

First-order current-induced textural transition of $^3\text{He-A}$ in a thin slab

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A three-dimensional fluctuation analysis valid at all temperatures is performed on the uniform texture of $^3\text{He-A}$ in a slab, in the presence of a uniform supercurrent parallel to the walls. A textural transition is confirmed at a critical phase gradient Q_c , above which the texture must vary periodically in the plane parallel to the walls if the temperature is sufficiently close to absolute zero. In the Ginzburg-Landau regime, the texture developed above Q_c varies only in the direction perpendicular to the slab, but is nonplanar. The transition is predicted to be first order as current is increased above critical, if the slab width is $\ll 10^{-3}$ cm, so that the orbital axis \hat{l} and the spin axis \hat{d} are effectively decoupled.

Recently, the hydrodynamic effect on the equilibrium texture of $^3\text{He-A}$ has attracted much attention. Several studies¹⁻⁴ have demonstrated that in spite of the anisotropic superfluid density favoring the alignment of the orbital axis \hat{l} with the flow, the uniform texture with \hat{l} parallel to the flow is only barely stable in the dipole-locked Ginzburg-Landau (GL) regime and actually becomes unstable against a helical distortion if dipole locking is switched off, or if the temperature is sufficiently close to absolute zero. These studies have assumed an open geometry, but may be applied to a slab geometry of width w if the condition $Qw \gg 1$ is imposed, where Q represents an overall phase gradient, which is proportional to the superfluid velocity V_s when \hat{l} is uniform.

In the opposite limit $Qw \ll 1$, the orientation of \hat{l} is dominated by the boundary condition that \hat{l} tends to be anchored normal to the walls.⁵ In the slab geometry, with planar walls located at $x=0$ and w , and a flow along the y axis, it has been shown^{6,7} that the \hat{l} axis remains uniformly along \hat{x} , until Q reaches a critical value Q_c , where a textural transition then takes place. These last two works, however, are susceptible to improvement, since (i) they have only considered stabilities against fluctuations depending only on x —a restriction which is clearly no longer satisfactory, in view of the findings of Ref. 1-4; (ii) For $Q > Q_c$ they have assumed that the system takes a planar texture, with \hat{l} confined in the xy plane—an assumption which is now known to be invalid because the gradient energy contains a term $-c_0(\nabla_s \cdot \hat{l})(\hat{l} \cdot \nabla \times \hat{l})$,^{1,2} which favors a $\nabla \times \hat{l} \neq 0$ in the general direction of \hat{l} , whenever $\nabla_s \cdot \hat{l} \neq 0$; (iii) Their studies are confined to the GL regime (i.e., when the temperature T and the transition temperature T_c satisfy $T_c - T \ll T_c$), and therefore need to be generalized to the whole temperature range below T_c ; (iv) The order of the transition was studied in

Ref. 6 but not in Ref. 7, and, as we shall show below, the true order of the transition was not correctly determined in Ref. 6, because of the planar textures it assumed for $Q > Q_c$.

In this work I first perform a three-dimensional fluctuation analysis on the $\hat{l} = \hat{x}$ texture in a slab, using the approach of Ref. 1 which is valid at all temperatures, and derive a general expression for Q_c . For $Q > Q_c$ it is established that the critical fluctuation has a periodic variation in the yz plane parallel to the walls, as well as the necessary x dependence, if temperature is sufficiently close to absolute zero; but in the GL regime, the critical fluctuation depends only on x , whether dipole locking is in operation or not, if only the strong-coupling effect is as small as estimated recently.⁸ Having established the validity of assuming only x dependence in the GL regime, I then study the order of the transition at Q_c , without restricting to planar textures for $Q > Q_c$. An exact Landau-type analysis reveals that the transition is nearly first order if $w \gg 10^{-3}$ cm, so that dipole locking is perfectly effective (except in negligible regions near the surfaces). The transition though, is unambiguously first order, with a discontinuous change in the texture, if $w \ll 10^{-3}$ cm when dipole locking is effectively switched off. On the other hand, the transition would always be second order, had planar textures been assumed for $Q > Q_c$.

For the fluctuation analysis, I begin with the same free energy as is used in Refs. 1 and 3 [cf., Ref. 1, Eq. (2), or Ref. 3, Eq. (1)], and the expansion

$$\nabla_s = Q\hat{y} + \delta\nabla_s, \quad \hat{l} = \hat{x} + \bar{\lambda} - \frac{1}{2}\lambda^2\hat{x},$$

where $\bar{\lambda} \cdot \hat{x} = 0$. As in Ref. 1, Eq. (3), I obtain

$$\delta\nabla_s = \frac{1}{2}(\lambda_y \nabla \lambda_z - \lambda_z \nabla \lambda_y) + \nabla\phi$$

valid to second order in $\bar{\lambda}$. To the same order, the

gradient free energy then reduces to

$$F_G = \int d^3r \left\{ \frac{1}{2} \rho_s [Q^2 + Q(\lambda_y \partial_y \lambda_z - \lambda_z \partial_y \lambda_y) + (\nabla \cdot \phi)^2] - \frac{1}{2} \rho_0 [\partial_x \phi + Q \lambda_y]^2 \right. \\ \left. - c_0 (\partial_x \phi + Q \lambda_y) (\partial_y \lambda_z - \partial_z \lambda_y) + \frac{1}{2} K_s (\nabla \cdot \bar{\lambda})^2 + \frac{1}{2} K_t (\partial_y \lambda_z - \partial_z \lambda_y)^2 + \frac{1}{2} K_b (\partial_x \bar{\lambda})^2 \right\}. \quad (1)$$

The boundary conditions are $\bar{\lambda} = \phi_x = 0$ at $x = 0$ and w , so we introduce the expansions

$$[\phi(\bar{r}), \bar{\lambda}(\bar{r})] = \frac{2}{\pi} \frac{1}{L_y L_z} \sum_{n,k,q} \left[\Phi_{nkq} \cos \left(\frac{n \pi x}{w} \right), \bar{\Lambda}_{nkq} \sin \left(\frac{n \pi x}{w} \right) \right] \exp(ikz + iqy). \quad (2)$$

The fluctuation free energy then becomes $\delta F = V^{-1} \sum_{nkq} \mathbf{A}^* \mathbf{M} \mathbf{A}^T$, where $\mathbf{A} \equiv (\Phi_{nkq}, \Lambda_{ynkq}, \Lambda_{zknq})$, tr means transpose, and

$$\mathbf{M} \equiv \begin{pmatrix} \rho_s^{\parallel} n^2 + \rho_s (\bar{q}^2 + \bar{k}^2) & n(\rho_0 \bar{Q} - ic_0 \bar{k}) & inc_0 \bar{q} \\ n(\rho_0 \bar{Q} + ic_0 \bar{k}) & K_b n^2 + K_s \bar{q}^2 + K_t \bar{k}^2 - \rho_0 \bar{Q}^2 & [(K_s - K_t) \bar{k} + i(\rho_s - c_0) \bar{Q}] \bar{q} \\ -inc_0 \bar{q} & [(K_s - K_t) \bar{k} - i(\rho_s - c_0) \bar{Q}] \bar{q} & K_b n^2 + K_s \bar{k}^2 + K_t \bar{q}^2 \end{pmatrix}, \quad (3)$$

where

$$\bar{Q} \equiv \frac{Qw}{\pi}, \quad \bar{q} \equiv \frac{qw}{\pi}, \quad \bar{k} \equiv \frac{kw}{\pi}, \quad \rho_s^{\parallel} \equiv \rho_s - \rho_0.$$

To find the independent fluctuation modes, we need to solve $\mathbf{M} \mathbf{A}^T = \lambda \mathbf{A}^T$, or $\det(\mathbf{M} - \lambda \mathbf{I}) = 0$. Since $\det \mathbf{M}$ is a linear function of \bar{Q}^2 , only one of the three eigenvalues can change sign in the range $0 < \bar{Q}^2 < \infty$, and therefore the condition $\det \mathbf{M} = 0$ uniquely determines \bar{Q}_c for each (n, k, q) . At $n = 1$, $k = q = 0$, it is found

$$\bar{Q}_c^2 = \frac{\rho_s^{\parallel} K_b}{\rho_s \rho_0}, \quad \frac{\partial \bar{Q}_c^2}{\partial \bar{k}^2} = \frac{\rho_0 K_b + \rho_s^{\parallel} K_t - c_0^2}{\rho_s \rho_0}, \quad (4) \\ \frac{\partial \bar{Q}_c^2}{\partial \bar{q}^2} = \frac{\rho_s [\rho_0 K_s - (\rho_s^{\parallel} - c_0)^2] + \rho_0^2 (K_b - K_s)}{\rho_s \rho_0^2}.$$

The parameters ρ_s , ρ_0 , K_b , etc. appearing in Eq. (4) have been evaluated by M. Cross in the weak-coupling approximation but with the Fermi-liquid corrections incorporated.⁹ In the low-temperature limit $T \rightarrow 0$, simple limiting behavior of these parameters may be obtained and used to evaluate the quantities in Eq. (4). This is presented in Appendix A, which reveals that in the limit $T \rightarrow 0$, $\partial \bar{Q}_c^2 / \partial \bar{q}^2$ is positive with or without dipole-locking and/or Fermi-liquid corrections, while $\partial \bar{Q}_c^2 / \partial \bar{k}^2$ is negative for the dipole-unlocked case only, if Fermi-liquid corrections are not made. After such corrections are made, $\partial \bar{Q}_c^2 / \partial \bar{k}^2$ becomes negative for the dipole-locked case as well (at all pressures). This means that a search over the whole (k, q) plane must be made in order to find the minimum \bar{Q}_c^2 , and the texture for $Q > Q_c$ must vary in at least one, but may be both directions parallel to the walls (besides the

obviously existing variation in the direction perpendicular to the walls). This finding is obviously related to the findings of Refs. 1–4. But note that in Refs. 1–4, which studied the limit $Qw \gg 1$, for which the wall effects may be neglected, the texture was found to have a periodic variation *along* the flow only, while here it is the wave number k defined in the direction *perpendicular* to the flow (and parallel to the walls) which favors a nonvanishing value. Further study employing numerical methods will be carried out to pinpoint the true texture about Q_c in the low-temperature limit. In the rest of this paper, I shall confine myself to the GL limit, where we have

$$\frac{\rho_s}{\gamma + 1} = \frac{\rho_0}{\gamma} = \rho_s^{\parallel} = \frac{c_0}{\gamma} = \frac{2K_s}{2\gamma + 3} = \frac{2K_t}{2\gamma + 3} = \frac{2K_b}{2\gamma + 3}, \quad (5a)$$

if dipole-locking is assumed, and

$$\frac{\rho_s}{\gamma + 1} = \frac{\rho_0}{\gamma} = \rho_s^{\parallel} = \frac{c_0}{\gamma} = 2K_s = 2K_t = \frac{2K_b}{2\gamma + 1}, \quad (5b)$$

if dipole locking is effectively switched off. The parameter γ has a value unity according to the weak-coupling theory, but strong-coupling effects can give it a pressure-dependent deviation from unity by about a few percent, according to a recent theoretical estimate.⁸ Thus Eq. (4) becomes

$$\bar{Q}_c^2 = \frac{2\gamma + 3}{2\gamma(\gamma + 1)}, \quad \frac{\partial \bar{Q}_c^2}{\partial \bar{k}^2} = \frac{5\gamma + 3}{2\gamma(\gamma + 1)}, \quad (6a) \\ \frac{\partial \bar{Q}_c^2}{\partial \bar{q}^2} = \frac{7\gamma - 2}{2\gamma^2} \quad (\text{dipole locking on})$$

and

$$\bar{Q}_c^2 = \frac{2\gamma + 1}{2\gamma(\gamma + 1)}, \quad \frac{\partial \bar{Q}_c^2}{\partial \bar{k}^2} = \frac{1}{2\gamma}, \quad (6b)$$

$$\frac{\partial \bar{Q}_c^2}{\partial \bar{q}^2} = \frac{3\gamma^2 + 3\gamma - 2}{2\gamma^2(\gamma + 1)} \quad (\text{dipole locking off}).$$

Equations (6) show that the $n=1$ branch of Q_c has a local minimum at $k=q=0$, for $\gamma > 0.286$ if dipole locking is on, and for $\gamma > 0.457$ if dipole locking is off. Next it is necessary to investigate whether a lower minimum of Q_c^2 might occur for $n > 1$ and/or at finite k and/or q . Numerical evaluation of $\det M$ shows that we can always put $n=1$, $k=0$, but the q dependence is less trivial. If dipole locking is on, $Q_c^2(\bar{q}^2)$ at $n=1$, $k=0$ is found to have only one minimum at any given $\gamma > 0$, but if dipole locking is off, $Q_c^2(\bar{q}^2)$ actually can have two minima in the narrow range

$$0.457 < \gamma < 0.575 \pm 0.025,$$

and the minimum at $\bar{q} \sim 1$ becomes lower than the minimum at $\bar{q} = 0$, when γ is decreased below the critical value 0.527 (cf, Fig. 1). This means a first-order textural transition as γ is varied, but unfortunately the required range of γ is unrealistic. However, I believe that a suitably applied magnetic field can move the transition into the physical range, as we have already found in the limit $Qw \gg 1$.¹⁰

Leaving the magnetic field effect for future studies, we now examine the order of the transition in the GL regime, as the current is increased beyond the value determined at $Q = Q_c$ for the uniform $\hat{l} = \hat{x}$ texture.¹¹ For this purpose, we introduce the representation for the order parameter

$$\hat{\Delta} = e^{iQy} \left(A_0 \hat{x} + A_+ \frac{\hat{y} + i\hat{z}}{2^{1/2}} + A_- \frac{\hat{y} - i\hat{z}}{2^{1/2}} \right), \quad (7)$$

$$A_0 = \sin \chi \exp(iS_1),$$

$$A_{\pm} = (1/2^{1/2}) (\cos \chi \pm 1) \exp(iS_1 \pm iS_2).$$

In view of the above fluctuation study, we can safely

$$\bar{f}_G = (\gamma \sin^2 \chi + 1) S_{1x}^2 + [\gamma (\sin^2 S_2 + \cos^2 \chi \cos^2 S_2) + 1] Q^2 + 2\gamma Q \sin \chi \cos S_2 (\cos \chi S_{1x} + S_{2x}) + 2 \cos \chi S_{1x} S_{2x} + (\gamma \cos^2 \chi + \frac{1}{2}) \chi_x^2 + (\frac{1}{2}) (1 + \cos^2 \chi) S_{2x}^2 + [(\gamma \sin^2 \chi + 1) (\chi_x^2 + \sin^2 \chi S_{2x}^2)], \quad (8)$$

where an irrelevant pure-divergence term has been dropped, and the last term in the bracket arises from the dipole locking of the spin axis \hat{d} with \hat{l} . Minimizing $\int \bar{f}_G d^3r$ with respect to S_1 gives a current conservation equation $dj_x/dx = 0$ and a boundary condi-

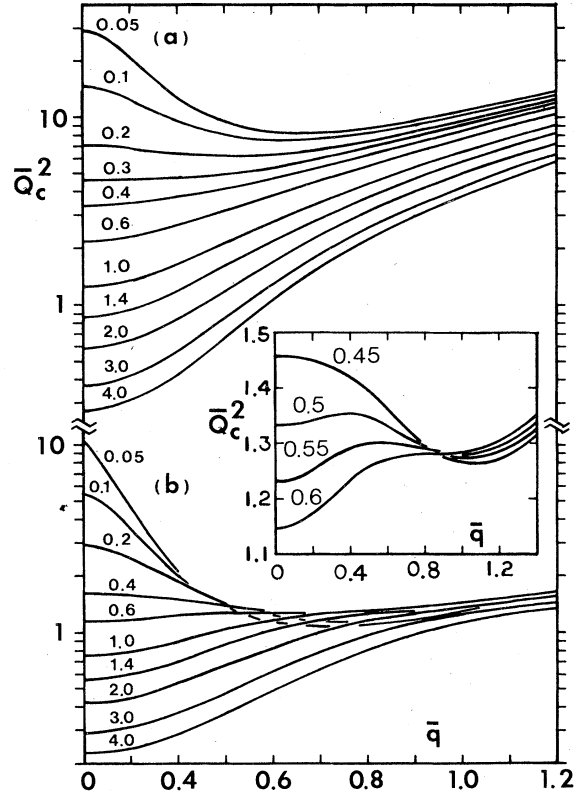


FIG. 1. Restricted to the Ginzburg-Landau regime $T_c - T \ll T_c$, when the parameters ρ_s , ρ_0 , K_b , etc., can all be expressed in terms of one parameter γ [cf, Eq. (5)], the quantity $Q_c^2 (n, \bar{q}, \bar{k})$ for $n=1$, $\bar{k}=0$ is plotted as a function of \bar{q} , for various values of γ (given right above the left end of the curves). The upper set of curves [case (a)] assumes dipole locking. The lower set of curves [case (b)] assumes no dipole locking. The inset gives further details of case (b), revealing that the curves have double minima in the narrow range $0.457 < \gamma < 0.575 \pm 0.025$.

assume that S_1 , χ , and S_2 depend on x only, at least for $Q - Q_c \ll Q_c$. Note that for planar textures ($\hat{l} \cdot \hat{z} = 0$) we would have $S_2 = 0$, but we shall not make this assumption here. The gradient energy density, after normalized by $2\Gamma_2 \Delta_0^{-2}$,¹¹ becomes

tion $j_x = 0$ at $x = 0$ and w . Together they may be solved to give

$$S_{1x} = -\frac{\cos \chi (S_{2x} + \gamma Q \sin \chi \cos S_2)}{(\gamma \sin^2 \chi + 1)}, \quad (9)$$

which may be used to eliminate S_{1x} from \bar{f}_G . Further minimization with respect to χ and S_2 would result in two nonlinear coupled differential equations which must be solved by numerical method, together with the boundary conditions $\chi = S_{2x} = 0$ at $x = 0$ and w . While this is currently being attempted, here I perform a Landau-type analysis for $Q - Q_c \ll Q_c$ by expanding $\bar{f}_G(\chi, S_2)$ to fourth order in its arguments, giving

$$\begin{aligned} \bar{f}_G = & (\gamma + 1)Q^2 + [(\gamma + \frac{1}{2})\chi_x^2 - \gamma(\gamma + 1)Q^2\chi^2] \\ & + \gamma(\gamma + 1)[(\gamma + \frac{1}{3})Q^2\chi^4 + 2Q\chi^3S_{2x} + Q^2\chi^2S_2^2] \\ & + (\gamma + \frac{1}{2})\chi^2S_{2x}^2 - \gamma\chi^2\chi_x^2 + (\chi_x^2 + \chi^2S_{2x}^2 + \gamma\chi^2\chi_x^2) \end{aligned} \quad (10)$$

Again, the last part in the parantheses arises from dipole locking. The minimization of this free energy for Q just above Q_c may be carried out in the standard fashion, giving

$$\chi = \alpha \sin\left[\frac{\pi x}{w}\right], \quad S_2 = \left[\frac{3\alpha}{4\bar{Q}_c}\right] \cos\left[\frac{\pi x}{w}\right], \quad (11)$$

$$\alpha^2 = \left[\frac{8\gamma(\gamma + 1)}{f(\gamma)}\right] (\bar{Q}^2 - \bar{Q}_c^2), \quad (12)$$

where $f(\gamma) = 3\gamma^2 + 13\gamma + 6$, if dipole locking is on; and $f(\gamma) = 3\gamma^2 - 3\gamma + 2$, if dipole locking is off. In the above equations, \bar{Q}_c is given by Eq. (6a) or (6b), for the two cases, respectively. The total free energy per unit area may then be evaluated, giving

$$\begin{aligned} \bar{F} & \equiv (w/2\pi^2) \int_0^w \bar{f}_G dx \\ & = \frac{1}{2}(\gamma + 1)\bar{Q}^2 - [\gamma^2(\gamma + 1)^2/f(\gamma)](\bar{Q}^2 - \bar{Q}_c^2)^2. \end{aligned} \quad (13)$$

The current density along \hat{y} may be obtained from $j_y = (2m/\hbar)(\partial f_G/\partial Q)$. Introducing the normalized total current

$$\bar{I} \equiv (\hbar\Delta_0^2/8\pi m\Gamma_2) \int_0^w j_y dx$$

(see Ref. 11) I find

$$\bar{I} = \bar{Q} \left[(\gamma + 1) - \left[\frac{4\gamma^2(\gamma + 1)^2}{f(\gamma)} \right] (\bar{Q}^2 - \bar{Q}_c^2) \right]. \quad (14)$$

The slope $\partial \bar{I}/\partial \bar{Q}$ is seen to change from $(\gamma + 1)$ for $Q < Q_c$, to either

$$(\gamma + 1)^2(6 - 5\gamma)/(3\gamma^2 + 13\gamma + 6),$$

if dipole locking is on; or

$$-(\gamma + 1)(5\gamma^2 + 7\gamma - 2)/(3\gamma^2 - 3\gamma + 2),$$

if dipole locking is off, for $Q > Q_c$. For the weak-

coupling case $\gamma = 1$, this is a change of slope from $+2$ for $Q < Q_c$, either to $+\frac{2}{11}$ if dipole locking is on, as found already in Ref. 11; or to -10 if dipole locking is off, for $Q > Q_c$. This means that as the current is increased toward the critical value $\bar{I}_c = (\gamma + 1)\bar{Q}_c$, the textural transition is second order but very close to being first order, if $w \gg 10^{-3}$ cm, so that dipole locking is on. It is unambiguously first order, with a discontinuous change of the texture, if $w \ll 10^{-3}$ cm, so that dipole locking is effectively switched off.¹² In the latter case, of course, one must still interpret the critical current $\bar{I}_c = (\gamma + 1)\bar{Q}_c$ as the "superheating" critical current, and expect the "thermodynamic" transition to occur at a lower "thermodynamic" critical current, the exact value of which can be determined only after numerical study is completed for all $Q > Q_c$. The orbital axis \hat{l} , however, can not reorient rapidly due to the Cross-Anderson dissipation mechanism.¹³ Large hysteresis should, therefore, be readily observable near this first-order textural transition. Note that had we assumed a planar texture for $Q > Q_c$, by putting $S_2 = 0$, we would have predicted for the slope $\partial \bar{I}/\partial \bar{Q}$ just above \bar{Q}_c the value $(\gamma + 1)^2/(3\gamma + 1)$ if dipole locking is on; and the value

$$(\gamma + 1)(2\gamma^2 + \gamma + 1)/(6\gamma^2 + 3\gamma + 1)$$

if dipole locking is off, which would mean second-order transition for both cases, and for *all* values of $\gamma > 0$. (The last slope was the value previously calculated by de Gennes and Rainer.⁶) Note also that if no more textural transition is to occur for $\bar{Q} > \bar{Q}_c$, then as $\bar{Q} \rightarrow \infty$, the texture should merge continuously into the one determined in Ref. 1-4, which is $\hat{l} \equiv \hat{y}$ (except within a distance $\sim Q^{-1}$ from the walls), if dipole locking is on; and is a helical texture varying periodically along y , if dipole locking is off. In the latter case, therefore, a periodic y dependence in the texture must develop as Q increases above Q_c , either continuously from zero, or through another textural transition. Finally, we note that should strong-coupling effect be capable of making $\gamma > 1.2$ at any pressure (below the solidification pressure), then even in the wide-channel dipole-locked case, the predicted transition would be first order.

A useful remark to add is, while both χ and S_2 are first-order small quantities just above Q_c , the deviation from a planar texture, as measured by $l_z = \sin\chi \sin S_2$, has actually second-order smallness! This is why in our fluctuation analysis, I actually obtained $\lambda_z = 0$, above the Q_c of $n = 1$, $k = q = 0$. This point reveals a curious advantage of the coordinates S_1 , χ , and S_2 used in the second half of this analysis.

I have attached an Appendix B at the end of this paper in order to clarify the procedure used here to discuss a first-order phase transition, and to reconcile it with the familiar Landau's picture.

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APPENDIX A: ZERO-TEMPERATURE LIMIT OF EQ. (4)

According to Cross,⁹ the parameters appearing in the gradient energy expression can all be expressed in terms of three parameters α , β , and γ . It can be readily verified that in the limit $T \rightarrow 0$, the values of these three parameters are approximately

$$\alpha \approx \frac{1}{4} [1 - O(\epsilon^4)], \quad \beta \approx \frac{1}{2} (1 - \epsilon^2),$$

and

$$\gamma \approx 3 [\ln(2 \bar{\gamma} / \epsilon^{1/2}) - \frac{4}{3}],$$

where

$$\epsilon \equiv (\pi T / \Delta_0)^2, \quad \Delta_0 \approx 2.029 k_B T_c,$$

is the zero-temperature maximum gap over the Fermi surface, and $\ln \bar{\gamma} = C \approx 0.5772$ is the Euler's constant. Using these parameter values, it can be shown that

$$\rho_s (\hbar/2m)^2 \approx A (1 + \frac{1}{3} F_{\uparrow}^{\dagger}) [1 - O(\epsilon)],$$

$$\rho_0 (\hbar/2m)^2 \approx A (1 + \frac{1}{3} F_{\uparrow}^{\dagger})^2 \epsilon,$$

$$c (\hbar/2m) \approx \frac{1}{2} A (1 + \frac{1}{3} F_{\uparrow}^{\dagger}) (1 - \epsilon),$$

and

$$c_0 (\hbar/2m) \approx A (1 + \frac{1}{3} F_{\uparrow}^{\dagger}) [1 - (1 + \frac{1}{6} F_{\uparrow}^{\dagger}) \epsilon],$$

where

$$A \equiv \rho (\hbar/2m)^2 / (1 + \frac{1}{3} F_{\uparrow}^{\dagger}).$$

Furthermore, if dipole locking is not in effect,

$$K_s \approx \frac{1}{4} A [1 - O(\epsilon)],$$

$$K_t \approx \frac{1}{12} A \{ (5 + F_{\uparrow}^{\dagger}) - [8 + 4F_{\uparrow}^{\dagger} + \frac{1}{3} (F_{\uparrow}^{\dagger})^2] \epsilon \},$$

and

$$K_b \approx A [\ln(4 \bar{\gamma}^2 / \epsilon) - \frac{1}{12} (26 - F_{\uparrow}^{\dagger})].$$

On the otherhand, if dipole locking is in effect, then

$$K_s \approx A (\frac{5}{4} + \frac{1}{3} F_{\uparrow}^{\dagger}) - O(\epsilon),$$

$$K_t \approx \frac{1}{12} A \{ (17 + F_{\uparrow}^{\dagger} + 4F_{\uparrow}^{\dagger}) - [4 + 2F_{\uparrow}^{\dagger} + \frac{1}{3} (F_{\uparrow}^{\dagger})^2] \epsilon \},$$

and

$$K_b \approx A [\ln(4 \bar{\gamma}^2 / \epsilon) - (14 - F_{\uparrow}^{\dagger} - 4F_{\uparrow}^{\dagger})].$$

(If the Fermi-liquid parameters F_{\uparrow}^{\dagger} and F_{\uparrow}^{\dagger} are set equal to zero, the bare values of these gradient-energy parameters can be regenerated.) The values of F_{\uparrow}^{\dagger} and F_{\uparrow}^{\dagger} at various pressures between zero and the melting pressure have been given by Ref. 14. In particular, $F_{\uparrow}^{\dagger} = 15.66$, $F_{\uparrow}^{\dagger} = -0.555$ at the melting pressure, and $F_{\uparrow}^{\dagger} = 6.04$, $F_{\uparrow}^{\dagger} = -0.670$ at zero pressure. We also note from the same source that F_{\uparrow}^{\dagger} is a monotonically increasing function of pressure, while F_{\uparrow}^{\dagger} peaks near $P \approx 18$ bar with a value $-0.500 < 0$. Using these pieces of information, one can establish the following limiting behavior for Eq. (4) as $T \rightarrow 0$:

1. Dipole-locked case

In this case, we find

$$\frac{\partial \bar{Q}_c^2}{\partial \bar{k}^2} = \frac{5 - 3F_{\uparrow}^{\dagger} + 4F_{\uparrow}^{\dagger}}{12(1 + \frac{1}{3} F_{\uparrow}^{\dagger})^2 \epsilon},$$

which is $-0.0952 \epsilon^{-1}$ at the melting pressure; $-0.145 \epsilon^{-1}$ at zero pressure; and is in fact negative at all pressures between them. However, if Fermi-liquid corrections are switched off, it would become $(\frac{5}{12}) \epsilon^{-1} > 0$.

$$\frac{\partial \bar{Q}_c^2}{\partial \bar{q}^2} = \frac{15 + 4F_{\uparrow}^{\dagger}}{12(1 + \frac{1}{3} F_{\uparrow}^{\dagger})^2 \epsilon},$$

which is clearly positive at all pressures, and is also positive without Fermi-liquid corrections.

2. Dipole-unlocked case

In this case, we find

$$\frac{\partial \bar{Q}_c^2}{\partial \bar{k}^2} = -\frac{7 + 3F_{\uparrow}^{\dagger}}{12(1 + \frac{1}{3} F_{\uparrow}^{\dagger})^2 \epsilon},$$

which is clearly negative at all pressures, and is also negative without Fermi-liquid corrections.

$$\frac{\partial \bar{Q}_c^2}{\partial \bar{q}^2} = \frac{1}{4(1 + \frac{1}{3} F_{\uparrow}^{\dagger})^2 \epsilon},$$

which is clearly positive at all pressures, and is also positive without Fermi-liquid corrections.

Combining the above results, we can conclude that at sufficiently low temperatures the critical fluctuation does not correspond to $n = 1$; $k = q = 0$, and therefore the texture for $Q > Q_c$ must have a periodic variation in the plane parallel to the walls.

APPENDIX B: REMARKS ON THE PROCEDURE USED FOR STUDYING A FIRST-ORDER TRANSITION

In this work, we have studied a first-order transition in terms of an *extensive* variable \bar{Q} and the corresponding (normalized) free energy $\bar{F}(T, \bar{Q}, \eta)$ [where we have used a single variable η to symbolize the functions $\chi(x)$ and $S_2(x)$]. Another approach would be to use the conjugate *intensive* variable $\bar{T} \equiv \partial \bar{F} / \partial \bar{Q} |_{T, \eta}$, and the corresponding "Gibbs energy" $\bar{G}(T, \bar{T}, \eta) \equiv \bar{F} - \bar{T} \bar{Q}$, with the \bar{Q} dependence eliminated in favor of \bar{T} . In the present case when $\eta = 0$ is a solution for $\bar{Q} \leq \bar{Q}_c$, and \bar{F} and \bar{G} are both even functions of η , Landau's picture of a *first-order* transition tells us that for \bar{T} very near $\bar{T}_c \equiv \bar{T}(\bar{Q} = \bar{Q}_c)$ (assuming that \bar{T} increases with \bar{Q} up to \bar{Q}_c), we must have the following expansion:

$$\bar{G}(\bar{T}, \eta) \simeq \bar{G}_0(\bar{T}) + a(\bar{T}_c - \bar{T})\eta^2 + b\eta^4 + \dots, \quad (\text{B1})$$

with $a > 0$, $b < 0$, so that for \bar{T} just below \bar{T}_c the point $\eta = 0$ is a very shallow minimum, sandwiched between two very closed by maxima at $\eta = \pm [a(\bar{T}_c - \bar{T})/|b|]^{1/2}$, and at least two more (lower) minima at some still larger $|\eta|$, allowing (\bar{Q}_c, \bar{T}_c) to be identified as the superheating critical point of the $\eta = 0$ branch. If the values of \bar{G} at all of its minima *and* maxima are used to define $\bar{G}_{\text{eq}}(\bar{T})$, it would give a familiar looped structure as shown, for example, in a textbook by Reif.¹⁵ [The unstable $\eta = 0$ branch for $\bar{T} > \bar{T}_c$ must be ignored for this purpose and the discussion below. We shall see that this ignored branch actually corresponds to a *maximum* of $\bar{F}(\bar{Q}, \eta)$, while all the branches kept in defining $\bar{G}_{\text{eq}}(\bar{T})$ corresponds to \bar{Q} -dependent *minima* in $\bar{F}(\bar{Q}, \eta)$ as a function η .] Since $\bar{Q} = -\partial \bar{G}_{\text{eq}}(T, \bar{T}) / \partial \bar{T} |_{T, \eta}$, this multivalued \bar{G}_{eq} also implies an S-shaped $\bar{Q}(\bar{T})$, which may be inverted into a function $\bar{T}(\bar{Q})$ that is *single valued*, but *non-monotonic*, like the greek letter μ without the last downward stroke. This nonmonotonic character of $\bar{T}(\bar{Q})$ is actually the *necessary and sufficient* condition for a first-order transition, since it follows from the

multivalued $\bar{G}_{\text{eq}}(\bar{T})$ curve, and also allows us to regenerate the latter by the prescription

$G_{\text{eq}} = - \int \bar{Q} d\bar{T} + \text{constant}$. [The Van der Waals nonmonotonic isotherm $P(v)$, after realizing the correspondence of $(-P, v)$ with (\bar{T}, \bar{Q}) , agrees with this picture.¹⁵] The function $\bar{T}(\bar{Q})$ may also be obtained *directly* from minimizing $\bar{F}(T, \bar{Q}, \eta)$ with respect to η , and then using $\bar{T} = \partial \bar{F}_{\text{eq}}(T, \bar{Q}) / \partial \bar{Q} |_{T, \eta}$. The two ways of determining η are necessarily equivalent, because

$$\frac{\partial \bar{F}}{\partial \eta} |_{T, \bar{Q}} = \frac{\partial \bar{G}}{\partial \eta} |_{T, \bar{T}}.$$

Furthermore, for \bar{Q} very near \bar{Q}_c , when the relevant η is small [corresponding to finding the *maxima* near $\eta = 0$ in Eq. (B1)], the following expansion may be used:

$$\bar{F}(\bar{Q}, \eta) = \bar{F}_0(\bar{Q}) + a'(\bar{Q}_c - \bar{Q})\eta^2 + b'\eta^4 + \dots \quad (\text{B2})$$

Here we must have $a' > 0$, $b' > 0$, so that new minima at $\eta \neq 0$, but very near $\eta = 0$, can appear when \bar{Q} is increased beyond \bar{Q}_c . The point $\eta = 0$, then, has to change from a local minimum to a local maximum at this point. Equation (10) of this paper is precisely a more complex version of Eq. (B2), with the properties described here, so when it predicts a negative $\partial \bar{T} / \partial \bar{Q}$ for $\bar{Q} = \bar{Q}_c + \epsilon$, we can be sure that the transition is first order.

It remains to comment on why \bar{T} should be identified as an intensive variable, while \bar{Q} is an extensive variable, and not vice versa. The key lies in the possibility for the system to split into stripes of different \bar{Q} (but the same \bar{T}), all aligned perpendicular to the current. This is in analogy with the liquid-gas system which has the possibility of forming a heterogeneous mixture of the gas and liquid phases, with different molar volume, but the same pressure P . In this sense, the current-induced textural transition behaves more like a one-dimensional system, because the heterogeneity occurs only in one dimension. Of course, the system has also the possibility of splitting into stripes aligned parallel to the current. Then all the stripes must have the *same* \bar{Q} , but *different* \bar{T} . If this were the favored situation, by the physical laws governing the system, then a first-order transition would have been predicted by *treating* \bar{Q} as an *intensive variable*, i.e., I would have found $b' < 0$, $b > 0$, and $\bar{F}_{\text{eq}}(\bar{Q})$ would be multivalued. The fact that this is not found in my study, rules out this possibility.

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- ¹²We have drawn this conclusion based on our belief that the curve $\bar{T}(Q)$, although turning into a downward direction at \bar{Q}_c , will eventually turn upward again and pass the critical value \bar{T}_c from below. We are presently attempting to verify this assumption by a numerical study. Then the sign change of $\partial \bar{T} / \partial Q$ at \bar{Q}_c implying a first-order textural transition may be understood in terms of an analogy with the Van der Waals theory of the first-order liquid-gas phase transition. The main feature of that theory is the prediction of nonmonotonic isotherms (with unstable $\partial(-p)/\partial v < 0$ regions) below a critical temperature T_c . Such curves must still be corrected by the "Maxwell construction", so as to allow for the formation of a heterogeneous liquid-gas mixture due to the unstable fluctuations of the homogeneous state. Similarly here, when $\partial \bar{T} / \partial Q$ changes sign at \bar{Q}_c , the assumption of a uniform texture and Q , independent of y and z becomes unphysical for a range of values of Q near and above Q_c , so that some sort of a generalized Maxwell construction is again required. The resulting inhomogeneous states (varying in the x - y plane) can, however, be fundamentally different from the helical states discovered in Refs. 3 and 4 for the limit $Qw \gg 1$, since the latter states are characterized by intrinsic periodicities independent of the size of the container. The former states, in the case of positive surface energies between the two end states of the Maxwell line, will contain large homogeneous regions (with size dependent on the size of the container), separated by interfacial regions of minimum allowed areas and narrow intrinsic widths. However, whether the surface energy involved is positive remains to be investigated.
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