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Continuum limit of the upper critical field H_{c2}^* for superconducting networks

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Recently Rammal *et al.* investigated the upper critical field H_{c2}^* for various superconducting networks and proposed the formula $H_{c2}^* = \frac{1}{2} d[\Phi_0/\pi\xi^2(T)] = dH_{c2}^{(\mathrm{bulk})}$ (with d the spatial dimension of the network) as the universal limiting behavior of $H_{c2}^*(T)$ as $T \to T_{c0}, H_{c2}^* \to 0$. We demonstrate with several examples that this is not true for networks with low point-group symmetry. We then propose that the above universal formula be changed to an inequality $H_{c2}^* \geq 2H_{c2}^{(\mathrm{bulk})}$, which reduces to an equality for two-dimensional networks with an n-fold rotational symmetry with $n \geq 3$. For three-dimensional networks $H_{c2}^*/H_{c2}^{(\mathrm{bulk})} = 3$ if the network has a cubic symmetry. Otherwise we find that the ratio can be larger or smaller than 3.

Recently, Rammal, Lubensky, and Toulouse[1] investigated the upper critical field H_{c2}^* as a function of the temperature T [or, equivalently, the inverse function $T_{c2}^*(H)$] for many finite and infinite superconducting networks; in particular, for infinite periodic networks, they proposed the formula

$$H_{c2}^*(T) = \frac{d}{2}\frac{\Phi_0}{\pi\xi^2(T)} = dH_{c2}^{(\mathrm{bulk})}(T) \tag{1}$$

as the universal behavior of $H_{c2}^*(T)$ in the continuum limit, i.e., as $T \to T_{c0}$ and $H_{c2}^* \to 0$. In the above, d is the dimension of the network, $\xi(T)$ is the temperature-dependent Ginzburg-Landau coherence length of the superconducting wires forming the network, $H_{c2}^{(\mathrm{bulk})}$ is the upper critical field of the *corresponding* bulk material [i.e., of the same $\xi(T)$], and T_{c0} is the superconducting transition temperature of the material at zero applied field. They checked Eq. (1) against two-dimensional square, triangular, and honeycomb lattices and three-dimensional simple cubic and centered cubic lattices, and then suggested that Eq. (1) is universally valid.

The purposes of this report are (1) to demonstrate with several examples (each containing at least one free parameter and therefore actually corresponding to a family of examples) that Eq. (1) is not true for networks with low point-group symmetry and (2) to propose that this equation be replaced by an inequality

$$H_{c2}^*(T) \geq 2H_{c2}^{(\mathrm{bulk})}(T) \ , \tag{2}$$

which reduces to an equality for two-dimensional networks with an n-fold rotational symmetry with $n \geq 3$. For three-dimensional networks we find $H_{c2}^*/H_{c2}^{(\mathrm{bulk})} = 3$ if the network has a cubic symmetry. Otherwise, we find that this ratio can be larger or smaller than 3 (but still not less than 2).

The first example we will discuss is a two-dimensional rectangular network with side lengths a and b (along the x and y directions, respectively, by definition) for each unit cell. Denoting the value of the superconducting order parameter at the node located at $\mathbf{r}_{nm} \equiv na\hat{\mathbf{e}}_x + mb\hat{\mathbf{e}}_y$ as $\Delta_{n,m}$, we obtain the following difference equation by the same procedure as used in Refs. 1 and 2:

$$\frac{\Delta_{n,m+1}e^{i2\pi n\Phi/\Phi_0} + \Delta_{n,m-1}e^{-i2\pi n\Phi/\Phi_0} - 2\Delta_{n,m}\cos[b/\xi(T)]}{\xi(T)\sin[b/\xi(T)]} + \frac{\Delta_{n+1,m} + \Delta_{n-1,m} - 2\Delta_{n,m}\cos[a/\xi(T)]}{\xi(T)\sin[a/\xi(T)]} = 0 \ , \tag{3}$$

where $\Phi_0 \equiv hc/2e$ is the flux quantum, $\Phi \equiv Hab$ is the flux through each unit cell, and we have assumed the Landau gauge $\mathbf{A} = (0, Hx, 0)$. Taking the continuum limit $T \to T_{co}, H \to 0$, when $\Phi \ll \Phi_0$ and (a and b) $\ll$ [$\xi(T)$ and $\xi_H \equiv (2eH/\hbar c)^{-1/2}$], Eq. (3) reduces to the differential equation

$$a\frac{\partial^2}{\partial x^2}\Delta + b\left(\frac{\partial}{\partial y} + i\frac{x}{\xi_H^2}\right)^2\Delta + \frac{a+b}{\xi^2(T)}\Delta = 0 \ . \tag{4}$$

This equation may be solved with the transformation $x = (a/b)^{1/4}x', y = (b/a)^{1/4}y'$, giving

$$\xi_H^{-2} = \left(\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}}\right)\xi^{-2}(T), \text{ or } H_{c2}^* = \left(\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}}\right)H_{c2}^{(\mathrm{bulk})} \ . \tag{5}$$

Thus the ratio $H_{c2}^{*}/H_{c2}^{(\text{bulk})}$ can take any value between 2 and ∞ with the lower bound reached when $a=b$, i.e., when the rectangular lattice reduces to a square lattice.

Our second example is a triangular lattice uniaxially compressed (or elongated) along one of the strand directions so that each triangular cell has two side lengths b and one side length a. Let the angle between a b side and an a side be denoted as θ, then $a=2b\cos\theta$. Choosing a coordinate system such that the a sides are along the x axis and the nodes are located at $\mathbf{r}_{nm}\equiv na\hat{\mathbf{e}}_x+mb\hat{\mathbf{e}}_\xi$ with $\hat{\mathbf{e}}_\xi\equiv\cos\theta\hat{\mathbf{e}}_x+\sin\theta\hat{\mathbf{e}}_y$, we obtain the difference equation

$$\frac{\Delta_{n,m+1}e^{i2\pi(2n+m+1/2)\Phi/\Phi_0}+\Delta_{n,m-1}e^{-i2\pi(2n+m-1/2)\Phi/\Phi_0}-2\Delta_{n,m}\cos[b/\xi(T)]}{\xi(T)\sin[b/\xi(T)]}+\frac{\Delta_{n-1,m+1}e^{i2\pi(2n+m-1/2)\Phi/\Phi_0}+\Delta_{n+1,m-1}e^{-i2\pi(2n+m+1/2)\Phi/\Phi_0}-2\Delta_{n,m}\cos[b/\xi(T)]}{\xi(T)\sin[b/\xi(T)]}+\frac{\Delta_{n+1,m}+\Delta_{n-1,m}-2\Delta_{n,m}\cos[a/\xi(T)]}{\xi(T)\sin[a/\xi(T)]}=0\ . \quad (6)$$

In the continuum limit it reduces to the differential equation

$$a\frac{\partial^2\Delta}{\partial x^2}+b\left(\frac{\partial}{\partial\xi}\bigg|_{\xi'}+i\frac{(\xi\cos\theta-\xi'\sin\theta)\sin\theta}{\xi_H^2}\right)^2\Delta+b\left(\frac{\partial}{\partial\eta}\bigg|_{\eta'}-i\frac{(\eta\cos\theta-\eta'\sin\theta)\sin\theta}{\xi_H^2}\right)^2\Delta+\frac{a+2b}{\xi^2(T)}\Delta=0\ , \quad (7)$$

where $\xi=x\cos\theta+y\sin\theta$, $\xi'=-x\sin\theta+y\cos\theta$, $\eta=-x\cos\theta+y\sin\theta$, and $\eta'=x\sin\theta+y\cos\theta$. After expressing $\partial/\partial\xi$ and $\partial/\partial\eta$ in terms of $\partial/\partial x$ and $\partial/\partial y$, this equation becomes

$$\cos\theta(1+\cos\theta)\frac{\partial^2}{\partial x^2}\Delta+\sin^2\theta\left(\frac{\partial}{\partial y}+i\frac{x}{\xi_H^2}\right)^2\Delta+\frac{1+\cos\theta}{\xi^2(T)}\Delta=0\ . \quad (8)$$

The solution of this equation gives

$$H_{c2}^{*}/H_{c2}^{(\text{bulk})}=[\cos\theta(1-\cos\theta)]^{-1/2}\ , \quad (9)$$

which again ranges between 2 and ∞. Of particular interest are the following special cases: (i) $\theta=60°$, corresponding to a equilateral triangular lattice, for which the right-hand side of Eq. (9) becomes 2, and (ii) $\theta=45°$, corresponding to a square lattice with one set of parallel diagonals (say, all from a lower left corner to an upper right corner) added in. For this case the right-hand side of Eq. (9) is equal to $[2(\sqrt{2}+1)]^{1/2}=2.197$. This case may also be verified directly without introducing the ξ and η variables. We have also worked out the case when both sets of parallel diagonals are added to a square lattice (without crossing at the centers of the squares), and found that Eq. (1) holds for this case.

Our third example is a square lattice with the cross-sectional areas of the horizontal strands S_1 being, in general, different from those of the vertical strands S_2. For such networks the de Gennes boundary condition [i.e., Eq (2) of Ref. 1; cf. also, Eq. (2.6) of Ref. 2] must be generalized to

$$\sum_p S_p\left(i\frac{\partial}{\partial l}+\frac{2\pi}{\Phi_0}A(l)\right)\Psi_p(l)\,|_{l=0}=0 \quad (10)$$

(in the notation of Ref. 1). We thus obtain the difference equation

$$S_2\{\Delta_{n,m+1}e^{i2\pi m\Phi/\Phi_0}+\Delta_{n,m-1}e^{-i2\pi m\Phi/\Phi_0}-2\Delta_{n,m}\cos[a/\xi(T)]\}+S_1\{\Delta_{n+1,m}+\Delta_{n-1,m}-2\Delta_{n,m}\cos[a/\xi(T)]\}=0\ . \quad (11)$$

In the continuum limit, this equation reduces to essentially the same equation as Eq. (4) except for the replacement of a by S_1 and b by S_2. Thus the solution is just Eq. (5) with the same replacements, which again satisfies Eq. (2) with the equality case corresponding to $S_1=S_2$.

In all three examples discussed above we find the system to behave in the continuum limit like an *anisotropic* superconductor unless the network has an n-fold rotational symmetry with $n\geq 3$, and Eq. (2) reduces to an equality only when such a symmetry is present.

Finally, we consider one three-dimensional network, viz., a three-dimensional simple rectangular network with side lengths a, b, and c for each unit cell. The applied field H is assumed to be along the c sides. The differential equation in the continuum limit for this case is

$$a\frac{\partial^2}{\partial x^2}\Delta+b\left(\frac{\partial}{\partial y}+i\frac{x}{\xi_H^2}\right)^2\Delta+c\frac{\partial^2}{\partial z^2}\Delta+\frac{a+b+c}{\xi^2(T)}\Delta=0\ , \quad (12)$$

and the solution is

$$H_{c2}^{*}=\left(\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}}+\frac{c}{\sqrt{ab}}\right)H_{c2}^{(\text{bulk})}\ , \quad (13)$$

which again satisfies Eq. (2). For $a=b=c$, corresponding to a cubic lattice, the proportionality coefficient reduces to 3, but otherwise it can be larger or smaller than 3.

We have not yet studied any other two- or three-dimensional periodic networks. However, we believe that the essential truth in association with the asymptotic behavior of $H_{c2}^{*}(T)$ in the limit $T\rightarrow T_{c0}$ has been revealed by the examples studied; namely, Eq. (1) holds only if the network has sufficient symmetry to ensure that it behaves as an isotropic superconductor in the continuum limit. In two-dimensional networks this requires the existence of two equivalent directions that are not collinear. In three-dimensional networks, this requires the existence of three equivalent directions that are not coplanar. Otherwise, the system acts as an anisotropic superconductor in the continuum limit, and Eq. (1) must be replaced by a proper inequality, viz., Eq. (2) without the equal sign.

In conclusion, we have shown in this work that infinite two- (or three-) dimensional periodic superconducting networks in the "continuum limit," i.e., when the longest strand in the network becomes much shorter than the Ginzburg-Landau coherence length, do not simply reduce to a genuinely continuous superconducting film (or bulk). In particular, a continuous superconducting film or bulk will automaticaly be isotropic if the underlying superconducting material is isotropic, whereas a superconducting network made with isotropic material will, according to the results of this analysis, be generally anisotropic unless there is sufficient symmetry in the network geometry to dictate otherwise. Of course, this whole analysis is based on a mean-field approach and the de Gennes boundary condition [i.e., Eq. (10)], so if there is anything wrong with the use of this approach, our results would be questionable. This possibility, however, is deemed unlikely since the approach has already enjoyed enormous success in explaining the rather complex features in the upper critical field measurements on superconducting networks in Ref. 3. It is therefore urged that experimentalists put our results to critical tests in order to ultimately confirm (or deny) our prediction. If it is confirmed, we would have a controllable way of tailoring the anisotropy properties of a superconductor which could have some practical usefulness.

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[1]R. Rammal, T. C. Lubensky, and G. Toulouse, Phys. Rev. B **27**, 2820 (1983).

[2]S. Alexander, Phys. Rev. B **27**, 1541 (1983).

[3]B. Pannetier, J. Chaussy, R. Rammal, and J. C. Villegier, Phys. Rev. Lett. **53**, 1845 (1984).