

UNDERSTANDING FRACTIONAL EQUIVALENCE AND THE DIFFERENTIATED  
EFFECTS ON OPERATIONS WITH FRACTIONS

A Thesis

by

EMILIE A. NAISER

Submitted to the Office of Graduate Studies of  
Texas A&M University  
in partial fulfillment of the requirements  
for the degree of

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December 2004

Major Subject: Curriculum and Instruction

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## ABSTRACT

Understanding Fractional Equivalence and the Differentiated Effects on Operations with Fractions. (December 2004)

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This study compared two representations for teaching fraction equivalence. It traced the implications of both representations on the student's comprehension of fractions as well as their ability to perform operations with fractions.

The participants in the study included 65 sixth grade students in three extant classrooms. Two classes were instructed using the textbook representation while the third class received instruction using a representation presented by Van de Walle and recommended by the National Council for Teaching Mathematics. Data were collected from pre-tests, post-tests, student work samples, field notes and a semi-structured interview.

Qualitative analyses were used to analyze the data. Items were coded for procedural and conceptual understanding and categorized into levels of proficiency.

Additionally, items involving operations with fractions were coded for error patterns. Conclusions were drawn about how the different representations affected student comprehension and faculty with fractions.

## DEDICATION

I dedicate this work to my students who constantly challenge me, inspire me and teach me.

## ACKNOWLEDGEMENTS

In the past few years, my support system has grown to include not only my wonderful mom, beautiful sister and close friends, but also my professors, my co-workers, and my students. I appreciate the guidance and encouragement from all of these people. The term family has taken on a much broader meaning, and I am thankful for all of you who have become a special part of my life. Your support has made this work possible.

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## TABLE OF CONTENTS

	Page
ABSTRACT.....	iii
DEDICATION.....	v
ACKNOWLEDGEMENTS.....	vi
TABLE OF CONTENTS.....	vii
LIST OF FIGURES.....	ix
LIST OF TABLES.....	x
INTRODUCTION .....	1
Statement of Problem.....	1
Rationale.....	2
Teaching Methods.....	3
Definitions.....	5
Overview.....	6
LITERATURE REVIEW .....	7
Teaching Fractions.....	7
Error Patterns with Fractions.....	8
The Learning Framework.....	10
Representations.....	12
Curriculum.....	13
Textbooks.....	14
National Council of Teaching of Mathematics....	17
METHODOLOGY .....	19
Participants.....	19
Classroom Lesson Descriptions.....	19
Instrumentation.....	21
Data Analyses.....	24
Credibility and Dependability.....	29

	Page
RESULTS .....	31
Student Comprehension of Fraction Equivalence..	31
Operations with Fractions.....	44
CONCLUSIONS .....	57
Comprehension of Fractions.....	57
Error Patterns in Operating with Fractions.....	68
REFERENCES.....	77
APPENDIX A.....	83
APPENDIX B.....	84
APPENDIX C.....	85
APPENDIX D.....	87
APPENDIX E.....	88
APPENDIX F.....	89
APPENDIX G.....	90
APPENDIX H.....	91
VITA.....	93



## LIST OF FIGURES

FIGURE	Page
1 Procedural Pre-test Results.....	33
2 Procedural Post-test Results.....	34
3 Error Pattern from Group B.....	39
4 Conceptual Pre-test Results.....	40
5 Conceptual Post-test Results.....	40
6 Student Representation from Group A.....	41
7 Student Representation from Group B.....	42
8 Example of Error Pattern - Cross Addition.....	48
9 Error Pattern when Multiplying Fractions.....	49

## LIST OF TABLES

TABLE		Page
1	Criteria for Levels of Procedural and Conceptual Comprehension.....	26
2	Coding Analysis of Pre-test from Group A.....	28
3	Comparative Results of Operations with Fractions.	45

## INTRODUCTION

## STATEMENT OF THE PROBLEM

Evaluations should be made of the various methods used for teaching fractional equivalency to identify the implications the methods have on the students' facility with fractions. What students already know about finding equivalent fractions can have positive and negative implications for students' comprehension of fraction concepts. Therefore, it is important to understand the possible effects on future learning. By identifying these implications, it is possible to make informed decisions about which strategy to use when teaching fractional equivalency. Furthermore, it is important to understand the consequences for each strategy when building upon this prior knowledge as students move to operations with fractions. Understanding the students' misconceptions of

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This thesis follows the format of the *Journal for Research in Mathematics Education*.

fractions as well as where the misconceptions stem, can help teachers prevent students from developing erroneous patterns.

#### RATIONALE

Thompson and Saldanha (2003) state that analyses of what students learn includes "tracing the implications that various understandings have for related future learning" (p. 95). They further their point by offering the following about textbook designers:

Designers always intend some understanding, whether or not they make it available for public scrutiny. We contend that mathematics education profits from efforts to both publicize and scrutinize those intentions. Such efforts increase the likelihood that the meanings we intend students to develop actually have the potential of being consistent with, and supportive of, the meanings, understandings, and ideas we hope they develop from them. (p. 95)

## TEACHING METHODS

In addition to a solid understanding of mathematics, teachers must also understand how children most successfully learn the subject (Ball, 1993). Teachers are faced with the responsibility of choosing how they will present material to their students. Often teachers seek recommendations from a methods textbook, teacher journal, or research article. However, regarding fractional equivalency these resources are not in agreement. Methods textbooks offer differing methodologies both within and among them. Furthermore, these textbook methodologies are often in conflict with the National Council of Teaching of Mathematics' (NCTM) recommendation as detailed in Van de Walle's (2001) book, *Elementary and Middle School Mathematics: Teaching Developmentally*. The sixth grade textbook used in this study, *Middle Grades Math Thematics*, demonstrates a method of dividing the fraction by a fraction equivalent to one whole (Billstein & Williamson, 1999). The second method offered by Van de Walle (2001) is to factor the numerator and denominator. Then, find and eliminate the common factor. "The search for a common factor keeps the process of writing an equivalent fraction

to one rule: Top and bottom numbers of a fraction can be multiplied by the same nonzero number" (p. 225). If there are varied methods to teaching fraction equivalency, then each method must possess some positive as well as negative aspects. Knowing these aspects can be useful to teachers who are deciding how to teach fraction equivalency.

The Texas Essential Knowledge and Skills (TEKS) is the state-mandated curriculum guidelines which establish what every student, from elementary school through high school, should know and be able to do. According to the TEKS, sixth grade students should be able to find equivalent forms of rational numbers including fractions, decimals, and percents. Students also are required to perform operations with fractions including addition and subtraction. It is important that the students' previous knowledge of fractions, including fraction equivalency, continues to help them as they learn about adding, subtracting, multiplying and dividing fractions.

In addition to the implications that occur when students perform operational procedures with fractions, is the students' comprehension of fraction equivalency itself. The varied methods of teaching fraction equivalency can

influence how the students represent and conceptualize fraction equivalency.

The following questions will be answered in this study:

1. What are the implications of student comprehension of fractions resulting from learning fraction equivalency using two different methods?
2. What error patterns and misconceptions do sixth grade students have when operating with fractions? And, are these errors implied from the methodology used to learn fraction equivalency?

#### DEFINITIONS

The following definitions are listed here in order to specify their meaning with regards to this study.

1. Overgeneralization: Jumping to a conclusion based on inadequate data (Ashlock, 2002)
2. Misconceptions: Incorrect features of student knowledge that is repeatable and explicit (Leinhardt, Zaslavsky, & Stein, 1990).

3. Error Patterns: "...systematic procedures that students learn but which most often do not provide the correct answer" (Ashlock, 2002, p. 9)

#### OVERVIEW

This study took place in a sixth grade classroom. The teacher taught a different method of finding equivalent fractions to two sections of sixth grade students. One of the methods was a textbook recommended method while the other followed NCTM's recommendations. Lessons on fraction equivalence were taught for approximately one week. This is considered the intervention. Throughout the course of the semester, as students encountered instruction in fractions they received the intervention strategies presented in the initial lesson. Students were assessed throughout the semester to monitor their comprehension of fraction equivalency and operational skills.



## LITERATURE REVIEW\*

## TEACHING FRACTIONS

The National Council of Teachers of Mathematics (NCTM) (2000), states that students in middle school should acquire a deep understanding of fractions and be able to use them competently in problem solving. However, it seems that just as students are struggling with fractions, so too are teachers feeling the frustration with teaching fractions effectively.

*The National Assessment of Educational Progress* reports show that fractions are "exceedingly difficult for children to master" (NAEP, 2001, p. 5). Additionally, students are frequently unable to remember prior experiences about fractions covered in lower grade levels (Groff, 1996). In an effort to increase the effectiveness teaching fractions, teachers iteratively review and modify the structure of their lessons on fraction concepts.

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students know and need to learn and then challenging and of "Effective Mathematics teaching requires understanding what supporting them to learn it well" (National Council of Teachers of Mathematics, 2000, p.16).

#### ERROR PATTERNS WITH FRACTIONS

Tirosh (2000) states that teachers should be familiar with the different types of cognitive processes (some of which are erroneous) that students use in learning fractions. If the teachers are aware of common errors, they can reflect on the design of their lesson to find ways to prevent these erroneous patterns from occurring.

Tirosh (2000) studied the comprehension of students' division of fractions. She categorized the mistakes made by children into three sections: (1) algorithmically based mistakes, (2) intuitively based mistakes, and (3) mistakes based on formal knowledge.

An algorithm is "a finite, step-by-step procedure for accomplishing a task that we wish to complete" (Usiskin 1998, p. 7). Algorithmically based mistakes are caused by incorrect rules, or "bugs" in the computation process. Kelly, Gersten and Carnine (1990) also conclude that a

numerous amount of student errors involving fractions result from confusing algorithms or inappropriate application of algorithms.

Intuitively based mistakes stem from intuitions already held about the subject. For example, a student learning about whole numbers may believe that when subtracting numbers you always subtract from the larger numbers. However, this is not the case in subtracting with integers. The student may intuitively believe that subtracting from a smaller number cannot be done. Tirosh (2000) asserts that students overgeneralize properties of operations with natural numbers to fractions. Williams and Ryan (2002) included overgeneralization on their list of common error patterns made by students. "Children may try to apply ideas they have about whole numbers to rational numbers and run into trouble" (National Research Council, 2001, p. 416).

Other mistakes can be made based on formal knowledge. This includes computational errors due to limited conceptions of fractions and insufficient familiarity with the properties of the operations (e.g., division is

commutative and therefore  $\frac{4}{2} = 2$  and  $\frac{2}{4} = 2$ ).

Another misunderstanding with fraction computation is that students do not think of the magnitude represented by each fraction. When given the problem to estimate  $\frac{12}{13}$  divided by  $\frac{7}{8}$ , only one third of U.S. 13- and 17-year-olds correctly estimated the answer. Both fractions can clearly be rounded to one whole resulting in an estimated sum of two. 28% of the 13-year-olds answered 19, and 27% answered 21 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981).

#### THE LEARNING FRAMEWORK

The National Research Council (2001) chose the term mathematical proficiency "to capture what we think it means for anyone to learn mathematics successfully" (p. 116). They continued on to describe mathematical proficiency as having five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. These strands are presented as interwoven and interdependent. Ashlock (2002) emphasizes the necessity of the first two strands specifically. "Both [conceptual understanding and procedural fluency] are necessary, but procedural learning must be based on

concepts already learned; procedural learning should be tied to conceptual learning and to real life applications” (p. 8).

The NRC (2001) describes conceptual understanding as “comprehension of mathematical concepts, operations, and relations” (p. 116). Other attributes of conceptual knowledge include students learning more than just facts and rules which are less likely to be retained. Instead, with conceptual knowledge the student can explain the mathematical idea, its importance, and can apply it to new situations.

Procedural fluency is the “skill in carrying out procedures flexibly, accurately, efficiently, and appropriately” (National Research Council, 2001, p. 116). It is important for students to know basic skills and computations and be able to carry them out efficiently and accurately. Often, conceptualization and procedural fluency are compared against each other, when in fact they are needed to support each other.

Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical

concepts with understanding, and using procedures can help strengthen and develop that understanding. (National Research Council, 2001, p. 122)

## REPRESENTATIONS

Representations are important in recording, analyzing, and communicating mathematical data, problems, and ideas (Preston, 2003) and serve as an important means to which students develop a conceptual understanding of mathematical ideas (Ball & Osborne, 1998; Hiebert & Wearne, 1986). Coulombe and Berenson (2001) call representations the language of mathematics. Friedlander and Tabach (2001) further assert that representations are "vehicles for learning and communication". In addition, they appeal to different styles of student learning. NCTM includes representations as one of the Standards stating that students should be able to use representations to model and interpret mathematics and to solve problems (National Council of Teaching of Mathematics, 2000). Additionally, the NRC (2001) describes what they consider to be an important indicator for conceptual learning as the student

"...being able to represent mathematical situations in different ways and knowing how different representation can be useful for different purposes" (p. 119). Representations are a powerful tool. It is important to use multiple representations to enhance student understanding. Also, it is beneficial to encourage students to create multiple ways of representing their mathematical ideas.

#### CURRICULUM

In the sixth grade, teachers must build on the student's prior knowledge of fractions while staying cognizant of what the students need to know to be successful in the future. According to the Texas Essential Knowledge and Skills (TEKS), as early as Kindergarten the students should be able to share a whole by dividing it into equal parts. In the first grade students should be able to describe the fractional part, such as three out of the four crayons are red. By fifth grade, the students are introduced to generating equivalent fractions and finding common denominators. The sixth grade TEKS include generating equivalent forms of numbers including fractions, decimals, and percents. In addition, the students should be

able to add and subtract fractions. In order to prepare the students for seventh grade, the students are also introduced to multiplying fractions. In the seventh grade the students learn to use fractions in problem solving situations requiring multiplication and division of fractions.

The alignment of the curriculum is intended to build upon the students' knowledge at each grade level. This can be a very effective way to promote student understanding. However, on the same note, if students carry misconceptions with them, it can largely impede their future learning because the topics build off each other so closely.

#### TEXTBOOKS

Teachers often use the textbook as the primary resource to plan mathematics instruction (Weiss, Banilower, McMahon, & Smith, 2001). Bush, Kulm and Surati (2000) add that "selecting textbooks is one of the most important decisions teachers make" (p. 34). Textbooks assist teachers in organizing and delivering instruction as well as serve as a source of problems for students to engage in and apply their knowledge. Reys, Chavez and Reys (2004) assign



textbooks with three roles: (1) determines the sequence that the teacher will present the material, (2) suggests the content to be taught, and (3) provides ideas and activities for engaging students' in the lesson.

In Texas, all schools follow a state mandated curriculum. The TEKS were developed by the Texas Education Agency to ensure that students receive instruction at the appropriate grade level. However, the state does not mandate a specific textbook for school use. School districts are responsible for adopting a textbook for their school. Schools usually adopt a new textbook every 5-7 years due to physical deterioration and modifications on the content.

Teachers have to decide daily what to teach, how to teach it and what activities to use. Often the primary resource for planning daily mathematics instruction is the textbook (Weiss, Banilower, McMahon, & Smith, 2001). "Educators place a great deal of trust in textbooks, so it is important that teachers and administrators regularly examine the content focus of district-adopted textbooks and the instructional strategies implicit within textbook lessons" (Reys, Chavez, & Reys, 2003, p. 63).

The textbook method analyzed in this study is from *Middle Grades Math Thematics*. The textbook offers a method for simplifying fractions, in Module 2, that includes dividing the numerator and denominator by a common factor (see Appendix A). The author, Jim Williamson (personal communication, March 9, 2004), explains the reasoning behind presenting this representation compared to the one presented by Van de Walle:

The reason we did not use it [Van de Walle's representation] is because it assumes that students know how to multiply fractions and that they understand multiplicative identities. These concepts are not covered until Module 4. We were also influenced by the traditional approach of teaching equivalent fractions before teaching operations. The rationale for the traditional approach is that equivalent fractions are closely related to the meaning of fractions and they are needed to do addition and subtraction. Also, students are traditionally, though needlessly, taught to express the results of computations in lowest terms.

## NATIONAL COUNCIL OF TEACHING OF MATHEMATICS

NCTM has developed ten Standards to describe a set of goals for mathematics instruction. The first five standards are focused in the areas of number and operations, algebra, geometry, measurement, and data analysis and probability. The next five describe standards for mathematical processes of problem solving, reasoning and proof, connections, communication, and representation. Through the Standards, NCTM describes the tools the students will need to be successful in the twenty-first century (National Council of Teaching of Mathematics, 2000).

In James Hiebert's article (2003), he concludes that the Standards are consistent with the best and most recent evidence on teaching and learning mathematics. These Standards are the backbone of Van de Walle's book, *Elementary and Middle School Mathematics: Teaching Developmentally*. The book serves as a resource for teachers and is based on the NCTM's Standards. Van de Walle offers alternative teaching strategies and activities that challenge traditional approaches. The method in the book for simplifying fractions is based on the same underlying structure as the textbook method. However, the

representation is different. In this representation, the numerator and denominator are factored. Then, the common factors are eliminated (see Appendix A).

## METHODOLOGY

The qualitative methods used in this study were designed to collect and analyze data from sixth grade students. The following sections give information about who participated in the study, how the data were collected, and how the data were analyzed.

### PARTICIPANTS

The participants in the study included 65 sixth grade students in three intact classrooms. The students all attended the same middle school in a mid-southern state. They comprised three different classes taught by the same mathematics teacher. A student assent form and parent consent form approved by the Institutional Review Board at Texas A&M University was obtained for each participant.

### CLASSROOM LESSON DESCRIPTIONS

Lessons on fraction equivalence were taught for approximately one week. This is considered the intervention. While the intervention occurred as a snapshot in time, each instance students encountered instruction in

fractions they received the intervention strategies presented at the onset. Therefore, for continuity within the groups across time, lessons were modified to ensure that the intervention was consistent and constrained to the appropriate group throughout fraction instruction. Of the three classes, two of the classes received instruction using the textbook representation. The textbook representation divides the fraction by a fraction equivalent to one. These two classes comprise the control group, Group A. The third class, Group B, received instruction using the representation presented by Van de Walle (2001). In this representation, the fraction is factored and the common factors are eliminated (See Appendix A). The fraction equivalence lesson began with students representing fractions using pictures. Students identified fractions that were equivalent. Next, the students practiced finding equivalent fractions using either the representations shown by the textbook or by Van de Walle depending on group membership. As the semester progressed, the students solved problems using equivalent fractions. For example, John has  $\frac{6}{10}$  of a candy bar. Mary has the same size bar, but hers is divided into five

pieces. John and Mary have the same amount of candy. Out of five candy pieces, how many does Mary have? Throughout the following lessons, including adding and subtracting fractions, students were building upon this prior knowledge of fraction equivalency given the treatment.

#### INSTRUMENTATION

Data were collected in the form of a pretest, posttest, semi-structured interviews, student work samples, and researcher field notes. The data were collected over a period of several months in order to monitor how the differing representations affected student learning of fractions. Throughout the semester the teacher remained consistent in presenting the methodology originally used in the fraction equivalence lesson. To ensure fidelity to the respective methods two persons familiar with the study, conducted observations and periodically lessons were videotaped for review by the researcher.

#### Fractional Equivalence Pre/Post Tests

The pre-test was administered one day prior to the fraction equivalence unit (the intervention). It consisted

of items on fraction equivalency and items on operations with fractions. The post-test was administered one day after the fraction equivalence unit. The post-test examined similar tasks within the same conditions excepting different numbers (see Appendix B).

### Observations

Structured observations of the students were made after the conclusion of the unit. The interviews were conducted one-on-one between a student and a clinical assistant to assess the student's conceptual understanding of fraction equivalence using a semi-structured interview script. The interviewer used written notes and audio tape to record the interactions. The interviewer generally followed a script (see Appendix C) that was the basis for the semi-structured interviews. The interviewer deviated veered from the script for specific cases which promised additional useful information about fractional equivalence. The purpose for the additional questions was to probe for more information from the student or to follow up on a topic initiated by the student that might be beneficial to



the study. The interviews consisted of a prompt and follow-up questions.

#### Work Samples

A purposeful sample of student work was collected during the course of the study. Samples included items from daily class work, quizzes, formal assessments, and student's self assessments. The work samples included numerical answers as well as open-ended responses with explanations of how the students derived their answers.

#### Field Notes

Additionally, field notes were taken by the researcher to record classroom observations and classroom discourse that was pertinent to the study. The notes were recorded immediately and filed into a journal kept by the researcher with notes describing the context of the observation. These notes were indexed to student work samples.

## DATA ANALYSES

Qualitative techniques were used to analyze the data. Initial coding of the data was conducted (similar to Miles & Huberman, 1994) in order to "find conditions among the participants, as a method of pointing out regularities in the setting" (Anfara, Brown, & Mangione, 2002, p. 32). Data were coded by "certain words, phrases, patterns of behavior, and subject's ways of thinking, and events [that] repeat and stand out" (Bogdan & Biklen, 1982, p. 166). Glaser and Strauss's (1967) constant comparison analysis was used to sort the data into categories. A code-mapping system was implemented to identify key aspects of the research questions. A data reduction strategy similar to Anfara et al. (2002) was used to provide insights based on the meta-categories evident from the data categorization. The various sources of data were used to triangulate the findings (see the subsection entitled CREDIBILITY AND DEPENDABILITY).

The first research question focuses on the students' comprehension of fractions. From the data collected, the items that focused on procedural or conceptual understanding of fractions were isolated. First the student

responses were coded for the procedure that they used. The categories of the procedures were emergent from the data. These categories are listed in the code map in Appendix D. The categories for the procedures show the different processes (correct and incorrect) that students used to find equivalent fractions. The categories were then ranked as a high, medium, or low level procedure based upon their accuracy (see Table 1). Codes F, P, H, M, J, and I all represent procedures that are correct. Therefore, they were considered a high level. Codes K, N, L, and Q were considered medium level. Code K produces the correct answer despite an error in the process. Code N produces one part of the answer but is missing another part completely. Code L and Q show some understanding of the process but contain minor errors. Codes G, and 0 are low level. G and 0 represent either incorrect answers or a blank response.

Next, the data were coded for conceptual comprehension of fraction equivalency. Following most procedural questions, students were asked to explain their answer using words or pictures. These explanations were coded to categorize the students' conceptual understanding. Again, the categories emerged from the data and are listed in

Appendix E. These codes were also divided into categories for high, medium and low level of conceptual understanding (see Table 1).

Table 1

*Criteria for Levels of Procedural and Conceptual Comprehension*

Procedural Levels	
High	Uses correct process and produces correct answer
Medium	Process contains minor errors or is missing steps, may produce correct or incorrect answer
Low	Incorrect process, incorrect answer or no response
Conceptual Levels	
High	Represents both fractions and makes a connection between the equivalent parts of the fractions
Medium	Represents both fractions but does not show a connection between the equivalent parts; or represents both fractions but does not use an equivalent whole unit
Low	Representation contains major errors or blank response

A response was coded as high if the student was able to represent both fractions accurately and show a connection between the equivalent parts. Code IV was considered a high level. Codes I, II, and III were

considered medium levels. Code I represented responses where the student simply described the algorithmic function of the procedure using words. Code II showed a representation of both fractions but no connection was made between the equivalent parts of the fraction. Code III was a representation of both fractions but the unit wholes were not equivalent. Codes V and 0 were considered low level because they consisted of either incorrect responses with major errors or blank responses.

The results from the pre-test and post-test were analyzed from both groups. The codes for each response were organized into a table to show the frequency of each category as well as the correlation of the procedural code with the conceptual code (see Table 2). The results were then translated from categories of strategies to the respective level of comprehension. Additional data from other examples of student work and interviews were analyzed in the same format to further investigate and validate the initial analysis.

To specifically address the second research question, student work on computations with basic operations of fractions was analyzed and coded for error patterns. Error patterns as described by Ashlock (2002) are "systematic

Table 2  
*Coding Analysis of Pre-test form Group A (Textbook)*  
*H = High Level, M = Medium Level, L = Low Level*

		Conceptual Understanding (see Appendix G)						
		<b>I</b>	<b>II</b>	<b>III</b>	<b>IV</b>	<b>V</b>	<b>0</b>	
		<i>M</i>	<i>M</i>	<i>M</i>	<i>H</i>	<i>L</i>	<i>L</i>	
Procedural Comprehension (see Appendix F)	<b>F</b>	<i>H</i>	5	2	0	0	0	0
	<b>G</b>	<i>L</i>	2	3	0	0	4	0
	<b>H</b>	<i>H</i>	1	1	2	0	1	0
	<b>I</b>	<i>H</i>	2	1	0	1	0	0
	<b>J</b>	<i>H</i>	1	0	0	2	0	0
	<b>K</b>	<i>M</i>	6	0	0	0	0	0
	<b>L</b>	<i>M</i>	1	0	0	0	0	0
	<b>M</b>	<i>H</i>	1	0	0	0	5	0
	<b>N</b>	<i>M</i>	0	0	0	0	0	0
	<b>P</b>	<i>H</i>	0	0	0	0	0	0
	<b>Q</b>	<i>M</i>	0	0	0	0	0	0
	<b>0</b>	0	0	1	0	0	0	8

procedures that students learn but which most often do not provide the correct answer" (p. 9). To find error patterns student work was analyzed. Additionally, follow up questions were used to clarify student cognition. The codes used for the error patterns are displayed in Appendix F. These codes emerged from the data. Once again, the error patterns used in the pre-tests and post-tests were compared for each group.

Next, student self-assessments were utilized to examine if the error patterns were implied from the specific lesson on fraction equivalence being studied.

Ashlock (2002) identifies self-assessment as one of the most important aspects of the assessment process as well as a critical tool for diagnosing error patterns. Examples of self-assessment questions are provided in Appendix G.

#### CREDIBILITY AND DEPENDABILITY

Creswell (2003) describes achieving credibility and dependability through triangulation. Triangulation is achieved by using "different data sources of information by examining evidence from the sources and using it to build a coherent justification for themes" (p. 196). The data

sources used in this study were interviews, samples of student work (tests and daily work), and field notes. During data analysis, a table was created to list the findings of the study and the related sources of data collection in order to verify that the findings emerged from multiple sources, thus constituting dependability. To validate the research design, Appendix H presents a matrix of how the data sources relate to the research questions. This matrix not only connects the data sources to the research questions, but it additionally highlights the triangulation of data from different sources to answer the questions.



## RESULTS

This section will begin with a report of the results for the procedural and conceptual comprehension of equivalent fractions. These results are intended to address the first research question: What are the implications of student comprehension of fractions resulting from learning fraction equivalency using two different methods? Following these results, the error patterns from operating with fractions will be reported in order to address the second research question: What error patterns and misconceptions do sixth grade students have when operating with fractions? And, are these errors implied from the methodology used to learn fraction equivalency?

### STUDENT COMPREHENSION OF FRACTION EQUIVALENCE

#### Procedural Comprehension

The data analysis began with a comparison of the Pre-test and Post-test results from Group A (textbook's representation) and Group B (Van de Walle's representation). The data was first coded based on the

procedure used by the student (see Appendix D). Then, based on these codes the data were categorized as a low, medium or high level of procedural comprehension (see Table 1). To obtain a high level of comprehension the answer had to be correct, thereby showing proficiency procedurally. The medium level meant that the student had minor errors or missing steps that led to a correct or incorrect answer. The low level contained major errors and an incorrect answer. Since data categorized in the medium or low levels meant an incorrect answer, these levels were considered not proficient procedurally.

On the pre-test, 32% of Group A showed procedural proficiency (high level of comprehension). Sixty-eight percent of the students in Group A were not procedurally proficient (medium or low levels). Group B's pre-test results were significantly lower. None of the students in Group B showed procedural proficiency. All of the students in Group B showed a lack of procedural proficiency. More specifically, 19% obtained a medium level and 81% obtained a low level (see Figure 1). The post-test results showed significant improvements by Group B compared to Group A. In Group A, only 16% of the students moved from non-proficient to proficient; in Group B, 48% of the students moved from

non-proficient to proficient. The results of the post-test for Groups A and B are compared in Figure 2.

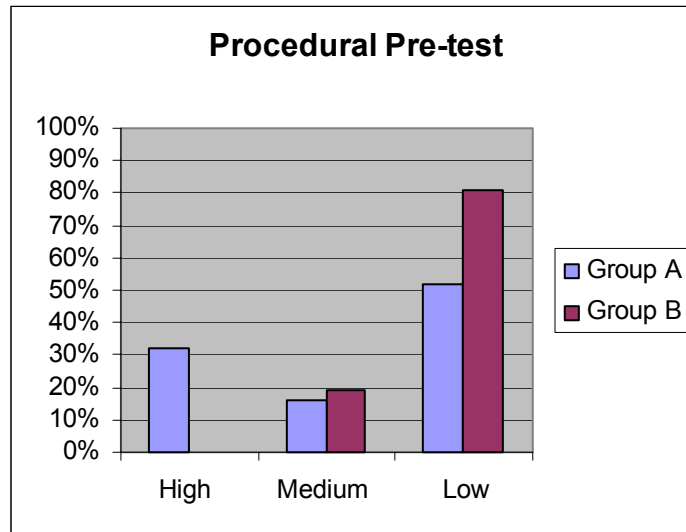


Figure 1. Procedural Pre-test Results

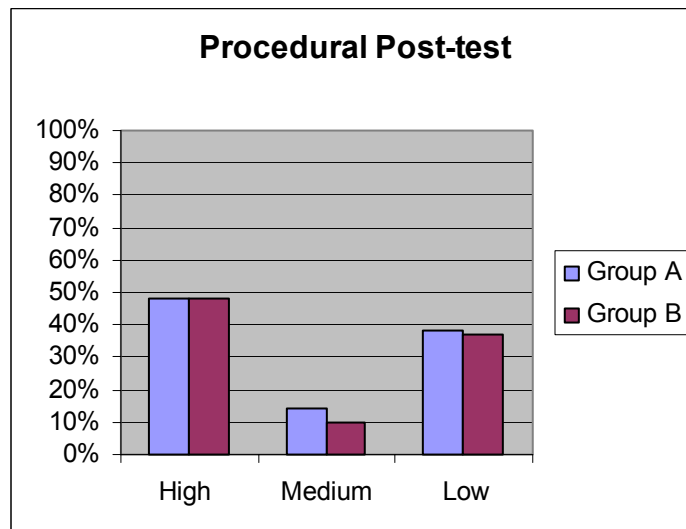


Figure 2. Procedural Post-test Results

While Group B showed impressive gains procedurally, the results from the post-test indicated that the majority of the students were not using the Van de Walle representation. Twenty-one percent of Group B used the method presented in the textbook even though they did not receive direct or indirect instruction using this method. None of the students in Group B generated equivalent fractions using the Van de Walle method of finding and eliminating a common factor. About 10% of the students in Group B did multiply by a fraction equivalent to one to generate an equivalent fraction.

During student interviews, a student from Group B commented, "you divide the numerator and the denominator to get a smaller fraction; well, the numbers get smaller but the fractions stay the same." Other students from Group B made similar comments. This helps explain why students in Group B reverted to dividing rather than Van de Walle's representation, which multiplies, when students wanted to simplify a fraction and obtain a numerator and denominator that were smaller numbers. Intuitively, the students thought of division. After evaluating more student work collected, the students in Group B used the Van de Walle method frequently; however, most students reverted to the

textbook representation which was not part of any of the lessons taught during the course of the study.

By analyzing field notes taken during the fraction lessons, it was evident from the discourse between the teacher and students that the students were noticing patterns. They would verbalize these patterns to the teacher often to explain their procedure or justify their answer. For example, when students were simplifying  $\frac{6}{12}$ , the teacher showed  $\frac{3 \times 2}{4 \times 2} = \frac{3}{4}$ . Students would recognize that this was the same as "halving the numerator and denominator", which is essentially dividing by  $\frac{2}{2}$ . When simplifying,  $\frac{9}{12} = \frac{3 \times 3}{3 \times 4}$ , students noticed the same answer could be obtained by dividing  $\frac{9}{12}$  by  $\frac{3}{3}$ . In student work collected during the study, students struggled with finding a common factor. For example, with  $\frac{12}{18}$ , the students would recognize that  $12 = 3 \times 4$  and  $18 = 2 \times 9$ , but they would become confused about what to do next because there was not a common factor to eliminate. The students were more comfortable recognizing a common divisor; such as 12 and 18

are both divisible by 2 ( $\frac{12}{18} \div \frac{2}{2} = \frac{6}{9}$ ). However, they were not

able to translate that into a multiplication problem of

$$\frac{12}{18} = \frac{6 \times 2}{9 \times 2} = \frac{6}{9}.$$

Most of the students in Group A, divided the numerator and denominator by a fraction equivalent to one to generate equivalent fractions. Additionally, students were able to multiply the numerator and denominator by a fraction equivalent to one whole to also obtain equivalent fractions. During interviews with Group A, one student was hesitant about multiplying to obtain an equivalent fraction. The student commented, "When you multiply the fraction gets bigger." After studying the interview prompt (see Appendix C), two students said, "Mrs. Cline and Mrs. Shark were right because they both divided to obtain an equivalent fraction while Ms. Nixon was wrong because her class multiplied." In fact, Ms. Nixon and Mrs. Shark were correct. Mrs. Cline was incorrect because she divided by a fraction not equivalent to one whole. Students in Group B were more comfortable multiplying to generate equivalent fractions. Unlike Group A, students in Group B recognized that as long as you multiplied by an equivalent of one, the

fractions will be equal. Group B responses also differed from Group A's, because the students used the Van de Walle method to prove that Mrs. Shark's and Ms. Nixon's classes used the correct procedure.

#### Procedural Errors

Both of the representations led to procedural errors noted in the tests and student work throughout the study. When dividing or multiplying the fraction by a fraction equivalent to one whole, some of the students obtained the correct answer but failed to write the procedure correctly.

For example,  $\frac{5}{15} \div 5 = \frac{1}{3}$  or  $\frac{2}{7} \times 2 = \frac{4}{14}$ . This error

pattern was more prevalent in Group A. In fact, during the pre-test and post-test, only Group A reported this procedural error.

Another error pattern emerged from the students exclusively in Group B. The students would factor the numerator and denominator, but added one whole instead of multiplying by the one whole. Figure 3 shows a sample of student work from Group B. The student is simplifying  $\frac{5}{15}$ .

Although the student recognized  $\frac{3}{3}$  as one whole, the student added the one instead of multiplying.

### Conceptual Comprehension

After the data were coded and analyzed on a procedural level, it was analyzed again based on conceptual understanding. It was recoded and analyzed based on the level of understanding for the concept. The data were

The image shows two handwritten mathematical expressions. On the left, there is a fraction  $\frac{5}{15}$  followed by a multiplication problem  $\frac{1 \times 4}{3 \times 5}$ . The numbers 4 and 5 in the numerator and denominator respectively are crossed out with a diagonal line. On the right, the mixed number  $1\frac{1}{3}$  is written and enclosed in a hand-drawn purple circle.

Figure 3. Error Pattern from Group B

categorized as a low, medium or high level of conceptual understanding (see Table 1). Conceptually, results from Group A and B's pre-tests were very similar. In Group A, 94% of the students performed at a low conceptual level



compared to Group B's 90%. In Group A, 6% of the students performed at a high conceptual level compared to Group B's 5% (see Figure 4). On the post-test, 60% more of the students in Group A performed at a medium level. 4% performed at the high level. Comparatively, Group B showed less progress. Only 24% more students performed at a medium level. Additionally, none of the students in Group B performed at a high level of conceptual comprehension (see Figure 5).

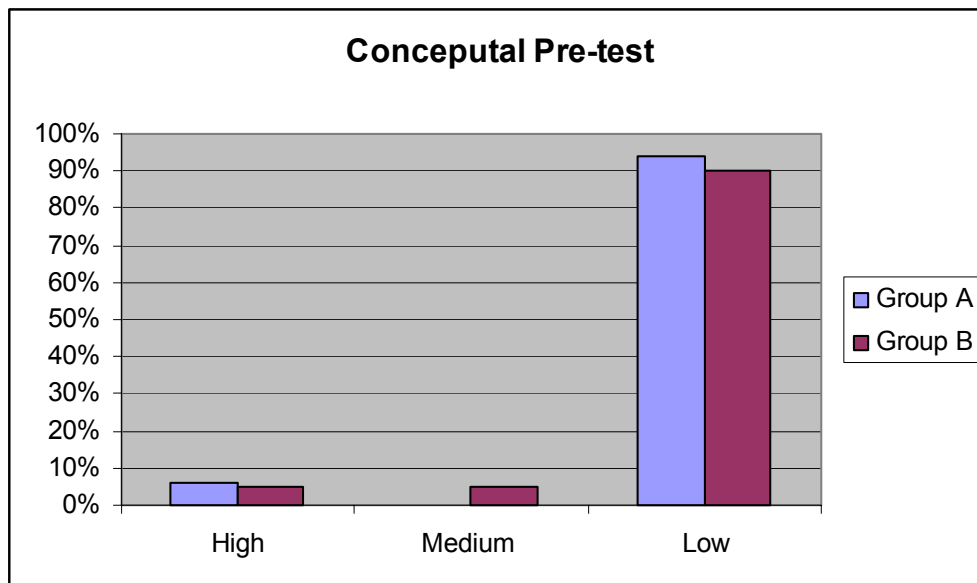


Figure 4. Conceptual Pre-test Results

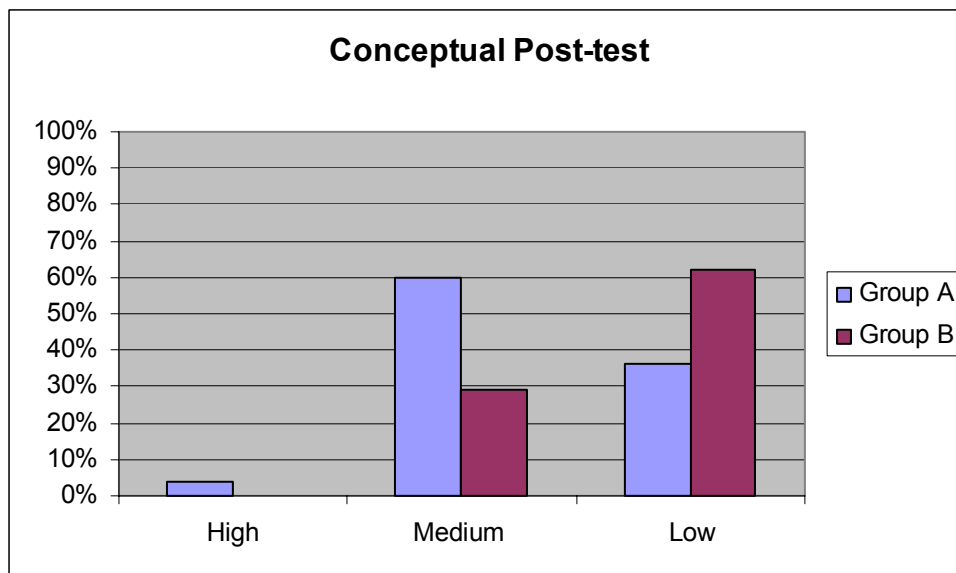


Figure 5. Conceptual Post-test Results

Once again, additional data collected during the study including student work, classroom observations, and interviews offered further insight to the conceptual understanding of fraction equivalence. During the interview, students were asked to draw a picture or a diagram to prove why the three parts of the interview problem were correct or incorrect. The student responses were coded using the same codes for the conceptual comprehension during the pre-test and post-test. 60% of the students performed at a medium level while 40% performed at a high level. In Group B, none of the students performed at

a high conceptual level. 67% performed at medium and 17% at low. These results are consistent with the results from the pre-test and post-test.

In Group A, one of the students drew a picture to explain how the fractions were equivalent and was also able to identify in the pictures where the division of the one whole was occurring. This student was able to link the procedural process to a conceptual understanding. Another student drew the representation shown in Figure 6 to show

that  $\frac{12}{15}$  equals  $\frac{24}{30}$ .

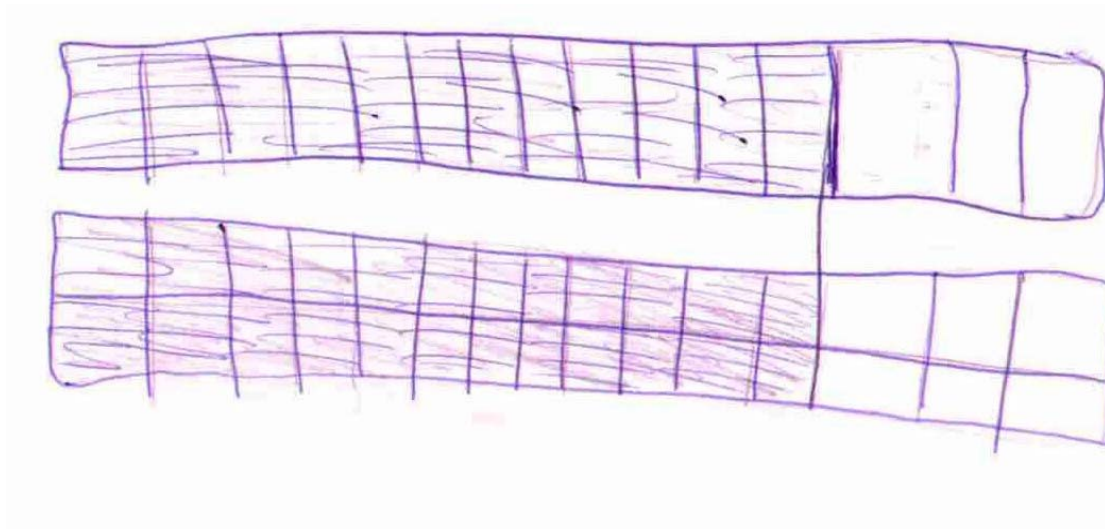


Figure 6. Student Representation from Group A

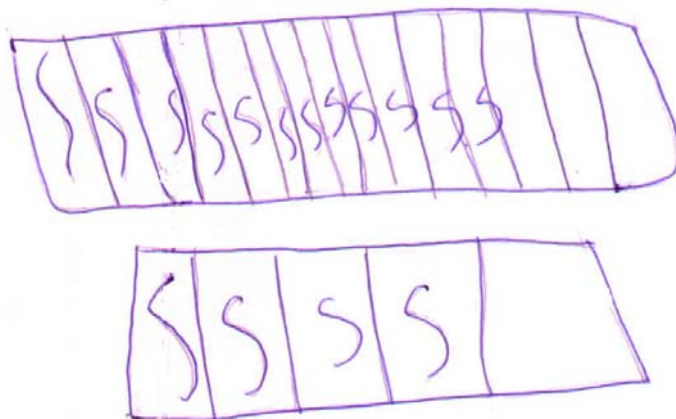


Figure 7. Student Representation from Group B

None of the students in Group B were able to correctly draw a picture to represent equivalent fractions. Most of them were able to recognize equivalent fractions represented in a diagram given to them by the interviewer. Some students in Group B attempted to draw pictures but the whole unit was not drawn equally in the two pictures. Figure 7 shows an example of student work from Group B. The student is trying to show that  $\frac{12}{15}$  equals  $\frac{4}{5}$ .

During interviews, more students in Group B recognized that  $\frac{3}{3}$  was equal to one whole and therefore could not be equal to  $\frac{12}{15}$  which is less than a whole. Students from Group A justified their answers by repeating that you must

"divide the top and bottom by the same number", referring to the numerator and the denominator as the top and bottom. Group B students answered similarly saying "whatever you do to the top you do to the bottom." However, only Group A students were able to prove why this was true. One student from Group B recognized that Mrs. Cline's class (dividing the numerator and denominator by different numbers) used the wrong procedure but did not know why.

The post-tests and other student work collected throughout the study showed consisted results with interview data about student conceptual knowledge of fraction equivalence. Students from Group A were able to represent equivalent fractions with a pictorial model. No students from Group B showed a correct picture or diagram. The majority of students from both groups described the algorithmic function of the procedure using words. Some students from Group A responded by drawing a representation of both fractions but made no connection between the representations. Also, Group A students drew the representation of both fractions, but did not make the unit whole the same size as shown earlier. Except for one instance, all students in Group B described the algorithmic function using words, gave an incorrect response with no

identifiable characteristics or left the response blank. Consistently throughout the data, students in Group B showed lower levels of conceptual understanding.

## OPERATIONS WITH FRACTIONS

### Manifested Error Patterns

The analysis for the second research question began by coding the pre-test and post-test for error patterns with operations with fractions. Both tests contained items requiring students to add, subtract, multiply and divide fractions. The error patterns emerged from the data and are listed in Appendix F. Error Patterns 6, 13, 14, and 15 did not emerge from the pre-test or post-test. These errors were found in additional student work that was collected. Table 3 compares the percentage of correct responses for both groups on the pre-test and post-test. It is important to note that some of the error patterns did result in correct answers.

TABLE 3

*Comparative Results of Operations with Fractions.  
Percentage of Correct Responses*

Group A (textbook)

	Addition	Subtraction	Multiplication	Division
Pre-test	3%	3%	37.5%	0%
Post-test	15%	21%	37%	37%

n = 40

Group B

	Addition	Subtraction	Multiplication	Division
Pre-test	0%	0%	44%	0%
Post-test	7%	13%	31.5%	31.5%

n = 25

The most frequent error pattern was students performing the operation straight across the fractions without finding a common denominator. For example, for addition the student would add the numerators and then add the denominators. For example,  $\frac{2}{3} + \frac{3}{4} = \frac{5}{7}$ . This was only an error pattern for addition and subtraction of fractions. It is a correct method for multiplication of fractions and consequently the correct responses for multiplication were high as a result of the usage of this pattern. For division, it is also a correct method although it is not the traditional way students are taught to divide. On the pre-test item, if the students divided the numerators and

then divided the denominators, the quotient was not a whole number. Some students divided across the fraction and then rounded off to the nearest whole number. For the post-test, because of the different numbers used it was possible for students to divide across the fraction and obtain the correct answer. This resulted in a correct response and is one reason why the percentage of correct answers on the division problems increased on the post-test. This pattern is connected to both interventions. For Group A, students were dividing the numerators and dividing the denominators across the fractions. While providing the correct answer for division problems, it is not the traditional way taught to students and does not always involve obtaining a numerator or denominator that is a whole number. In Group B, students were multiplying the numerators and multiplying the denominators. This is mathematically correct and the traditional way of teaching students to multiply fractions.

On the pre-test for Group A, the pattern of operating across the fractions constituted 45% of the error patterns. It was most common when students added, subtracted, and multiplied. On the pre-test for Group B, this pattern constituted almost 60% of the errors. Again, it was most common for addition, subtraction, and multiplication. On



the post-test for Group A and Group B, the pattern was recorded 40% and 39% of the time, respectively. This indicated a decline for this pattern in both groups.

Another identified error pattern used for multiplying fractions was cross multiplication. Cross multiplication was an emergent error pattern from both groups. This error included students multiplying the numerator of one fraction by the denominator of the other fraction. One of the cross products was recorded as the numerator and the other product was recorded as the denominator. Usually, students recorded the larger number as the denominator. Prior to the pre-test students had not cross multiplied in this sixth grade mathematics class, however, they had used this technique previously in fifth grade. Following the pre-test, students in both groups did use cross multiplication to compare fractions. No connection between cross multiplication and the two methods for finding equivalent fractions was found through student self assessments and other data collected.

Similar to cross multiplication, other error patterns included cross addition, cross subtraction, and cross division. These error patterns occurred in both groups. For example, cross addition meant that a student added the

numerator of one fraction to the denominator of the other fractions and recorded the sums as the numerator and denominator for their answer. Usually, they recorded the larger number as the denominator. Examples of this error pattern are shown in Figure 8.

Solve the following problems.

$$1. \frac{3}{4} \times \frac{2}{3} = \frac{6}{6}$$

$$2. \frac{2}{5} \times \frac{1}{3} = \frac{5}{6}$$

Figure 8. Example of Error Pattern - Cross Addition

No evidence was found that this error pattern was linked to the methods used for finding equivalent fractions. A decline in this pattern was noted over the period of time throughout the data collected. The following paragraphs reveal other error patterns that emerged throughout the course of the study due to new material covered in the class. It seems that the students replaced this error pattern with either a correct answer due to new

knowledge learned or a different error pattern due to a new misconception that formed along the course of the study.

Another pattern noted involved finding a common denominator for the fractions. This pattern was a correct strategy for adding and subtracting. However, it proved to be erroneous for multiplication and division problems because after finding the common denominator the students carried over the common denominator into their answer (see Figure 9).

The image shows handwritten mathematical work. On the left, the problem  $\frac{5}{10} \times \frac{1}{2} = \frac{25}{10}$  is written. To the right, two incorrect solutions are shown. The first is  $\frac{1 \times 5}{2 \times 5} = \frac{5}{10}$ , where the numerator is  $1 \times 5$  and the denominator is  $2 \times 5$ . The second is  $\frac{5 \times 1}{10 \times 1} = \frac{5}{10}$ , where the numerator is  $5 \times 1$  and the denominator is  $10 \times 1$ .

Figure 9. Error Patter when Multiplying Fractions

This pattern was recorded more often in the post-test compared to the pre-test for all four operations. Between

the pre-test and the post-test, students learned about adding and subtracting fractions. This explains the increased frequency of the pattern. It also explains why student percentages of correct responses in addition and subtraction rose between the pre-test and post-test. Consequently, it is a possible reason for student percentages of correct responses declining for multiplication.

The additional patterns coded were not as frequent and not as obviously connected to a specific method used during teaching the fraction unit. For example, one of the error patterns included adding instead of subtracting. This was most likely a careless error made due to misreading the problem and was not consistent enough to investigate further. Some of the patterns were due to misconceptions that students had previously about fractions. The state curriculum identifies adding and subtraction fractions as a sixth grade learning goal and multiplying and dividing fractions as a seventh grade learning goal. Until sixth grade, students have had very little to no experience in operating with fractions. One such misconception about fractions was noted in field notes during a one-on-one interaction between a student and the teacher. The student

inquired about dividing fractions and when the topic would be studied. The teacher curiously asked the student what he knew about dividing fractions. The student replied confidently, "Dividing fractions is easy. You just switch the numbers and multiply." Looking for more information the teacher gave the student a few division problems to try.

The first problem stated,  $\frac{2}{3} \div \frac{1}{4}$ . The student answered,  $\frac{3}{8}$ .

The following problem was  $\frac{2}{9} \div \frac{1}{8}$ , for which the student

answered  $\frac{9}{16}$ . The student solved three more division

problems and consistently used the error pattern. This student's misconception about division was that you "switched" some of the numbers. The student failed to "switch" the correct fraction. The error pattern displayed was that the student inverted the dividend instead of the divisor before multiplying the fractions. While this misconception stems from another experience outside of the intervention being studied, it does give insight to some of the other error patterns that emerged from the student data where students make mistakes using an algorithm that has little meaning to them. For example, cross multiplication

is an algorithm that students do not fully comprehend and are unable to apply appropriately.

In addition to the pre-tests and post-tests, additional student work was analyzed. As noted earlier, from the student work additional error patterns emerged that were not present on the pre-test or post-test. However, the frequency of the error patterns remained consistent with the findings from the analysis of the pre-tests and post-tests. Performing the operation straight across the fraction remained the most frequent error pattern. This error pattern shows a very intuitive way to operate with numbers. Almost all of the students' previous experience with operating with numbers has involved students solving problems with whole numbers by starting at the left and working to the right. The students are not viewing the fractions individually as numbers that are less than a whole, but rather they are separating the numerators from the denominators to make it look like a more familiar problem. By separating the numerators and the denominators, the student has two problems to solve that look like common mathematics problems that they have encountered previously.

## Linking Error Pattern with Instructional Intervention

The second part of the research question seeks to connect the error pattern with the method of teaching equivalent fractions. Some brief statements were mentioned about this in the previous paragraphs, but this section will continue to bring more detail about the subject. Several strategies were used to link the error pattern to the particular methods of study.

Self-assessments by the students gave insight to what prior knowledge the student was using to solve the problem (see Appendix G). Specifically for operating with fractions, two questions were asked: (1) What mathematics did you use to solve these problems? (2) If you had to teach your little brother or sister how to add fractions, what would you tell them?

For adding fractions, sample responses from questions one included the following: finding a common denominator, adding, how to find a common factor, how to simplify your answer, how to cross multiply, and how to multiply to get a common denominator. This self-assessment is helpful because by analyzing student work, you can not only see the error pattern, but you can also gain insight to what previous

knowledge the student is using. For example, a common error pattern for subtracting fractions was to subtract the denominator of one fraction and the numerator of the other fraction. This seemed to resemble cross multiplication. When asked what mathematics the student used to solve the problem in the self-assessment, the student listed cross multiplication. Thus, providing direct evidence of where the misconception was rooted. Pertinent to this study was whether or not the error patterns were connected to the methods of teaching equivalent fractions. Therefore, the student work and self-assessments were analyzed to find clues that would help make these specific connections. From the student responses several connections were inferred.

First, when dividing fractions a student from Group A responded to the question about what mathematics he/she used to solve the problem. "I divided the numerator by the numerator and the denominator by the denominator." By reflecting on past experiences that this student has had in mathematics lesson, the method of finding equivalent fractions used with Group A directly relates to this statement. Consequently, a greater percentage of students in Group A used this strategy and correctly answered the division problems.



Additionally, another difference was noted throughout samples of student work from daily assignments. Group A showed a better ability to apply their knowledge of equivalent fractions in diverse settings. Here is a description of a specific instance that was recorded in the observation notes. This took place midway through the course of the study. The students had already received instruction on finding equivalent fractions as well as adding and subtracting fractions. As a review, during a group assignment students were given a problem that incorporated adding two fractions. When adding the fractions  $\frac{4}{12} + \frac{2}{3}$ , the student calculated,  $\frac{4}{12} \div \frac{4}{4} = \frac{1}{3}$ . With 3 as the common denominator, the student added  $\frac{1}{3} + \frac{2}{3}$  to obtain the correct answer. Students in Group A showed the ability throughout student work samples to find common denominators by either multiplying or dividing by a fraction equivalent to one whole. However, in Group B no evidence was found in any of the work samples of students dividing to find a common denominator. This showed that with regards to finding common denominators to solve

addition and subtraction problems the students in Group A had a higher level of procedural fluency.

Additionally, pre-test and post-test comparisons offer further evidence. For Group A, the percent of correct responses for the multiplication problems decreased by 0.5% while Group B's percentages dropped by 12.5%. Students in The main difference between Group A and Group B was that for finding equivalent fractions Group B's was solely based on multiplication of fractions. From the self-assessment, no direct connection was evident between the multiplication of fractions during the method of study and the multiplication of fractions in a basic operational problem. The post-test results further support that conclusion by showing that although the students had more experience with multiplying fractions, they were not able to make the connections and be as successful with operations with fractions involving multiplication.

## CONCLUSIONS

### COMPREHENSION OF FRACTIONS

From the results, several conclusions can be drawn to help answer the research questions. Here is an overview of what led to these conclusions. The first question was "What are the implications for student comprehension of fractions resulting from learning fraction equivalency using two different methods?" To answer this question, the data were analyzed for two different components: procedural comprehension and conceptual comprehension. Using a specific set of criteria, the data were categorized into levels of comprehension (low, medium and high) to describe the proficiency of the student's work. The high level was deemed proficient while the medium and low levels were considered non-proficient. Data were collected from pre-tests, post-tests, field notes, student work samples, and semi-structured interviews.

## Procedural Comprehension

The students who used the textbook representation, Group A, showed a higher procedural proficiency. Group B received instruction using Van de Walle's representation and reported a higher procedural proficiency. However, even though they did not receive instruction using the textbook representation, the overwhelming majority of the students in Group B used the textbook representation to solve the problems. "During experiences with a concept or a process a student focuses on whatever the experiences appear to have in common, and connects that information to information already known" (Ashlock, 2002, p. 14). The students in Group B made connections from the Van de Walle representation to previous experiences and constructed their own representation for finding equivalent fractions, which was in fact, the textbook representation. To simplify a fraction, students in Group B more readily equated obtaining smaller numbers in the fraction with division as implied from the interview results. From previous knowledge with division, they understood that dividing obtained a smaller number. This shows that the textbook representation had a stronger connection to past experiences, thereby

promoting proficiency when using the representation to generate equivalent fractions. As teachers, building on prior knowledge is an important tool to use in the classroom with all mathematical topics. In this case, making these connections and finding a representation that was sensible to the students promoted better procedural proficiency.

Additionally, students in Group A equated division with obtaining a smaller number. Often in early grades, teachers can reinforce rules that are applicable in their grade level, but may not be true all the time. In this case, students believed that division was the way to get smaller numbers. They also confused smaller numbers in the fraction with a smaller value. When operating with whole numbers, this is true. However, with rational numbers this is not always the case. Smaller numbers in a fraction can often mean an equivalent if not greater value than a fraction that contains larger numbers. Instruction that emphasizes the role of dividing by a fraction equivalent to one whole can help alleviate the misconception that division results in a smaller value. By reinforcing the identity property of one and emphasizing or even rewriting the fraction as one whole, students can challenge this

misconception. In Group A, the intervention included the division of a fraction equivalent to one whole; however, while students knew that they should divide the numerator and denominator by the same number, they did not always understand why. As teachers use the textbook representation of dividing by a fraction equivalent to one whole, it is important to not separate the numbers as dividing numerators and dividing denominators. Instead, emphasis should be placed on the fraction equivalent to one whole and the properties of dividing by one.

Teachers must be sensitive to their mathematical language and explanations so as not to impede the learning of the students in later grade levels. For example, in elementary school teachers often present the number line starting at 0 as the smallest number. Students are taught that you cannot subtract  $5 - 9$ . Later, when a student is introduced to negative numbers, the students are plagued with the rule that you must subtract a smaller number from a larger number. These small inconsistencies in our teaching can lead to challenges for students in higher grade levels when the teacher is introducing integers and rational numbers. It could be helpful to introduce a broader version of the number line at an earlier grade

level. While students are building their basic addition and subtraction skills using concrete examples and representations such as the number line, it may be helpful to expose them to more integers and rational numbers.

Procedural fluency refers to knowledge of procedures as well as how to use them appropriately and flexibly (National Research Council, 2001). Group A (using the textbook representation (see Appendix A) showed greater flexibility in applying the procedure. For example, when operating with fractions, students in Group A used multiplication and division to find a common denominator and generate equivalent fractions. This flexibility actually shows a higher level of understanding, because not only can the student use the procedure, but they can also apply it to new and different situations flexibly. Teachers should provide students with the opportunity to use procedures in a variety of contexts. Otherwise, the student may be left with a rigid understanding of only specific times and ways that the procedure can be used. An effective way to promote this learning is to contextualize the problem and expose students to real world applications of the procedure (Perlmutter, Bloom, Rose, & Rogers, 1997).

## Conceptual Comprehension

The strong procedural fluency exhibited by the students who used the textbook representation supports the findings of this section. Procedural skills and conceptual understanding are interwoven (National Research Council, 2001). It should come as no surprise that the textbook representation which yielded high procedural comprehension would also indicate a higher conceptual level of comprehension. Conceptual understanding helps students avoid procedural errors as well as modify and adapt the procedures to new situations (National Research Council, 2001). "Understanding the concepts and reasoning involved in an algorithm does lead to a more secure mastery of that procedure" (Ashlock, 2002, p. 8).

Data analyzed for conceptual understanding included pictures and diagrams drawn by the students to explain their procedure and prove their results. During semi-structured interviews Group A was able to draw representations of equivalent fractions and link the representations to the procedure. No students from Group B were able to do this. These student representations help teachers understand the "students' ways of interpreting and



thinking about mathematics" (National Council of Teaching Mathematics, 2000, p. 68). From the student representations collected during the interviews, it was shown that Group A had a better conceptual comprehension than Group B.

It is important to note that some manipulatives or drawings lend themselves to one particular method of representation (Watanabe, 2002). When representing equivalent fractions, field notes from the lessons show that the teacher modeled fractions predominantly by using shapes cut into fractional pieces. For example, to represent equivalent fractions the class would draw a rectangle to represent one whole. Then, the rectangle was divided and shaded accordingly to represent the fraction. Next, a congruent rectangle was drawn to represent the same size whole. This time the rectangle was divided into twice as many pieces as the first. It was then shaded so that the shaded pieces were equivalent to the shaded pieces in the original rectangle. Quite simply, the pieces of the original rectangle were divided by 2 in order to form the second rectangle. Thus, the representation with shapes reflects the textbook representation most closely. The Van de Walle method, however, does not lend itself as easily to a representation. This may have contributed to the

students' inability to conceptualize and represent the method pictorially.

The lack of conceptualization shown by the representations might further support the need to make connections to previous learning. Typically, in elementary grades fractions are represented using shapes and shading a part of a whole. Students become familiar with the part to whole representation and can easily extend the representation to show equivalent fractions as represented by the textbook method.

Reflecting on the students results on the conceptualization of equivalent fractions based on the student representations, two implications can be drawn. First, making connections is a powerful tool for student learning. Second, it is possible the students were not exposed to enough of a variety of representations to be able to represent the Van de Walle method adequately. Presenting various modes of representation can increase student understanding. Additionally, students have the opportunity to find a representation that make the most sense to them. Better yet, students should be encouraged to create their own representations (National Research Council, 2001).

### Comprehension of Fractions Equivalent to One Whole

Students from Group B (using Van de Walle representation) showed a better understanding of the fractions equal to one whole. In the interviews, students from Group B identified the fraction  $\frac{3}{3}$  as equivalent to one whole; therefore, students reasoned that it could not be equivalent to  $\frac{12}{15}$ . None of the students from Group A used this justification. Furthermore, procedural errors emergent in Group A's work confirmed the lack of understanding of fractions equivalent to one whole. The most common error from students in Group A was writing the divisor as a whole number rather than a fraction equivalent to one. For example,  $\frac{5}{15} \div 5 = \frac{1}{3}$  or  $\frac{2}{7} \times 2 = \frac{4}{14}$ . Tirosh (2000) classifies this mistake as an intuitively based mistake and states the following: "The primitive, partitive model of division imposes three constraints on the operation of division. . ." p. 7. The first of these three "constraints" being that the divisor must be a whole number. This procedural error indicates a gap in the conceptual understanding of this process as well as a faulty reliance on a previous experience with division. This again emphasizes the importance of teachers in early

grade levels to be aware of the misunderstandings that they may be reinforcing when teaching basic skills such as division.

On the other hand, the representation used by Van de Walle emphasizes the function of a fraction equivalent to one whole. In this representation the common factors are eliminated and rewritten as one. Consequently, students are repeatedly linking the fraction equivalent to one with the whole number one. Being aware of the strengths and weaknesses of each representation can guide teachers when planning their instruction. In this case, a teacher choosing to use the textbook method should find ways to reinforce the concept of a fraction equivalent to one whole. One idea could be for the students to check their answer by rewriting the divisor (the fraction equivalent to one whole) as the whole number one, somewhat similar to what happens in the Van de Walle representation. This would help students identify the divisor as a fraction equal to one whole which would hopefully eliminate the procedural error of writing the divisor as a whole number. It would also emphasize the identity property of one that this procedure is based on. Any number divided by one results in a number of the same value despite the fact that the

numbers in the second fraction may be smaller than the numbers in the original fraction.

Another weakness was noticed from the textbook representation used by Group A. In field notes and interviews, students from Group A often used the phrase "whatever you do to the top, you do to the bottom" to explain the procedure for finding equivalent fractions. This shows a separation of the numerators and denominators as two different problems. The National Research Council (NRC) (2001) confirms that interpreting rational numbers as numbers, although basic, is often overlooked. Students are accustomed to thinking of rational numbers as parts of a whole which can lead to an "inadequate foundation for building proficiency" (NRC, 2001, p. 235). The NRC continues to point out that even the symbolic nature of a fraction contributes to the misunderstanding of rational numbers. A fraction looks like a whole number over another whole number and leads students to think of them as two different numbers. Students in Group A seem to be separating the division of the numerators from the division of the denominators, thus, creating two separate division problems with whole numbers. This deemphasizes and nearly eliminates the function of the textbook representation

which is to divide a fraction by another fraction equivalent to one whole. Identifying fractions on a number line may be a useful tool to help students see how the rational numbers fit into the whole number system which is more familiar to them. It could also strengthen their number sense skills so that the students could make better estimations and predications about the reasonableness of their answers.

#### ERROR PATTERNS IN OPERATING WITH FRACTIONS

The second research question was "What error patterns and misconceptions do sixth grade students have when operating with fractions? And, are these errors implied from the methodology used to learn fraction equivalency?" Ashlock (2002) states "errors are a positive thing in the process of learning. . . an opportunity to reflect and learn" (p. 9). To begin the process of reflection and learning, data were collected from items that involved operation with fractions: adding, subtracting, multiplying, and dividing. The data were coded for error patterns. Then, using self-assessments the error patterns were linked to

the two representations used for teaching fraction equivalency.

#### Operating Across the Fraction

The most common error pattern for operations with fractions was for students to perform the operation horizontally across the fractions. For example, when adding fractions, students would add the numerators and then add the denominators. Once again the separation of the numerators and the denominators creates two separate addition problems of whole numbers.

Group A exhibited these same misconceptions about rational numbers when finding equivalent fractions using the textbook representation as discussed in the previous section. Consequently, Group A had a higher percentage of this error pattern when dividing fractions showing a link between the intervention and their ability to divide fractions. Although the correct answer can be obtained by dividing the numerator and then dividing the denominator, it is not the traditional way of teaching division of fractions. Students from Group A obtained the correct answer by operating across the fraction until division

problems did not produce a whole number quotient. "Children may try to apply ideas they have about whole numbers to rational number and run into trouble" (National Research Council, 2001, p. 416). Students then began to adjust their method by rounding off the answer or using some other modification. However, Van de Walle's representation used by Group B multiplies across the fraction which is mathematically correct and will not need to be relearned in the future. The textbook representation proved to have more negative implications on student's ability to perform operations with fractions. As discussed in the previous section, by giving students a better understanding of rational numbers and correcting the students' tendency to separate the numerators from the denominators, this error pattern may be avoided. This misconception about rational numbers is proving to be not only a problem when finding equivalent fractions but also when operating with fractions.

#### Overgeneralizing

As noted earlier, overgeneralizing is described by Ashlock (2002) as jumping to a conclusion based on



inadequate data. Tirosh (2000) adds that students overgeneralize properties of operations with natural numbers to fractions. Overgeneralizing contributed to the error patterns on operations with fractions. The most prevalent error pattern when multiplying with fractions was cross multiplication. This error pattern was found in both Group A and Group B. Students had previously learned cross multiplication in order to compare and order fractions. The students modified this method in order to add, subtract, and divide fractions. Instead of multiplying across the numerator and denominator, they would add or subtract or divide. Then, in almost all the cases, the larger number was recorded as the denominator.

These observations can lead to several conclusions. First addressed will be the students tendency to use the larger number as the denominator. Students usually learn a fraction as a part of a whole. When confronted with an improper fraction it does not fit their preconceived interpretation of fractions. It is hard to understand how  $5/4$  could be "5 parts out of 4." Therefore, students adjust the numbers to what is more familiar to them which is the larger number as the denominator. Thompson and Saldanha (2003) assert the following: "We see a strong possibility

that nonintroductory lessons about fractions are largely meaningless to many students participating in them" p. 95. From this quote, it should be noted that teachers should make the fraction lessons more meaningful to students because it is building a foundation for future experiences with fractions. It is possible that the "part of a whole" interpretation of fractions was emphasized in instruction without a variety of meaningful representations and applications that extended into at least an informal knowledge of improper fractions or mixed numbers.

Secondly, it highlights the fact that students relied on a basic procedural understanding of cross multiplication and a weak conceptual understanding of the algorithm. The students are applying the procedure in the wrong situation. Cross multiplication should only be taught to students if they can conceptualize why they are using it. It can become a shortcut to learning that is taught by repeated drills and memorization rather than focusing on a solid understanding of comparing fractions. When teaching students to compare fractions, the lesson should begin with students estimating the order using benchmark fractions. The students should be able to represent the fractions with a diagram or picture to prove their answers. Additionally,

the students can use a common denominator to order and compare fractions. For the sake of time, many teachers jump to the cross multiplication of fractions and leave students without a conceptual understanding of the topic and thereby leading to erroneous patterns when applying the algorithm. In the long run, cross multiplication is ironically a shortcut that leads to long term negative implications and the need for review and re-teaching episodes. Cross multiplication is only one example of many algorithms that are over generalized by students. Similar patterns of overgeneralizations can be avoided by emphasizing procedural and conceptual comprehension.

#### Emergent Error Patterns

Several error patterns emerged from the data that were not prevalent on the pre-test items, such as, finding a common denominator. Over the course of this study the students were given instruction on adding and subtracting fractions. Included in this instruction was how to find a common denominator. On the post-test this became an error pattern used when multiplying and dividing fractions.

Carpenter et al. (1976) stated that "students making this

error have learned the process of finding common denominators, but they do not understand when it needs to be applied" p. 139. On the post-test students correctly used finding a common denominator to add and subtract fractions and then tried to also apply it to multiplying and dividing fractions. Although, the correct answer could be derived this way, students were not applying it correctly. The most common error was to carry the common denominator into the answer. While correct when adding and subtracting, it is erroneous in a multiplication or division problem.

Another error pattern observed on student work triangulated by field notes was confusion about the division algorithm. The students inverted the dividend instead of the divisor, thereby, showing lack of conceptual knowledge to support the procedure resulting in errors. Tirosh (2000) categorizes this error as an algorithmically based mistake and describes it as resulting from the rote memorization of the algorithm. "When an algorithm is viewed as a meaningless series of steps, students may forget some of these steps or change them in ways that lead to errors" p. 7. An algorithm as a "meaningless series of steps" has proven to be a common thread to error patterns. Division

has been a controversial topic based on when it should be taught and how it should be taught. The algorithm is based on algebraic reasoning that students in middle school may not be familiar with. In this case, teachers must come up with more creative and meaningful ways to help students conceptualize the division of fractions. In a study on pre-service teachers and their understanding of the division of fractions, it was shown that many of the pre-service teachers did not properly understand why the algorithm for division of fractions worked (Tirosh, 2000). Teachers must challenge their own knowledge and the ways that they were taught to learn mathematics years ago when rules and rote memorization were more prevalent. With the better understanding of how children learn, teachers must strive to deemphasize these meaningless rules and promote a solid conceptual and procedural understanding. In the case of division, instruction should begin with a concrete representation and slowly progress to the abstract. After using a concrete model of dividing fractions students will be able to slowly move to a more symbolic representation. Hopefully, this will lead to less errors in dividing fractions and eliminate the reliance on a meaningless algorithm. It may be possible for students to derive the

invert and multiply algorithm on their own making it more meaningful to them or it may lead to students choosing a different method altogether such as finding a common denominator. Either way it will be rooted in a solid understanding of the division of fractions that is meaningful to the student. It is also important to embed this understanding into contexts that relate to the students and show examples of real applications of the knowledge in various situations, creating an even stronger understanding of the concept.

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## APPENDIX A

*Fraction Equivalence Representations*

Math Thematics (1999)

$$\frac{7 \div 7}{21 \div 7} = \frac{1}{3}$$

NCTM Recommendation as presented by Van de Walle (2001)

$$\frac{7}{21} = \frac{1 \times 7}{3 \times 7} = \frac{1}{3} \times 1 = \frac{1}{3}$$

## APPENDIX B

*Post-test*

## Section I

1. Write two fractions equivalent to  $\frac{5}{15}$ .
2. Use a picture or words to explain your answer.
3. Write two fractions equivalent to  $\frac{9}{12}$ .
4. Use a picture or words to explain your answer.

## Section II

5. Add  $\frac{2}{6} + \frac{4}{10}$ .
6. Subtract  $\frac{6}{10} - \frac{4}{8}$ .
7. Multiply  $\frac{6}{8} \times \frac{2}{6}$ .
8. Divide  $\frac{6}{8} \div \frac{2}{4}$ .
9. Add  $\frac{15}{25} + \frac{3}{5}$ .
10. Divide  $\frac{4}{12} \div \frac{3}{6}$ .

## APPENDIX C

*Interview*

**The sixth graders are finding equivalent fractions for  $\frac{12}{15}$ .**

In **Mrs. Cline's** class, the students divided the numerator by 4 and the denominator by 5.

$$\frac{12 \div 4}{15 \div 5} = \frac{3}{3}$$

In **Mrs. Shark's** class, the students divided the numerator and the denominator by 3.

$$\frac{12 \div 3}{15 \div 3} = \frac{4}{5}$$

In **Ms. Nixon's** class, the students multiplied the numerator and denominator by 2.

$$\frac{12 \times 2}{15 \times 2} = \frac{24}{30}$$

1. **Which class or classes answered the question correctly? Incorrectly?**
  - a. If they say Mrs. Cline is wrong Ask: **Why is Mrs. Cline's wrong?**
  - b. If the student says because you have to perform the same operation to the numerator and the denominator ("what you do to the top, you do to the bottom") then ASK: **Why is it important to do the same thing to the numerator and denominator?**
  - c. If they say Mrs. Cline is *right* Ask: **Can you show me proof that it is right (picture, representation)?**  
*First, give them a blank sheet of paper. If they cannot get started successfully, then give them diagram to help. If they are still struggling, offer fraction strips.*
  - d. If they say Ms. Nixon is wrong Ask: **Why is Ms. Nixon's wrong?**
  - e. If they say Mrs. Shark is wrong Ask: **Why is Mrs. Shark wrong?**

2. **When you divide the numerator and denominator by the same number, what happens to the fraction (*gets smaller, larger, stays the same*)?**
  - a. If the students respond smaller then Ask: **Do the numbers get smaller or does the fraction get smaller?** If the student answers the fraction gets smaller then move to R1
  - b. If the students respond the number gets smaller but the fraction stays the same (or equivalent statement) then Ask: **How can you prove that?**
  - c. If the students respond the fraction gets smaller then Ask: **Why do you think the fraction gets smaller? Can you draw me an example?**
3. **When you multiply the numerator and denominator by the same number, what happens to the fraction?**
  - a. If the student responds that it gets bigger, Ask: **Do the numbers get bigger or does the fraction get bigger?**
  - b. If the student says the fraction gets bigger then move to R2
  - c. If the student responds the numbers get bigger but the fraction stays the same (or equivalent statement) then Ask: **How can you prove that?**
  - d. If the students respond the fraction gets bigger then Ask: **Why do you think the fractions gets bigger? Can you draw me an example?**



## APPENDIX D

*Code Map for Procedural Comprehension of Fraction Equivalence*

F	Numerator and denominator are divided by the same factor
G	Answer is wrong - no identified error pattern
H	Simplified the fraction and/or doubled the numerator and denominator, showed no work
I	Only simplified the fraction, no work shown
J	Cross multiplied
K	Divided or multiplied by a whole number not equal to one, but obtained correct answer (ex. $1/5$ multiplied by 3 = $3/15$ ; $6/9$ divided by 3 = $2/3$ )
L	Divided or multiplied the numerator and denominator by different numbers
M	Multiplied by number equivalent to one in order to obtain answers, no work shown
N	One correct answer, one incorrect answer
P	Factored the numerator and denominator, then eliminated the common factor
Q	Factored the numerator and denominator, but added one whole instead of multiplying by one whole (ex. $3/15 = 1 \times 3/5 \times 3 = 1 \ 1/5$ )
0	No answer

## APPENDIX E

*Code Map for Conceptual Comprehension of Fraction Equivalence*

I	Described algorithmic function of procedure using words
II	Drew a representation of both fractions but made no connections between the representations
III	Drew a representation of both fractions, but did not make the unit whole the same size
IV	Drew a representation of both fractions and showed that the same amount is shaded in both representations
V	Incorrect response with no identifiable characteristics
0	No answer

## APPENDIX F

*Code Map for Error Patterns on Operations with Fractions*  
 2. (a) *What error patterns and misconceptions do sixth grade students have when operating with fractions?*

<b>Teaching Method</b>	
A	Textbook
B	Van de Walle
<b>Error Pattern</b>	
0	no answer
1	Cross multiply
2	Perform operation straight across fraction
4	operate across and round off to nearest whole number
5	found a common denominator
6	cross subtract
7	multiplied numerators/added denominators
8	no discernable pattern
9	$\frac{1}{2}$ (on division problem)
10	correct answer
11	adding instead of subtracting
12	add numerators and multiply denominators
13	cross divide
14	subtract instead of divide
15	add numerators and divide denominators
<b>Instrument</b>	
T1	Pre-test
T2	Post-test
S	Student work
I	Interview
J	Self-assessment
F	Field Notes
<b>Operation</b>	
A	Add
Sb	Subtract
D	Division
M	Multiplication

## APPENDIX G

*Example of Student Self Assessment Questions*

1. What does someone need to know to be able to do this assignment?
2. What mathematics did you use to solve these problems?
3. Did you use any drawings or manipulatives to help you solve the problems? If so, can you describe how you used them?
4. How do you know your answer is correct?

Note: Questions adapted from Ashlock (2002)

## APPENDIX H

*Research Questions in Relation to Data Sources*

Research question	Data source
1. What are the implications of student comprehension of fractions resulting from learning fraction equivalency using two different methods?	I1, I2, I3, T1, T3, S, J, F
2. (a) What error patterns and misconceptions do sixth grade students have when operating with fractions?	I2, I3, T2, T3, S, J, F
2. (b) And, are these errors implied from the methodology used to learn fraction equivalency?	T3, S, J, F

I = Interview (*number indicates the specific question on the interview*)

T = Pretest/Posttest (*number indicates specific section of test; T3 specifies open-ended questions on test*)

S = Samples of student work (*classwork, quizzes, etc.*)

J = Self-assessment/student open ended questions

F = Field notes (*including follow up questions from student work*)

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