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Power-law fading of the frustration effect in a periodic rectangular superconducting network with increasing aspect ratio

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It is shown that the same amount of total frustration imposed on a rectangular superconducting network produces a monotonically diminishing effect as the aspect ratio a/b of the network is increased. Various types of power-law behavior are found in the limit of $a/b \to \infty$, the most interesting one being that the slope discontinuities of T_{c2}^{*}/T_{c0} at rational (p/q) flux quanta per unit cell approach zero according to the power law $(a/b)^{-q}$, so that the higher-q cusps fade away faster than the lower-q ones.

"Frustration"[1] is one of the most fundamental synthesizing ideas in the contemporary conceptual framework of condensed-matter physics. It underlies the basic physics of a very broad "superclass" of materials that are now all termed "glass." Thus, such novel terms as protons glass,[2] electron glass,[3] and gauge or superconductive glass,[4] etc., have been introduced into the modern vocabulary of condensed-matter physics, in addition to the more familiar terms such as structural glass, metallic glass, and spin glass. To put it in very simple terms, frustration means the existence of mutually conflicting demands from different terms in a many-body Hamiltonian, so that the system must strike a delicate compromise to find its lowest-energy eigenstate. Usually, frustration occurs in a random distribution due to quenched disorder, but in some systems frustration can also occur uniformly. A simple example of the latter is a triangular array of Ising spins with nearest-neighbor antiferromagnetic bonds. These bonds give conflicting demands to any odd number of spins that form a closed path in the lattice, including all triplets of spins that form the elementary triangles.

Experimentally by Pennetier, Chaussy, Rammal, and Urllegier[4] and theoretically by Alexander,[5] and by Rammal, Lubensky, and Toulouse,[6] etc., the upper critical field $H_{c2}^{*}(T)$ [or the inverse function $T_{c2}^{*}(H)$] of a two-dimensional, periodic, square, superconducting network has been studied as an example of *tunable* uniform frustration. For example, simply by changing a magnetic field applied perpendicular to the network, one can tune the amount of frustration that is imposed on every closed superconducting path in the network as a result of the Aharonov-Bohm effect.[7] However, whenever the total flux Φ through each unit cell of area A_0 is a rational fraction p/q of the flux quantum $\Phi_0 \equiv hc/2e$, some closed paths in the network will enclose an integer multiple of Φ_0, and therefore will not be frustrated, implying a relatively more favorable situation for the nucleation of superconductivity in the system. Thus T_{c2}^{*} should be relatively less suppressed from its zero field value T_{c0} for this value of H. With this understanding in mind, Pennetier *et al.*[4] measured T_{c2}^{*}/T_{c0} as a function of Φ/Φ_0, and observed a dense distribution of cusps (or slope discontinuities) of various sizes at all rational values of Φ/Φ_0 which agree very well with theoretical predictions.[5,6] While this result is extremely interesting, it is clear that the *effect* of frustration to the *whole* network is not *monotonically* tuned when Φ/Φ_0 is increased, because to any closed path of area nA_0 in the network, frustration is a *periodic* function of Φ/Φ_0 of period $1/n$.

The purpose of this Rapid Communication is to report a different kind of tuning of frustration, where it is the *effect* of frustration to the *whole* system that is *monotonically* tuned, as if the system is made more and more immune to the presence of frustration. The system we have studied is only a slight generalization of the system studied by Pennetier *et al.*[4] Namely, we study a two-dimensional *rectangular* superconducting network of various aspect ratio a/b. We find that by simply increasing a/b from unity to infinity (or, equivalently, by decreasing a/b from unity to zero), the *effect* of frustration to the *whole* system can be *monotonically* reduced. For a direct quantitative measure of the effect of frustration to the whole system, we shall again use the general behavior of $(1-T_{c2}^{*}/T_{c0})$, but in particular we shall look at the sizes of the slope discontinuities of this quantity at the rational values of Φ/Φ_0. We wish to emphasize that since we have kept the product $ab(\equiv A_0)$ constant as we vary a/b, we have kept the amount of frustration imposed on any individual closed path in the network constant. It should be further noted that there is a one-to-one correspondence of all closed paths in any two rectangular networks of different a/b. We may therefore say that the networks of different a/b ratios have been subject to the same amount of *total* frustration (as a *cause*) at any given values for $A_0 \equiv ab$ and H, and yet the *effect* of frustration to these systems as measured by the said slope discontinuities, for example, is still found to be a monotonically decreasing function of a/b. Our result is based on a numerical study, which not only establishes this monotonic dependence, but further reveals many interesting types of power-law behavior in the asymptotic regime. In particular, we find that as $a/b \to \infty$, the size of the slope discontinuity at $\Phi/\Phi_0 = p/q$ decays to zero according to the power law $(a/b)^{-q}$. This means that those cusps in the T_{c2}^{*}/T_{c0} vs Φ/Φ_0 curve associated with a larger integer *denominator* q fade away faster than those associated with a smaller q, leaving a less "jerky" T_{c2}^{*}/T_{c0} vs Φ/Φ_0 curve at larger values of a/b.

The theory of superconducting networks, which is based on the phenomenological Ginzburg-Landau theory, plus the de Gennes boundary conditions[8] at the nodes of the networks, has been reviewed in many previous publications.[5,6,9] Briefly, one must first solve the Ginzburg-Landau equation on each strand [linearized for obtaining $T_{c2}^{*}(H)$] in terms of the values of the order parameter at its two end nodes. Applying the de Gennes boundary conditions at a typical node then gives a linear difference equation. For a rectangular network, this difference equation has been derived previously,[10] which reads

$$\frac{\Delta_{n,m+1}\exp(i2\pi n\Phi/\Phi_0)+\Delta_{n,m-1}\exp(-i2\pi n\Phi/\Phi_0)-2\Delta_{n,m}\cos[b/\xi(T)]}{\xi(T)\sin[b/\xi(T)]}+\frac{\Delta_{n+1,m}+\Delta_{n-1,m}-2\Delta_{n,m}\cos[a/\xi(T)]}{\xi(T)\sin[a/\xi(T)]}=0\ , \tag{1}$$

where the rectangular unit cell has length a along x, and length b along y. $\xi(T)=\xi_0(1-T_{c2}^{*}/T_{c0})^{-1/2}$ is the coherence length of the strands, $\Delta_{n,m}$ is the value of the order parameter at the (n,m)th node with n measured along x and m along y, $\Phi\equiv Hab$ is the flux through each elementary rectangle, and we have worked in the Landau gauge $A=(0,Hx,0)$. After letting $\Delta_{nm}=\bar{\Delta}_n\exp(ikmb)$, Eq. (1) reduces to a one-dimensional difference equation, which is then solved numerically.

In Fig. 1, we have plotted $[(a/b)^{1/2}+(b/a)^{1/2}]\times[ab/2\pi\xi^2(T)]\propto(1-T_{c2}^{*}/T_{c0})$ as a function of $\Phi/\Phi_0\propto H$ for $a/b=1$, 2, 3, 5, 10, 25, and 100. The factor $[(a/b)^{1/2}+(b/a)^{1/2}]$ is a normalization factor chosen to render all curves to have the same initial slope of unity at $\Phi/\Phi_0=0$, in agreement with a recent theory by one of us (C.R.H.).[10] The fact that this normalization factor does not make all curves fall on a single universal curve shows that this scaling behavior is obeyed only in the limit $\Phi/\Phi_0\to 0$, $T_{c2}^{*}\to T_{c0}$, where $a,b\ll\xi(T)$. As a matter of fact, in Fig. 2 we have plotted the values of $\ln[ab/\xi^2(T)]$ at several rational values of Φ/Φ_0 vs $|\ln(a/b)|$—the absolute sign indicates the symmetry of this system with respect to the transformation of $a\rightleftarrows b$—revealing a universal asymptotic slope of -1, instead of $-\frac{1}{2}$ which was predicted in Ref. 10 for the behavior near $\Phi/\Phi_0=0$. We can understand this as a crossover behavior. As $a/b\to\infty$ at any fixed Φ/Φ_0, which is not an integer, eventually a crossover into the regime $a\gtrsim\xi(T)$ must occur, whereas Ref. 10 is a theory for the limit $a,b\ll\xi(T)$.

In Fig. 3, we have plotted the numerical derivative

$$d\{[(a/b)^{1/2}+(b/a)^{1/2}](ab/2\pi\xi^2)\}/d(\Phi/\Phi_0)$$

vs Φ/Φ_0 for four values of a/b, viz., 1, 2, 5, and 10. It is clear from these plots that the curves become less jerky as a/b increases, with the slope discontinuities associated with a larger denominator integer q fading away faster than those associated with a smaller q.

To analyze the quantitative aspects of this behavior, we have first plotted in Fig. 4 the logarithms of the left and right derivatives of $ab/\xi^2(T)$ with respect to Φ/Φ_0 at several rational values (p/q) of Φ/Φ_0, namely, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{1}{11}$, and $\frac{1}{40}$, again vs $|\ln(a/b)|$. Note that since the left derivative goes through zero for some rational values of Φ/Φ_0, we have to plot the logarithm of its *absolute* value, which explains the singular dips of some of the curves. This plot reveals that as $a/b\to\infty$, the left and right derivatives both approach zero as $(a/b)^{-2}$ for all rational values of Φ/Φ_0. This means that the leading asymptotic term in $ab/\xi^2(T)$ as $a/b\to\infty$, which behaves as $C_1(a/b)^{-1}$, must be independent of Φ/Φ_0, or in other words, insensitive to frustration.

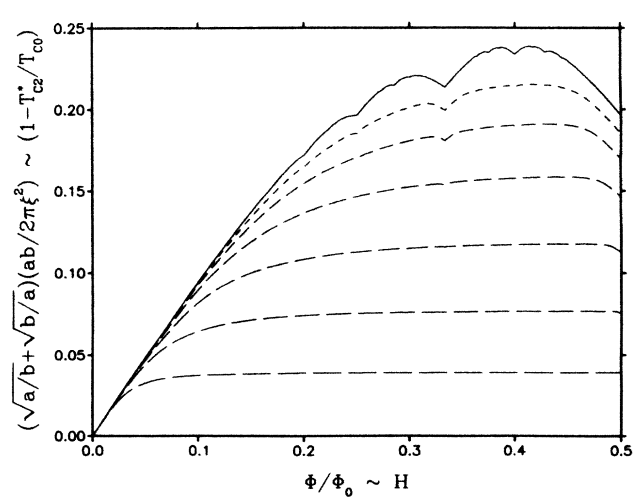

FIG. 1. Plot of $(\sqrt{a/b}+\sqrt{b/a})(ab/2\pi\xi^2)\propto(1-T_{c2}^{*}/T_{c0})$ vs $\Phi/\Phi_0\propto H$ for $a/b=$ (from the top down) 1, 2, 3, 5, 10, 25, and 100.

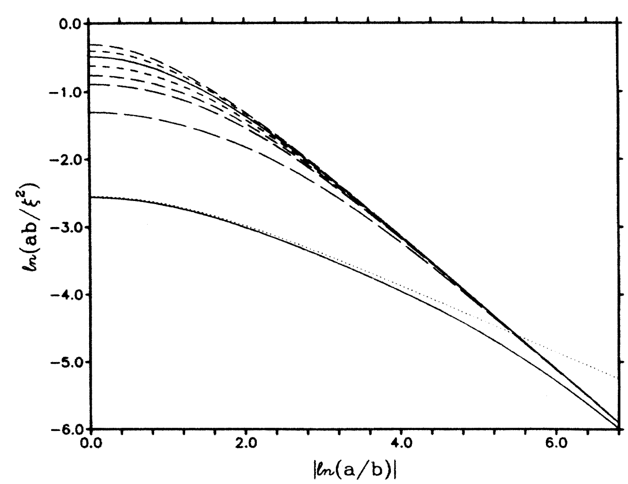

FIG. 2. Plot of $\ln(ab/\xi^2)$ at $\Phi/\Phi_0=$ (from the top curve down) $\frac{2}{5}$, $\frac{1}{3}$, $\frac{1}{2}$ $(=\frac{1}{4})$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{11}$, and $\frac{1}{40}$ vs $|\ln(a/b)|$. The dotted line is obtained by multiplying the slope at $\Phi/\Phi_0=0$ as predicted in Ref. 11 by $\frac{1}{40}$.

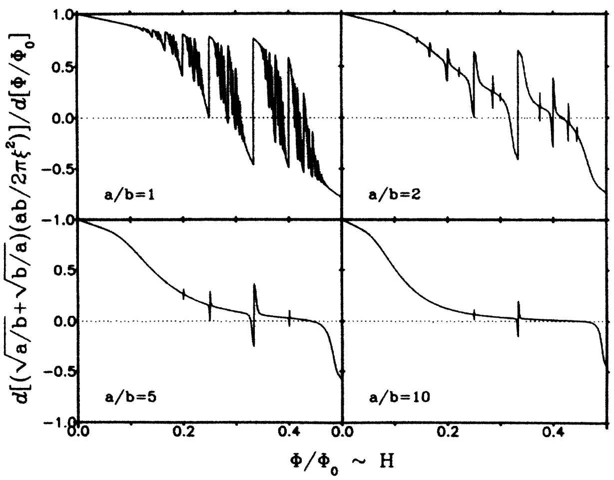

FIG. 3. Plot of the numerical derivative of $(\sqrt{a/b}+\sqrt{b/a})(ab/2\pi\xi^2)$ with respect to Φ/Φ_0 vs Φ/Φ_0 for $a/b=1$, 2, 5, and 10.

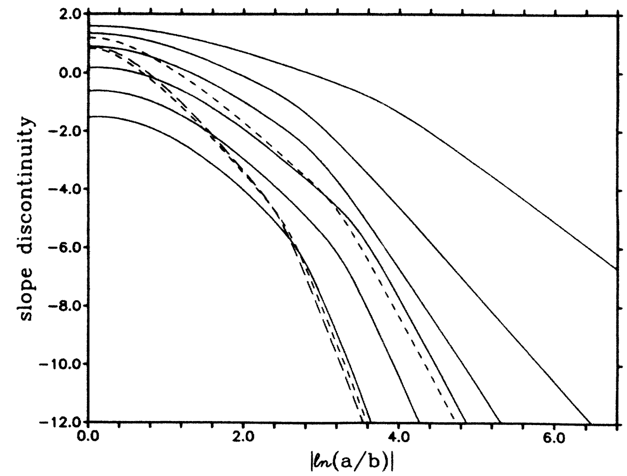

FIG. 5. Plot of the logarithms of the slope discontinuities (i.e., the right derivative minus the left derivative) of ab/ξ^2 at $\Phi/\Phi_0=$ (from the top curve down on the right side of the figure) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$ as functions of $|\ln(a/b)|$. The curves for $\frac{2}{5}$, $\frac{2}{7}$, and $\frac{3}{7}$ are given in dashed lines to enhance clarity.

This result may also be seen from Fig. 2 where all curves agree not only in slopes but also in values as $a/b\rightarrow\infty$.

Finally, we obtain the most interesting finding of this work by plotting in Fig. 5 the logarithms of the slope *discontinuities* at several rational values of Φ/Φ_0, i.e., we plot the logarithms of

$$\left.\frac{d(ab/\xi^2)}{d(\Phi/\Phi_0)}\right|_{p/q+\epsilon}-\left.\frac{d(ab/\xi^2)}{d(\Phi/\Phi_0)}\right|_{p/q-\epsilon},$$

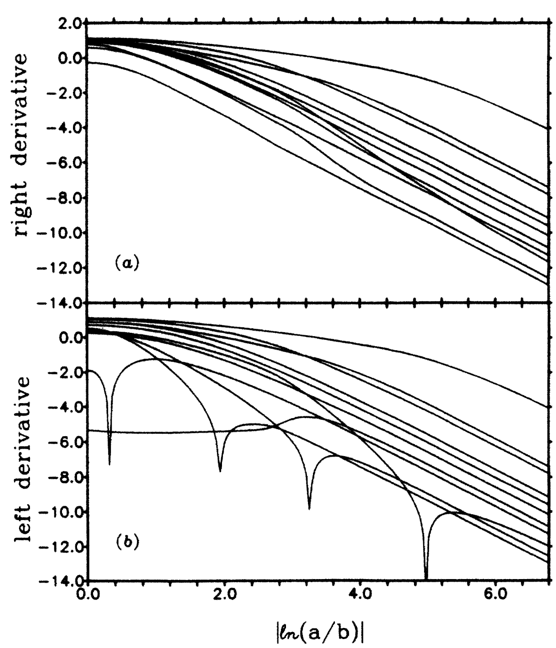

FIG. 4. Plot of the logarithms of (a) the right derivatives, and (b) the left derivatives of ab/ξ^2 with respect to Φ/Φ_0 at $\Phi/\Phi_0=$ (from the top curve down on the right-hand side of each subfigure) $\frac{1}{40}$, $\frac{1}{2}$, $\frac{1}{11}$, $\frac{1}{7}$, $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{2}{7}$, $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{7}$ vs $|\ln(a/b)|$.

at several values of p/q as functions of $|\ln(a/b)|$, which reveals that as $a/b\rightarrow\infty$, these slope discontinuities behave as $(a/b)^{-q}$. This result quantifies the finding from Fig. 3 that the ab/ξ^2 vs Φ/Φ_0 curve becomes less jerky as a/b increases toward infinity, with the slope discontinuities associated with larger q's fading away faster than those associated with smaller q's.

We can first crudely understand the diminishing influence of frustration as a/b increases with the following argument: Frustration exists only if there are closed paths in the system. Thus frustration effects should totally disappear if one cuts all the short bonds of length b (or, equivalently, if the order parameter is suppressed on these short bonds). Since the total length (or mass) of all of the short bonds per unit area is only b/a times that of the long bonds, it is clearly easier for the system to accommodate frustration at larger values of a/b. (See explicit examples given later for $\Phi/\Phi_0=\frac{1}{2}$ and $\frac{1}{4}$, which will show explicitly how the system copes with frustration in the limit $a/b\rightarrow\infty$.)

More quantitatively, we can understand the falloff of ab/ξ^2 at noninteger values of Φ/Φ_0 according to $(a/b)^{-1}$ as $a/b\rightarrow\infty$, as simply due to the fact that as $b/\xi\rightarrow 0$, a/ξ must approach $\pi/2$ for values of Φ/Φ_0 which are not within $O((b/\xi)^{1/2})$ of any integer including zero. This limiting behavior is born out in Fig. 1. To derive this result, we note that the ground state of Eq. (1) corresponds to $k=0$, and $\bar{\Delta}_n$ even. We can, therefore, recast this equation into the form

$$\bar{\Delta}_1=\cos(a/\xi)\bar{\Delta}_0+[\sin(a/\xi)/\sin(b/\xi)][\cos(b/\xi)-1]\bar{\Delta}_0\ , \qquad (2)$$

$$\bar{\Delta}_{n(>1)}=2\cos(a/\xi)\bar{\Delta}_{n-1}+2[\sin(a/\xi)/\sin(b/\xi)]\times[\cos(b/\xi)-\cos(2n\pi\Phi/\Phi_0)]\bar{\Delta}_{n-1}-\bar{\Delta}_{n-2}\ . \qquad (3)$$

As $b/\xi\rightarrow 0$, the second equation will make $\bar{\Delta}_{n>1}$ diverge if Φ/Φ_0 is not within $O((b/\xi)^{1/2})$ of an integer, unless $\bar{\Delta}_1$ is infinitesimal, which is satisfied with $a/\xi \rightarrow \pi/2$. The fact that this limiting value is independent of Φ/Φ_0 also explains why the left and right derivatives of ab/ξ^2, with respect to Φ/Φ_0 at any rational values for the latter, must decay to zero with a power of b/a larger than unity. Although the most natural power is then two, we do not yet have a direct argument as to why it is so, nor why the slope discontinuities at $\Phi/\Phi_0 = p/q$ must decay to zero as $(a/b)^{-q}$ as $a/b \rightarrow \infty$. (However, see the conclusion.)

There is another numerical surprise discovered in this study, which can be easily understood. By shifting all of the curves in Fig. 1 vertically so they all have vertical coordinates zero at $\Phi/\Phi_0 = \frac{1}{2}$, it is found that all curves merge again at $\Phi/\Phi_0 = \frac{1}{4}$ with again a vertical coordinate of zero. This implies that ab/ξ^2 at $\Phi/\Phi_0 = \frac{1}{2}$ and $\frac{1}{4}$ are equal for *all* values of a/b. This can be easily understood as follows: For $\Phi/\Phi_0 = \frac{1}{2}$, $\bar{\Delta}_n$ becomes a periodic function of period two. In other words, $\bar{\Delta}_2 = \bar{\Delta}_0$, so Eqs. (2) and (3) reduce to two coupled equations from which we find

$$\sin^2(a/\xi) + \sin^2(b/\xi) = \sin^2[(a+b)/\xi] \ , \quad (4)$$

and

$$\bar{\Delta}_1/\bar{\Delta}_0 = \{\sin[(a+b)/\xi] - \sin(a/\xi)\}/\sin(b/\xi) \ . \quad (5)$$

On the other hand, for $\Phi/\Phi_0 = \frac{1}{4}$, we have $\bar{\Delta}_4 = \bar{\Delta}_0$ and $\bar{\Delta}_3 = \bar{\Delta}_1$ (since $\bar{\Delta}_n$ is even). Equations (2) and (3) then also give Eq. (4) as the eigencondition, and $(\bar{\Delta}_2/\bar{\Delta}_0)^{1/2} = (\bar{\Delta}_1/\bar{\Delta}_0)$ is also given by Eq. (5). Equation (4) has the solution $a/\xi = \pi/2 - b/\xi$, which is consistent with our earlier conclusion that $a/\xi \rightarrow \pi/2$ as $b/\xi \rightarrow 0$ for all Φ/Φ_0 that are not within $O((b/\xi)^{1/2})$ of an integer. In addition, the right-hand side of Eq. (5) approaches $b/2\xi \ll 1$ as $b/\xi \rightarrow 0$, confirming that the system indeed sacrifices a good fraction of the short bonds in order to cope with frustration in this limit.

In summary, we have demonstrated with a numerical study that the same amount of frustration imposed on a rectangular superconducting network produces diminishing effects as the aspect ratio a/b of the network is increased toward infinity. Furthermore, various types of power-law behavior are found in the limit $a/b \rightarrow \infty$, including the most interesting result that the slope discontinuity of $(1 - T_{c2}^*/T_{c0})$ vs Φ/Φ_0 at $\Phi/\Phi_0 = p/q$ decays as $(a/b)^{-q}$, and therefore fade away faster for larger q. Simple understandings are obtained for some of our findings, but not yet on this most interesting result. However, since our explicit solutions at $\Phi/\Phi_0 = \frac{1}{2}$ and $\frac{1}{4}$ show spontaneously broken translational symmetry, with a discrete set of degenerate ground states, solitons clearly play an important role when Φ/Φ_0 is only infinitesimally away from a rational value p/q. Attempting to understand the $(a/b)^{-q}$ behavior within this framework is currently underway.

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