordinates of that row being zero. Similarly, for $i = 1, \dots, \ell$, the even rows will be $N(2i, 1) = \ell p^i + 1$, $N(2i, 2) = -\ell p^i + 1$, and $N(2i, 2i + 2) = 2\ell$, again all other coordinates being zero. It is clear that the rows of N are linearly independent and it is the proper dimension. A small bit of arithmetic

verifies that the rows of N are orthogonal to the columns of B . Finally we can remove any row of N to get our desired \mathcal{A} matrix. We remove one row because B should be the null space of \hat{A} rather than the null space of \mathcal{A} . Therefore \mathcal{A} can have the form

$$\begin{bmatrix} -lp+1 & lp+1 & 2l & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ lp+1 & -lp+1 & 0 & 2l & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -lp^2+1 & lp^2+1 & 0 & 0 & 2l & 0 & \cdots & 0 & 0 & 0 & 0 \\ lp^2+1 & -lp^2+1 & 0 & 0 & 0 & 2l & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & \cdots & & & & \\ -lp^{\ell-1}+1 & lp^{\ell-1}+1 & 0 & 0 & 0 & 0 & \cdots & 2l & 0 & 0 & 0 \\ lp^{\ell-1}+1 & -lp^{\ell-1}+1 & 0 & 0 & 0 & 0 & \cdots & 0 & 2l & 0 & 0 \\ -lp^\ell+1 & lp^\ell+1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2l & 0 \end{bmatrix}$$

6. SUMMARY

This entire thesis is on \mathcal{A} -discriminant varieties and amoebae. In a nutshell, it is an exploration of the structure of these objects in various settings. I dare say it was a successful exploration. We now have a more in depth understanding of the contour of the complex amoeba in two dimensions. We easily know what its slope is at any given point. We know how many cusps it can have. We know how to force cusps. We know how to efficiently search for systems with extremal behavior.

In the non-Archimedean setting we know even more new information about the \mathcal{A} -discriminant amoeba. In all dimensions, for the first time we can efficiently describe its structure. In all dimensions, for the first time we can bound how complicated the topology can be. In two dimensions, not only can we bound the number of connected components in the complement, but we can construct families that admit that bound asymptotically.

We have proved many new bounds for various types of \mathcal{A} -discriminant varieties and amoebae. But the work of mathematics is never finished; every new answer leads to more questions. This is true even in the simple case of $k = 3$, when looking at the number of connected components in the complement. In the p -adic world we have found a $O(n^2)$ bound that is at least asymptotically tight, but what is the true, tight bound? We are worse off when we look at the real setting. We have a similar bound and it appears to be tight, but we have no proof.

Then we move on to the case $k > 3$. Here we have new bounds on the same topological question, $O(n^{2(k-1)(k-2)})$, in the non-Archimedean world, but it seems the bounds could be improved. What are the asymptotic bounds in this case? How complicated can they be? On the other hand, in the real setting, we do not even

know how to bound them polynomially. The more one knows the more one realizes how little they know. This is the beauty of the scientific world—every time she reveals her secrets she also reveals more questions to ponder.

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APPENDIX A

REAL SAGE CODE: $K = 3$

The following code will plot the real \mathcal{A} -discriminant amoeba for the Rusek-Shih example. In the box $[-10, 10]^2$.

The code is included for completeness. It can be downloaded (with extra comments) here:

<https://raw.githubusercontent.com/krusek/mathematics/master/sage/amoeba.sage>

```
1 A = [[6,0,0,0,3,1],[0,3,1,6,0,0],[1,1,1,0,0,0]]
2 show(amoeba(A, 1/100), xmin=-10,ymin=-10,xmax=10,ymax=10)

1 # This will simply calculate the real A-discriminant
2 # amoeba from the input A matrix as a list.
3
4 var('n')
5
6 # This will not run as quickly if the entries of B are
7 # irrational. That is, it will leave the irrational part,
8 # irrational.
9 def f_from_B(B):
10     linears = map(lambda b: b[0]*cos(n)+b[1]*sin(n), B)
11     logs = map(lambda l: l.abs().log(), linears)
12     mult = lambda b: map(lambda i: b[i]*logs[i],
13         range(len(logs)))
14     f0 = sum(mult(map(lambda b: b[0], B)), 0)
```

```

15  f1 = sum(mult(map(lambda b: b[1], B)), 0)
16  return lambda m: [f0.subs({n:1.0*m}),
17                    f1.subs({n:1.0*m})]
18
19  # This gets the zeros of the linear forms of B and
20  # converts them to angles (in [0, pi) ) Then it sorts them.
21  def get_asymptotes(B):
22      sgn = lambda b: -1 if b < 0 else 1
23      at2 = lambda b: arctan2(sgn(b[0])*b[0], -sgn(b[0])*b[1])
24      BB = map(lambda b: at2(b), B)
25      BB.sort()
26      return BB
27
28  def amoeba_from_B(B, count):
29      ass = get_asymptotes(B)
30      f = f_from_B(B)
31      if min(ass) > 1e-6:
32          ass = [0] + ass
33      if math.pi - max(ass) > 1e-6:
34          ass.append(float(math.pi))
35      lines = []
36      for i in range(len(ass)-1):
37          s = ass[i] + 1/count
38          e = ass[i+1] - 1/count
39          k = ceil((e-s)/math.pi/2*count)+2

```

```

40     step = (e-s)/(k-1)
41     l = map(lambda i: f(s+i*step), range(k))
42     lines.append(line(l))
43     return sum(lines)
44
45 def get_B_list(A):
46     Ah = A + [[1]*len(A[0])]
47     Am = matrix(Ah).transpose()
48     Bm = Am.integer_kernel().basis_matrix().transpose()
49     B = map(lambda b: list(b), list(Bm))
50     return B
51
52 def amoeba(A, angle):
53     B = get_B_list(A)
54     return amoeba_from_B(B, ceil(2*math.pi/angle+1))

```

APPENDIX B

NON-ARCHIMEDEAN SAGE CODE: $K = 3$

The following code will plot the 3-adic \mathcal{A} -discriminant amoeba for the Rusek-Shih example. In the box $[-4, 25] \times [-25, 4]$.

The code is included for completeness. It can be downloaded (with extra comments) here:

<https://github.com/krusek/mathematics/raw/master/sage/trop2d.sage>

```

1 A = [[6,0,0,0,3,1],[0,3,1,6,0,0],[1,1,1,0,0,0]]
2 show(amoeba(A, 3), xmin=-4,ymin=-25,xmax=25,ymax=4, \
3     aspect_ratio=1)

1 def amoeba_from_B(B, p):
2     Z = filter(lambda b: b[1][0] != 0, enumerate(B))
3     Q = Qp(p)
4     Bm = matrix(B)
5     rval = []
6     for z in Z:
7         bb = B[z[0]]
8         BB = map(lambda b: [b[0]/bb[0], \
9             b[1]-b[0]*bb[1]/bb[0]], B)
10        QL = map(lambda b: [Q(b[0]).valuation(), \
11            Q(b[1]).valuation()], BB)
12        FQ = lambda r: map(lambda b: min(b[0]+r, b[1]), QL)
13        FI = lambda x: (matrix(FQ(x))*Bm).list()
```

```

14
15     zv = map(lambda b: b[1]-b[0], QL)
16     zv = filter(lambda zz: zz != Infinity and \
17                 zz != -Infinity, zv)
18     zv = [-100, 100] + list(set(zv))
19     zv.sort()
20
21     l = map(lambda i: FI(zv[i]), range(len(zv)))
22     l = line(l)
23     rval.append(l)
24     return rval
25
26 def get_B_list(A, transpose=False):
27     Ah = A + [[1]*len(A[0])]
28     Am = matrix(Ah).transpose()
29     Bm = Am.integer_kernel().basis_matrix()
30     if transpose:
31         Bm = Bm.transpose()
32     B = map(lambda b: list(b), list(Bm))
33     return B
34
35 def amoeba(A, p):
36     B = get_B_list(A, True)
37     return amoeba_from_B(B, p)

```

APPENDIX C

NON-ARCHIMEDEAN SAGE CODE: GENERAL K

The following code will plot the 3-adic \mathcal{A} -discriminant amoeba for the plane example. In the box $[-10, 10]^3$.

The code is included for completeness. It can be downloaded (with extra comments) here:

<https://github.com/krusek/mathematics/raw/master/sage/tropdd.sage>

```
1 A = [[0, 1, 2, 3, 4]]
2 box = Polyhedron(list(get_verts(10, 3)))
3 a = amoeba(A, 3)
4 show(sum(lambda p: box.intersection(p).show(), a))

1 def getlist(n, i, l, ex=[0]):
2     for j in range(i, n):
3         if ex.count(j) > 0:
4             continue
5         if l == 1:
6             yield [j]
7         else:
8             for k in getlist(n, j+1, l-1):
9                 yield [j] + k
10
11 def get_ieq(l, i, j):
12     # l[i] >= l[j]
```

```

13  ieq = [0] * len(l)
14  ieq[i] = -1
15  ieq[j] = 1
16  ieq[0] = -l[i]+l[j]
17  return ieq
18
19  def get_tropical_line_complement(l1):
20      l = [l1[-1]] + l1[:-1]
21      I = filter(lambda j: l[j] != Infinity, range(len(l)))
22      p = []
23      for ii in range(len(I)):
24          i = I[ii]
25          for jj in range(ii + 1, len(I)):
26              j = I[jj]
27              eqs1 = [get_ieq(l, i, j)]
28              eqs2 = [get_ieq(l, j, i)]
29              for kk in range(len(I)):
30                  k = I[kk]
31                  if kk == ii or kk == jj:
32                      continue
33                  eqs1.append(get_ieq(l, i, k))
34                  eqs2.append(get_ieq(l, j, k))
35              pp = Polyhedron(ieqs=eqs1)
36              if p.count(pp) == 0:
37                  p.append(pp)

```



```

38     pp = Polyhedron(ieqs=eqs2)
39     if p.count(pp) == 0:
40         p.append(pp)
41     return p
42
43 def get_verts(bound, n):
44     if n == 0:
45         yield []
46     else:
47         for v in get_verts(bound, n-1):
48             yield [bound] + v
49             yield [-bound] + v
50
51 def amoeba_from_B(B, prime, bound=10):
52     verts = map(lambda v: v, get_verts(bound, len(B[0]) - 1))
53     amoeba = []
54     Q = Qp(prime)
55     val = lambda l: map(lambda q: Q(q).valuation(), l)
56     Bm = matrix(B)
57     for II in getlist(len(B), 0, len(B[0]) - 1, [0]):
58         print II
59         p = [Polyhedron(verts)]
60         I = II + [0]
61         D = map(lambda i: B[i], I)
62         Dm = matrix(D)

```

```

63     print Dm
64     if Dm.determinant() == 0:
65         continue
66     BD = Bm*Dm.inverse()
67     BL = map(lambda l: list(l), list(BD))
68     QL = map(lambda l: val(l), BL)
69     mn = lambda l, r: min(map(lambda i: l[i]+r[i], \
70         range(len(l))))
71     FQ = lambda r: map(lambda l: mn(l,r), QL)
72     FI = lambda x: (matrix(FQ(x + [0]))*Bm).list()
73
74     CC = map(lambda l: \
75         get_tropical_line_complement(l), QL)
76     print len(CC)
77     CC = filter(lambda c: c != [], CC)
78     print len(p)
79     for C in CC:
80         p = map(lambda c: map(lambda pp: \
81             c.intersection(pp), p), C)
82         p = sum(p, [])
83         p = filter(lambda pp: pp.dim() == C[0].dim(), p)
84     print len(p)
85     for pp in p:
86         v = pp.vertices()
87         v = map(lambda vv: FI(vv), v)

```

```

88         amoeba.append(Polyhedron(v))
89     return amoeba
90
91 def get_B_list(A, transpose=False):
92     Ah = A + [[1]*len(A[0])]
93     Am = matrix(Ah).transpose()
94     Bm = Am.integer_kernel().basis_matrix()
95     if transpose:
96         Bm = Bm.transpose()
97     B = map(lambda b: list(b), list(Bm))
98     return B
99
100 def amoeba(A, p):
101     B = get_B_list(A, True)
102     return amoeba_from_B(B, p, 100)

```